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AUTHOR: John Scott Bradley, B.Sc. (McMaster University)

M.Sc. (University of British Columbia)

SUPERVISOR: Doctor H.P. Heinig

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WEIGHTED NORM INEQUALITIES
AND HOMOGENEOUS SPACES

ABSTRACT

This thesis considers weighted norm inequalities. We characterize those pairs of weight functions for which a mixed norm version of the Hardy inequalities hold and apply these results to certain well known operators.

The two weight problem for weak boundedness of certain fractional maximal functions is solved and we give a new necessary condition for strong type boundedness of the fractional maximal and fractional integral operators. Under an additional assumption our condition is shown to be sufficient.

Many of these results are true in the setting of the homogeneous spaces of Calderón. Proofs of this together with some L log L type results are given.

The space of functions of bounded mean oscillation (BMO) is defined on a homogeneous space. Under certain conditions BMO and BMO^r $(0<r<\infty)$ are shown to be equivalent and the intermediate spaces $(L^p, BMO^s)_{\theta,q}$, $0<\theta<1$, $0<p,q,s<\infty$, are characterized. We also prove a weighted interpolation theorem for analytic families of operators.

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INTRODUCTION

In this thesis we study the Hardy-Littlewood maximal function, the fractional maximal function and other related integral operators. If T is such an operator, we investigate its action on weighted Lebesgue spaces. Specifically we consider the following problem: For what numbers p and q, and for what non-negative weight functions u and v is the map $f \to T(f)$ bounded from L_u^p to L_v^q ? That is, when does the inequality

$$(*) \qquad ||Tf||_{v,q} \leq C||f||_{u,p}$$

hold? Conversely, if inequality (*) holds, what can we say about the weight functions u and v ?

If $u = v \equiv 1$, $p = q > 1$ and $Tf = \tilde{f}$ the conjugate function of f, this inequality is the well known theorem of Marcel Riesz. From it one can show that the partial Fourier sum $S_n[f]$ of a function f converges to f in the L^p norm

$$||S_n[f] - f||_p \to 0 \quad \text{as} \quad n \to \infty.$$

For $p = 1$ and $p = +\infty$, this fails although the case $p = 1$ leads to the important notions of weak integrability and weak boundedness.

Abstractions of the M. Riesz theorem are the interpolation theorems of Riesz-Thorin and Marcinkiewicz. These results were further generalized in the general theory of interpolation by Lions-Peetre, Calderón, and many others. In this thesis we obtain a weighted interpolation theorem for analytic families of operators (Chapter 4).

Suppose f is a locally integrable function and consider its mean value

$$(**) \qquad \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt.$$

Lebesgue's theorem asserts that this average tends to $f(x)$ almost everywhere as $h \to 0$. Instead of the limit, Hardy and Littlewood [33] considered the supremum of (**) with f replaced by $|f|$ and $h > 0$. This is the Hardy-Littlewood maximal function which has been the object of much study. The basic property, namely the L^p-boundedness, $p > 1$, of the maximal function was effectively utilized in the study of convergence of orthogonal systems. A significant generalization of the Hardy-Littlewood result was obtained by Muckenhoupt [55]. He proved that (*) holds for this operator, with $p = q > 1$ and $u = v$, if and only if the weight belongs to the (Muckenhoupt) class A_p. Since the maximal function and its n-dimensional analogue pointwise dominate a large class of convolution operators ([69, p.62]) weighted norm estimates of many singular integral operators were obtained ([1],[15],[42]). Unquestionably the Hardy-Littlewood maximal function plays an important role in modern analysis, finding applications in such diverse areas as:

(i) complex function theory (one of the original reasons it was introduced [33]),

(ii) dominated pointwise convergence of singular integrals ([13],[14],[75],[77]),

(iii) conjugate function theory ([77]),

(iv) harmonic functions ([61],[66]),

(v) differentiation of integrals ([27],[64]),

(vi) ergodic theory ([11],[26],[76]),

(vii) differentiation theorems for groups ([21]),

(viii) weighted mean convergence of orthogonal series ([51],[52],[57]).

The fractional maximal function

$$\sup_{x\in I} \frac{1}{|I|^{1-\alpha}} \int_I |f(t)|\, dt \quad , \qquad 0\le\alpha<1,$$

I an interval of $\mathbb{R}$ with measure $|I|$, may be considered as an extension of the Hardy-Littlewood maximal function. Its importance lies in the fact that it dominates the fractional integral operator or Riesz potential. The (L^p,L^q) boundedness of the fractional maximal function can therefore be utilized to prove a weighted norm inequality for the fractional integral operator. In this work we define the fractional maximal function on homogeneous spaces, and obtain corresponding inequalities.

The thesis is motivated by recent studies of

Muckenhoupt, Muckenhoupt and Wheeden, and Fefferman and Stein, as well as the work of Calderón on the maximal function. We incorporate and synthesize many of their ideas in this work.

The plan of the thesis is as follows: In Chapter 1 we solve the previously stated problem (*) for the operators

$$\int_0^x f(t)\,dt \quad \text{and} \quad \int_x^\infty f(t)\,dt$$

if $1 \leq p \leq q \leq \infty$. These results are then used to prove generalizations and extensions of Hardy's inequality and are applied to other well known operators. Chapter 2 considers the corresponding problem for the fractional maximal function. A characterization of pairs of weights for which a weak type inequality corresponding to (*) holds is obtained when p and q are related by $\frac{1}{q} = \frac{1}{p} - \alpha$. In addition, necessary conditions for the inequality (*) are given.

We study the fractional maximal function defined on homogeneous spaces in Chapter 3. The program of Muckenhoupt and Wheeden [58] is shown to carry over to the setting of weighted homogeneous spaces.

In Chapter 4 functions of bounded mean oscillation (BMO) are discussed. We estimate the Peetre K-functional for the pair (L^p, BMO) and characterize the spaces $(L^p, BMO)_{\theta,q}$, $0<p,q<\infty$, $0<\theta<1$. Moreover a weighted extension of a result of Fefferman and Stein [23, Theorem 5] makes it possible to prove a weighted interpolation theorem for analytic families of operators.

Finally, we include an appendix which contains notation used throughout, as well as standard theorems used in the thesis. It is hoped that this addition makes this thesis self-contained.

CHAPTER 1

HARDY INEQUALITIES

1.1 Introduction

The classical Hardy inequality ([29],[30]) states that for all non-negative f and $p > 1$

(1.1) $$\int_0^\infty \left[\frac{1}{x}\int_0^x f(t)\,dt\right]^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty [f(t)]^p\,dt.$$

This inequality was introduced in an attempt to simplify the proofs then known of "Hilbert's double series theorem" ([29], [34, Theorem 315]). A more general inequality ([34, Theorem 330]) asserts that for $p \ge 1$, $ap > 1$,

(1.2) $$\int_0^\infty \left[x^{-a}\int_0^x f(t)\,dt\right]^p dx \le \left(\frac{p}{ap-1}\right)^p \int_0^\infty [t^{-a+1}f(t)]^p\,dt,$$

and the constant is best possible. A similar inequality holds for $\int_x^\infty f(t)\,dt$.

This result is important because it is the basis of the proof of the Hardy-Littlewood maximal inequality

(1.3). $$\int_0^\infty [Mf(x)]^p\,dx \le \left(\frac{2p}{p-1}\right)^p \int_0^\infty |f(x)|^p\,dx,$$

where $p > 1$ and $f \in L^p(0,\infty)$. Here Mf denotes the one-sided

Hardy-Littlewood maximal function

$$\text{(1.4)} \qquad Mf(x) = \sup_{t\geq 0} \frac{1}{t-x} \int_x^t |f(s)|\, ds.$$

The last inequality has many applications and will be discussed in detail in Chapter 2.

Variations on (1.2) find widespread use in interpolation theory (see for example [70, p 197],[5],[35],[36],[62],[65]).

Muckenhoupt [54] gave a new proof of a characterization due to Tomaselli [73], Talenti [71], and Artola [3] of pairs of weight functions (u,v) satisfying

$$\text{(1.5)} \qquad \left\{\int_0^\infty |u(x)\int_0^x f(t)\ dt|^p\ dx\right\}^{\frac{1}{p}} \leq C \left\{\int_0^\infty |f(x)\ v(x)|^p\ dx\right\}^{\frac{1}{p}}.$$

They showed that (1.5) holds if and only if

$$\text{(1.6)} \qquad \sup_{r>0}\left\{\int_r^\infty |u(x)|^p\ dx\right\}^{\frac{1}{p}} \left\{\int_0^r |v(x)|^{-p'}\ dx\right\}^{\frac{1}{p'}} = K < \infty,$$

and $K \leq C \leq K(p)^{\frac{1}{p}}(p')^{\frac{1}{p'}}$, where C is the least constant in (1.5). Further

$$\text{(1.7)} \qquad \left\{\int_0^\infty |u(x)\int_x^\infty f(t)\ dt|^p\ dx\right\}^{\frac{1}{p}} \leq C\left\{\int_0^\infty |f(x)\ v(x)|^p\ dx\right\}^{\frac{1}{p}}$$

was characterized.

In the next section the characterizations of Muckenhoupt et al. are generalized to include mixed norms. Our principal results are Theorems 1.1 and 1.2. They are then applied to prove certain variants of Hardy's inequality which generalize extensions of Levinson [48]. In Section 1.3 the weak type results of Andersen and Muckenhoupt [2] are summarized and compared with the strong type results of Section 1.2. The final section of this chapter contains applications to certain well known operators.

1.2 Hardy Inequalities With Mixed Norm.

Recently ([8].) we have shown that the results of Muckenhoupt et al. can be extended to the mixed norm case. The main results are given by the next two theorems.

Theorem 1.1. Let $1\le p\le q\le\infty$ and suppose that u and v are non-negative. Then

$$\left(\int_0^\infty \left|u(x)\int_0^x f(t)\,dt\right|^q dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |f(x)\ v(x)|^p\, dx\right)^{\frac{1}{p}} \tag{1.8}$$

holds if and only if

$$\sup_{r>0}\left(\int_r^\infty u(x)^q\, dx\right)^{\frac{1}{q}}\cdot\left(\int_0^r v(x)^{-p'}\, dx\right)^{\frac{1}{p'}} = K < \infty. \tag{1.9}$$

Furthermore $K\le C\le K(p)^{\frac{1}{q}}(p')^{\frac{1}{p'}}$, $p\ne 1$, where C is the least constant for which (1.8) holds. If $p = 1$, $K = C$.

Proof. Without loss of generality assume that f is non-negative.

First suppose that (1.8) holds. If $f(x) = 0$, $x > r$, a reduction in the intervals of integration in (1.8) yields

$$\left(\int_r^\infty u(x)^q\, dx\right)^{\frac{1}{q}}\left(\int_0^r f(x)\, dx\right) \le C\left(\int_0^r [f(x)\ v(x)]^p\, dx\right)^{\frac{1}{p}}. \tag{1.10}$$

If $p > 1$, substituting $f(x) = v(x)^{-p'}$ in (1.10) gives (1.9) with $K \le C$.

If $p = 1$ and $0 < \operatorname{ess\,sup}_{0<x<r} v(x)^{-1} < \infty$, let $E = \{x: v(x)^{-1} \ge -\frac{1}{n} + \operatorname{ess\,sup}_{0<t<r} v(t)^{-1}\}$ and $f(x) = \chi_E(x)\, v(x)^{-1}$. Substituting this into (1.10) gives

$$\left(\int_r^\infty u(x)^q\, dx\right)^{\frac{1}{q}} \left\{-\frac{1}{n} + \operatorname{ess\,sup}_{0<x<r} v(x)^{-1}\right\} \le C.$$

We obtain (1.9) upon letting n tend to ∞. If $p = 1$ and $\operatorname{ess\,sup}_{0<x<r} v(x)^{-1} = 0$, (1.9) holds for any K because of the convention $0 \cdot \infty = 0$. If $\operatorname{ess\,sup}_{0<x<r} v(x)^{-1} = \infty$, there exists an f such that $\int_0^r f(x)\, v(x)\, dx < \infty$ while $\int_0^r f(x)\, dx = \infty$. Using this f in (1.10) forces $\int_r^\infty u(x)^q\, dx = 0$ and (1.9) follows by convention.

To prove that (1.9) implies (1.8) in the case $1<p\le q<\infty$ we define $h(t) = \left\{\int_0^t v(s)^{-p'}\, ds\right\}^{1/(pp')}$. Then by Hölder's inequality and Minkowski's integral inequality (see appendix)

$$\begin{aligned} I &\equiv \int_0^\infty [u(x) \int_0^x f(t)\, dt]^q\, dx \\ &\le \int_0^\infty u(x)^q \left\{\int_0^x [f(t)\, v(t)\, h(t)]^p\, dt\right\}^{\frac{q}{p}} \left\{\int_0^x [v(s)\, h(s)]^{-p'}\, ds\right\}^{\frac{q}{p'}} dx \end{aligned}$$

$$\leq \left\{\int_0^\infty [f(t)v(t)h(t)]^p \left(\int_t^\infty u(x)^q \left(\int_0^x [v(s)h(s)]^{-p'} ds\right)^{\frac{q}{p'}} dx\right)^{\frac{p}{q}} dt\right\}^{\frac{q}{p}}.$$

Performing the innermost integration yields

$$\left(\int_0^x [v(s)\ h(s)]^{-p'}\ ds\right)^{\frac{q}{p'}} = (p')^{\frac{q}{p}}\left[\left(\int_0^x v(s)^{-p'}\ ds\right)^{\frac{1}{p'}}\right]^{\frac{q}{p'}},$$

which by (1.9) is bounded by

$$K^{\frac{q}{p'}}\ (p')^{\frac{q}{p'}}\left(\int_x^\infty u(s)^q\ ds\right)^{-\frac{1}{p'}}.$$

Hence

$$I \leq (Kp')^{\frac{q}{p'}}\left\{\int_0^\infty [f(t)\ v(t)\ h(t)]^p \left(\int_t^\infty u(x)^q \left(\int_x^\infty u(s)^q\ ds\right)^{-\frac{1}{p'}} dx\right)^{\frac{p}{q}} dt\right\}^{\frac{q}{p}},$$

and again evaluating the inner integral and applying (1.9) we obtain

$$\left(\int_t^\infty u(x)^q \left(\int_x^\infty u(s)^q\ ds\right)^{-\frac{1}{p'}} dx\right)^{\frac{p}{q}} = p^{\frac{p}{q}}\left(\int_t^\infty u(s)^q\ ds\right)^{\frac{1}{q}}$$
$$\leq Kp^{\frac{p}{q}}\left(\int_0^t v(s)^{-p'}\ ds\right)^{-\frac{1}{p'}} = Kp^{\frac{p}{q}}\ h(t)^{-p}.$$

Consequently

$$I \leq K^q p(p')^{\frac{q}{p'}}\left\{\int_0^\infty [f(t)\ v(t)]^p\ dt\right\}^{\frac{q}{p}}$$

which proves (1.8) with $C \leq K(p)^{\frac{1}{q}}(p')^{\frac{1}{p'}}$.

If $p = 1$ and $q < \infty$, applying Minkowski's integral inequality to the left side of (1.8) yields

$$\left\{\int_0^\infty [u(x) \int_0^x f(t)\, dt]^q\, dx\right\}^{\frac{1}{q}}$$

$$\leq \int_0^\infty f(t) \left\{\int_t^\infty u(x)^q\, dx\right\}^{\frac{1}{q}}$$

which by (1.9) is dominated by

$$K\int_0^\infty f(t)\, v(t)\, dt.$$

If $1 \leq p \leq q = \infty$, Hölder's inequality and (1.9) yield

$$u(x)\int_0^x f(t)\, dt$$

$$\leq u(x) \left\{\int_0^x v(t)^{-p'}\, dt\right\}^{\frac{1}{p'}} \left\{\int_0^x [f(t)\, v(t)]^p\, dt\right\}^{\frac{1}{p}}$$

$$\leq Ku(x) \left\{\operatorname*{ess\,sup}_{t>x} u(t)\right\}^{-1} \left\{\int_0^x [f(t)\, v(t)]^p\, dt\right\}^{\frac{1}{p}}$$

$$\leq K\left\{\int_0^\infty [f(t)\, v(t)]^p\, dt\right\}^{\frac{1}{p}}.$$

We point out that if $p = q$ the constants obtained are best possible. If u is the characteristic function of $[1,2]$ and v is 1 on $[0,1]$ and ∞ elsewhere then $K = C = 1$

(take $f = \chi_{[0,1]}$ in (1.8)), while, if $p = q$, $C = Kp^{\frac{1}{p}}(p')^{\frac{1}{p'}}$ in the classical case [54].

The following dual result is obtained analogously.

<u>Theorem 1.2</u>. Suppose that $1 \le p \le q \le \infty$ and that u and v are non-negative. Then

$$(1.11) \qquad \left(\int_0^\infty |u(x)\int_x^\infty f(t)\ dt|^q\ dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |f(x)\ v(x)|^p\ dx\right)^{\frac{1}{p}},$$

if and only if

$$(1.12) \qquad \sup_{r>0}\left(\int_0^r u(x)^q\ dx\right)^{\frac{1}{q}} \left(\int_r^\infty v(x)^{-p'}\ dx\right)^{\frac{1}{p'}} = K < \infty.$$

In addition $K \le C \le K(p)^{\frac{1}{q}}(p')^{\frac{1}{p'}}$, $p \ne 1$. If $p = 1$, $K = C$.

As an immediate corollary we obtain the following result.

<u>Corollary 1.3</u>. Let $1 \le p \le q < \infty$. If $aq > 1$, $bq < 1$ and f is non-negative,

$$(1.13) \qquad \left(\int_0^\infty (x^{-a}\int_0^x f(t)\ dt)^q\ dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (x^{-a+\frac{1}{q}+\frac{1}{p'}}\ f(x))^p\ dx\right)^{\frac{1}{p}}$$

and

$$(1.14)\quad \left(\int_0^\infty (x^{-b}\int_x^\infty f(t)\,dt)^q dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty (x^{-b+\frac{1}{q}+\frac{1}{p'}} f(x))^p \, dx\right)^{\frac{1}{p}}.$$

The case $1<p<q$, $a = \frac{1}{q} + \frac{1}{p'}$, is due to Hardy and Littlewood [32] while this corollary can be found in its entirety in Flett ([24],[25]).

The inequalities for $p < 1$ are reversed as the following result of Hardy shows.

Theorem 1.4 [34, Theorem 347]. Let $0<p<1$, $ap>1$, $bp<1$ and f be non-negative. Then

$$(1.15)\quad \left(\int_0^\infty [x^{-a}\int_0^x f(t)\,dt]^p \,dx\right)^{\frac{1}{p}} > \left(\frac{p}{|ap-1|}\right)\left(\int_0^\infty [x^{-a+1} f(x)]^p \,dx\right)^{\frac{1}{p}}$$

and

$$(1.16)\quad \left(\int_0^\infty [x^{-b}\int_x^\infty f(t)\,dt]^p \,dx\right)^{\frac{1}{p}} > \left(\frac{p}{|bp-1|}\right)\left(\int_0^\infty [x^{-b+1} f(x)]^p \,dx\right)^{\frac{1}{p}},$$

unless $f \equiv 0$. Further, the constants are best possible.

Analogous results have been obtained by Beesack [4] in the case $p < 0$.

One can, in fact, show that there are no non-trivial pairs of weight functions (u,v) for which inequalities of the form (1.8) or (1.11) hold if $0<p<1$, $p<q\leq\infty$. This is

an immediate consequence of the following result which can be found in [74]:

Theorem 1.5. Let $0<p<1$, $p<q\le\infty$ and $0<r<\infty$, $0<s\le\infty$. Then there does not exist any non-trivial linear operator from $L(p,r)$ to $L(q,s)$. (For a definition of $L(\cdot,\cdot)$ see Appendix.)

We now study variants of Hardy's inequality along the lines of Levinson [48]. For this purpose we assume that ϕ is a twice differentiable, non-negative functon on $\mathbb{R}$. Further we suppose that there exists a $q > 1$ such that $\phi\phi'' \ge (1-\frac{1}{q})(\phi')^2$.

Theorem 1.6. Let ϕ be as above, $\frac{1}{q} \le r \le 1$, and $a > 0$. If u and v are non-negative functions satisfying

$$(1.17) \qquad \sup_{t>0}\left(\int_t^\infty \frac{u(x)}{x^{aq}}\,dx\right)^{\frac{1}{q}} \left(\int_0^t v(x)^{-1/(qr-1)}\,dx\right)^{(qr-1)/qr} = K < \infty,$$

then

$$(1.18) \qquad \int_0^\infty u(x)\,\phi\left(ax^{-a}\int_0^x t^{a-1} f(t)\,dt\right)dx \le C\left(\int_0^\infty v(x)\,[x^{q(a-1)}\phi(f(x))]^r\,dx\right)^{\frac{1}{r}}.$$

Proof. Let $\psi(x) = \phi(x)^{\frac{1}{q}}$ and set $p = qr$. Now

$\psi'' = \frac{1}{q}(-\frac{1}{q'})\phi^{-1-1/q'}(\phi')^2 + \frac{1}{q}\phi^{-1/q'}\phi''$ is non-negative since $\phi\phi'' \geq \frac{1}{q'}(\phi')^2$. Hence ψ is convex and we can use Jensen's inequality followed by Theorem 1.1 to obtain

$$\left(\int_0^\infty u(x)[\psi(ax^{-a}\int_0^x t^{a-1}f(t)\,dt)]^q\,dx\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_0^\infty u(x)x^{-aq}\left(\int_0^x t^{a-1}\psi(f(t))\,dt\right)^q dx\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_0^\infty v(t)\,[t^{a-1}\psi(f(t))]^p\,dt\right)^{\frac{1}{p}}$$

$$= C\left(\int_0^\infty v(t)\,[t^{(a-1)q}\phi(f(t))]^r\,dt\right)^{\frac{1}{p}}.$$

This completes the proof.

