SPACE TIME FINITE ELEMENTS AND DYNAMICS

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by

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ABSTRACT

Variational methods using the least action principle are used to set up equations for the vibration of systems. Using variational methods, finite elements of structures can be formulated on the space time domain to allow calculation of transient vibratory response to initial conditions and forcing functions. This work attempts to systematize such formulations, discuss limitations in the methods, and compare the accuracy of the two simplest methods to conventional finite difference techniques. Examples are given for single degree of freedom systems, the stretched string, simply supported beam, and plate. No method superior to the Newmark β method has been developed.

LIST OF SYMBOLS

IN ORDER OF APPEARANCE.

β	AN ARBITRARY CONSTANT AFFECTING THE ACCURACY AND STABILITY OF A NUMERICAL INTEGRATION OPERATOR
M	MASS MATRIX
C	DAMPING MATRIX
K	STIFFNESS MATRIX
f(t)	FORCING FUNCTION
Δt	TIME INCREMENT
v	VELOCITY
a	ACCELERATION
x	POSITION
u	DISPLACEMENT FIELD FOR STRUCTURE
α_i	A POLYNOMIAL COEFFICIENT
$\{\alpha\}$	VECTOR OF THESE COEFFICIENTS
$\{P\}$	VECTOR OF POWER TERMS IN DISPLACEMENT POLYNOMIAL
C	MATRIX OF VECTORS P
$\{\delta_i\}$	VECTOR OF DISPLACEMENTS OF NODE i.

x_i	COORDINATE(S) of NODE i
$\{\delta\}_e$	VECTOR OF NODAL DISPLACEMENTS
N	SHAPE FUNCTION MATRIX
δ	VARIATIONAL OPERATOR
L	ACTION
E	ELASTICITY MATRIX
φ	A TIME DOMAIN INTERPOLATION FUNCTION
ψ	A SPACE DOMAIN INTERPOLATION FUNCTION
∂	A DIFFERENTIAL OPERATOR MATRIX
B	MATRIX CONVERTING NODAL DISPLACEMENT TO INTERNAL STRAINS
σ	AN INTERNAL STRESS
ϵ	AN INTERNAL STRAIN
E	ELASTICITY
V	SPATIAL VOLUME
γ	A PERTURBATION OF A VIBRATORY SYSTEM
ϕ	FREQUENCY OF VIBRATION OF NUMERICAL TECHNIQUE
λ	AMPLITUDE FACTOR OF PERTURBATION, ONE TIME STEP
ξ	NORMALIZED TIME
Q	MODAL MATRIX, AN ORTHOGONAL MATRIX

ω	FREQUENCY OF OSCILLATION
T	STRING TENSION
σ	DENSITY
M_x	MOMENT, X - AXIS
M_y	MOMENT, Y - AXIS
M_{xy}	MOMENT, TORSION
E I	ELASTICITY x MOMENT OF INERTIA
D	RIGIDITY OF PLATE

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CHAPTER I

1.1 Introduction.

In the dynamic analysis of continuous systems, analytic solutions for motion, or other dependent variables, are possible for only certain simplified boundary conditions. Equilibria of quantities associated with a differential element of the domain of the system lead to forms of differential equations : ordinary differential, partial differential, integro-differential, and of various orders. For real problems with domains of complex properties and boundary conditions, the engineer may resort to numerical solution, obtaining satisfactory information about the particular system concerned.

The finite element method descretizes a continuous system into smaller units of the domain. Properties of these units (elements) may be described by differential equations, but their interaction with respect to the dependent variables of these differential equations is obtained by finite summation over all the elements. Solution for these dependent variables is obtained by the solution of the resultant systems of algebraic or ordinary differential equations.

The usefullness of finite elements defined on the space time domain for dynamics problems has been investigated in this study. Systematic ways to set up element matrices using

Hamilton's principle have been found, and a computer program to automate construction of an element matrix from first principles has been written. Methods applicable to transient structural vibrations including damping have been investigated. Stability and accuracy comparisons with finite difference methods have been carried out. Limitations in space time element techniques have been found, and several methods affecting the techniques have been used to achiev optimum quality. Quality is a result of accuracy, stability, and necessary computer storage.

Applications to motion of single degree of freedom systems, stretched strings, beams, and plates have been carried out. Elements can be created on the space and time domains simultaneously or on the time domain alone for any structural model. The former generalized space time element has not been found advantageous for structural dynamics.

1.2 Literature survey.

The form of the fundamental differential equation of structural vibration is shared with those of electromagnetic waves, acoustics, fluid surface waves, LCR circuits, and many other linear problems [1,2]°. Direct integration in time of the equation

$$[M]\ddot{x} + [C]\dot{x} + [K]x = f(t) \qquad (1.1)$$

has commonly been done by finite difference approximations [1] which result in recursion formulae.

In 1969 the extension of finite element into the time dimension was suggested by Zienkiewicz and Parekh [3], and by Oden [4] in a paper exemplifying aspects of the general method of finite elements [5]. Oden showed the possibility of solution in a recursion scheme form, and its preferability to simultaneous solution at a variety of times becomes quickly obvious. The matrix storage becomes restricted to one layer of elements in time per time step. Simultaneously, Fried [6] and Argyris and Scharpf [7] brought out papers detailing in virtually identical fashion applications of finite element descretization, phenomena using Hamilton's principle as a suitable variational statement of properties over an element. The tech-

° Numbers in square brackets designate references at the end of the thesis.

nique is good for any mass and stiffness matrix pair, because of the possibility of decoupling [3] equations into single degree of freedom systems for which analysis can be readily derived.

Motion approximations based on a variational principle were first used by Pian and Obrien [9], with Hamilton's principle of least action in an extended form allowing variation of the displacement field at the extremities of the time interval. Pricewise continous approximation of the displacement field in the time domain and substitution into Hamiltons least action functional were carried out by Fried [6] developing space time finite elements. Geradin [10] constructed a consistent integration operator similarly, but in the extended form of Pian & Obrien [9]. An integration operator is defined as a transformation on the acceleration, velocity, and displacements vectors at time t_n, and possibly earlier times, to the acceleration, velocity, and displacement vectors at time t_{n+1} [11]. Acceleration and velocity may only be implicit for some operators. Geradin's technique was allowed to circumvent the sketchily explained boundary condition problem in the methods of Fried, and of Argyris & Scharpf. Geradin reported the first accuracy and stability analysis of the space-time technique and created an absolute stability scheme for his technique. Absolute stability means the solution, in our case displacement, will not diverge for any size of time step .

A broad review of most conventional integration operators for transient structural response is the classification study of Geradin [12], which concisely gives various techniques and includes :

Reduction of the equations of motion to a system of 2N first order equations, and their solution.

Multistep methods specialized to second order equations - many known techniques.

The variational approach.

Stabilities and accuracies of those techniques.

The time integration techniques for equation (1.1) of the finite difference method seem to fall into two classes [1]. Finite difference approximations give the velocity and accelerations at the various time steps for substitution into the governing differential equation (1.1). For example at step n

$$v_n = (x_{n+1} - x_{n-1}) / 2\Delta t \qquad (1.2a)$$

$$a_n = (x_{n+1} - 2x_n + x_{n-1}) / \Delta t^2 \qquad (1.2b)$$

lead to the well known central difference operator

$$x_{n+1} = \left[\frac{M}{\Delta t^2} + \frac{C}{2\Delta t}\right]^{-1} \left[\left(\frac{2M}{\Delta t^2} - K\right) x_n + \left(\frac{C}{2\Delta t} - \frac{M}{\Delta t^2}\right) x_{n-1}\right] \qquad (1.3)$$

upon substitution, which lacks absolute stability.

In the absence of damping, C , the class of techniques not involving the product $M^{-1} K$ at the new time step x_{n+1}, as

classified as implicit. The explicit class is exemplified by far less matrix storage space if the mass matrix is diagonal, and a far faster run on the computer. Krieg [1] exemplifies both types and shows the impossibility of finding an absolutely stable two step explicit operator. The implicit operators, however, can have this absolute stability. Newmark [9] developed an implicit method, now called the "Newmark β method" in which stability is controlled by a parameter . It reduces to Eqn. (1.5) when $\beta = 0$. A recent analysis of existing techniques is that of Bathe & Wilson [8] which shows the superiority of the Newmark β method to other common techniques. This conclusion is echoed by Geradin in his classification study [12], and by Nickell [11] in his survey, which gives a good introduction to the definitions involved. Details of Newmark's method, with attention to alternate derivation, interpretation, and starting procedure are given by Chan, Cox, and Benfield [13], and the Newmark method is employed in the NASTRAN package [14], with $\beta = 1/3$ for stability in nonlinear problems.

Different types of approximation replacing the first and second derivatives have been proposed, resulting in differing degrees of accuracy [13]. A stability proof of a four point backwards difference technique is given by Johnson [15], using root locus analysis, and absolute stability can result. Leech et al [16] did a similar work to that of Krieg [1] on the explicit operators, and reached the same conclusion ; upper bounds on time step size were unavoidable. Equivalence

of the matrix equations to decoupled problems [8] with diagonalized matrices lends explanation to the result that time step size be [illegible] the shortest eigenperiod [16]. This equivalence leads Bathe & Wilson to conclude [3] that the most critical factor affecting choice of time step size for a stable implicit method be frequency content in the forcing function. Modes above the maximum frequency will not respond in any event, and time step size choice can place them in the region attenuated by the implicit method [11]. The effect of attenuation is, in fact, beneficial for it removes the "noise" of the higher modes' responses, and can stabilize nonlinear vibration studies [14]. High frequency response is important for many wave propagation problems, however, especially where discontinuities in velocity or acceleration persist. Time steps can become so small that the central difference method with a lumped mass matrix becomes attractive [11].

In the stability study of Nickell [17] the "Gurtin Method" is developed, using a Laplace transform, a modification is developed to stabilize it, and another to remove the artificial damping of the techniques, if desired. In general, artificial damping and period dilation are induced by all implicit approximation techniques ; these are minimized in the "Gurtin Method" and the result is slightly superior to the Newmark method, $\beta = 1/4$.

Extensions of stability studies to the nonlinear transient response problem have been carried out by Mu et al [18], by Belytschko & Schoeberle [19], and a variety of works by Argyris et al [20]. Belytschko achieved an unconditionally stable algorithm for the Newmark β method, dependent on the boundedness of the internal energy of deformation, and the kinetic energy.

Forced response calculations for modally decomposed systems were the forerunners of the modern direct integration operators. The emphasis was purely on the lowest modes of the system [11]. The integration operators have a cutoff in response above a certain frequency ; for a stable method the response does not become influenced by modes above this frequency; hence the need for the stability studies finding the cutoff rate and frequencies.

Matrix storage space and speed of the algorithm are the other influence upon choice of a method. The Newmark operator is [13] in the absence of a forcing function

$$\left[M + \frac{C\,\Delta t}{2} + \beta\,\Delta t^2 K \right] x_{n+1} = \left[2M - (1-2\beta)\,\Delta t^2 K \right] x_n - \left[M - \frac{C\,\Delta t}{2} + \beta \Delta t^2 K \right] x_{n-1} \qquad (1.4)$$

It can be seen that in the absence of damping the cofactors of x_{n+1} and x_{n-1} are identical ; this means that for undamped problems the Newmark β method requires less storage (and time) than any other implicit scheme.

Finite element approximation as given by Fried [6] used position and velocity as nodal degrees of freedom, Argyris & Scharpf [7] considering these a necessity. The method allowed insertion of the usual mass and stiffness matrices created by standard finite element procedures. Oden [4] and later Ricardo [21] in a paper detailing initial condition and force application, and Kaczkowski [22] successfully employed a "linear time element" using only nodal displacement as a degree of freedom. The size of the matrices necessitated by Fried's technique and artificial damping have lead to an early dismissal of its usefullness [23,24], but the linear time element need use no more storage than the explicit finite difference techniques with a consistent mass matrix. In this thesis work however, it was found that what is achieved is merely a special case of the Newmark β method. Application of the method of space-time elements to heat conduction has been mentioned by Fried [6] and carried out by Köhler & Pittr [25] who also examine the use of time domain elements of three and four nodes. Application of space time elements to nonlinear vibration have been carried out by Argyris & Chan [26], and in other work by the Argyris group [20,27].

The boundary conditions [6,10] of Hamilton's functional limited the scope of the present work. A number of modifications to Fried's and Geradin's techniques have been developed and optimized for accuracy. More elaborate modifications could grow from this work.

CHAPTER II

THEORETICAL BACKGROUND

2.1 An element and its interpolation function.

We consider, in finite element methods for structural analysis, a displacement field we shall call u. We may also wish to consider spatial derivatives of this displacement. They are organized in a vector $\{u\}$.

$$\{u\} = \begin{Bmatrix} u \\ u' \\ u'' \\ \vdots \end{Bmatrix} \qquad (2.1)$$

We shall let the prime (') denote a spatial derivative, and and a dot ($\dot{} = \frac{d}{dt}$) denote a time derivative.

Over a finite element we normally model the displacement with a polynomial ; for example in one dimension :

$$u = a_1 + a_2 x + a_3 x^2 \quad \ldots\ldots \qquad (2.2)$$

or

$$u = \{1 \quad x \quad x^2 \quad \ldots\} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{Bmatrix} \qquad (2.3)$$

or

$$u = \{P\}\{a\} \tag{2.4}$$

where P is the vector $\{1 \quad x \quad x^2 \quad \ldots\}$

a is the vector $\{a_1 \; a_2 \; a_3 \; \ldots\}^T$ or

coefficient vector.

For the mentioned spatial derivatives, then :

$$u' = P'a \tag{2.5}$$

$$u'' = P''a$$

etc.

