CARBON FIBRE REINFORCED

PLASTIC

AN INTRODUCTION TO CARBON FIBRE REINFORCED PLASTIC

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SCOPE AND CONTENTS: The development and properties of carbon fibre are reviewed together with the properties and applications of carbon fibre reinforced plastic. Techniques for fabricating this material and certain design problems are discussed and future developments considered.

Some simple predictions of the elastic moduli of fibre reinforced composites are compared with experimental data and a modification for the prediction of the longitudinal shear modulus suggested. Typical values for the elastic moduli of carbon fibre reinforced plastic are presented.

A method for predicting the behaviour of laminated composite structures, which is more realistic than conventional netting analysis, is described and a computer programme for determining the strength and stiffness of such structures included.

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PREFACE

The aim of this thesis is to provide an introduction to the use of carbon fibre reinforced plastic.

The thesis is divided into four sections: section A deals with the general background of carbon fibres and serves to provide a designer with the information necessary to decide whether the material will be of use to him and in what capacity. It covers such things as basic properties, comparison with other fibres, forms of material available, present fabrication techniques, and some associated design problems. The remainder of the thesis aims to supply some of the basic tools that the designer will require when he begins to work with the material.

One of the first tools the designer needs is a knowledge of the elastic moduli of the material. Only one of the five independent elastic moduli of this material is generally available and section B serves to provide typical values to give a designer some feel for the material. With a composite of this nature there are several variables which affect the moduli - fibre content, resin type and fibre type - and to make full use of this relatively expensive material it may often be necessary to vary these to give the moduli desired for a particular application, and thus a reliable method of predicting the moduli is required. Section B

compares three commonly used prediction methods, first in their standard form and then modified to take account of the anisotropy of carbon fibres. Numerical results for various fibre contents and fibre types are given and the suitability of the various prediction methods is discussed.

Having found the elastic properties of a unidirectional composite, the next requirement is a method for predicting the behaviour of a combination of several unidirectional layers with different orientations. Section C presents such a method for analysing laminated composites which is far more realistic than conventional netting analysis. The method is not widely used or even known at present, and this is the justification for including it.

Section D contains the references, the computer programme used in section B for predicting the elastic moduli and a computer programme for analysing composite structures. This latter programme has been developed from an existing programme to simplify its use and the modifications are explained and discussed in this section.

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LIST OF PRINCIPAL SYMBOLS

A	in-plane stiffness matrix for a laminate
A'	in-plane compliance matrix for a laminate
B	stiffness coupling matrix for a laminate
B'	compliance coupling matrix for a laminate
C	anisotropic stiffness matrix
D	flexural stiffness matrix for a laminate
D'	flexural compliance matrix for a laminate
E	isotropic Young's modulus
E_L	longitudinal Young's modulus
E_T	transverse Young's modulus
G	isotropic shear modulus
G_{LT}	longitudinal shear modulus
G_T	transverse shear modulus
h	laminate thickness
K	plate curvature
M	moment resultants
N	stress resultants
Q	stiffness matrix for a lamina in natural coordinate system
$\bar{Q}$	stiffness matrix for a lamina in an arbitrary coordinate system
S	shear strength of a unidirectional composite

T	difference between lamination and operating temperature
v	volume fraction of a constituent material in composite
X	axial strength of a unidirectional composite
Y	transverse strength of a unidirectional composite
α	thermal expansion matrix
β	G^f/G^m
γ	shear strain
ε	normal strain
ζ	measure of reinforcement afforded by fibres
θ	fibre orientation
ν	isotropic Poisson's ratio
ν_{LT}	major Poisson's ratio
ν_{TL}	minor Poisson's ratio
ν_{TT}	transverse Poisson's ratio
σ	normal stress
τ	shear stress

Superscripts:

f	fibre
m	matrix
o	in-plane component of strain

SECTION A: CARBON FIBRE AND ITS USES

A1. INTRODUCTION

Today's rapidly expanding technology, especially in the field of aerospace, is continually creating demands for new and better materials. Sir Barnes Wallis is attributed as having said: "I can design an aircraft to fly at ten thousand miles an hour - can you give me the materials?", and this is indicative of the problems that face the materials engineer today.

With the possible exception of beryllium, no metal available today is sufficiently stiff and light to meet all future design requirements. In bulk form the majority of metals and alloys have a specific stiffness (ratio of elastic modulus to density) of about 100×10^6 in, and this value is not expected to be improved significantly in the foreseeable future.

Ceramic whiskers - tiny, single crystal fibres - offer a significant increase in specific strength and stiffness compared to metals in both bulk and whisker form. Some properties of several whiskers listed by Sutton [1]*

* Footnote: References appear in Appendix [4]

are shown in Table [i], together with values for some bulk metals as given in [2].

Table [i] Properties of various whiskers and bulk metals

Material	Density lb/in^3	Tensile Strength 10^3 lbf/in^2	Young's Modulus 10^3 lbf/in^2	Specific Strength 10^6 in	Specific Modulus 10^7 in
Whiskers					
Ceramic					
Al_2O_3	0.143	3000	62	21.2	43.4
BeO	0.103	1900	50	18.4	48.5
B_4C	0.091	2000	70	21.9	76.9
SiC	0.115	3000	70	26.1	60.8
Si_3N_4	0.115	2000	55	17.4	47.8
Graphite	0.060	2845	102	47.4	170.0
Metal					
Cr	0.260	1290	35	5.0	13.4
Ni	0.324	560	31	1.7	9.6
Bulk Metal					
Cr	0.260	120	36	0.46	13.8
Ni	0.324	46	32	0.14	9.9
Ti (99%)	0.163	80	17	0.49	10.1
Be (QMV)	0.067	42	42	0.63	62.6

Whiskers are of little structural use until they are bonded together in a matrix material. In the resulting composite the whiskers, aligned in one direction, carry the bulk of the applied load, while the matrix binds them together, spaces them, protects them from mechanical and chemical damage and distributes the load to the individual whiskers.

It will be some while before whisker materials can be widely used, however, because of the difficulties of growing whiskers reproducibly on a large scale and of aligning and spacing the whiskers in a suitable matrix. Also the problem of treating the surfaces of the whiskers to promote wetting and bonding to the matrix remains to be solved.

A more practicable form of reinforcing material at present is a continuous fibre. The fibre can then be woven into matting or tapes which are often more convenient, or used in filament winding processes. Continuous glass filaments have been available for some years and glass fibre reinforced plastic is commonly used for making shells - boat hulls, car bodies, filament wound piping and rocket motor cases etc. But glass fibre reinforced plastic has a specific stiffness no better than most metals, being about 110×10^6 in (although it does possess an appreciably better specific strength).

Thus there is a need for high modulus, high strength continuous fibres. The only suitable fibres that are available in commercial quantities at present are boron,

graphite and carbon, and it is with the latter that this thesis is primarily concerned. The words carbon and graphite may generally be used interchangeably when referring to these fibres. However, the British usually refer to their fibres as carbon, while the Americans use the word graphite. Thus for convenience the use of carbon herein will be restricted to the British fibres and graphite will refer to the U. S. fibres. Particular emphasis is placed on the carbon fibres as these at present seem to be attracting most attention from both British and U. S. industries.

A2. DEVELOPMENT OF CARBON FIBRES

A2.1. Historical Development

Carbon fibres of some form have been in use for many years: Edison's first electric light bulb filaments were made using carbonized bamboo fibres over seventy years ago. Carbon fibres made by pyrolizing rayon or cellulose have been used for, among other things, ablative purposes in rocket nozzles. These fibres had poor mechanical properties, however, having specific strengths and stiffnesses of about 0.8×10^6 in and 10×10^7 in respectively.

The first serious work aimed at making carbon fibres for structural purposes was started at the Wright Patterson Air Force Base in Ohio in the late 1950's, where they developed a technique which enabled them to produce graphite fibres from a viscose thread having a specific stiffness of up to 40×10^7 in on occasions. Union Carbide did further work on the process and developed it commercially, and in 1964 placed 'Thornel' 25, with a specific stiffness of 40×10^7 in, on the market in limited quantities at a price of over $1000/lb. Union Carbide have since added 'Thornel' 50 to their range which is a stronger and stiffer material, and more will be said of this later.

Meanwhile some work of a similar nature was proceeding in Japan. A report by Shindo [3] in 1961 showed that carbon fibres with a specific stiffness of about 30×10^7 in could be produced from polyacrylonitrile yarn by a heat treatment process. The Osaka Institute, for whom Shindo was working, patented the process but did not try to improve on these properties to produce a material for structural use.

In Great Britain, Rolls-Royce, who had considerable experience in using glass fibre reinforced plastics and had been investigating various reinforcing systems, began to concentrate on carbon fibre - resin systems in 1962. But it was the Royal Aircraft Establishment (RAE) Farnborough, beginning work on carbon fibres in 1963, who obtained the master patent when Watt, Phillips and Johnson applied for British Patent No. 1,110,791 in April 1964. The RAE process was different to that of Shindo, though it used a similar precursor, and produced fibres with a specific stiffness of about 100×10^7 in.

In 1965 RAE asked the Ministry of Technology to scale up the process as fibre production was taking up so much time that investigation of the fibre properties was being hindered. Because of its furnaces and other facilities, the Atomic Energy Research Establishment (AERE) Harwell, was awarded a contract to produce RAE type fibres and were soon producing fibres as good as those at RAE.

Thus the Ministry of Technology, aware of the potential of this material, invited two companies to further scale up the process and develop it commercially. The companies involved were Courtaulds, a large textile firm and sole producer of the precursor material in England, both then and now, and Morganite Research and Development, who have considerable experience in carbon and high temperature technology.

The first work to be published concerning the RAE fibres was in May 1966, by Watt et al [4], and gave some fibre properties, an indication of the structure and some information on the properties and preparation of carbon fibre reinforced plastic. This was followed in July 1966 by an article by Standage and Prescott [5] of the Advanced Research Department, Rolls-Royce Ltd., announcing that they had prepared carbon fibres of high elastic modulus in continuous lengths and relatively large quantities. Typical stiffness and strength distributions for the fibre were given as well as values for carbon fibre reinforced epoxy and polyimide resins. Rolls-Royce had collaborated with the government research laboratories and were using a process based on the RAE one, but much original research was done in order to turn the technique into a tonnage process and to produce large amounts of carbon fibre reinforced resin, known within the company as 'Hyfil'.

In February 1967, Moreton et al [6] of RAE reported the effect of the heat treatment temperature on the strength

and stiffness of carbon fibre. This showed that for fibres requiring a high stiffness heat treatment at 2500 ^{o}C or more was necessary, while fibres with maximum strength were produced with a heat treatment temperature of 1500^{o}C - 1600^{o}C. Thus two types of carbon fibre became available - a high modulus carbon fibre and a high strength carbon fibre.

The adhesion between fibres and resin matrix was poor with the early carbon fibres, resulting in low interlaminar shear strengths, but recently a surface treatment operation, which neither coats nor contaminates the surface of the fibre, has been introduced which has resulted in much higher values for interlaminar shear strength.

A2.2 Method of Manufacture

The method of manufacture of RAE type carbon fibres is necessarily somewhat clouded, but Gunston [7] has given a good outline of the batch technique originally used and this is summarized here for completeness.

The precursor of the carbon fibres is a special grade of Courtelle which is made by squirting the liquid polymer through a spinneret having 10,000 microscopic holes. The filaments set in a bath to form a staple fibre, or tow, and after further processing are wound on a bobbin.

To make carbon fibres the tow is unwound from the bobbin on to a rigid frame so that it is evenly loaded with

about 300 m of tow under precise tension and lateral spacing to give the correct weight of fibre in the resulting sheet. On each side of the frame the parallel tows are stitched across the top and bottom to form a secure web.

Then several frames are loaded into a furnace in which the tows are heated in air to a temperature below 300°C. They become oxidised and try to shrink; this increases the tension in the fibres and helps orientate the fibre molecules. The frames are then unloaded and each web cut off just outside the stitching.

The resulting 'warp sheets' are stacked in refractory boxes and then heated in another furnace to at least 1000°C for several hours until all that remains is carbon. Finally the fibres are subjected to a precise heat treatment in an inert atmosphere to improve their mechanical properties. For high strength fibres this heat treatment temperature is about 1500°C, while for high modulus fibres it is over 2500°C.

Continuous plants are now in operation but these are subject to very close commercial security. The main principles, however, are shown in Fig. [1], which is based on a sketch in Gunston's article [7].

A2.3 Structure and Properties

It appears [8 and 9] that the structure of carbon fibres is a consequence of the structure of the parent organic

fibre. The RAE type fibres consist of long primary units lying parallel to the fibre axis, and these primary units are bonded together to form a stretched network of branched fibrils that apparently run the full length of the fibre [8].

X-ray diffraction photographs [9] suggest that the fibres consist of highly orientated graphite crystallites about 50 Å in size, the angle of orientation being very close to the fibre axis. It is this preferred orientation which gives the fibre its high modulus; this can be seen in Fig. [2] which shows a plot of the orientation of the fibre against the mean Young's modulus of a fibre bundle [4].

The strength and modulus of these carbon fibres is dependent on the heat treatment temperature [6] as shown in Fig. [3]. High modulus fibres [type I or HM] are heat treated at above 2500°C, while high strength fibres [type II or HT] require a temperature of only about 1500°C. Some properties of the commercially available RAE type fibres obtained from [10] and [11] are shown in Table [ii]. The tests were performed on 5 cm gauge lengths; tests on 1 cm gauge lengths have given average strengths 12% higher than for 5 cm lengths [6]. This is due to the scatter of fibre strengths; the chance of including a weak spot in the test piece being less when using a shorter specimen.

Table [ii] Properties of various RAE type carbon fibres

Fibre Type	Filament Diameter Microns	Density lb/in^3	Tensile Strength $10^3 lbf/in^2$	Young's Modulus $10^6 lbf/in^2$	Specific Strength 10^6 in	Specific Modulus 10^7 in
Modmor I	7.5	0.072	200-300	55-65	2.8-4.1	76-90
Modmor II	7.5	0.063	350-450	35-45	5.6-7.1	56-71
Grafil A	7.9	0.0628	275-325	28-35	4.4-5.2	44-55
Grafil HT	7.8	0.0635	350-450	35-42	5.5-7.0	55-66
Grafil HM	7.5	0.0700	250-325	50-60	3.5-4.6	70-85

The consistency in quality of fibre manufacture is indicated [10] by results from 40 successive production batches of metre length 'Modmor' type I fibre: the standard deviation was only 4×10^6 lbf/in^2 for the Young's modulus and 23×10^3 lbf/in^2 for the ultimate tensile strength. More detailed information on the quality control and test methods used by Morganite Research and Development is given by Blakelock and Lovell [12].

Until recently carbon fibres have tended to give low values of adhesion within a matrix, which limits the composite properties; this fact made many people regard carbon as an unsuitable reinforcing material and helps to explain the concentration on boron in the U. S. a few years ago. A surface treatment has been developed, however, which improves the

adhesion, and stereoscan electron micrographs [12] shown in Fig. [4] demonstrate this improved bonding between fibre and resin. Measurement of the interlaminar shear strength of a composite also provides an indication of the degree of bonding between fibre and matrix, good bonding giving a higher interlaminar shear strength. Table [iii] shows typical values for fibre-resin composites for treated and untreated fibre [11], the treated fibre being denoted by a suffix S.

Table [iii] Interlaminar shear strengths for composites with treated and untreated fibre

Fibre	Grafil HT	Grafil HT-S	Grafil HM	Grafil HM-S
Interlaminar Shear Strength lbf/in^2	5000-7000	10000-12000	3000-5000	8000-12000

Clearly the resin used has an important effect on the absolute value of the interlaminar shear strength obtained.

A2.4 Cost

The present cost of carbon fibre is largely dependent on the length and amount of fibre required. Table [iv] shows the latest available prices. Clearly the effectiveness of carbon fibre must not be judged on present prices, however, but on its cost in, say, five or ten years' time, when it is

being used and produced in bulk.

Table [iv] Present cost of carbon fibre

Fibre Type	Length	Quantity	Price per lb
Modmor	1 metre	<2.2 lb	$260
Modmor	1 metre	>110 lb	$130
Modmor	1000-1500 ft	small	$410
Grafil HT	48 in	< 11 lb	$143
Grafil HT	48 in	> 11 lb	$130
Grafil HM	48 in	< 11 lb	$169
Grafil HM	48 in	> 11 lb	$156
Grafil HT or HM	1000 ft.	< 11 lb	$430
Grafil HT or HM	1000 ft.	> 11 lb	$390

There have been many estimates of the future cost of carbon fibres. Possibly the lowest was suggested by the Atomic Energy Research Establishment at Harwell, who suggest a figure of about $5 per lb. Gunston [7] has suggested that a more realistic figure might be $12 per lb. The fibre manufacturers do not appear to be quite so optimistic, however.

Morganite Research and Development are producing fibre at the rate of a few tons a year at present, but Mr. V. Dembo, New Products Investigation Manager, expected [13] that by the mid seventies they would be producing about one hundred tons

of fibre a year at a cost of about $35-50 per lb.

Mr. D. J. Willats, General Manager of Courtoulds Carbon Fibres Unit, recently disclosed [14] that although people talk of the enormous potential demand for carbon fibres, most orders at present are for experimental quantities of a few pounds, and that their output at present is only a few tons a year. He expected the price to fall to about $70-85 per lb for continuous fibres over the next three years. In ten years' time the price might drop to about $25 per lb assuming an output of five hundred tons a year. Mr. Willats attributed the high cost to two main factors: (i) the high cost of the precursor material which is about $5 per lb and (ii) the fact that the unit output is so low, i. e. there are very high capital costs. Also overall output even at five hundred tons a year is small in comparison with a normal textile fibre, say Courtelle, which is produced at the rate of ten thousand tons a year.

A3. COMPARISON WITH OTHER FIBRES

A3.1 General

Table [v] shows some typical properties of various commercially available reinforcing fibres. It can be seen by comparison with the values shown for bulk metals in Table [1] that boron, 'Thornel' graphite and RAE type carbon fibre offer a substantial increase in specific stiffness and strength over conventional materials.

Table [v] Typical properties of some commercially available fibres

Fibre	Density lb/in^3	Tensile strength $10^3 lbf/in^2$	Young's modulus $10^6 lbf/in^2$	Specific strength 10^6 in	Specific modulus 10^7 in
E-glass	0.092	500*	10.5	5.4 *	11
S-glass	0.090	600*	12.5	6.7 *	14
Asbestos	0.116	300	25	2.6	22
Boron[t]	0.095	460	60	4.8	63
Thornel 50	0.059	285	50	4.8	85
Modmor I	0.072	250	60	3.5	84
Modmor II	0.063	400	40	6.3	64

t on tungsten core

* value for virgin filament; finishing, sizing and handling may cause up to 50% loss in strength

A3.2. Boron

Boron is considerably more expensive than carbon having a present price of about $300 per lb; even in ten years the price is not expected to fall much below $150 per lb [15]. Also because of its large cross section (100 microns compared to 8 microns for carbon) it is less suitable for small radius applications.

A large amount of money has been spent on the development of boron fibres, however, and there is consequently much more design data presently available for boron than for carbon. Thus boron may be expected to continue to find applications while more experience is gained with carbon, but in a few years, carbon should take over a large portion of these applications.

Boron may still find applications where the effects of thermal expansion in a metal-composite structure are important. The coefficient of thermal expansion of boron is much more compatible with that of metals compared with carbon which has a negative coefficient of thermal expansion along the fibre.

A3.3. 'Thornel' Graphite

The specific strength and stiffness of 'Thornel' 50 graphite are comparable to RAE type carbon fibre as can be seen in Table [v], but the latter is said to be more consistent

in continuous lengths: the properties of 'Thornel' tend to be discontinuous along the fibre. The bonding between fibre and matrix is also better with RAE type fibre, producing a greater interlaminar shear strength in the resulting composite.

A3.4. Other Graphite Fibres

Other graphite fibres have recently become available from H. I. Thompson and Co. and the Great Lakes Carbon Corporation, both in the U. S. These have a disadvantage which is common to all the U. S. fibres, namely an irregular cross-section; the British fibres have circular cross-sections. This difference stems from the essentially different manufacturing processes used in Britain and the U. S.

A4. RANGE OF CARBON FIBRE PRODUCTS AVAILABLE

Carbon fibre is available in the following forms:

[i] Short staple fibre:

Carbon fibre is available as a short staple fibre cut to any length from $\frac{1}{4}$ inch to 12 inches, and is useful in the manufacture of thermosetting and thermoplastic moulding materials. It can also be used to manufacture fibrous webs, e. g. mats, tapes and felts.

[ii] Long staple fibre:

This may be one metre or 48 inches in length and consists of a twist free tow of 10,000 filaments.

[iii] Continuous filament tow:

Lengths of up to 3,000 feet of twist free 10,000 filament tow are available for such uses as filament winding.

[iv] Flock:

Carbon fibre is available in flock form having a nominal length of 1/10 inch, and can be used as a reinforcement in thermosetting resin systems.

[v] Preimpregnated unidirectional warp sheet:

Carbon fibres preimpregnated with one of several resins are available. The 10,000 filament tows are laid parallel and run the length of the sheet. The sheets are available in a variety of sizes and thicknesses from 0.001 in to 0.030 in. This form of carbon fibre is ideal for use in matched moulds.

[vi] Preimpregnated unidirectional tape:

This is available in a standard size 3 inches wide by 100 yards long. Narrower widths and lengths up to 300 yards are available, however.

[vii] Random fibre mats:

Random fibre mats can be supplied lightly bound with a resin compatible binder to provide cohesion for handling, or preimpregnated with resin.

[viii] Thermoplastic moulding and extrusion compounds:

A range of carbon fibre filled thermoplastic materials is available in the form of granules suitable for injection moulding and extrusion. Nylon 66 and Polypropylene are available as standard products containing 20% by weight of 'Grafil' A. Other fibre contents in a variety of thermoplastics can also be supplied.

[ix] Thermosetting moulding compounds:

A range of moulding compounds is available based on B-stage thermosetting resins containing carbon fibre. The material is supplied as a coarse ground fibrous mass and is suitable for processing on conventional compression moulding equipment.

A5. PROPERTIES OF CARBON FIBRE REINFORCED PLASTIC

A5.1. General

As it was mentioned in the introduction, carbon fibre must be bonded together in a matrix material before it becomes a useful structural material. At present plastics and resins are the most widely used matrix materials because of the experience gained using them in glass reinforced plastic. Carbon fibre has been shown to be compatible with the following thermo-setting resins [16]:

Unsaturated polyesters
Amino, polyamide and anhydride cured epoxides
Phenolic and silicone resins
Friedel-craft resins
Polyphenylenes (Monsanto)
Silphenylenes (I.C.I.)
Polyimides (Du Pont and RAE types).

The properties of the reinforced plastic will clearly be dependent on the type and content of matrix and the type and treatment of fibre used. Typical properties are shown in Tables [vi] and [vii], however, for some typical polyester and epoxy resin reinforced with Modmor I carbon fibre [10]. Detailed descriptions of the mechanical property measurements on Modmor carbon fibre-resin composites are contained in [17].

Table [vi] Properties of carbon fibre/polyester resin composites

50% by volume of untreated 'RAE' type I fibre	
Flexural strength	120,000 lbf/in^2
Flexural modulus	20 x 10^6 lbf/in^2
Tensile strength	130,000 lbf/in^2
Tensile modulus	28 x 10^6 lbf/in^2
Coefficient of thermal expansion per °C: along fibre	-0.73 x 10^{-6}
across fibre	29 x 10^{-6}
Thermal conductivity	0.04 cal/s cm°C
Tensile fatigue at 30,000 ± 20,000 lbf/in^2	>20 x 10^6 cycles without failure
Tensile/compressive fatigue at 0 ±40,000 lfb/in^2	>20 x 10^6 cycles without failure
Impact strength (Izod)	19.8 (unnotched) 18.7 (notched)

Table [vii] Properties of carbon fibre/epoxy resin composites

60% by volume of treated 'Modmor' type I fibre	
Flexural strength	120,000-125,000 lbf/in^2
Flexural modulus	$25\text{-}28 \times 10^6$ lbf/in^2
Tensile modulus	$30\text{-}40 \times 10^6$ lbf/in^2
Shear strength	7,000-10,000 lbf/in^2

A5.2 Effect of Temperature

Carbon fibre reinforced plastics are precluded from use in high temperature applications by the temperature limit of the resin. A standard type Shell epoxy resin (828/DDM/HT973) is limited to 100°C for example, wile a Ciba epoxy novalac resin (LY558/HT973) is limited to 220°C. As resins capable of withstanding higher temperatures become available, so the temperature limit of carbon fibre reinforced plastic will rise. The carbon fibre itself is unaffected by temperature rises up to about 1500°C in a non-oxidising atmosphere, and oxidation does not usually present a problem since the fibres are protected from the atmosphere in a composite.

A5.3. Effect of Water

Unlike glass fibre reinforced plastics which may lose

15% or more of their strength under humid conditions because of weakening of the glass/resin interface, carbon fibre composites suffer little degradation of mechanical properties even when subjected to prolonged immersion in water. With certain resins, however, loss in strength up to 5% may be experienced.

A5.4. Hardness and Machinability

The hardness of carbon fibre composites is largely dependent on the hardness of the resin matrix. This is generally quite low and the composites may thus be machined by any of the conventional techniques used for machining metals and plastics.

It has been found that the turning of these composites is best achieved dry and at high speeds, the speed being limited only by the tendency of the tool to overheat. Tools should be very sharp, tungsten carbide or diamond tipped tools being preferable, to prevent tearing of the fibre from within the matrix. For heavy cuts a positive top rake of about 20° with side and front clearances of 15° is desirable, while for finishing cuts no top rake is necessary.

A5.5. Specific Electrical Resistance

The specific electrical resistance of carbon fibre along the axis is about 775 micro-ohm-cm at 25°C, falling

to about 660 micro-ohm-cm at 180°C. In a composite the resistance along the direction of the fibres is proportional to the volume of fibre. Across the fibre direction the resistance is much greater and tends to be more variable.

A5.6 Friction and Wear

Lancaster [18] in a recent survey of the progress in the development of self-lubricating bearing materials has drawn attention to the beneficial effect of reinforcing polymer bearing materials with carbon fibres. He has shown that carbon fibre reinforced resins appear to exhibit anisotropy in their friction and wear characteristics and Fig. [5] shows some results for reinforced polyester sliding against steel: the coefficients of friction are given adjacent to each curve. It is concluded from the low wear rate obtained when the fibres are normal to the sliding interface that the fibres are then supporting part of the load. Lancaster has also shown that addition of randomly orientated chopped fibres can be very effective in reducing the friction and wear of several different polymers as shown in Table [viii].

Table [viii] Effect of reinforcement by carbon fibres on the friction and wear rate of various polymers sliding against mild steel
load, 1.2 kg; speed, 54 cm/s; surface finish ≈ 6 μin c.l.a.

Polymer	Wear rate, $cm^3/cm\ kg \times 10^{-10}$		Coeffieient of friction	
	No fibre	30 per cent wt fibre	No fibre	30 per cent wt fibre
Friedel-Crafts	860	0.13	0.56	0.24
Polyester (17449)	170	0.11	0.50	0.28
Polyvinylchloride	500	0.65	0.43	0.32
P.t.f.e.	460	1.6	0.25	0.25
Polypropylene	220	1.75	0.47	0.34
Polymethylmethacrylate	180	0.5	0.59	0.24
Polycarbonate	41	1.2	0.61	0.25
Nylon 66	10	1.1	0.40	0.35
Polyethylene	4.2	0.9	0.68	0.27
Polymide	2.1	0.4	0.68	0.35

A6. SOME PRESENT APPLICATIONS AND FABRICATION METHODS

A6.1. General

Carbon fibre components may be fabricated using any of the techniques developed in glass fibre technology. These methods may be broadly classified as moulding, filament winding and laminating from prepreg sheets. Each of these methods has its advantages and has been successfully used in association with carbon fibre. A few of these applications will now be described.

A6.2. Moulded Structures

One method of forming moulded structures, which encompasses hand lay-up, vacuum bag moulding and pressure bag moulding, involves laying the fibre to form a random mat in a resin matrix. This method has been widely used in the manufacture of glass fibre boat hulls, motor-car bodies etc., and has the advantage of being inexpensive since it is simple and requires unskilled labour. It does not use the fibre efficiently, however, and consequently the strength and modulus of the resulting component are lower than when using unidirectional fibres. Another disadvantage of this method is that it is best achieved with a woven fabric: the weave

crimps the fibre which induces stress concentrations and the effective strength and modulus are again reduced. Development is in progress, nevertheless, on a fabric in which the cross weave is nylon and this may help to alleviate the problem.

Fig. [6] shows the ultra-lightweight nose section of the Lola G.T. type T 70 Mark 3B [Group 4] racing car, made by Specialised Mouldings Ltd., Huntingdon, England. This is a moulded structure of glass fibre reinforced polyester resin containing a network of Courtaulds Grafil HM carbon fibres. This network achieves a 20% weight saving while producing a stiffer structure. A car with a similar nose section won the 'Daytona' 24 hour race in February 1969.

Another type of moulding is injection moulding where the molten plastic is injected into a cavity. This method is widely used for mass producing small plastic components. Adding short, [1/4 inch] carbon fibres to the liquid resin before injection increases the rigidity of the resultant component as well as giving greater dimensional stability, higher heat distortion temperatures and improved friction and wear characteristics. Alternatively the short fibres may be incorporated into a plastic 'dough' which is then moulded. Fig. [7] shows a dough moulded carbon fibre reinforced "rabbit" made by the Atomic Energy Research Establishment at Harwell. These "rabbits" are used for nuclear in-pile experiments.

A6.3. Filament Wound Structures

Fabricating components by winding continuous filaments with a resin to act as a binder and matrix is a process that has been used with glass filaments for some years. It is of particular use for making circular or conical section components such as tubes and pressure vessels, although it is now being used in the manufacture of more complex shapes such as wing flaps. Generation of these complex shapes requires specialized machinery and is likely to become an entirely automatic numerically controlled process. The advantages of this method, such as the ability to produce an integrated structure and being able to orientate the fibres in such a way as to place the strength and stiffness where required, are expected, nonetheless, to make this method important in the manufacture of airframe and aero-engine components in the future.

In the U.K., Imperial Metal Industries have experience in winding glass fibre rocket motor cases and were one of the first companies to produce filament wound carbon fibre structures. A selection of filament wound tubes, bottles and rods from I.M.I. is shown in Fig. [8]. They have also wound carbon fibre rocket motor cases and some experimental data on the burst level of carbon fibre pressure vessels has been given by Jones [19].

One interesting test structure made by I.M.I. is shown

in Fig. [9]. It is a circular cylinder with circumferential and longitudinal stiffeners. The entire structure was made from carbon fibre reinforced plastic and all but the longitudinal ribs filament wound in one piece. The circumferential ribs have a special shape so that the helically wound fibres remain straight.

Bristol Aerojet, also in the U. K., again have considerable experience of filament winding rocket motor cases, and a 6 inch diameter solid propellant rocket motor case made by Bristol Aerojet from substandard material as a winding trial is shown in Fig. [10]. Small scale pressure vessels have been filament wound from type 11 material using epoxy resin and the results of tests on these are given by Trigg [20]. This paper also gives a good account of the development of rocket motor systems and the advantages of using carbon fibre therein.

Another form of construction, similar to filament winding, is tape winding, in which a number of parallel tows preimpregnated with resin to form a tape up to 3 inches in width and from 5 to 30 thousandths of an inch in thickness are wound over a frame or mandrel. This method has the advantage of allowing quicker lay down rates and permits a further stage of inspection, i. e. the tape can be inspected before use. It has several disadvantages, nonetheless: to be efficient a high uniformity of tension within the tape is

necessary and this tends to make the tapes expensive; winding several tows (up to 40 is already common) under individual tension may be easier. Also tape winding is less suitable for winding parts of large curvature.

Nevertheless, tape winding looks to be a promising method for the manufacture of components such as wing flaps and other relatively large flat parts. Yurenka and Parks [21], describe a programme initiated at McDonnel Douglas to develop an automatic boron prepreg* tape lay-up machine for orienting and laminating the skins of a Douglas A-4 Skyhawk flap and the Boeing Corporation have also spent a considerable amount of money on developing tape lay-up techniques.

A6.4. Structures Made from Prepreg

The third main method of fabrication is to laminate parts from sheets of prepreg carbon fibres. The sheets may be cut to shape with scissors or, for mass production, with cutters rather like pastry cutters. Several sheets, orientated to give the desired directional properties, are placed together. in a matched metal mould and subjected to a carefully controlled cycle of temperature and pressure to form the shape and cure the resin.

* 'prepreg' is a corruption of the work 'preimpregnated' and although it does not as yet appear in the Oxford English dictionary, it is a commonly used word in this subject.

This method is most suitable for fabricating relatively thin solid parts and has the advantage of being a semi-automatic process once the pre-form shapes and moulding cycle are determined. Also when removed from the mould the part requires very little finishing. The method may be restricted to parts where a large number is required, however, since the moulds are expensive to manufacture.

Rolls-Royce are using this method for the manufacture of all their high duty composite components and one area of application which has received much publicity is in the fan blades of the RB211 turbo fan shown in Fig. [11]. By using carbon fibre reinforced plastic for the fan blades in the low pressure compressor they are able to produce a more efficient blade with a higher surge margin on a weight to weight basis with titanium. Alternatively, considering titanium and composite blades of the same aero-dynamic efficiency, it is possible to produce a much lighter blade, and consequently lighter supporting structure, by using the composite. Furthermore, these blades are cheaper. The arrangement and shapes of the various layers of carbon fibre prepreg used to manufacture these blades is shown in Fig. [12] and a description of the manufacturing process is given by Weil [22]. Weil also gives a detailed account of the reasons for choosing this material together with some of the problems that had to be overcome in order to do so.

A slightly different way of using prepreg sheets is to sandwich aluminum honeycomb core between two sets of prepreg sheets to form panelling for such things as aircraft flooring. Each set might contain one unidirectional sheet, two orthogonal sheets or any other combination required to give the desired properties. This idea is used in the satellite structure shown in Fig. [13] which was recently built by I.M.I. to develop fabrication techniques and provide a comparison with conventional metal structures. The principle of construction is rather interesting and is described by Jones [19], together with the manufacturing details. Basically it consists of four filament wound octagonal spines as the structural members and panels are formed by sandwiching aluminum honeycomb core between carbon fibre reinforced plastic cylindrical shapes of rectangular or triangular cross section as shown in Fig. [14]. The optimum design indicates a weight saving of 38% of the equivalent metal structure and if tests at RAE show clear advantages, this could replace the all metal construction on the flight hardware of the X-3 payload intended to be launched by a Black Arrow.

A6.5. Other Applications

Other applications which are already being tested include stiffening conventional metal components. The British Aircraft Corporation are following this approach and are now testing reinforced components for the new, wide-bodied BAC3-11,

as reported by McElhinney et al [23]. Development work is progressing on reinforcing the flanges of conventional metal beams used in aircraft for such purposes as supporting the passenger floor and stiffening pressure carrying bulkheads.