Setting $u(x) \equiv 1$ and $v(x) = x^{-aqr+r+qr-1}$ in (1.18) we obtain:

<u>Corollary 1.7</u>. Let ϕ and r satisfy the conditions of Theorem 1.6 and let $aq > 1$. Then

$$(1.19)\qquad \int_0^\infty \phi(ax^{-a}\int_0^x f(t)\,t^{a-1}\,dt)\,dx \leq C\left(\int_0^\infty [x\phi(f(x))]^r \frac{dx}{x}\right)^{\frac{1}{r}}.$$

If $a = r = 1$ this is Theorem 1 of [48]. If $a = r = 1$ and $\phi(x) = \exp(x)$ Corollary 1.7 is due to Pólya (see for example [30]).

Replacing f by $\log|f|$ and setting $\phi(x) = \exp(x)$ in Corollary 1.7 yields the following.

Corollary 1.8. Let $0<r\leq 1$, $a>0$. Then

$$\int_0^\infty \exp\left(ax^{-a}\int_0^x \log|f(t)|\,t^{a-1}\,dt\right)dx \leq C\left(\int_0^\infty |xf(x)|^r \frac{dx}{x}\right)^{\frac{1}{r}}, \tag{1.20}$$

unless $f \equiv 0$.

The case $a = r = 1$ is due to Knopp [44] and appears in [34, Theorem 335].

Analogous results can be obtained if the $\int_0^x$ in Theorem 1.6 is replaced by $\int_x^\infty$.

In the following theorem inequality (1.18) is reversed.

Theorem 1.9. Let $0<p<1$ and ϕ be a twice differentiable, non-negative function on $\mathbb{R}$ such that ϕ'' is non-positive and $\phi\phi'' \leq (1-\frac{1}{p})(\phi')^2$. If $f(x)\in\mathrm{Dom}(\phi)$, $0<x<\infty$, then

$$\left(\int_0^\infty \phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)dx\right) \geq C\int_0^\infty \phi(f(x))\,dx. \tag{1.21}$$

The function $\phi(x) = \ln(e^{\frac{1}{p}} + x)$ satisfies the conditions of this theorem.

Proof. The proof is essentially that of Theorem 1.6 with $a = 1$, except that instead of Theorem 1.1 we apply Theorem 1.4. We omit the details.

1.3 Weak Type Results.

Recently Andersen and Muckenhoupt [2] have studied "weak type Hardy inequalities". We discuss their results briefly in this section and contrast them with the strong type results just obtained.

The following terminology is needed:

Definition 1.10. Let α be real and define the operators P_α and Q_α by

(1.22) $$P_\alpha f(x) = x^{-\alpha}\int_0^x f(t)\,dt,\ x > 0,$$

and

(1.23) $$Q_\alpha f(x) = x^{-\alpha}\int_x^\infty f(t)\,dt,\ x > 0.$$

Definition 1.11. For $0<p,q\leq\infty$ and (u,v) a pair of non-negative weight functions on $(0,\infty)$ we say that (u,v) is a strong type (p,q) weight pair for an operator T if

(1.24) $$\left(\int_0^\infty |Tf(x)\ u(x)|^q\,dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty |f(x)\ v(x)|^p\,dx\right)^{\frac{1}{p}}$$

and a weak type (p,q) weight pair for T if

(1.25) $$\left(\int_{\{|Tf|>y\}} u(x)^q \, dx\right)^{\frac{1}{q}} \le \frac{C}{y} \left(\int_0^\infty |f(x).v(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Inequality (1.25) is "weaker" than (1.24) in the sense that (1.24) implies (1.25).

Theorem 1.1 says that (u,v) is a strong type (p,q) weight pair for $P_\alpha, -\infty < \alpha < \infty$, $1 \le p \le q \le \infty$, if and only if

$$\sup_{r>0}\left(\int_r^\infty \left(\frac{u(x)}{x^\alpha}\right)^q dx\right)^{\frac{1}{q}} \left(\int_0^r v(x)^{-p'} dx\right)^{\frac{1}{p'}} = K < \infty,$$

while Theorem 1.2 contains the corresponding result for Q_α.

The principal weak type results are as follows:

<u>Theorem 1.12 [2]</u>. Suppose $1 \le p \le q < \infty$ and $\alpha \le 0$. Then (u,v) is a weak type (p,q) pair for P_α if and only if

(1.26) $$B(\alpha) = \sup_{r>0} r^{-\alpha}\left(\int_r^\infty u(x)^q \, dx\right)^{\frac{1}{q}}\left(\int_0^r v(x)^{-p'} \, dx\right)^{\frac{1}{p'}}$$

is finite.

<u>Theorem 1.13 [2]</u>. Suppose $1 \le p \le q < \infty$ and $\alpha > 0$. Let

(1.27) $$B(\alpha,a) = \sup_{r>0}\left(\int_r^\infty \left(\frac{r}{x}\right)^a \left(\frac{u(x)}{x^\alpha}\right)^q dx\right)^{\frac{1}{q}} \left(\int_0^r v(x)^{-p'} \, dx\right)^{\frac{1}{p'}}.$$

If $B(\alpha,a)$ is finite for some $a > 0$ then (u,v) is a weak type (p,q) weight pair for $P_{-\alpha}$. Conversely, if (u,v) is a weak type (p,q) weight pair for P_α, then $B(\alpha,a)$ is finite for all $a > 0$.

Corollary 1.14 [2]. Suppose $1 < p \leq q \leq \infty$, $\alpha > 0$. Then the following are equivalent:

(1.28) $\left(x^{\frac{a-1}{q}}, x^{\frac{a-1}{p}}\right)$ is a weak type (p,q) weight pair for P_α.

(1.29) $\left(x^{\frac{a-1}{q}}, x^{\frac{a-1}{p}}\right)$ is a strong type (p,q) weight pair for P_α.

(1.30) $a < p$ and $a(p-q) = pq(\alpha-1)$.

The results for Q_α are similar with the roles of $\alpha > 0$ and $\alpha \leq 0$ reversed.

Comparison of Theorems 1.12 and 1.13 with Theorem 1.1 shows that if (u,v) is a strong type (p,q) weight pair for P_α then it is a weak type pair, while, if $\alpha = 0$ strong and weak type are equivalent. However, it is clear that weak and strong type do not, in general, coincide, because the pair (x^{a-1}, x^{a-1}) is a weak type $(1,1)$ pair for P_1 if $a \leq 1$ but a strong type $(1,1)$ pair for P_1 only if $a < 1$. The functions $u(x) = [x/\log(1+x)]^{1/2}$, $v(x) = (1+x)^{1/2}$ form a weak, but not a strong, type $(2,2)$ pair for P_1 ([2]).

As the following theorem shows weak and strong type coincide in certain cases.

<u>Theorem 1.15 [2]</u>. If $1 < p < \infty$, $\alpha > 0$ and $w(x)$ is non-negative on $(0,\infty)$, then $(w(x), x^{1-\alpha}w(x))$ is a weak type (p,p) pair for P_α if and only if $(w(x), x^{1-\alpha}w(x))$ is a strong type (p,p) pair for P_α.

1.4 Applications.

We conclude this chapter with a few elementary applications of the previous results. We begin by proving and extension of a result of Rooney ([63, Theorem 6]) concerning boundedness properties of the Laplace transform of f, that is,

$$\mathscr{L}f(x) = \int_0^\infty e^{-xt} f(t)\, dt, \; x > 0,$$

provided the integral converges.

Theorem 1.16. Let u and v be non-negative weight functions with the property that for some $1 < p \leq q < \infty$

$$(1.31) \qquad \sup_{r>0} \left(\int_r^\infty \left[\frac{u(x)}{x}\right]^q dx\right)^{\frac{1}{q}} \left(\int_0^r v(x)^{-p'}\, dx\right)^{\frac{1}{p'}} < \infty$$

and

$$(1.32) \qquad \sup_{r>0} \left(\int_0^r u(x)^q\, dx\right)^{\frac{1}{q}} \left(\int_r^\infty [xv(x)]^{-p'}\, dx\right)^{\frac{1}{p'}} < \infty.$$

Then

$$(1.33) \qquad \left(\int_0^\infty \left|\frac{u(x)}{x}\mathscr{L}f\left(\frac{1}{x}\right)\right|^q dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty |f(x)\, v(x)|^p\, dx\right)^{\frac{1}{p}}.$$

<u>Proof</u>. Since $x^{-1}e^{-t/x} \leq e^{-1}t^{-1}$, $t \geq x$, it follows that

$$\frac{1}{x}\mathscr{L}f\left(\frac{1}{x}\right) = \frac{1}{x}\int_0^\infty e^{-t/x} f(t)\, dt$$

$$\leq \frac{1}{x}\int_0^x f(t)\, dt + e^{-1}\int_x^\infty \frac{f(t)}{t}\, dt.$$

Hence

$$\left(\int_0^\infty \left[\frac{u(x)}{x}\mathscr{L}f\left(\frac{1}{x}\right)\right]^q dx\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^\infty \left(\frac{u(x)}{x}\int_0^x f(t)\, dt\right)^q dx\right)^{\frac{1}{q}} + e^{-1}\left(\int_0^\infty \left(u(x)\int_x^\infty \frac{f(t)}{t}\, dt\right)^q dx\right)^{\frac{1}{q}},$$

which by Theorems 1.1 and 1.2 is dominated by

$$C\left(\int_0^\infty [f(t)\ v(t)]^p\, dt\right)^{\frac{1}{p}}.$$

In particular, if $u(x) = x^\alpha$ and $v(x) = x^\beta$ with $-\frac{1}{q} < \alpha < 1 - \frac{1}{q}$ and $\beta = \alpha - 1 + \frac{1}{q} + \frac{1}{p'}$, the conclusion of the theorem holds. If $p = q$ and $\alpha = \beta = 0$ this result is due to Hardy ([30], [31]). For $p = q$ and $-\frac{1}{p} < \alpha = \beta < 0$ the result may be found in [63].

We denote by f^* the non-increasing rearrangement of f. Let $f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)\, dt$ be the integral average of f^*.

We need the following special case of a result of Jodeit and Torchinsky [40, Theorem 4.7].

Theorem 1.17. Let $2 \le p < \infty$ and T be a sublinear operator of types $(1,\infty)$ and $(2,2)$. Then there exists a constant C, such that for every $f \in L^1 + L^2$

(1.34) $$\int_0^x (Tf)^*(t)^p \, dt \le C \int_0^x \left(\frac{1}{t} f^{**}(t)\right)^p dt, \quad x > 0.$$

An immediate consequence of this and Corollary 1.3 (with $a = 2$) is:

Theorem 1.18. Let T be a sublinear operator of types $(1,\infty)$ and $(2,2)$. If $p \ge 2$, $1 < r \le p$, and $f \in L^1 + L^2$, then

(1.35) $$\left(\int_0^\infty (Tf)^*(x)^p \, dx\right)^{\frac{1}{p}} \le C\left(\int_0^\infty x^{-\frac{r}{p'} - 1} f^*(x)^r \, dx\right)^{\frac{1}{r}}.$$

Theorem 1.19. Let T be a linear operator of types $(1,\infty)$ and $(2,2)$. Suppose that $1 < r \le 2q < \infty$ and u,v,w are non-negative functions. If $p > 1$ and $r \le 2p \le 2q$ such that

(1.36) $$\sup_{s>0}\left(\int_0^s [xu(x)]^q \, dx\right)^{\frac{1}{q}} \left(\int_s^\infty v(x)^{-p'} \, dx\right)^{\frac{1}{p'}} < \infty,$$

and

(1.37) $$\sup_{s>0}\left(\int_s^\infty \left[\frac{v(x)}{x^2}\right]^p dx\right)^{\frac{1}{2p}} \left(\int_0^s w(x)^{-r'} \, dx\right)^{\frac{1}{r'}} < \infty,$$

then it follows that

$$(1.38)\quad \left\{\int_0^\infty [u(\tfrac{1}{x})\,(Tf)^*(x)^2]^q \frac{dx}{x^2}\right\}^{\frac{1}{q}} \le C\left\{\int_0^\infty [w(x)f^*(x)]^r\,dx\right\}^{\frac{2}{r}}.$$

Proof. It follows immediately from Theorem 1.17 that

$$x(Tf)^*(x)^2 \le C\int_0^x \left(\tfrac{1}{t}\,f^{**}(\tfrac{1}{t})\right)^2 dt$$

$$\text{i.e. } (Tf)^*(\tfrac{1}{x})^2 \le C\,x\int_x^\infty f^{**}(s)^2\,ds.$$

Now, since u and v satisfy (1.36), Theorem 1.2 applies so that

$$\begin{aligned}
&\left\{\int_0^\infty [u(x)(Tf)^*(\tfrac{1}{x})^2]^q\,dx\right\}^{\frac{1}{q}}\\
&\le C\left\{\int_0^\infty [xu(x)\int_x^\infty f^{**}(s)^2\,ds]^q\,dx\right\}^{\frac{1}{q}}\\
&\le C\left\{\int_0^\infty [v(x)f^{**}(x)^2]^p\,dx\right\}^{\frac{1}{p}}\\
&= C\left\{\int_0^\infty \left[\frac{\sqrt{v(x)}}{x}\int_0^x f^*(s)\,ds\right]^{2p} dx\right\}^{\frac{1}{p}}\\
&\le C\left\{\int_0^\infty [w(x)f^*(x)]^r\,dx\right\}^{\frac{2}{r}},
\end{aligned}$$

where the last inequality follows from (1.37) and Theorem 1.1. This completes the proof.

A routine calculation shows that in the case $r = 2p = 2q$ the functions $u(x) = x^{\alpha}$, $v(x) = x^{\alpha+2}$, $w(x) = x^{(\alpha+2)/2}$, where $\alpha p < -1 < (\alpha+1)p$, satisfy the conditions of Theorem 1.19 and hence we have

<u>Corollary 1.20</u>. Let $1 < p < \infty$ and suppose α satisfies $\alpha p < -1 < (\alpha+1)p$. Then, if T is of type $(1,\infty)$ and $(2,2)$,

$$(1.39) \quad \left\{\int_0^\infty [x^{-\alpha}(Tf)^*(x)^2]^p \frac{dx}{x^2}\right\}^{\frac{1}{p}} \leq C\left\{\int_0^\infty [x^{\alpha+2}f^*(x)^2]^p \, dx\right\}^{\frac{1}{p}}.$$

CHAPTER 2

HARDY-LITTLEWOOD FRACTIONAL MAXIMAL FUNCTIONS

2.1 Introduction.

In this chapter we study boundedness of the fractional maximal function and fractional integral operator between weighted Lebesgue spaces. One of the results (Theorem 2.4) is a characterization of those pairs of weights for which the fractional maximal function satisfies a weak type (p,q) inequality. Although a characterization for the strong type (p,q) inequalities is not available we give a necessary condition on the weights which is required for the (p,q) boundedness of these operators (Theorems 2.7, 2.8). A further assumption insures that these conditions are also sufficient and in a special case a complete characterization is obtained (Theorem 2.12).

The work of this chapter is motivated by the results of Muckenhoupt [55] and Muckenhoupt and Wheeden [59]. Their results for the maximal function and Hilbert transform are extended to the fractional maximal function and fractional integral operator.

The plan of this chapter is as follows: Section 2.2 collects notation and provides a brief survey of recent work. In Section 2.3 we obtain the characterization of those weights (u,v) for which the fractional maximal function satisfies a weak type (p,q) estimate. The next section contains necessary conditions for strong type (p,q) inequalities for the fractional maximal and fractional integral operators. Finally, in Section 2.5 we remark on the n-dimensional analogues of these results.

2.2 Notation and Survey.

If f is a locally integrable function on $\mathbb{R}$, the fractional maximal function $M_\alpha f$, $0 \leq \alpha < 1$ is defined by

$$(2.1) \qquad M_\alpha f(x) = \sup_{I \ni x} \frac{1}{|I|^{1-\alpha}} \int_I |f(t)|dt,$$

where the supremum is taken over all intervals I containing x and $|I|$ denotes the Lebesgue measure of I.

If $\alpha = 0$, $M_0 f = Mf$ the Hardy-Littlewood maximal function. The L^p-boundedness of $f \to Mf$ is the well known theorem of Hardy and Littlewood [33] and many generalizations of this result are known. For example Stein [68] proved that for $1 < p < \infty$

$$(2.2) \qquad \int_{\mathbb{R}} |Mf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}} |f(x)|^p w(x)\, dx$$

with $w(x) = |x|^\alpha$, $-\frac{1}{p} < \alpha < 1 - \frac{1}{p}$. Fefferman and Stein [22] showed that (2.2) holds whenever $Mw(x) \leq Cw(x)$ almost everywhere. Recently Muckenhoupt provided a complete characterization of those weights satsifying (2.2).

Theorem 2.1 [55]. Let $1 < p < \infty$ and $w(x) \geq 0$, then (2.2) holds if and only if

$$(2.3)\qquad \sup_I \left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-1/(p-1)}\,dx\right)^{p-1} = K < \infty.$$

Inequality (2.3) defines the familiar A_p-class of Muckenhoupt and we say $w \in A_p$ if $w(x) \geq 0$ and (2.3) is satisfied. If $p = 1$, (2.3) is to be interpreted as

$$(2.4)\qquad Mw(x) \leq Cw(x) \quad \text{a.e.}$$

To complete the definition we say $w \in A_\infty$ if there exist constants $K, \delta > 0$, such that for any measurable $E \subseteq I$,

$$(2.5)\qquad \frac{|E|_w}{|I|_w} \leq K\left(\frac{|E|}{|I|}\right)^\delta ,$$

where $|E|_w = \int_E w(x)\,dx$.

If M is replaced by the Hilbert transform

$$(2.6)\qquad Hf(x) = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\,dy,$$

Theorem 2.1 still holds ([39]). For extensions of Theorem 2.1 to more general singular integrals see [1],[14],[15],[42]. It is interesting to note that Helson and Szegö [37] had obtained a characterization of weights for which $H: L^2(\mathbb{R}, w(x)\,dx) \to L^2(\mathbb{R}, w(x)\,dx)$ is bounded. They had shown that $w = \exp(u + Hv)$ where $u, v \in L^\infty$ and $\|v\|_\infty < \frac{\pi}{2}$. A characterization of those weights (u,v), such that,

$$(2.7) \qquad \int_{\mathbb{R}} |Hf(x)|^2 u(x)\, dx \le C \int_{\mathbb{R}} |f(x)|^2 v(x)\, dx$$

was recently obtained by Cottar and Sadosky [19],[20], but the conditions on the weights appear totally different from the A_2 condition and will not be discussed here.

In general the problem of characterizing the pairs of weights (u,v) satisfying

$$(2.8) \qquad \int_{\mathbb{R}} |Mf(x)|^p u(x)\, dx \le C \int_{\mathbb{R}} |f(x)|^p v(x)\, dx$$

is open. Of course, if f is decreasing, and $\int_{\mathbb{R}}$ is replaced by $\int_0^\infty$, Theorem 1.1 (with $p = q$) solves the problem with (u,v) satisfying

$$(2.9) \qquad \sup_{r>0} \left(\int_r^\infty \frac{u(x)}{x^p}\, dx\right) \left(\int_0^r v(x)^{-1/(p-1)}\, dx\right)^{p-1} < \infty.$$

The corresponding problem with inequality (2.8) replaced by the weaker inequality

$$(2.10) \qquad \int_{\{Mf>y\}} u(x)\, dx \le \frac{C}{y^p} \int_{\mathbb{R}} |f(x)|^p v(x)\, dx$$

is completely solved [55] (see also Theorem 2.4). Inequality (2.10) holds if and only if

$$(2.11) \qquad \sup_I \left[\frac{1}{|I|}\int_I u(x)\,dx\right]\left[\frac{1}{|I|}\int_I v(x)^{-1/(p-1)}\,dx\right]^{p-1} = K < \infty.$$

If (2.11) is satisfied we say that $(u,v) \in A_p$.

It is not difficult to see that $(u,v) \in A_p$ is necessary if (2.8) holds, but not sufficient ([59]). The situation for (2.7) is the same but in view of the close relationship between the maximal function and the Hilbert transform it is rather surprizing that the functions $u(x) = x^{-1}|\log x|^{-5/2}\chi_{(0,1/2]}(x)$, $v(x)^{-1} = x|\log x|^{3/2}\chi_{(0,1/2]}(x)$ and $f(x) = \chi_{(0,1/2]}(x)$, satisfy (2.11) and (2.8) but not (2.7) with $p = 2$.

The relationship that exists between the maximal function and the Hilbert transform is reflected in the fractional maximal function (2.1) and the fractional integral operator T_α defined for $0 < \alpha < 1$ by

$$(2.12) \qquad T_\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}}\,dy,$$

whenever the integral converges. To complete the definition we will define T_0 to be the Hilbert transform H (2.6). From this point on we ignore the constant factor $\Gamma(\alpha)^{-1}$.

It is known that $T_\alpha f$ is, in some sense, dominated by $M_\alpha f$. This is used - as we show in Chapter 3 - to force norm estimates for the fractional integral operator whenever norm estimates for the fractional maximal function are known. Thus the following theorem of Muckenhoupt and Wheeden is crucial.

Theorem 2.2 [58]. Let $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, then

$$\left(\int_{\mathbb{R}} |M_\alpha f(x)\ u(x)|^q\ dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}} |f(x)\ u(x)|^p\ dx\right)^{\frac{1}{p}} \tag{2.13}$$

and for $0 \leq \alpha < 1$, $1 \leq p < \frac{1}{\alpha}$,

$$\left(\int_{\{M_\alpha f>y\}} u(x)^q\ dx\right)^{\frac{1}{q}} \leq Cy^{-1}\left(\int_{\mathbb{R}} |f(x)\ u(x)|^p\ dx\right)^{\frac{1}{p}} \tag{2.14}$$

if and only if

$$\sup_I\left(\frac{1}{|I|}\int_I u(x)^q\ dx\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int_I u(x)^{-p'}\ dx\right)^{\frac{1}{p'}} = K < \infty, \tag{2.15}$$

with the usual modification if $p = 1$.

As suggested by the previous remark, a corresponding theorem for the fractional integral can be obtained from this result.

Condition (2.15) suggests the following definition, which is required in the next section.