In general for the displacement vector $u = \{u, u', u'' \ldots\}^T$

$$\{u\} = [C] \tag{2.6}$$

Where

$$[C] = \begin{bmatrix} P \\ P' \\ P'' \\ \vdots \end{bmatrix} \tag{2.7}$$

If we insert the coordinates of an element's corner nodes in the C matrix, we get the displacement at that node ; we shall call it δ_i where i specifies the node's number.

$$\{\delta_i\} = \{u(x_i)\} \tag{2.8}$$

so :

$$\{\delta_i\} = [C(x_i)]\{\alpha\} \tag{2.9}$$

The vector of all nodal displacements for an element is called $\{\delta\}_e$

$$\{\delta\}_e = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \vdots \end{Bmatrix} = [A]\{\alpha\} \tag{2.10}$$

where :

$$[A] = \begin{bmatrix} C(x_1) \\ C(x_2) \\ C(x_3) \\ \vdots \end{bmatrix} \tag{2.11}$$

The number of polynomial terms in $\{P\}$ and therefore the number in $\{\delta\}$, and the number of terms in $\{\delta\}_e$ (degrees of freedom) are chosen to make $[A]$ a square nonsingular matrix. Therefore :

$$\{\alpha\} = [A]^{-1}\{\delta\}_e \tag{2.12}$$

$$\{u\} = [C][A]^{-1}\{\delta\}_e \tag{2.13}$$

$$\{u\} = [N]\{\delta\}_e \tag{2.14}$$

Where :

$$[N] = [C][A]^{-1} \tag{2.15}$$

is the "shape function" matrix ; each term in N is a polyno-

mial. It should be noted that :

$$u = P A^{-1} \delta_e \qquad (2.16a)$$

$$u' = P' A^{-1} \delta_e \qquad (2.16b)$$

indicating the relationship between the rows of N . If the time dimension were included in the original polynomial (2.2), it would be found that :

$$\dot{u} = \dot{P} A^{-1} \delta_e \qquad (2.17a)$$

$$\ddot{u} = \ddot{P} A^{-1} \delta_e \qquad (2.17b)$$

in which case nodes of the element would be found at different points in space-time, all the nodes not in one single flat plane in the space-time domain, otherwise the A matrix would be singular. Thus, we have a finite element of space-time. It will be found that the best procedure for calculating structural motion, as for any causual system evolving in time, is to use elements of constant shape in the spatial domain, and defined by flat hyperplanes perpendicular to the time axis. A recursion form of solution for the state in time results.

2.3 The variational principle, for vibrations.

Hamilton's Principle states :

$$\delta L = \delta \int_{t_1}^{t_2} (T - V)\, dt = 0 \qquad (2.18)$$

For a more detailed exposition see Lanczos [28], or Courant [29].

Direct insertion of mass and stiffness matrices in the kinetic and potential energy terms of the variational statement were used by Fried [6], Argyris & Scharpf [7], Zienkiewicz [2], Geradin [11], and Oden [30], and for the linear time element by Ricardo [21]. We attempt to provide an automated technique for the procedure.

Two approaches exist ; one substitutes directly into Hamilton's functional (2.18) as above ; the other considers the equilibrium of net forces acting on an element and assumes minimum action, in the manner of Kaczkowski [22] or Oden's example [4].

In the first case :

$$L = \frac{1}{2} \int_{t_1}^{t_2} (\dot{u}^T [M] \dot{u} - u^T [K] u)\, dt = 0 \qquad (2.19)$$

where

$$u = N \delta_e \qquad (2.20a)$$

$$\dot{u} = \dot{N} \delta_e \qquad (2.20b)$$

and

M = mass matrix

K = stiffness matrix

$\dot{u}$ = collected velocity terms of $\{u\}$, or derived via $[N]$

u = collected position terms of $\{u\}$.

A matrix multiplication producing the integrand is :

$$\{u \quad \dot{u}\} \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} u \\ \dot{u} \end{Bmatrix} \tag{2.21}$$

and we shall define

$$[E] = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \tag{2.22}$$

At this point it has become obvious that the shape function matrix $[N]$ includes terms for time dependent displacement. To allow the above expressions, the time dependent part of a term N_{ij} must have been separable from the spatial part, if K and M are for descretized structures.

$$N_{ij} = \varphi(t) \cdot \psi(x,y,\ldots) \tag{2.23}$$

We can express

$$\begin{Bmatrix} u \\ \dot{u} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \frac{\partial}{\partial t} \end{Bmatrix} u = \partial u \tag{2.24}$$

so we define

$$\{\partial\} = \begin{Bmatrix} 1 \\ \frac{\partial}{\partial t} \end{Bmatrix} \tag{2.25}$$

We see $\{\partial\}$ is an operator matrix.

$$\partial u = \partial N \delta_e = B \delta_e \tag{2.26}$$

so

$$B = \partial N \tag{2.27}$$

and in general, B converts nodal displacements $\{\delta\}_e$ into internal movements, or "strains". Equilibrium within the element is calculated with respect to these "strains". Behaviour of the entire structure is represented with the algebraic equations created upon finite summation over all the nodes. This implies that u and $\dot{u}$ are strains in the space time domain. This further implies that E is an "elasticity" matrix by analogy with conventional finite element technique. The integrand of (2.18) becomes :

$$\delta_e^T B^T E B \delta_e \tag{2.28}$$

Upon taking variational derivatives, we get :

$$\int_{t_1}^{t_2} (B^T E B) \, dt \, \delta_e = 0 \qquad (2.29)$$

Since we are dealing with general stiffness and mass matrices, the terms $\{\delta_i\}$ may include spatial derivatives, with the result that we may represent the stretched string, beam, plate, or any structure.

The second approach is that of the general space-time finite element, which does not use any predetermined mass or stiffness matrix. The equilibria over an element, which in general was used in deriving the partial differential equation describing its vibration (for example , plate vibrations), are organized and put into a matrix formation, this time including mass acceleration reaction terms. Integration by parts with respect to time, and taking variational derivations lead us to a repeat of the least action principle. Again

$$u = [N] \delta_e \qquad (2.14)$$

and in the general case the polynomial terms in $[N]$ can have time, t, and the spatial dimensions, x, y, ... mixed. Strains in the space time domain (displacements in a differential element which lead to forces) are defined by

$$\epsilon = \partial N \, \delta_e \tag{2.30a}$$

$$\epsilon = B \, \delta_e \tag{2.30b}$$

Conceptually, strains are related to "forces" acting on the element (stresses) :

$$\sigma = E \epsilon = E B \delta_e \tag{2.31}$$

Stresses now include inertial reaction to motion ; this explains presence of the mass or density term in the matrix $[E]$. The taking of the partial integration with respect to time [26], [20] can be implied° by associating the mass or density term with $\ddot{u}$ and setting up the sign properly. We finally end up with

$$\delta \int_{t_1}^{t_2} \int_V \sigma^T \epsilon \, dt \, dV = 0 \tag{2.32}$$

V being the volume in space of the element.

The two approaches are equivalent, the first being a special case of the second ; when terms in N are separable into time and space cofactors, the volume integral is taken first, and rows and columns are properly arranged, the mass

° The by-product [26,10] in $M\dot{u}\,\delta u\Big|_{t_1}^{t_2}$ is the source of present limitations with this work.

and stiffness matrices, $\left[M\right]$ and $\left[K\right]$, come out and we have the specialized first case.

The matrix $\int_{t_1}^{t_2} B^T E B \, dt$ of the first case has the structure

$$\begin{array}{c} \\ t_1 \\ t_2 \end{array} \begin{array}{c} \begin{array}{cc} t_1 & t_2 \end{array} \\ \begin{bmatrix} K_1 & T_2 \\ T_1 & K_2 \end{bmatrix} \end{array} \tag{2.33}$$

when the nodes for the degrees of freedom are organized

$$\delta_e = \begin{Bmatrix} \delta_{t_1} \\ \delta_{t_2} \end{Bmatrix} . \tag{2.34}$$

Given the specified layering of elements with respect to the time axis, upon assembly we get a matrix of the form :

$$\begin{bmatrix} K_1 & T_2 & 0 & 0 & \\ T_1 & K_{1+2} & T_2 & 0 & \\ 0 & T_1 & K_{1+2} & T_2 & \\ 0 & 0 & T_1 & \ddots & \ddots \\ & \cdots & & \ddots & \vdots \end{bmatrix} \begin{Bmatrix} \delta_{t_1} \\ \delta_{t_2} \\ \delta_{t_3} \\ \delta_{t_4} \\ \vdots \end{Bmatrix} \tag{2.35}$$

Which leads to the following recursion scheme :

starting : $\delta_2 = -T_2^{-1} K_1 \delta_1$ (2.36)

generally : $\delta_i = -T_2^{-1} K_{1+2} \delta_{i-1} - T_2^{-1} T_1 \delta_{i-2}$ (2.37)

When velocities were incorporated as nodal degrees of freedom by Fried [6] and other subsequent workers, it was as follows :

$$u = \left[N(t)\right] \begin{Bmatrix} \delta_e \\ \dot{\delta}_e \end{Bmatrix} \tag{2.38}$$

where $\begin{Bmatrix} \delta_e \\ \dot{\delta}_e \end{Bmatrix}$ is displacement and spatial derivative is $d/dt \ (\delta_e)$

$$\dot{u} = \left[\dot{N}(t)\right] \begin{Bmatrix} \delta_e \\ \dot{\delta}_e \end{Bmatrix} \tag{2.40}$$

Terms in $\left[N(t)\right]$ come from Hermite interpolation.

When put in matrix formation Hamilton's functional looks as follows :

$$\delta \int_{t_1}^{t_2} \{ \delta_e \quad \dot{\delta}_e \} \left[\begin{Bmatrix} 1 \\ \frac{\partial}{\partial t} \end{Bmatrix} \left[N\right] \right]^T \begin{bmatrix} -K & 0 \\ 0 & +M \end{bmatrix} \left[\begin{Bmatrix} 1 \\ \frac{\partial}{\partial t} \end{Bmatrix} \left[N\right] \right] \begin{Bmatrix} \delta \\ \dot{\delta}_e \end{Bmatrix} dt \tag{2.41}$$

2.3 Development of space time finite element matrices.

Rearrangement of rows and columns is normally needed for a recursion scheme ;

$\{\delta_e \quad \dot{\delta}_e\}^T$ are reorganized to give $\{\delta_{t=1} \quad \dot{\delta}_{t=1} \quad \delta_{t=2} \quad \dot{\delta}_{t=2}\}^T$.

For further development we shall consider M and K to represent a point mass on a spring (single degree of freedom system) ; the results can be generalized [8] to allow these to be matrices.

Following Geradin [10], we can create the matrix defined by expression (2.41), partitionable as

$$\left[\begin{array}{c:c} K_1 & T_2 \\ \hdashline T_1 & K_2 \end{array}\right] \left\{\begin{array}{c} \delta_1 \\ \dot{\delta}_1 \\ \hdashline \delta_2 \\ \dot{\delta}_2 \end{array}\right\} \qquad (2.42)$$

With $\{\delta_1 \quad \dot{\delta}_1\}^T$ as an initial condition, there is sufficient information to predict

$$\delta_2 = -T_2^{-1} K_1 \quad \delta_1 \qquad (2.43a)$$

or

$$\delta_2 = -K_2^{-1} T_1 \quad \delta_1 \qquad (2.43b)$$

Geradin uses equation (2.43b) as his operator. Because of the symmetry of time (2.43a) can also be used. Technique (2.43a) has numerically induced negative damping, while (2.43b) has positive damping. Geradin proceeds with a stability analysis in the following manner :

A perturbation γ would oscillate :

$$\gamma = \alpha e^{i\phi t} \tag{2.44a}$$

For the numerical technique

$$\gamma = \alpha e^{i\phi(n\Delta t)} \qquad n = 1, 2, 3, \ldots \tag{2.44b}$$

where Δt is time step size ; n is number of steps.

$$\gamma = \alpha \lambda^n \tag{2.45}$$

where $$\lambda = e^{i\phi\Delta t} \tag{2.46}$$

Substituting into (2.43b)

$$\alpha \cdot \lambda^{n+1} = -K_2^{-1} T_1 \alpha \lambda^n \tag{2.47}$$

yielding

$$\lambda I = -K_2^{-1} T_1 \tag{2.48}$$

or $$K_2^{-1} T_1 + \lambda I = 0 \tag{2.49}$$

Solving for the magnitude of $e^{i\phi\Delta t}$ tells us if the oscillation diverges or is stable ; hence we solve the

eigenvalue problem

$$\det (K_2^{-1} T_1 + \lambda I) = 0 \qquad (2.50)$$

for the magnitude of λ.

Geradin did not consider (43a). The solution can be set up as :

$$K_1 \delta_1 + T_2 \delta_2 = 0 \qquad (2.51)$$

$$T_1 \delta_1 + K_2 \delta_2 = 0 \qquad (2.52)$$

We can take a linear combination of the two

$$(\beta K_1 + (1-\beta) K_2) \delta_1 + (\beta T_2 + (1-\beta) K_2) \delta_2 = 0 \qquad (2.53)$$

where β is an arbitrary weight. Then, instead of equation (2.50) we get :

$$\det \left| (\beta T_1 + (1-\beta) K_2)^{-1} (\beta K_1 + (1-\beta) T_1) + \lambda I \right| = 0 \qquad (2.54)$$

When $\beta = 1/2$, within the range of stability, no induced damping is present, and the accuracy is excellent, as is seen upon solving this equation for a range of values Δt.