Ian Proctor Metal Masts Ltd. in the U.K. have made lighter and stiffer metal masts for the Flying Dutchman class of racing dinghy by reinforcing a smaller section alloy mast with carbon fibres.

At least one golf club has been manufactured from carbon fibre reinforced plastic.

A7. SOME DESIGN PROBLEMS

The most obvious design problem associated with carbon fibre reinforced plastic is its anisotropy, i.e. its strength and stiffness are dependent on direction. The designer working with this material cannot rely on conventional analyses based on isotropic elasticity but must, at present, go back to first principles and rework each analysis using the theory of anisotropic elasticity. Each analysis must also be more thorough, since secondary stresses which may sometimes be neglected when working with isotropic materials may now become important: for example, a transverse stress of the order of 3% of the longitudinal stress could reasonably be ignored when dealing with mild steel, say, but for unidirectional carbon fibre reinforced plastic the transverse strength may be only about 3% of the longitudinal strength and thus the transverse stress becomes significant. This anisotropy may be used to advantage, nonetheless. The designer is now able to add a further variable to his optimization sequence: the materials optimization. By varying the fibre content and the number and orientation of layers in a structure he is able to produce a material with appropriate strength and stiffness in each direction.

The negative coefficient of thermal expansion of carbon

fibre reinforced plastic in the fibre direction creates problems for the designer, especially when using carbon fibre and metal components together. McElhinney et al [23] report that when reinforcing the flanges of aluminum beams by wet lay-up of individual tows, subsequent curing (at temperatures over 100°C) produced distortion and large residual stresses. The problem was overcome in this case by bonding precured carbon fibre reinforced plastic to the beam with a low-temperature-cure adhesive instead of using the wet lay-up technique. Further problems arise, of course, if the component has to operate at varying temperatures. Moreover, residual thermal stresses result whenever prepreg laminates are bonded together at temperatures different from their operating temperatures as shown in the following sections. These may be quite large.

Carbon fibres do not possess a plastic range but remain elastic to failure. Because of this, unidirectional carbon fibre reinforced plastic remains elastic to failure in the fibre direction, though Petit and Wadoups [24] have shown that the transverse stress-strain response is non-linear and that consequently the response of laminated composites may also be non-linear. This non-linearity is generally small, nevertheless, compared to normal engineering materials such as mild steel or aluminum.

This essential lack of yield creates many problems for the designer. Pearce [25], in an article which attempts to acquaint designers with some of the ways in which carbon fibre reinforced plastic differs from both metals and glass reinforced plastic, draws attention to the fact that high modulus carbon fibre reinforced plastic has an elongation at failure of only about 0.5%. An engineer would normally regard this material as brittle, being used to working with materials having elongations of about 20% at failure, but Pearce points out that he unconsciously accepts a limiting strain of about 0.4% anyway, as this is the strain at proof or yield stress for common engineering materials. If he looks at stresses, however, using similar safety factors in the two materials results in a higher load in the carbon fibre reinforced plastic structure, with the added advantage that if the structure is overloaded short of the ultimate there will be no detrimental effect and it will recover elastically.

But, as it was mentioned earlier, the stress analysis must be much more thorough. Any local overloading in conventional structures merely results in local plastic deformation, and the load is diffused into the rest of the structure: in carbon fibre reinforced plastic structures, because the material does not yield, local overloading can result in failure. This inability to distribute concentrated loads over the structure becomes very important when considering how to attach

other parts; for instance, how does one attach engines and landing gear to a carbon fibre reinforced wing?

For similar reasons, joining components can present problems, especially when joining a composite part to a metal one. Various methods are being investigated and adhesive bonding may provide an answer in some cases. The basic solution, however, is a complete change in the approach to a design. Instead of relying on traditional ideas and methods, the designer must rid himself of all conception of the design as it now is and start right from the beginning. While it may be prudent at present to simply make or reinforce existing parts using this new material in order to gain experience, if the material is to be utilized to its full potential, whole structures must be redesigned right from the start in such a way as to eliminate any attachment or joining problems.

Another problem, though not strictly design, is that of testing the components once made. Sample checks on strength, resin content, stiffness and interlaminar shear strength can be made but these are destructive tests. Non-destructive testing is more difficult since few of the present techniques are applicable to this material, but the subject is receiving much attention. Rolls-Royce (Composite Materials) Ltd. have used acoustic tests to check the homogeneity of glass fibre reinforced plastic and Yurenka and Parks [21] describe two methods which were used at McDonnel Douglas to

test the A-4 Skyhawk flaps referred to earlier. Radiographic (X-Ray) examinations were used to determine the position of internal details, the presence of any foreign matter in the bond area, and any crushed or damaged core. Another test used was to apply a coating containing thermally sensitive phosphors to the material. When the coating is heated and viewed under an ultra-violet light, the phosphors fluoresce in inverse proportion to the heat sink area underneath. Since the heat sink of the panel varies greatly from bonded to unbonded areas, this technique accurately discloses any disbond in the panel. While it is possible to discover areas of disbonding, however, it is still extremely difficult to determine areas of poor bonding.

A8. FUTURE DEVELOPMENTS

The possible applications of carbon fibre reinforced plastic are too numerous to cover in detail here. Peters [26], in an introductory article on this material, suggests besides the aerospace field such diverse areas as chemical engineering, bearing materials, competition yachts and racing craft, model aircraft, telecommunications (radar scanners, dishes and masts), submersibles and stiffening overhead EHT and MV lines. Some description of each of these uses is given in this article. Gunston [7] also lists and describes numerous possible applications including prostheses, sporting goods (skis, fishing rods, oars, etc.), glassware and marine and hydrospace uses.

The more immediate applications are clearly going to be in the field of aerospace, however, where the weight saving can justify the present high cost of the material; it has been suggested [e.g. (23)] that on a typical subsonic aircraft the value to the operator may range from $80 to $1,300 per lb of weight saved depending on the aircraft utilization. Peters [26] suggests a figure of about $250 in the case of the Concorde and proposes ten areas where carbon fibre reinforced plastic could be used to save weight. These are shown in Fig [15] which appeared in Peters' article.

Hieronymus [27] in a recent survey of the progress of some American aircraft companies in the use of composites reports that the largest programme devoted exclusively to carbon is at the Aircraft Division of the Northrop Corporation. They will fabricate and ground test the main landing gear strut door, speed brake, leading edge flap, horizontal and vertical stabilizers of the F-5 over the next three years under an Air Force contract of a little less than $4 million. Testing of the main landing gear strut door should be completed by the end of 1969 and the speed brake, which will be made with chopped fibres in an epoxy matrix using matched moulds, should be tested by the middle of 1970. The leading edge flap and horizontal stabilizer will comprise full depth honeycomb structures covered with a one piece carbon skin. The most difficult structure, the vertical stabilizer is scheduled for fabrication in late 1971 and test completion in early 1972. It will utilize carbon fibre reinforced epoxy skins with spars and substructure and is representative of the structures common to most aircraft wings. The programme includes no flight testing, but Northrop may undertake this at its own expense.

This programme is indicative of the rate of progress to be expected over the next few years in the aircraft industry. Prime structures of commercial aircraft will probably

be reinforced with carbon fibre by about 1975. Other applications will increase as the price of fibre drops and the extent of knowledge of its usage increases.

Development of different matrices for use with carbon fibres is progressing and resins capable of withstanding 400°C should be available by about 1973. Metal matrices will be necessary in higher temperature applications and work is in progress on the reinforcement of nickel with carbon fibres. This should have a temperature limit of about 1000°C, but some estimates do not expect this material to reach its full potential before 1978.

An important question asked by the potential users of carbon fibre is whether this fibre is going to continue to be used for a reasonable period of time, or is some other newer and better material going to become available. One can never be certain, but it appears that in the foreseeable future there are only two improvements likely to be offered: better fibres and whiskers. Any improvement in fibres is likely to be small, and when improved fibres do become available, none of the present experience with carbon will be lost as it will almost certainly be directly applicable to the new fibre. Of the many possible whiskers, silicon carbide may be one of the first to become available. Evans and Parrat [28] have reported a programme of work which demonstrated the feasibility of producing these whiskers cheaply and of using them successfully, and licences have been issued to some manufacturers to develop

the process. But work to scale up production of carbon fibres was started over four years ago, so even if the work progresses smoothly one would not expect SiC whiskers to be commercially available in large quantities for at least three or four years. Also, once they are available, it may take a little longer to develop fabrication methods as with carbon fibres there was the experience gained with glass and boron to be drawn upon. The applications are also likely to be different: SiC whiskers do not appear to be very suitable for reinforcing resins, but are most likely to be used to reinforce aluminum especially in high temperature applications. Thus whiskers are likely to supplement carbon fibres rather than replace them.

A9. CONCLUSIONS

Carbon fibres are now commercially available and appear to offer advantages over other commercially available fibres. The advantages to be gained from using carbon fibre reinforced plastic, mainly an increase in specific stiffness, justify the expense of development programmes: this has been shown in particular by Rolls-Royce Ltd.

No substantially better fibre is likely to become available for a considerable time and whiskers, which may be an economic proposition by 1974 at the earliest, will tend to supplement rather than replace carbon fibres. Any development work on the use of carbon fibre reinforced plastic would, in any case, be applicable to another fibre and most would be applicable also to whiskers.

Fabrication methods that will be widely used are filament and tape winding and forming prepreg sheets in matched moulds, though some use of chopped fibres in thermo-plastics is to be expected. Military aircraft with composite prime structures should be flying by 1973 and commercial aircraft by 1975. Also by this time carbon fibre reinforced plastic will have become much more competitive in areas other than aerospace.

Carbon fibre reinforced plastic offers some problems to designers, principally because of its anisotropy and lack

of a plastic range. At the present time the effect of the anisotropy is to make the analysis much more complicated, but in time, as designers become more familiar with the theory of anisotropic elasticity and computer programmes for analysing composite structures become available, this problem will reduce in magnitude. The difficulty of attaching and joining composite structures, caused by the lack of a plastic range, can be alleviated by better design. This will only come by taking a fresh look at what the component should do and then designing without regard to traditional designs. For this reason it may be a bad idea to use carbon fibre reinforced plastic to just stiffen or replace existing sub-components as this will tend to give set ideas on its usage.

SECTION B: ELASTIC MODULI OF FIBRE REINFORCED COMPOSITES

B1. INTRODUCTION

B1.1. Stress-Strain Relationships for an Othortropic Body

If through each point of a body it is possible to draw a plane possessing the property that any two directions symmetric with respect to this plane are equivalent with respect to the elastic properties, the body is said to possess a plane of elastic symmetry. A body which has three orthogonal planes of elastic symmetry at each point is called orthogonally-anisotropic, or for brevity, orthotropic.

The stress-strain relationships for an orthotropic body may be expressed in the form [29]:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} \quad \text{----(1)}$$

Where σ and τ denote the normal and tangential components of stress, and ε and γ denote the normal and tangential components of strain respectively. $C_{ij} = C_{ji}$ and are the moduli of

elasticity. As shown, there are a maximum of 9 independent elastic moduli for an orthotropic body.

B1.2. Transverse Isotropy

If an axis of elastic symmetry of rotation passes through each point of an orthotropic body then the body is said to be transversely isotropic. All directions in the planes normal to the axis of symmetry are equivalent with respect to the elastic properties and the body is isotropic in these planes.

Clearly an isotropic matrix reinforced with isotropic fibres all orientated in one direction which form a hexagonal or random array in the plane normal to the fibres will be transversely isotropic with the plane normal to the fibres being isotropic. Carbon and graphite fibres are not generally isotropic, however, but are approximately transversely isotropic, with the cross sectional plane of the fibre being isotropic. Nevertheless, an isotropic matrix reinforced with these fibres will still be transversely isotropic.

The number of elastic moduli are reduced from nine for a general orthotropic body to five for a transversely isotropic body by the following relationships [29]:

$$C_{12} = C_{13}; \quad C_{22} = C_{33}; \quad C_{55} = C_{66};$$
$$\text{and } C_{44} = \frac{1}{2}\,(C_{33} - C_{23})$$

where the x-axis is perpendicular to the plane of isotropy. Then the stress-strain relationships may be written:

$$\sigma_x = C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{12}\varepsilon_z \qquad \text{----(2)}$$

$$\sigma_y = C_{12}\varepsilon_x + C_{22}\varepsilon_y + C_{23}\varepsilon_z \qquad \text{----(3)}$$

$$\sigma_z = C_{12}\varepsilon_x + C_{23}\varepsilon_y + C_{22}\varepsilon_z \qquad \text{----(4)}$$

$$\tau_{yz} = \tfrac{1}{2}(C_{22} - C_{23})\gamma_{yz} \qquad \text{----(5)}$$

$$\tau_{xz} = C_{66}\gamma_{xz} \qquad \text{----(6)}$$

$$\tau_{xy} = C_{66}\gamma_{xy} \qquad \text{----(7)}$$

Denoting the 'engineering' elastic constants as:

E_L longitudinal Young's modulus (i.e. in x direction)

E_T transverse Young's modulus (i.e. in plane of isotropy)

G_T shear modulus in plane of isotropy

G_{LT} longitudinal shear modulus (i.e. in x-y or x-z plane)

ν_{LT} major Poisson's ratio (i.e. $-\varepsilon_y/\varepsilon_x$ or $-\varepsilon_z/\varepsilon_x$ for pure tension in x direction)

ν_{TL} minor Poisson's ratio (i.e. $-\varepsilon_x/\varepsilon_y$ or $-\varepsilon_x/\varepsilon_z$ for pure tension in y or z directions respectively)

ν_{TT} transverse Poisson's ratio (i.e. $-\varepsilon_y/\varepsilon_z$ or $\varepsilon_z/\varepsilon_y$ for pure tension in z or y directions respectively)

It can easily be shown that these are related to the five elastic moduli by the expessions:

$$E_L = \frac{(C_{11}C_{22} + C_{11}C_{23} - 2C_{12}^2)}{(C_{22} + C_{23})} \qquad ----(8)$$

$$E_T = \frac{(C_{22}-C_{23})(C_{11}C_{22}+C_{11}C_{23}-2C_{12}^2)}{(C_{11}C_{22}-C_{12}^2)} \qquad ----(9)$$

$$G_T = \frac{1}{2}(C_{22}-C_{23}) \qquad ----(10)$$

$$G_{LT} = C_{66} \qquad ----(11)$$

$$\nu_{LT} = \frac{C_{12}}{(C_{22}+C_{23})} \qquad ----(12)$$

$$\nu_{TL} = \frac{C_{12}(C_{22}-C_{23})}{(C_{11}C_{22}-C_{12}^2)} \qquad ----(13)$$

$$\nu_{TT} = \frac{(C_{11}C_{23}-C_{12}^2)}{(C_{11}C_{22}-C_{12}^2)} \qquad ----(14)$$

Since there are only five independent elastic constants there must exist two relationships between the seven engineering constants used above. These are:

$$E_T = 2\,G_T\,(1+\nu_{TT}) \qquad ----(15)$$

and

$$\nu_{TL} = \nu_{LT}\,E_T/E_L \qquad ----(16)$$

By consideration of the inverse of C_{ij} it can easily be shown [30] that the elastic moduli may be expressed in terms of the engineering constants as follows:

$$C_{11} = \left(\frac{1}{E_T} - \frac{\nu_{TT}}{E_T}\right)/X \qquad \text{----(17)}$$

$$C_{22} = \left(\frac{1}{E_T E_L} - \frac{\nu^2_{LT}}{E_L^2}\right)/\left(\frac{1}{E_T} + \frac{\nu_{TT}}{E_T}\right)X \qquad \text{----(18)}$$

$$C_{12} = \nu_{LT} \; / \; E_L X \qquad \text{----(19)}$$

$$C_{23} = \left(\frac{\nu_{TT}}{E_T E_L} + \frac{\nu^2_{LT}}{E_L^2}\right)/\left(\frac{1}{E_T} + \frac{\nu_{TT}}{E_T}\right)X \qquad \text{----(20)}$$

$$C_{66} = G_{LT} \qquad \text{----(21)}$$

Where $$X = \frac{1}{E_L}\left(\frac{1}{E_T}(1-\nu_{TT}) - \frac{2}{E_L}(\nu^2_{LT})\right) \qquad \text{----(22)}$$

B1.3. Composite Elastic Moduli

With many different matrices and fibre reinforcements available and a variable fibre content, it becomes useful to be able to predict the properties of the composite in terms of the constituent material properties and the proportions of each material. The remainder of this section outlines and compares some methods by which the five elastic constants of the composite may be predicted from a knowledge of the two elastic constants of the isotropic matrix, two or five elastic constants of the isotropic or transversely isotropic fibre and the volume fraction of fibre in the composite.

B1.4. Some Previous Work

Chamis and Sendeckyj [31] have recently presented a very thorough critique on the theories predicting the thermoelastic properties of unidirectional fibrous composites covering over one hundred articles in which the method of approach varies from simple netting analysis to complex statistical methods. Concise descriptions and evaluations of these techniques are provided and current trends discussed. Numerical results for a few theories are presented together with a limited amount of experimental data, and it is shown that, in general, values predicted for E_T and G_{LT} are too low. It appears that the simple methods at present still provide at least as good agreement with experimental data as the more sophisticated methods, and so only a few of the simpler methods are presented here.

The method of Hashin and Rosen [32], though not the simplest of methods, is included more for general reference purposes than because of any agreement with experimental data as it is the method most often cited for comparison by other authors.

Whitney and Riley [33] used a method which may be considered somewhat analogous to that of Hashin and Rosen, but is much less rigorous mathematically and is written to appeal to the engineer rather than the mathematician. It is included here for this reason and also because Whitney [34] has extended

this method for use with anisotropic fibres and Blakslee et al [35] have found good agreement with experimental data from graphite fibre reinforced plastic using this latter method. References [33] and [34] contain numerous errors, however, and thus the results presented for these methods are not necessarily those given in the published papers, but are results which have been derived using these methods.

A further review of the more important prediction methods is contained in [36]. This reference suggests that for design purposes a simple but approximately precise formula is desirable for rapid calculation of the composite properties, and that the Halpin-Tsai equations give both quick and accurate results. Thus these equations are also included in this section and compared with other predictions and some experimental data.

B2. SOME METHODS FOR PREDICTING COMPOSITE ELASTIC MODULI

B2.1. Method of Hashin and Rosen

Hashin and Rosen [32] have derived bounds and expressions for the five elastic moduli of materials reinforced with hollow circular fibres by a variational method. Exact results have been obtained for hexagonal arrays of identical fibres and approximate results for a random array of fibres, which may have unequal cross sections, and modifications for use with solid fibres are included. The composite is assumed macroscopically homogeneous and the matrix and fibre are assumed linearly elastic, homogeneous and isotropic.

The results for a random array of solid fibres are presented as they have a much simpler form and their bounds coincide except for G_{TT}. Also this arrangement is the one which generally occurs in practice.

The bulk modulus governing plane-strain deformation in the yz - plane, K_T is used in the analysis and is related to the previously defined constants by the relationship:

$$E_T = \frac{4G_T K_T}{K_T + \Psi G_T} \qquad \text{----(23)}$$

where $$\Psi = 1 + \frac{4K_T \nu_{LT}^2}{E_L} \qquad \text{----(24)}$$

Then

$$K_T = k^m \frac{\phi[1 + (1-2\nu^m)v^f] + (1-2\nu^m)v^m}{(\phi v^m + v^f + 1-2\nu^m)} \qquad ----(25)$$

where v^m and v^f are the volume fractions of matrix and fibre respectively and

$$\phi = \frac{k^f}{k^m} \qquad ----(26)$$

where k^m and k^f are the plane-strain bulk moduli of matrix and fibre respectively and are given by

$$+ \frac{2\nu G}{(1-2\nu)} \qquad ----(27)$$

Then ET can be evaluated from equations (23) and (24).

The bounds on the transverse shear modulus are given by:

$$G_T^{(+)} = G^m \left(1 - \frac{2(1-\nu^m)}{(1-2\nu^m)} v^f \bar{A}_4^{\varepsilon}\right) \qquad ----(28)$$

and $$G_T^{(-)} = G^m \Big/ \left(1 + \frac{2(1-\nu^m)}{(1-2\nu^m)} v^f \bar{A}_4^{\sigma}\right) \qquad ----(29)$$

where the superscripts (+) and (-) refer to upper and lower bound solutions respectively, and $\bar{A}_4^{\varepsilon}$ and $\bar{A}_4^{\sigma}$ are each the solutions of systems of six linear simultaneous equations as given in Appendix [1].

The longitudinal shear modulus is given by:

$$G_{LT} = G^m \frac{\beta(1+v^f) + v^m}{\beta v^m + (1+v^f)} \qquad ----(30)$$

where $$\beta = \frac{G^f}{G^m} \qquad ----(31)$$

The longitudinal Young's modulus is given by:

$$E_L = v^f E^f + v^m E^m \qquad ----(32)$$

and the major Poisson's ratio by:

$$\nu_{LT} = \frac{v^f E^f L_1 + v^m E^m L_2 \nu^m}{v^f E^f L_3 + v^m E^m L_2} \qquad ----(33)$$

where

$$\left.\begin{aligned} L_1 &= 2\nu^f (1-\nu^{m^2}) v^f + v^m(1+\nu^m)\nu^m \\ L_2 &= v^f(1-\nu^f - 2\nu^{f^2}) \\ L_3 &= 2(1-\nu^{m^2})v^f + (1+\nu^m)v^m \end{aligned}\right\} \qquad ----(34)$$

B2.2. Method of Whitney and Riley

Whitney and Riley [33] have derived approximate expressions for four of the elastic moduli of a material reinforced with solid circular fibres and suggest the use of Hashin and Rosen's result for the fifth. The same assumptions as regards the properties of the constituents are made as before

and it is assumed that the fibres are packed in such a way as to make the composite transversely isotropic. The repeating element consists of a single filament embedded in a matrix cylinder of finite outer radius and the analysis consists of finding the stresses in the cylinders for various surface loadings and using the results in an energy balance to yield the elastic constants.

The expression derived for the longitudinal Young's modulus is the same as given in equation (32) and the use of equations (30) and (31) is suggested for the longitudinal shear modulus.

The major Poisson's ratio is given by the relationship:

$$\nu_{LT} = \nu^m - \frac{2(\nu^m-\nu^f)(1-\nu^m)E^f v^f}{E^m(1-\nu^m)L^f + [L^m v^f + (1+\nu^m)]E^f} \qquad ----(35)$$

where $L = 1-\nu-2\nu^2$ ----(36)

The plane strain bulk modulus is given by:

$$k_T = \frac{[(k^f+G^m)k^m + (k^f-k^m)G^m v^f]}{[(k^f+G^m) - (k^f-k^m)v^f]} \qquad ----(37)$$

where $k = \frac{E}{2L}$ ----(38)

which is equivalent to equation (27).

As an approximation it is assumed that the transverse Poisson's ratio is given by the law of mixtures relationship:

$$\nu_{TT} = \nu^f v^f + \nu^m (1-v^f) \qquad ----(39)$$

Then the transverse Young's modulus can be obtained from the expression:

$$E_T = \frac{2k_T(1-\nu_{TT})E_L}{E_L + 4k_T\nu_{LT}^2} \qquad ----(40)$$

and the transverse shear modulus can then be obtained from equation (15).

B2.3. Whitney's Extension to Include Anisotropic Fibres

Whitney [34] , by applying the equations of anisotropic elasticity to the fibre, extended the method above to include the effect of transversely isotropic fibres, with the plane of isotropy being normal to the axis of a fibre. Using subscripts as defined in B1.2. to distinguish between longitudinal and transverse fibre properties, the expression for the determination of the longitudinal composite Young's modulus becomes:

$$E_L = v^f E_L^f + v^m E^m \qquad ----(41)$$

while the major Poisson's ratio is given by the relationship:

$$\nu_{LT} = \nu^m - \frac{2(\nu^m - \nu_{LT}^f)(1-\nu^{m2})E_T^f v^f}{E^m(1-\nu^m)L^f + [L^m v^f + (1+\nu^m)]E_T^f} \qquad ----(42)$$

where L^m is given by equation (36) and

$$L^f = 1 - \nu_{TT}^f - 2\left(\frac{E_T^f}{E_L^f}\right)\nu_{LT}^{f\,2} \qquad ----(43)$$

The plane strain bulk modulus, k_T, is given by equation (37) where k^m is given by equation (38) and

$$k^f = \frac{E_T^f}{2L^f} \qquad ----(44)$$

Then, using the approximation

$$\nu_{TT} = \nu_{TT}^f v^f + \nu^m (1-v^f) \qquad ----(45)$$

the transverse Young's modulus can be obtained from equation (40).

The expression for the longitudinal shear modulus is unchanged except for the substitution of G_{LT}^f for G^f and is given by equations (30) and (31).

B2.4. The Halpin-Tsai Equations

It has been suggested [36] that it is often advantageous for design purposes to be able to rapidly calculate accurate estimates for the elastic moduli of fibre reinforced composites, and that a simple but approximately precise formula to interpolate the existing exact machine calculations available in the current literature is to be desired. Halpin and Tsai [37] have shown that Herman's solution [38] generalizing Hill's self consistent model [39] can be reduced to

very simple approximate forms: these are known as the Halpin- Tsai equations and aim to satisfy the need expressed above.

The expression for the longitudinal Young's modulus is the same as given in equation (41) while the major Poisson's ratio can be approximated by the relationship:

$$\nu_{LT} = \nu_{LT}^{f} v^{f} + \nu^{m}(1-v^{f}) \qquad \text{----(46)}$$

The expressions for the remaining three independent elastic moduli can be reduced to the form:

$$\frac{\rho}{\rho^{m}} = \frac{(1+ \zeta\eta v^{f})}{(1- \eta v^{f})} \qquad \text{----(47)}$$

where $$\eta = \frac{(\rho^{f}/\rho^{m}-1)}{(\rho^{f}/\rho^{m}+\zeta)} \qquad \text{----(48)}$$

and ρ = composite moduli, E_T, G_{LT}, or G_T

ρ^{m} = corresponding matrix moduli

ρ^{f} = corresponding fibre moduli

ζ = a measure of the reinforcement which depends upon the boundary conditions

It is suggested in [36] that for estimating G_{LT}, the ζ factor is given by

$$\zeta = 1 \qquad \text{----(49)}$$

while for E_T it is given by

$$\zeta = 2 \qquad \text{----(50)}$$

The expression for estimating G_T becomes identical to that derived by Herman if ζ is given by

$$\zeta = \frac{1}{(3-4\nu^m)} \qquad \text{----(51)}$$

B2.5. A Modification for the Longitudinal Shear Modulus

The estimates for the ζ factors given above were obtained from comparison with numerical micromechanics solutions employing formal elasticity theory. It is said [36] that by setting ζ =1 and 2, equation (47) yields results for G_{LT} and E_T respectively that duplicate those of Adams and Doner [40,41], obtained by a lengthy numerical procedure, for all ratios of ρ^f/ρ^m. On inspection, however, it is clear that with ζ=1, equation (47) reduces to the same expression as derived by Hashin and Rosen [32] for a random array of solid fibres [equation (30)]: this expression is generally regarded as giving poor agreement with experimental results [31, 36], the predictions being too low. Referring to Table II of [40], which compares the numerical results of Adams and Doner with the predictions of [32], it appears that equation (47) with ζ=1 only gives reasonable agreement with [40] for fibre volume fractions up to about 0.5, and that using this equation for a volume fraction of 0.75 would give a value about 30% too low when G^f_{LT}/G^m = 20 (this is a typical modulus

ratio for a glass or carbon-resin system).

Since it has been shown [42] that the predictions of [40] agree very well with experimental data for both carbon and glass fibre reinforced plastics, it would seem to be appropriate to determine a function for ζ that makes equation (47) more nearly approximate these results. The considerations governing the choice of the function were (i) that it should increase G/G^m for high values of v^f; (ii) that G/G^m should still reduce to the correct values at $v^f=0$ and 1; (iii) that it should be a simple function capable of rapid calculation. These three conditions are met by choosing the function to be:

$$\zeta = 1 + 40(v^f)^{10} \qquad \text{----(52)}$$

There is no apparent theoretical justification for choosing this function, and this approximation is no more than simple curve fitting. However, substitution of equation (52) into equations (48) and (47) gives results which approximate to Adams and Doner's solution to a much greater degree than is obtained using $\zeta = 1$, as can be seen in Fig. [16]. The maximum discrepancy for fibre volume fractions of up to 0.75 (which is a common practical limit) is of the order of 10%, which is quite acceptable for engineering estimates, compared with discrepancies of up to 50% when using $\zeta = 1$.

It is undoubtedly possible to determine a function for ζ which will offer an even closer approximation to the results of [40]. Such a function would generally be more complex than the one given in equation (52), however, and would no longer offer a simple rapid estimate of the modulus, whereas the expression for ζ given above is a simple piece of mental arithmetic assuming access to a set of logarithmic tables. Furthermore it is difficult to justify a more accurate approximation until a greater quantity of reliable experimental data becomes available.

B3. NUMERICAL RESULTS AND DISCUSSION

B3.1. Comparison of Numerical Results for Glass Fibre Reinforced Plastic with Experimental Data

Comparison of theoretical predictions with experimental data is somewhat difficult because of the limited amount of reliable data that is currently available. Furthermore the data that is available tends to be from samples with slightly different constituent properties which further complicates the process. The results presented in this section are, with the exception of Fig. [17], for an E glass-epoxy resin system with the constituent properties shown in Table [ix].

Table [ix] Elastic properties of E-glass and epoxy resin

	E-glass	Epoxy Resin
E	10.6×10^6 lbf/in^2	0.5×10^6 lbf/in^2
ν	0.22	0.35
$G = \frac{E}{2(1+\nu)}$	4.34×10^6 lbf/in^2	0.185×10^6 lbf/in^2

This particular system was chosen since it is one of the most common and provides therefore a useful comparison with the carbon fibre systems presented later.

To show the effect of the modification to the Halpin-Tsai equation for the longitudinal shear modulus, presented in B2.5, however, a different system was used. Fig. [17] shows some experimental data obtained from [42] for an E-glass, polyester resin system, together with Adams and Doner's predictions and the predictions based on equation (47) with $\zeta = 1$ and $\zeta = 1+40(v^f)^{10}$. This particular set of data was used as it was obtained from pure torsion of a solid circular cylindrical rod with the fibres oriented parallel to the longitudinal axis. This is recognized as being the best method of determining the shear modulus, though it has not often been used as specimens are usually prepared from prepreg from which it is much easier to produce plates than solid rods. The data would seem to be reliable since there is very little observed scatter, in contrast to the results obtained from other methods. It can be seen from Fig. [17] that the modified Halpin-Tsai equation approximates Adams and Doner's numerical solution quite closely and that it fits the experimental data to within about 10%, the approximation underestimating the experimental values. The modification clearly improves the accuracy of the prediction for values of v^f above about 0.5 and does not have a deleterious effect below this value.

Figures [18] to [22] show the predictions for the five independent elastic constants for the glass fibre epoxy

resin system described above with some experimental data obtained from [43]. Fig. [18] shows that the rule of mixtures prediction used by each of the methods approximates the data to within about 10%, and that the data is distributed both above and below the prediction.

From Fig. [19] it is clear that there is quite a spread in both the predictions and the experimental data for the transverse Young's modulus. Hashin and Rosen's lower bound yields the lowest prediction while Halpin and Tsai's and Whitney and Riley's prediction are very similar to Hashin and Rosen's upper bound solution, but the error is as much as 25% in some cases.

The prediction shown in Fig. [20] for the major Poisson's ratio using the method of Hashin and Rosen is the same as when using Whitney and Riley's method. It gives slightly lower values than the rule of mixtures as used by Halpin and Tsai, but the maximum discrepancy is only about 3%. The experimental data is somewhat limited but is generally lower than both predictions, the maximum difference between the rule of mixtures prediction and the experimental data being about 10%.

Fig. [21] shows the predictions for the longitudinal shear modulus of the glass fibre reinforced epoxy system. The set of experimental data shown is subject to large variations probably because of the method of testing.

Nonetheless, it is clear that the modified Halpin-Tsai equation gives a better approximation to it than Hashin and Rosen's prediction.

The Halpin-Tsai prediction for the transverse shear modulus lies between Hashin and Rosen's upper and lower bounds, as shown in Fig. [22], while Whitney and Riley's prediction lies somewhat above the upper bound. There is no experimental data with which to compare these predictions, however, as it is very difficult to measure the transverse shear modulus.

B3.2. Numerical Results for Carbon Fibre Reinforced Plastic

In order to give some idea of the variation in the elastic properties of carbon fibre reinforced epoxy resin with fibre content, some results have been calculated using the Halpin-Tsai equations (the modified form for the longitudinal shear modulus) and these results are shown in Fig. [23] to [27]. The Halpin-Tsai equations were used for the predictions as these were shown in the previous section to give reasonable approximations to the experimental data while being of a very simple form. Since the elastic moduli of the carbon fibre itself are not currently available, with the exception of the longitudinal Young's modulus, estimates for both the high modulus and high strength fibres were obtained by comparison with some estimated and measured values for

Thornel given in [35] and [36]. The figures for Thornel have been shown to be reasonable by comparing predicted composite values with experimental data [35]. The estimates used for the carbon fibre herein are shown in Table [x], while the moduli of the epoxy resin are given in Table [ix].

Table [x] Estimated elastic properties of carbon fibres

	High modulus fibre	High Strength fibre
E_L	60×10^6 lbf/in^2	40×10^6 lbf/in^2
E_T	1.2×10^6 lbf/in^2	1.5×10^6 lbf/in^2
G_T	0.5×10^6 lbf/in^2	0.6×10^6 lbf/in^2
G_{LT}	4.0×10^6 lbf/in^2	4.0×10^6 lbf/in^2
ν_{LT}	0.2	0.2

The term 'high modulus' refers only to the longitudinal Young's modulus of the fibre, and it can be seen from Fig. [23] to [27] that the only elastic modulus of the composite that is greater with the high modulus fibre reinforcement is the longitudinal Young's modulus. Both the transverse shear modulus and the transverse Young's modulus of the composite are greater for the high strength fibre, while the longitudinal shear modulus is the same for both

since the fibre modulus was assumed to be the same for both types.

B3.3 Effect of the Fibre Anisotropy

To show the effect of neglecting the anisotropy of carbon fibre, some predictions of the elastic moduli have been calculated using the same method as in the previous section, and assuming the fibre properties to be:

$$E_L^f = 60 \times 10^6 \text{ lbf/in}^2 \text{ and } \nu_{LT}^f = 0.2$$

Clearly the longitudinal Young's modulus and the major Poisson's ratio will be unaffected as they depend upon the longitudinal fibre properties which are unchanged. The transverse properties are affected, however, and Fig. [28] shows the variation of the transverse Young's modulus with fibre content for the high modulus fibre reinforced epoxy resin calculated using the Halpin-Tsai equations assuming both isotropic and anisotropic fibre. Because of the reduced value of the transverse fibre modulus in the anistropic case, the modulus of the resulting composite is also reduced, and it can be seen from Fig. [28] that for a fibre content of 0.70 assuming the fibre to be isotropic results in a predicted transverse modulus four times too large.