Definition 2.3. A pair of weight functions (u,v) is said to belong to $A_{p,q}$, $1 \leq p,q \leq \infty$, if

$$(2.16)\qquad \sup_I \left(\frac{1}{|I|}\int_I u(x)^q\,dx\right)^{\frac{1}{q}} \left(\frac{1}{|I|}\int_I v(x)^{-p'}\,dx\right)^{\frac{1}{p'}} = K < \infty$$

for $1 < p \leq \infty$, $1 \leq q \leq \infty$. If $p = 1$ the condition is

$$(2.17)\qquad \sup_I \left(\frac{1}{|I|}\int_I u(x)^q\,dx\right)^{\frac{1}{q}} \operatorname*{ess\,sup}_{x\in I} \frac{1}{v(x)} = K < \infty.$$

If $u = v$ we write $u \in A_{p,q}$ and note that if $p = q$, $(u,v) \in A_{p,p}$ if and only if $(u^p, v^p) \in A_p$.

2.3 Characterization for Weak Types.

The characterization given by the next theorem includes the result of Muckenhoupt and Wheeden [58] and is known if $p = q$ ([55, Theorem 1]). The proof is essentially as in [58].

Theorem 2.4. Let $0 \le \alpha < 1$, $1 \le p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$. Then u and v are weight functions satisfying

$$\left(\int_{\{M_\alpha f > y\}} u(x)^q \, dx\right)^{\frac{1}{q}} \le C y^{-1} \left(\int_{\mathbb{R}} |f(x) \, v(x)|^p \, dx\right)^{\frac{1}{p}}, \tag{2.18}$$

if and only if $(u,v) \in A_{p,q}$ (2.16).

Proof. First we prove necessity. In the case $p > 1$ we fix I and set $J = \int_I v(x)^{-p'} \, dx$.

If $J = 0$, $(u,v) \in A_{p,q}$ because of the convention $0 \cdot \infty = 0$.

If $0 < J < \infty$, let $f(x) = v(x)^{-p'} \chi_I(x)$. Then $M_\alpha f(x) \ge J|I|^{\alpha-1}$ on I and by (2.18) with $y = J|I|^{\alpha-1}$ we obtain

$$\left(\int_I u(x)^q \, dx\right)^{\frac{1}{q}} \le \left(\int_{\{M_\alpha f > J|I|^{\alpha-1}\}} u(x)^q \, dx\right)^{\frac{1}{q}}$$

$$\le C|I|^{1-\alpha} J^{-1} \left(\int_I v(x)^{-p \cdot p' + p} \, dx\right)^{\frac{1}{p}}.$$

Since J is finite, it follows that $v(x) > 0$ almost everywhere on I and hence the integral on the right-hand side equals $J^{1/p}$. Multiply both sides of this inequality by $J^{1/p'}$ and (2.16) follows because $I - \alpha = 1/q + 1/p'$.

If $J = \infty$, then $v(x)^{-1} \notin L^{p'}(I)$. Thus there is a $g \in L^p(I)$ such that $g(x) = 0$ outside of I and $\int_I g(x)\ v(x)^{-1}\ dx = \infty$. (To see that such a g exists assume the converse, that is, assume that the linear functional $Tg \equiv \int_I g(x)\ v(x)^{-1}\ dx$ is finite for every $g \in L^p$. Define the functions $v_n(x) = \min\ (n, v(x)^{-1})$, $n = 1,2,3,\ldots$ and the operators $T_n g \equiv \int_I g(x)\ v_n(x)\ dx$. Since $v_n \in L^{p'}(I)$ it follows by the Riesz representation theorem that T_n is a continuous linear functional on $L^p(I)$ and $T_n g \to Tg$. The Banach-Steinhaus Theorem implies that T is continuous and the Riesz representation theorem shows $v(x)^{-1} \in L^{p'}(I)$. This contradicts the assumption that $J = \infty$.) Set $f(x) = g(x)\ v(x)^{-1}$, then $|f(x)\ v(x)|^p = g(x)^p \in L^1(I)$, but $M_\alpha f \equiv \infty$ on I. It follows from (2.18) that

$$\left(\int_I u(x)^q\ dx\right)^{\frac{1}{q}} \le Cy^{-1}\left(\int_I |f(x)\ v(x)|^p\ dx\right)^{\frac{1}{p}}$$

holds for any $y > 0$. Consequently $\int_I u(x)^q\ dx = 0$ and again (2.16) holds by the convention $0 \cdot \infty = 0$.

If $p = 1$, we again fix I. If $\operatorname{ess\,inf}_{y \in I} v(y) = \infty$, (2.16) holds for any K by convention. Otherwise let $\varepsilon > 0$; then there exists an $E \subseteq I$, $|E| > 0$, such that $v(x) \leq \operatorname*{ess\,inf}_{y \in I} v(y) + \varepsilon$ for all $x \in E$. Let $f(x) = \chi_E(x)$, then $M_\alpha f(x) \geq |E| \cdot |I|^{\alpha-1}$ on I. By (2.18), with $y = |E| \cdot |I|^{\alpha-1}$, we have

$$\left(\int_I u(x)^q \, dx\right)^{\frac{1}{q}} \leq C \frac{|I|^{1-\alpha}}{|E|} \int_E v(x) \, dx$$

$$\leq C|I|^{1-\alpha}\left(\varepsilon + \operatorname*{ess\,inf}_{y \in I} v(y)\right),$$

but since ε was arbitrary, (2.16) follows.

To prove the sufficiency we begin by defining $E_{y,m} = \{x \in (-m,m): |M_\alpha f(x)| > y\}$ where $m > 0$. For every $x \in E_{y,m}$, there exists an interval I_x containing x such that $|I_x|^{\alpha-1}\int_{I_x} |f| > y$. Hence $E_{y,m} \subseteq \bigcup_x I_x$ and since $E_{y,m}$ is bounded we can apply a Besicovitch type covering lemma (see for example [27]) to choose a sequence $\{I_k\}_{k=1}^{\infty}$ such that $E_{y,m} \subseteq \bigcup_k I_k$ and $\sum_{k=1}^{\infty} \chi_{I_k}(x) \leq 2$. Thus

$$\int_{E_{y,m}} u(x)^q \, dx$$

$$\leq \sum_k \int_{I_k} u(x)^q \, dx$$

$$\leq \sum_k \int_{I_k} u(x)^q \, dx \, \frac{y^{-q}}{|I_k|^{q(1-\alpha)}} \left(\int_{I_k} |f(t)| \, dt\right)^q.$$

If $1 < p < \infty$ we can apply Hölder's inequality so the last expression is dominated by

$$y^{-q}\sum_k \int_{I_k} \frac{u(x)^q\, dx}{|I_k|^{q(1-\alpha)}} \left(\int_{I_k} v(x)^{-p'}\, dx\right)^{\frac{q}{p'}} \left(\int_{I_k} |f(x)\; v(x)|^p\, dx\right)^{\frac{q}{p}}$$

which by hypothesis is dominated by

$$Cy^{-q} \sum_k \left(\int_{I_k} |f(x)\; v(x)|^p\, dx\right)^{\frac{q}{p}}$$

$$\leq 2^{\frac{q}{p}}\, Cy^{-q} \left(\int_{\mathbb{R}} |f(x)\; v(x)|^p\, dx\right)^{\frac{q}{p}}.$$

Taking q^{th} roots and letting $m \to \infty$ proves the result.

The case $p = 1$ is handled in much the same manner.

Corollary 2.5. Let $1 \leq p < s < \infty$ and $(u,v) \in A_p$, then

(2.19) $$\int_{\mathbb{R}} |Mf(x)|^s\, u(x)\, dx \leq C \int_{\mathbb{R}} |f(x)|^s\, v(x)\, dx.$$

Proof. Although this result follows immediately by interpolating between the case $\alpha = 0$ of Theorem 2.4 and the fact that $M: L^\infty \to L^\infty$, we will give a more direct proof.

Fix $y > 0$ and set $f_1(x) = f(x)$ if $|f(x)| > y/2$ and 0 otherwise. Set $f_2(x) = f(x) - f_1(x)$. Notice that $|f_2(x)| \leq y/2$ and hence $Mf_2(x) \leq y/2$. Therefore $\{Mf>y\} \subseteq \{Mf_1>y/2\}$ and so by Theorem 2.4

$$\int_{\{Mf>y\}} u(x)\ dx \leq \frac{C}{y^p} \int_{\mathbb{R}} |f_1(x)|^p\ v(x)\ dx$$

$$= \frac{C}{y^p} \int_{\{|f(x)|>y/2\}} |f(x)|^p\ v(x)\ dx.$$

Consequently

$$\int_{\mathbb{R}} |Mf(x)|^s\ u(x)\ dx$$

$$= s \int_0^\infty y^{s-1} |\{Mf>y\}|_u\ dy$$

$$\leq Cs \int_0^\infty y^{s-1}\ y^{-p} \int_{\{|f|>y/2\}} |f(x)|^p\ v(x)\ dx\ dy$$

which by Fubini's theorem is

$$= C\int_{\mathbb{R}} |f(x)|^p\ v(x) \int_0^{2|f|} y^{s-p-1}\ dy\ dx$$

$$= C\int_{\mathbb{R}} |f(x)|^s\ v(x)\ dx.$$

2.4 Necessary Conditions.

We introduce here a condition which is necessary for strong type boundedness of both M_α and T_α and under additional assumptions is also sufficient.

Definition 2.6. Let u,v be non-negative functions on $\mathbb{R}$ and $1 < p,q < \infty$. We say that the pair (u,v) satisfies the $C_{p,q}$ condition (that is $(u,v) \in C_{p,q}$) if

$$(2.20) \qquad \left(\int_{\mathbb{R}} \left[\frac{u(x)}{(|I|+|x-x_I|)^{1/q+1/p'}}\right]^q dx\right)^{\frac{1}{q}} \left(\int_I v(x)^{-p'}\, dx\right)^{\frac{1}{p'}} \leq K < \infty$$

for every interval I. Here x_I is the centre of I.

If $p = q$ this reduces to the C_p condition of Muckenhoupt and Wheeden ([59]) with $u(x)$ and $v(x)$ replaced by $u(x)^q$ and $v(x)^p$ respectively.

If we restrict the range of integration in (2.20) to I, then $x \in I$ and $|x-x_I| < |I|$, which shows that (2.20) implies $A_{p,q}$.

Theorem 2.7. Assume that $0 \leq \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, and that there exists a constant C such that

$$(2.21)\qquad \left(\int_{\mathbb{R}} |M_\alpha f(x)\ u(x)|^q\ dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}} |f(x)\ v(x)|^p\ dx\right)^{\frac{1}{p}}.$$

Then $(u,v) \in C_{p,q}$.

Proof. Fix I and set $J = \int_I v(x)^{-p'}\,dx$. If $J = 0$, $(u,v) \in C_{p,q}$ by the convention $0\cdot\infty = 0$.

If $0 < J < \infty$ define $f(x) = v(x)^{-p'}\chi_I(x)$. Then $M_\alpha f(x) \geq (|I| + |x-x_I|)^{\alpha-1}J$, where x_I is the centre of I. Substituting this into the left-hand side of (2.21) yields

$$\left(\int_{\mathbb{R}} \left[\frac{Ju(x)}{(|I|+|x-x_I|)^{1-\alpha}}\right]^q dx\right)^{\frac{1}{q}} \leq C\left(\int_I v(x)^{-p'}\ dx\right)^{\frac{1}{p}} = CJ^{\frac{1}{p}}.$$

This implies $(u,v) \in C_{p,q}$.

If $J = \infty$, $v(x)^{-1} \notin L^{p'}(I)$ and hence there is a $g \in L^p(I)$ such that $v(x)^{-1} g(x) \notin L^1(I)$. Let $f(x) = g(x)\ v(x)^{-1}\chi_I(x)$, then the right-hand side of (2.21) is finite and $M_\alpha f \equiv +\infty$. It follows that $u(x) = 0$ almost everywhere and since $0\cdot\infty = 0$ this implies $(u,v) \in C_{p,q}$.

Theorem 2.8. Suppose that $u,v \geq 0$, $0 \leq \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$ and that

$$(2.22)\qquad \left(\int_{\mathbb{R}} |T_\alpha f(x)\ u(x)|^q\ dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}} |f(x)\ v(x)|^p\ dx\right)^{\frac{1}{p}}.$$

Then $(u,v) \in C_{p,q}$.

<u>Proof</u>. Fix I and set $J = \int_I v(x)^{-p'}\,dx$. If $J = 0$, $(u,v) \in C_{p,q}$ as before.

If $J = \infty$, let $E \subseteq I$ be any subinterval for which $\int_E v(x)^{-p'}\,dx = \infty$. Let $g \in L^p$ be such that $g(x) = 0$ if $x \notin E$ but $g(x)\,v(x)^{-1} \notin L^1(E)$. Let $f(x) = g(x)\,v(x)^{-1}\chi_E(x)$. Then the right-hand side of (2.22) is finite for this f and $|T_\alpha f(x)| = \infty$ for $x \notin E$ since $|x - y| \le |E| + |x - x_E|$. Therefore $u(x) = 0$ almost everywhere outside of E. Repeatedly bisecting I produces E's of arbitrarily small length such that $|T_\alpha f(x)| = \infty$ if $x \notin E$. It follows that $u(x) = 0$ almost everywhere on I. But $0\cdot\infty = 0$, which implies $(u,v) \in C_{p,q}$.

If $0 < J < \infty$, write $I = [a, a+h]$ and choose $r > 0$ so that $\int_a^{a+r} v(x)^{-p'}\,dx = \frac{J}{2}$. Now set $f(x) = v(x)^{-p'} \times \chi_{[a,\, a+r]}(x)$ so that if $x > a + r$, $|T_\alpha f(x)| \ge (|I| + |x - x_I|)^{\alpha-1} J/2$. Substituting this into (2.22) yields

$$\left(\int_{a+r}^{\infty}\left[\frac{u(x)}{(|I|+|x-x_I|)^{1-\alpha}}\right]^q dx\right)^{\frac{1}{q}} \frac{J^{\frac{1}{p'}}}{2} \le C.$$

Similarly, if $f(x) = v(x)^{-p'}\chi_{[a+r,a+h]}(x)$ and $x < a + r$, $|T_\alpha f(x)| \ge (|I|+|x-x_I|)^{\alpha-1} J/2$, from which it follows that

$$\left(\int_{-\infty}^{a+r}\left[\frac{u(x)}{(|I|+|x-x_I|)^{1-\alpha}}\right]^q dx\right)^{\frac{1}{q}} \frac{J^{\frac{1}{p'}}}{2} \le C.$$

Adding the last two inequalities shows that $(u,v) \in C_{p,q}$.

If $\alpha = 0$ this is [59, Theorem 2] and the next theorem extends [59, Theorem 3].

<u>Theorem 2.9</u>. Suppose that $u,v \geq 0$, $0 \leq \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$ and

$$(2.23) \qquad \left(\int_{\{|T_\alpha f|>y\}} u(x)^q \, dx\right)^{\frac{1}{q}} \leq Cy^{-1}\left(\int_{\mathbb{R}} |f(x) \; v(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Then

$$(2.24) \qquad \left(\frac{1}{|I|}\int_I u(x)^q \, dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} \left[\frac{v(x)}{(|I|+|x-x_I|)^{1/q+1/p'}}\right]^{p'} dx\right)^{\frac{1}{p'}} \leq K < \infty.$$

That is $(v^{-1}, u^{-1}) \in C_{q',p'}$.

<u>Proof</u>. Begin by setting $I = [a, a+h]$ and define

$$B(x) = \left[\frac{v(x)^{-1}}{(|I|+|x-x_I|)^{1/q+1/p'}}\right]^{p'} \quad \text{and} \quad J = \int_{\mathbb{R}} B(x) \, dx.$$

If $J = 0$, the conclusion follows because of the convention $0\cdot\infty = 0$.

If $0 < J < \infty$ choose an r satisfying $\int_{-\infty}^{r} B(x) \, dx = J/2$ and define $f(x) = [v(x)^p \, (|I| + |x-x_I|)^{1-\alpha}]^{-1/(p-1)} \chi_{[r,\infty)}(x)$. Let $x \in [a,r]$ and $t \in [r,\infty)$, then $0 \leq t - x \leq |t-x_I| + |x_I-x| \leq |t-x_I|+|I|$. Thus for $x \in [a,r]$ we have

$$T_\alpha f(x) = \int_r^\infty \frac{f(t)}{(t-x)^{1-\alpha}}\, dt \geq \int_r^\infty B(t)\, dt = J/2,$$

since $1 - \alpha = \frac{1}{q} + \frac{1}{p'}$. Substituting into (2.23) (with $y = J/2$) we have

$$\left(\int_a^r u(x)^q\, dx\right)^{\frac{1}{q}} \leq CJ^{\frac{1}{p}-1} . \tag{2.25}$$

A similar argument with

$$f(x) = [v(x)^p(|I| + |x-x_I|)^{1-\alpha}]^{-1/(p-1)}\chi_{(-\infty,r]}(x)$$

shows that for $x \in [r,a+h]$

$$T_\alpha f(x) = \int_{-\infty}^r \frac{f(t)}{(x-t)^{1-\alpha}}\, dt \geq \int_{-\infty}^r B(t)\, dt = J/2.$$

From this and (2.23) we see

$$\left(\int_r^{a+h} u(x)^q\, dx\right)^{\frac{1}{q}} \leq CJ^{\frac{1}{p}-1}. \tag{2.26}$$

Upon adding (2.25) and (2.26) we see

$$\left(\int_a^{a+h} u(x)^q\, dx\right)^{\frac{1}{q}} \leq CJ^{-\frac{1}{p'}},$$

from which we obtain (2.24).

If $J = \infty$, set $v_n(x) = v(x) + \frac{1}{n}$. Note

$$\int_{\mathbb{R}} \left[\frac{v_n(x)^{-1}}{(|I|+|x-x_I|)^{1-\alpha}} \right]^{p'} dx \le C \int_{\mathbb{R}} \frac{dx}{(|I|+|x-x_I|)^{p'(1-\alpha)}}$$

is finite, and (2.23) holds with v replaced by v_n with the same constant. Now we can use the case $0 < J < \infty$ to show that (2.24) holds with v replaced by v_n and the constant independent of n. Next let n tend to ∞ and apply the monotone convergence theorem. This proves (2.24).

Next we show that under some additional assumptions on the functions u and v our necessary conditions are also sufficient.

<u>Theorem 2.10</u>. Let $0 \le \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$ and suppose $u, v \ge 0$. If there exists an $A > 0$ such that $u(x) \le A\, v(y)$ for $0 < \frac{x}{4} \le y \le 4x$ and

$$(2.27) \quad \left(\int_0^\infty \left[\frac{u(x)}{(|I|+|x-x_I|)^{1/q+1/p'}} \right]^q dx\right)^{\frac{1}{q}} \left(\int_I v(x)^{-p'}\, dx\right)^{\frac{1}{p'}} \le K < \infty$$

for all intervals $I \subseteq [0,\infty)$, then there exists a $C > 0$ such that for every f with support in $[0,\infty)$

$$(2.28) \qquad \left(\int_0^\infty |M_\alpha f(x)\ \tilde{u}(x)|^q\, dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |f(x)\ v(x)|^p\, dx\right)^{\frac{1}{p}}.$$

<u>Proof</u>. Since M_α is subadditive, Minkowski's inequality shows that the left side of (2.28) is dominated by 3 times the

sum of

$$\left(\sum_{-\infty}^{\infty}\int_{2^n}^{2^{n+1}} |M_\alpha(f\chi_{[0,2^{n-1}]})(x)\ u(x)|^q\ dx\right)^{\frac{1}{q}}, \tag{2.29}$$

$$\left(\sum_{-\infty}^{\infty}\int_{2^n}^{2^{n+1}} |M_\alpha(f\chi_{[2^{n-1},2^{n+2}]})(x)\ u(x)|^q\ dx\right)^{\frac{1}{q}}, \tag{2.30}$$

and

$$\left(\sum_{-\infty}^{\infty}\int_{2^n}^{2^{n+1}} |M_\alpha(f\chi_{[2^{n+2},\infty)})(x)\ u(x)|^q\ dx\right)^{\frac{1}{q}}, \tag{2.31}$$

and hence it suffices to show that (2.29), (2.30), and (2.31) are dominated by the right side of (2.28).

We first suppose that I is any interval such that $I \cap [2^n, 2^{n+1}] \neq \emptyset$ and $\int_I |f\chi_{[0,2^{n-1}]}| \neq 0$. Clearly $|I| > 2^{n-1}$ and, since $\int_0^{2^{n-1}} |f|$ is an upper bound for any integral of $|f\chi_{[0,2^{n-1}]}|$, it follows that (2.29) is bounded by

$$\left(\sum_{-\infty}^{\infty}\int_{2^n}^{2^{n+1}} [(2^{n-1})^{\alpha-1}\int_0^{2^{n-1}} |f(t)|\ dt\ u(x)]^q\ dx\right)^{\frac{1}{q}}.$$

This is in turn bounded by $4^{1-\alpha}$ times

$$\left(\sum_{-\infty}^{\infty}\int_{2^n}^{2^{n+1}} \left[\frac{u(x)}{x^{1-\alpha}}\int_0^x |f(t)|\ dt\right]^q dx\right)^{\frac{1}{q}}$$

$$= \left(\int_0^{\infty}\left[\frac{u(x)}{x^{1-\alpha}}\int_0^x |f(t)|\ dt\right]^q dx\right)^{\frac{1}{q}}.$$

Now, if $r > 0$ and we choose $I = [0,r]$, (2.27) implies

$$\left(\int_r^\infty \left[\frac{u(x)}{(r+|x-r/2|)^{1-\alpha}}\right]^q dx\right)^{\frac{1}{q}} \left(\int_0^r v(x)^{-p'} dx\right)^{\frac{1}{p'}} \leq K < \infty,$$

and hence

$$\left(\int_r^\infty \left[\frac{u(x)}{x^{1-\alpha}}\right]^q dx\right)^{\frac{1}{q}} \left(\int_0^r v(x)^{-p'} dx\right)^{\frac{1}{p'}} \leq K < \infty$$

so by Theorem 1.1 we see that (2.29) is dominated by the right side of (2.28).