To develop a "linear time element" with only displacement "u" at the nodes, we shall normalize the element on the interval (-1 to +1). Then we set

$$\xi = \left(t - \left(t_1 + t_2 \right) / 2 \right) / \left(t_1 - t_2 \right) \qquad (2.55)$$

where t_1 and t_2 are the time extremities on the time interval.

$$2a = t_2 - t_1 \qquad (2.56a)$$

$$\Delta t = 2a \qquad (2.56b)$$

$$-1 \leq \xi \leq +1 \qquad (2.57)$$

Consequently

$$\frac{d}{dt} = \frac{1}{a} \frac{d}{d\xi} \qquad (2.58)$$

$$\int dt = a \int d\xi \qquad (2.59)$$

This leads us to the interpolation

$$H = \left[\left(1-\xi \right) \quad \left(1+\xi \right) \right] \qquad (2.60)$$

Substitution into Hamilton's functional (2.41) leads to

$$\int_{-1}^{+1} \begin{bmatrix} (1-\xi) & (1+\xi) \\ -\frac{1}{a} & +\frac{1}{a} \end{bmatrix} \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} (1-\xi) & (1+\xi) \\ -\frac{1}{a} & +\frac{1}{a} \end{bmatrix} (a\, d\xi) \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{2Ka}{3} - \frac{M}{2a} & \frac{Ka}{3} + \frac{M}{2a} \\ \frac{Ka}{3} + \frac{M}{2a} & \frac{2Ka}{3} - \frac{M}{2a} \end{bmatrix} \tag{2.61}$$

$$= \begin{bmatrix} K_1 & T_2 \\ T_1 & K_2 \end{bmatrix} \tag{2.62}$$

The recursion schemes resulting from assembly are subject to the problems documented in the text by Fox & Mayers [31].

In the case of no damping, we see that

$$T_1 = T_2 \tag{2.63}$$

so our recursion scheme is

$$\delta_i = \left(-T_2^{-1} K_{1+2} \right) \delta_{i-1} - \delta_{i-2} \tag{2.64}$$

The introduction of viscous damping brings the attendant problem of justification of its use in the variational principle. There is no general theoretical model for structural damping ; in general engineers try to approximate it with a viscous damping yielding equivalent energy dissipation.

Simple consideration shows the damping force to be

$$- C \dot{u} \tag{2.65}$$

In the differential equation describing a vibratory system

$$M \ddot{u} + C \dot{u} + K u = F \tag{2.66}$$

ignoring the force, for the integrand of Hamilton's functional we get

$$-M \ddot{u} \delta u + C \dot{u} \delta u + K u \delta u \tag{2.67}$$

which is often given in the form

$$\delta (1/2 \dot{u}^T M \dot{u} - u^T C \dot{u} - 1/2 u^T K u) \tag{2.68}$$

with no derivative taken with respect to $C \cdot \dot{u}$ in the term $u^T C \dot{u}$.

This nondifferentation of C $\dot{a}$ is noted by Fried [6], and taken up in a paper by Levinson [32].

Production of the terms (2.67) and (2.68) will come from defining the "elasticity" matrix as

$$E = \begin{bmatrix} -K & -C \\ 0 & M \end{bmatrix} \tag{2.69}$$

at this point the sign of E does not really matter. Hence, nonsymmetric finite element matrices result.

In general, $X^T X$ is a symmetric matrix. We get $B^T E B$ and if E is non-symmetric, we lack the symmetric result.

For the linear time element, taking the integrals, we get :

$$\begin{bmatrix} \frac{2Ka}{3} - \frac{C}{2} - \frac{M}{2a} & \frac{Ka}{3} + \frac{C}{3} + \frac{M}{2a} \\ \frac{Ka}{3} - \frac{C}{2} + \frac{M}{2a} & \frac{2Ka}{3} + \frac{C}{2} - \frac{M}{2a} \end{bmatrix} \tag{2.70}$$

The similar approaches for the time element modelling position and velocities at the nodes are documented by Fricd, Argyris & Scharpf, Zienkiewicz, Oden, and Geradin. They have

$$\delta_e = \begin{Bmatrix} u_1 \\ \dot{u}_1 \\ u_2 \\ \dot{u}_2 \end{Bmatrix} \tag{2.71}$$

Upon the case of the appropriate Hermitian interpolation functions in properly organized expression (2.41), the finite element matrix, modified by Geradin's inclusion of the boundary terms

$$M\,\dot{u}\,\delta u\,\Big|_{t_1}^{t_2} \tag{2.72}$$

is

$$\begin{bmatrix}
\left(\frac{156}{420}\Delta t K-\frac{1}{2}C-\frac{6}{5\Delta t}M\right) & \left(\frac{22}{420}\Delta t^2K+\frac{\Delta t}{10}C-\frac{11}{10}M\right) & \left(\frac{54}{420}\Delta t K+\frac{1}{2}C+\frac{6}{5\Delta t}M\right) & \left(\frac{-13}{420}\Delta t^2K-\frac{\Delta t}{10}C-\frac{1}{10}M\right) \\
\left(\frac{22}{420}\Delta t^2K-\frac{\Delta t}{10}C-\frac{1}{10}M\right) & \left(\frac{4}{420}\Delta t^3K-\frac{2}{15}\Delta t M\right) & \left(\frac{13}{420}\Delta t^2K-\frac{\Delta t}{10}C+\frac{1}{10}M\right) & \left(\frac{-3}{420}\Delta t^3K+\frac{\Delta t^2}{60}C+\frac{\Delta t}{30}M\right) \\
\left(\frac{54}{420}\Delta t K-\frac{1}{2}C+\frac{6}{5\Delta t}M\right) & \left(\frac{13}{420}\Delta t^2K+\frac{\Delta t}{10}C+\frac{1}{10}M\right) & \left(\frac{156}{420}\Delta t K+\frac{1}{2}C-\frac{6}{5\Delta t}M\right) & \left(\frac{-22}{420}\Delta t^2K+\frac{\Delta t}{10}C+\frac{11}{10}M\right) \\
\left(\frac{-13}{420}\Delta t^2K+\frac{\Delta t}{10}C-\frac{1}{10}M\right) & \left(\frac{-3}{420}\Delta t^3K-\frac{\Delta t^2}{60}C+\frac{\Delta t}{30}M\right) & \left(\frac{-22}{420}\Delta t^2K-\frac{\Delta t}{10}C+\frac{1}{10}M\right) & \left(\frac{4}{420}\Delta t^3K-\frac{2}{15}\Delta t M\right)
\end{bmatrix} \tag{2.73}$$

2.- Development of a damping matrix.

At this point we shall digress to develop a damping matrix that will allow study of the space time finite element techniques. Given the mass and stiffness matrices, we can develop the modal matrix Q. For any eigenvalue,

$$\det (M^{-1} K - \lambda I) = 0 \qquad (2.74)$$

there will be an associated eigenfunction (displacement vector) satisfying the above equation. The full set of vectors, properly normalized, gives the square matrix Q.

Then the modal matrix can be used to decouple the equations :

$$Q^T M Q = \hat{M} \quad \text{a diagonal matrix} \qquad (2.75a)$$

$$Q^T K Q = \hat{K} \quad \text{a diagonal matrix} \qquad (2.75b)$$

If the damping matrix were proportional to either K or M

$$C \propto M \qquad (2.76a)$$

or $$C \propto K \qquad (2.76b)$$

$$Q^T C Q = \hat{C} \quad \text{which is a diagonal matrix.} \qquad (2.77)$$

$$C = Q^{T^{-1}} \; C_* \; Q^{-1} \tag{2.78}$$

since, for a single degree of freedom system

$$c = 2 \xi \sqrt{k\, m} \tag{2.79}$$

ξ = damping rate

C_* can be constructed from M_* and K_* to give the desired degree of damping in any mode. Then C can be developed in full form with equation (2.78).

2.5 Stabilities and accuracies.

We shall now do a stability analysis of the linear time element technique. For the recursion formula

$$\delta_i = F(M, K, \Delta t)\,\delta_{i-1} - \delta_{i-2} \tag{2.80}$$

we have obtained :

$$\delta_i = -\left[\left(\frac{K\Delta t}{6} + \frac{M}{\Delta t}\right)^{-1}\left(\frac{2K\Delta t}{3} - \frac{2M}{\Delta t}\right)\right]\delta_{i-1} - \delta_{i-2} \tag{2.81}$$

As noted in the first chapter, equivalence of the problem to the modaly decomposed problem allows study of stability and accuracy to the technique with a single degree of freedom system. Proceeding as in equation (2.46) and substituting in equation (2.81) we get

$$\lambda^2 = F\lambda - 1 \tag{2.82a}$$

Hence

$$\lambda = \frac{+F \pm \sqrt{F^2 - 4}}{2} \tag{2.82b}$$

Analysis of (2.36b) yields

$$|\lambda| = \begin{cases} 1 & \text{for } -2 \le F \le +2 \\ >1 & \text{for } F < -2 \end{cases} \qquad (2.85)$$

This corresponds to Δt being roughly 1/2 of the period of the system, above which the solution diverges. Since the analysis is for any mode, Δt must be less than half the shortest natural period for the system.

The technique implied by equation (2.44a) can further be used to find the periodicity resulting from the numerical technique. From the theory of complex variables, where

$$\lambda = e^{i\phi\Delta t} \qquad (2.46)$$

we get

$$|\mathrm{Re}\,\lambda| = |\cos\phi\Delta t| \qquad (2.84a)$$

$$|\mathrm{Im}\,\lambda| = |\sin\phi\Delta t| \qquad (2.84b)$$

So

$$\phi = \frac{1}{\Delta t} \tan^{-1} \left| \frac{\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda} \right| \qquad (2.35)$$

This technique is just as true for any order of model, as in expression (2.50). Periodicity errors of the various methods will be investigated in the next chapter.

2.6 Analogy with finite difference methods.

The central difference technique (1.3) is an obvious alternative to (2.81), and in the absence of damping is

$$\delta_i = (2I - \Delta t^2 \; M^{-1} \; K) \, \delta_{i-1} - \delta_{i-2} \qquad (2.86)$$

The two differ in accuracy properties (periodicity error) but are of exact amplitude ($\lambda = 1$) within their respective ranges of stability, both having Δt_{max} near 1/2 of the natural period of the system. Obviously (2.86) is superior when M is diagonal, because of the resulting matrix sizes.

The following, however, we note :

Newmark's β method (1.4), in the absence of damping and forcing function, can be written :

$$\left(M+\beta\Delta t^2 K\right) \delta_i = 2 \left(M-\Delta t^2(1/2-\beta)K\right) \delta_{i-1} - \left(M+\beta\Delta t^2 K\right) \delta_{i-2} \qquad (2.87a)$$

The linear time element method, equation (2.81), can be put in the form :

$$\left(M + 1/6\ \Delta t^2\ K\right) \delta_i = 2\ \left(M - \Delta t^2\ (1/3)\ K\right) \delta_{i-1} - \left(M + 1/6\ \Delta t^2\ K\right) \delta_{i-2} \qquad (2.87b)$$

and the central difference technique, equation (1.3), is :

$$\left(M\right) \delta_i = 2\ \left(M - \Delta t^2\ (1/2)\ K\right) \delta_{i-1} - \left(M\right) \delta_{i-2} \qquad (2.87c)$$

The last two are merely special cases of the first (β = 1/6 and 0 respectively) and the work was covered, implicitly, eighteen years ago. The minimum periodicity error possible seems with a hybrid between (2.87b) and (2.87c) ; this is simply β = 1/12.

Chan et.al. [13] note that the significance of β can be shown by its correspondence to the variation in acceleration during the time interval :

β = 1/4 corresponds to a constant acceleration during the time interval equal to the mean of the initial and final accelerations ;

β = 1/6 corresponds to a linear variation of acceleration in the interval ;

$\beta = 0$ will be similar to the central difference method.

At this point it is informative to record the Houbolt method, as studied by Johnson [15]

$$\left(\frac{2M}{\Delta t^2} + K \right) \delta_i = F(t) + \left(\frac{M}{\Delta t^2} \right) \cdot (5 \delta_{i-1} - 4 \delta_{i-2} + \delta_{i-3}) \tag{2.88}$$

which, besides requiring a little more vector manipulation has less accuracy than the Newmark β method [12], and the "Gurtin method" [17] in its stabilized form :

$$\left(M + \frac{\Delta t^2}{12} K \right) \delta_i = \frac{\Delta t^2}{12} \left(F_i + 1/2 \, F_{i-1} \right) + \left(M + \frac{\Delta t^2}{24} K \right) \delta_{i-1} + \left(1/2 \; M \right) \delta_{i-2} \tag{2.89}$$

This cannot be used as written ; the time step taken, $(t_i - t_{i-1})$ is half of Δt as written ; the reader should consult reference [17] of Nickell's work [17]. The artificial damping of the technique can be reduced with a method in Nickell's paper, but the accuracy increase over Newmark's β method is too slight to justify the increase in matrix storage space, in the case of no damping.

2.7 Modification of Geradin's method.

The variational method conventionally assumes no variation of displacement at the ends of the time interval. This means that the term $m\dot{x}\delta x\Big|_{t_1}^{t_2}$ drops out of the expression (2.18) in the work of Fried [6] and of Argyris & Scharpf [7], who use this as a reason for dropping lines 1 and 3 from expression (2.73) and being left with an operator in the shape

$$\begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{Bmatrix} \delta_i \\ \dot{\delta}_i \end{Bmatrix} + \begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{Bmatrix} \delta_{i-1} \\ \dot{\delta}_{i-1} \end{Bmatrix} \qquad (2.90)$$

Geradin modified the method by keeping (2.72) and used expression (2.43b) which has the form of (2.90) above, which we shall rewrite as

$$\left[X\right]\left\{\delta_i\right\} + \left[Y\right]\left\{\delta_{i-1}\right\} = 0 \qquad (2.92)$$

in the homogeneous case.

Matrix X is $2N \times 2N$ for a structure of N degrees of freedom, after constraints. The following method is preferable to inversion of X to solve for $\left\{\delta_i\right\}$.