Fig. [29] shows that the predictions for the longitudinal shear modulus of the composite are also larger when the fibre is assumed to be isotropic. The difference is not as great as for the transverse Young's modulus, however, because the fibre shear modulus is assumed to be very similar for isotropic or anisotropic fibres.

B4. CONCLUSIONS

It has been shown that the simple Halpin-Tsai equations for predicting the elastic moduli of fibre reinforced composites give good agreement with experimental data for glass fibre reinforced epoxy resin in the case of the longitudinal Young's modulus, the major Poisson's ratio and the transverse Young's modulus. The agreement is at least as good as provided by some other common, but more complex, methods and the Halpin-Tsai equations are thus to be preferred.

The modified form of the Halpin-Tsai equation for the longitudinal shear modulus gives excellent agreement with experimental data for glass fibre reinforced polyester resin, and gives better agreement in general than analytical methods. Numerical procedures may give slightly closer approximations but the simplicity of the modified Halpin-Tsai equation makes it ideal for rapid design estimates.

There is no experimental data with which to compare the predictions for the transverse shear modulus, but the appropriate Halpin-Tsai equation gives values consistent with those of other predictions and may therefore be used until more information becomes available.

Some typical values for the elastic moduli of both high modulus and high strength carbon fibre reinforced epoxy resin have been given and only in the case of the longitudinal

Young's modulus does the high modulus fibre give a higher composite modulus. If the fibre is assumed to be isotropic the predictions for the transverse Young's modulus are very much higher than when anisotropic fibre properties are assumed.

SECTION C: ANALYSES OF COMPOSITE STRUCTURES

C1. INTRODUCTION

Most fibre composite structures today are effectively built up from a series of thin sheets containing unidirectional fibres, and this will continue to be the principal method of construction for some years to come. Thus, although its use might at first sight appear to be rather limited, a method for analysing just laminated composites is quite adequate when dealing with fibre reinforced composite structures. Furthermore, extension of this method to include random chopped fibre composites, which are the only other fibre composites likely to be used, is possible.

Essentially, the method consists of finding the stress-strain behaviour of a unidirectional layer and then using this, together with the thickness, number and orientations of the layers of a structure, to predict the stress-strain behaviour of the structure. Applying the external loading system, the stress-strain state of the structure is determined and then referred back to the individual layers, where a failure criteria is used to determine whether the individual layers can withstand the given loading system.

This method was first presented by Tsai [43,44,45] though not very clearly, and has very recently been presented in some depth [36]. This section attempts to present a clear outline of the method sufficient to enable a design engineer to use the computer programme given in section D3, with confidence and understanding, without going to the depth required by a research engineer.

C2. THEORY OF LAMINATED COMPOSITES

C2.1. Characterization of a Lamina

In chapter B1.2. it was shown that a unidirectional fibre reinforced composite can be characterized in general by five independent elastic constants, and the stress-strain relationships are given as:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{23} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{22}-C_{23}) & 0 & 0 \\ & & & & C_{66} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} \quad -(53)$$

Since a lamina is usually thin, a state of plane

stress within each lamina is assumed, whence

$$\sigma_z = \tau_{yz} = \tau_{xz} = 0 \qquad \text{----(54)}$$

Substituting (54) into (53) gives

$$\gamma_{yz} = \gamma_{xz} = 0$$

and

$$\varepsilon_z = -\frac{C_{12}}{C_{22}}\varepsilon_x - \frac{C_{23}}{C_{22}}\varepsilon_y \qquad \text{----(55)}$$

and then substitution of (55) into (53) yields

$$\left.\begin{aligned} \sigma_x &= (C_{11} - \frac{C_{12}^2}{C_{22}})\varepsilon_x + (C_{12} - \frac{C_{12}C_{23}}{C_{22}})\varepsilon_y \\ \sigma_y &= (C_{12} - \frac{C_{12}C_{23}}{C_{22}})\varepsilon_x + (C_{22} - \frac{C^2_{23}}{C_{22}})\varepsilon_y \\ \gamma_{xy} &= C_{66}\gamma_{xy} \end{aligned}\right\} \text{----(56)}$$

Defining

$$\left.\begin{aligned} Q_{11} &= (C_{11} - \frac{C_{12}^2}{C_{22}}) \\ Q_{12} &= (C_{12} - \frac{C_{12}C_{23}}{C_{22}}) \\ Q_{22} &= (C_{22} - \frac{C_{23}^2}{C_{22}}) \end{aligned}\right\} \text{----(57)}$$

and

$$Q_{66} = C_{66}$$

the stress-strain relationships may be expressed as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \qquad ----(58)$$

It may be noted that there are only four independent elastic constants necessary to characterize a lamina, the constant governing shear in the yz plane being unnecessary.

Considering successively pure tension in the x and y directions and then pure shear in the xy plane, it can be easily shown that the elastic constants given above may be expressed in terms of engineering constants by the following relationships:

$$Q_{11} = \frac{E_L}{(1 - \nu_{TL}\nu_{LT})} \qquad ----(59)$$

$$Q_{22} = \frac{E_T}{(1-\nu_{TL}\nu_{LT})} \qquad ----(60)$$

$$Q_{12} = \nu_{LT}Q_{22} = \nu_{TL}Q_{11} \qquad ----(61)$$

and $$Q_{66} = G_{LT} \qquad ----(62)$$

To take account of thermal strains that arise when the operating temperature is different from the laminating temperature (at which a stress free state is assumed to exist), one notes that the total strain may be considered as the sum of the mechanical and free thermal strains, i.e.

$$\varepsilon_i = \varepsilon_i^m + \alpha_i T \qquad \text{----(63)}$$

where α_i is the appropriate thermal expansion coefficient and T is the difference between the lamination and operating temperature [$T=(T_{oper}-T_{lam})$]

$$\therefore \varepsilon_i^m = \varepsilon_i - \alpha_i T \qquad \text{----(64)}$$

For an orthotropic material the thermal expansion matrix is given as [46],

$$\alpha_i = \begin{bmatrix} \alpha_x & 0 & 0 \\ 0 & \alpha_y & 0 \\ 0 & 0 & \alpha_z \end{bmatrix} \qquad \text{----(65)}$$

Then for a plane stress system it is easily shown that the overall stress-strain relationships, taking account of thermal effects may be written:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} (\varepsilon_x - \alpha_x T) \\ (\varepsilon_y - \alpha_y T) \\ \gamma_{xy} \end{bmatrix} \qquad \text{----(66)}$$

This is the basic equation for a limina with the fibres oriented along the x direction. For laminae with fibres not oriented along the x direction, the stiffness and thermal expansion matrices given above must be transformed to different coordinate axis. Then for fibres oriented at any angle θ to the reference coordinates, xy, equation (66) becomes:

$$\begin{bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau}_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} (\bar{\varepsilon}_x - \bar{\alpha}_x T) \\ (\bar{\varepsilon}_y - \bar{\alpha}_y T) \\ (\bar{\gamma}_{xy} - \bar{\alpha}_{xy} T) \end{bmatrix} \qquad ----(67)$$

where $\bar{Q}_{ij}$ and $\bar{\alpha}_i$ are given by the usual transformations [36] i. e.

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta \\
\bar{Q}_{22} &= Q_{11}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta \\
\bar{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta+\cos^4\theta) \\
\bar{Q}_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta+\cos^4\theta) \\
\bar{Q}_{16} &= (Q_{11}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta + (Q_{12}-Q_{22}+2Q_{66})\sin^3\theta\cos\theta \\
\bar{Q}_{26} &= (Q_{11}-Q_{12}-2Q_{66})\sin^3\theta\cos\theta + (Q_{12}-Q_{22}+2Q_{66})\sin\theta\cos^3\theta
\end{aligned} \qquad ----(68)$$

and
$$\begin{aligned} \bar{\alpha}_x &= \alpha_x\cos^2\theta + \alpha_y\sin^2\theta \\ \bar{\alpha}_y &= \alpha_x\sin^2\theta + \alpha_y\cos^2\theta \\ \bar{\alpha}_{xy} &= 2\sin\theta\cos(\alpha_y - \alpha_x) \end{aligned} \qquad \text{----(69)}$$

In order to clarify the development of the remainder of the equations relating to laminated composites the effect of thermal strains will not be included in the following derivations. They can, however, be dealt with quite simply in the same way as the mechanical strains, and for completeness their effect is included in the final results.

The constitutive equations for the p^{th} layer for the remainder of this section will be denoted by:

$$[\sigma]_p = [Q]_p[\varepsilon]_p \qquad \text{----(70)}$$

for the xy coordinate system (fibres coincident with x direction) and

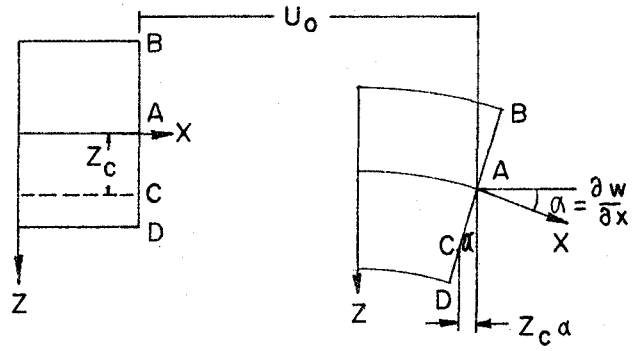
$$[\bar{\sigma}]_p = [\bar{Q}]_p[\bar{\varepsilon}]_p \qquad \text{----(71)}$$

for the $\bar{x}\bar{y}$ coordinate system (oriented at an angle θ to the xy system).

2.2. Strain-Displacement Relationships

Consider the deformation of a section of a laminate in the xz plane as shown below. It is assumed that point A

at the geometrical midplane undergoes a displacement u_o in the x direction and that the normal to the midplane remains straight.



a) initial cross-section

b) deformed cross-section

Then the displacement in the x direction of any point C on the normal is given by

$$u_c = u_o - z_c \frac{\partial w}{\partial x} \qquad \text{----(72)}$$

where z_c is the z coordinate of C measured from the midplane and w is the displacement the z direction.

Since

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \text{----(73)}$$

for small deformations, then

$$\varepsilon_x = \frac{\partial}{\partial x}\left(u_o - z\frac{\partial w}{\partial x}\right)$$

i.e. $$\varepsilon_x = \frac{\partial u_o}{\partial x} - \frac{\partial^2 w}{\partial x^2} \qquad \text{----(74)}$$

But the midplane strain, ε_x^o, is defined as

$$\varepsilon_x^o = \frac{\partial u_o}{\partial x} \qquad \text{----(75)}$$

and the plate curvature, k_x, is defined as

$$k_x = -\frac{\partial^2 w}{\partial x^2} \qquad \text{----(76)}$$

Thus, substituting equations (75) and (76) into (74)

$$\varepsilon_x = \varepsilon_x^o + zk_x$$

Similarly it can be shown that

$$\varepsilon_y = \varepsilon_y^o + zk_y$$

and $$\gamma_{xy} = \gamma_{xy}^o + zk_{xy}$$

Thus $$[\varepsilon] = [\varepsilon^o] + z[k] \qquad \text{----(77)}$$

Hence the strains at any point in a laminate may be determined from the midplane strains, which are known functions of the midplane displacements, the plate curvatures, which are known functions of the deflection (w), and the z coordinate. Then equation (71) may be rewritten

$$[\bar{\sigma}]_p = [\bar{Q}]_p [\bar{\varepsilon}^o] + z[\bar{Q}]_p[\bar{k}] \qquad \text{----(78)}$$

C2.3. Stress and Moment Resultants

Since the stress in a laminate varies from layer to layer, it is more convenient to use an equivalent system of stress and moment resultants on a laminate. Three stress and three moment resultants are defined which together form a statically equivalent stress system, but which is applied to the geometrical midplane. These six quantities are defined as:

$$N_x = \int_{-h/2}^{h/2} \sigma_x \, dz \qquad \text{----(79)}$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y \, dz \qquad \text{----(80)}$$

$$N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \, dz \qquad \text{----(81)}$$

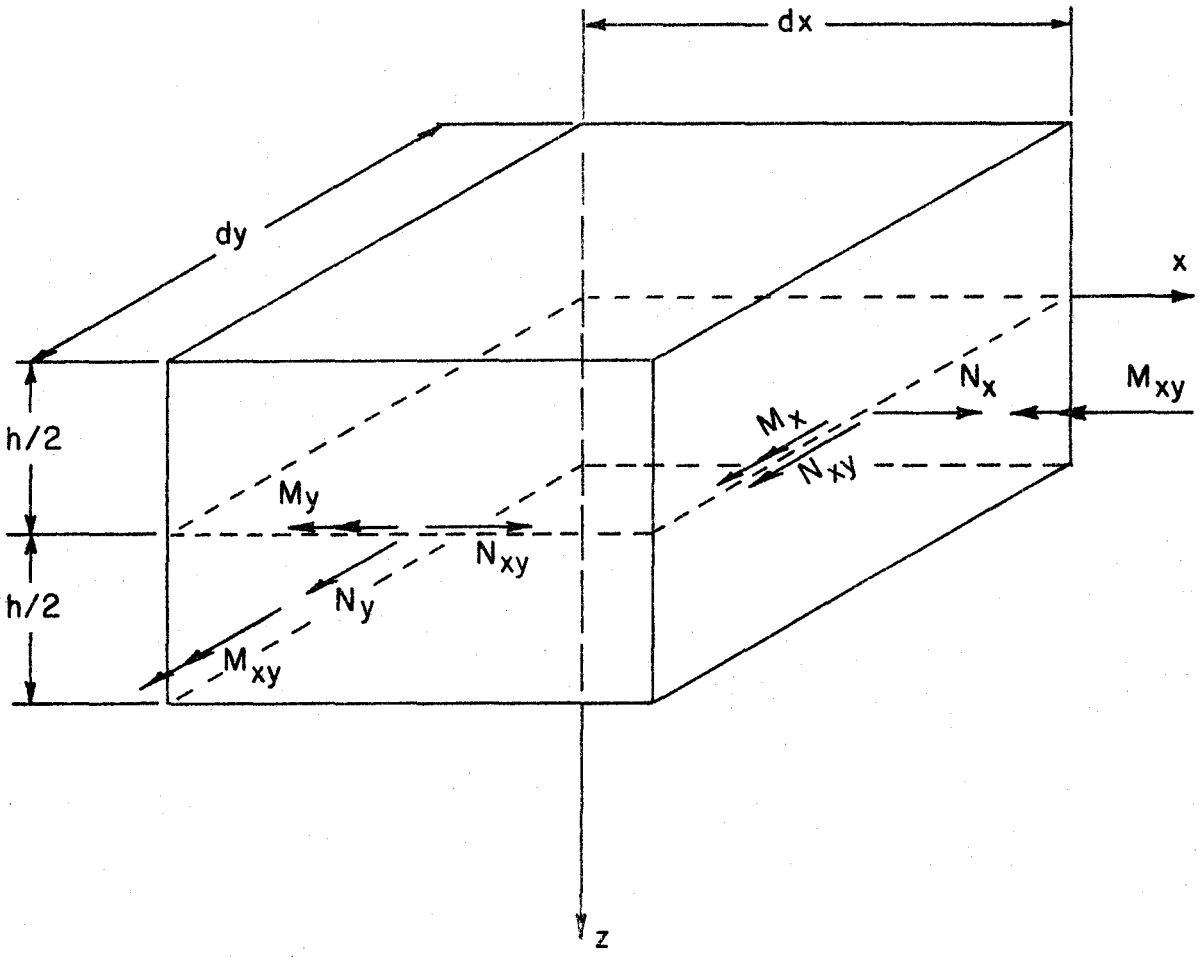
$$M_x = \int_{-h/2}^{h/2} \sigma_x \, z dz \qquad \text{----(82)}$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y \, z dz \qquad \text{----(83)}$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \, z dz \qquad \text{----(84)}$$

where h is the thickness of the laminate.

Clearly the stress resultants, N, have the dimensions of force per unit length and the moment resultants, M, moment per unit length. The positive directions of these resultants are shown below.



C2.4. Laminate Constitutive Equations

Equations (79-81) can be expressed in the form

$$[N] = \int_{-h/2}^{h/2} [\bar{\sigma}]\, dz \qquad \text{----(85)}$$

Substituting equation (78) into (85) and separating the continuous integral into an integral over n layers yields

$$[N] = \sum_{p=1}^{n} \left[\int_{h_{p-1}}^{h_p} [\bar{Q}]_p [\bar{\varepsilon}^o]\, dz + \int_{h_{p-1}}^{h_p} [\bar{Q}]_p [\bar{k}]\, z dz \right] \qquad (86)$$

where h_{p-1} and h_p are the z coordinates of the p^{th} layer. But since $[\bar{Q}]$, $[\bar{\varepsilon}^o]$ and $[\bar{k}]$ are independent of z.

$$[N] = \sum_{p=1}^{n} \left[[\bar{Q}]_p [\bar{\varepsilon}^o] \int_{h_{p-1}}^{h_p} dz + [\bar{Q}]_p [\bar{k}] \int_{h_{p-1}}^{h_p} z dz \right] \qquad \text{--(87)}$$

Furthermore $[\bar{\varepsilon}^o]$ and $[\bar{k}]$ are not functions of p and thus equation (87) may be reduced to the form:

$$[N] = [A]\,[\bar{\varepsilon}^o] + [B]\,[\bar{k}] \qquad \text{----(88)}$$

where

$$A_{ij} = \sum_{p=1}^{n} (\bar{Q}_{ij})_p \, (h_p - h_{p-1}) \qquad \text{----(89)}$$

and

$$B_{ij} = \frac{1}{2}\sum_{p=1}^{n}(\bar{Q}_{ij})_p\,(h_p^2 - h_{p-1}^2) \qquad ----(90)$$

Similarly equations (82-84) can be expressed in the form

$$[M] = \int_{-h/2}^{h/2}[\bar{\sigma}]\,z\,dz \qquad ----(91)$$

which may be reduced to the form

$$[M] = [B]\,[\bar{\varepsilon}^{o}] + [D]\,[\bar{k}] \qquad ----(92)$$

where $$D_{ij} = \frac{1}{3}\sum_{p=1}^{n}(\bar{Q}_{ij})_p(h_p^3 - h_{p-1}^3) \qquad ----(93)$$

and B_{ij} is defined as before.

Combining equations (88) and (92), the total plate constitutive equation can be written as:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{bmatrix} \bar{\varepsilon}^{o} \\ \bar{k} \end{bmatrix} \qquad ----(94)$$

By simple matrix manipulation it can be shown that equation (94) can also be expressed in the form

$$\begin{bmatrix} \bar{\varepsilon}^{o} \\ M \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix}\begin{bmatrix} N \\ \bar{k} \end{bmatrix} \qquad ----(95)$$

where

$$\left.\begin{aligned} [A^*] &= [A]^{-1} \\ [B^*] &= -[A]^{-1}[B] \\ [C^*] &= [B]\,[A]^{-1} \\ [D^*] &= [D] - [B]\,[A]^{-1}[B] \end{aligned}\right\} \qquad ----(96)$$

This form of equation is used in plate and shell formulation. Complete inversion of equation (94) yields:

$$\begin{bmatrix} \bar{\varepsilon}^{o} \\ \bar{k} \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix} \qquad \text{----(97)}$$

where

$$\left.\begin{aligned} [A'] &= [A^*] - [B^*]\,[D^*]^{-1}\,[C^*] \\ [B'] &= [B^*]\,[D^*]^{-1} \\ [C'] &= [D^*]^{-1}\,[C^*] \\ [D'] &= [D^*]^{-1} \end{aligned}\right\} \text{----(98)}$$

Equations (94), (95) and (97) are the most useful forms of the laminate constitutive equations; they can be obtained from the elastic properties of each lamina, the stacking sequence and some simple matrix calculations.

The effect of temperature variations can be included by defining thermal stress and moment resultants in a similar way as for the mechanical stress and moments, i. e.

$$N_x^T = \int_{-h/2}^{h/2} (\bar{Q}_{11}\bar{\alpha}_x + \bar{Q}_{12}\bar{\alpha}_y + \bar{Q}_{11}\bar{\alpha}_{xy})\, T\, dz \qquad \text{---(99)}$$

etc. Then N_x etc. in equations (94), (95) and (97) is replaced by $\bar{N}_x$ etc., where

$$\bar{N}_x = N_x + N_x^T \qquad \text{---(100)}$$

C2.5. Effect of Coupling Terms

Equation (92) indicates that for a general laminated plate the bending moments arise from, or are given in terms of, the midplane strains and the plate curvatures. That is stretching (or compressing) the midplane as well as enforcing curvatures results in bending moments. Similarly equation (88) shows that normal stresses are given in terms of both the strains and curvatures. Hence under a pure tensile load, for example, a general laminate will twist as well as stretch. One consequence is that tensile testing of laminated specimens requires special equipment as normal methods do not allow twisting to occur; thus a bending moment would be imposed and so the load would no longer be purely tensile. Also temperature variations will induce bending of a general laminate because of these coupling terms.

C3. SOME SPECIAL LAMINATES

Two classes of laminates are worthy of particular attention because of the simplifications that are introduced in the constitutive equation when they are used. One of these is known as a midplane symmetric laminate and, as its name implies, for each layer above the midplane there is an identical layer (both in orientation and properties) at the same distance below the midplane. Now the terms in the [B] matrix are given by equation (90) as:

$$B_{ij} = \frac{1}{2} \sum_{p=1}^{n} (\bar{Q}_{ij})_p \, (h_p^2 - h_{p-1}^2)$$

Clearly B_{ij} is even in h_p and thus for a midplane symmetric laminate [B] is identically zero. Besides considerably simplifying the constitutive equation and thus making composite analysis much easier, such laminates are free from the bending-stretching coupling present in non-symmetric laminates. Thus thermal variations, for example, will not cause warping of the laminate.

Another laminate deserving attention is one in which for every lamina at a positive orientation θ there is another with the same properties (including thickness) at a negative orientation θ. Now $\bar{Q}_{16}$ (and $\bar{Q}_{26}$) has the same absolute value

for a positive orientation θ as for a negative orientation, but with an opposite sign. Thus from equation (89) it is clear that for this type of laminate A_{16} and A_{26} are zero. This means that such a laminate will be orthotropic with respect to the inplane forces and strains.

Clearly it is also possible to construct a laminate in which [B] is identically zero and $A_{16}=A_{26}=0$. Such a laminate is orthotropic with respect to the inplane problem.

C4. LAMINATE STRENGTH ANALYSIS

The basic assumption involved in a laminate strength analysis is that a strength criterion for a lamina under a state of plane stress exists and that the criterion is valid for any orientation of the lamina in the laminate. The criterion generally used is based on Hill's [47] generalization of the Mises (distortional energy) isotropic yield criterion. Tsai [44] proposed that failure by yielding and ultimate strength could be considered synonymous for fibre reinforced composites, and that the strength criterion for plane stress could then be written

$$\left(\frac{\sigma_x}{X}\right)^2 - \left(\frac{X}{Y}\right)\left(\frac{\sigma_x}{X}\right)\left(\frac{\sigma_y}{Y}\right) + \left(\frac{\sigma_y}{Y}\right)^2 + \left(\frac{\tau_{xy}}{S}\right)^2 = 1 \qquad \text{----(101)}$$

where X and Y are the tensile or compressive strengths in the x and y directions respectively of the lamina and S is the shear strength. These three quantities are usually determined experimentally.

To estimate whether a particular laminate can withstand some given loading system, the stress and moment resultants are found from a load analysis of the overall structure and then the state of strain in the laminate can be obtained using equation (97). Then using equation (78) the stress state in each lamina is determined in the structural

($\bar{x}\bar{y}$) axis system. Since equation (101) applies only to the natural (xy) axis system, the stresses must be transformed to this system by the usual stress transformation equations before substitution into the strength criterion. The laminate will withstand the load if the left hand side of equation (101) is less than one for each and every lamina. This procedure may be repeated for loadings up to the point where one lamina fails, at which point the laminate might fail or the other laminae might be able to carry the additional load. What load the failed lamina is capable of taking is not well understood at this time, however, and for design purposes it is usually assumed that the failed lamina is incapable of taking any load. The actual failing load is generally considered to be the load at which the first lamina fails, but may also be calculated as the maximum load, if as each lamina fails it is assumed to contribute nothing to the laminate apart from holding the other laminae together.

As might be expected it is also possible to determine the maximum values of the stress and moment resultants for particular loading configurations by obtaining the lamina stresses in terms of these resultants and then substituting in the strength criterion and solving for the resultants. One complication with this procedure is that the tensile and compressive strengths of a lamina may not be equal, particularly

in the transverse direction. This means that the values of X and Y in equation (101) depend upon the signs of σ_x and σ_y. This is easily taken into account in a computer programme, however.

C5. COMPARISON WITH NETTING ANALYSIS

Netting analysis has been and is still widely used in some sectors of industry to analyse filament wound and prepreg structures. The method assumes that the filaments are perfectly flexible and the matrix perfectly compliant. Thus netting analysis, in a very approximate manner, takes into account the [A] matrix but ignores the [B] and [D] matrices.

As for the strength analysis, netting analysis assumes that all the load is taken by the fibres in the fibre direction. This would suggest that for a unidirectional 45° lamina, for example, the uniaxial strength would be about 0.7 of the strength in the fibre direction, while for a typical fibre reinforced composite, the theory presented above suggests a figure of only about 0.1. Furthermore, netting analysis predicts the strength in the fibre direction to be given by the simple rule of mixtures relationship, which is not confirmed experimentally.

A comparison between the two analyses for the strength of a unidirectional composite is shown in Fig. [30] with experimental data for glass fibre reinforced plastic taken from [44]. The values for X, Y and S used in the computation were 150×10^3, 4×10^3 and 6×10^3 lbf/in^2 respectively, and the fibre strength and volume fraction were

taken as 400×10^3 lbf/in^2 and 0.6. Clearly the continuum analysis gives a much better approximation to the experimental data.

As a further example, Fig. [31] shows the stiffness and strength of a typical cross-ply glass fibre reinforced plastic composite. The experimental data is again taken from [44] and it may be noted that a further limitation of netting analysis is that it gives no information as to when individual layers will fail.

C6. CONCLUDING REMARKS

It is now accepted in major American aerospace industries that the type of analysis presented in this section is more realistic than conventional netting analysis, and its adoption by industries still using netting analysis is to be encouraged. The approach may also be used to investigate the vibration and stability etc. of laminated structures (e.g. [48]).

A computer programme which encompasses all of the above results is discussed and listed in section D3.

SECTION D: APPENDICES

D1. DETERMINATION OF $\bar{A}_4^{\varepsilon}$ AND $\bar{A}_4^{\sigma}$

The constant $\bar{A}_4^{\varepsilon}$ used in the expression for the upper bound of the transverse shear modulus presented by Hashin and Rosen is given by the solution of the following system of equations.

$$\bar{A}_1^{\varepsilon} + \frac{1}{v^f}\bar{A}_2^{\varepsilon} + v^f{}^2\bar{A}_3^{\varepsilon} + v^f\bar{A}_4^{\varepsilon} = 1 \quad -(102)$$

$$-\frac{(3-4v^m)}{(3-2v^m)v^f}\bar{A}_2^{\varepsilon} - 2v^f{}^2\bar{A}_3^{\varepsilon} + \frac{v^f}{(1-2v^m)}\bar{A}_4^{\varepsilon} = 0 \quad -(103)$$

$$\bar{A}_1^{\varepsilon} + \bar{A}_2^{\varepsilon} + \bar{A}_3^{\varepsilon} + \bar{A}_4^{\varepsilon} - \bar{B}_1^{\varepsilon} - \bar{B}_2^{\varepsilon} = 0 \quad -(104)$$

$$-\frac{(3-4v^m)}{(3-2v^m)}\bar{A}_2^{\varepsilon} - 2\bar{A}_3^{\varepsilon} + \frac{1}{(1-2v^m)}\bar{A}_4^{\varepsilon} + \frac{(3-4v^f)}{(3-2v^f)}\bar{B}_2^{\varepsilon} = 0 \quad -(105)$$

$$\bar{A}_I^{\varepsilon} + \frac{3}{(3-2v^m)}\bar{A}_2^{\varepsilon} - 3\bar{A}_3^{\varepsilon} + \frac{1}{(1-2v^m)}\bar{A}_4^{\varepsilon} - \beta\bar{B}_1^{\varepsilon} - \frac{\beta 3}{(3-2v^f)}\bar{B}_2^{\varepsilon} = 0 \quad -(106)$$

$$-\frac{1}{(3-2v^m)}\bar{A}_2^{\varepsilon} + 2\bar{A}_3^{\varepsilon} - \frac{1}{(1-2v^m)}\bar{A}_4^{\varepsilon} + \frac{\beta}{(3-2v^f)}\bar{B}_2^{\varepsilon} = 0 \quad -(107)$$

The constant $\bar{A}_4^\sigma$ used in the corresponding lower bound is given by an analagous solution with equations (102) and (103) replaced by:

$$\bar{A}_1^\sigma + \frac{3}{(3-2\nu^m)v^f}\bar{A}_2^\sigma - 3\bar{A}_3^\sigma + \frac{v^f}{(1-2\nu^m)}\bar{A}_4^\sigma = 0 \quad ---(108)$$

$$- \frac{1}{(3-2\nu^m)v^f}\bar{A}_2^\sigma + 2v^{f^2}\bar{A}_3^\sigma - \frac{v^f}{(1-2\nu^m)}\bar{A}_4^\sigma = 0 \quad ---(109)$$

D2. PROGRAMME FOR DETERMINING ELASTIC CONSTANTS

D2.1. Description of Programme

The computer programme listed in D2.4. evaluates the engineering elastic constants for a material reinforced with isotropic and anisotropic fibres using the methods described in section B. The constants may be evaluated for more than one volume fraction for any set of data and several sets of data may be processed in one run. Two alphanumeric arrays are provided, SYSTEM and UNITS, so that the type of composite and the units in which the constants are expressed may be printed out. Other parameters are adequately defined in the listing.

D2.2. Typical Input

A typical data input deck is shown below.

Parameter	Value	Format
ISO	0	
NHR	0	
NHT	0	(4I1)
NWR	0	
EM	.500000E+06	
VM	.35	(E12.6,F12.0)

Parameter	Value	Format
EFL VFLT	.600000E+08 .20	(E12.6,F12.0)
UNITS (J)	UNITS-LBF/SQ.IN	(8A10)
SYSTEM(J)	HIGH MODULUS CARBON FIBRES USING SOME ASSUMED PROPERTIES	(8A10)
VOLF	.5	(F12.0)
VOLF	.7	(F12.0)
VOLF	-0.9	(F12.0)
ISO NHR NHT NWR	1 1 0 0	(4I1)
EM VM	.500000E+06 .35	(E12.6,F12.0)
EFL EFT GFT GFLT VFLT	.600000E+08 .120000E+07 .500000E+06 .400000E+07 .20	(4E12.6,F12.0)
UNITS(J)	UNITS-LBF/SQ.IN	(8A10)
SYSTEM(J)	HIGH MODULUS CARBON FIBRE USING SOME ASSUMED PROPERTIES	(8A10)
VOLF	.5	(F12.0)
VOLF	.7	(F12.0)
VOLF	-2.0	(F12.0)

D2.3. Typical Output

The output corresponding to the input deck above is shown below.