Next, let $u_n = \operatorname{ess\,sup}_{x\in[2^n,2^{n+1}]} u(x)$. Thus (2.30) is bounded by

$$\left(\sum_{-\infty}^{\infty}\int_{2^n}^{2^{n+1}} [u_n M_\alpha(f\chi_{[2^{n-1},2^{n+2}]})(x)]^q dx\right)^{\frac{1}{q}}$$

$$= \left(\sum_{-\infty}^{\infty}\int_{2^n}^{2^{n+1}} [M_\alpha(u_n f\chi_{[2^{n-1},2^{n+2}]})(x)]^q dx\right)^{\frac{1}{q}}$$

which by Theorem 2.2 (with $u \equiv 1$) is dominated by

$$\left(\sum_{-\infty}^{\infty}\left(\int_{2^{n-1}}^{2^{n+2}} |u_n f(x)|^p dx\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$

Since $q/p \geq 1$ it follows that

$$\left(\sum_{-\infty}^{\infty}\left(\int_{2^{n-1}}^{2^{n+2}} |u_n f(x)|^p dx\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

$$= \left\{ \left[\sum_{-\infty}^{\infty} \left(\int_{2^{n-1}}^{2^{n+2}} |u_n f(x)|^p \, dx \right)^{\frac{q}{p}} \right]^{\frac{p}{q}} \right\}^{\frac{1}{p}}$$

$$\leq \left(\sum_{-\infty}^{\infty} \int_{2^{n-1}}^{2^{n+2}} |u_n f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

To see that $u_n \leq Av(y)$ if $2^{n-1} \leq y \leq 2^{n+2}$, first observe that for $x \in [2^n, 2^{n+1}]$ either $2^{n-1} \leq y \leq x \leq 2^{n+1}$ or $2^n \leq x \leq y \leq 2^{n+2}$. From this it readily follows that $\frac{x}{4} \leq y \leq 4x$, and therefore we can apply the hypothesis $u(x) \leq Av(y)$, $\frac{x}{4} \leq y \leq 4x$. This completes the estimate of (2.30).

To extimate (2.31) we begin by finding a pointwise estimate of $M_\alpha(f\chi_{[2^{n+2},\infty)})(x)$. Fix n and set

$$S = S_n = \sup_{k \geq n+2} 2^{k(\alpha-1)} \int_{2^k}^{2^{k+1}} |f(t)| \, dt.$$

Then $\int_{2^k}^{2^{k+1}} |f| \leq S2^{k(1-\alpha)}$ for $k \geq n + 2$. Now, for $y \geq 2^{n+2}$ choose $j \in \mathbb{Z}$, such that, $2^j \leq y \leq 2^{j+1}$. Then, for $x \in [2^n, 2^{n+1}]$

$$\frac{1}{(y-x)^{1-\alpha}} \int_{2^{n+2}}^{y} |f| \leq \frac{1}{(2^j - 2^{n+1})^{1-\alpha}} \sum_{k=n+2}^{j} \int_{2^k}^{2^{k+1}} |f|$$

$$\leq \frac{1}{(2^j - 2^{n+1})^{1-\alpha}} \sum_{k=n+2}^{j} 2^{k(1-\alpha)} S.$$

Since $2^j - 2^{n+1} \geq 2^{j-1}$, the last sum is dominated by $S \sum_{k=n+2}^{j} 2^{(k-j+1)(1-\alpha)} \leq S \, 2^{1-\alpha}/(2^{1-\alpha}-1)$.

Hence $M_\alpha(f\chi_{[2^{n+2},\infty)})(x) \le S\, 2^{1-\alpha}/(2^{1-\alpha}-1)$ which implies (2.31) is bounded by a constant times

$$\left(\sum_{-\infty}^{\infty}\int_{2^n}^{2^{n+1}} [\sup_{k\ge n+2} 2^{k(\alpha-1)}\int_{2^k}^{2^{k+1}} |f(t)|\, dt\; u(x)]^q\, dx\right)^{\frac{1}{q}}$$

$$\le \left(\sum_{-\infty}^{\infty} \sup_{k\ge n+2} \left[2^{k(\alpha-1)} \int_{2^k}^{2^{k+1}} |f(t)|\, dt\right]^q \int_{2^n}^{2^{n+1}} u(x)^q\, dx\right)^{\frac{1}{q}}.$$

Replacing $\sup\limits_{k\ge n+2}$ with $\sum\limits_{k=n+2}^{\infty}$ and interchanging the order of summation we see this is dominated by

$$\left(\sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{k-2} \left[2^{k(\alpha-1)} \int_{2^k}^{2^{k+1}} |f(t)|\, dt\right]^q \int_{2^n}^{2^{n+1}} u(x)^q\, dx\right)^{\frac{1}{q}}$$

$$= \left(\sum_{k=-\infty}^{\infty} 2^{kq(\alpha-1)}\left[\int_{2^k}^{2^{k+1}} |f(t)|\, dt\right]^q \int_0^{2^{k-1}} u(x)^q\, dx\right)^{\frac{1}{q}}$$

By Hölder's inequality this is less than

$$\left(\sum_{k=-\infty}^{\infty}\left[\int_{2^k}^{2^{k+1}} |f(t)v(t)|^p dt\right]^{\frac{q}{p}} \left(\int_{2^k}^{2^{k+1}} v(t)^{-p'} dt\right)^{\frac{q}{p'}} \left(\int_0^{2^{k-1}} 2^{kq(\alpha-1)} u(x)^q dx\right)\right)^{\frac{1}{q}}$$

Choosing $I = [2^k, 2^{k+1}]$ in (2.27), shows that the product of the last two integrals is bounded. Hence the last expression is dominated by

$$\left(\sum_{-\infty}^{\infty}\left[\int_{2^k}^{2^{k+1}} |f(t)\; v(t)|^p\, dt\right]^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

$$\le \left(\int_0^{\infty} |f(t)\; v(t)|^p\, dt\right)^{\frac{1}{p}}.$$

This completes the proof.

<u>Theorem 2.11</u>. Let $0 \le \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, and $u,v \ge 0$ satisfying

$$\left(\int_{-\infty}^{0}\left[\frac{u(x)}{(|I|+|x-x_I|)^{1/q+1/p'}}\right]^q dx\right)^{\frac{1}{q}} \left(\int_I v(x)^{-p'}\, dx\right)^{\frac{1}{p'}} \le K < \infty, \tag{2.32}$$

for all intervals $I \subseteq [0,\infty)$. Then, if $f(x) = 0$ for all $x < 0$ there exists a $C > 0$, such that

$$\left(\int_{-\infty}^{0} |M_\alpha f(x)\ u(x)|^q\, dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |f(x)\ v(x)|^p\, dx\right)^{\frac{1}{p}}. \tag{2.33}$$

<u>Proof</u>. The left-side of (2.33) is bounded by twice the sum of

$$\left(\sum_{-\infty}^{\infty} \int_{-2^{n+1}}^{-2^n} |M_\alpha(f\chi_{[0,2^n]})\ (x)\ u(x)|^q\, dx\right)^{\frac{1}{q}} \tag{2.34}$$

and

$$\left(\sum_{-\infty}^{\infty} \int_{-2^{n+1}}^{-2^n} |M_\alpha(f\chi_{[2^n,\infty)})\ (x)\ u(x)|^q\, dx\right)^{\frac{1}{q}}. \tag{2.35}$$

To estimate (2.34) we proceed in much the same manner as we did with (2.29). Let I be any interval such that $\int_I |f\chi_{[0,2^n]}| \neq 0$ and $I \cap [-2^{n+1},-2^n] \neq \emptyset$. Then $|I| > 2^n$ and $\int_I |f\chi_{[0,2^n]}| \le \int_0^{2^n} |f|$ so that (2.34) is bounded by

$$\left(\sum_{-\infty}^{\infty} \int_{-2^{n+1}}^{-2^n} [2^{n(\alpha-1)} \int_0^{2^n} |f(t)|dt\, u(x)]^q\, dx\right)^{\frac{1}{q}}$$

which is in turn dominated by

$$\left(\int_0^{\infty}\left[\frac{u(-x)}{x^{1-\alpha}}\int_0^x |f(t)|\, dt\right]^q dx\right)^{\frac{1}{q}}.$$

If we choose $I = [0,r]$, $r > 0$, (2.32) yields

$$\left(\int_r^{\infty}\left[\frac{u(-x)}{x^{1-\alpha}}\right]^q dx\right)^{\frac{1}{q}} \left(\int_0^r v(x)^{-p'}\, dx\right)^{\frac{1}{p'}} \le K\, \infty$$

and we can therefore apply Theorem 1.1 to show that (2.34) is dominated by the right-side of (2.33).

Inequality (2.35) is treated in the same manner as (2.31), that is, we show that for $x \in [-2^{n+1}, 2^n]$,

$$M_\alpha(f\chi_{[2^n,\infty)})\,(x) \le KS$$

where $S = \sup_{k \ge n} \int_{2^k}^{2^{k+1}} |f|$. Consequently (2.35) is bounded by

$$\left(\sum_{n=-\infty}^{\infty} \int_{-2^{n+1}}^{-2^n} \sup_{k \ge n}\left[2^{k(\alpha-1)}\int_{2^k}^{2^{k+1}} |f(t)|\, dt\, u(x)\right]^q dx\right)^{\frac{1}{q}}$$

$$\le \left(\sum_{k=-\infty}^{\infty} 2^{kq(\alpha-1)} \sum_{n=-\infty}^{k} \left[\int_{2^k}^{2^{k+1}} |f(t)|\, dt\right]^q \int_{-\infty}^{-2^k} u(x)^q\, dx\right)^{\frac{1}{q}}$$

which by Hölder's inequality is

$$\leq \left(\sum_{k=-\infty}^{\infty}\left[\int_{2^k}^{2^{k+1}} |f(t)v(t)|^p dt\right]^{\frac{q}{p}}\left(\int_{2^k}^{2^{k+1}} v(t)^{-p'} dt\right)^{\frac{q}{p'}}\left(\int_{-\infty}^{-2^k} 2^{kq(\alpha-1)} u(x)^q dx\right)\right)$$

The result follows by applying (2.32) with $I = [2^k, 2^{k+1}]$ and noting that

$$\left(\sum_k\left[\int_{2^k}^{2^{k+1}} |f(t)\ v(t)|^p\ dt\right]^{\frac{q}{p}}\right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^{\infty} |f(t)\ v(t)|^p\ dt\right)^{\frac{1}{p}}.$$

Since M_α is subadditive and $f = f\chi_{[0,\infty)} + f\chi_{(-\infty,0]}$ we can combine the last two theorems and obtain:

Theorem 2.12. Let $0 \leq \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$. Suppose that u and v are non-negative functions having the property that there exists an $A > 0$, such that, $u(x) \leq Av(y)$ whenever $0 < \frac{x}{4} \leq y \leq 4x$ or $4x \leq y \leq \frac{x}{4} < 0$. If u, v also satisfy the $C_{p,q}$ - condition then

(2.36) $$\left(\int_{\mathbb{R}} |M_\alpha f(x)\ u(x)|^q\ dx\right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}} |f(x)\ v(x)|^p\ dx\right)^{\frac{1}{p}}.$$

In the case $p = 1$ a complete characterization can be obtained.

Theorem 2.13. Let $0 \leq \alpha < 1$, $\frac{1}{q} = 1 - \alpha$ and u and v be non-negative functions. If T is either T_α or M_α,

$$(2.37) \qquad \left(\int_{\mathbb{R}} |Tf(x)\ u(x)|^q\, dx\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}} |f(x)\ v(x)|\, dx\right)$$

holds if and only if

$$(2.38) \qquad \left(\int_{\mathbb{R}} \frac{u(y)^q}{|x-y|}\, dy\right)^{\frac{1}{q}} \le Cv(x) \quad \text{a.e.}$$

<u>Proof</u>. Suppose that (2.37) holds. Let I be an interval with centre x_I, $a = \operatorname{ess\,inf}_{y\in I} v(y)$, and $\varepsilon > 0$. If $E = \{x \in I: v(x) \le a + \varepsilon\}$, define $f(x) = |E|^{-1}\chi_E(x)$. Clearly, if $y \notin I$, both $M_\alpha f(y)$ and $T_\alpha f(y)$ are greater than $(|x_I-y| + |I|)^{\alpha-1}$ so our hypothesis shows that

$$\left(\int_{y\notin I}\left[\frac{u(y)}{(|x_I-y|+|I|)^{1-\alpha}}\right]^q dy\right)^{\frac{1}{q}} \le C\int_{\mathbb{R}} |f(y)v(y)|dy \le C(a + \varepsilon),$$

but since ε was arbitrary and $\frac{1}{q} = 1 - \alpha$ it follows that

$$\left(\int_{y\notin I} \frac{u(y)^q}{|x_I-y|+|I|}\, dy\right)^{\frac{1}{q}} \le C \operatorname{ess\,inf}_{y\in I} v(y).$$

Now choosing $I = [x-h, x+h]$ and letting h tend to zero we obtain (2.38).

To prove the sufficiency we note from the definitions that both $M_\alpha f(x)$ and $T_\alpha f(x)$ are dominated by $\int_{\mathbb{R}} \frac{|f(y)|}{|x-y|^{1-\alpha}}\, dy$ and hence by Minkowski's integral inequality we see that the left-side of (2.37) is dominated by

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|f(y)\ u(x)|}{|x-y|^{1-\alpha}}\, dy\right)^{q} dx\right)^{\frac{1}{q}}$$

$$\leq \int_{\mathbb{R}} |f(y)| \left(\int_{\mathbb{R}} \frac{|u(x)|^{q}}{|x-y|}\, dx\right)^{\frac{1}{q}} dy$$

$$\leq C \int_{\mathbb{R}} |f(y)\ v(y)|\, dy,$$

where the last inequality follows by (2.38).

2.5 Extensions to n Dimensions, $n \geq 1$.

The results discussed in this chapter were stated and proved on $\mathbb{R}^1$. In many cases these results can be extended to $\mathbb{R}^n$, $n \geq 1$, with a few relatively minor modifications.

To do this we redefine

$$\bar{M}_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(t)| \, dt, \; 0 \leq \alpha < n,$$

where the supremum is taken over all cubes Q containing x with sides parallel to the co-ordinate axes. The operator T_α is defined only for $0 < \alpha < n$ by

$$T_\alpha f(x) = \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy.$$

T_0 was defined to be the Hilbert transform on $\mathbb{R}^1$ and, as such, has no ready extension to $\mathbb{R}^n$.

The A_p and $A_{p,q}$ conditions remain unchanged except to replace intervals in $\mathbb{R}^1$ with cubes in $\mathbb{R}^n$ while we define the $C_{p,q}$ condition to be

$$\left(\int_{\mathbb{R}^n} \left[\frac{u(x)}{(|Q|+|x-x_Q|^n)^{1/q+1/p'}}\right]^q dx\right)^{\frac{1}{q}} \left(\int_Q v(x)^{-p'} dx\right)^{\frac{1}{p'}} \leq K < \infty.$$

The condition (2.38) becomes

$$\left(\int_{\mathbb{R}} \frac{u(y)^q}{|x-y|^n}\, dy\right)^{\frac{1}{q}} \leq C\, v(x) \quad \text{a.e.}$$

With these few minor modifications the only results which we were unable to extend to $\mathbb{R}^n$ are the sufficiency results contained in Theorems 2.10, 2.11, and 2.12.

CHAPTER 3

THE MAXIMAL FUNCTION ON HOMOGENEOUS SPACES

3.1 Introduction and Preliminary Results.

A homogeneous space is a triple (X, ρ, μ) where X is a metric space with measure μ such that the space of continuous functions of compact support is a dense subspace of the space of integrable functions. The space X is also endowed with a "distance function" ρ (not necessarily the original metric) satisfying

(3.1) $\rho(x,x) = 0$;

(3.2) $\rho(x,y) = \rho(y,x) > 0$ if $x \neq y$;

(3.3) There is a constant $c \geq 1$ such that $\rho(x,y) \leq c[\rho(x,z) + \rho(z,y)]$ for all $x,y,z \in X$;

(Such a ρ is called quasi-metric.)

(3.4) Given a neighbourhood N of a point x there is an $\varepsilon > 0$ such that the ρ-ball $B(x,\varepsilon) = \{y: \rho(x,y) \leq \varepsilon\}$ is contained in N;

(3.5) The balls $B(x,r) = \{y: \rho(x,y) \leq r\}$ are measurable and the measure $|B(x,r)| = \mu(B(x,r))$ is a continuous function of r for each x, and there is a constant $c > 1$ such that μ satisfies a "doubling condition" i.e. $|B(x,2r)| \leq c|B(x,r)| < \infty$ for all r and x. For convenience we assume that the constant here coincides with the one in (3.3)

These spaces were introduced in a slightly different form by Coifman and Weiss [17] and Coifman and de Guzmán [16] to study singular integrals. They have also been studied by several other authors, including Coifman and Weiss [18], Macías [49], and Macías and Segovia [50]. Homogeneous spaces in the form discussed in this chapter were introduced by Calderón [12]. He used them to extend results of Kurtz [47] which showed that Theorem 2.1 holds in $\mathbb{R}^n$ if both the maximal operator M and the classes A_p are defined in terms of families of rectangles satisfying some minimal restrictions.

In this chapter we discuss some general properties of homogeneous spaces before going on to summarize Calderón's results [12] for the maximal operator M in Section 3.2. In Section 3.3 we show that many of the results of the previous chapter concerning fractional maximal and fractional integral operators also hold in this setting. We also prove some $L \log L$ estimates in Theorems 3.12 and 3.15.

Examples of homogeneous spaces ([17],[18]) are the following:

(1) $X = \mathbb{R}^n$, $\rho(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}$, μ is Lebesgue measure.

(2) $X = \mathbb{R}^n$, $\rho(x,y) = \sum_{i=1}^{n} |x_i - y_i|^{\alpha_i}$, where $\alpha_1,\ldots,\alpha_n$ are positive numbers and μ is Lebesgue measure. These spaces are called non-isotropic.

(3) X and ρ as in example (2) and $d\mu(x) = [\rho(x,0)]^{\beta}\, dx, \beta > 0$, where dx is the element of Lebesgue measure.

(4) X a topological space such that each point $x \in X$ has a neighbourhood basis $\mathcal{U}_x$, such that, $\cap \mathcal{U}_x = \{x\}$ and for any $U, V \in \mathcal{U}_x$ either $U \subseteq V$ or $V \subseteq U$. Let μ be a measure satisfying

(i) $\mu(\{x\}) = 0$ for all $x \in X$,

(ii) there exists a $c > 0$, such that, for any $U \in \mathcal{U}_x$, $y \in U$, there exists a $V \in \mathcal{U}_y$ with $U \subseteq V$ and $\mu(V) \leq c\mu(U)$.

Define

$$\rho(x,y) = \inf\{\mu(U) : U \in \mathcal{U}_x,\ y \in U\} + \inf\{\mu(v) : V \in \mathcal{U}_y,\ x \in V\}.$$

One of the difficulties involved in the study of homogeneous spaces as we have defined them is the lack of a clear relation between the radius r and the measure $|B(x,r)|$ of the ball $B(x,r)$. Thus we introudce the following:

Definition 3.1 [17], [49]. If (X,ρ,μ) is a homogeneous space, we call ρ a measure distance if there exist positive constants C_1 and C_2, such that,

$$C_1 \min(r,\mu(X)) \leq |B(x,r)| \leq C_2 \min(r,\mu(X))$$

for all $r > 0$ and $x \in X$.

We can endow any homogeneous space with a measure distance δ which is, in some sense, equivalent to the original quasi-metric ρ.

To do this we need the following sequence of lemmas.

Lemma 3.2 [12]. There is a constant γ such that $|B(x,ar)| \leq ca^{\gamma}|B(x,r)|$, $a \geq 1$, where c is the constant in (3.5).

Proof. Let $2^{k-1} \leq a < 2^k$, $k \geq 1$, an integer. Then $k \leq 1 + \log_2 a$ and from condition (3.5) we obtain

$$|B(x,ar)| \leq |B(x,2^k r)| \leq c^k |B(x,r)| \leq c^{1+\log_2 a}|B(x,r)| = ca^{\gamma}|B(x,r)$$

with $\gamma = \log_2 c$.

Lemma 3.3 [12]. For each x, $|B(x,r)|$ is a continuous, non-decreasing function of r, and $|B(x,r)| > 0$ for $r > 0$, unless μ vanishes identically.

Proof. We need only show that $|B(x,r)| > 0$ for $r > 0$.

This is immediate however, for if $|B(x,r)| = 0$ it follows by Lemma 3.2 that $|B(x,ar)| = 0$ for all $a \geq 1$ and hence μ vanishes identically.

Lemma 3.4. If μ does not vanish identically, then $\mu(X) < \infty$ if and only if X is bounded (i.e. $X \subseteq B(x,r)$ for some $r>0$).

Proof. If X is bounded (3.5) implies $\mu(X) < \infty$.

To prove the reverse implication we proceed by contradiction. Assume $\mu(X) < \infty$ and X is not bounded. Fix $x_0 \in X$ and choose $r_1 > 0$ so large that $|B(x_0,r_1)| > K\mu(X)$, where $c(5c^2)^\gamma/(1 + c(5c^2)^\gamma) < K < 1$. This is possible because $\lim_{r\to\infty}|B(x_0,r)| = \mu(X)$. Since X is not bounded there exists an $x \in X$ such that $\rho(x,x_0) > 3cr_1$. Choose $r_0 > r_1$ satisfying $3cr_0 < \rho(x,x_0) < 4cr_0$. Then $|B(x_0,r_0)| > K\mu(x)$. In addition $B(x_0,r_0) \cap B(x,r_0) = \emptyset$ and hence $|B(x,r_0)| \leq (1 - K)\ \mu(X)$. Furthermore $B(x_0,r_0) \subseteq B(x,5c^2r_0)$ from which we obtain

$$K\mu(X) \leq |B(x_0,r_0)| \leq |B(x,5c^2r_0)| \leq c(5c^2)^\gamma\ |B(x,r_0)|$$

$$\leq c(5c^2)^\gamma\ (1-k)\ \mu(X).$$

Consequently $K \leq c(5c^2)^\gamma/(1+c(5c^2)^\gamma)$, a contradiction.

Lemma 3.5 [12]. Let F be a family of balls of bounded radii.

Then there exists a countable subfamily of disjoint spheres $B(x_i,r_i)$ such that each sphere in F is contained in one of the balls $B(x_i,br_i)$ where $b = 3c^2$ and c is the constant in (3.3).