We represent (2.90) as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} \delta_i \\ \dot{\delta}_i \end{Bmatrix} + \begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{Bmatrix} \delta_{i-1} \\ \dot{\delta}_{i-1} \end{Bmatrix} = 0 \qquad (2.93)$$

So :

$$A\ \delta_i + B\ \dot{\delta}_i + E\ \delta_{i-1} + F\ \dot{\delta}_{i-1} = 0 \qquad (2.94a)$$

$$C\ \delta_i + D\ \dot{\delta}_i + G\ \delta_{i-1} + H\ \dot{\delta}_{i-1} \quad 0 \qquad (2.94b)$$

$$\delta_i = -A^{-1}\ B\ \dot{\delta}_i - A^{-1}\ E\ \dot{\delta}_{i-1} - A^{-1}\ F\ \dot{\delta}_{i-1} \qquad (2.95)$$

Substituting in (2.94b) :

$$C\ (\ -A^{-1}\ B\ \dot{\delta}_i - A^{-1}\ E\ _{i-1} - A^{-1}\ F\ \dot{\delta}_{i-1}\ )$$

$$+\ D\ \dot{\delta}_i + G\ \delta_{i-1} + H\ \dot{\delta}_{i-1} = 0 \qquad (2.96)$$

Then :

$$\dot{\delta}_i = -\ (\ D - CA^{-1}\ B\ )^{-1}\ (\ G - CA^{-1}\ E\ )\ \delta_{i-1}$$

$$-\ (\ D - CA^{-1}\ B\ )^{-1}\ (\ H - CA^{-1}\ F\ )\ \dot{\delta}_{i-1} \qquad (2.97)$$

If we define

$$W = -\left(D - CA^{-1} B \right)^{-1} \left(G - CA^{-1} E \right) \qquad (2.98a)$$

$$X = -\left(D - CA^{-1} B \right)^{-1} \left(H - CA^{-1} F \right) \qquad (2.98b)$$

Then

$$\dot{\delta}_i = W \delta_{i-1} + X \dot{\delta}_{i-1} \qquad (2.99)$$

Then :

$$A \delta_i + B W \delta_{i-1} + B X \dot{\delta}_{i-1} + E \delta_{i-1} + F \dot{\delta}_{i-1} = 0 \qquad (2.100)$$

$$\delta_i = -A^{-1} \left(B W + E \right) \delta_{i-1} - A^{-1} \left(B X + F \right) \dot{\delta}_{i-1} \qquad (2.101)$$

Defining

$$Y = -A^{-1} \left(B W + E \right) \qquad (2.102a)$$

$$Z = -A^{-1} \left(B X + F \right) \qquad (2.102b)$$

we get $$\delta_i = Y \delta_{i-1} + Z \dot{\delta}_{i-1} \qquad (2.103a)$$

$$\dot{\delta}_i = W \delta_{i-1} + X \dot{\delta}_{i-1} \qquad (2.103b)$$

and the total work is :

Invert (A) (banded symmetric, N x N)

Invert ($D - CA^{-1} B$) (full matrix N x N)

Several matrix multiplications.

W, X, Y, and Z are N x N full matrices.

This is much preferable to inverting the 2N x 2N matrix in (2.90). The final storage occupied is the same : (2N x 2N), four times that of the Newmark β method without damping, if no Cholesky decomposition methods° are used. In a computer of finite size, this limits space time finite element applications to only the smaller problems, if relative accuracies rule in their favour.

In the next chapter the accuracy of this order of space time finite element is explored.

° A symmetric matrix can be broken ($A = LL^T$) where L is a triangular matrix ; only L need be stored ; if A is also banded, L is banded.

A^{-1} is in general full. Back transformation is done with respect to L, rather than inversion.

2.8 Finite difference modification of Geradin's method.

The method (2.90) (and its analogy,(2.103)) can be subject to the following modification :

The term $\dot{\delta}_{i-1}$, and the term $\dot{\delta}_i$ can be represented by finite difference approximations based on δ_i, δ_{i-1}, δ_{i-2}, and the set of matrices stored becomes only half the present number.

We derive the finite difference equations :

The 3 points can be fit by a quadratic :

$$y = a + bx + cx^2 \qquad (2.104a)$$

or $$\delta = a + bt + ct^2 \qquad (2.104b)$$

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{bmatrix} 1 & -t & t^2 \\ 1 & 0 & 0 \\ 1 & t & t^2 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \qquad (2.105)$$

if they are equally spaced, and the points are located at $-t$, 0, $+t$ respectively.

Then

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1}{2t} & 0 & \frac{1}{2t} \\ \frac{1}{2t^2} & \frac{-1}{t^2} & \frac{1}{2t^2} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} \qquad (2.106)$$

That is

$$a = \delta_2 \tag{2.107a}$$

$$b = -\frac{1}{2t}\delta_1 + \frac{1}{2t}\delta_3 \tag{2.107b}$$

$$c = \frac{1}{2t^2}\delta_1 - \frac{1}{t^2}\delta_2 + \frac{1}{2t^2}\delta_3 \tag{2.107c}$$

Then

$$\dot{\delta} = b + 2c \tag{2.108}$$

So

$$\dot{\delta}_1 = -\frac{3}{2t}\delta_1 + \frac{2}{t}\delta_2 - \frac{1}{2t}\delta_3 \tag{2.109a}$$

$$\dot{\delta}_2 = -\frac{1}{2t}\delta_1 + \frac{1}{2t}\delta_3 \tag{2.109b}$$

$$\dot{\delta}_3 = \frac{1}{2t}\delta_1 - \frac{2}{t}\delta_2 + \frac{3}{2t}\delta_3 \tag{2.109c}$$

Let A, C, B, D represent the terms in any row, or combination of rows in (2.73) ; we have

$$A\,\delta_{i-1} + C\,\dot{\delta}_{i-1} + B\,\delta_i + D\,\dot{\delta}_i = 0$$

Two families of finite difference approximation can result.

a. In one δ_i is equivalent to δ_3 above.

b. In the other δ_i is equivalent to δ_2 above, and the approximations are made accordingly, and the terms regrouped to solve for δ_i. An equation of the form

$$\delta_i = X\,\delta_{i-1} + Y\,\delta_{i-2} \tag{2.110}$$

will result, X and Y are N x N matrices.

Hybrids of the above two types have also been tried, in this work, for optimal accuracy while remaining stable.

Geradin [10] developed an absolute stability scheme for his technique by finding displacement at a later time $t_i + \alpha\Delta t$, α is arbitrary.

Using equation (2.92) we modify to get

$$X'\,\delta_{i+\alpha} + Y'\,\delta_i = 0 \tag{2.111}$$

with

$$X' = X(\alpha\Delta t) \tag{2.112a}$$

$$Y' = Y(\alpha\Delta t) \tag{2.112b}$$

Placing the Hermitian interpolation functions in matrices properly, Geradin gets

$$A_1\,\delta_{i+1} + A_0\,\delta_i = \delta_{i+\alpha} \tag{2.113}$$

and the stability equation for this modification can take the form

$$\det \left| \lambda X' A_1 + (Y' - X' A_0) \right| = 0 \qquad (2.114)$$

This feature was added to the modification of Geradin's technique in this work.

A similar move to develop absolute stability for equation (2.110) is to interpolate from δ_{i+1} , δ_i , δ_{i-1} to $\delta_{i+\alpha}$, δ_i , $\delta_{i-\alpha}$. This is done by substituting $+\alpha t$, 0 , $-\alpha t$ in equation (2.108). We then rearrange to again solve for δ_{i+1} in terms of δ_i , δ_{i-1} , or δ_i from δ_{i-1} , δ_{i-2} , (there being no difference).

We get

$$\dot{\delta}_{i+\alpha} = \frac{2\alpha+1}{2\Delta t} \delta_{i+1} - \frac{2\alpha}{\Delta t} \delta_i + \frac{2\alpha-1}{2\Delta t} \delta_{i-1} \qquad (2.115a)$$

$$\dot{\delta}_{i-} = -\frac{2\alpha-1}{2\Delta t} \delta_{i+1} + \frac{2\alpha}{\Delta t} \delta_i - \frac{2\alpha+1}{2\Delta t} \delta_{i-1} \qquad (2.115b)$$

When $\alpha = 1$, we get equation (2.109).

Additionally

$$\delta_{i+\alpha} = \delta_{i+1} \left(\frac{\alpha}{2} + \frac{\alpha^2}{2} \right) + \delta_i \ (1 - \alpha^2) + \delta_{i-1} \left(-\frac{\alpha}{2} + \frac{\alpha^2}{2} \right) \qquad (2.116a)$$

$$\delta_{i-\alpha} = \delta_{i+1}\left(\frac{\alpha}{2}+\frac{\alpha^2}{2}\right) + \delta_i\,(1-\alpha^2) + \delta_{i-1}\left(\frac{\alpha}{2}+\frac{\alpha^2}{2}\right) \qquad (2.116b)$$

The two families are now

$$a.\quad B\,\delta_{i+\alpha} + D\,(\dot{\delta}_{i+\alpha}) + A\,\delta_i + C\,\dot{\delta}_i = 0 \qquad (2.117a)$$

$$b.\quad B\,\delta_i + D\,\dot{\delta}_i + A\,\delta_{i-\alpha} + C\,\dot{\delta}_{i-\alpha} = 0 \qquad (2.117b)$$

Substitutions lead to equations in δ_{i+1} , δ_i , δ_{i-1} ; we shift to δ_i , δ_{i-1} , δ_{i-2} , and get :

Case a :

$$\delta_i\left(B\left(\frac{\alpha}{2}+\frac{\alpha^2}{2}\right) + D\left(\frac{2\alpha+1}{2t}\right) + C\left(\frac{1}{2t}\right)\right)$$
$$+\delta_{i-1}\left(B\,(1-\alpha^2) + D\left(-\frac{2\alpha}{2t}\right) + A\right)$$
$$+\delta_{i-2}\left(B\left(-\frac{\alpha}{2}+\frac{\alpha^2}{2}\right) + D\left(\frac{2\alpha-1}{2t}\right) - C\left(\frac{1}{2t}\right)\right) = 0 \qquad (2.118a)$$

Case b :

$$\delta_i\left(A\left(-\frac{\alpha}{2}+\frac{\alpha^2}{2}\right) + D\left(\frac{1}{2t}\right) - C\left(\frac{2\alpha-1}{2t}\right)\right)$$
$$+\delta_{i-1}\left(B + A\,(1-\alpha^2) + C\left(\frac{2\alpha}{t}\right)\right)$$
$$+\delta_{i-2}\left(A\left(\frac{\alpha}{2}+\frac{\alpha^2}{2}\right) + D\left(-\frac{1}{2t}\right) - C\left(\frac{2\alpha+1}{2t}\right)\right) = 0 \qquad (2.118b)$$

Accuracy of frequency (φ , eqn. (2.85)) and amplitude, λ , have been optimized with respect to blends of the above two equations and α , under the constraints of stability.

Whole new families of approximations can be developed by approximating δ_i , δ_{i-1} , $\dot{\delta}_i$, $\dot{\delta}_{i-1}$ with parameters free for manipulation to provide stability. A large number of three step equations can be created in this way. One of the simplest, which was studied, is of this form :

$$\dot{\delta}_i = \frac{3}{2\Delta t}\delta_i - \frac{2}{\Delta t}\delta_{i-1} + \frac{1}{2\Delta t}\delta_{i-2} \qquad (2.119a)$$

$$\dot{\delta}_{i-1} = \frac{3}{2\Delta t}\delta_{i-1} - \frac{2}{\Delta t}\delta_{i-2} + \frac{1}{2\Delta t}\delta_{i-3} \qquad (2.119b)$$

Using forward difference, central difference, and backward difference equations will produce a variety of methods.

Another interesting possibility is to approximate equation (2.103a) :

$$\delta_i = Y\,\delta_{i-1} + Z\,\dot{\delta}_{i-1} \qquad (2.103a)$$

by methods such as

$$\delta_i = Y\,\delta_{i-1} + Z\,(\delta_i - \delta_{i-2})/2\Delta t$$

so that we get

$$\delta_i \left(1 - \frac{Z}{2\Delta t}\right) + \delta_{i-1}\, Y + \delta_{i-2}\left(-\frac{Z}{2\Delta t}\right) = 0 \tag{2.120}$$

or to modify this with methods such as those used in developing the Newmark β method, or the α technique (2.115, 2.116).

2.9 Basis for programs written.

Early in this work it was hoped to create some generalized space time elements, after Kaczkowski [20], incorporating $\dot{u}$ and similar terms as nodal degrees of freedom, so that for a plate element we might use

$$\{u\} = \begin{Bmatrix} u \\ u_x \\ u_y \\ u_{xy} \\ u_t \end{Bmatrix} \tag{2.121}$$

The problem that assembled equations as in (2.36) would not work with boundary conditions at the ends of the time

interval $[7, 10]$ prevented this development in this work ; its solution may lead to interesting results in problems of engineering-physics, without leading to matrices of undesirable size as in (2.73).

In order to develop the matrices for any of the above work, a computer program was written to create rectangular finite element matrices with the general method in the first part of the chapter. The dimensions of the element, degrees of freedom at each node, and polynomial $\{P\}\{\alpha\}^T$ are selected.

From this the matrix $[C]$ is formed.

From node coordinates (appropriately normalized) $[A]$ is formed.

Then

$$N = C A^{-1} \qquad (2.122)$$

The matrix operator $[\partial]$ is read in,

$$B = \partial N \qquad (2.123)$$

Then the element matrix is formed :

$$\int_V B^T E B \, dV \qquad (2.124)$$

having specified matrix $[E]$.

The polynomials are organized, differentiated, multiplied, and integrated by the program. The program has created space time elements, stiffness matrices, consistent mass matrices, and elements of higher order for the single degree of freedom mass spring systems.

It was later realized a more efficient way to proceed would be to find

$$A^{-1^T} \left\{ \int_V (\partial C)^T \; E \; (\partial C) \; dV \right\} A^{-1} \qquad (2.125)$$

especially when various elements were to be formed in real applied work.