PREDICTION OF ELASTIC CONSTANTS FOR HIGH MODULUS CARBON FIBRE USING SOME ASSUMED PROPERTIES

UNITS - LBF/SQ.IN

ANISOTROPIC FIBRES

FIBRE PROPERTIES ARE

EL= 6.000E+07 ET= 1.200E+06 GT= 5.000E+05 GLT= 4.000E+06 VLT= .200 VTL= .004 VTT= .20

MATRIX PROPERTIES ARE

E = 5.000E+05 G = 1.852E+05 V = .350

VOLUME FRACTION OF FIBRE = .50

METHOD	EL	ET	GT	GLT	VLT	VTL	VT
HALPIN	3.125E+07	7.838E+05	2.885E+05	5.004E+05	.275	.007	.35
WHITNEY	3.045E+07	9.783E+05	3.836E+05	4.953E+05	.273	.009	.27

VOLUME FRACTION OF FIBRE = .70

METHOD	EL	ET	GT	GLT	VLT	VTL	VT
HALPIN	4.215E+07	9.298E+05	3.529E+05	1.083E+06	.245	.005	.31
WHITNEY	4.215E+07	1.063E+06	4.269E+05	8.381E+05	.244	.006	.24

PREDICTION OF ELASTIC CONSTANTS FOR HIGH MODULUS CARBON FIBRE USING SOME ASSUMED PROPERTIES

UNITS - LBF/SQ.IN

ISOTROPIC FIBRES

FIBRE PROPERTIES ARE

E = 6.000E+07 G = 2.500E+07 V = .200

MATRIX PROPERTIES ARE

E = 5.000E+05 G = 1.852E+05 V = .350

VOLUME FRACTION OF FIBRE = .50

METHOD	EL	ET	GT	GLT	VLT	VTL	VT
		1.710E+06	6.209E+05			.015	.55
HASHIN	3.925E+07			5.448E+05	.265		
		1.233E+06	3.476E+05			.011	.37
HALPIN	3.925E+07	1.928E+06	4.790E+05	5.516E+05	.275	.018	1.01
WHITNEY	3.925E+07	1.989E+06	7.300E+05	5.448E+05	.265	.017	.27

VOLUME FRACTION OF FIBRE = .70

METHOD	EL	ET	GT	GLT	VLT	VTL	VT
		3.231E+06	1.232E+06			.018	.52
HASHIN	4.215E+07			1.008E+06	.237		
		2.219E+06	7.265E+05			.012	.31
HALPIN	4.215E+07	3.729E+06	8.601E+05	1.440E+06	.245	.022	1.14
WHITNEY	4.215E+07	3.542E+06	1.422E+06	1.008E+06	.237	.020	.24

D2.4. Listing

The Fortran IV computer listing is shown overleaf. The programme was processed on a C.D.C. '6400' digital computer.

```
C
C     PROGRAMME TO EVALUATE ENGINEERING ELASTIC CONSTANTS FOR A
C     MATERIAL REINFORCED WITH ISOTROPIC OR TRANSVERSELY ISOTROPIC
C     FIBRES FROM A KNOWLEDGE OF THE FIBRE AND MATRIX ELASTIC CONSTANTS
C     AND THE CONTENT OF FIBRE BY VOLUME.THREE DIFFERENT PREDICTION
C     METHODS ARE AVAILABLE WITHIN THIS PROGRAMME.
C
C     EM    - MATRIX YOUNG'S MODULUS
C     GM    - MATRIX SHEAR MODULUS
C     VM    - MATRIX POISSON RATIO
C     EFL   - FIBRE LONGITUDINAL YOUNG'S MODULUS
C     EFT   - FIBRE TRANSVERSE YOUNG'S MODULUS
C     GFT   - FIBRE SHEAR MODULUS IN PLANE OF ISOTROPY
C     GFLT  - FIBRE LONGITUDINAL SHEAR MODULUS
C     VFLT  - FIBRE MAJOR POISSON RATIO
C     VFTL  - FIBRE MINOR POISSON RATIO
C     VFTT  - FIBRE TRANSVERSE POISSON RATIO
C     EL    - COMPOSITE LONGITUDINAL YOUNG'S MODULUS
C     ET    - COMPOSITE TRANSVERSE YOUNG'S MODULUS
C     GT    - COMPOSITE SHEAR MODULUS IN PLANE OF ISOTROPY
C     GLT   - COMPOSITE LONGITUDINAL SHEAR MODULUS
C     VLT   - COMPOSITE MAJOR POISSON RATIO
C     VTL   - COMPOSITE MINOR POISSON RATIO
C     VTT   - COMPOSITE TRANSVERSE POISSON RATIO
C     VOLF  - VOLUME FRACTION OF FIBRE.  ALSO CONTROL CHARACTER
C             IF VOLF GT 0 PROGRAMME IS EXECUTED USING INITIAL DATA
C             IF VOLF LT 0 BUT GT -1 A NEW SET OF DATA IS READ IN
C             IF VOLF LT -1 PROGRAMME IS TERMINATED
C     ISO   - CONTROL CHARACTER, =0 FOR ISOTROPIC FIBRES
C                                =1 FOR ANISOTROPIC FIBRES
C     NWR   - CONTROL CHARACTER, =0 IF WHITNEY'S METHOD TO BE USED
C                                =1 OTHERWISE
C     NHR   - CONTROL CHARACTER, =0 IF HASHIN'S METHOD TO BE USED
C                                =1 OTHERWISE
C     NHT   - CONTROL CHARACTER, =0 IF HALPIN'S METHOD TO BE USED
C                                =1 OTHERWISE
      COMMON EM,GM,VM,EFL,EFT,GFT,GFLT,VFLT,VFTL,VFTT,VOLF,ISO
      DIMENSION UNITS(8),SYSTEM(8),A(6,6),Y(6)
  100 READ(5,1) ISO,NHR,NHT,NWR
    1 FORMAT(4I1)
      READ(5,2) EM,VM
    2 FORMAT(E12.6,F12.0)
      IF(ISO.EQ.1) GO TO 110
      READ(5,2) EFL,VFLT
      EFT=EFL
      GFLT=EFL/(2.0*(1.0+VFLT))
      GFT=GFLT
      GO TO 120
  110 READ(5,3) EFL,EFT,GFT,GFLT,VFLT
    3 FORMAT(4E12.6,F12.0)
  120 READ(5,4) (UNITS(J),J=1,8),(SYSTEM(I),I=1,8)
    4 FORMAT(8A10)
      GM=EM/(2.0*(1.0+VM))
      VFTL=VFLT*EFT/EFL
      VFTT=EFT/(2.0*GFT)-1.0
      WRITE(6,10) (SYSTEM(I),I=1,8),(UNITS(J),J=1,8)
   10 FORMAT(1H1,10X,37H PREDICTION OF ELASTIC CONSTANTS FOR ,8A10,////,
     112X,8A10,////)
      IF(ISO.EQ.1) GO TO 200
      WRITE(6,11) EFL,GFT,VFLT
   11 FORMAT(11X,17H ISOTROPIC FIBRES,///,11X,21H FIBRE PROPERTIES ARE,/
     1/,18X,4H E =,E1 . 3,2X,4H G =,E1 . 3,2X,4H V =,F5.3,/)
```

```
      GO TO 300
200   WRITE(6,12) EFL,EFT,GFT,GFLT,VFLT,VFTL,VFTT
12    FORMAT(11X,19H ANISOTROPIC FIBRES,///,11X,21H FIBRE PROPERTIES ARE
     1,//,18X,4H EL=,E10.3,2X,4H ET=,E10.3,2X,4H GT=,E10.3,2X5H GLT=,E10
     1.3,2X,5H VLT=,F5.3,2X,5H VTL=,F5.3,2X,5H VTT=,F5.3,/)
300   WRITE(6,13) EM,GM,VM
13    FORMAT(11X,22H MATRIX PROPERTIES ARE,//,18X,4H E =,E10.3,2X,4H G =
     1,E10.3,2X,4H V =,F5.3,///)
400   READ(5,5) VOLF
5     FORMAT(F12.0)
      IF(VOLF.LT.0.0) GO TO 500
      WRITE(6,14) VOLF
14    FORMAT(1H-,10X,27H VOLUME FRACTION OF FIBRE =,F4.2,//)
      WRITE(6,15)
15    FORMAT(11X,8H METHOD ,7X,2HEL,14X,2HET,14X,2HGT,15X,3HGLT,12X,3HVL
     1T,9X,3HVTL,9X,3HVTT,/)
      IF(NHR.EQ.0) CALL HASHIN
      IF(NHT.EQ.0) CALL HALPIN
      IF(NWR.EQ.0) CALL WHITNE
      GO TO 400
500   IF(VOLF.LT.-1.0) STOP
      GO TO 100
      END
```

```
      SUBROUTINE HASHIN
C     SUBROUTINE TO DETERMINE ELASTIC CONSTANTS USING METHOD OF
C     HASHIN AND ROSEN
C     ETU  - UPPER BOUND ON ET
C     GTU  - UPPER BOUND ON GT
C     VTLU - UPPER BOUND ON VTL
C     VTTU - UPPER BOUND ON VTT
C     ETL  - LOWER BOUND ON ET
C     GTL  - LOWER BOUND ON GT
C     VTLL - LOWER BOUND ON VTL
C     VTTL - LOWER BOUND ON VTT
      COMMON EM,GM,VM,EFL,EFT,GFT,GFLT,VFLT,VFTL,VFTT,VOLF,ISO
      DIMENSION A(5,6),Y(6)
      IF(ISO.EQ.0) GO TO 100
      WRITE(6,9)
    9 FORMAT(11X,6H HASHIN ,4X,32H THIS METHOD IS NOT APPLICABLE TO ANIS
     1OTROPIC FIBRES)
      RETURN
  100 EL=EFL*VOLF+EM*(1.0-VOLF)
      PL1=2.0*VFLT*(1.0-VM*VM)*VOLF+(1.0-VOLF)*(1.0+VM)*VM
      PL2=VOLF*(1.0-VFLT-2.0*VFLT*VFLT)
      PL3=2.0*(1.0-VM*VM)*VOLF+(1.0+VM)*(1.0-VOLF)
      VLT=(VOLF*EFL*PL1+(1.0-VOLF)*EM*PL2*VM)/(VOLF*EFL*PL3+(1.0-VOLF)*E
     1M*PL2)
      GLT=(GFLT*(1.0+VOLF)+GM*(1.0-VOLF))*GM/(GFLT*(1.0-VOLF)+GM*(1.0+VO
     1LF))
      PKF=GFLT+2.0*VFLT*GFLT/(1.0-2.0*VFLT)
      PKM=GM+2.0*VM*GM/(1.0-2.0*VM)
      PHI=PKF/PKM
      PK=PKM*(PHI*(1.0+(1.0-2.0*VM)*VOLF)+(1.0-2.0*VM)*(1.0-VOLF))/(PHI*
     1(1.0-VOLF)+(VOLF+(1.0-2.0*VM)))
      PSI=1.0+4.0*PK*VLT*VLT/EL
      TEMP=1.0/(1.0-2.0*VM)
      CONST=(3.0-4.0*VM)/(3.0-2.0*VM)
      FCONST=(3.0-4.0*VFLT)/(3.0-2.0*VFLT)
      ETA=GFLT/GM
C     DEFINE COEFFICIENTS A AND Y AS IN APPENDIX 1 OF HASHINS PAPER
C     FOR UPPER BOUND SOLUTION
      A(1,1)=1.0
      A(1,2)=1.0/VOLF
      A(1,3)=VOLF*VOLF
      A(1,4)=VOLF
      A(1,5)=0.0
      A(1,6)=0.0
      A(2,1)=0.0
      A(2,2)=-CONST/VOLF
      A(2,3)=-2.0*VOLF*VOLF
      A(2,4)=VOLF*TEMP
      A(2,5)=0.0
      A(2,6)=0.0
      A(3,1)=1.0
      A(3,2)=1.0
      A(3,3)=1.0
      A(3,4)=1.0
      A(3,5)=-1.0
      A(3,6)=-1.0
      A(4,1)=0.0
      A(4,2)=-CONST
      A(4,3)=-2.0
      A(4,4)=TEMP
      A(4,5)=0.0
      A(4,6)=FCONST
```

```
      A(5,1)=1.0
      A(5,2)=3.0/(3.0-2.0*VM)
      A(5,3)=-3.0
      A(5,4)=TEMP
      A(5,5)=-ETA
      A(5,6)=-3.0*ETA/(3.0-2.0*VFLT)
      A(6,1)=0.0
      A(6,2)=-1.0/(3.0-2.0*VM)
      A(6,3)=2.0
      A(6,4)=-TEMP
      A(6,5)=0.0
      A(6,6)=ETA/(3.0-2.0*VFLT)
      Y(1)=1.0
      Y(2)=0.0
      Y(3)=0.0
      Y(4)=0.0
      Y(5)=0.0
      Y(6)=0.0
      CALL SOLVE(A,Y,ID,6,6)
C     SOLVE IS A LIBRARY SUBROUTINE TO SOLVE A SET OF SIMULTANEOUS
C     EQUATIONS A*X=Y.Y IS INITIALLY RHS AND FINALLY SOLUTION VECTOR
      GTU=GM*(1.0-2.0*(1.0-VM)/(1.0-2.0*VM)*VOLF*Y(4))
      ETU=4.0*GTU*PK/(PK+PSI*GTU)
      VTLU=VLT*ETU/EL
      VTIL=ETU/(2.0*GTU)-1.0
C     REDEFINE A AND Y FOR LOWER BOUND SOLUTION
      A(1,1)=1.0
      A(1,2)=(3.0/(3.0-2.0*VM))/VOLF
      A(1,3)=-3.0*VOLF*VOLF
      A(1,4)=VOLF*TEMP
      A(1,5)=0.0
      A(1,6)=0.0
      A(2,1)=0.0
      A(2,2)=(-1.0/(3.0-2.0*VM))/VOLF
      A(2,3)=2.0*VOLF*VOLF
      A(2,4)=-VOLF*TEMP
      A(2,5)=0.0
      A(2,6)=0.0
      A(3,1)=1.0
      A(3,2)=1.0
      A(3,3)=1.0
      A(3,4)=1.0
      A(3,5)=-1.0
      A(3,6)=-1.0
      A(4,1)=0.0
      A(4,2)=-CONST
      A(4,3)=-2.0
      A(4,4)=TEMP
      A(4,5)=0.0
      A(4,6)=FCONST
      A(5,1)=1.0
      A(5,2)=3.0/(3.0-2.0*VM)
      A(5,3)=-3.0
      A(5,4)=TEMP
      A(5,5)=-ETA
      A(5,6)=-3.0*ETA/(3.0-2.0*VFLT)
      A(6,1)=0.0
      A(6,2)=-1.0/(3.0-2.0*VM)
      A(6,3)=2.0
      A(6,4)=-TEMP
      A(6,5)=0.0
```

```
      A(6,6)=ETA/(3.0-2.0*VFLT)
      Y(1)=1.0
      Y(2)=0.0
      Y(3)=0.0
      Y(4)=0.0
      Y(5)=0.0
      Y(6)=0.0
      CALL SOLVE(A,Y,ID,6,6)
      GTL=GM/(1.0+2.0*(1.0-VM)/(1.0-2.0*VM)*VOLF*Y(4))
      ETL=4.0*GTL*PR/(PR+PSI*GTL)
      VTLL=VLT*ETL/EL
      VTTU=ETL/(2.0*GTL)-1.0
      WRITE(6,10) ETU,GTU,VTLU,VTTU,EL,GLT,VLT,ETL,GTL,VTLL,VTTL
   10 FORMAT(38X,E10.3,6X,E10.3,36X,F5.3,7X,F5.3,/,12X,6HHASHIN,4X,E10.3
     1,39X,E10.3,7X,F5.3,/,38X,E10.3,6X,E10.3,36X,F5.3,7X,F5.3)
      RETURN
      END
```

```
      SUBROUTINE HALPIN
C     SUBROUTINE TO DETERMINE ELASTIC CONSTANTS USING HALPIN-TSAI
C     EQUATIONS
      COMMON EM,GM,VM,EFL,EFT,GFT,GFLT,VFLT,VFTL,VFTT,VOLF,ISO
      EL=EM*(1.0-VOLF)+EFL*VOLF
      VLT=VM*(1.0-VOLF)+VFLT*VOLF
      ETA=(EFT/EM-1.0)/(EFT/EM+2.0)
      ET=EM*(1.0+2.0*ETA*VOLF)/(1.0-ETA*VOLF)
      ZETA=1.0+40.0*VOLF**10
      ETA=(GFLT/GM-1.0)/(GFLT/GM+ZETA)
      GLT=GM*(1.0+ZETA*ETA*VOLF)/(1.0-ETA*VOLF)
      ZETA=1.0/(3.0-4.0*VM)
      ETA=(GFT/GM-1.0)/(GFT/GM+ZETA)
      GT=GM*(1.0+ZETA*ETA*VOLF)/(1.0-ETA*VOLF)
      VTT=ET/(2.0*GT)-1.0
      VTL=VLT*ET/EL
      WRITE(6,10)EL,ET,GT,GLT,VLT,VTL,VTT
   10 FORMAT(11X,8H HALPIN ,3X,E10.3,6X,E10.3,6X,E10.3,7X,E10.3,7X,F5.3,
     17X,F5.3,7X,F5.3)
      RETURN
      END
```

```
      SUBROUTINE WHITNE
C     SUBROUTINE TO DETERMINE ELASTIC CONSTANTS USING METHOD OF
C     WHITNEY AND RILEY
      COMMON EM,GM,VM,EFL,EFT,GFT,GFLT,VFLT,VFTL,VFTT,VOLF,ISO
      PLF=1.0-VFTT-2.0*EFT/EFL*VFLT*VFLT
      PLM=1.0-VM-2.0*VM*VM
      DENOM=EM*(1.0-VOLF)*PLF+(PLM*VOLF+(1.0+VM))*EFT
      VLT=VM-2.0*(VM-VFLT)*(1.0-VM*VM)*EFT*VOLF/DENOM
      EL=EM+(EFL-EM)*VOLF
      VTT=VFTT*VOLF+VM*(1.0-VOLF)
      GLT=((GFLT+GM)+(GFLT-GM)*VOLF)*GM/((GFLT+GM)-(GFLT-GM)*VOLF)
      PKF=EFT/(2.0*PLF)
      PKM=EM/(2.0*PLM)
      PK=((PKF+GM)*PKM+(PKF-PKM)*GM*VOLF)/(PKF+GM-(PKF-PKM)*VOLF)
      ET=(2.0*PK*(1.0-VTT)*EL)/(EL+4.0*PK*VLT*VLT)
      VTL=VLT*ET/EL
      GT=ET/(2.0*(1.0+VTT))
      WRITE(6,10)EL,ET,GT,GLT,VLT,VTL,VTT
   10 FORMAT(11X,8H WHITNEY,3X,E10.3,6X,E10.3,6X,E10.3,7X,E10.3,7X,F5.3,
     17X,F5.3,7X,F5.3)
      RETURN
      END
```

D3. PROGRAMME FOR ANALYSING COMPOSITE STRUCTURES

D3.1. Description of Programme

The programme listed in D3.6. is based on the one that appears in [45]. The only modifications are to the input and output and these are described below. The logic of the programme is not described as full documentation is available in [45] if it is required. For normal usage, knowledge of the input and output formats and the theory in section C should be adequate.

Essentially the programme calculates the in-plane and coupling stiffness and compliance matrices and then performs a strength analysis for any one stress or moment resultant not equal to zero. The output shows the maximum stress or moment resultant that each surface of each layer is capable of withstanding for any input temperature difference.

D3.2. Modification to Original Programme

The input to Tsai's original programme required the elastic properties of each layer referred to the structural axes, together with the orientation of each layer. Since a laminate is normally made from layers of similar material, it is possible to calculate all the elastic properties of the layers from just the four engineering constants of a

unidirectional lamina and the orientations, and the programme has thus been modified accordingly. This makes the input far simpler, there being less data to read in; furthermore one is more used to working with the engineering constants, and they are more easily available.

The facility of reading in the elastic constants for each layer is retained, however, for laminates constructed from laminae of different materials. This is achieved via the control character NENG which is defined below.

Because it is possible for a lamina to have different tensile and compressive strengths, it is not possible to compute the failing stresses directly. For each lamina, one set of values of σ_x and σ_y is computed for each quadrant in the xy stress plane using the appropriate strength values for each quadrant. Then, in the original programme, the stress or moment resultant corresponding to this set of values is printed out together with the quadrant number assigned. From the signs of σ_x and σ_y it is determined to which quadrant these results belong, and this number is then also printed out. The user must then check all the results to find a valid solution, i.e. one where the solution is in the same quadrant as originally assigned.

The second modification to the programme was to enable this last step to be performed within the programme, and so simplify its use. All that the user is now required to do is

check which layer will fail first and on what face. This has not been incorporated into the programme as it generally is desirable to know which layer will fail first and just how strong the other layers are.

D3.3. Input Parameter Definitions

N	total number of layers
THTA	defined only for angle ply composites is the fibre orientation in degrees
LPP	defines the particular case under consideration LPP = 1 implies a cylinder or pressure vessel LPP = 2 implies a plate
JJ	defines the heading to be printed JJ = 1 implies cross ply (i.e. orthogonal plies) JJ = 2 implies angle ply JJ = 3 implies general laminate
RMM	cross ply ratio (total thickness of all layers oriented in one direction to total thickness of layers oriented in orthogonal direction)
NENG	defines type of input data to be used NENG = 0 implies elastic constants of each layer are to be read in NENG = 1 implies engineering elastic constants for one unidirectional ply are to be read in
THETA(K)	fibre orientation of the kth layer in radians
EL	longitudinal Young's modulus of unidirectional lamina
ET	transverse Young's modulus of unidirectional lamina
GLT	longitudinal shear modulus of unidirectional lamina
VLT	major Poisson's ratio of unidirectional lamina

C(I,J,K) elastic constant Q_{ij} for kth layer

UNIALP(1) longitudinal thermal expansion coefficient of a unidirectional lamina

UNIALP(2) transverse thermal expansion coefficient of a unidirectional lamina

ALPHA(I,K) thermal expansion coefficient α_i for kth layer

H(K) thickness of kth layer

KQR control character
KQR = 1 implies only stiffness analysis
KQR = 0 implies strength analysis to be performed also and that following data should also be read in

LL defines the loading conditions

For a plate:

LL = 1 implies $N_x \neq 0$
LL = 2 implies $N^x \neq 0$
LL = 3 implies $N^y_{xy} \neq 0$
LL = 4 implies $M^x_{xy} \neq 0$
LL = 5 implies $M^x \neq 0$
LL = 6 implies $M^y_{xy} \neq 0$

For a cylinder or pressure vessel:

LL = 1 implies $N_x \neq 0$
LL = 2 implies $N^x \neq 0$
LL = 3 implies $2N^y_x = N_y$

NM number of input values of temperature

T(J) jth input temperature difference = operating temperature - lamination temperature

XA(K) axial tensile strength of kth layer

YA(K) transverse tensile strength of kth layer

XP(K) axial compressive strength of kth layer

YP(K) transverse compressive strength of kth layer

S(K) shear strength of kth layer

TITLE alphanumeric description of case under consideration

D3.4. Typical Input

A data input deck for a two layer 15° angle ply composite is shown below for the case $N_x \neq 0$.

Parameter	Value	Format
N	2	
THTA	15.0	
LPP	2	(I2,F5.0,2I1,F12.0,I1)
JJ	2	
RMM	1.0	
NENG	1	
THETA(1)	.261800E+00	(6E12.6)
THETA(2)	-.261800E+00	
EL	.302500E+08	
ET	.783800E+06	(6E12.6)
GLT	.500400E+06	
VLT	.275000E+00	
UNIALP(1)	-.730000E-06	(6E12.6)
UNIALP(2)	.290000E-04	
H(1)	.5	(6F12.0)
H(2)	.5	
KOR	0	
LL	1	(2I1,I2)
NM	3	
T(1)	-200.0	
T(2)	0.0	(6F12.6)
T(3)	200.0	
XA(1)	.130000E+06	(6E12.6)
XA(2)	.130000E+06	
YA(1)	.120000E+05	(6E12.6)
YA(2)	.120000E+05	
XP(1)	.130000E+06	(6E12.6)
XP(2)	.130000E+06	
YP(1)	.200000E+05	(6E12.6)
YP(2)	.200000E+05	

Parameter	Value	Format
S(1)	.100000E+05	(6E12.6)
S(2)	.100000E+05	
TITLE	N1 not equal to 0.0	(12A6)

D3.5. Typical Output

The output corresponding to the input deck above is shown below.

ANGLE-PLY THETA = 15.00 DEGREES ALL LAYERS INTACT
2 LAYERS (N = 2)

PROPERTIES OF A UNIDIRECTIONAL LAYER

EL= 3.0250E+07 LBF/SQ.IN ET= 7.8320E+05 LBF/SQ.IN GLT= 5.0040E+05 LBF/SQ.IN VLT= .28

ALPHA(L)=-7.3000E-07 PER DEG.F ALPHA(T)= 2.9000E-05 PER DEG.F

LAMINATE PROPERTIES

ICKNESS LAYERS NCHES)	COORDINATES OF LAYER SURFACES (INCHES)		COEFS. OF STIFFNESS MATRIX (10+6 LB./IN.SQ.)						COEFS. OF THERMAL EXPANSION (10-6 IN./IN./DEG.F.)		
T(K)	Z(K)	Z(K+1)	C(1,1)	C(1,2)	C(2,2)	C(6,1)	C(6,2)	C(6,6)	ALPHA(1)	ALPHA(2)	ALPHA(6)
.5000	-.5000	0.0000	26.5403	2.0073	.9718	-6.7932	-.5878	2.2917	1.2615	27.0085	14.8650
.5000	0.0000	.5000	26.5403	2.0073	.9718	6.7932	.5878	2.2917	1.2615	27.0085	-14.8650

A (10+6 LB./IN.)			A* (10-6 IN./LB.)			A PRIME (10-6 IN./LB.)			COEF. OF THERMAL FORC (LB./IN./DEG.	
63	2.6873	0.0000	.0447	-.0922	0.0000	.0820	-.0751	0.0000	N1-T	-13.2851
73	.9716	0.0000	-.0922	1.2196	0.0000	-.0751	1.2258	0.0000	N2-T	20.0395
00	0.0000	2.2917	0.0000	0.0000	.4364	0.0000	0.0000	1.0177	N3-T	0.0000

B (10+6 LB.)			B* (10+0 IN.)			B PRIME (10-6 1/LB.)			COEF. OF THERMAL MOME (LB./DEG.F.	
00	0.0000	1.6083	-0.0000	-0.0000	-.0623	0.0000	0.0000	-.7606	M1-T	0.0000
00	0.0000	.1470	-0.0000	-0.0000	-.0226	0.0000	0.0000	-.2758	M2-T	0.0000
63	.1470	0.0000	-.7411	-.0641	-0.0000	-.7606	-.2758	0.0000	M3-T	-2.4050

H*
(10+0 IN.)

0.0000	0.0000	.7411
0.0000	0.0000	.0641
.0623	.0226	0.0000

D (10+6 LB.IN.)			D* (10+6 LB.IN.)			D PRIME (10-6 1/LB.IN.)		
17	.1673	0.0000	.9532	.0524	0.0000	1.1043	-.9008	0.0000
73	.0814	0.0000	.0524	.0716	0.0000	-.9008	14.7100	0.0000
00	0.0000	.1510	0.0000	0.0000	.0819	0.0000	0.0000	12.2120

Z (IN.)	STRESS COMPONENT	COEF. OF N1 (1/IN.)	COEF. OF N2 (1/IN.)	COEF. OF N6 (1/IN.)	COEF. OF M1 (1/IN.SQ.)	COEF. OF M2 (1/IN.SQ.)	COEF. OF M6 (1/IN.SQ.)	COEF. OF TEMP (LB./IN.SQ./F
				-- LAYER 1 --				
500	SIGMA 1	-.2917	-.4683	10.3698	-8.5834	-.9367	-40.9256	-42.1036
	2	-.1118	.9595	.8973	-.2236	-6.0811	-3.3214	-3.6434
	6	.2905	.1053	-2.6644	1.7430	.6320	.1621	9.4693
000	SIGMA 1	2.2917	.4683	-10.0930	5.1668	1.8733	-40.9256	42.1036
	2	.1118	1.0405	-.7534	.4471	.1621	-3.3214	3.6434
	6	-.5810	-.2107	-2.5834	-1.7430	-.6320	.1621	-18.9386
				-- LAYER 2 --				
000	SIGMA 1	2.2917	.4683	10.3698	-5.1668	-1.8733	-21.2931	42.1036
	2	.1118	1.0405	.8973	-.4471	-.1621	-2.0627	3.6434
	6	.5810	.2107	-2.6644	-1.7430	-.6320	5.1668	18.9386
500	SIGMA 1	-.2917	-.4683	-.2768	8.5834	.9367	-21.2931	-42.1036
	2	-.1118	.9595	-.1340	.2236	6.0811	-2.0627	-3.6434
	6	-.2905	-.1053	-.0811	1.7430	.6320	5.1668	-9.4693

	AXIAL TENSILE STRENGTH (PSI)	AXIAL COMPRESSIVE STRENGTH (PSI)	TRANSVERSE TENSILE STRENGTH (PSI)	TRANSVERSE COMPRESSIVE (PSI)
00	1.300000E+05	1.300000E+05	1.200000E+04	2.000000E+04
00	1.300000E+05	1.300000E+05	1.200000E+04	2.000000E+04

SHEAR STRENGTH (PSI)
1.000000E+04
1.000000E+04

CASE N1 NOT EQUAL TO 0.0

-- LAYER 1 --

	TEMPERATURE (DEG.F)	STRESS OR MOMENT PER INCH OF THICKNESS (LBF/IN OR LBF)	
Z= -.50			
	-200.0	4.655635E+04	-4.844656E+04
	0.0	4.760401E+04	-4.771445E+04
	200.0	4.838835E+04	-4.673484E+04
Z= 0.00			
	-200.0	5.350718E+04	-4.785781E+04
	0.0	5.156797E+04	-5.124507E+04
	200.0	4.842993E+04	-5.336784E+04

-- LAYER 2 --

	TEMPERATURE (DEG.F)	STRESS OR MOMENT PER INCH OF THICKNESS (LBF/IN OR LBF)	
Z= 0.00			
	-200.0	5.350718E+04	-4.785781E+04
	0.0	5.156797E+04	-5.124507E+04
	200.0	4.842993E+04	-5.336784E+04
Z= .50			
	-200.0	4.655635E+04	-4.844656E+04
	0.0	4.760401E+04	-4.771445E+04
	200.0	4.838835E+04	-4.673484E+04