Proof. Let M be a bound for the radii of the spheres in F and let $0 < a < 1$ be such that

$$3c^2 = c^2(1 + \frac{1}{a}) + \frac{c}{a}.$$

Since $c > 1$, such an a exists. Now, for each integer k, $k > 0$, construct inductively a family of spheres with the following properties:

(1) $B(x_{i,k},r_{i,k}) \in F$, $a^kM < r_{i,k} \leq a^{k-1}M$;

(2) the $B(x_{i,h},r_{i,h})$ are disjoint for $h \leq k$;

(3) for each k the family is maximal with respect to properties (1) and (2).

Obviously such a family exists.

Now let $B(x,r) \in F$. If $a^kM < r \leq a^{k-1}M$, then $B(x,r)$ intersects one of the spheres $B(x_{i,h},r_{i,h})$, $h \leq k$. But then $r_{i,h} > ar$ and therefore, if $z \in B(x,r)$ and $y \in B(x_{i,h},r_{i,h}) \cap B(x,r)$, by (3.3) we have

$$\rho(z,x_{i,h}) \leq c[\rho(z,x) + \rho(x,x_{i,h})]$$

$$\leq cr + c[\rho(x,y) + \rho(y,x_{i,h})]$$

$$\leq cr + c^2(r + r_{i,h})$$

$$\leq c\,\frac{r_{i,h}}{a} + c^2\left(\frac{r_{i,h}}{a} + r_{i,h}\right)$$

$$= 3c^2 r_{i,h} = br_{i,h},$$

by the definition of a ; that is $z \in B(x_{i,h}, r_{i,h})$

Lemma 3.6. For every $x \in X$, $\lim_{r\to 0^+} |B(x,r)| = 0$ and $\mu(\{x\}) = 0$.

We are now able to construct the quasi-distance δ and prove that it is a measure distance.

Define

$$\delta(x,y) = \inf\{|B_\rho|:\ x,y \in B_\rho\}$$

where the infimum is taken over the set of all ρ-balls $B_\rho (= B_\rho(z,s) = \{y: \rho(z,y) \leq s\}$ for some z and s) which contain both x and y.

It is clear that δ is a quasi-metric and that the δ-balls $B_\delta(x,r) = \{y: \delta(x,y) \leq r\}$ can be written as $B_\delta(x,r) = \cup\{B_\rho : x\in B_\rho, |B_\rho| \leq r\}$. Since $|B_\rho(x,s)|$ is a continuous function of s it follows that $\min(r,\mu(X)) \leq |B_\delta(x,r)|$. To show that δ is a measure distance it therefore suffices to show that $|B_\delta(x,r)| \leq C \min(r,\mu(X))$ for some C. Now $y \in B_\delta(x,r)$, if and only if, there exists a $z \in X$ and an $s > 0$ such that $x,y \in B_\rho(z,s)$ and $|B_\rho(z,s)| \leq r$. It is obvious that $B_\rho(z,s) \subseteq B_\rho(x,2cs) \subseteq B_\rho(z,(2c)^2 s)$ and that $|B_\rho(z,(2c)^2 s)| \leq c(2c)^{2\gamma}|B_\rho(z,s)| \leq c(2c)^{2\gamma} r$, from which it follows that $|B_\rho(x,2cs)| \leq c(2c)^{2\gamma} r$. Taking $t = \min(\mathrm{dia}(X), \sup\{s: |B_\rho(x,2cs)| \leq c(2c)^{2\gamma} r\})$ we see that $y \in B_\rho(x,2ct)$. Since $t < \infty$ we can apply Lemma 3.5 to the ρ-balls of $B_\delta(x,r) = \cup\{B_\rho: x \in B_\rho, |B_\rho| \leq r\}$ to obtain a $B_\rho(z,s) \subseteq B_\delta(x,r)$ such that $|B_\rho(z,s)| \leq r$ and $B_\delta(x,r) \subseteq B_\rho(z,3c^2 s)$. Then

$$|B_\delta(x,r)| \leq |B_\rho(z,3c^2 s)| \leq c(3c^2)^\gamma |B_\rho(z,s)| \leq c(3c^2)^\gamma r$$

which completes the proof.

It is immediate from the preceding discussion that the topologies induced by ρ and δ are equivalent.

The quasi-metric δ is treated in [50] by different techniques.

We remark that the quasi-metrics ρ and δ are not merely multiples of each other, that is, there are no positive constants C_1 and C_2 satisfying $C_1\delta \leq \rho \leq C_2\delta$. To see this we consider $X = \mathbb{R}^2$, $\rho(x,y) = |x-y|$ ordinary Euclidean distance and μ Lebesgue measure. Then $\delta(x,y) = \frac{\pi}{4}|x-y|^2 = \frac{\pi}{4}\rho(x,y)^2$ and our remark follows.

3.2 Maximal Inequalities.

The fractional maximal function $M_\alpha f(x)$, $0 \leq \alpha < 1$, if defined for locally integrable f by

$$(3.6) \qquad M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha}} \int_B |f|\,d\mu,$$

where the supremum is taken over all ρ-balls containing x.

If $X = \mathbb{R}$, $\rho(x,y) = |x-y|$, $d\mu(x) = dx$ this is the fractional maximal function defined in the previous chapter, while, if $X = \mathbb{R}^n$, $d\mu(x) = dx$, $\rho(x,y) = \sup_{1 \leq i \leq n} |x_i - y_i|$ this becomes the fractional maximal function defined in Section 2.5.

Some authors ([12]) study the centered maximal function

$$(3.7) \qquad M_\alpha^c f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha}} \int_{B(x,r)} |f|\,d\mu.$$

Since $M_\alpha^c f(x) \leq M_\alpha f(x) \leq CM_\alpha^c f(x)$ it makes little difference for our purposes which maximal function we use. The first inequality is trivial. To see the second, let $B(z,r)$ be any ρ-ball containing x. Then $B(z,r) \subseteq B(x,2cr) \subseteq B(z,(2c)^2 r)$ and hence

$$\frac{1}{|B(z,r)|^{1-\alpha}} \int_{B(z,r)} |f|\,d\mu \leq \frac{[c(2c)^{2\gamma}]^{1-\alpha}}{|B(x,2cr)|^{1-\alpha}} \int_{B(x,2cr)} |f|\,d\mu \leq CM_\alpha^c f(x).$$

As one might expect the key to weighted boundedness of the maximal operator M is an extension of the A_p-condition to this setting.

Definition 3.7 [12]. Let $1 \le p \le \infty$ and $w \ge 0$. We say that $w \in A_p$ if

(3.8) $$\sup_B \left(\frac{1}{|B|}\int_B w\,d\mu\right)\left(\frac{1}{|B|}\int_B w^{-1/(p-1)}\,d\mu\right)^{p-1} = K_w < \infty,\ 1 < p < \infty;$$

(3.9) $$\left(\frac{1}{|B|}\int_B w\,d\mu\right) \le K_w \operatorname*{ess\,inf}_{x\in B} w(x),\ p = 1;$$

(3.10) there exists $\delta > 0$ such that for any measurable $E \subseteq B$

$$\frac{|E|_w}{|B|_w} \le K_w\left(\frac{|E|}{|B|}\right)^{\delta},\ p = \infty$$

where $|A|_w = \int_A w\,d\mu$.

We next list some of the properties of the A_p-condition (see [12] for details).

(3.11) Let $w \in A_p$, $1 \le p < \infty$, and E be any subset of the sphere B. Then

$$\frac{|E|_w}{|B|_w} \ge K_w^{-1}\left(\frac{|E|}{|B|}\right)^{p}.$$

(3.12) Suppose $E \subseteq B$ and $|E| < \delta|B|$. Then

$|E|_w \leq [1 - K_w^{-1} (1 - \delta)^p] \; |B|_w$, whenever $w \in A_p$, $1 \leq p < \infty$.

(3.13) Let f be integrable on any ball. Then

$$\lim_{r \to 0^+} \frac{1}{|B(x,r)|_w} \int_{B(x,r)} f w d\mu = f(x)$$

μ-almost everywhere. In particular $|f| \leq M f$ almost everywhere.

(3.14) If $w \in A_p$, $p \geq 1$, then there exists an $r > 1$ such that

$$\left[\frac{1}{|B|} \int_B w^r \, d\mu \right]^{\frac{1}{r}} \leq \frac{C}{|B|} \int_B w \, d\mu.$$

(3.15) If $w \in A_p$, $p > 1$, then there exists a p_0, $1 < p_0 < p$, depending on p, w , and the constant c of (3.3) and (3.5), such that, $w \in A_r$, $p_0 < r \leq \infty$.

Norm boundedness of the operator M is treated in the next section.

3.3 The Fractional Case.

Turning our attention to the fractional maximal operator $M_\alpha f$, $0 \le \alpha < 1$, we first show the following:

Theorem 3.8. Let $0 \le \alpha < 1$, $1 \le p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$. A non-negative function u satisfies

$$(3.16) \qquad \left(\int_{\{M_\alpha f > y\}} u^q d\mu\right)^{\frac{1}{q}} \le Cy^{-1}\left(\int_X |f\cdot u|^p d\mu\right)^{\frac{1}{p}}, \quad y > 0,$$

if and only if

$$(3.17) \qquad \left(\frac{1}{|B|} \int_B u^q d\mu\right)^{\frac{1}{q}} \left(\frac{1}{|B|} \int_B u^{-p'} d\mu\right)^{\frac{1}{p'}} \le K < \infty,$$

with the appropriate modifications if $p = 1$.

Proof. Inequality (3.17) follows from (3.16) by the same argument as that used in Theorem 2.4.

To prove that (3.17) implies (3.16) we define, for $N > 0$,

$$M_{\alpha,N} f(x) = \sup_{\substack{B \ni x \\ \text{dia} B \le N}} \frac{1}{|B|^{1-\alpha}} \int_B |f| d\mu,$$

and $E_{y,N} = \{x: M_{\alpha,N} f(x) > y\}$, $y > 0$. Now, for every $x \in E_{y,N}$ there exists a B_x containing x, dia $B_x \le N$, such that

$|B_x|^{\alpha-1} \int_{B_x} |f| d\mu > y$. Hence $E_{y,N} \subseteq \bigcup_x B_x$ and we can apply Lemma 3.5 to obtain a countable disjoint subfamily $B_i = B(x_i, r_i)$ such that each $B_x \subseteq B(x_i, br_i)$ for some i.

As observed in [7], u satisfies (3.17) if and only if $u^q \in A_s$ where $s = 1 + \frac{q}{p'}$. Then applying (3.11) (with $w = u^q$) and Lemma 3.2 we see

$$\begin{aligned}\int_{E_{y,N}} u^q \, d\mu &\le \sum_i \int_{B(x_i, br_i)} u^q \, d\mu \\ &\le Kc^s b^{\gamma s} \sum_i \int_{B(x_i, r_i)} u^q \, d\mu \\ &\le Kc^s b^{\gamma s} \sum_i \int_{B_i} u^q \, d\mu \, \frac{y^{-q}}{|B|^{q(1-\alpha)}} \left(\int_{B_i} |f| d\mu\right)^q .\end{aligned}$$

The remainder of the proof follows exactly that of Theorem 2.4.

If $\alpha = 0$ this is Lemma 6 of [12]. As a corollary we obtain Theorem 3 of [12].

<u>Corollary 3.9</u>. If $w \in A_p$, $p > 1$, then for $p \le q \le \infty$

$$\int_X |Mf|^q \, w \, d\mu \le C \int_X |f|^q \, w \, d\mu. \tag{3.18}$$

If $p = 1$, (3.18) holds for $1 < q \le \infty$, and for $y > 0$

$$\int_{\{Mf>y\}} w \, d\mu \le Cy^{-1} \int_X |f| \, w \, d\mu. \tag{3.19}$$

Proof. Inequality (3.19) is the case $\alpha = 0$, $p = 1$ of (3.16)

To prove (3.18), note that if $\alpha = 0$ and $w = u^p$, condition (3.17) says $w \in A_p$. By (3.15), $w \in A_r$, some $r < p$ and so (3.16) holds with p and q replaced by r. Since $M: L^\infty \to L^\infty$ we can apply the Marcinkiewicz interpolation theorem to arrive at (3.18).

Inequality (3.17) is the $A_{p,q}$-condition over balls.

Definition 3.10. A non-negative function u is said to belong to the class $A_{p,q}$, $1 \leq p,q < \infty$, if it satisfies (3.17).

Theorem 3.11. Suppose that $0 \leq \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$ and $u \in A_{p,q}$. Then there exists a constant C, such that

$$(3.20) \quad \left(\int_X |M_\alpha f(x) u(x)|^q d\mu(x)\right)^{\frac{1}{q}} \leq C \left(\int_X |f(x) u(x)|^p d\mu(x)\right)^{\frac{1}{p}}.$$

If $X = \mathbb{R}^n$ this can be found in [58]. If $\alpha = 0$ this is due to Calderón [12] and is given in Corollary 3.9 above.

Proof. The proof is essentially the same as [58, (2.4)].

Define $w(x) = u(x)^q$, then $w \in A_s$, $s = 1 + \frac{q}{p'}$. Hence, by (3.15), $w \in A_r$, some $1 < r < s$. The number r

can be written in the form $r = 1 + \frac{q_0}{p_0}$, where $\frac{1}{q_0} = \frac{1}{p_0} - \alpha$, $1 < p_0 < p$. By Theorem 3.8

$$(3.21) \qquad \left(\int_{\{M_\alpha f > y\}} w \, d\mu\right)^{\frac{p_0}{q_0}} \leq C y^{-p_0} \int_X |f|^{p_0} w^{\frac{p_0}{q_0}} \, d\mu .$$

Next, define the sublinear operator T by $Tg(x) = M_\alpha(gw^\alpha)(x)$. Then setting $f(x) = g(x)\, w(x)^\alpha$ transforms (3.21) into

$$(3.22) \qquad \int_{\{Tg > y\}} w \, d\mu \leq C y^{-q_0} \left(\int_X |g|^{p_0} w \, d\mu\right)^{\frac{q_0}{p_0}} .$$

In a similar manner we may choose p_1 such that $p < p_1 < \frac{1}{\alpha}$ and define q_1 by $\frac{1}{q_1} = \frac{1}{p_1} - \alpha$. The number $t = 1 + q_1/p_1$ is greater than s and hence $w \in A_t$. Therefore Theorem 3.8 shows that (3.21) is true if we replace p_0 and q_0 by p_1 and q_1 respectively. Consequently

$$(3.23) \qquad \int_{\{Tg > y\}} w \, d\mu \leq C y^{-q_1} \left(\int_X |g|^{p_1} w \, d\mu \right)^{\frac{q_1}{p_1}} ,$$

where T is the same operator as in (3.22). By the Marcinkiewicz interpolation theorem it follows that

$$(3.24) \qquad \left(\int_X |Tg|^q w \, d\mu \right)^{\frac{1}{q}} \leq C \left(\int_X |g|^p w \, d\mu\right)^{\frac{1}{p}} .$$

Replacing $g(x) = f(x)\, w(x)^{-\alpha}$ and $w(x) = u(x)^q$ gives the desired result.

When $p = 1$ we have the following result:

<u>Theorem 3.12</u>. Let $0 < \alpha < 1$, $\frac{1}{q} = 1 - \alpha$, $u \in A_{1,q}$ and f have support in some fixed ball B. Then

$$(3.25) \qquad \left(\int_B |M_\alpha f \cdot u|^q \, d\mu\right)^{\frac{1}{q}}$$

$$\leq C\left(|B|_w + \int_B |f| \log^+ |f\, u^{1-q}| \, u \, d\mu\right),$$

where $w = u^q$.

<u>Proof</u>. The argument is similar to the ones used in [75] and [77].

Let $g(x) = f(x)\, w(x)^{-\alpha}$ and define $E_n = \{x \in B: 2^n \leq |g(x)| \leq 2^{n+1}\}$, $n = 0,1,2,\ldots$ and $g_n(x) = g(x)\chi_{E_n}(x)$. Since $u \in A_{1,q}$ it follows that $w \in A_1$ and so we can apply (3.14) to show that $u \in A_{p_0,q_0}$ for some $p_0 > 1$, $\frac{1}{q_0} = \frac{1}{p_0} - \alpha$. In this case the interpolation argument used in Theorem 3.11 leads to (3.20) with constant $C = C_t = 0(1/(1 - t))$ as $t \to 1$ where $\frac{1}{p} = t + \frac{1-t}{p_0}$, $\frac{1}{q} = \frac{t}{q_1} + \frac{1-t}{q_0} = \frac{1}{p} - \alpha$. (See appendix). Now define $\frac{1}{p_n} = t_n + \frac{1-t_n}{p_0}$, $\frac{1}{q_n} = \frac{1}{p_n} - \alpha$, where $t_n = n/(n+1)$. Then, if $Tg(x) = M_\alpha(gw^\alpha)(x)$ is the operator used in the proof of Theorem 3.11, we can apply Minkowski's and Hölder's inequalities followed by (3.24) to find

$$\left(\int_B |Tg|^q \, w \, d\mu\right)^{\frac{1}{q}}$$

$$\leq \sum_0^\infty \left(\int_B |Tg_n|^q \, w \, d\mu\right)^{\frac{1}{q}}$$

$$\leq \sum_0^\infty |B|_w^{\frac{1}{q}} \left(\frac{1}{|B|_w} \int_B |Tg_n|^{q_n} \, w \, d\mu\right)^{\frac{1}{q_n}} \qquad \text{(since } q_n > q\text{)}$$

$$\leq \sum_0^\infty |B|_w^{1-1/p_n} \, C_{t_n} \left(\int_B |g_n|^{p_n} \, w \, d\mu\right)^{\frac{1}{p_n}}.$$

Since $C_{t_n} = 0(1/(1-t_n))$ as $n \to \infty$ this last expression is dominated by

$$C|B|_w \sum_0^\infty (n+1) \; |B|_w^{-\frac{1}{p_n}} \, 2^{n+1} \left(\int_{E_n} w \, d\mu\right)^{\frac{1}{p_n}}$$

$$= C|B|_w \sum_0^\infty (n+1) \left(\frac{|E_n|_w}{|B|_w}\right)^{\frac{1}{p_n}} 2^{n+1}.$$

Thus we have

$$\left(\int_B |Tg|^q \, w \, d\mu\right)^{\frac{1}{q}} \leq C|B|_w \sum_0^\infty (n+1) \left(\frac{|E_n|_w}{|B|_w}\right)^{\frac{1}{p_n}} 2^{n+1}. \tag{3.26}$$

Let N be the set of those n for which $(|E_n|_w/|B|_w)^{\frac{1}{p_n}} \leq 3^{-(n+1)}$. Let N' be the complement of N in the non-negative integers. If $n \in N'$, noting that $\frac{1}{p_n} - 1 = \frac{1-p_0}{p_0(n+1)} < 0$, it is clear that there exists a constant C with $(|E_n|_w/|B|_w)^{\frac{1}{p_n}-1} \leq C$. From this and (3.26) it follows that

$$\left(\int_B |Tg|^q \, w \, d\mu\right)^{\frac{1}{q}}$$

$$\leq C|B|_w \sum_{n\in N} (n+1) \left(\tfrac{2}{3}\right)^{n+1} + C \sum_{n\in N'} (n+1)\, 2^{n+1} |E_n|_w$$

$$\leq C\left(|B|_w + \int_B |g| \, (\log^+ |g|) \, w \, d\mu\right).$$

Recalling the definitions of T, g and w completes the proof.

When $X = \mathbb{R}^n$ this is [75, Proposition 1].

Corollary 3.13. Let $0 < \alpha < 1$, $\frac{1}{q} = 1 - \alpha$, $u \in A_{1,q}$ and f have support in some fixed ball B. Then, if $w = u^q$,

$$(3.27) \qquad \left(\int_B |M_\alpha f|^q \, d\mu\right)^{\frac{1}{q}} \leq C\left(|B|_w + \int_B |f| \log^+ |f| \, u^q \, d\mu\right).$$

Proof. Theorem 3.12 (with $u \equiv 1$) implies

$$\left(\int_B |M_\alpha f|^q \, d\mu\right)^{\frac{1}{q}} \leq C\left(|B| + \int_B |f| \log^+ |f| \, d\mu\right)$$

$$= C\frac{|B|}{|B|_w}\left(|B|_w + \frac{|B|_w}{|B|} \int_B |f| \log^+ |f| \, d\mu\right)$$

and, since $w \in A_1$, it follows that this expression is dominated by

$$C\,\frac{|B|}{|B|_w}\left(|B|_w + (\operatorname*{ess\,inf}_{x\in B} u(x)^q)\int_B |f|\,\log^+|f|\,d\mu\right)$$

$$\le C\left(|B|_w + \int_B |f|\,\log^+|f|\,u^q\,d\mu\right);$$

and the proof is complete.

Now, since $u \in A_{1,q}$ it follows that $w \in A_\infty$ and hence we obtain from (3.26)

$$\left(\int_B |Tg|^q\,w\,d\mu\right)^{\frac{1}{q}}$$

$$\le C\,|B|_w \sum_0^\infty (n+1)\left(\frac{|E_n|}{|B|}\right)^{\frac{\delta}{p_n}} 2^{n+1}$$

$$= C\,\frac{|B|_w}{|B|}\left(|B| \sum_0^\infty (n+1)\left(\frac{|E_n|}{|B|}\right)^{\frac{\delta}{p_n}} 2^{n+1}\right).$$

In the special case that $\delta \ge 1$ this is dominated by

$$C\,\frac{|B|_w}{|B|}\left(|B| \sum_0^\infty (n+1)\left(\frac{|E_n|}{|B|}\right)^{\frac{1}{p_n}} 2^{n+1}\right).$$

Defining N and N' as in Theorem 3.12 we obtain

$$\left(\int_B |Tg|^q\,w\,d\mu\right)^{\frac{1}{q}}$$

$$\le C\,\frac{|B|_w}{|B|}\left(|B| \sum_{n\in N} (n+1)\left(\tfrac{2}{3}\right)^{n+1} + \sum_{n\in N'} (n+1)\,2^{n+1}\,|E_n|\right)$$

$$\leq C \frac{|B|_w}{|B|} \left(|B| + \int_B |g| \log^+ |g| \, d\mu \right).$$

Substituting for T, g, and w yields

Corollary 3.14. Let $0 < \alpha < 1$, $\frac{1}{q} = 1 - \alpha$, $u \in A_{1,q}$ and f have support in some fixed ball B. If $w = u^q \in A_\infty$ with $\delta \geq 1$, it follows that

(3.28)
$$\left(\int_B |M_\alpha f \cdot u|^q \, d\mu \right)^{\frac{1}{q}}$$
$$\leq C \frac{|B|_w}{|B|} \left(|B| + \int_B |f| \log^+ |fu^{1-q}| \, u^{1-q} \, d\mu \right).$$

We note that since $|B|_w / |B| \leq K \operatorname{ess\,inf}_{y \in B} v(y)$. (3.28) implies (3.25) in this case.