Studies of the use of space time finite elements were further enhanced by a program written to calculate mode shapes and natural frequencies of a structure from conventional mass and stiffness matrices. The program uses a well known procedure [33] in order to simplify matrix algebra :

$$M \{\ddot{x}\} + K \{x\} = 0 \qquad (2.126)$$

using

$$\{x\} = \{|x|\} \sin \omega t \qquad (2.127)$$

we get

$$K \{|x|\} = \omega^2 \quad M \quad \{|x|\} \tag{2.128}$$

or

$$M \{|x|\} = \frac{1}{\omega^2} \quad K \quad \{|x|\} \tag{2.129}$$

$$K^{-1} M \{x\} = \frac{1}{\omega^2} \{x\} \tag{2.130}$$

obviously an eigenvalue problem.

Let

$$L L^T = K \qquad \text{(Cholesky decomposition)} \tag{2.131}$$

Then

$$L^{T^{-1}} L^{-1} = K^{-1} \tag{2.132}$$

$$L^{T^{-1}} L^{-1} M x = \frac{1}{\omega^2} x \tag{2.133}$$

$$L^{-1} M (L^{T^{-1}} L^T) x = \frac{1}{\omega^2} L^T x \tag{2.134}$$

$$L^{-1} M L^{T^{-1}} (L^T x) = \frac{1}{\omega^2} (L^T x) \tag{2.135}$$

Set

$$L^T x = r \tag{2.136}$$

and we get

$$L^{-1} \; H \; L^{T^{-1}} \quad r \quad = \quad \frac{1}{\omega^2} \quad r \qquad (2.137)$$

If we start

$$L^{-1} \; M \quad = \quad B \qquad (2.138)$$

$$L \quad B \quad = \quad M \qquad (2.139)$$

which can be solved column by column for B, after the decomposition of K.

$$B \; L^{T^{-1}} \quad = \quad A \qquad (2.140)$$

or

$$L \; A \quad = \quad B^T \qquad (2.141)$$

since A is symmetric.

Then

$$A \quad r \quad = \quad \frac{1}{\omega^2} \quad r \qquad (2.142)$$

and A is symmetric.

Demonstration of symmetry :

$$L^{-1^T} = L^{T^{-1}} \tag{2.143}$$

Then

$$A = (L^{-1} M) L^{T^{-1}} \tag{2.144}$$

$$A^T = L^{-1} (M L^{T^{-1}}) \tag{2.145}$$

$$= L^{-1} (M^T L^{T^{-1}}) \quad M \text{ being symmetric} \tag{2.146}$$

$$= L^{-1} M^T L^{T^{-1}} \tag{2.147}$$

$$= L^{-1} M L^{T^{-1}} = A \tag{2.148}$$

Also, it is known $X^T X$ is symmetric, X a real general matrix. So, $X^T M X$ is symmetric if M is symmetric. Finally,

$$L^T x = r \tag{2.149}$$

solves for the true mode shapes.

2.10 A start point for further modification.

A final note on another form of future development is to note (2.92)

$$X\,\delta_i + Y\,\delta_{i-1} = 0 \qquad (2.150)$$

and try such innovations as

$$X\,\delta_{i-1} + Y\,\delta_{i-2} = 0 \qquad (2.151)$$

to form

$$X\,\delta_i + (X + Y)\,\delta_{i-1} + Y\,\delta_{i-2} = 0 \qquad (2.152)$$

This leads, substituting (2.46) the stability study technique, to

$$\lambda^2 X + \lambda(X + Y) + Y = 0 \qquad (2.153)$$

a quadratic eigenvalue problem. This is taken up by references [9] and [22] of Geradin's classification study [12].

CHAPTER III

3.1 Element Matrix.

Formulation of an element matrix is the first step in proceeding with the work. Aside from the two space - time elements of the previous chapter, few element matrices lend themselves to easy manual computation. The computer program DINO has been written to create an element matrix by the system described in Sections 2.1 and 2.8 . The rectangular elements can exist in from one to four dimensions, and the polynomial $\{P\}\{\alpha\}^T$, therefore element matrix width, can have up to 32 terms. In its present form, the program is only meant for the creation of one element, which can be assembled with copies of itself for straightforward academic studies. The program sets up its own interpolation function matrix N , and could be used for this end only, by any interested person. The interpolation is between nodes at which any number of derivatives may be specified, in addition to displacement. It should be noted that calculation of N becomes numerically unstable for high order derivatives.

3.2 The String.

We can create a string element with two degrees of freedom per node :

$$u = \begin{Bmatrix} u \\ u_x \end{Bmatrix} \tag{3.1}$$

$$u = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \tag{3.2}$$

Figure 11 illustrates this model for the string element. Nodal coordinates, normalized, are at (-1,+1). Restorative force is $T u_x$ for small displacements where :

- The x subscript denotes differentiation with respect to this subscript
- T is tension, and we are using the theory of the stretched string.

Then, following methods for stiffness matrices :

$$\delta \frac{1}{2} \int_V \epsilon^T \sigma \, dV = 0 \tag{3.3}$$

we set

$$\epsilon = u_x \tag{3.4a}$$

$$\sigma = T u_x \tag{3.4b}$$

$$\epsilon = B \delta_e = \partial N \delta_e = \partial \begin{Bmatrix} u \\ u_x \end{Bmatrix} \tag{3.5}$$

We can use either

$$\partial = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \end{bmatrix} \tag{3.6a}$$

or

$$\partial = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{3.6b}$$

$$E = [T] \tag{3.7}$$

which gives the terms for

$$\int_V B^T E B \, dV \, \delta_e \tag{3.8}$$

This is sufficient information for the program to produce a stiffness matrix.

To get the mass matrix we let

$$E = [\rho] \tag{3.9}$$

where ρ is the density of the length of the string, and

$$\partial = [1 \quad 0] \tag{3.10}$$

since we are interested in mass to be moved vertically.

When these elements were used in the mode and frequency program, FEMVIB, frequencies predicted were in excellent agreement with theory, and demonstrated that the programs created were working properly. Table 1 presents natural frequencies as calculated for a four element string ; density = 1 unit/unit length, tension = 1 unit, and of 3 units total length.

TABLE 1 STRING VIBRATION

	CALCULATED	THEORY
ω_1	.392700	.392699
ω_2	.785500	.785398
ω_3	1.17919	1.17809
ω_4	1.58114	1.57080
ω_5	1.98506	1.96350

3.3 The Beam.

The easiest way to proceed is to know the elastic energy stored by a displacement of the beam. This is [34] :

$$\frac{1}{2}\int_1 M_x \, u_{xx} \, dx \tag{3.11}$$

$$M_x = E\,I\,u_{xx} \tag{3.12}$$

$$\sigma = M_x = E\,I\,u_{xx} \tag{3.13a}$$

$$\epsilon = u_{xx} \tag{3.13b}$$

$$\partial = \left[\frac{\partial^2}{\partial x^2} \quad 0 \quad 0 \quad \ldots\right] \tag{3.14}$$

We can expect high accuracy if we choose a six degree of freedom element : u, u_x, u_{xx} at each of two nodes. Figure 11 shows the element.

If we let

$$\delta_e = \begin{Bmatrix} \delta_1 \\ \delta_{1_x} \\ \delta_{1_{xx}} \\ \delta_2 \\ \delta_{2_x} \\ \delta_{2_{xx}} \end{Bmatrix} \tag{3.15}$$

then

$$\partial = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & 0 & 0 \end{bmatrix} \tag{3.16}$$

Nodal coordinates are again (-1,+1).

$$E = [E\ I] \tag{3.17}$$

Mass matrix again comes from

$$E = [\rho] \tag{3.18}$$

$$\partial = [1 \quad 0 \quad 0] \tag{3.19}$$

Polynomial is :

$$a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4 + a_6x^5 \tag{3.20}$$

Frequencies for simply supported and cantilevered beams were computed, and gave excellent agreement with theory [35]. Figure 1 shows the first two modes of vibration as predicted.

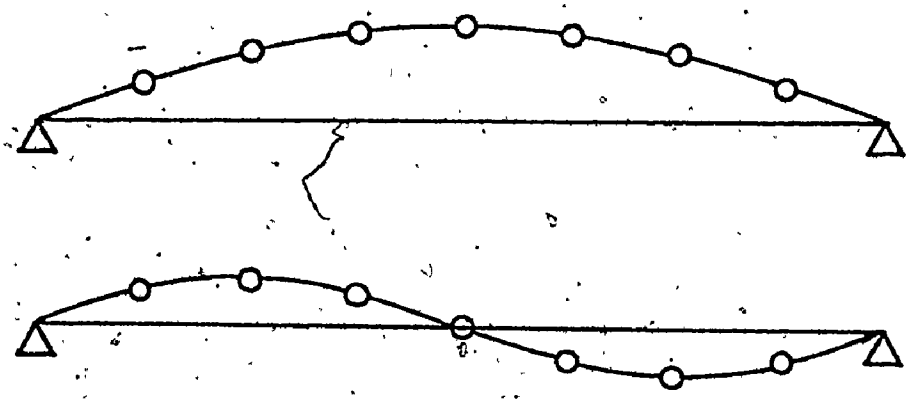
Table 2 presents frequencies calculated.

TABLE 2 BEAM VIBRATION

	CALCULATED	THEORY
ω_1	.0385531	.0385531
ω_2	.1542126	.1542126
ω_3	.3469783	.3469783
ω_4	.6165509	.6168503
ω_5	.9638352	.9638286

This is with $EI = 1.$, $\rho = 1.$, 8 elements ; total length 16 units.

TWO MODES



DOUBLE HINGED
BEAM
8 ELEMENT
6 DEG FREE/ELEMENT

FIGURE 1

3.4 The Plate.

In a plate, the energy of deformation is :

$$U = \frac{1}{2} \int_A (M_x\, u_{xx} + M_y\, u_{yy} + 2\, M_{xy}\, u_{xy})\, dA \qquad (3.21)$$

So :

$$\partial = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & 0 & 0 & \cdots \\ \frac{\partial^2}{\partial y^2} & 0 & 0 & \cdots \\ \frac{\partial^2}{\partial x \partial y} & 0 & 0 & \cdots \end{bmatrix} \qquad (3.22)$$

$$E = \begin{bmatrix} D & D\nu & 0 \\ D\nu & D & 0 \\ 0 & 0 & 2D(1-\nu) \end{bmatrix} \qquad (3.23)$$

$$\epsilon = \begin{Bmatrix} u_{xx} \\ u_{yy} \\ u_{xy} \end{Bmatrix} \qquad (3.24)$$

$$\sigma = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \qquad (3.25)$$

For a high accuracy plate study, we can select :

$$\delta_i = \begin{Bmatrix} \delta_i \\ \delta_{i_x} \\ \delta_{i_y} \\ \delta_{i_{xx}} \\ \delta_{i_{yy}} \\ \delta_{i_{xy}} \end{Bmatrix} \tag{3.26}$$

The element is presented in Figure 11.

The polynomial coefficient set found to work was, for $x^n y^m$, (n,m) pairs as follows :

00, 10, 01, 20, 11, 02, 30, 21, 12, 03, 40, 31,
22, 13, 04, 50, 41, 32, 23, 14, 05, 51, 33, 15.

The nodes being at

(-1,-1)
(-1,+1)
(+1,-1)
(+1,+1)

The mass matrix comes from

$$\partial = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.27}$$

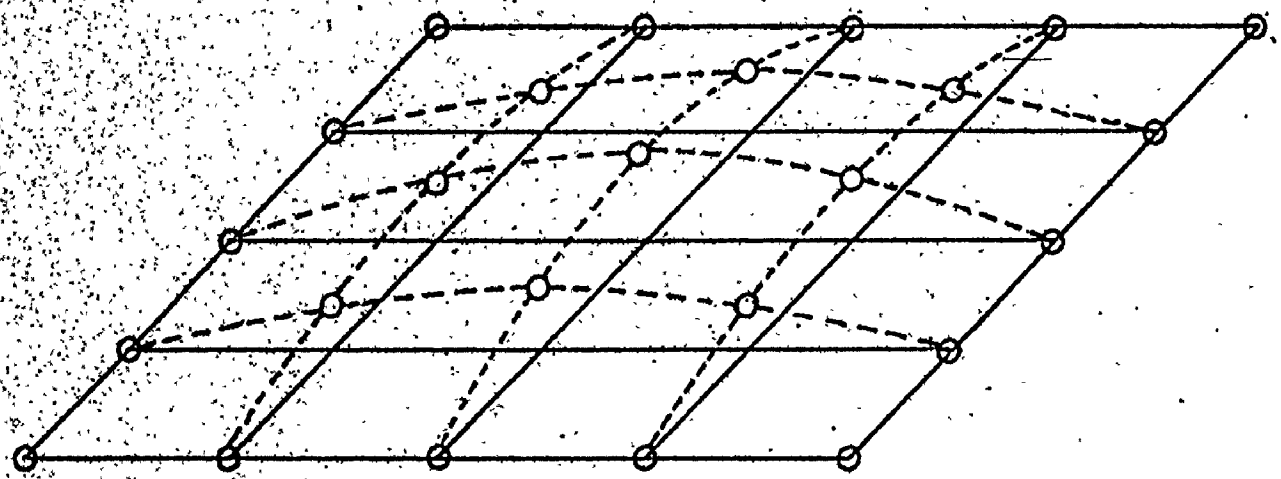
$$E = [\rho] \tag{3.28}$$

with ρ being per unit area.