D3.6. Listing

The Fortran IV computer listing is shown overleaf. The programme was processed on a CDC '6400' digital computer.

```
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      COMMON THETA(50),N,TM(3,3),LPP,LL,PCNO(3,50,2),RB(3,50,2),PCNT(3,5
     10,2),PCNTR(3,50,2),PCMO(3,50,2),PCMT(3,50,2),PCMTR(3,50,2),RC(3,50
     1,2),PCT(3,50,2),RS(3,2),RD(3,2),XA(50),S(50),XP(50),YA(50),YP(50),
     1CVS(4),CVP(4),CTS(4),NM,SOL(4,50,2),T(50),SIGMX(2),SIGMY(2),IQUAD(
     14,50,2),PRB(3,50),CNO(3,50),CNTR(3,50),CNT(3,50),PRC(3,50),CT(3,50
     1),TITLE(10),JK,Z(55)
      DIMENSION ALPHA(3,50),H(50),A(3,3),B(3,3),D(3,3),C(3,3,50),HS(50),
     1HC(50),AN(3,6),X(3,3),ASTAR(3,3),BSTAR(3,3),HSTAR(3,3),DSTAR(3,3),
     1DPRI(3,3),BPRI(3,3),APRI(3,3),SUM(3,50),TSUM(3),TADD(3),RNT(3),RMT
     1(3),SASR(3),DSUM(3,55),CSUM(3,50,2),TC(3,3),UNIALP(2)
C N = NO.OF LAYERS
C MAXIMUM VALUE OF N IS N = 50
C THTA IS ANGLE IN DEGREES FOR HEADING,NEEDED ONLY FOR ANGLE PLY
C LPP = 1 IMPLIES PRESSURE VESSEL OR CYLINDER
C LPP = 2 IMPLIES PLATE
C JJ=1IMPLIES CROSS-PLY
C JJ=2 IMPLIES ANGLE-PLY
C JJ=3 IMPLIES GENERAL LAMINATE
C RMM=CROSS-PLY RATIO
C NENG=1  IMPLIES ENGINEERING CONSTANTS OF ONE  UNIDIRECTIONAL PLY
C         ARE INPUT DATA
C NENG=0  IMPLIES ELASTIC CONSTANTS OF EACH LAYER ARE INPUT DATA
   10 READ(5,101) N,THTA,LPP,JJ,RMM,NENG
  101 FORMAT(I2,F5.0,2I1,F12.0,I1)
      READ (5,103) (THETA(K),K=1,N)
      IF(NENG.EQ.0) GO TO 8
      READ(5,103) EL,ET,GLT,VLT
      READ(5,103) (UNIALP(J),J=1,2)
      VTL=VLT*ET/EL
      TEMP=1.0-VLT*VTL
      TC(1,1)=EL/TEMP
      TC(1,2)=VLT*ET/TEMP
      TC(1,3)=0.0
      TC(2,1)=TC(1,2)
      TC(2,2)=ET/TEMP
      TC(2,3)=0.0
      TC(3,1)=0.0
      TC(3,2)=0.0
      TC(3,3)=GLT
      DO 6 K=1,2
      RM=COS(-THETA(K))
      RN=SIN(-THETA(K))
      RPMN = RM * RN
      RM2=RM*RM
      RN2=RN*RN
      RM4=RM**4
      RN4=RN**4
      RPMN2=RPMN*RPMN
      RM3N=RM**3*RN
      RN3M=RN**3*RM
      C(1,1,K)=2.0*RPMN2*(TC(1,2)+2.0*TC(3,3))+RM4*TC(1,1)+RN4*TC(2,2)
      C(1,2,K)=RPMN2*(TC(1,1)+TC(2,2)-4.0*TC(3,3))+(RM4+RN4)*TC(1,2)
      C(1,3,K)=RM3N*(-TC(1,1)+TC(1,2)+2.0*TC(3,3))+RN3M*(-TC(1,2)-2.0*TC
     1(3,3)+TC(2,2))
      C(2,1,K)=C(1,2,K)
      C(2,2,K)=RN4*TC(1,1)+2.0*RPMN2*TC(1,2)+RM4*TC(2,2)+4.0*RPMN*RPMN*T
     1C(3,3)
      C(2,3,K)=RN3M*(-TC(1,1)+TC(1,2)+2.0*TC(3,3))+RM3N*(-TC(1,2)-2.0*TC
     1(3,3)+TC(2,2))
```

```
      C(3,3,K)=RPMN2*(TC(1,1)+TC(2,2)-2.0*TC(1,2))+(RM*RM-RN*RN)**2*TC(3
     1,3)
      ALPHA(1,K)=RM2*UNIALP(1)+RN2*UNIALP(2)
      ALPHA(2,K)=RN2*UNIALP(1)+RM2*UNIALP(2)
    6 ALPHA(3,K)=2.0*RPMN*(UNIALP(2)-UNIALP(1))
      GO TO 9
    8 READ (5,103) ((ALPHA(I,K),I=1,3),K=1,N)
      READ (5,103) (C(1,1,K),C(1,2,K),C(2,2,K),C(3,1,K),C(3,2,K),C(3,3,K
     1),K=1,N)
  103 FORMAT (6E12.6)
      DO 7 K=1,N
      C(2,1,K) = C(1,2,K)
      C(1,3,K) = C(3,1,K)
      C(2,3,K) = C(3,2,K)
    7 CONTINUE
    9 READ (5,102) (H(K),K=1,N)
  102 FORMAT (6F12.0)
      TOTAL = 0.0
      DO 20 K = 1,N
   20 TOTAL = TOTAL + H(K)
      Z(1) = - TOTAL / 2.0
      MM = N + 1
      DO 30 K = 2,MM
      KM = K - 1
   30 Z(K) = Z(KM) + H(KM)
      IF(JJ.EQ.2) GO TO 40
      IF(JJ.EQ.3) GO TO 60
      WRITE(6,104) RMM,N,N
  104 FORMAT (1H1,37X,9HCROSS-PLY,4X,3HM =,F5.3,17HALL LAYERS INTACT/50X
     1,I2,1X,12HLAYERS (N = ,I2,1H))
      GO TO 70
   40 WRITE (6,105) THTA,N,N
  105 FORMAT (1H1,33X,9HANGLE-PLY,4X,8HTHETA = ,F5.2,1X,7HDEGREES,4X,17H
     1ALL LAYERS INTACT/52X,I2,1X,12HLAYERS (N = ,I2,1H))
      GO TO 70
   60 WRITE (6,107) N,N
  107 FORMAT (1H1,41X,16HGENERAL LAMINATE,4X,17HALL LAYERS INTACT/50X,I2
     1,1X,12HLAYERS (N = ,I2,1H))
   70 IF(NENG.EQ.1) WRITE(6,99) EL,ET,GLT,VLT,(UNIALP(J),J=1,2)
   99 FORMAT(1H-,///,40X,37H PROPERTIES OF A UNIDIRECTIONAL LAYER,///,10
     1X,4H EL=,E11.4,10H LBF/SQ.IN,4X,4H ET=,E11.4,10H LBF/SQ.IN,4X,5H G
     1LT=,E11.4,10H LBF/SQ.IN,4X,5H VLT=,F5.2,///,20X,10H ALPHA(L)=,E11.
     14,10H PER DEG.F,8X,10H ALPHA(T)=,E11.4,10H PER DEG.F,///////,48X,2
     10H LAMINATE PROPERTIES)
      WRITE (6,108)
  108 FORMAT (1H0//2X,5HLAYER,2X,9HTHICKNESS,2X,14HCOORDINATES OF/3X,3HN
     10.,3X,9HOF LAYERS,2X,14HLAYER SURFACES,15X,26HCOEFS. OF STIFFNESS
     1MATRIX,14X,27HCOEFS. OF THERMAL EXPANSION/9X,8H(INCHES),6X,8H(INCH
     1ES),22X,17H(10+6 LB./IN.SQ.),22X,21H(10-6 IN./IN./DEG.F.)//4X,1HK,
     16X,4HH(K),5X,4HZ(K),4X,6HZ(K+1),3X,6HC(1,1),3X,6HC(1,2),3X,6HC(2,2
     1),3X,6HC(6,1),3X,6HC(6,2),3X,6HC(6,6),2X,8HALPHA(1),1X,8HALPHA(2),
     11X,8HALPHA(6)//)
      DO 75 K = 1,N
      KP = K + 1
   75 WRITE (6,109) K,H(K),Z(K),Z(KP),C(1,1,K),C(1,2,K),C(2,2,K),C(3,1,K
     1),C(3,2,K),C(3,3,K),ALPHA(1,K),ALPHA(2,K),ALPHA(3,K)
  109 FORMAT (3X,I2,3X,F9.4,F9.4,F9.4,-6PF9.4,-6PF9.4,-6PF9.4,-6PF9.4,-6
     1PF9.4,-6PF9.4,6PF9.4,6PF9.4,6PF9.4)
      DO 80 K = 1,N
```

```
 80 HC(K) = (Z(KP) ** 3) - (Z(K) ** 3)
    DO 100 I = 1,3
    DO 100 J = 1,3
    A(I,J) = 0.0
    B(I,J) = 0.0
    D(I,J) = 0.0
    DO 90 K = 1,N
    A(I,J) = A(I,J) + (C(I,J,K) * H(K))
    B(I,J) = B(I,J) + (C(I,J,K) * HS(K))
 90 D(I,J) = D(I,J) + (C(I,J,K) * HC(K))
    B(I,J) = B(I,J) / 2.0
    D(I,J) = D(I,J) / 3.0
100 CONTINUE
    L = 0
    DO 200 I = 1,3
    DO 200 J = 1,3
200 AN(I,J) = A(I,J)
210 DO 220 I = 1,3
    DO 220 J = 4,6
220 AN(I,J) = 0.0
    DO 230 I = 1,3
    J = I + 3
230 AN(I,J) = 1.0
    IF (L .EQ. 1) GO TO 270
    CALL MATS (AN,X,3,3,MATERR)
    IF (MATERR) 240,240,235
235 WRITE (6,110) ((A(I,J),I=1,3),J=1,3)
110 FORMAT (1H0,20HMATRIX A IS SINGULAR//(3(-6PF8.4)))
    GO TO 10
240 CALL MATMPY (X,B,BSTAR,3,3,3)
    DO 250 I = 1,3
    DO 250 J = 1,3
    ASTAR(I,J) = X(I,J)
250 BSTAR(I,J) = - BSTAR(I,J)
    CALL MATMPY (B,X,HSTAR,3,3,3)
    CALL MATMPY (HSTAR,B,DSTAR,3,3,3)
    CALL MATSBT (D,DSTAR,3,3)
    DO 260 I = 1,3
    DO 260 J = 1,3
260 AN(I,J) = DSTAR(I,J)
    L = 1
    GO TO 210
270 CALL MATS (AN,DPRI,3,3,MATERR)
    IF (MATERR) 290,290,280
280 WRITE (6,111) ((DSTAR(I,J),I=1,3),J=1,3)
111 FORMAT (1H0,24HMATRIX DSTAR IS SINGULAR//(3(-6PF8.4)))
    GO TO 10
290 CALL MATMPY (BSTAR,DPRI,BPRI,3,3,3)
    CALL MATMPY (BPRI,HSTAR,APRI,3,3,3)
    CALL MATSBT (ASTAR,APRI,3,3)
    DO 300 I = 1,3
    DO 300K = 1,N
    SUM(I,K) = 0.0
    DO 300 J = 1,3
300 SUM(I,K) = SUM(I,K) + (C(I,J,K) * ALPHA(J,K))
    DO 320 I = 1,3
    TSUM(I) = 0.0
    TADD(I) = 0.0
    DO 310 K = 1,N
    TSUM(I) = TSUM(I) + (SUM(I,K) * H(K))
```

```
      RNT(I) = TSUM(I)
  320 RMT(I) = TADD(I) / 2.0
      IF (LPP .EQ. 2) GO TO 370
      DO 330 K = 1,N
      DO 330 I = 1,3
      CNO(I,K) = 0.0
      CNT(I,K) = 0.0
      CNTR(I,K) = 0.0
      DO 330 J = 1,3
      CNO(I,K) = CNO(I,K) + (C(I,J,K) * ASTAR(J,1))
      CNT(I,K) = CNT(I,K) + (C(I,J,K) * ASTAR(J,2))
  330 CNTR(I,K) = CNTR(I,K) + (C(I,J,K) * ASTAR(J,3))
      DO 340 I = 1,3
      SASR(I) = 0.0
      DO 340 J = 1,3
  340 SASR(I) = SASR(I) + (ASTAR(I,J) * RNT(J))
      DO 360 K = 1,N
      DO 360 I = 1,3
      CT(I,K) = 0.0
      DO 350 J = 1,3
  350 CT(I,K) = CT(I,K) + (C(I,J,K) *SASR(J))
  360 CT(I,K) = CT(I,K) - SUM(I,K)
      GO TO 420
  370 DO 375 K = 1,N
      DO 375 I = 1,3
      DO 375 LR = 1,2
      PCNO(I,K,LR) = 0.0
      PCNT(I,K,LR) = 0.0
      PCNTR(I,K,LR) = 0.0
      PCMO(I,K,LR) = 0.0
      PCMT(I,K,LR) = 0.0
  375 PCMTR(I,K,LR) = 0.0
      DO 380 K = 1,N
      DO 380 I = 1,3
      DO 380 J = 1,3
      DO 380 LR = 1,2
      KP = K
      IF (LR .EQ. 2) KP = KP + 1
      PCNO(I,K,LR) = PCNO(I,K,LR) + (C(I,J,K) * (APRI(J,1) + (Z(KP) * BP
     1RI(J,1))))
      PCNT(I,K,LR) = PCNT(I,K,LR) + (C(I,J,K) * (APRI(J,2) + (Z(KP) * BP
     1RI(J,2))))
      PCNTR(I,J,LR) = PCNTR(I,K,LR) + (C(I,J,K) * (APRI(J,3) + (Z(KP) *
     1BPRI(J,3))))
      PCMO(I,K,LR) = PCMO(I,K,LR) + (C(I,J,K) * (BPRI(J,1) + (Z(KP) * DP
     1RI(J,1))))
      PCMT(I,K,LR) = PCMT(I,K,LR) + (C(I,J,K) * (BPRI(J,2) + (Z(KP) * DP
     1RI(J,2))))
  380 PCMTR(I,J,LR) = PCMTR(I,K,LR) + (C(I,J,K) * (BPRI(J,3) + (Z(KP) *
     1DPRI(J,3))))
      MM = N + 1
      DO 390 K = 1,MM
      DO 390 I = 1,3
      DSUM(I,K) = 0.0
      DO 390 J = 1,3
  390 DSUM(I,K) = DSUM(I,K) + ((APRI(I,J) + (Z(K) * BPRI(I,J))) * RNT(J)
     1) + ((BPRI(I,J) + (Z(K) * DPRI(I,J))) * RMT(J))
      DO 410 K = 1,N
      DO 410 I = 1,3
      CSUM(I,K,1) = 0.0
```

```
      DO 400 J = 1,3
      CSUM(I,K,1) = CSUM(I,K,1) + (C(I,J,K) * DSUM(J,K))
      KP = K + 1
400   CSUM(I,K,2) = CSUM(I,K,2) + (C(I,J,K) * DSUM(J,KP))
      PCT(I,K,1) = CSUM(I,K,1) - SUM(I,K)
410   PCT(I,K,2) = CSUM(I,K,2) - SUM(I,K)
420   WRITE (6,112)
112   FORMAT (1H1////16X,1HA,31X,2HA*,27X,7HA PRIME, 7X,22HCOEF. OF THER
     1MAL FORCE/10X,14H(10+6 LB./IN.),18X,14H(10-6 IN./LB.),18X,14H(10-6
     1 IN./LB.),11X,16H(LB./IN./DEG.F.)//)
      WRITE (6,113) (A(I,1),A(I,2),A(I,3),ASTAR(I,1),ASTAR(I,2),ASTAR(I,
     13),APRI(I,1),APRI(I,2),APRI(I,3),I,RNT(I),I=1,3)
113   FORMAT (1X,-6PF10.4,-6PF10.4,-6PF10.4,    6PF10.4,6PF10.4,6PF10.4,
     1   6PF10.4,6PF10.4,6PF10.4,5X,1HN,I1,3H-1 ,1X,6PF10.4)
      WRITE (6,114)
114   FORMAT (1H0//16X,1HB,31X,2HB*,27X,7HB PRIME, 7X,23HCOEF. OF THERMA
     1L MOMENT/12X,10H(10+6 LB.),22X,10H(10+6 IN.),21X,12H(10-6 1/LB.),1
     14X,12H(LB./DEG.F.)//)
      WRITE (6,115) (B(I,1),B(I,2),B(I,3),BSTAR(I,1),BSTAR(I,2),BSTAR(I,
     13),BPRI(I,1),BPRI(I,2),BPRI(I,3),I,RMT(I),I=1,3)
115   FORMAT (1X,-6PF10.4,-6PF10.4,-6PF10.4,    6PF10.4,6PF10.4,6PF10.4,
     1   6PF10.4,6PF10.4,6PF10.4,5X,1HM,I1,3H-1 ,1X,6PF10.4)
      WRITE (6,116)
116   FORMAT (1H0//48X,2HH*/44X,10H(10+6 IN.)//)
      WRITE (6,117) (HSTAR(I,1),HSTAR(I,2),HSTAR(I,3),I=1,3)
117   FORMAT (33X,3F10.4)
      WRITE (6,118)
118   FORMAT (1H0//16X,1HD,31X,2HD*,27X,7HD PRIME/10X,13H(10+6 LB.IN.),1
     19X,13H(10+6 LB.IN.),18X,15H(10-6 1/LB.IN.)//)
      WRITE (6,119) (D(I,1),D(I,2),D(I,3),DSTAR(I,1),DSTAR(I,2),DSTAR(I,
     13),DPRI(I,1),DPRI(I,2),DPRI(I,3),I=1,3)
119   FORMAT (1X,-6PF10.4,-6PF10.4,-6PF10.4,2X,-6PF10.4,-6PF10.4,-6PF10.
     14,2X,6PF10.4,6PF10.4,6PF10.4)
      IF (LPP .EQ. 1) GO TO 450
      WRITE (6,120)
120   FORMAT (1H1//7X,1HZ,8X,6HSTRESS,3X,11HCOEF. OF N1,2X,11HCOEF. OF N
     12,2X,11HCOEF. OF N6,2X,11HCOEF. OF M1,2X,11HCOEF. OF M2,2X,11HCOEF
     1. OF M6,2X,14HCOEF. OF TEMP./5X,5H(IN.),4X,9HCOMPONENT,4X,7H(1/IN.
     11),6X,7H(1/IN.),6X,7H(1/IN.),4X,10H(1/IN.SQ.),3X,10H(1/IN.SQ.),3X,1
     10H(1/IN.SQ.),5X,15H(LB./IN.SQ./F.)//)
      DO 430 K = 1,N
      KP = K + 1
      WRITE (6,121) K
121   FORMAT (50X,9H-- LAYER ,I2,3H --//)
      WRITE (6,122) Z(K),(PCNO(I,K,1),PCNT(I,K,1),PCNTR(I,K,1),PCMO(I,K,
     11),PCMT(I,K,1),PCMTR(I,K,1),PCT(I,K,1),I=1,3),Z(KP),(PCNO(I,K,2),P
     1CNT(I,K,2),PCNTR(I,K,2),PCMO(I,K,2),PCMT(I,K,2),PCMTR(I,K,2),PCT(I
     1,K,2),I=1,3)
122   FORMAT (3X,F8.4,4X,7HSIGMA 1,4X,F8.4,5F13.4,6X,F8.4/21X,1H2,4X,F8.
     14,5F13.4,6X,F8.4/21X,1H6,4X,F8.4,5F13.4,6X,F8.4/)
430   CONTINUE
440   CALL PARTWO
      GO TO 470
450   WRITE (6,123)
123   FORMAT (1H1//31X,6HSTRESS,3X,11HCOEF. OF N1,2X,11HCOEF. OF N2,2X,1
     111HCOEF. OF N6,2X,14HCOEF. OF TEMP./29X,9HCOMPONENT,4X,7H(1/IN.),6X
     1,7H(1/IN.),6X,7H(1/IN.),4X,15H(LB./IN.SQ./F.)//)
      DO 460 K = 1,N
      WRITE (6,124) K
```

```
125 FORMAT (30X,7HSIGMA 1,4X,F8.4,2F13.4,6X,F8.4/36X,1H2,4X,F8.4,2F13.
   14,6X,F8.4/36X,1H6,4X,F8.4,2F13.4,6X,F8.4/)
460 CONTINUE
    GO TO 440
470 CONTINUE
    STOP
    END
```

```
      SUBROUTINE MATS (A,X,N,M,MATERR)
      DIMENSION A(3,6),X(3,3)
      MATERR = 0
      MM = N + M
      DO 50 I = 2,N
      II = I - 1
      DO 50 J = 1,II
      IF (A(I,J) .EQ. 0.0) GO TO 50
      IF ((ABS(A(J,J)) - ABS(A(I,J))) .LT. 0.0) GO TO 10
      R = A(I,J) / A(J,J)
      GO TO 30
   10 R = A(J,J) / A(I,J)
      DO 20 K = 1,MM
      B = A(J,K)
      A(J,K) = A(I,K)
   20 A(I,K) = B
   30 JJ = J + 1
      DO 40 K = JJ,MM
   40 A(I,K) = A(I,K) - (R *A(J,K))
   50 CONTINUE
      IF ((ABS(A(N,N)) - 1.0E-10) .GT. 0.0) GO TO 70
   60 WRITE (6,101) N,N
  101 FORMAT (26HO                    ELEMENT(,I2,1H,,I2,1H),38H VERY SMALL
     1. CASE DELETED BY MATS      )
      MATERR = 1
      GO TO 100
   70 DO 90 J = 1,M
      KK = N + J
      X(N,J) = A(N,KK) / A(N,N)
      DO 90 I = 2,N
      JJ = N - I + 1
      B = 0.0
      II = N - I + 2
      DO 80 K = II,N
   80 B = B + (A(JJ,K) * X(K,J))
      IF ((ABS(A(JJ,JJ)) - 1.0E-10) .LE. 0.0) GO TO 60
   90 X(JJ,J) = (A(JJ,KK) - B) / A(JJ,JJ)
  100 RETURN
      END
```

```
      SUBROUTINE MATMPY (A,B,C,L,M,N)
      DIMENSION A(3,3),B(3,3),C(3,3)
      DO 20 I = 1,L
      DO 20 J = 1,N
      SUM = 0.0
      DO 10 LL = 1,M
   10 SUM = SUM + (A(I,LL) * B(LL,J))
   20 C(I,J) = SUM
      RETURN
      END
```

```
      SUBROUTINE MATSBT (A,B,M,N)
      DIMENSION A(3,3),B(3,3)
      DO 10 I = 1,M
      DO 10 J = 1,N
      C = B(I,J)
   10 B(I,J) = A(I,J) - C
      RETURN
      END
```

```
      SUBROUTINE PARTWO
      COMMON THETA(50),N,TM(3,3),LPP,LL,PCNO(3,50,2),RB(3,50,2),PCNT(3,5
     10,2),PCNTR(3,50,2),PCMO(3,50,2),PCMT(3,50,2),PCMTR(3,50,2),RC(3,50
     1,2),PCT(3,50,2),RS(3,2),RO(3,2),XA(50),S(50),XP(50),YA(50),YP(50),
     1CVS(4),CVP(4),CTS(4),NM,SOL(4,50,2),T(50),SIGMX(2),SIGMY(2),IQUAD(
     14,50,2),PRB(3,50),CNO(3,50),CNTR(3,50),CNT(3,50),PRC(3,50),CT(3,50
     1),TITLE(10),JK,Z(55)
   10 READ (5,101) KQR,LL,JK,NM
  101 FORMAT (3I1,I2)
C KQR = 0 IMPLIES SUBROUTINE IS TO CONTINUE READING
C KQR = 1 IMPLIES RETURN TO THE MAIN PROGRAM
C LL IMPLIES CASE UNDER CONSIDERATION
C FOR PLATE
C    LL = 1 IMPLIES N1 NOT EQUAL TO 0.0
C    LL = 2 IMPLIES N2 NOT EQUAL TO 0.0
C    LL = 3 IMPLIES N6 NOT EQUAL TO 0.0
C    LL = 4 IMPLIES M1 NOT EQUAL TO 0.0
C    LL = 5 IMPLIES M2 NOT EQUAL TO 0.0
C    LL = 6 IMPLIES M6 NOT EQUAL TO 0.0
C FOR CYLINDER
C    LL = 1 IMPLIES N1 NOT EQUAL TO 0.0
C    LL = 2 IMPLIES N6 NOT EQUAL TO 0.0
C    LL = 3 IMPLIES 2N1 = N2
C NM = NO. OF INPUT VALUES OF TEMPERATURE
C MAXIMUM VALUE OF NM = 50
      IF (KQR .EQ. 1) GO TO 570
      READ (5,102) (T(K),K=1,NM)
  102 FORMAT (6F12.6)
      READ (5,103) (XA(K),K=1,N)
  103 FORMAT (6E12.6)
      READ (5,103) (YA(K),K=1,N)
      READ (5,103) (XP(K),K=1,N)
      READ (5,103) (YP(K),K=1,N)
      READ (5,103) (S(K),K=1,N)
      READ(5,104) TITLE
  104 FORMAT (12A6)
   20 WRITE (6,105)
  105 FORMAT (1H1,1X,1H2,3X,22HAXIAL TENSILE STRENGTH,2X,26HAXIAL COMPRE
     1SSIVE STRENGTH,3X,27HTRANSVERSE TENSILE STRENGTH,2X,31HTRANSVERSE
     1COMPRESSIVE STRENGTH/1X,4H(IN),9X,5H(PSI),22X,5H(PSI),23X,5H(PSI),
     126X,5H(PSI)//)
      DO 30 K = 1,N
      WRITE (6,106) Z(K),XA(K),XP(K),YA(K),YP(K)
  106 FORMAT (F8.4,3X,E13.6,12X,E13.6,16X,E13.6,18X,E13.6)
   30 CONTINUE
      WRITE (6,107) (S(K),K=1,N)
  107 FORMAT (1H0,52X,14HSHEAR STRENGTH/57X,5H(PSI)//(52X,E13.6))
      TEMP = -0.7777777E-77
      DO 560 K = 1,N
      RM = COS(THETA(K))
      RN = SIN(THETA(K))
      TM(1,1) = RM * RM
      TM(1,2) = RN * RN
      RPMN = RM * RN
      TM(1,3) = 2.0 * RPMN
      TM(2,1) = TM(1,2)
      TM(2,2) = TM(1,1)
      TM(2,3) = - TM(1,3)
      TM(3,1) = - RPMN
      TM(3,2) = RPMN
      TM(3,3) = TM(1,1) - TM(1,2)
```

```
      DO 550 J = 1,2
      IF (LPP .EQ. 1) GO TO 300
      GO TO (50,70,90,210,230,250),LL
   50 DO 60 I = 1,3
   60 RB(I,K,J) = PCNO(I,K,J)
      GO TO 270
   70 DO 80 I = 1,3
   80 RB(I,K,J) = PCNT(I,K,J)
      GO TO 270
   90 DO 100 I = 1,3
  100 RB(I,K,J) = PCNTR(I,K,J)
      GO TO 270
  210 DO 220 I = 1,3
  220 RB(I,K,J) = PCMO(I,K,J)
      GO TO 270
  230 DO 240 I = 1,3
  240 RB(I,K,J) = PCMT(I,K,J)
      GO TO 270
  250 DO 260 I = 1,3
  260 RB(I,K,J) = PCMTR(I,K,J)
  270 DO 280 I = 1,3
  280 RC(I,K,J) = PCT(I,K,J)
      DO 290 I = 1,3
      RS(I,1) = RB(I,K,J)
  290 RS(I,2) = RC(I,K,J)
      GO TO 400
  300 IF (J .EQ. 2) GO TO 550
      GO TO (310,330,350),LL
  310 DO 320 I = 1,3
  320 PRB(I,K) = CNO(I,K)
      GO TO 370
  330 DO 340 I = 1,3
  340 PRB(I,K) = CNTR(I,K)
      GO TO 370
  350 DO 360 I = 1,3
  360 PRB(I,K) = (0.5 * CNO(I,K)) + CNT(I,K)
  370 DO 380 I = 1,3
  380 PRC(I,K) = CT(I,K)
      DO 390 I = 1,3
      RS(I,1) = PRB(I,K)
  390 RS(I,2) = PRC(I,K)
  400 CALL MATMPY (TM,RS,RD,3,3,2)
      S1 = RD(1,1) ** 2
      S2 = RD(1,1) * RD(2,1)
      S3 = RD(2,1) ** 2
      S4 = RD(3,1) ** 2
      S5 = 2.0 * RD(1,1) * RD(1,2)
      S6 = (RD(1,2) * RD(2,1)) + (RD(1,1) * RD(2,2))
      S7 = 2.0 * RD(2,1) * RD(2,2)
      S8 = 2.0 * RD(3,1) * RD(3,2)
      S9 = RD(1,2) ** 2
      S10 = RD(1,2) * RD(2,2)
      S11 = RD(2,2) ** 2
      S12 = RD(3,2) ** 2
      R1 = XA(K) / YA(K)
      R2 = XP(K) / YA(K)
      R3 = XP(K) / YP(K)
      R4 = XA(K) / YP(K)
      SQ = S(K) ** 2
      YAS = YA(K) ** 2
```

```
      XAS = XA(K) ** 2
      YPS = YP(K) ** 2
      XPS = XP(K) ** 2
      XY = XA(K) * YA(K)
      XPYP = XP(K) * YP(K)
      XYP = XA(K) * YP(K)
      XPY = XP(K) * YA(K)
      CVS(I) = (S1 / XAS) - (S2 / (R1 * XY)) + (S3 / YAS) + (S4 / SW)
      CVS(2) = (S1 / XPS) - (S2 / (R2 * XPY)) + (S3 / YAS) + (S4 / SW)
      CVS(3) = (S1 / XPS) - (S2 / (R3 * XPYP)) + (S3 / YPS) + (S4 / SW)
      CVS(4) = (S1 / XAS) - (S2 / (R4 * XYP)) + (S3 / YPS) + (S4 / SW)
      CVP(1) = (S5 / XAS) - (S6 / (R1 * XY)) + (S7 / YAS) + (S8 / SW)
      CVP(2) = (S5 / XPS) - (S6 / (R2 * XPY)) + (S7 / YAS) + (S8 / SW)
      CVP(3) = (S5 / XPS) - (S6 / (R3 * XPYP)) + (S7 / YPS) + (S8 / SW)
      CVP(4) = (S5 / XAS) - (S6 / (R4 * XYP)) + (S7 / YPS) + (S8 / SW)
      CTS(1) = (S9 / XAS) - (S10 / (R1 * XY)) + (S11 / YAS) + (S12 / SW)
      CTS(2) = (S9 / XPS) - (S10 / (R2 * XPY)) + (S11 / YAS) + (S12 / SW
     1)
      CTS(3) = (S9 / XPS) - (S10 / (R3 * XPYP)) + (S11 / YPS) + (S12 / S
     1W)
      CTS(4) = (S9 / XAS) - (S10 / (R4 * XYP)) + (S11 / YPS) + (S12 / SW
     1)
      DO 470 I = 1,4
      DO 470 JL = 1,NM
      DISC = ((CVP(I) * T(JL)) ** 2) - (4.0 * CVS(I) * (CTS(I) * (T(JL)
     1** 2) - 1.0))
  410 IF (DISC .LT. 0.0) GO TO 420
      SOL(I,JL,1) = (- (CVP(I) * T(JL)) + SQRT(DISC)) / (2.0 * CVS(I))
      SOL(I,JL,2) = (- (CVP(I) * T(JL)) - SQRT(DISC)) / (2.0 * CVS(I))
      GO TO 430
  420 SOL(I,JL,1) = TEMP
      SOL(I,JL,2) = TEMP
  430 DO 470 IL = 1,2
      SIGMX(IL) = (RD(1,1) * SOL(I,JL,IL)) + (RD(1,2) * T(JL))
      SIGMY(IL) = (RD(2,1) * SOL(I,JL,IL)) + (RD(2,2) * T(JL))
      IF (SIGMX(IL) .GE. 0.0 .AND. SIGMY(IL) .GE. 0.0) GO TO 440
      IF (SIGMX(IL) .LT. 0.0 .AND. SIGMY(IL) .GT. 0.0) GO TO 450
      IF (SIGMX(IL) .LT. 0.0 .AND. SIGMY(IL) .LT. 0.0) GO TO 460
      IQUAD(I,JL,IL) = 4
      GO TO 470
  440 IQUAD(I,JL,IL) = 1
      GO TO 470
  450 IQUAD(I,JL,IL) = 2
      GO TO 470
  460 IQUAD(I,JL,IL) = 3
  470 CONTINUE
      IF(J.EQ.2) GO TO 565
      WRITE(6,108) TITLE
  108 FORMAT (1H1,47X,7HCASE    ,10A6)
      WRITE(6,110) K
  110 FORMAT (1HO//53X,9H-- LAYER ,I2,3H --/)
      WRITE(6,117)
  117 FORMAT(30X,12H TEMPERATURE,14X,39H STRESS OR MOMENT PER INCH OF TH
     1ICKNESS,/,32X,8H (DEG.F),28X,16H (LBF/IN OR LBF),//)
      IF(LPP.EQ.2) WRITE(6,119) Z(K)
  119 FORMAT(23X,3H Z=,F5.2)
      GO TO 566
  565 KP=K+1
      WRITE(6,119) Z(KP)
  566 DO 550 JL=1,NM
      DO 550 JJ=1,2
```

```
      DO 550 I=1,4
      IF(IQUAD(I,JL,JJ).NE.I) GO TO 550
      IF(JJ.EQ.2) GO TO 555
      II=I
      JJJ=JJ
      GO TO 550
  555 WRITE(6,118) T(JL),SOL(II,JL,JJJ),SOL(I,JL,JJ)
  118 FORMAT(33X,F7.1,17X,E13.6,12X,E13.6)
  550 CONTINUE
  560 CONTINUE
  570 CONTINUE
      RETURN
      END
```

D4. REFERENCES

1. W.H. Sutton, "Whisker Composite Materials - a Prospectus for the Aerospace Designer", Astronautics and Aeronautics, Vol. 4 (1966), p. 46.

2. Metals Handbook, Vol. 1, American Society for Metals (1961).

3. A. Shindo, "Studies on Graphite Fibre", Report 317, Government Industrial Research Institute, Osaka, (1961).

4. W. Watt, L.N. Phillips and W. Johnson, "High-strength, High-modulus Carbon Fibres", The Engineer, Vol. 221 (1966), p. 815.

5. A.E. Standage and R. Prescott, "High Elastic Modulus Carbon Fibre", Nature, Vol. 211 (1966), p. 169.

6. R. Moreton, W. Watt and W. Johnson, "Carbon Fibres of High Strength and High Breaking Strain", Nature, Vol. 213 (1967), p. 690.

7. W. T. Gunston, "Carbon Fibres", Science Journal, Vol. 5 (1969), p. 39.

8. D. V. Badami, J.C. Joiner and G.A. Jones, "Microstructure of High Strength, High Modulus Carbon Fibres", Nature, Vol. 215 (1967) p. 386.

9. W. Johnson and W. Watt, "Structure of High Modulus Carbon Fibres", Nature, Vol. 215 (1967), p. 384.

10. The New World of 'Modmor' High Modulus Carbon Fibres, New Products Data Sheet, Morganite Research and Development Ltd., (1968).

11. 'Grafil' Data Sheet FSL3, Carbon fibres unit, Courtaulds Ltd., (1969).

12. H.D. Blakelock and D.R. Lovell, "High Modulus Reinforcing Carbon", 24th Ann. Conf. (1969) S.P.I. Reinforced Plastics and Composites Division.

13. V. Dembo, Private Communication, June 1969.

14. D. J. Willats, Private Communication, May 1969.

15. J. N. Fleck and E.J. Jablonowski, "What we can expect in the 1970's", Metal Progress, Vol. (1969), p. 99.

16. L.N. Phillips, "Carbon Fibre Reinforced Plastics: a personal view", Fibre Science and Technology, Vol. 1 (1968), p. 3.

17. H.D. Blakelock and K.C.A. Blasdale, "Mechanical Property Measurements on Modmor Carbon Fibre-Resin Composites", Morganite Research and Development Ltd., (1968).

18. J.K. Lancaster, "Composite Self-Lubricating Bearing Materials" Proc.Inst.Mech.E., Vol.182(1967-68), Part 1, No. 2, p.33

19. M.W. Jones, "Carbon Fibre Reinforced Plastic Components for Aerospace Use", Canadian Aeronautics and Space Journal, Vol. 15 (1969), p. 211.

20. T.A. Trigg, "Carbon Fibre Composites in Rocket Motor Systems", Journal of the British Interplanetary Society, Vol. 22 (1969), p. 337.

21. S. Yurenka and J.T. Parks, "An Automatic Fabrication Process for Composite Aircraft Structures", SAMPE Vol. 14 (1968), p. II-1A-3.

22. J.D.D. Weil, "Advanced Materials - Application", Rolls-Royce Ltd. (1968).

23. D.M. McElhinney, A.W. Kitchenside, and K. A. Rowland, "Carbon Fibres take to the Air", New Scientist, Vol. 43 (1969), p. 282.

24. P.H. Petit and M.E. Waddoups, "A Method of Predicting the non-linear behaviour of Laminated Composites", J. Composite Materials, Vol. 3 (1969), p. 2.

25. D.G. Pearce, "Designing for Carbon Fibre reinforced Plastics, Design Engineering (U.K.) Feb. 1969, p. 99.

26. D.M. Peters, "Carbon Fibre Reinforced Plastics", Design Engineering (U.K.) Sept. 1968, p. 3.

27. W.S. Hieronymus, "Carbon Composite Program Gains", Aviation Week and Space Technology, Aug. 18, 1969, p. 51.

28. C.C. Evans and N.J. Parrat, "Whiskers - the Strength of the Single Crystal" New Scientist, Vol. 43, (1969) p. 68.

29. S.G. Lekhnitskii, Theory of Elasticity of an Anistropic Elastic Body, Holden Day Inc., (1963).

30. R.F.S. Hearmon, An Introduction to Applied Anisotropic Elasticity, Oxford University Press, (1961).

31. C.C. Chamis and G.P. Sendeckyj, "Critique on Theories predicting Thermoelastic Properties of Fibrous Composites", J. Composite Materials, Vol. 2 (1968), p. 332.

32. Z. Hashin and B.W. Rosen, "The Elastic Moduli of Fibre reinforced Materials", J. Applied Mechanics, Vol. 31 (1964), p. 223. [Errata - J. Applied Mechanics, Vol. 32 (1965), p. 219.]

33. J.M. Whitney and M.B. Riley, "Elastic Properties of Fiber Reinforced Composite Materials", J.AIAA, Vol. 4 (1966), p. 1537.

34. J.M. Whitney, "Elastic Moduli of Composite Materials reinforced with Orthotropic Filaments", AFML-TR-65-411, (1966).

35. O.L. Blakslee, A.A. Pallozzi, W.A. Doig, G.B. Spence and D.P. Hanley, "Fabrication, Testing and Design Studies with 'Thornel' Graphite Fiber Epoxy Resin Composites", SAMPE Vol. 12 (1967), p.AC-6.

36. J.E. Ashton, J.C. Halpin and P.H. Petit, Primer on Composite Materials: Analysis, Technomic (1969).

37. J.C. Halpin and S.W. Tsai, "Environmental Factors in Composite Materials Design", AFML-TR-67-423.

38. J.J. Hermans, "The Elastic Properties of Fiber reinforced Materials when the Fibres are Alligned", Proc. Konigl. Nederl. Akad. Van Weteschappen Amsterdam, Vol. B70, No. 1 (1967), p. 1.

39. R. Hill, "Theory of Mechanical Properties of Fibre-strengthened Materials: III. Self-consistent Model", J. Mech. Phys. Solids, Vol. 13 (1965), p. 189.

40. D.F. Adams and D.R. Doner, "Longitudinal Shear loading of a unidirectional Composite", J. Composite Materials, Vol. 1 (1967), p. 4.

41. D.F. Adams and D.R. Doner, "Transverse Normal Loading of a unidirectional Composite", J. Composite Materials, Vol. 1 (1967), p. 152.

42. R.D. Adams, M.A.O. Fox, R.J.L. Flood, R.J. Friend and R. L. Hewitt, "The Dynamic Properties of unidirectional Carbon and Glass Fiber reinforced Plastics in Torsion and Flexure", J. Composite Materials, Vol. 3 (1969), p. 594.

43. S.W. Tsai, "Structural Behaviour of Composite Materials", NASA-CR-71, (1964).

44. S.W. Tsai, "Strength Characteristics of Composite Materials," NASA-CR-224, (1965).

45. S.W. Tsai, D.F. Adams and D.R. Doner, "Analyses of Composite Structures", NASA-CR-620, (1966).

46. H.S. Carslaw and J.C. Jaeger, Conduction of Heat in Solids, Oxford University Press, (1959).

47. R. Hill, The Mathematical Theory of Plasticity, Oxford University Press, (1950).

48. J.E. Ashton and M.E. Waddoups, "Analysis of Anisotropic Plates", J. Composite Materials, Vol. 3 (1969), p. 148.

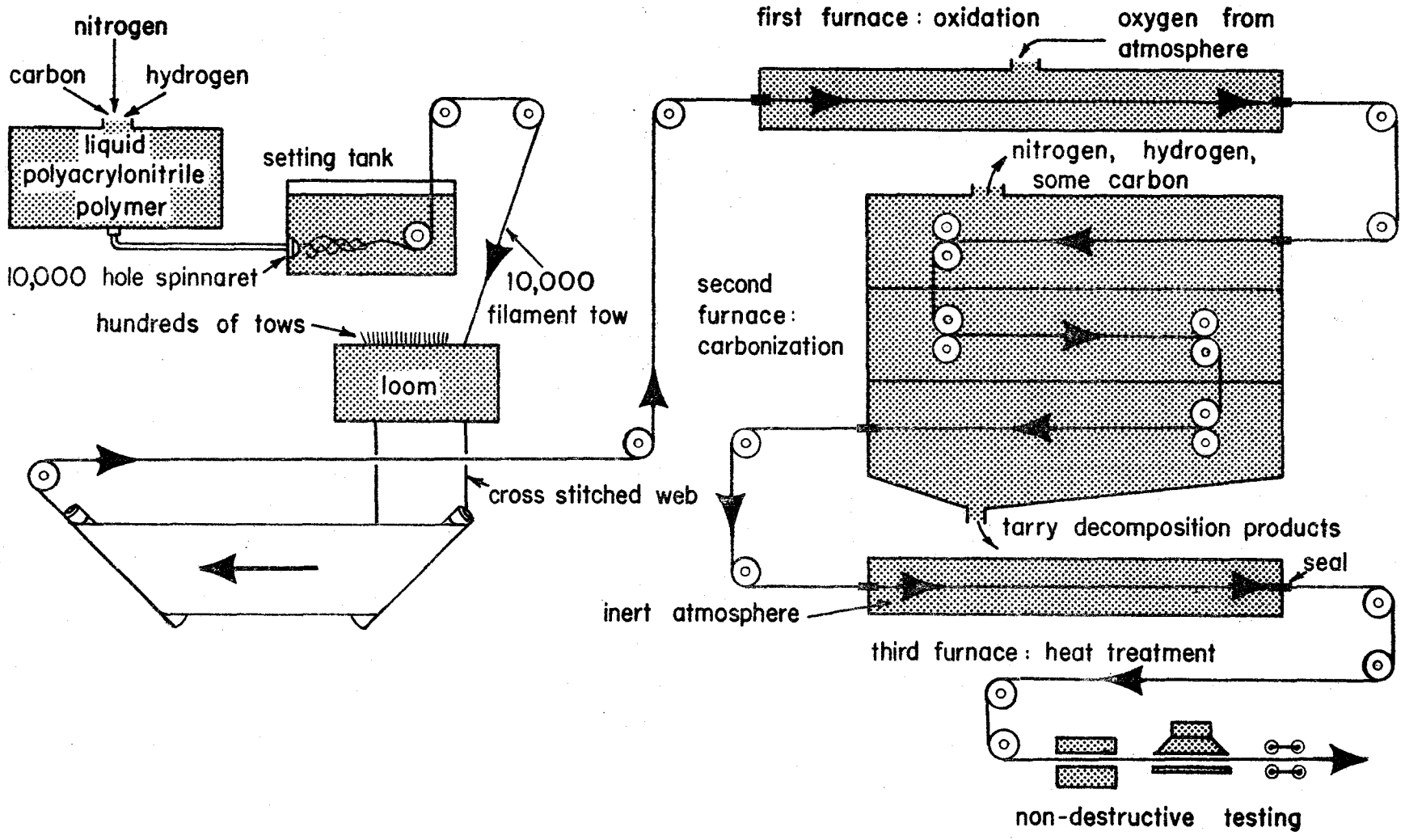


FIGURE 1 Principles of a continuous carbon fibre production process.