We next discuss an analogue of the fractional integral operator T_α which was treated in Chapter 2. For this purpose assume that there are positive constants C and γ such that $|B(x,r)| \leq Cr^\gamma$ for all r and x. (We may, if necessary, replace the quasi-metric ρ with the measure distance δ. In this case γ would be 1.)

We then define T_α, $0 < \alpha < \gamma$, by

(3.29)
$$T_\alpha f(x) = \int_X f(y) \, \rho(x,y)^{\alpha-\gamma} \, d\mu(y).$$

Theorem 3.15. Assume $0 < \alpha < \gamma$, $1 < p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, and

$u \in A_{p,q}$. Then

$$(3.30) \qquad \left(\int_X |T_\alpha f\cdot u|^q \, d\mu\right)^{\frac{1}{q}} \leq C \left(\int_X |f\cdot u|^p \, d\mu\right)^{\frac{1}{p}} .$$

If $p = 1$ and the support of f is a subset of B then

$$(3.31) \qquad \left(\int_B |T_\alpha f\cdot u|^q \, d\mu\right)^{\frac{1}{q}}$$

$$\leq C \left(|B|_w + \int_B |f| \log^+ |fu^{1-q}| \, u \, d\mu\right) .$$

Proof. Due to the complexity of the notation involved we prove only the case $\gamma = 1$. Minor changes give the result if $\gamma \neq 1$.

Choose $\tau > 0$ and observe

$$T_\alpha f(x) = \int_X f(y) \, \rho(x,y)^{\alpha-1} \, d\mu(y)$$

$$= \int_{\rho(x,y)<\tau} + \int_{\rho(x,y)>\tau}$$

$$\equiv I_1 + I_2 .$$

Define the annuli

$$S_n = \{y: 2^{-(n+1)} \tau \leq \rho(x,y) \leq 2^{-n}\tau\}, n \in \mathbb{Z} .$$

Then

$$|I_1| \leq \sum_0^\infty \int_{S_n} |f(y)|\ \rho(x,y)^{\alpha-1}\, d\mu(y)$$

$$\leq \sum_0^\infty (2^{-(n+1)}\ \tau)^{\alpha-1} \int_{S_n} |f(y)|\, d\mu(y)$$

$$\leq \sum_0^\infty 2^{1-\varepsilon}(2^{-n})^{\varepsilon}\tau^{\varepsilon} \frac{1}{(2^{-n}\tau)^{1-\alpha+\varepsilon}} \int_{B_n} |f(y)|\ d\mu(y),$$

where $\varepsilon > 0$ and $B_n = \{y\colon \rho(x,y) \leq 2^{-n}\ \tau\}$. Thus $|I|_1(x) \leq C_\varepsilon \tau^\varepsilon\ (M_{\alpha-\varepsilon}f)(x)$.

By a similar argument

$$|I_2| \leq \sum_1^\infty (2^{n-1}\ \tau)^{\alpha-1} \int_{B_{-n}} |f(y)|\ d\mu(y)$$

$$\leq C'_\varepsilon\ \tau^{-\varepsilon}\ (M_{\alpha+\varepsilon}f)(x).$$

Set $\tau_\varepsilon = [(M_{\alpha+\varepsilon}f)(x)/(M_{\alpha-\varepsilon}f)(x)]^{\frac{1}{2}}$, then by what we have shown

$$(3.32) \qquad |T_\alpha f(x)| \leq C\ [(M_{\alpha+\varepsilon}f)(x)\ (M_{\alpha-\varepsilon}f)(x)]^{\frac{1}{2}}.$$

Define $\frac{1}{q_\varepsilon} = \frac{1}{p} - (\alpha+\varepsilon)$ and $\frac{1}{\bar{q}_\varepsilon} = \frac{1}{p} - (\alpha-\varepsilon)$. Integrating (3.32) and applying Hölder's inequality yields

$$(3.33) \qquad \int_X |T_\alpha f\cdot u|^q\ d\mu$$

$$\leq C \int_X \left[(M_{\alpha+\varepsilon}f)\cdot(M_{\alpha-\varepsilon}f)\right]^{\frac{q}{2}} u^q\ d\mu$$

$$\leq C \left(\int_X |(M_{\alpha+\varepsilon}f) \cdot u|^{q_\varepsilon} d\mu\right)^{\frac{q}{2q_\varepsilon}} \left(\int_X |(M_{\alpha-\varepsilon}f) \cdot u|^{\bar{q}_\varepsilon} d\mu\right)^{\frac{q}{2\bar{q}_\varepsilon}} .$$

Now, noting that $u \in A_{p,q}$ if and only if $u \in A_s$, $s = 1 + \frac{q}{p'}$, we can use (3.14) to show

$$\left(\frac{1}{|B|} \int_B u^{q_\varepsilon} \, d\mu\right)^{\frac{1}{q_\varepsilon}} \leq \left(\frac{1}{|B|} \int_B u^q \, d\mu\right)^{\frac{1}{q}}$$

for sufficiently small ε , while Hölder's inequality shows

$$\left(\frac{1}{|B|} \int_B u^{\bar{q}_\varepsilon} \, d\mu\right)^{\frac{1}{\bar{q}_\varepsilon}} \leq \left(\frac{1}{|B|} \int_B u^q \, d\mu\right)^{\frac{1}{q}}.$$

These two inequalities show that $u \in A_{p,q_\varepsilon}$ and $u \in A_{p,\bar{q}_\varepsilon}$ respectively so that we can apply Theorem 3.11 to the operators $M_{\alpha+\varepsilon}$ and $M_{\alpha-\varepsilon}$ to obtain

$$\left(\int_X |(M_{\alpha+\varepsilon}f) \cdot u|^{q_\varepsilon} \, d\mu\right)^{\frac{q}{2q_\varepsilon}} \leq C \, ||f \cdot u||_p^{\frac{q}{2}}$$

and

$$\left(\int_X |(M_{\alpha-\varepsilon}f) \cdot u|^{\bar{q}_\varepsilon} \, d\mu\right)^{\frac{q}{2\bar{q}_\varepsilon}} \leq C||f \cdot u||_p^{\frac{q}{2}}$$

from which it follows that

$$\int_X |T_\alpha f \cdot u|^q \, d\mu \leq C||f \cdot u||_p^q.$$

This completes the proof of (3.30)

To prove (3.31) we begin by obtaining (3.33) with $\frac{1}{q_\varepsilon} = 1 - (\alpha+\varepsilon)$ and $\frac{1}{\bar{q}_\varepsilon} = 1 - (\alpha-\varepsilon)$. The argument above shows that $u \in A_{1,q_\varepsilon}$ and $u \in A_{1,\bar{q}_\varepsilon}$ so that we may apply Theorem 3.12 to obtain

$$\left(\int_B |(M_{\alpha+\varepsilon}f) \cdot u|^{q_\varepsilon}\, d\mu\right)^{\frac{q}{2q_\varepsilon}} \leq C\left(|B|_w + \int_B |f| \log^+ |fu^{1-q}|\, u\, d\mu\right)^{\frac{q}{2}}$$

and

$$\left(\int_B |(M_{\alpha-\varepsilon}f) \cdot u|^{\bar{q}_\varepsilon}\, d\mu\right)^{\frac{q}{2\bar{q}_\varepsilon}} \leq C\left(|B|_w + \int_B |f| \log^+ |fu^{1-q}|\, u\, d\mu\right)^{\frac{q}{2}}.$$

Combining this last group of inequalities leads to (3.31).

CHAPTER 4

INTERPOLATION OF OPERATORS

4.1 Introduction

If (X,ρ,μ) is a homogeneous space, we say that a locally integrable function f is of bounded mean oscillation (i.e. $f \in$ BMO) if

$$(4.1) \qquad ||f||_* = \sup_B \inf_c \frac{1}{|B|} \int_B |f-c| \, d\mu < \infty,$$

where the supremum is taken over all balls B and the infimum is taken over all constants c.

We point out that $||\cdot||_*$ is not a norm since $||f||_* = 0$ implies f is constant almost everywhere. However, if BMO is defined to be the space of equivalence classes of functions differing by a constant then BMO becomes a Banach space.

Some authors (notably [23] and [41]) define BMO to be the space of all locally integrable functions f satisfying

$$(4.2) \qquad \sup_B \frac{1}{|B|} \int_B |f-f_B| \, d\mu < \infty,$$

where $f_B = \frac{1}{|B|} \int_B f \, d\mu$ is the integral average of f over B. It is easy to show that the two definitions are equivalent. For any ball B,

$$\inf_c \frac{1}{|B|}\int_B |f-c|\, d\mu \le \frac{1}{|B|}\int_B |f-f_B|\, d\mu,$$

while, for any ball B and constant c,

$$\frac{1}{|B|}\int_B |f-f_B|\, d\mu \le \frac{1}{|B|}\int_B |f-c|\, d\mu + \frac{1}{|B|}\int_B |f_B-c|\, d\mu \le \frac{2}{|B|}\int_B |f-c|\, d\mu.$$

In the case $X = \mathbb{R}^n$ the space BMO was introduced by John and Nirenberg [41] to study regularity properties of solutions of elliptic partial differential equations. Recently, Fefferman and Stein [23] have shown that BMO is the dual of the Hardy space H^1. Further, BMO is essentially the range of certain singular integral operators such as the Hilbert and Riesz transforms. These, and other properties, have resulted in a great deal of recent interest in BMO.

In the next section we introduce the spaces BMO^r, $0 < r < \infty$. Extending a theorem of John and Nirenberg [41] it is possible to show that, for a large class of homogeneous spaces, BMO and BMO^r are equivalent. In Section 4.3 we characterize the intermediate spaces $(L^p, BMO^r)_{\theta,q}$, $0 < p, r, q < \infty$, $0 < \theta < 1$, by estimating the Peetre K-functional and in Section 4.4 a weighted interpolation theorem for analytic families of operators is proven.

4.2 The Spaces BMO^r $(0 < r < \infty)$.

As we observed, (4.1) and (4.2) define the same space and yield equivalent norms. We generalize the definition of BMO in the following:

Definition 4.1. A function f belongs to the class BMO^r, $0 < r < \infty$, if

$$(4.3) \qquad ||f||_{*,r} = \sup_B \left(\inf_c \frac{1}{|B|} \int_B |f-c|^r \, d\mu \right)^{\frac{1}{r}} < \infty.$$

The expression (4.2) cannot be extended in this manner if $0 < r < 1$. In (4.3) f need only be a member of L^r_{loc} (in fact it is clear that $f \in L^r_{loc}$ is a necessary but not sufficient condition for $f \in BMO^r$) while any attempt to generalize (4.2) requires that f_B exist which, in turn, requires that $f \in L^1_{loc}$.

If $X = \mathbb{R}^n$, John and Nirenberg [41] showed that $BMO = BMO^r$ if $1 \leq r < \infty$, while Hanks [28] proved that this is still true if $0 < r \leq 1$.

We show that this is also the case for a large class of homogeneous spaces.

Definition 4.2. If (X,ρ,μ) is a homogeneous space we say that the measure μ is uniform with respect to ρ if

$\liminf_{r\to\infty}\ \inf_{x\in X} |B(x,r)| = \mu(X)$, where as usual $|B(x,r)| = \mu(B(x,r)) = \mu(\{y\in X: \rho(x,y) \le r\})$.

Equivalently, μ is a uniform if there exits a continuous, non-decreasing $f: [0,\infty) \to [0,\infty)$ satisfying

(i) $f(0) = 0$,

(ii) $\lim_{r\to\infty} f(r) = \mu(X)$,

(iii) $|B(x,r)| \ge f(r)$ for all $x\in X$, $r>0$.

For example, any translation invariant measure is uniform as is any measure distance or any finite measure.

The following is a modification of the well known Calderón-Zygmund decomposition lemma ([69]).

Lemma 4.3. Suppose (X,ρ,μ) is a homogeneous space with uniform measure μ and f is an integrable function on X. Let $a > 0$ be a constant such that $\mu(X)^{-1}\int_X |f| < a$. Then there exists a family of balls $\mathcal{B} = \{B_\alpha\}$ such that

(4.4) $$a < \frac{1}{|B_\alpha|}\int_{B_\alpha} |f|\, d\mu \le ca, \text{ for all } \alpha,$$

where c is the constant of the doubling condition (3.5),

(4.5) $$|f(x)| \le a \text{ for almost all } x \notin \bigcup_\alpha B_\alpha .$$

Furthermore, there exists a countable disjoint subfamily $B_i = B(x_i, r_i)$ of $\mathscr{B}$ such that each $B_\alpha \in \mathscr{B}$ is contained in one of the balls $B(x_i, br_i)$ where $b = 3c^2$.

<u>Proof</u>. Choose $r > 0$ so large that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f| \leq a, \text{ ~~for all~~ } x \in X.$$

This is possible because μ is uniform. Single out one such $B(x_0,r)$ and consider $B(x_0, \frac{r}{2})$. Then either

$$\frac{1}{|B(x_0,r/2)|}\int_{B(x_0,r/2)} |f| \leq a \tag{4.6}$$

or

$$\frac{1}{|B(x_0,r/2)|}\int_{B(x_0,r/2)} |f| > a. \tag{4.7}$$

In the case (4.6) we proceed to decrease the radius of the ball $B(x_0,r/4)$, $B(x_0,r/8)$, ...until we are forced (if ever) into case (4.7)

In the second case we do not reduce the ball any further. Note that $|B(x_0,r)| \leq c|B(x_0,r/2)|$ by (3.5) and thus

$$a < \frac{1}{|B(x_0,r/2)|}\int_{B(x_0,r/2)} |f| \leq \frac{c}{|B(x_0,r)|}\int_{B(x_0,r)} |f| \leq ca.$$

Hence we select $B(x_0, r/2)$ to be one of our balls B_α. We denote by $\mathscr{B}$ the collection of all balls of the form $B(x, 2^{-n}r)$ obtained from (4.7) as we allow x to range over X.

Now we note that if $x \notin \cup\{B_\alpha : B_\alpha \in \mathscr{B}\}$,

$$\frac{1}{|B(x,2^{-n}r)|}\int_{B(x,2^{-n}r)} |f| \le a, \; n = 1,2,3,\ldots$$

and since the family $\{B(x,s)\}_{s>0}$ differentiates the integral (3.13) we see that $|f(x)| \le a$ for almost all $x \notin \bigcup_\alpha \{B_\alpha : B_\alpha \in \mathscr{B}\}$.

Since the balls we have constructed are all of bounded radius, it follows by Lemma 3.5 that there exists a countable subfamily of disjoint spheres $B_i = B(x_i, r_i)$ of $\mathscr{B}$, such that, each ball of $\mathscr{B}$ is contained in some $B(x_i, br_i)$ where $b = 3c^2$.

Remark 4.4. Let $\mathscr{B}^a = \{B_\alpha^a\} = \{B^a(x_\alpha, r_\alpha)\}$ be the collection of balls corresponding to the constant a. If $a_1 < a_2$ are two constants, it follows that $B_\alpha^{a_2}$ is contained in some $B_\alpha^{a_1}$. Hence $B_\alpha^{a_2}$ is contained in some $B^{a_1}(x_i, br_i)$ but it does not follow that $B^{a_2}(x_i, r_i)$ is contained in some $B^{a_1}(x_i, r_i)$ (since the proof of Lemma 3.5 involves a maximal argument.)

Remark 4.5. Examining the choice of the $B_\alpha = B(x_\alpha, r_\alpha)$ in

$\mathscr{B}$, we see that $\overline{B(x_\alpha, 2^n r_\alpha)} \not\in \mathscr{B}$, $n = 1,2,\ldots$, and , in fact,

$$\frac{1}{|B(x_\alpha, 2^n r_\alpha)|} \int_{B(x_\alpha, 2^n r_\alpha)} |f| \leq a, \; n = 1,2,3,\ldots$$

(This follows from (4.6) if $2^n r_\alpha \leq r$ and from the fact that the uniformity of μ allows us to choose r so large that $\frac{1}{|B(x,r)|} \int_X |f| \leq a$.)

Now let $B_0 = B(x_0, r_0)$ be a fixed ball in X and consider the new space $(B_0, \rho|B_0, \mu|B_0)$ obtained by restricting ρ and μ to B_0. The balls of this space (denoted by $B'(x,r)$) are given for $x \in B_0$ by

$$B'(x,r) = \{y \in B_0 : \rho(x,y) \leq r\} = B(x,r) \cap B_0 .$$

In general B_0 with this inherited structure does not form a homogeneous space as the following example shows.

Let $X = \mathbb{R} - (0,1)$, $\rho(x,y) = |x - y|$ and μ be Lebesgue measure. Then (X, ρ, μ) is a homogeneous space; but if $B_0 = B(1,1) \subseteq X$ then $(B_0, \rho|B_0, \mu|B_0)$ is not. To see this, note that $B_0 = B(1,1) = \{y : |y-x| \leq 1\} = \{0\} \cup [1,2]$ and $B'(0,1/2) = B(0,1/2) \cap B_0 = \{0\}$. Hence $|B'(0,1/2)| = 0$ so Lemma 3.3 requires μ to vanish identically on B_0, a contradiction to the definition of μ.

There is, however, a large class of homogeneous spaces for which B_0 is a homogeneous space.

<u>Definition 4.6</u>. A homogeneous space (X,ρ,μ) is said to be full if there exists a $K > 0$ such that for $x \in B(x_0,r_0)$, $x_0 \in X$, $r_0 > 0$, there is a $z \in X$ satisfying $B(z,r/K) \subseteq B(x,r) \cap B(x_0,r_0)$ for every $r \in (0,2cr_0)$.

If (X,ρ,μ) is a full homogeneous space and $B_0 = B(x_0,r_0) \subseteq X$ then $(B_0, \rho|B_0, \mu|B_0)$ is a homogeneous space. To see this we note that the only defining property of a homogeneous space which is not immediately inherited from (X,ρ,μ) is the doubling condition (3.5). To prove this does hold, assume $0 < r < 2cr_0$. Then $|B'(x,2r)| \leq |B(x,2r)| \leq c|B(x,r)| \leq c|B(z,2cr)| \leq c^2(2cK)^\gamma |B(z,r/K)| \leq c^2(2cK)^\gamma |B'(x,r)|$. If $r \geq 2cr_0$, $B'(x,r) = B_0$ and the doubling condition still holds.

$\mathbb{R}^n$ together with Euclidean distance and Lebesgue measure is a full homogeneous space, but $(\mathbb{R} - (0,1), |\cdot|, dx)$ is not.

If (X,ρ,μ) is a full homogeneous space and $f \in BMO^s(X,\rho,\mu)$ then, for any $B_0 = B(x_0,r_0) \subseteq X$, $f|B_0 \in BMO^s(B_0, \rho|B_0, \mu|B_0)$. To see this let c be any constant and consider

$$\frac{1}{|B'(x,r)|}\int_{B'(x,r)} |f-c|^s$$

$$\leq \frac{1}{|B(z,r/K)|}\int_{B(z,2cr)} |f-c|^s$$

$$\leq \frac{c(2cK)^\gamma}{|B(z,2cr)|}\int_{B(z,2cr)} |f-c|^s$$

and our observation follows.

We will need one more covering lemma.

Lemma 4.7 [17, p.72]. Let $f \in L(X,d\mu)$ be a function of bounded support and $a > \mu(X)^{-1} \int_X |f|$. Then there exists a family of balls $B(x_i,r_i)$ such that

$$|f(x)| \le ca \text{ for almost all } x \notin \bigcup_i B(x_i,r_i), \tag{4.8}$$

$$\frac{1}{|B(x_i,r_i)|} \int_{B(x_i,r_i)} |f| \, d\mu \le ca, \tag{4.9}$$

$$\sum_i |B(x_i,r_i)| \le \frac{c}{a} \int_X |f| \, d\mu, \tag{4.10}$$

(4.11) there exists a positive integer A such that

$$\sum_i \chi_{B(x_i,r_i)}(x) \le A.$$

The next theorem shows that, under certain conditions, $BMO = BMO^r$.

Theorem 4.8. Suppose (X,ρ,μ) is a full homogeneous space. If $||f||_{*,r} = 1$, $0 < r < \infty$, and, for each ball B the constant minimizing $\int_B |f-c|^r$ is denoted c_B, it follows that

$$|\{x \in B: |f-c_B| > t\}| \le C_1 e^{-C_2 t} |B|, \quad t > 0, \tag{4.12}$$

for each B , where C_1 and C_2 are positive constants.

Proof. For any ball B set $E_t = \{x\in B: |f-c_B|^r > t\}$. Clearly

$$|E_t| \le \int_{E_t} \frac{|f-c_B|^r}{t^r}\, d\mu$$

$$\le \frac{1}{t^r}\frac{|B|}{|B|}\int_B |f-c_B|^r\, d\mu$$

$$\le \frac{1}{t^r}|B|\cdot||f||^r_{*,r} = \frac{1}{t^r}|B|.$$

Next, suppose $F(t)$ is the minimal function, such that, $|E_t| \le F(t)\cdot|B|$ for all t, B and B'. Obviously such a function exists and is dominated by Ct^{-r}.

Now fix $B = B_0$, take $s \ge (Ac)^{\frac{1}{r}}$ and $t > 2c^{\frac{1}{r}}s$, where A is the constant of (4.11). Since $(B_0, \rho|B_0, \mu|B_0)$ and $f|B_0$ satisfy the conditions of Lemma 4.7 we can decompose B_0 into balls $B_i = B(x_i, r_i)$ such that

(4.13) $$\frac{1}{|B_i|}\int_{B_i} |f-c_{B_0}|^r \le cs^r ,$$

(4.14) $|f-c_{B_0}| \le sc^{\frac{1}{r}}$ almost everywhere on $B_0 - \bigcup_i B_i$,

(4.15) $$\sum_i |B_i| \le \frac{c}{s^r}\int_{B_0} |f-c_{B_0}|^r ,$$

(4.16) there exists a positive integer A such that $\sum_i \chi_{B_i}(x) \le A$ for all $x\in B_0$.