Reasonable results were obtained for predicted natural frequencies of a simply supported plate. Figure 2 shows the first mode of vibration. The plate is of 16 elements, 25 nodes, 8 units x 8 units size ; $\rho = 1.$, $D = 1.$

TABLE 3 PLATE VIBRATIONS

	CALCULATED	THEORY
ω_1	.31208	.30843
ω_2	.77411	.77106
ω_3	.77411	.77106
ω_4	1.23961	1.23370
ω_5	1.52063	1.54213



SIMPLY SUPPORTED
PLATE VIBRATION
(FIRST MODE)
24 DEG FREE/ELEMENT

FIGURE 2

3.5 Time elements.

For time elements for dynamic problems, the approaches are straightforward :

$$\text{Polynomial} = a_1 + a_2 t + a_3 t^2 + \ldots \tag{3.29}$$

$$\delta_i = \begin{Bmatrix} \delta_i \\ \dot{\delta}_i \\ \ddot{\delta}_i \\ \vdots \end{Bmatrix} \tag{3.30}$$

$$E = \begin{bmatrix} -K & -C \\ 0 & M \end{bmatrix} \tag{3.31}$$

$$\partial = \begin{bmatrix} 1 & 0 & 0 & \ldots \\ \dfrac{\partial}{\partial t} & 0 & 0 & \ldots \end{bmatrix} \tag{3.32}$$

However, as noted, boundary value problems limited the present work.

3.6 The General Space Time Element.

The procedure for organizing the terms is easiest if terms for "ϵ" and "σ" are collected out of consideration for the terms which should enter the action integral (2.18). In that case, a possible generalized element for the string would have :

$$\delta_i = \left\{ \delta_i \right\} \tag{3.33}$$

that is, displacement at each node the only degree of freedom. With four nodes for the string interval in space time, the polynomial is :

$$u = \alpha_1 + \alpha_2 x + \alpha_3 t + \alpha_4 xt \tag{3.34}$$

$$\epsilon = \begin{Bmatrix} u_x \\ u_t \end{Bmatrix} \tag{3.35}$$

$$\sigma = \begin{Bmatrix} -T u_x \\ \rho u_t \end{Bmatrix} \tag{3.36}$$

$$E = \begin{bmatrix} -T & 0 \\ 0 & \rho \end{bmatrix} \tag{3.37}$$

and we take

$$\int_t \int_V B^T \; E \; B \; dt \; dV \tag{3.38}$$

This generalized element was not researched further, because of the lack of a facility to determine accuracy of the implicit mass and stiffness, and the lack of a way to insure absolute stability. The method might still be of interest for systems of other than physical structures.

3.7 Results for Direct Integration Techniques.

The accuracies of Newmark's β method (2.87a) including the linear time element and the central difference operator have been studied and compared with the Gurtin method. Studied and optimized have been blends of (2.43a) and (2.43b), based on the work done by Geradin and his stability techniques (2.113). Accuracies of methods in Section 2.7 have been optimized. This is done by plotting values of amplitude, λ, constrained to be stable, while looking for optimal accuracy of ϕ, the frequency predicted for the numerical techniques.

As previously pointed out, Geradin proceeded to use the last two rows of (2.73). Any two rows can be selected; Argyris and Scharpf used rows two and four. Further modifications can be made by adding two rows, and using this new row with the sum of the two other rows. The best result comes from rows two and four, and is presented. We present graphs of $|\lambda|$ for optimal accuracy of ϕ under the constraint of stability imposed by the α technique. These are followed by Figure 2 comparing errors in ϕ. λ and therefore ϕ are calculated using equation (2.114).

Figure 3 is a plot of $|\lambda|$ as a function of ratio of time interval to period of oscillation, for Geradin's method. It can be seen that λ breaks into two real roots in two places, and exceeds 1 for large time steps, there-

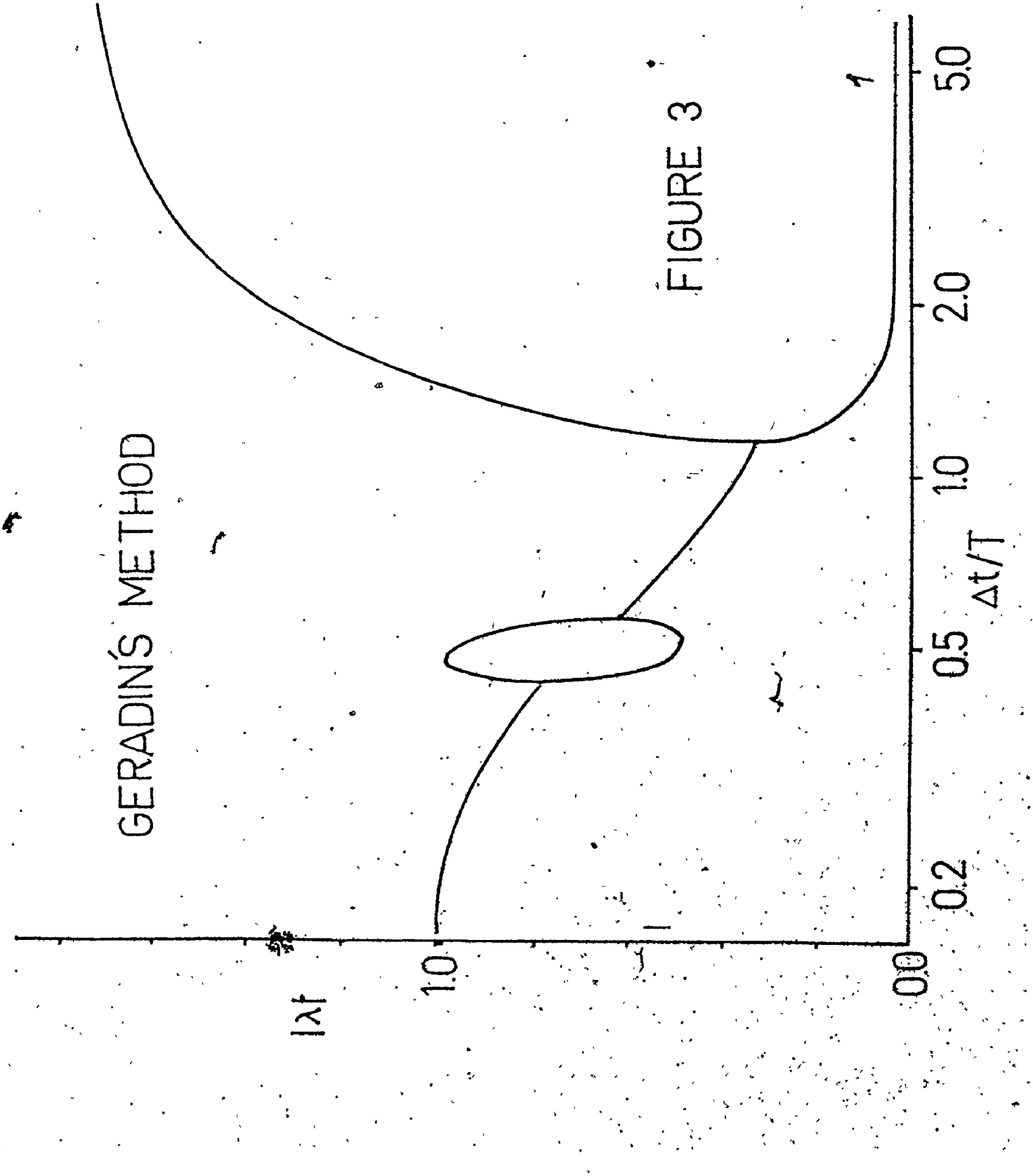


FIGURE 3

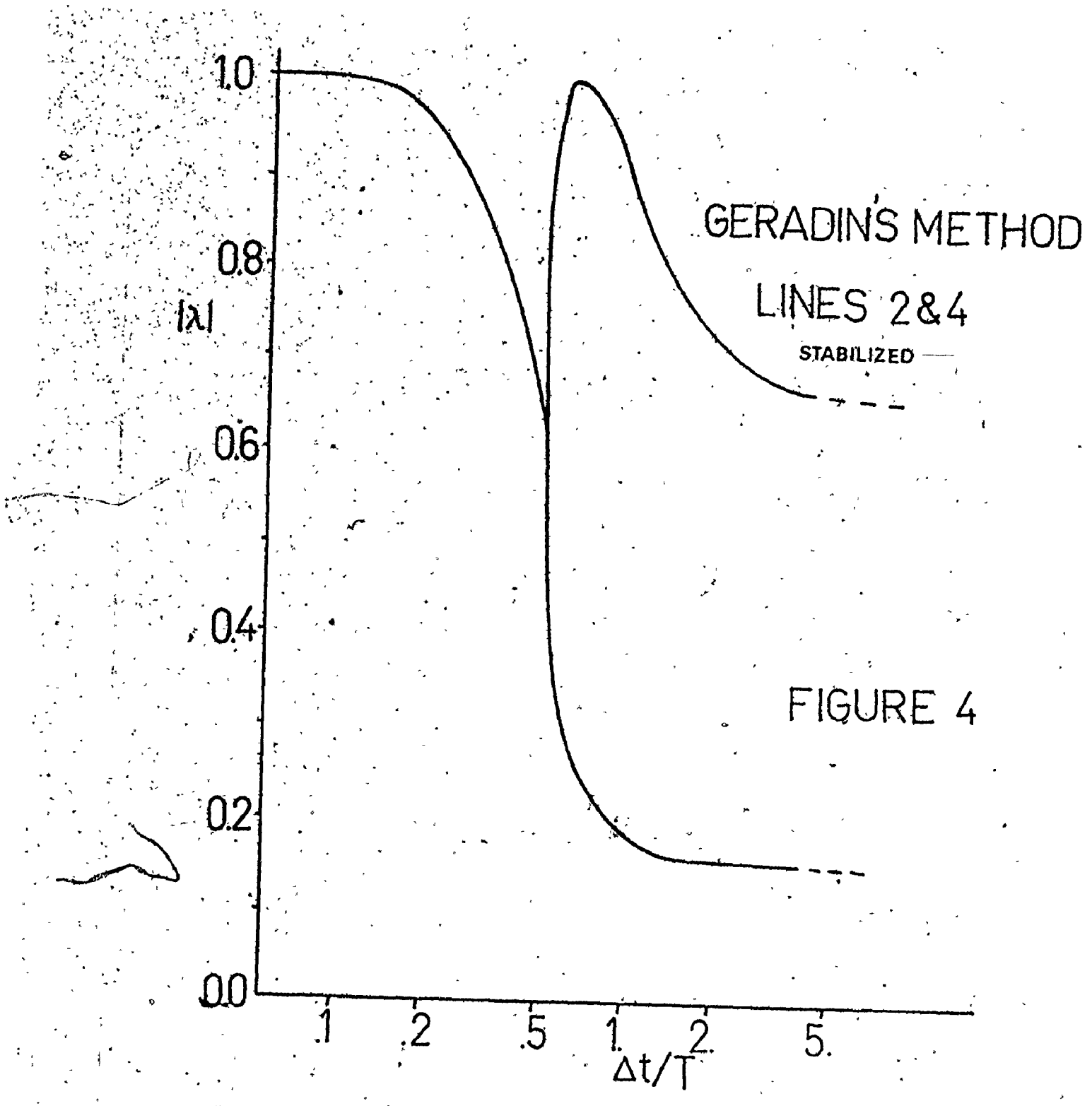
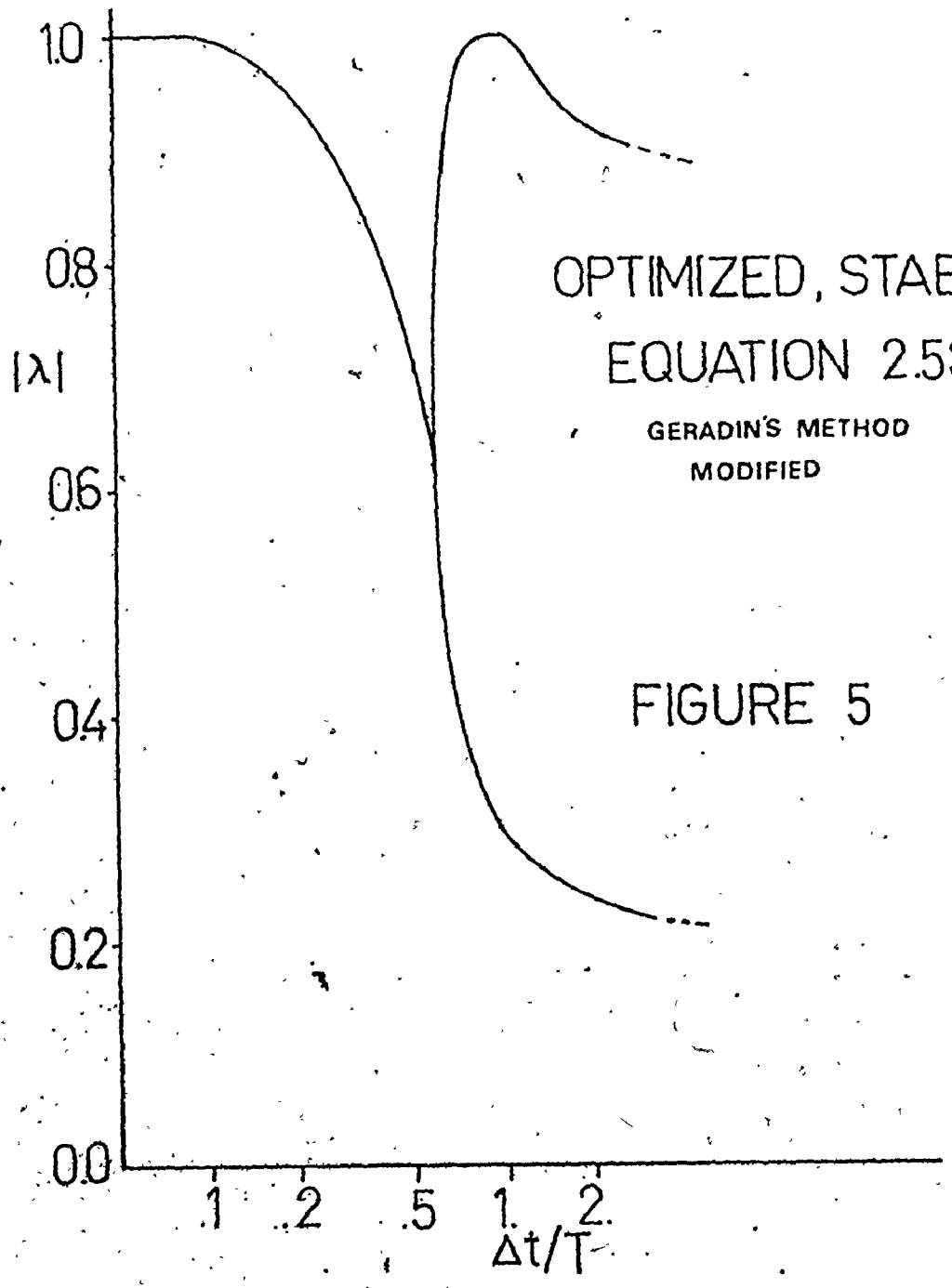