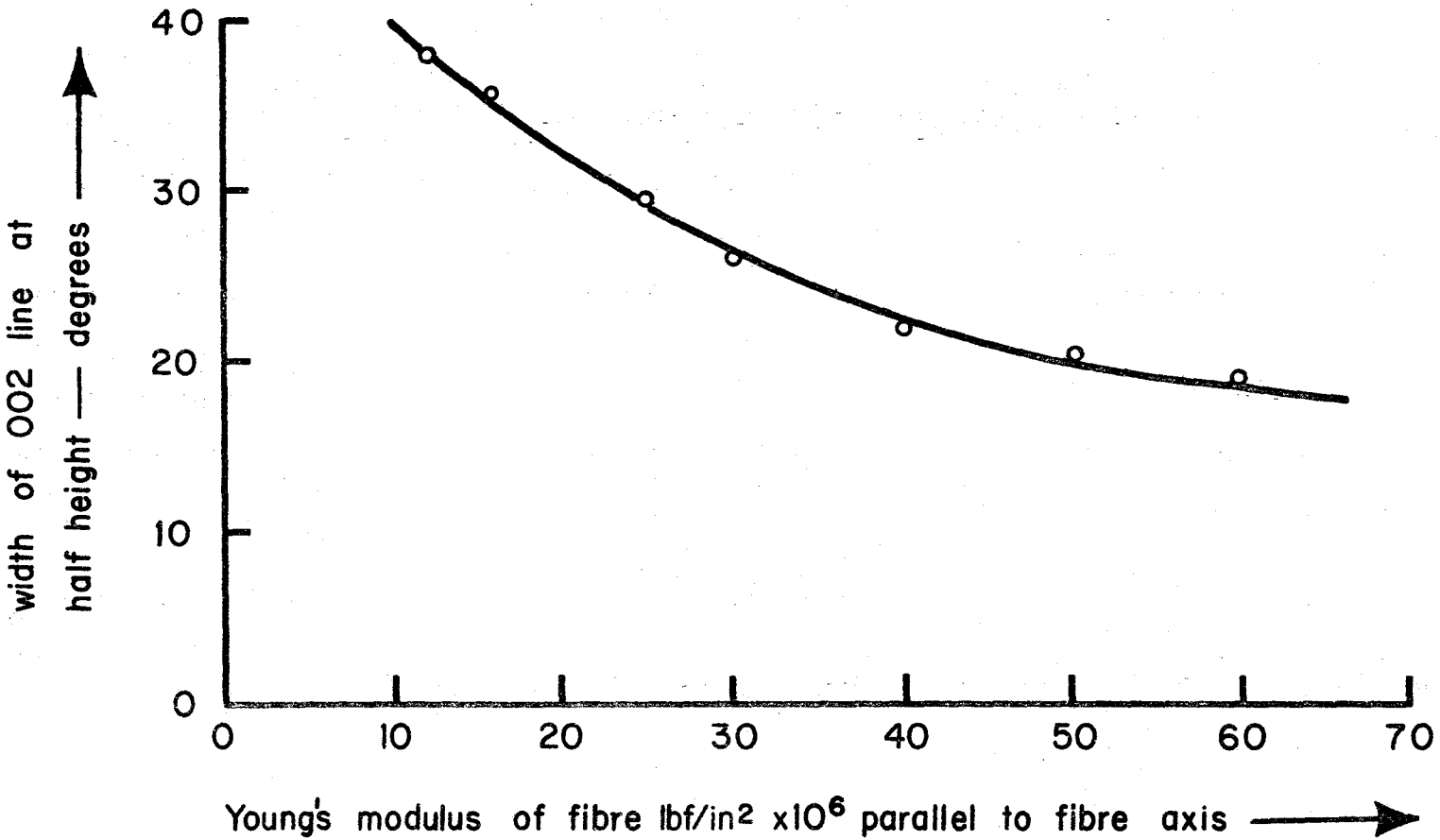


FIGURE 2 Relationship between relative degree of preferred orientation of graphite crystallites and elastic modulus of carbon fibres.

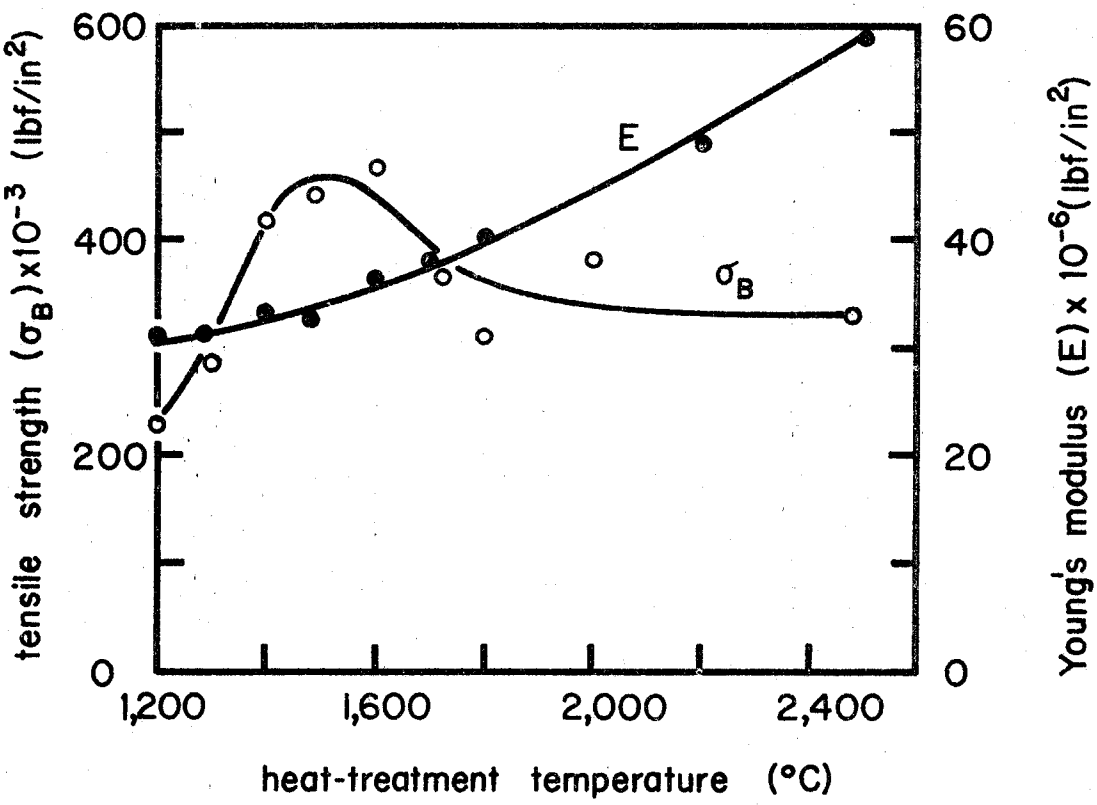
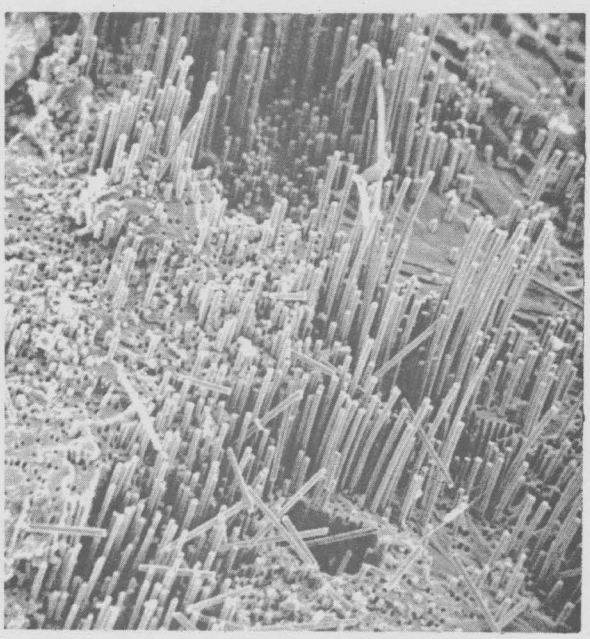
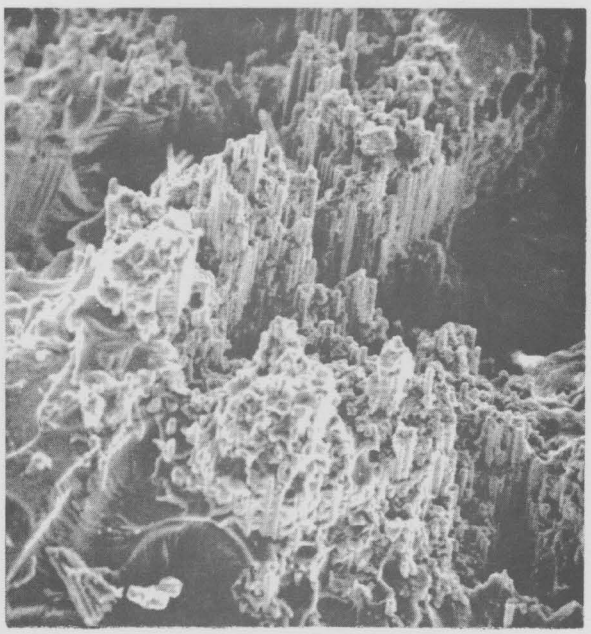


FIGURE 3 Effect of heat-treatment temperature on the mechanical properties of carbon fibres at room temperature



untreated fibre



treated fibre

FIGURE 4 Stereoscan electron micrographs of fractured composites. (courtesy of Morganite Research and Development Ltd.)

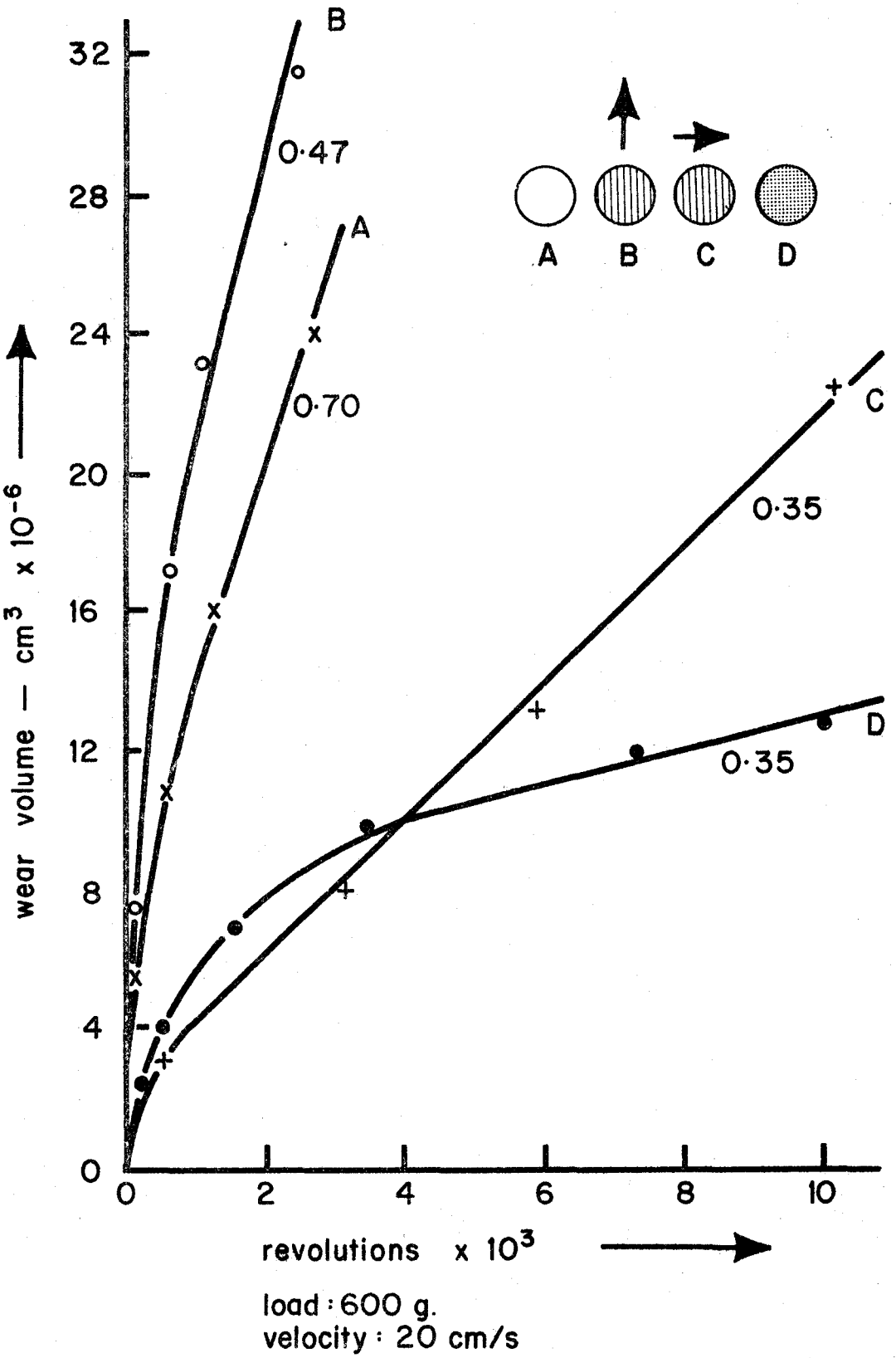


FIGURE 5 Wear of polyester resin, reinforced with carbon fibres sliding against tool steel

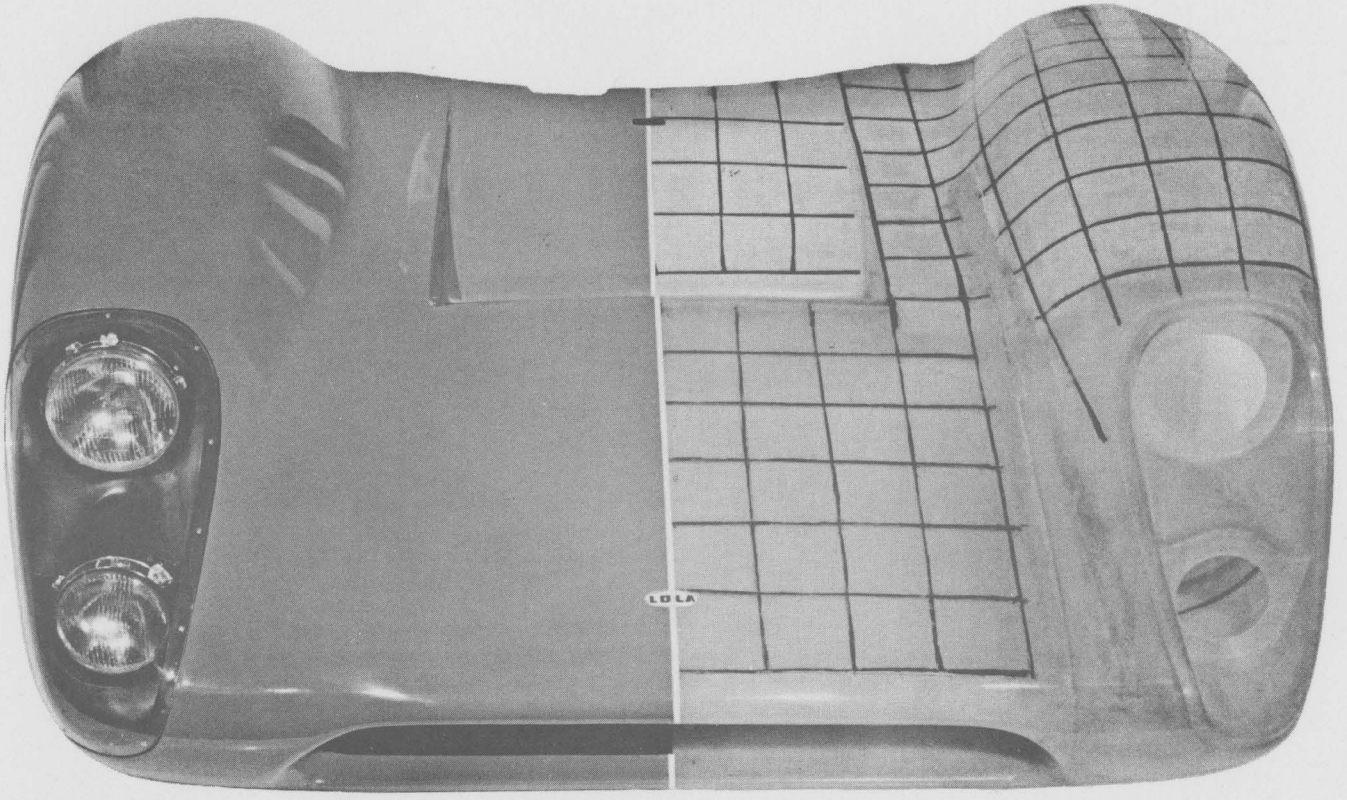


FIGURE 6 Nose section of a Lola racing car. (courtesy of Courtaulds Ltd., Specialised Mouldings Ltd., and Lola Cars Ltd.)

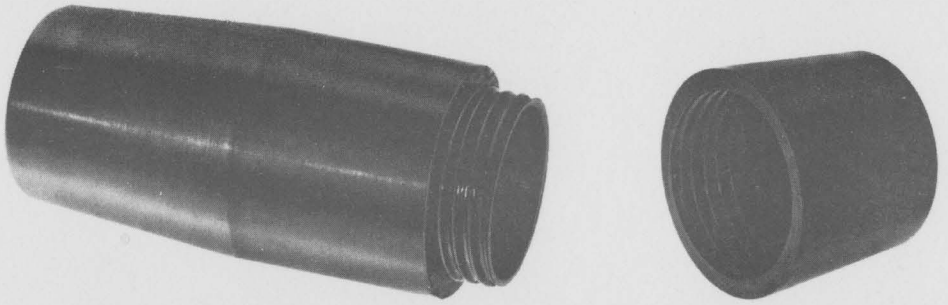


FIGURE 7 Carbon fibre "rabbitt". (courtesy of AERE, Harwell)

FIGURE 8 Some filament wound structures from IMI (courtesy of IMI Ltd.)

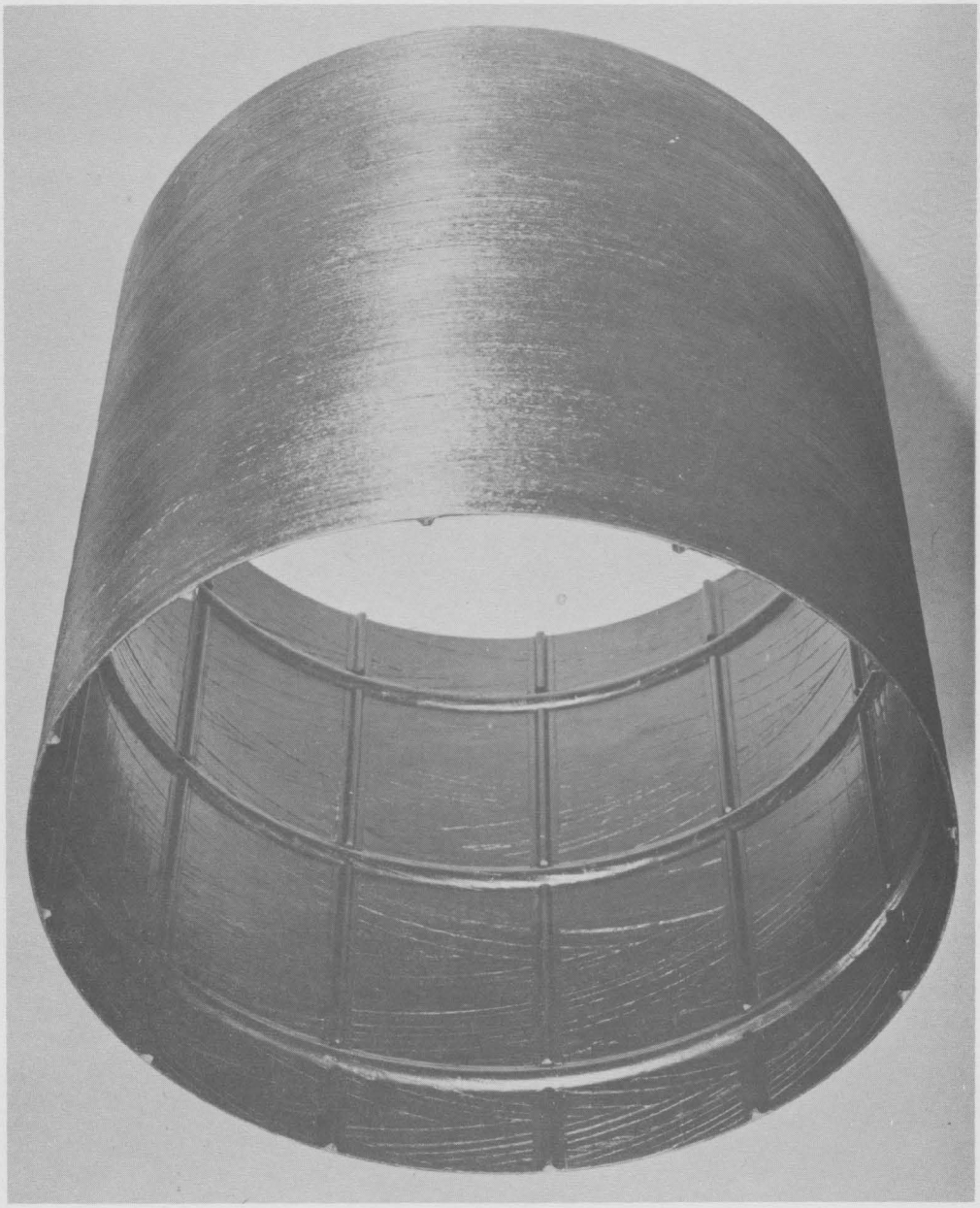


FIGURE 9 Filament Wound Ribbed Cylinder
(courtesy of I.M.I. Ltd.)

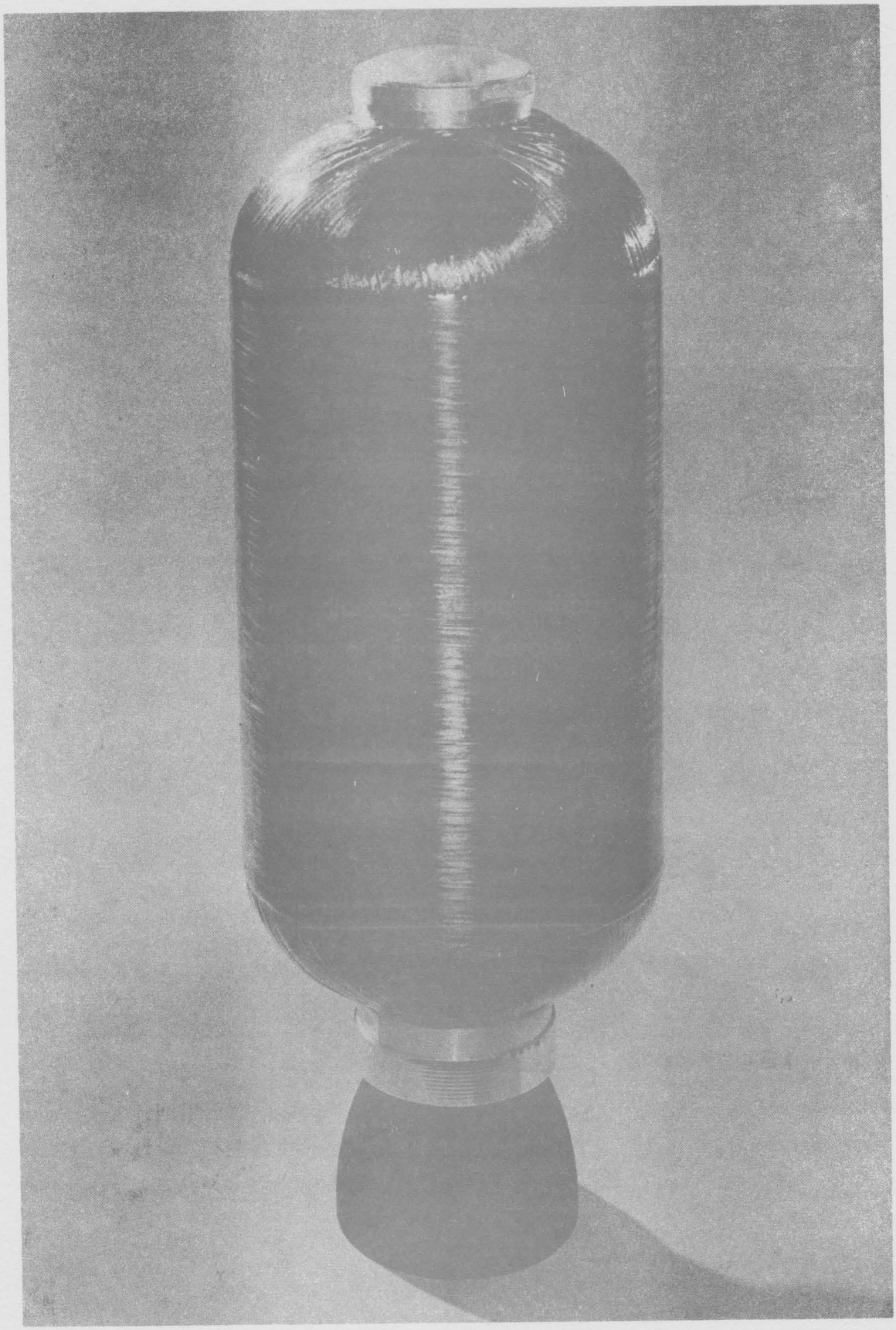


FIGURE 10 6 inch diameter carbon fibre rocket motor case (courtesy of Bristol Aerojet Ltd.)

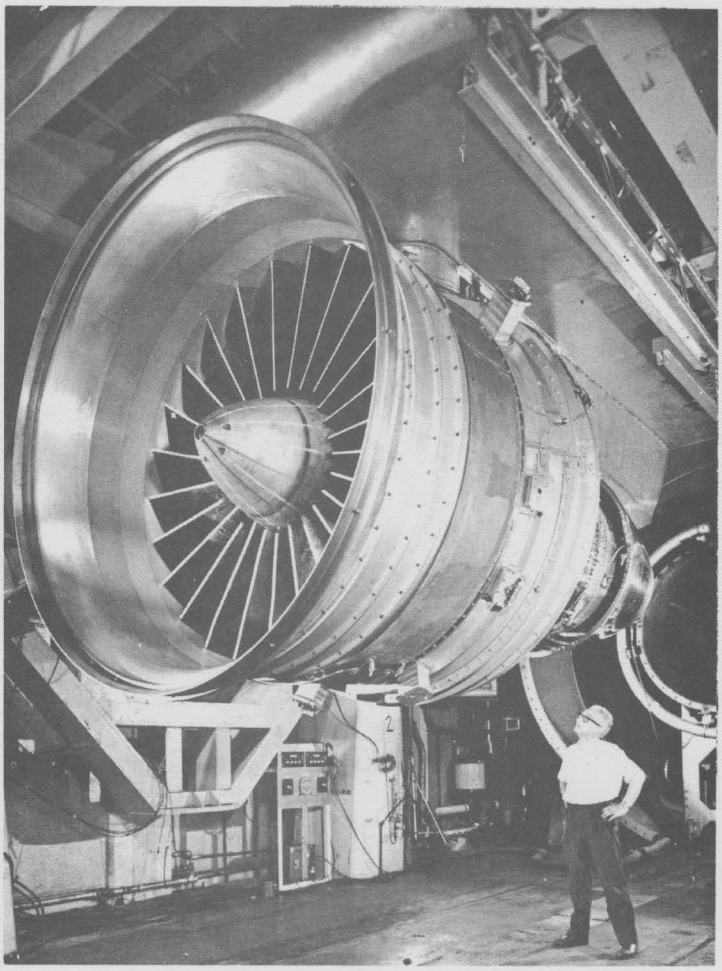


FIGURE 11 Rolls-Royce R.B.211 turbo fan engine on test.
(courtesy of Rolls-Royce (Composite Materials) Ltd.)

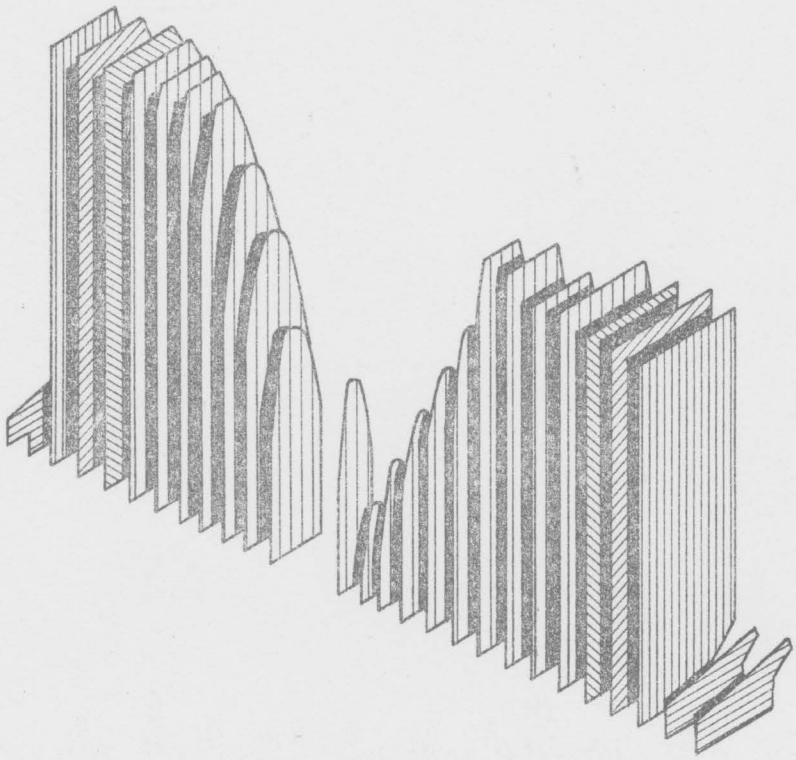


FIGURE 12 Typical layup of composite rotor blade

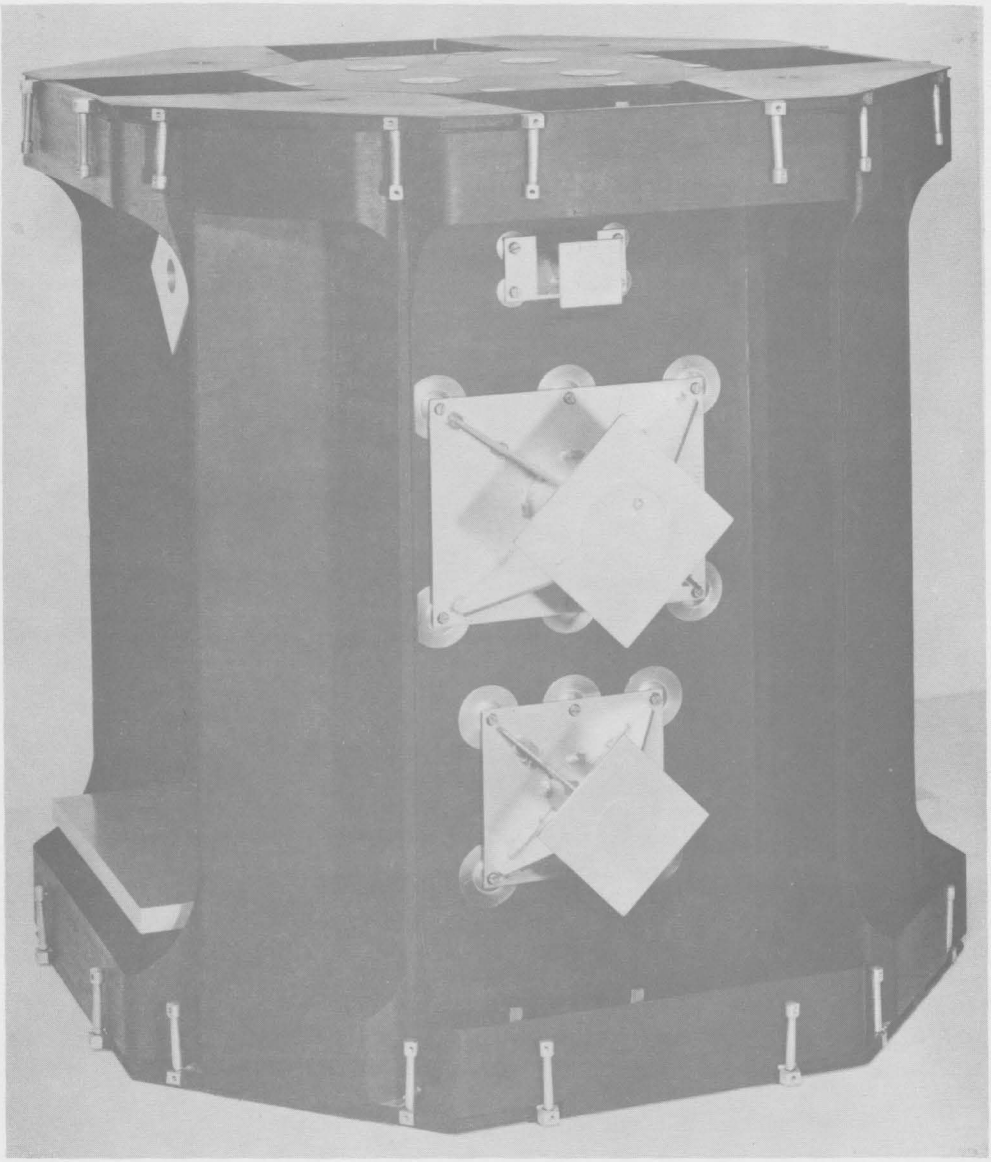


FIGURE 13 Carbon fibre reinforced plastic satellite structure (courtesy of IMI Ltd.)