Now since $t > 2c^{\frac{1}{r}}s$, (4.14) implies

$$|E_t| \le \sum_i |\{x \in B_i : |f-c_{B_i}| > t\}|$$

$$\le \sum_i |\{x \in B_i : |f-c_{B_0}| + |c_{B_i}-c_{B_0}| > t\}|.$$

Noting that

$$\frac{1}{|B_i|}\int_{B_i} |f-c_{B_i}|^r \le \frac{1}{|B_i|}\int_{B_i} |f-c_{B_0}|^r,$$

it follows from the triangle inequality and (4.13) that

$$|c_{B_i}-c_{B_0}|^r \le \frac{2^{r+1}}{|B_i|}\int_{B_i} |f-c_{B_0}|^r < 2^{r+1} cs^r.$$

Therefore, by (4.15),

$$|E_t| \le \sum_i |\{x \in B_i : |f-c_{B_i}| > t - 2^{(r+1)/r} c^{\frac{1}{r}} s\}|$$

$$\le \sum_i F(t-2^{(r+1)/r} c^{\frac{1}{r}}s)\ |B_i|$$

$$\le F(t-2^{(r+1)/r} c^{\frac{1}{r}}s)\ \frac{c}{s^r}\int_{B_0} |f-c_{B_0}|^r$$

$$\le c\ \frac{F(t-(2^{r+1}c)^{\frac{1}{r}}s)}{s^r}\ \frac{|B_0|}{|B_0|}\int_{B_0} |f-c_{B_0}|^r$$

$$\le c\ \frac{F(t-(2^{r+1}c)^{\frac{1}{r}}s)}{s^r}\ |B_0|$$

since $||f||_{*,r} = 1$. From the last inequality we see that

$$F(t) \le c\,\frac{F(t-(2^{r+1}c)^{\frac{1}{r}}s)}{s^r}.$$

If we write $s = (ec)^{\frac{1}{r}}$ we have

$$F(t) \le \frac{F(t-2^{(r+1)/r}(c^2e)^{\frac{1}{r}})}{e}.$$

From the definition of F it is apparent that F is a decreasing function, bounded above by 1. If $F(t) \le C_1 \exp(-t/(2^{r+1}c^2e)^{1/r})$ on some interval of length $(2^{r+1}c^2e)^{\frac{1}{r}}$, the same estimate must hold for all larger values of t. It is clear that for $0 < \alpha \le t \le (2^{r+1}c^2e)^{1/r} + \alpha$ we can find a constant C_1, such that, $t^{-r} \le C_1 \exp(-t/(2^{r+1}c^2e)^{\frac{1}{r}})$ and it follows that $F(t) \le C_1 \exp(-t/(2^{r+1}c^2e)^{1/r})$.

Now, since $||\cdot||_{*,r}$ is homogeneous

$$|\{x\in B_0:\ |f-c_{B_0}| > t\}| \le C_1\, e^{-C_2 t/||f||_{*,r}}\, |B_0|$$

whenever $f \in BMO^r$.

Consequently

$$\frac{1}{|B|}\int_B |f-c_B|\, d\mu = \frac{1}{|B|}\int_0^\infty |\{x\in B:\ |f-c_B| > t\}|\, dt$$

$$\le \frac{1}{|B|}\int_0^\infty C_1 e^{-C_2 t/||f||_{*,r}}\, |B|\, dt$$

$$= \ ||f||_{*,r} \frac{C_1}{C_2} \int_0^\infty e^{-t}\, dt.$$

Thus $||f||_* \le \frac{C_1}{C_2} ||f||_{*,r}$.

To prove the reverse inequality we note that the above argument, with $r = 1$, yields

$$|\{x \in B_0 : \ |f - c_{B_0}| > t\}| \ \le \ C_1 \ e^{-C_2 t/||f||_*} \ |B_0|.$$

Therefore

$$\frac{1}{|B|} \int_B |f - c_B|^r \, d\mu \ = \ \frac{r}{|B|} \int_B t^{r-1} \ |\{x \in B: \ |f - c_B| > t\}| \ dt$$

$$\le \ \frac{C_1 r}{|B|} \int_0^\infty t^{r-1} \ e^{-C_2 t/||f||_*} \ dt$$

$$\le \ C \ r \ ||f||_*^r \int_0^\infty t^{r-1} \ e^{-t} \ dt$$

$$= \ C \ T^1(r+1) \ ||f||_*^r \ .$$

4.3 BMO and Interpolation.

The BMO-norm of a function f can be considered as a measure of the oscillation of f on the space X. The corresponding local oscillation is described by the "sharp function" defined, for $0 < r < \infty$, by

$$f_r^{\#}(x) = \sup_{B \ni x} \left(\inf_c \frac{1}{|B|} \int_B |f-c|^r \, d\mu \right)^{\frac{1}{r}}, \tag{4.17}$$

where the supremum is taken over all balls B containing x. This was first introduced in the case $r = 1$ by Fefferman and Stein [23] and in the case $0 < r < \infty$ by Hanks [28].

Clearly $f_r^{\#} \in L^{\infty}$ if and only if $f \in BMO^r$ and $||f_r^{\#}||_{\infty} = ||f||_{*,r}$.

We define a corresponding r^{th} power maximal function

$$M^r f(x) = \sup_{B \ni x} \left(\frac{1}{|B|} \int_B |f|^r \, d\mu \right)^{\frac{1}{r}}. \tag{4.18}$$

Note that $M^1 f(x) \sim Mf(x)$, the Hardy-Littlewood maximal function.

It is evident from these definitions that $f_r^{\#}(x) \leq M^r f(x)$, $0 < r < \infty$, and consequently $||f||_{*,r} = ||f_r^{\#}||_{\infty} \leq ||f||_{\infty}$. It follows that L^{∞} is continuously embedded in BMO^r.

The sharp function is subadditive:

$$(4.19) \qquad (f+g)^{\#}_r(x) \le 2^{\frac{1}{r}} (f^{\#}_r(x) + g^{\#}_r(x)) .$$

This follows immediately from

$$\sup_{B \ni x} \left(\inf_c \frac{1}{|B|} \int_B |f+g-c|^r \right)^{\frac{1}{r}}$$

$$\le \sup_{B \ni x} \left(\frac{1}{|B|} \int_B |f+g-c_f-c_g|^r \right)^{\frac{1}{r}}$$

$$\le 2^{\frac{1}{r}} [f^{\#}_r(x) + g^{\#}_r(x)] ,$$

where c_f and c_g are the values minimizing $\int_B |f-c|^r$ and $\int_B |g-c|^r$ respectively.

We will also use the well known fact that

$$(4.20) \qquad M^r: L(p,q) \to L(p,q) , \ 0 < r < p < \infty \text{ and } 0 < q \le \infty.$$

This can be found in Stein [69] in the case $X = \mathbb{R}^n$, $r = 1$. His proof works here with minimal modifications.

As already noted $f^{\#}_r \le M^r f$ and hence $||f^{\#}_r||_p \le ||M^r f||_p$ for all r and p. Under certain conditions this inequality can be reversed.

Theorem 4.9. Let μ be a uniform measure and suppose $f \in L^{p_0}(X,d\mu)$, $0 < r < p_0 < \infty$ and $p_0 \le p < \infty$. If $\mu(X) = \infty$

then

(4.21) $$||M^r f||_p \le C||f_r^{\#}||_p ,$$

where C is a constant independent of f.

If $X = \mathbb{R}^n$, $r = 1$, this is due to Fefferman and Stein [23] and $X = \mathbb{R}^n$, $0 < r < 1$, the result is due to Hanks [28, Proposition 1]. Our argument parallels the ones used there but differs in detail.

<u>Proof</u>. We prove only the case $0 < r < p_0 \le 1$, $p_0 \le p < \infty$ since obvious modifications to the proof give the result in the remaining cases.

Given $a > 0$ we apply Lemma 4.3 to find balls $\{B_\alpha^a\}$ with the properties

(4.22) $$a < \left(\frac{1}{|B_\alpha^a|}\int_{B_\alpha^a} |f|^r \, d\mu\right)^{\frac{1}{r}} \le c^{\frac{1}{r}} a, \text{ for all } \alpha;$$

(4.23) $|f(x)| \le a$ almost all $x \notin \bigcup_\alpha B_\alpha^a$;

(4.24) there exists a countable subfamily $\{B_i^a = B^a(x_i, r_i)\}$ of $\{B_\alpha^a\}$ which are pairwise disjoint, such that, each B_α^a is contained in some $B^a(x_i, br_i)$ where $b = 3c^2$ and c is the constant from the "triangle inequality" (3.3).

Further the decomposition may be taken such that each $B_\alpha^{a_1}$ (and hence each $B_i^{a_1}$) is contained in some $B^{a_2}(x_i, br_i)$, whenever $a_2 < a_1$.

Define $\sigma(a) = \sum_j |B_j^a|$ and let $B_0 = B_{j_0}^{aH}(x_{j_0}, r_{j_0})$, where $0 < H < 1$ is to be determined later. If A is any positive real number, choose $n \in \mathbb{Z}^+$ to be the smallest possible natural number such that $2^n \geq b = 3c^2$. If $\beta = 2^n$ we can clearly replace b in (4.24) by β. Now we estimate $\sum_{B_j^a \subseteq \beta B_0} |B_j^a|$, where we write βB_0 for $B_{j_0}^{aH}(x_{j_0}, \beta r_{j_0})$. Let $A > 0$ and consider the two cases:

<u>Case 1</u>: $\beta B_0 \subseteq \{x: f_r^\#(x) > \frac{a}{A}\}$. Then

$$\sum_{B_j^a \subseteq \beta B_0} |B_j^a| \leq |\{x: f_r^\#(x) > \frac{a}{A}\} \cap \beta B_0| . \tag{4.25}$$

<u>Case 2</u>: $\beta B_0 \nsubseteq \{x: f_r^\#(x) > \frac{a}{A}\}$. In this case there is some $x \in \beta B_0$ such that $f_r^\#(x) \leq \frac{a}{A}$. But this implies that

$$\frac{1}{|\beta B_0|} \int_{\beta B_0} |f - c_{\beta B_0}|^r \leq (\frac{a}{A})^r ,$$

where $c_{\beta B_0}$ is the value minimizing the integral. In fact $c_{\beta B_0}$ satisfies

$$\frac{1}{|\beta B_0|} \int_{\beta B_0} |f - c_{\beta B_0}|^r \leq \frac{1}{|\beta B_0|} \int_{\beta B_0} |f|^r$$

and hence by the triangle inequality and (4.22)

$$|c_{\beta B_0}|^r \le \frac{2}{|\beta B_0|} \int_{\beta B_0} |f|^r \le 2[c^n H a]^r = \frac{a^r}{2}, \text{ where}$$

$n = \log_2\beta$ and $H = 1/(4c^n)$. As a result

$$\frac{1}{|B_j^a|} \int_{B_j^a} |f-c_{\beta B_0}|^r \ge \left(\frac{1}{|B_j^a|} \int_{B_j^a} |f|^r\right) - |c_{\beta B_0}|^r \ge a^r - \tfrac{1}{2}a^r = \frac{a^r}{2}.$$

This implies

(4.26)
$$\begin{aligned} \sum_{B_j^a \subseteq \beta B_0} |B_j^a| &\le \frac{2}{a^r} \sum_{B_j^a \subseteq \beta B_0} \int_{B_j^a} |f-c_{\beta B_0}|^r \\ &\le \frac{2}{a^r} \int_{\beta B_0} |f-c_{\beta B_0}|^r \\ &\le \frac{2}{A^r} |\beta B_0| \\ &\le \frac{2c^n}{A^r} |B_0| \end{aligned}$$

where the last inequality follows by the doubling condition (3.5). Combining the two estimates (4.25) and (4.26) and summing over the balls B_j^{aH} we see that

(4.27)
$$\sigma(a) \le D_{f_r^\#}\left(\frac{a}{A}\right) + \frac{2c^n}{A^r}\, \sigma(aH),$$

where $D_g(t) = |\{x: |g(x)| > t\}|,\ t > 0$, is the distribution functions of g.

Next we show that

$$(4.28) \qquad \sigma(a) \le D_{M^r f}(a) \le K_1\, \sigma(K_2 a)$$

for some positive constants K_1 and K_2.

The first inequality is obvious since $x \in B_j^a$ implies, by (4.22), $a < \left(|B_j^a|^{-1} \int_{B_j^a} |f|^r \right)^{1/r} \le M^r f(x)$. To prove the second let $\tilde{B}_j$ be the ball with the same center as B_j^a but expanded by a factor of 2c. Suppose $x \notin \bigcup_j B_j^a$ and choose any ball B such that $x \in B$; then

$$\int_B |f|^r = \int_{B \cap (\bigcup_j B_j^a)} |f|^r + \int_{B - (\bigcup_j B_j^a)} |f|^r .$$

The second integral is majorized by $a^r|B|$ since $|f| \le a$ almost everywhere on the complement of $\bigcup_j B_j^a$ by (4.23). For the first we note that if $B \cap B_j^a \ne \emptyset$ and $B \not\subseteq \tilde{B}_j$ (since $x \in B$) then $B_j^a \subseteq \tilde{\tilde{B}}$ where $\tilde{\tilde{B}}$ is the ball with the same center as B but whose radius is expanded by a factor of $4c^2$. Therefore we can estimate the first integral by

$$\begin{aligned} \int_{B \cap (\bigcup_j B_j^a)} |f|^r &\le \sum_{B_j^a \subseteq \tilde{\tilde{B}}} \int_{B_j^a} |f|^r \\ &\le \sum_{B_j^a \subseteq \tilde{\tilde{B}}} c a^r |B_j^a| \\ &\le c a^r |\tilde{\tilde{B}}| \end{aligned}$$

$$\leq c^2(4c^2)^\gamma a^r|B|,$$

where the last inequality follows by Lemma 3.2 and $\gamma = \log_2 c$.

Hence, for any such B,

$$\int_B |f|^r \leq (1 + c^2(4c^2)^\gamma) a^r|B|$$

which implies $M^r f(x) \leq (1 + c^2(4c^2)^\gamma)^{1/r} a$. Consequently

$$\{x: M^r f(x) > (1 + c^2(4c^2)^\gamma)^{\frac{1}{r}} a\} \subseteq \bigcup_j B_j^a ,$$

and therefore

$$D_{M^r f}((1 + c^2(4c^2)^\gamma)^{\frac{1}{r}} a) \leq (2bc)^\gamma \sigma(a), \text{ where } b = 3c^2.$$

Hence we see that $D_{M^r f}(a) \leq K_1 \sigma(K_2 a)$, where $K_1 = (2bc)^\gamma$ and $K_2 = (1 + c^2(4c^2)^\gamma)^{-1/r}$, proving (4.28). Combining this result with the estimate (4.27) we obtain

$$\text{(4.29)} \qquad \frac{1}{K_1} D_{M^r f}(a/K_2) \leq D_{f_r^\#}(\tfrac{a}{A}) + \frac{2c^n}{A^r} D_{M^r f}(a\, H).$$

Now define

$$I_N = p\int_0^N a^{p-1} D_{M^r f}(a)\, da,$$

then $I_N < \infty$ for $p_0 \leq p < \infty$, because $M^r: L^p \to L^p$,

$r < p < \infty$. By (4.29)

$$I_N \leq K_1 p \int_0^N a^{p-1} D_{f_r^\#}\left(\frac{K_2 a}{A}\right) da + \frac{2c^n K_1 p}{A^r} \int_0^N a^{p-1} D_{M^r f}(aK_2 H)\, da$$

$$= K_1 p \left(\frac{A}{K_2}\right)^p \int_0^{K_2 N/A} a^{p-1} D_{f_r^\#}(a)\, da + \frac{2c^n K_1 p}{A^r (K_2 H)^p} \int_0^{NHK_2} a^{p-1} D_{M^r f}(a)\, da.$$

Choose $A = A_0$ so large that $2c^n K_1 p/(A^r (K_2 H)^p) < \frac{1}{2}$. Then

$$\frac{1}{2} I_N \leq K_1 \left(\frac{A_0}{K_2}\right)^p p \int_0^{K_2 N/A_0} a^{p-1} D_{f_r^\#}(a)\, da.$$

Letting $N \to \infty$ completes the proof.

We also know $f_r^\# \leq M^r f$, $M^r : L^p \to L^p$, $0 < r < p \leq \infty$, and $f \leq M^r f$ almost everywhere. Taken together with our last theorem this shows

(4.30) $$K_1 ||f||_p \leq ||f_r^\#||_p \leq K_2 ||f||_p,$$

whenever $f \in L^{p_0}$, $0 < r \leq p_0 \leq p < \infty$, $p > r$.

<u>Theorem 4.10</u>. Let (X, ρ, μ) be a homogeneous space with uniform measure μ such that $\mu(X) = \infty$ and $f \in L^{p_0} + BMO^s$, $0 < r < p_0 < \infty$, $r \leq s$. Then

(4.31) $$K(t, f; L^{p_0}, BMO^s) \geq C\left(\int_0^{t^{p_0}} (f_r^\#)^{*p_0}(x)\, dx\right)^{\frac{1}{p_0}}.$$

Here g^* is the decreasing equimeasurable rearrangement of g and K is the Peetre K-functional.

If $X = \mathbb{R}^n$, $p_0 < 1$, $r < s = 1$, this result is [28, Proposition 2] and the proof given there works in our setting.

<u>Proof</u>. Note that if $f = f_0 + f_1$, $f_0 \in L^{p_0}$, $f_1 \in BMO^s$, then

$$||f_0||_{p_0} + t||f_1||_{*,s} \geq 2^{-\frac{1}{p_0}}(||f_0||_{p_0}^{p_0} + t^{p_0}||f_1||_{*,s}^{p_0})^{\frac{1}{p_0}}.$$

In fact, if $p_0 \geq 1$ this last inequality is true without the factor $2^{-1/p_0}$. Hence it suffices to show that $\int_0^{t^{p_0}} (f_r^{\#})^{*p_0}(x)dx$ is majorized by $C \inf\limits_{f=f_0+f_1} (||f_0||_{p_0}^{p_0} + t^{p_0}||f_1||_{*,s}^{p_0})$. If $f_0 \in L^{p_0}$, and $f_1 \in BMO^s$ the subadditive properties of the sharp and $*$-operators show

$$\int_0^{t^{p_0}} (f_r^{\#})^{*p_0}(x)\ dx \leq 2^{\frac{p_0}{r}}\int_0^{t^{p_0}} [(f_0)_r^{\#*}(\tfrac{x}{2}) + (f_1)_r^{\#*}(\tfrac{x}{2})]^{p_0}\ dx$$

$$\leq 2^{p_0+p_0/r} \int_0^{t^{p_0}} [(f_0)_r^{\#*p_0}(\tfrac{x}{2}) + (f_1)_r^{\#*p_0}(\tfrac{x}{2})]\ dx$$

$$\leq 2^{p_0+p_0/r+1}\left\{||(f_0)_r^{\#}||_{p_0}^{p_0} + \int_0^{t^{p_0}} (f_1)_r^{\#*p_0}(\tfrac{x}{2})\ dx\right\}$$

since $(f_0)_r^{\#*}$ is equidistributed with $(f_0)_r^{\#}$. Now by Hölder's inequality

$$\|g_r^{\#}\|_\infty \le \|g_s^{\#}\|_\infty = \|g\|_{*,s}.$$

Hence

$$\int_0^{t^{p_0}} (f_r^{\#})^{*p_0}(x)\,dx$$

$$\le 2^{p_0+p_0/r+1}\left(\|(f_0)_r^{\#}\|_{p_0}^{p_0} + t^{p_0}\|f_1\|_{*,s}^{p_0}\right)$$

which by (4.30) is

$$\le C\left(\|f_0\|_{p_0}^{p_0} + t^{p_0}\|f_1\|_{*,s}^{p_0}\right).$$

The proof is complete.

Remark 4.11. Recently Bennett and Sharpley [6] have proven a sharper result in the case $X = \mathbb{R}^n$, $0 < p_0 < \infty$, namely

$$K(t,f;L^{p_0},BMO) \sim t(f_{p_0}^{\#})^*(t).$$

We quote the following lemma without proof.

Lemma 4.12 [28]. If A_0 and A_1 are quasi-normed spaces continuously embedded in a topological vector space A then $A_0 \cap A_1$ is a dense subset of the intermediate space $(A_0,A_1)_{\Theta,q}$ for $0 < \Theta < 1$, $0 < q < \infty$.

We are now in a position to characterize the intermediate space $(L^{p_0}, BMO)_{\Theta,q}$.

<u>Theorem 4.13</u>. Let $0 < r < p_0 < \infty$, $r \le s$, $0 < \Theta < 1$, $0 < q \le \infty$, and $p = p_0/(1-\Theta)$. Then

$$(4.32) \qquad (L^{p_0}, BMO^s)_{\Theta,q} = L(p,q),$$

if μ is a uniform measure and satisfies $\mu(X) = \infty$.

<u>Proof</u>. Since $||f||_{*,s} \le ||f||_\infty$, $L(p,q) = (L^{p_0},L^\infty)_{\Theta,q} \subseteq (L^{p_0},BMO^s)_{\Theta,q}$. Hence it suffices to show the opposite inclusion.