1.0
0.8
|λ|
0.6
0.4
0.2
0.0
.1
.2
.5
1.
2
5.
Δt/T
GERADIN'S METHOD
LINES 2&4
STABILIZED

FIGURE 4



OPTIMIZED, STABILIZED

EQUATION 2.53

GERADIN'S METHOD

MODIFIED

FIGURE 5

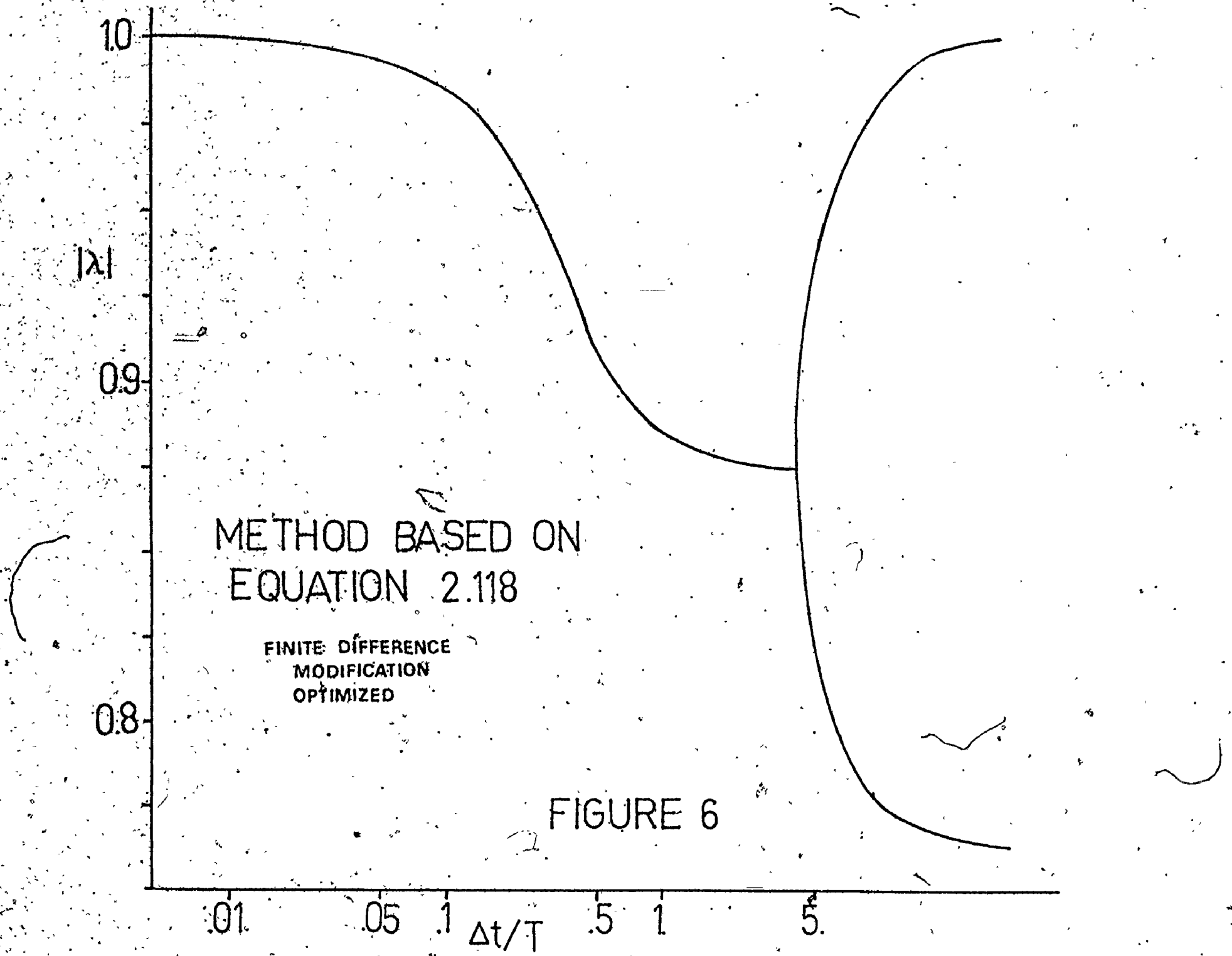

1.0
|λ|
0.9
0.8
METHOD BASED ON
EQUATION 2.118
FINITE DIFFERENCE
MODIFICATION
OPTIMIZED
.01
.05
.1
Δt/T
.5
1.
5.

FIGURE 6

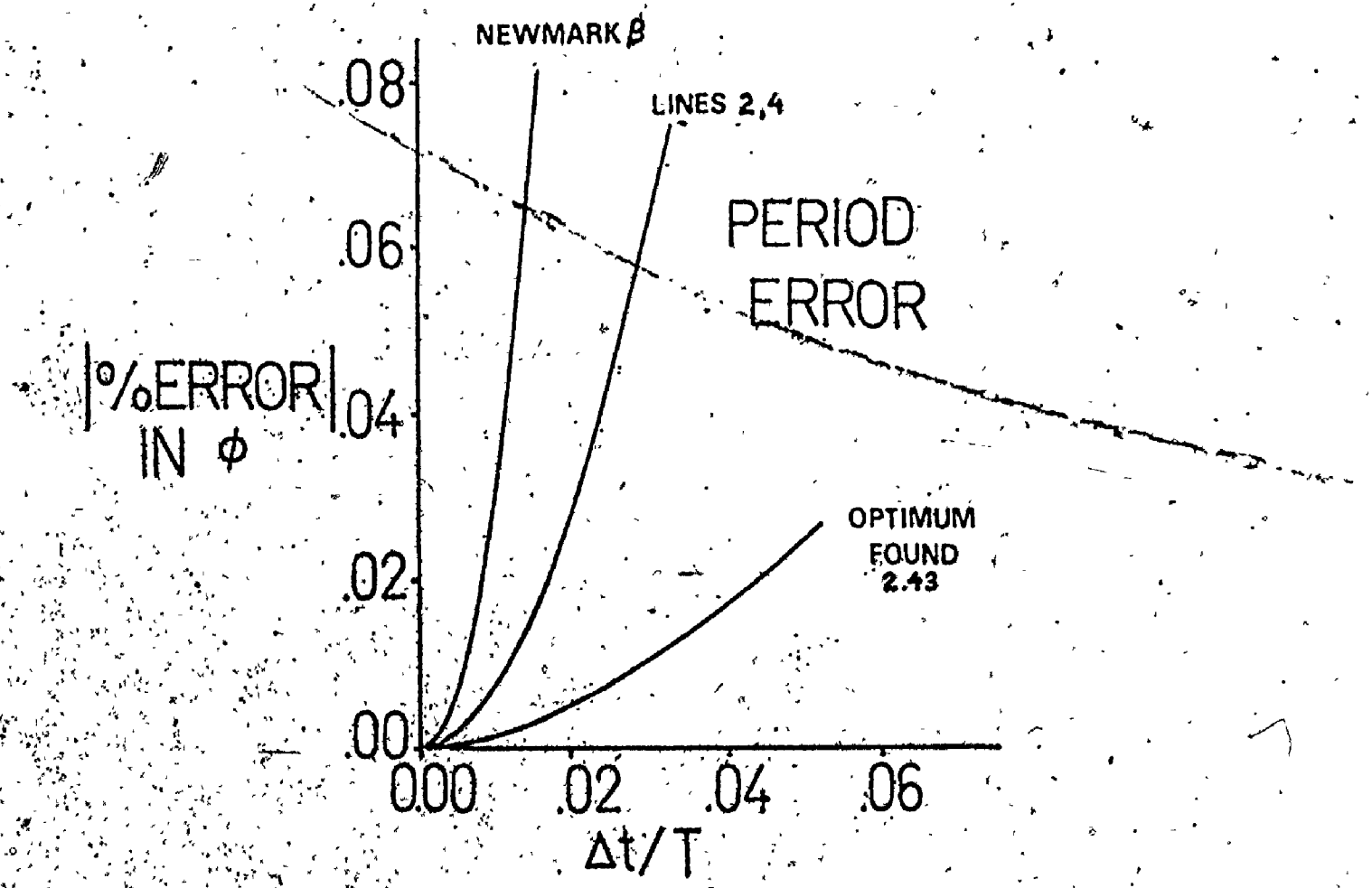

NEWMARK β
LINES 2,4
PERIOD
ERROR
OPTIMUM
FOUND
2.43
|%ERROR|
IN φ
.08
.06
.04
.02
.00
0.00
.02
.04
.06
Δt/T

FIGURE 7

fore is not inherently stable. Figure 4 is a plot of $|\lambda|$ stabilized based on lines 2 & 4 of Equation 2.73 as used by Argyris & Scharpf instead of lines 3 & 4 as used by Geradin. This method produced the best accuracy of ϕ . Figure 5 is the value of $|\lambda|$ associated with the optimum of Equation 2.53, an improvement upon the previous two methods. Figure 6 is $|\lambda|$ for the finite difference simplification of the method of Geradin. Method (2.118a,2.118b) can be applied to any row or combination of rows from (2.73). Results have been optimized for values of α and blends of techniques (a) and (b), a two degree of freedom optimization problem. The results are all similar, and have accuracy of ϕ almost identical to that of the Newmark β method, but the disadvantage that there are two full matrices even with no damping. The Newmark β method remains simpler and superior. Another modification is that α be different for techniques (a) and (b), a three degree of freedom problem, but no great accuracy increase occured.

Attempts with equation (2.120) all failed with meaninglessly scattered points.

A test of Geradin's method was performed for a simple mass spring system with relatively large time steps, and the motion was as anticipated ; oscillatory with correct frequency and slight induced damping. The result is presented in Figure 8.

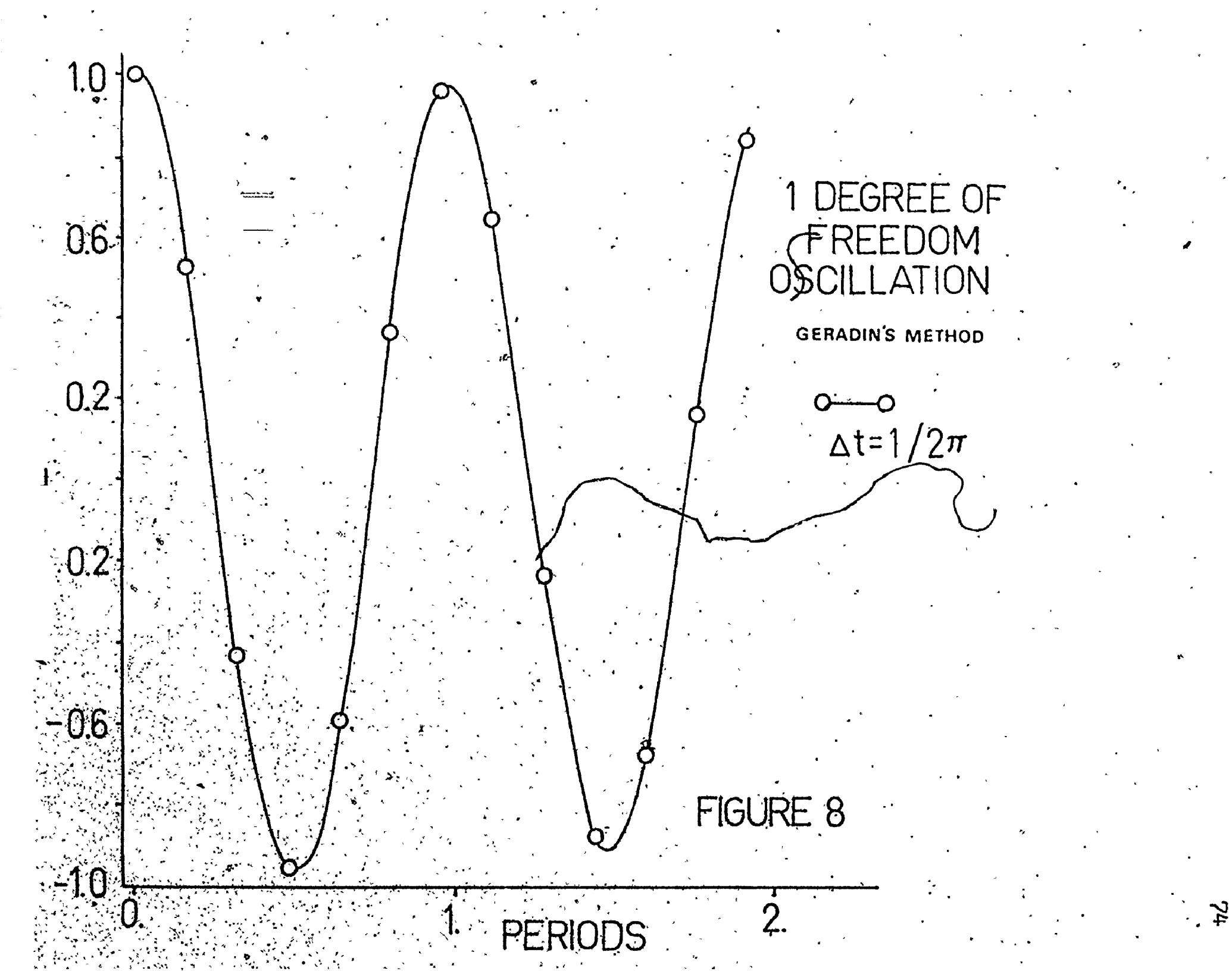

1.0
0.6
0.2
0.2
-0.6
-1.0
0.
1.
2.
PERIODS
1 DEGREE OF FREEDOM OSCILLATION
GERADIN'S METHOD
Δt=1/2π

FIGURE 8

Frequencies of oscillation for displaced strings, beams, and plates agreed well with theory ; transient behavior with initial displacement equal to a natural mode was made with the linear time element technique, and then the Newmark β method in general, and motion was correct. It occured at the natural frequency for the given mode, and mode shape was maintained. An initial velocity problem also ran appropriately, at the correct frequency, and reaching the correct amplitude. Figure 9 is a graph of the motion of a node of a square plate given first mode initial displacement, and is stable, using the Newmark method, $\beta = 1/4$. Figure 10 is a graph of motion of a node of a simply supported beam, with $\beta = 1/6$, and too large a time step for stability.