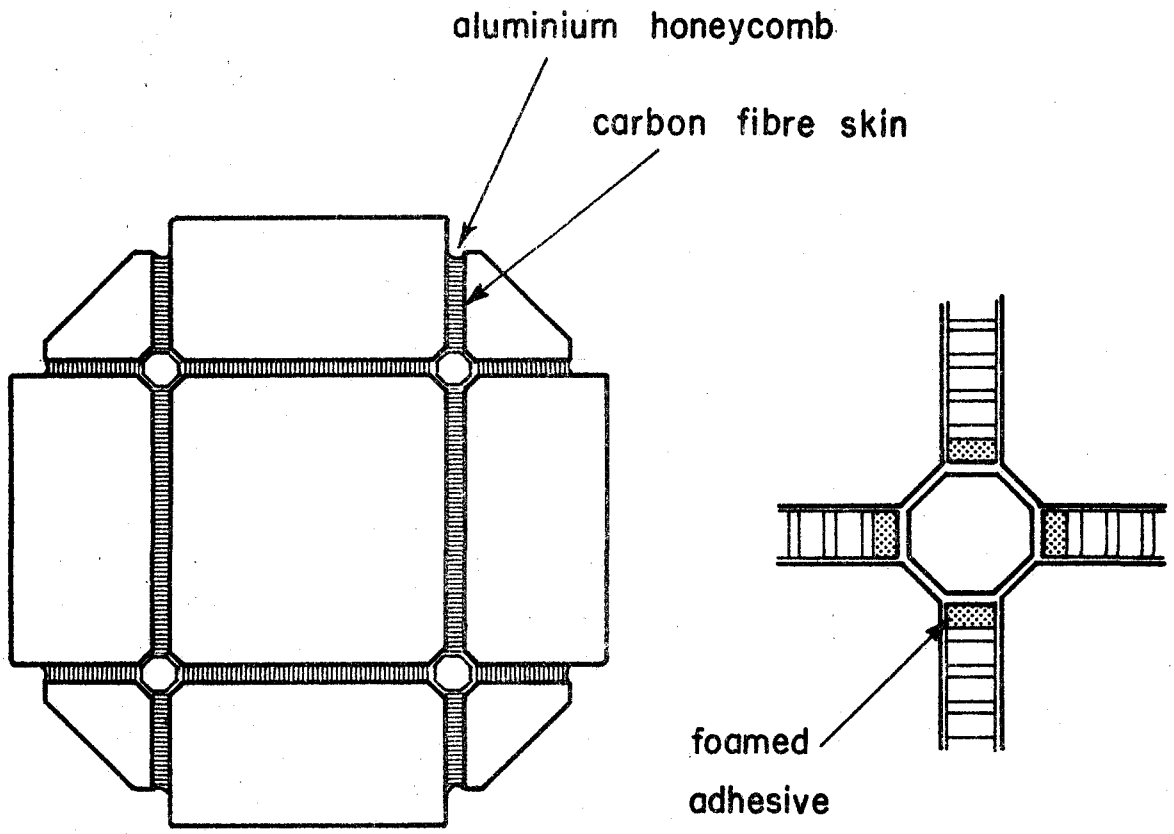


FIGURE 14 Diagram of cross section of satellite structure shown in figure 13

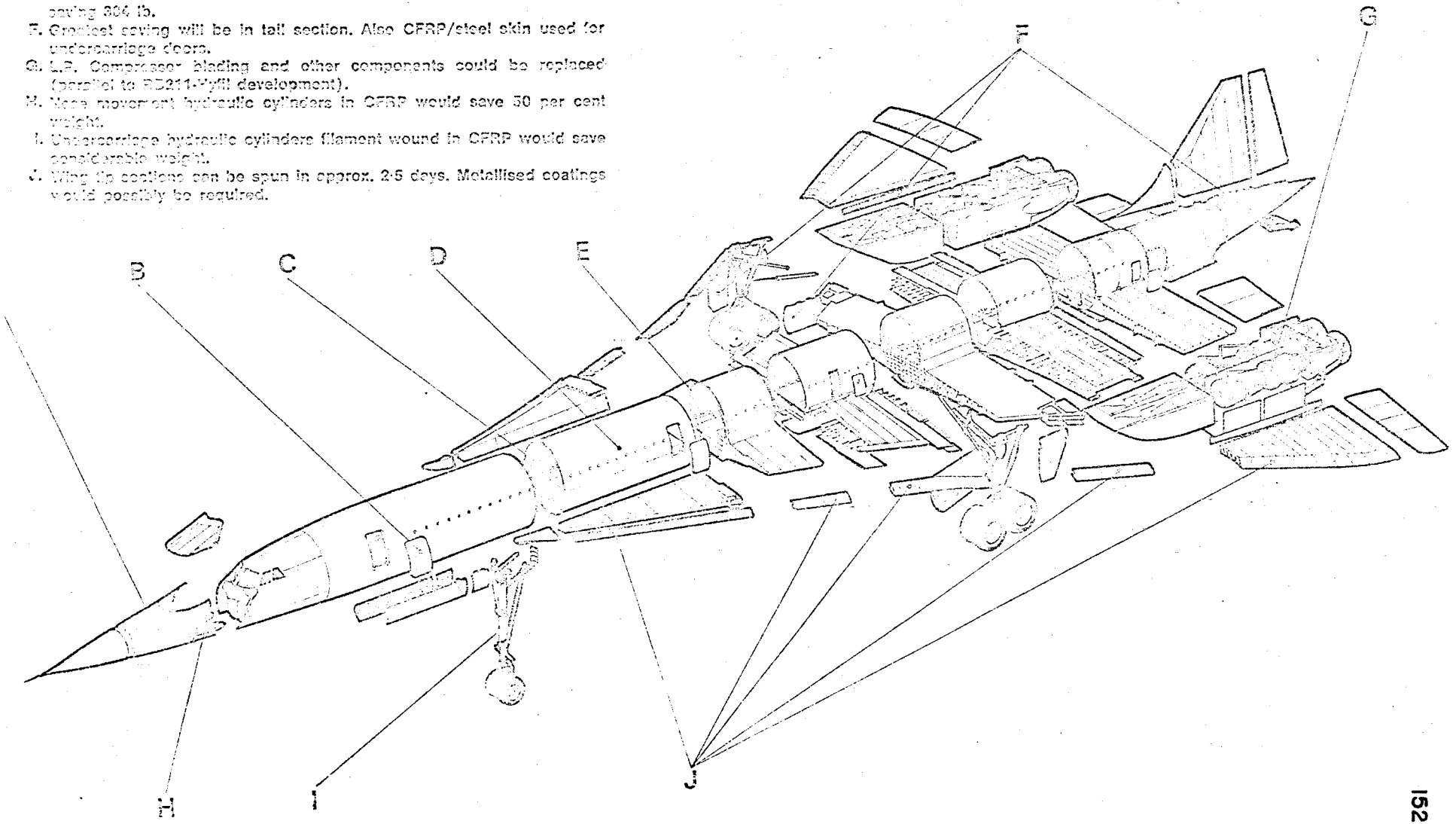
10 Areas of Weight Saving

A. Nose section (not radome section) in CFRP would reduce weight by 20 per cent.
B. Door stiffeners and internal facings would save approx. 20 per cent in weight.
C. Stringers: RAE Farnborough have constructed CFRP stringers with 25 per cent weight saving.
D. Internal low stressed panels in CFRP would save 20-30 per cent of the weight.
E. An estimated 3 lb could be lifted from each of 128 seats. Total weight saving 384 lb.
F. Greatest saving will be in tail section. Also CFRP/steel skin used for undercarriage doors.
G. L.P. Compressor blading and other components could be replaced (parallel to RB211-Hyfil development).
H. Nose movement hydraulic cylinders in CFRP would save 50 per cent weight.
I. Undercarriage hydraulic cylinders filament wound in CFRP would save considerable weight.
J. Wing tip sections can be spun in approx. 2·5 days. Metallised coatings would possibly be required.

Concorde

One lb saved in weight has a cost effectiveness of £100 over the life of the aircraft.

This is an imaginative estimate and illustration of the future impact of CFRP. The present temperature limit of the material is 200° C. Laboratory work is continuing—to lift this range to 1000°C. We have listed 10 areas only where weight saving could be effected but the applications are innumerable. CFRP scientists forecast an overall weight saving of 20-25 per cent and this could be an influential factor in the recent discussions concerning a stretched Concorde.



GURE 15 Ten areas of weight saving in the Concorde (courtesy "Design Engineering" U.K.)

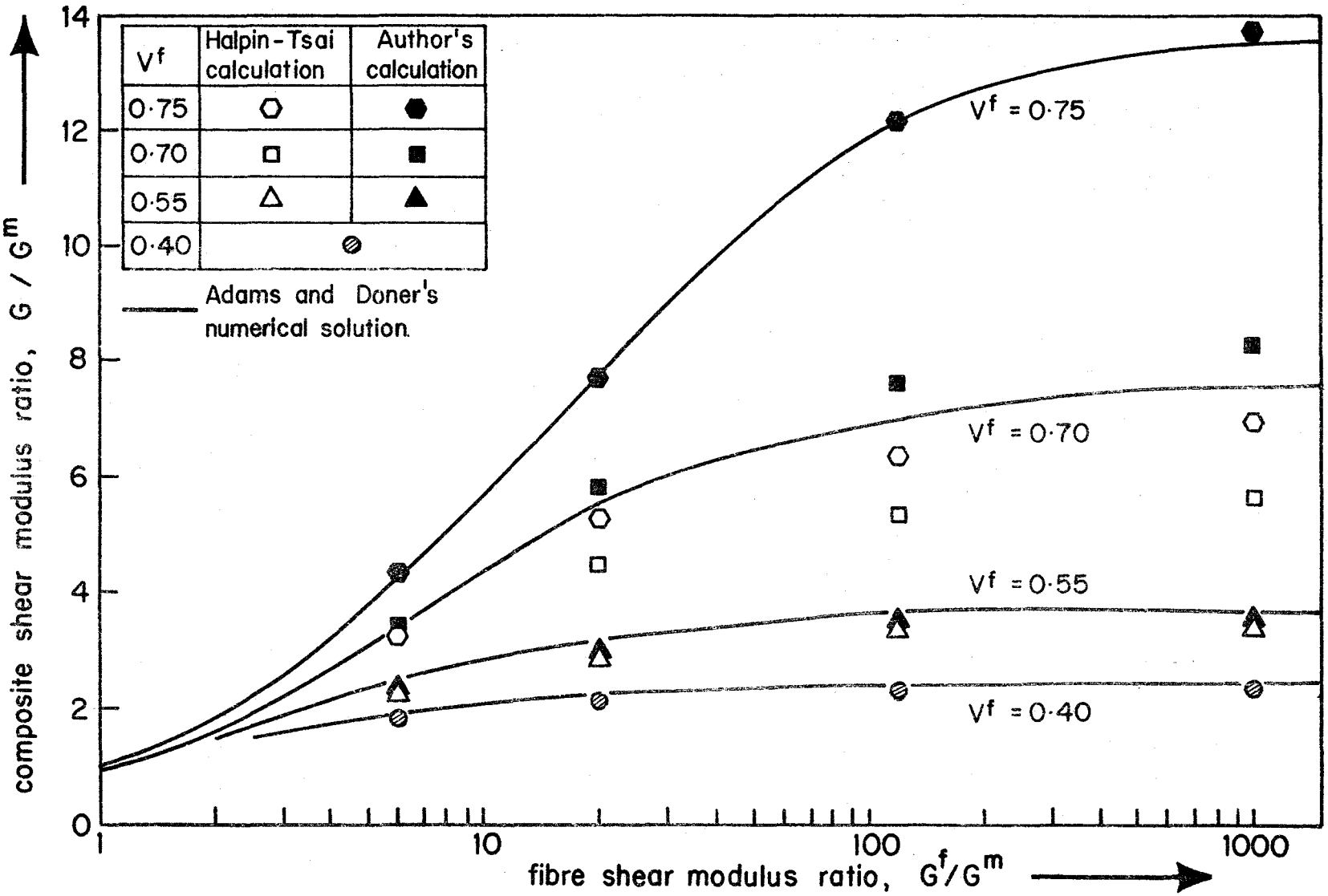


FIGURE 16 Comparison of Halpin-Tsai and present calculations with Adams and Doner's numerical solution.

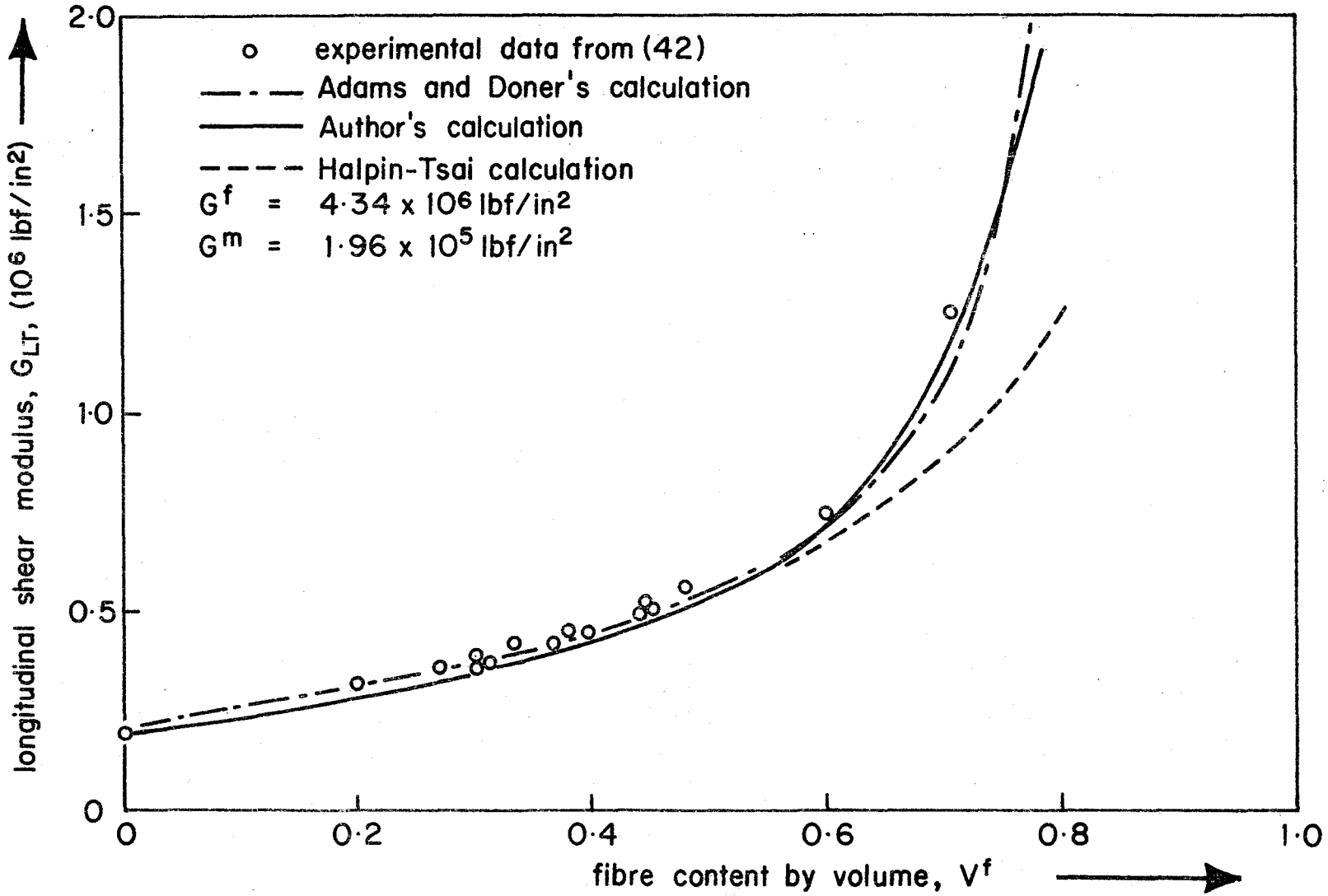


FIGURE 17 G_{LT} versus V^f for an E-glass polyester resin system.

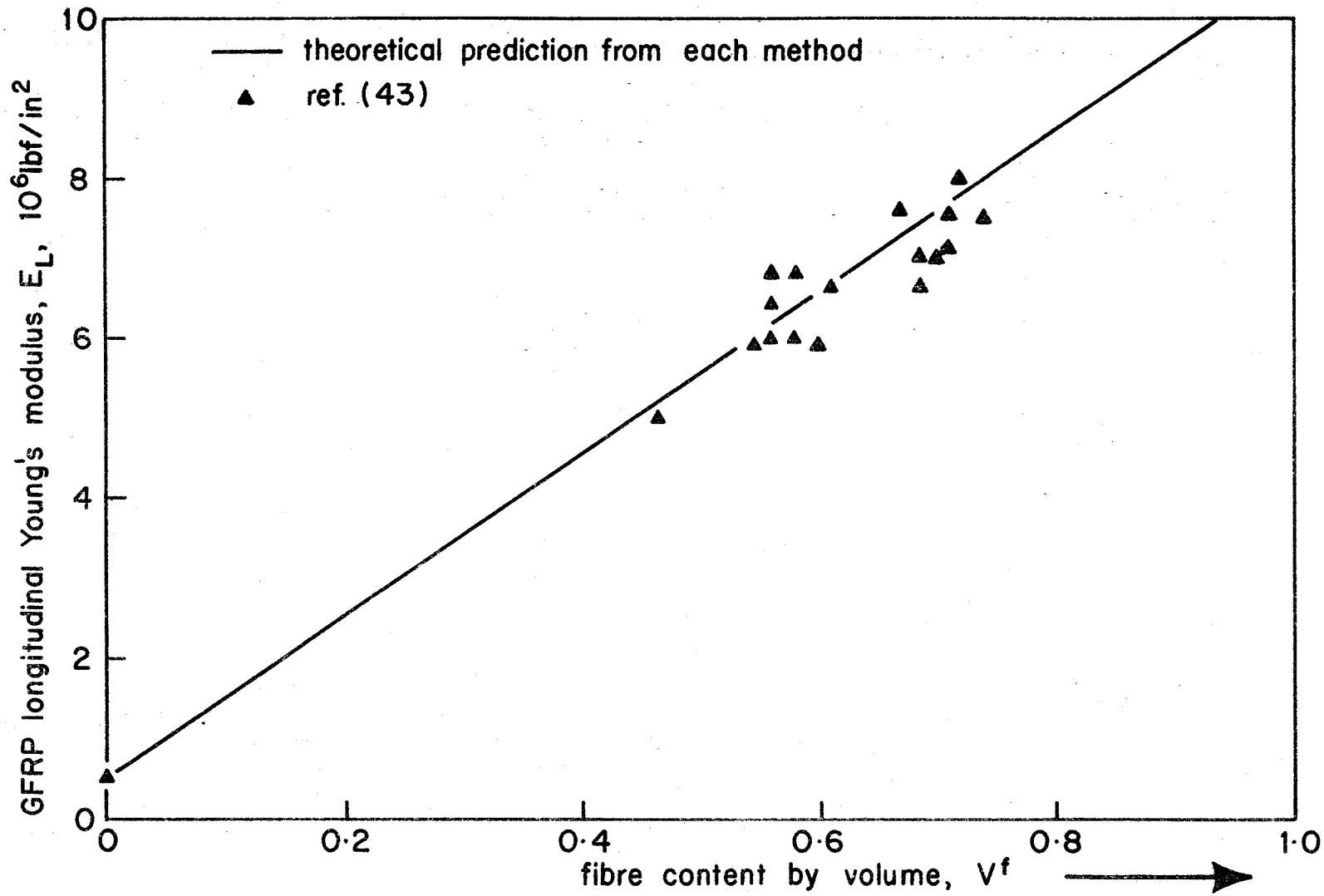


FIGURE 18 E_L versus V^f for GFRP

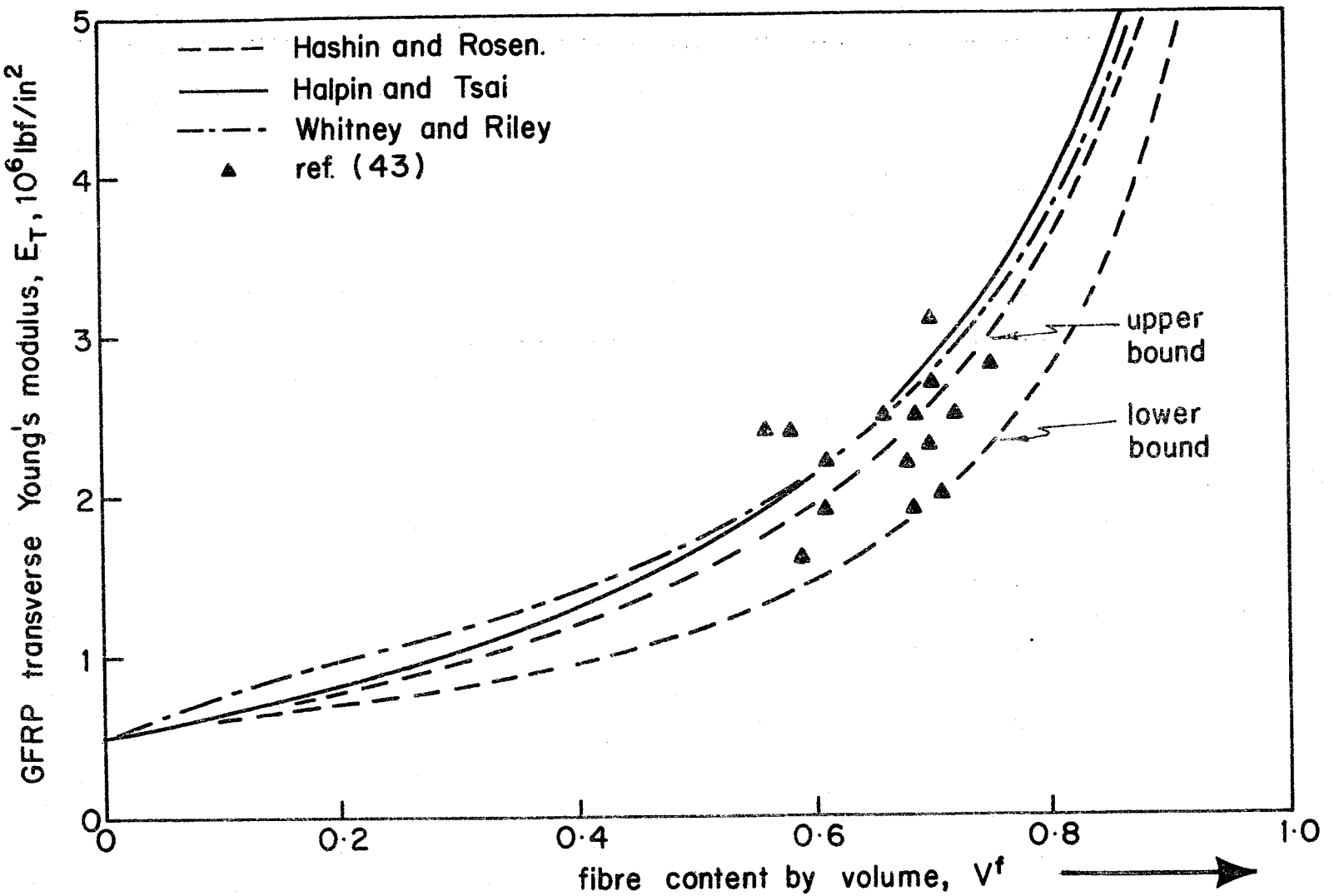


FIGURE 19 E_T versus V^f for GFRP

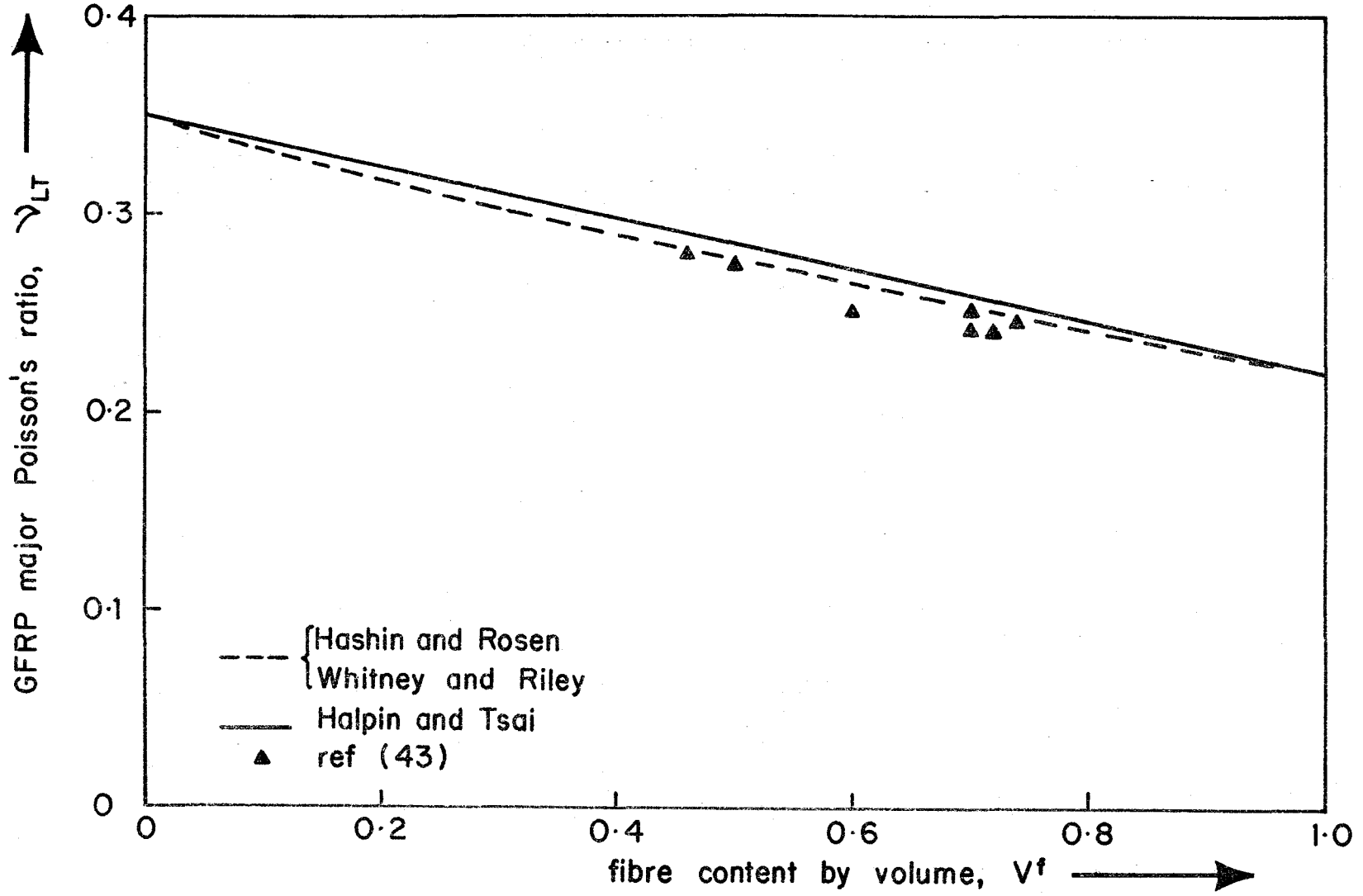


FIGURE 20 ν_{LT} versus V^f for GFRP

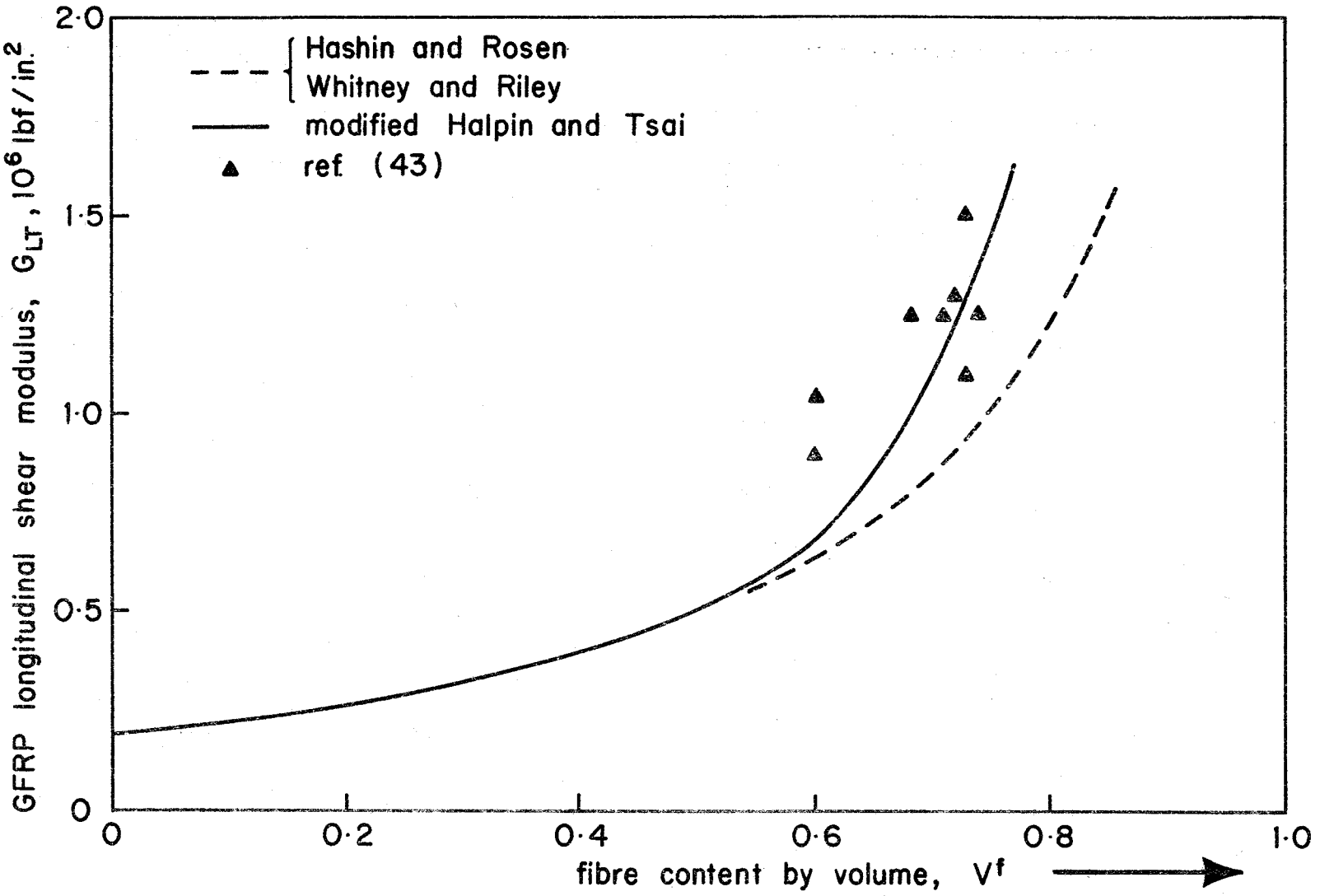


FIGURE 21 G_{LT} versus V^f for GFRP

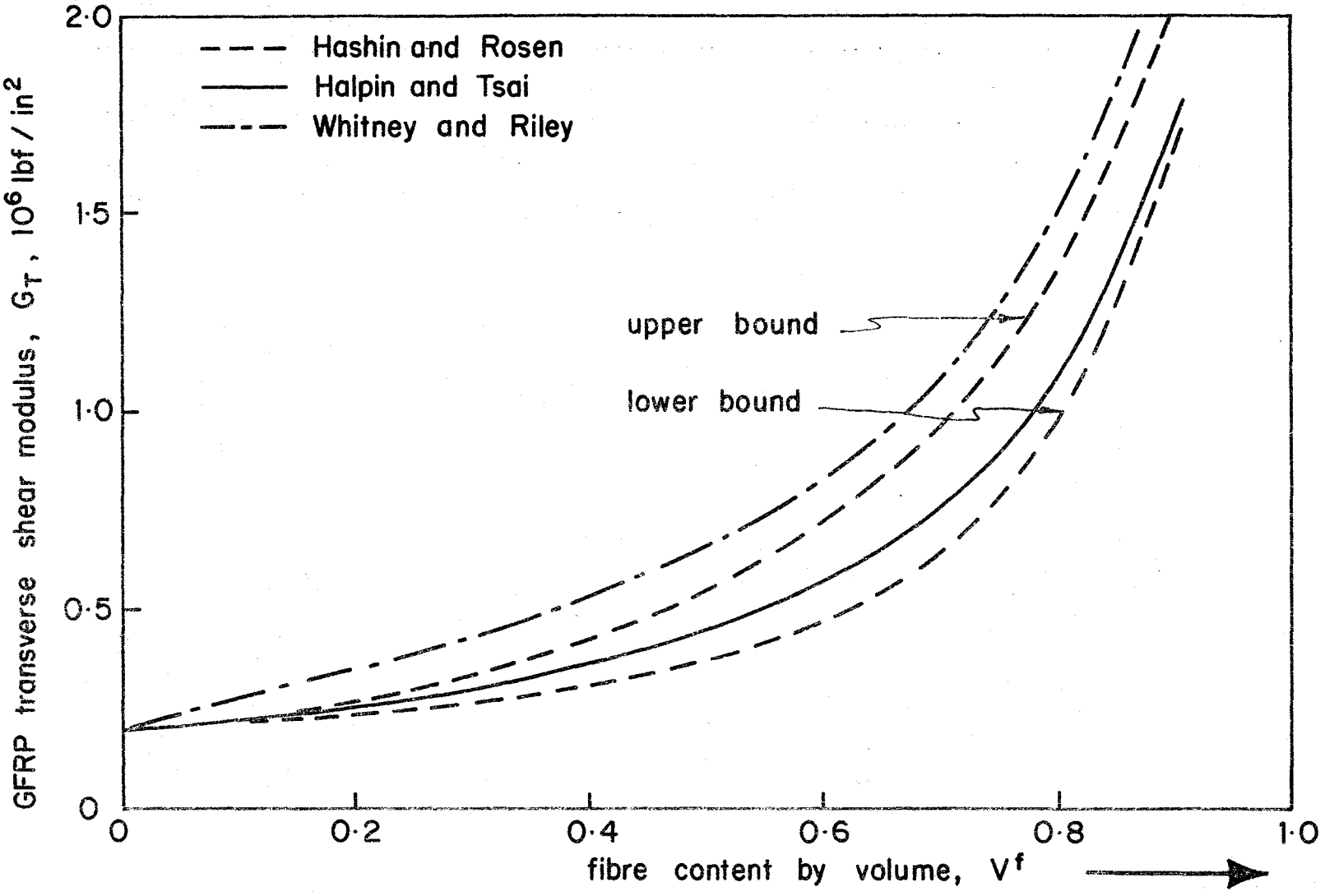


FIGURE 22 G_T versus V^f for GFRP

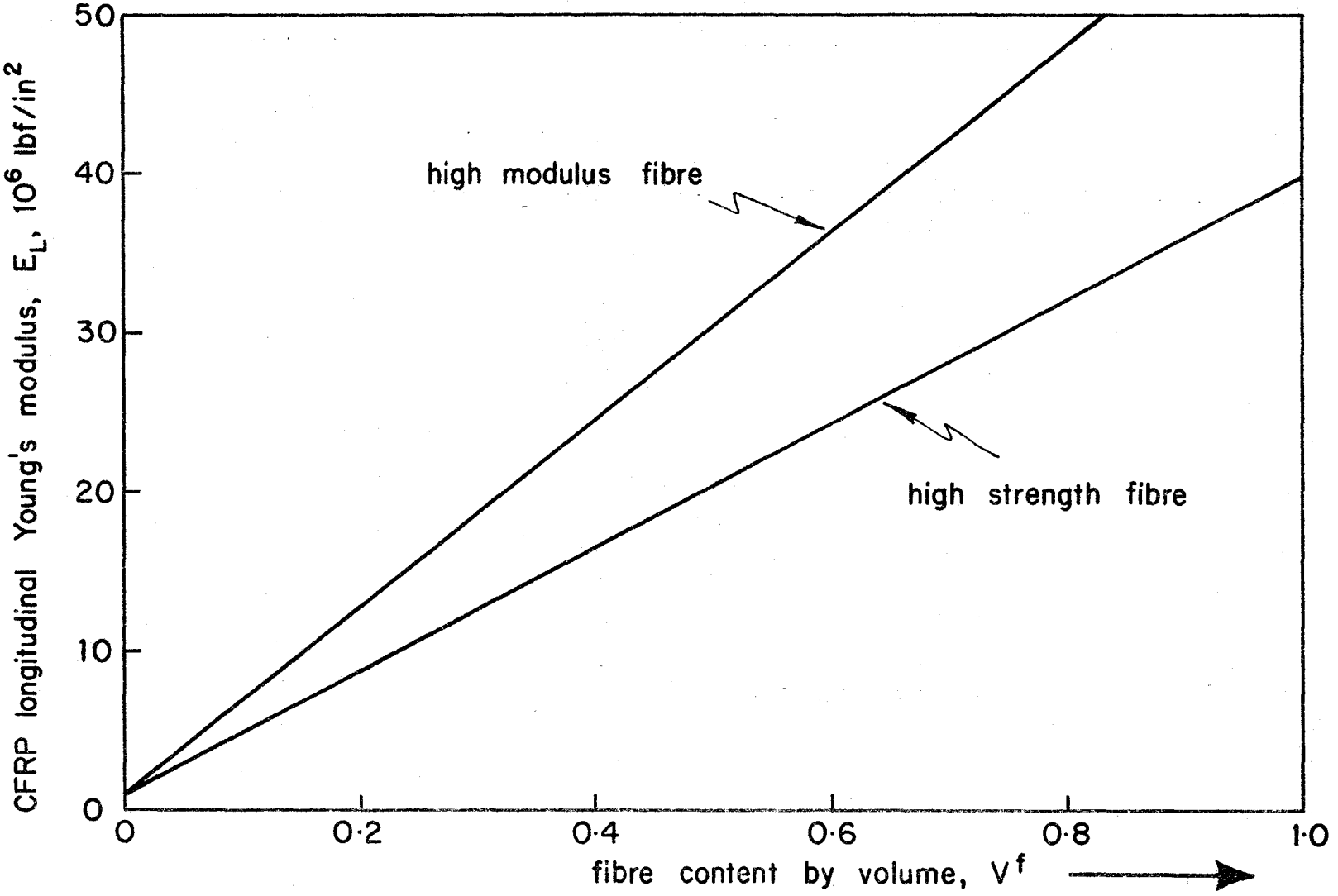


FIGURE 23 E_L versus V^f for CFRP

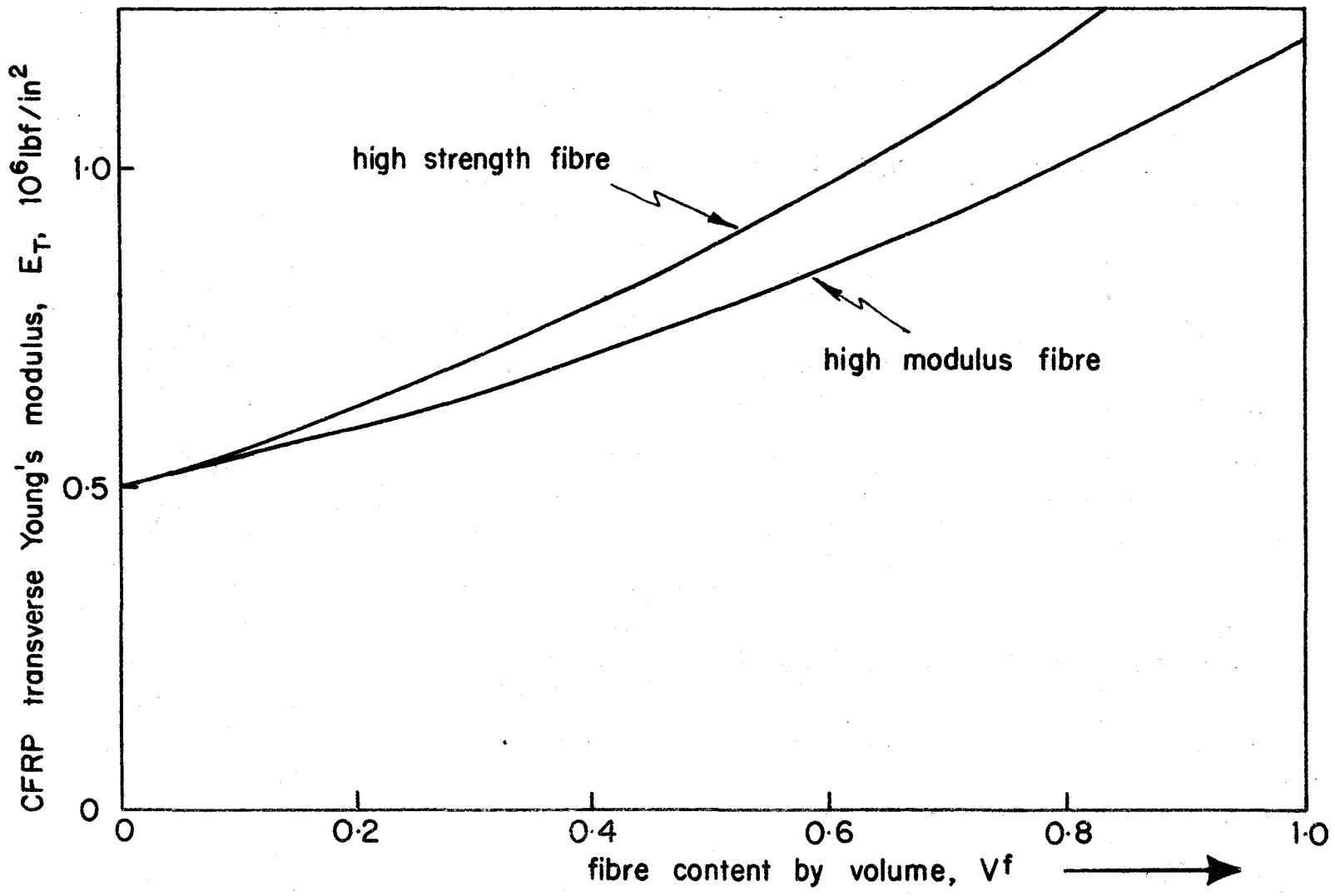


FIGURE 24 E_T versus V^f for CFRP

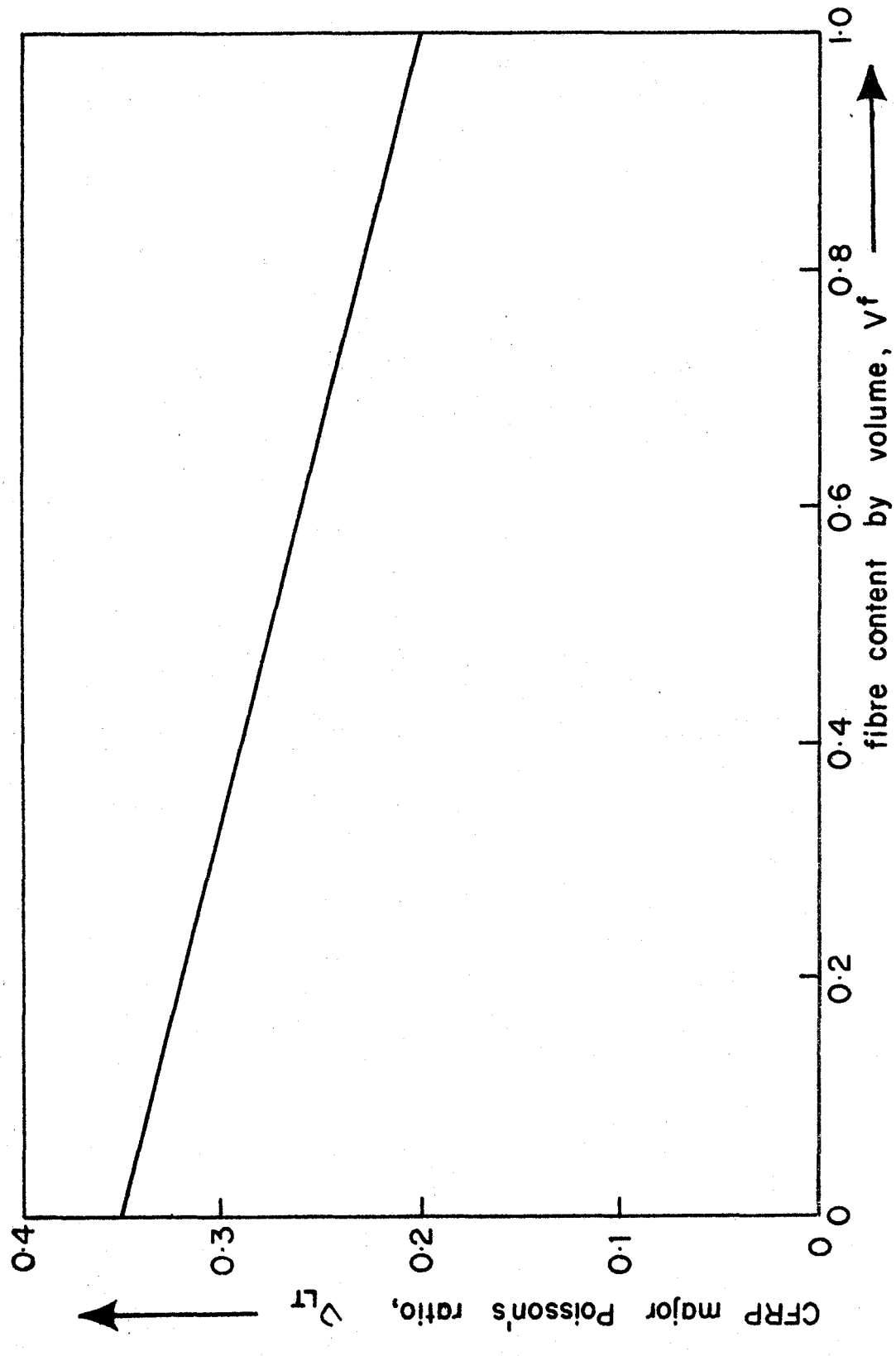


FIGURE 25 ν_{LT} versus V^f for CFRP

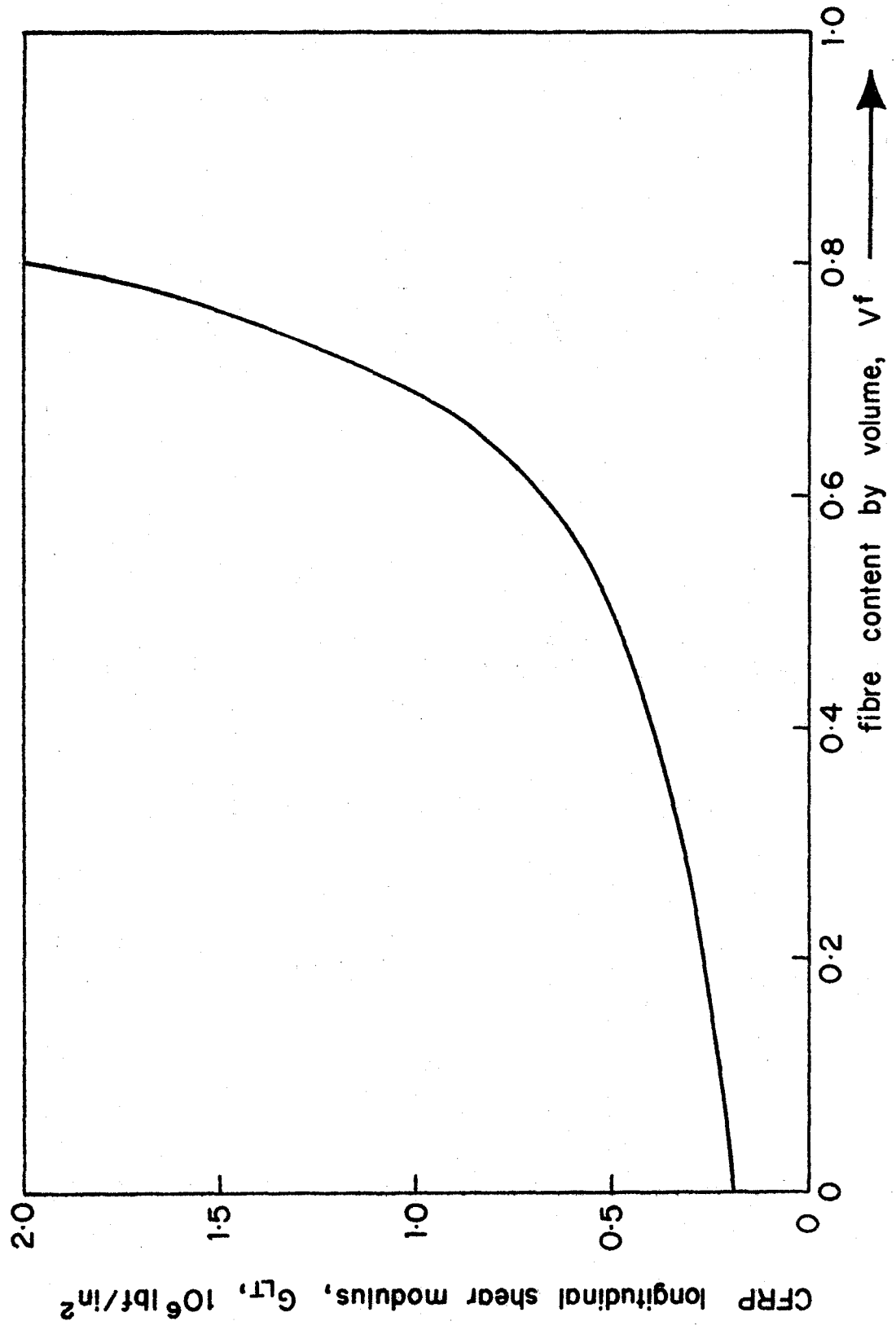


FIGURE 26 G_{LT} versus Vf for CFRP.

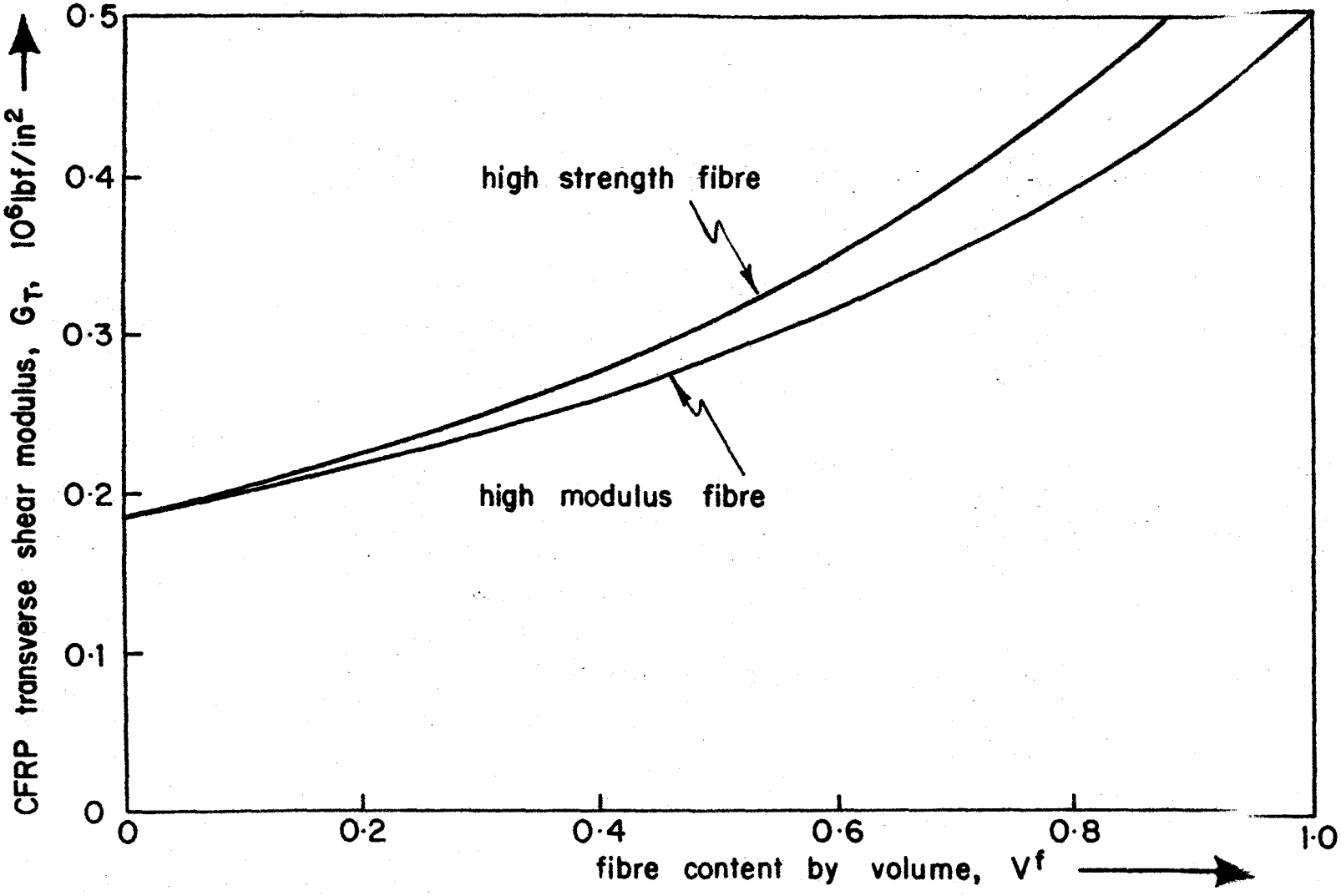


FIGURE 27 G_T versus V^f for CFRP.

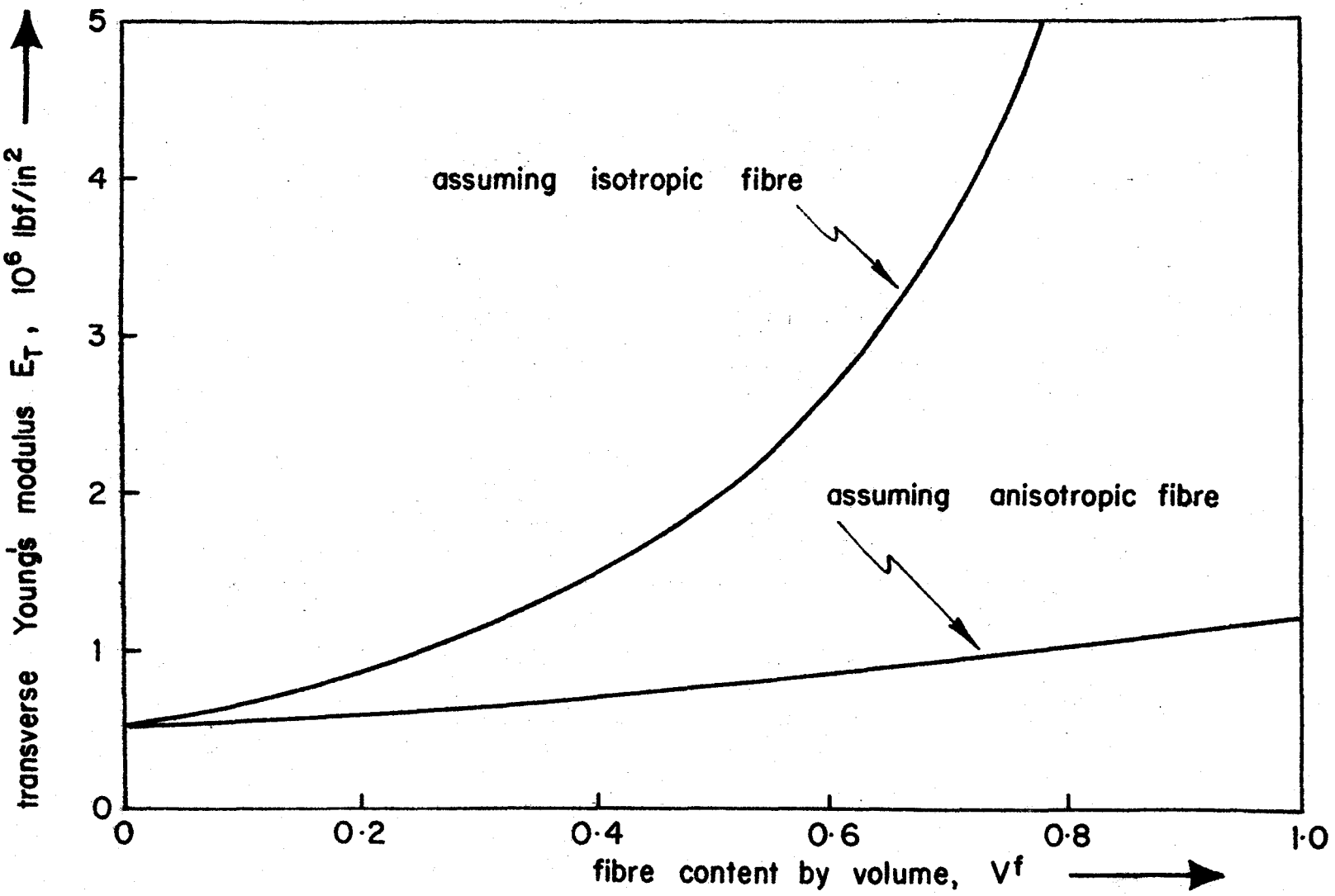


FIGURE 28 Effect of fibre anisotropy on E_T for high modulus fibre.

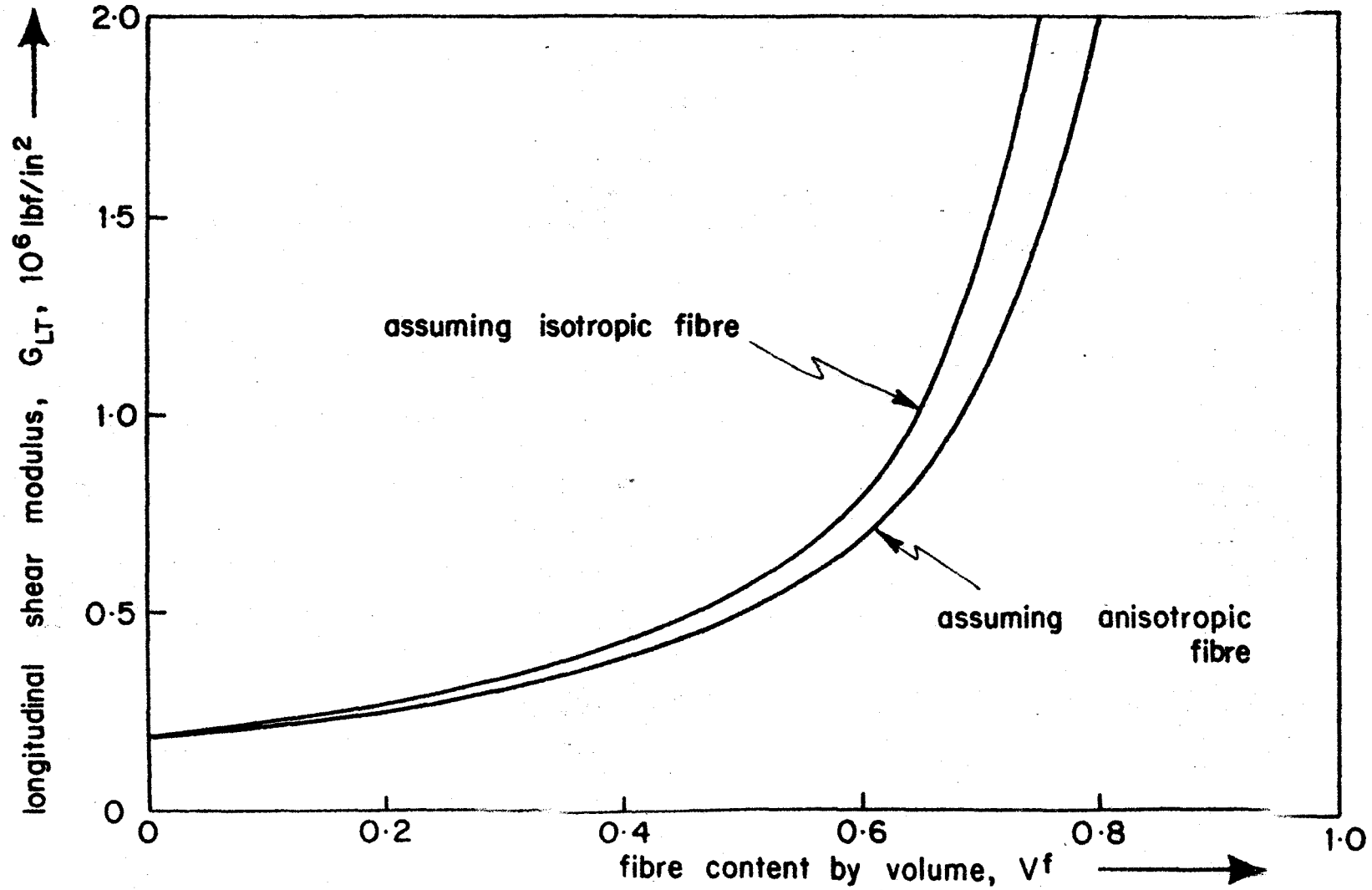


FIGURE 29 Effect of fibre anisotropy on G_{LT} for high modulus fibres.

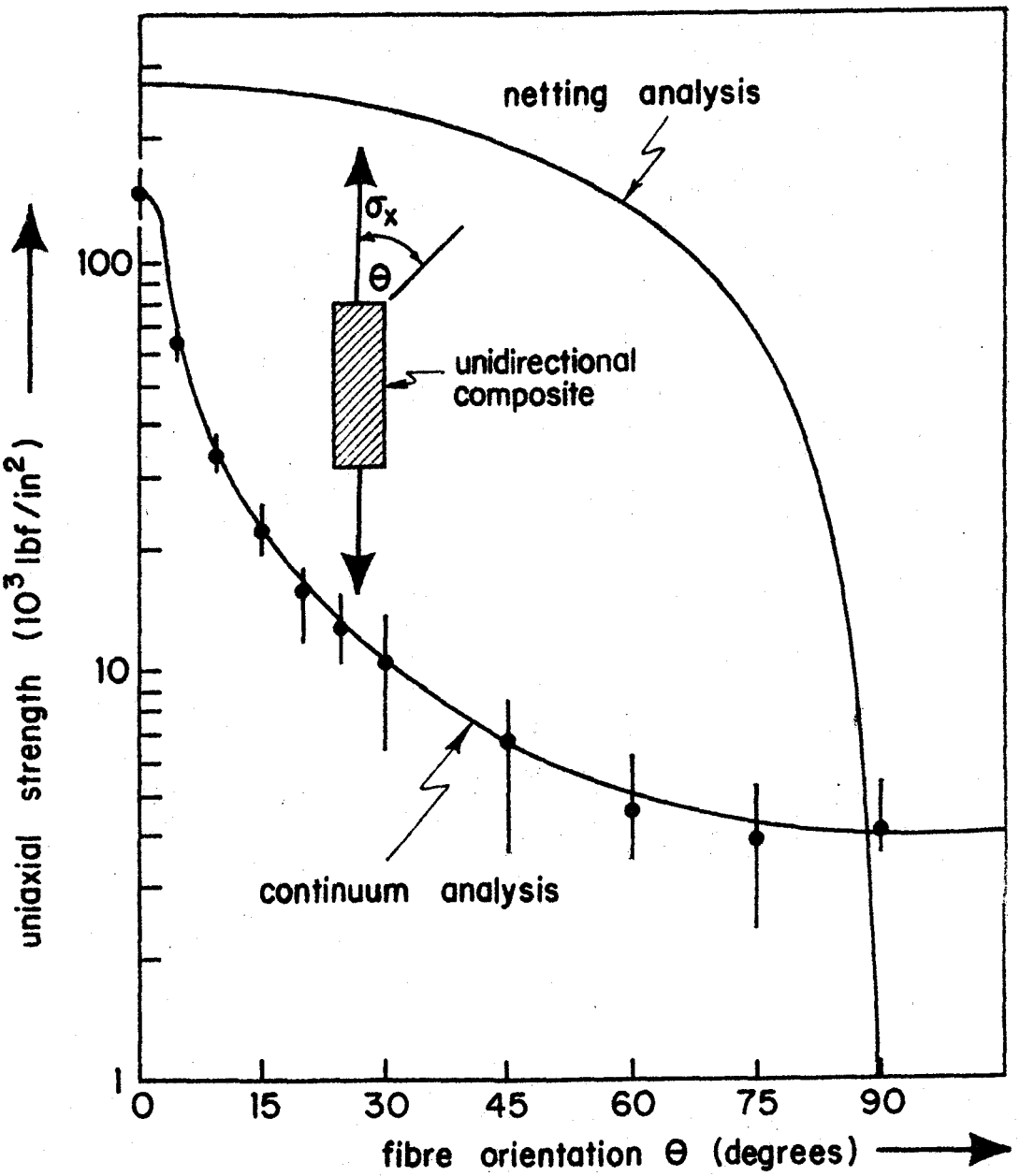


FIGURE 30 Strength of unidirectional composites.

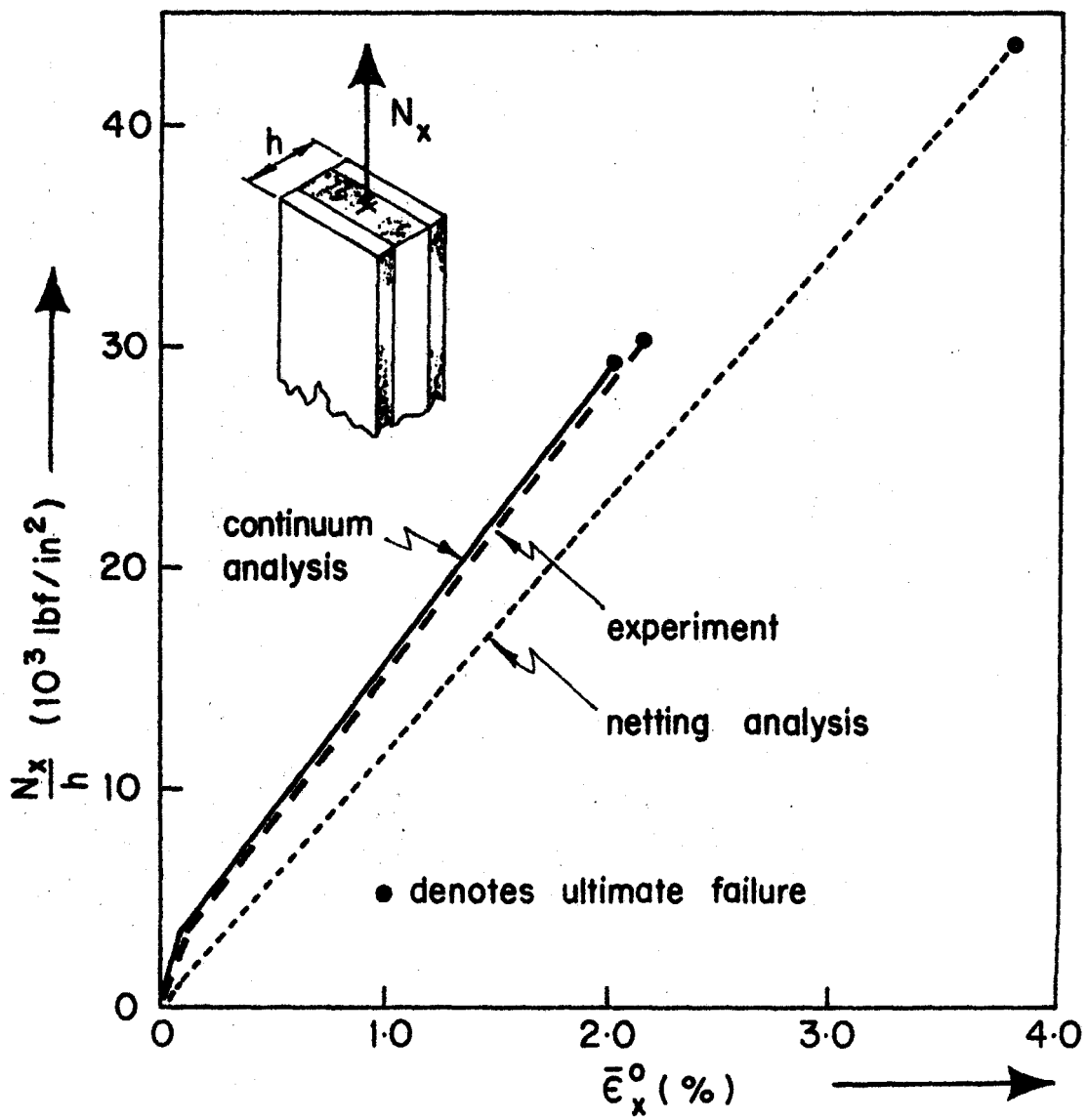


FIGURE 31 Strength of a typical cross-ply composite.