Figure 11 shows elements as used for the string, beam, and plate studies. They are normalized so the nodes are at (-1,+1) in space ; the scaling factor to the real positions at (-1,+1) are a part of the computer program DINO's imputs.

The maintainance of mode shape is a demonstration of the fact that the transient analysis methods work. The programs prepared are easily modified to incorporate a damping matrix and a forcing function. In this way any structural system can be modelled. The accuracy of the period should be considered and the time steps kept small

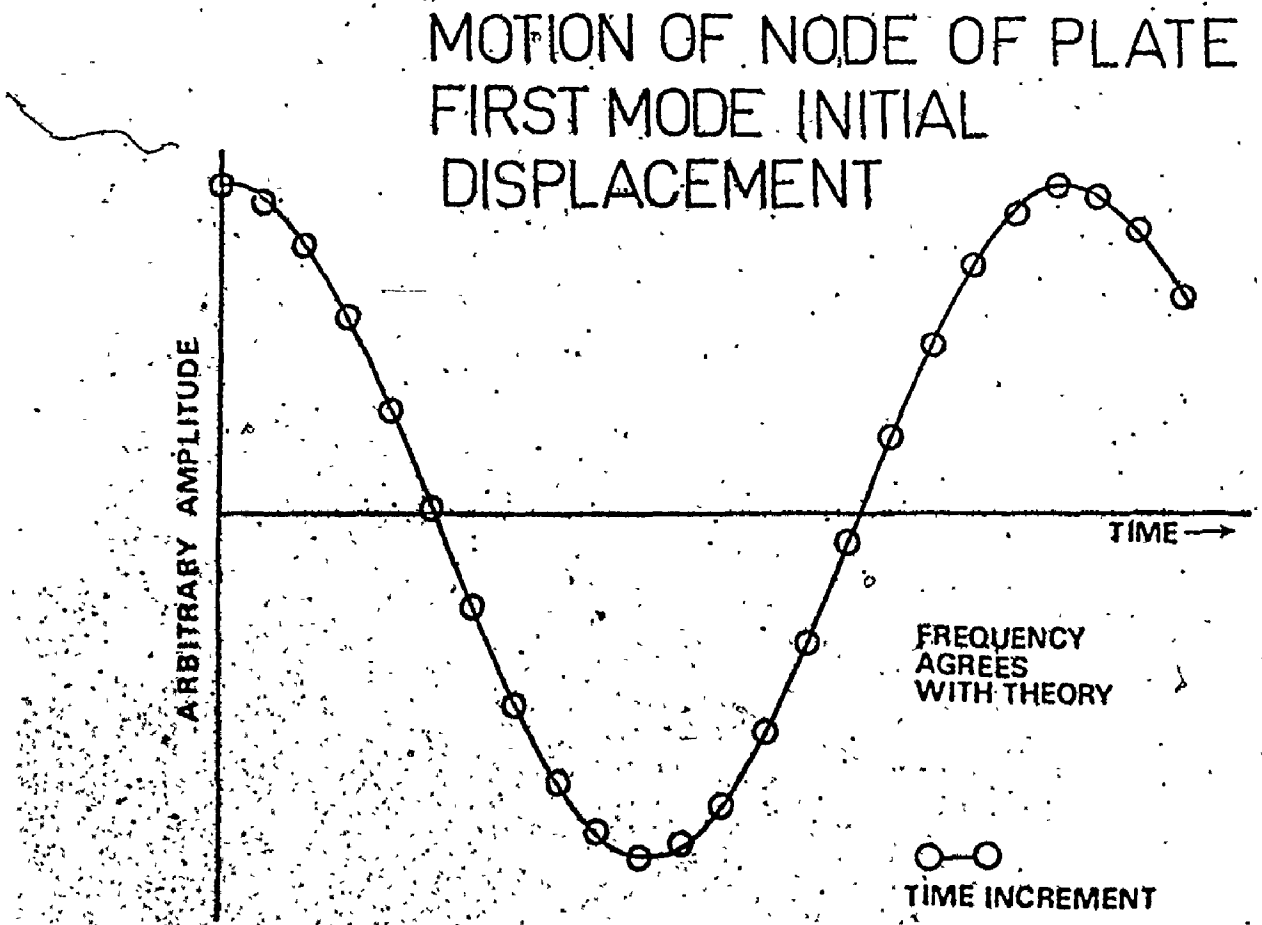
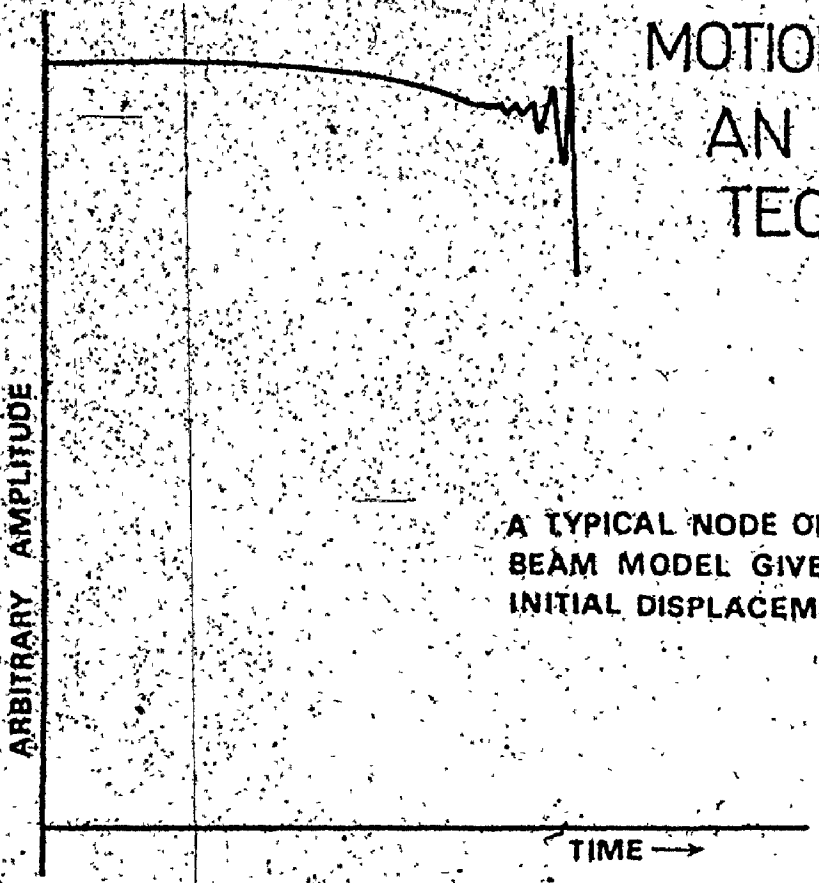

MOTION OF NODE OF PLATE
FIRST MODE INITIAL
DISPLACEMENT
ARBITRARY AMPLITUDE
TIME
FREQUENCY
AGREES
WITH THEORY
TIME INCREMENT

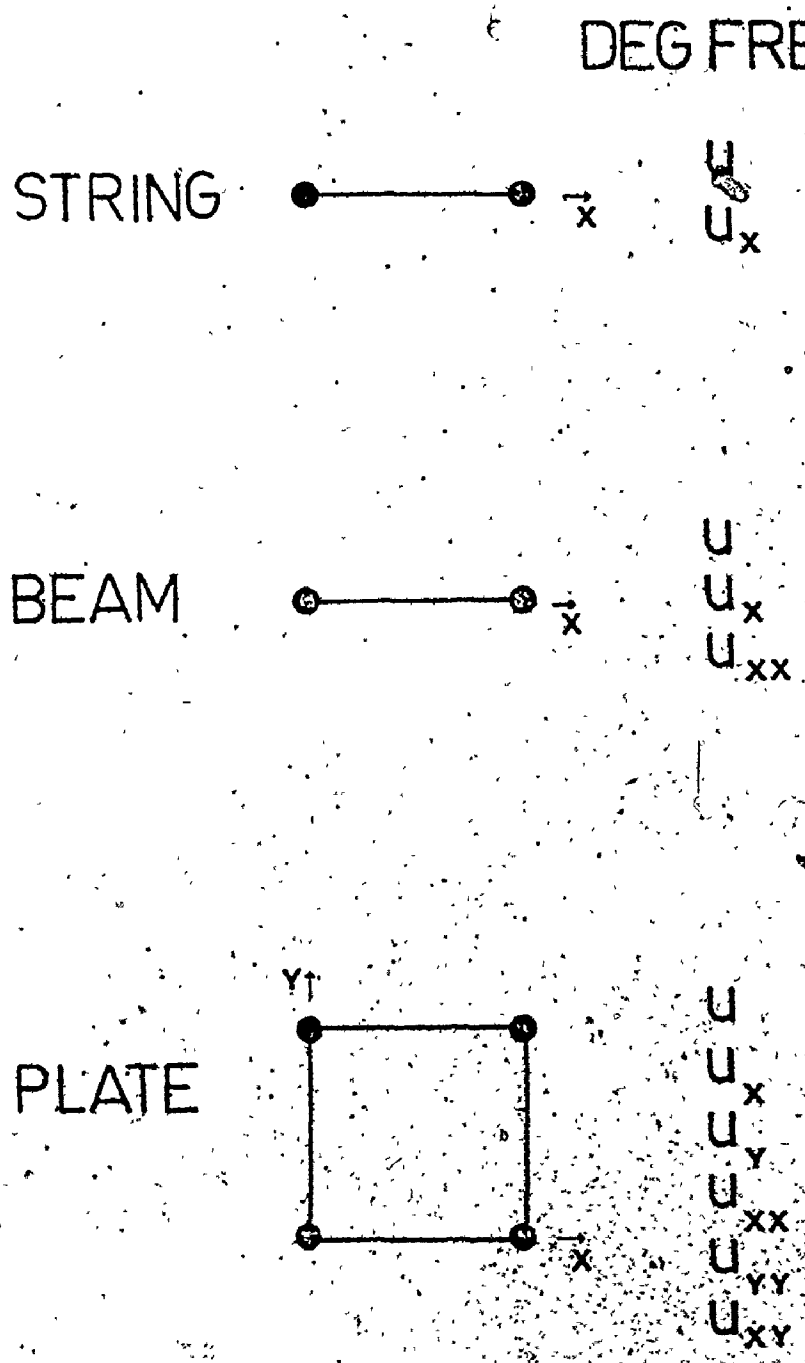
FIGURE 9



MOTION OF NODE
AN UNSTABLE
TECHNIQUE

A TYPICAL NODE OF THE
BEAM MODEL GIVEN
INITIAL DISPLACEMENT

FIGURE 10



THE ELEMENTS

FIGURE 11

enough that for the study time of interest, shift in phase will not lead to inaccurate response, within the time that a displacement is naturally damped out. This becomes a second criterion for time step size selection beyond forcing function harmonic content. Its consideration will lead to more accurate studies, and more optimal computer run time costs.

CHAPTER IV

Conclusions.

Major computer programs have been written to create individual finite elements, for vibration mode analysis, and to predict transient vibrations with the Newmark β method. Many simpler programs experimented with direct integration operators.

Program DINO created mass and stiffness matrices for strings, beams, and plates and were used in program VIB to obtain natural modes in agreement with theory. Correct transient vibratory response followed a modal displacement and the initial velocity problem in program FEMVIB. Space time elements in agreement with hand derivable elements also were created, and damping was properly introduced. Attempts to use the generalized space time elements are undesirable for structural dynamics because of the boundary value problem and because stability conditions would be unique to every structure.

Inspection of Figure 7, Period Errors, gives a view of desirable methods, when tempered by consideration of matrix storage space required. For a given level

of error it is seen that the method of Geradin - Argyris and Scharpf allows double the time step of Newmark, however double the storage space is required in the case of damping, four times in the case of no damping, and the advantage is lost. The fact that $|\lambda|$ in Geradin's method, Figure 3, breaks prematurely into two real roots was not noted by Geradin, and leads to the "poor" optimum found and represented in Figure 7, where we must constrain $|\lambda|$ to remain ≤ 1. This optimum allows 5 x the time step of Newmark's β method, reasonable for small systems with no damping, but unlikely to be adopted in view of the complications. The finite difference substitution into the finite element technique, represented in Figure 6, leads to an accuracy almost identical to Newmark's β method, but lacking its simplicity. This work finds the Newmark β method the simplest method to employ and of competitive accuracy.

The advancement of uses of generalized space time elements awaits insight into the boundary value problem ; clues offered in the published work of Argyris & Scharpf were found insufficient.

That the space time finite element methods can work has been demonstrated with simple and complex systems, and considerations and comparisons have been given, and give more guidance than any previous work.

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