We begin by considering the case $q = p$ and $f \in L^{p_0} \cap BMO^s$. By Theorem 4.10

$$||f||_{(L^{p_0},BMO^s)_{\Theta,p}} = \left\{\int_0^\infty [t^{-\Theta}K(t,f;\ L^{p_0},BMO^s)]^p\ \frac{dt}{t}\right\}^{\frac{1}{p}}$$

$$\ge \left\{\int_0^\infty [t^{-\Theta}C\left(\int_0^{t^{p_0}} (f_r^\#)^{*p_0}(x)\ dx\right)^{\frac{1}{p_0}}]^p\ \frac{dt}{t}\right\}^{\frac{1}{p}}$$

$$= C\left\{\int_0^\infty [\xi^{-\Theta/p_0}\left(\int_0^\xi (f_r^\#)^{*p_0}(x)\ dx\right)^{\frac{1}{p_0}}]^p\ \frac{1}{p_0}\ \frac{d\xi}{\xi}\right\}^{\frac{1}{p}},$$

where $\xi = t^{p_0}$,

$$\ge Cp_0^{-1/p}\left\{\int_0^\infty [\xi^{-\frac{\Theta}{p_0}+\frac{1}{p_0}}\left(\frac{1}{\xi}\int_0^\xi (f_r^\#)^{*p_0}(x)\,dx\right)^{\frac{1}{p_0}}]^p\ \frac{d\xi}{\xi}\right\}^{\frac{1}{p}}$$

$$\geq \; C p_0^{-1/p} \int_0^\infty [\xi^{\frac{1}{p}} (f_r^\#)^*(\xi)]^p \frac{d\xi}{\xi}^{\frac{1}{p}}$$

since $(f_r^\#)^*$ is non-increasing.

Applying (4.30), we obtain

$$||f||_{(L^{p_0},BMO^s)_{\Theta,p}} \geq C p_0^{-1/p} ||f_r^\#|| \geq C_1 C ||f||_p.$$

Consider a sequence $\{f_n\}$ in $L^{p_0} \cap BMO^s$ which is Cauchy in the intermediate space norm. Then the last inequality shows that $\{f_n\}$ is Cauchy in L^p and hence has a limit $f \in L^p$ which must also be the limit of the f_n's in $(L^{p_0},BMO^s)_{\Theta,p}$ since $L^p \subseteq (L^{p_0},BMO^s)_{\Theta,p}$. By Lemma 4.12, $L^{p_0} \cap BMO$ is dense in $(L^{p_0},BMO^s)_{\Theta,p}$, so $||f||_{(L^{p_0},BMO^s)_{\Theta,p}} \geq CC_1 ||f||_p$, whenever $f \in (L^{p_0},BMO^s)_{\Theta;p}$. This shows that $(L^{p_0},BMO^s)_{\Theta,p} \subseteq L^p$ and consequently $(L^{p_0},BMO^s)_{\Theta,p} = L^p$.

Now, since $(L^{p_0},L^\infty)_{\Theta,q} = L(p,q)$ (Kree [46]), reiteration yields $(L^{p_1},L^{p_2})_{\Theta,q} = L(p_3,q)$ where $\frac{1}{p_3} = \frac{1-\Theta}{p_1} + \frac{\Theta}{p_2}$. Again reiterating we find

$$(L^{p_0},BMO^s)_{\Theta,q} = (L^{p_0},(L^{p_0},BMO^s)_{\Theta_1,\frac{p_0}{1-\Theta_1}})_{\Theta_2,q},$$

where $0 < \Theta_1, \Theta_2 < 1, \Theta = \Theta_1\Theta_2$.

Therefore,

$$(L^{p_0}, BMO^s)_{\Theta,q} = (L^{p_0}, L^{p_0/(1-\Theta)})_{\Theta_2,q} = L(p,q)$$

and reiterating once more, the proof is complete.

As an immediate consequence of this we obtain

Corollary 4.14. Let T be a sublinear operator such that

$$T: \; L^{p_1} \to L^{p_0}$$

$$T: \; \dot{L}^{p_2} \to BMO^s.$$

Then

$$T: \; L(p_3,q) \to L(p,q) \; ,$$

where $0 < p_1, p_2 < \infty$, $0 < p_0 < \infty$, $0 < q \leq \infty$, $0 < \Theta < 1$ and $\frac{1}{p_3} = \frac{1-\Theta}{p_1} + \frac{\Theta}{p_2}$, $p = \frac{p_0}{1-\Theta}$.

In the case $X = \mathbb{R}^n$, $s = 1$, this is [28, Remark 2].

4.4 Analytic Families of Operators and A_p.

Throughout the remainder of this chapter (X,ρ,μ) is a homogeneous space with uniform measure such that $\mu(X) = \infty$. We denote by L^p_w, $0 < p \leq \infty$, w a non-negative weight function, the set of all functions f, such that,

$$\|f\|_{w,p} = \begin{cases} \left(\int_X |f|^p \, w d\mu\right)^{\frac{1}{p}} & , 0 < p < \infty \\ \operatorname{ess\,sup}_{x\in X} |f(x)| & , p = \infty , \text{ w.r.t. } d\nu = w d\mu \end{cases}$$

is finite.

Fefferman and Stein [23] used Theorem 4.9 (in the case $X = \mathbb{R}^n$, $r = 1$) to prove interpolation theorems for analytic families of operators. In [9] we used a weighted analogue of Theorem 4.9 to prove certain weighted estimates with the weight in A_p, $1 < p < \infty$.

We observe that the proof of (4.29) utilized only the doubling property (3.5) of μ so that (4.29) still holds if we replace μ by ν, where $d\nu(x) = w(x)\, d\mu(x)$ and $w \in A_p$.

The "flat function" is defined for locally integrable f by

$$f^b(x) = \sup_{B\ni x} \frac{1}{|B|} \int_B |f-f_B| \, d\mu ,$$

where $f_B = \frac{1}{|B|}\int_B f$. Since $f_1^{\#}(x) \le f^b(x) \le 2f_1^{\#}(x)$ it follows that we can deduce the following variant on Theorem 4.9

<u>Theorem 4.15</u>. Suppose $1 < p < \infty$ and $f \in L_w^{p_0}(X,d\mu)$ for some p_0, $1 < p_0 \le p$. If $w \in A_{p_0}$ and $f^b \in L_w^p(X,d\mu)$, then $(Mf) \in L_w^p(X,d\mu)$ and

$$||Mf||_{w,p} \le C_p ||f^b||_{w,p}.$$

The next interpolation theorem extends [9, Theorem 3] to homogeneous spaces.

<u>Theorem 4.16</u>. Suppose T_z, $z = \xi + i\tau$, is a family of operators, analytic in $0 < \xi < 1$, continuous in $0 \le \xi \le 1$, and uniformly bounded there. If $w \in A_2$ and for $f \in L_w^2 \cap L^\infty$,

$$||T_{i\tau}(f)||_* \le K_0 ||f||_\infty$$

and

$$||T_{1+i\tau}(f)||_{w,2} \le K_1 ||f||_{w,2},$$

then

$$||T_\Theta(f)||_{w,q} \le K_\Theta ||f||_{w,q},$$

for all $f \in L_w^2 \cap L_w^q$, where $q = 2/\theta$, $0 < \theta < 1$.

Proof. Let $F(y) = (T_z f)(y)$ and $B(x)$ be a function from X to balls in X such that $x \in B(x)$. Suppose $\eta(x,y)$ is a measurable function on $X \times X$ such that $|\eta(x,y)| = 1$ and define

$$(U_z f)(x) = \frac{1}{|B(x)|}\int_{B(x)} (F(y) - F_{B(x)})\eta(x,y)d\mu(y),$$

where $F_{B(x)} = \frac{1}{|B(x)|}\int_{B(x)} F(y)d\mu(y)$.

Clearly $|(U_z F)(x)| \le F^b(x)$ and $\sup |(U_z F)(x)| = F^b(x)$, where the supremum is taken over all possible functions $B(x)$ and $\eta(x,y)$. Since $w \in A_2$ it follows by Theorem 4.15 and Corollary 3.9 that $||F^b||_{w,2} \le 2||MF||_{w,2} \le C||F||_{w,2}$ and since $F = T_z f$ is analytic it is clear that

$$\int_X (U_z F)(x)\ g(x)\ w(x)\ d\mu(x)\ ,\ g \in L_w^2,$$

is analytic in $0 < \xi < 1$, continuous and bounded in $0 \le \xi \le 1$.

Recalling that $w \in A_p$ implies $w > 0$ almost everywhere, it follows by hypothesis that

$$||U_{i\tau}(F)||_{w,\infty} = ||U_{i\tau}(F)||_\infty \le ||F^b||_\infty \le 2||F||_\infty = 2||T_{i\tau}(f)||_*$$

$$\leq 2K_0||f||_\infty = 2K_0||f||_{w,\infty}$$

and

$$||U_{1+i\tau}(F)||_{w,2} = ||F^b||_{w,2} \leq 2||MF||_{w,2} \leq C||F||_{w,2}$$

$$= C||T_{1+i\tau}(f)||_{w,2} \leq CK_1||f||_{w,2} ,$$

so by an interpolation theorem of Stein and Weiss [70,p.205] we obtain

$$||U_\Theta(F)||_{w,q} \leq (2K_0)^{1-\Theta}(CK_1)^\Theta||f||_{w,q} ,$$

where $q = 2/\Theta$, $0 < \Theta < 1$. Taking supremums over all functions $B(x)$ and n yields

$$||F^b||_{w,q} \leq K_\Theta||f||_{w,q} ,$$

where $K_\Theta = (2K_0)^{1-\Theta}(CK_1)^\Theta$. Finally by Theorem 4.15 we obtain

$$||T_\Theta(f)||_{w,q} \leq ||M(T_\Theta(f))||_{w,q} \leq C_p||T_\Theta(f)^b||_{w,q}$$

$$\equiv C_p||F^b||_{w,q} \leq K_\Theta C_p||f||_{w,q},$$

which completes the proof.

Corollary 4.17. Let $\{T_z\}$ and g satisfy the conditions of Theorem 4.16. If in addition for $f \in L^p_w$, $1 < p < 2$, and $0 < \theta < 1$,

$$\int_X (T_\theta f)(x)\ g(x)\ d\mu(x) = \int_X f(x)\ \overline{(T_\theta g)(x)}\ d\mu(x)$$

then

$$||T_\theta f||_{w,p} \leq K_\theta ||f||_{w,p} ,$$

whenever $w \in A_p$.

Proof. If $w \in A_p$ then $v \equiv w^{-1/(p-1)} \in A_{p'}$.
Hence by Hölder's inequality and Theorem 4.16 we obtain

$$\begin{aligned} ||T_\theta f||_{w,p} &= \sup_{||g||_{v,p'} \leq 1} \left|\int_X (T_\theta f)(x)\ g(x)\ d\mu(x)\right| \\ &= \sup_{||g||_{v,p'} \leq 1} \left|\int_X f(x)\ \overline{(T_\theta g)(x)}\ w(x)^{\frac{1}{p}}\ w(x)^{-\frac{1}{p}}\ d\mu(x)\right| \\ &\leq ||f||_{w,p} \sup_{||g||_{v,p'} \leq 1} ||T_\theta g||_{v,p'} \\ &\leq K_\theta ||f||_{w,p} \end{aligned}$$

and hence the result.

APPENDIX

A 1 Introduction.

The purpose of this appendix is to provide a convenient reference for the reader. The definitions and results listed here are well known and consequently no attempt has been made to credit the original work where this material appears.

The standard reference work for this area is Zygmund's "Trigonometric Series" [77]. The bulk of the material presented in this chapter can be found there; however, other standard texts will occasionally be referenced.

The last section is a very brief discussion of the A_p-condition of Muckenhoupt and the sources quoted there are readily available.

We begin by establishing a few conventions:

(i) If $0 < p \leq \infty$ the number p' will be defined by $\frac{1}{p} + \frac{1}{p'} = 1$. If $p = 1$, $p' = \infty$ and if $p = \infty$, $p' = 1$.

(ii) If $p = \infty$ the expression $\left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}$ will be interpretted as $\operatorname{ess\,sup}_{x \in X} |f(x)|$.

(iii) All the functions and sets that we consider will be assumed measurable and all the weight functions will be non-negative.

(iv) The characteristic function of a set E is denoted χ_E and is given by

$$\chi_E(x) = \begin{cases} 1 & , x \in E \\ 0 & , x \notin E. \end{cases}$$

(v) If E is any (measurable) set $|E|$ will denote the measure of E and if w is a weight function $|E|_w$ will denote the w-measure of E i.e. $|E|_w = \int_E w(x)\, dx$.

(vi) For simplicity of notation, when no confusion can occur, we will write $\{|Tf| > y\}$ for $\{x: |Tf(x)| > y\}$ and $\int_E f\, d\mu$ or $\int_E f$ instead of $\int_E f(x)\, d\mu(x)$.

(vii) $0 \cdot \infty = 0$.

(viii) $\frac{1}{\infty} = 0$, $\frac{1}{0} = +\infty$.

(ix) Throughout C and K will denote constants, independent of f, but may differ at different appearances.

A 2 Elementary Concepts.

Definition. If $0 < p \leq \infty$, the Lebesgue space $L^p(X,\mu)$ is the set of all measurable functions f on X such that

$$||f||_p = \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} & , 0 < p < \infty, \\ \operatorname*{ess\,sup}_{x \in X} |f(x)| & , p = \infty, \end{cases}$$

is finite.

Theorem (Hölder's Inequality). Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\int_X |fg| \, d\mu \leq ||f||_p \, ||g||_{p'} .$$

An immediate consequence of this is the following:

Corollary. If $\mu(X) < \infty$ and $0 < p \leq q \leq \infty$, then $L^q \subseteq L^p$ and

$$||f||_p \leq ||f||_q \, (\mu(X))^{\frac{1}{p} - \frac{1}{q}} .$$

__Definition.__ Let $(a_n)_{n=0}^{\infty}$ be a sequence (either real or complex valued). If $0 < p \leq \infty$ we say that $a = (a_n) \in \ell^p$ if

$$\|a\|_p = \begin{cases} \left(\sum_{n=0}^{\infty} |a_n|^p\right)^{\frac{1}{p}} & , 0 < p < \infty, \\ \sup_{0 \leq n < \infty} |a_n| & , p = \infty, \end{cases}$$

is finite.

These spaces are also nested, but in the reverse direction to the spaces L^p.

__Theorem [45, p.137].__ If $0 < p \leq q \leq \infty$, then $\ell^p \subseteq \ell^q$ and

$$\|a\|_q \leq \|a\|_p$$

__Theorem (Minkowski's inequality)__. Let $1 \leq p \leq \infty$ and $f, g \in L^p$. Then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

The continuous analogue can be found in [77, p. 19].

__Theorem (Minkowski's integral inequality).__ Let $1 \leq p < \infty$. Then

$$\left(\int_Y \left(\int_X |f(x,y)|\ dx\right)^p dy\right)^{\frac{1}{p}}$$

$$\leq \int_X dx \left(\int_Y |f(x,y)|^p\ dy\right)^{\frac{1}{p}}.$$

Corollary. Let f, g, z be non-negative functions on $[0,\infty)$ and suppose z is increasing. Then, for $p \geq 1$,

$$\int_0^\infty \left(\int_0^{z(x)} f(y)\ dy\right)^p g(x)\ dx$$

$$\leq \left(\int_0^\infty f(y) \left(\int_{z^{-1}(y)}^\infty g(x)\ dx\right)^{\frac{1}{p}} dy\right)^p.$$

Theorem (Fubini's Theorem). Let (X_i, μ_i), $i = 1,2$, be complete measure spaces and μ be the product measure on $X_1 \times X_2$. If f is an integrable function on $X_1 \times X_2$ then all three integrals

$$\int_{X_1 \times X_2} f(x,y)\ d\mu(x,y)\ ,\ \int_{X_1} \int_{X_2} f(x,y)\ d\mu_2(y)\ d\mu_1(x),$$

$\int_{X_2}\int_{X_1} f(x,y)\ d\mu_1(x)\ d\mu_2(y)$ exist and are equal.

Theorem (Jensen's inequality). Let ϕ be a convex function on $[0,\infty)$. Then, if f is a non-negative integrable function on the finite measure space (X,μ),

$$\phi\left(\frac{1}{\mu(X)} \int_X f\ d\mu\right) \leq \frac{1}{\mu(X)} \int_X \phi(f)\ d\mu.$$

If ϕ is concave the inequality is reversed.

The dual of the space L^p, $1 \leq p < \infty$, is the space $L^{p'}$ as the following theorem shows.

<u>Theorem (Riesz Representation Theorem)</u>. Let T be a bounded linear functional on L^p, $1 \leq p < \infty$. Then there is a function $g \in L^{p'}$ such that

$$T(f) = \int_X f\, g$$

and $||T|| = ||g||_{p'}$.

<u>Definition</u>. Let $0 < p, q \leq \infty$ and $L^p(X,\mu)$ and $L^q(Y,\nu)$ be Lebesgue spaces. If T is a (sub)linear operator mapping L^p into L^q we say T is of strong type (p,q) if $||Tf||_q \leq K||f||_p$ for all $f \in L^p$. We define the (p,q) norm of T to be the infimum of all such K's and write

$$T\colon L^p \to L^q$$

to mean T maps L^p continuously into L^q.

<u>Definition</u>. Let f be a measurable function on some measure space (X,μ). The distribution function of f is defined by

$$D_f(t) = \mu(\{x\colon |f(x)| > t\}) \ , \ t > 0.$$

Definition. Let f be a measurable function on (X,μ). Its non-increasing equimeasurable rearrangement f^* is given for $x > 0$ by

$$f^*(x) = \inf \{t: D_f(t) \le x\},$$

where D_f is the distribution function of f.

It is well known [70] that

(i) D_f and f^* are non-increasing and continuous on the right;

(ii) $D_f(f^*(t)) \le t$ for all $t > 0$;

(iii) f and f^* are equimeasurable, that is, $D_f(t) = D_{f^*}(t)$, $t > 0$;

(iv) if $0 < p < \infty$,

$$\begin{aligned} ||f||_p^p &= \int_X |f|^p \, d\mu \\ &= \int_0^\infty f^*(t)^p \, dt \\ &= p \int_0^\infty t^{p-1} D_f(t) \, dt; \end{aligned}$$

(v) $\operatorname{ess\,sup}_{x \in X} |f(x)| = \sup_{0<t<\infty} f^*(t) = ||f||_\infty$;

(vi) $(f+g)^*(x) \le f^*(\frac{x}{2}) + g^*(\frac{x}{2})$, $x > 0$;

(vii) $D_{f+g}(x) \le D_f(\frac{x}{2}) + D_g(\frac{x}{2})$, $x > 0$.

Definition. Let the domain of the subadditive operator T contain all finite linear combinations of characteristic functions of sets of finite measure and all truncations of all its members. When $0 < p \leq \infty$ and $0 < q < \infty$ we say that T is of weak type (p,q) if there exists a constant C such that

$$D_{Tf}(y) \leq \left(\frac{C||f||_p}{y}\right)^q , \ y > 0,$$

or equivalently

$$t^{\frac{1}{q}}(Tf)^*(t) \leq C||f||_p , \ t > 0.$$

In the case where $q = \infty$ we define weak type to be the same as strong type.

It is easy to see that strong type implies weak type but not conversely. For example the operator $Tf(x) = \frac{1}{x}\int_0^x f(t)\,dt$ is of weak type $(1,1)$ but not strong type $(1,1)$. It is interesting to note that T is of strong type (p,p), $p > 1$, by the classical Hardy inequality (1.1).

Definition. The Lorentz space $L(p,q)$ is the space of all measurable functions f such that

$$||f||^*_{p,q} = \left\{\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\right\}^{\frac{1}{q}} < \infty,$$

for $0 < p < \infty$, $0 < q < \infty$, and

$$||f||^*_{p,q} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty,$$

when $0 < p \leq \infty$, $q = \infty$.

It should be mentioned that even for $1 \leq p < \infty$, $1 \leq q < \infty$ or $1 \leq p \leq \infty$, $q = \infty$, $||\cdot||^*_{p,q}$ is not always a norm since the triangle inequality may fail. However, $||\cdot||^*_{p,q}$ can be used to introduce a topology on the space which, in most cases, can be shown to be that of a Banach space ([10],[70]).

Replacing $f^*(t)$ with its integral average $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$ transforms $||\cdot||^*_{p,q}$ into a norm.

Theorem. If $f \in L(p,q_1)$ and $q_1 \leq q_2$ then $||f||^*_{p,q_2} \leq ||f||^*_{p,q_1}$ and consequently

$$L(p,q_1) \subseteq L(p,q_2).$$

We note that if $p = q$,

$$||f||^*_{p,p} = \left\{\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^p \frac{dt}{t}\right\}^{\frac{1}{p}} = ||f||_p,$$

and hence $L(p,p) = L^p$.

The following corollary of the Banach-Steinhaus theorem can be found in [38, 14.24].

<u>Theorem</u>. Let A and B be Banach spaces and let $(T_n)_{n=1}^{\infty}$ be a pointwise convergent sequence of bounded linear transformations from A into B. Then the mapping $T: A \to B$ defined by

$$T(x) = \lim_{n\to\infty} T_n(x)$$

is a bounded linear transformation.

We use several covering lemmas. One lemma of particular importance is the following "Besicovitch type" ([27, Theorem 1.1]).

<u>Theorem</u>. Let A be a bounded set in $\mathbb{R}^n$. For each $x \in A$ let $Q(x)$ be a closed cubic interval centered at x. Then we can choose a possibly finite sequence $(Q_k)_{k=1}^{\infty}$ from the given intervals $(Q(x))_{x\in A}$ such that:

(i) the set A is covered by the sequence, i.e. $A \subseteq \bigcup_k Q_k$;

(ii) no point of $\mathbb{R}^n$ is in more than Θ_n (a number depending only on n) cubes of the sequence (Q_k),

$$\text{i.e. } \sum_k \chi_{Q_k}(x) \leq \Theta_n \quad , \; x \in \mathbb{R}^n;$$

(If $n = 1$ we may take $\Theta_n = 2$).

(iii) the sequence (Q_k) can be distributed in ε_n (a number that depends only on n) families of disjoint cubes.

A 3 Interpolation Theory.

An extremely important tool in harmonic analysis is interpolation theory. The two best known results in this area are the Theorems of Riesz-Thorin and Marcinkiewicz given below.

<u>Theorem (Riesz-Thorin)</u>. Let T be a linear operator of type (p_0,q_0) and (p_1,q_1), $1 \le p_i, q_i \le \infty$, with norms K_0 and K_1 respectively. Let $0 \le \Theta \le 1$ and define $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $\frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$. Then T is of type (p,q) and the (p,q) norm of T is at most $K_0^{1-\Theta} K_1^{\Theta}$.

<u>Theorem (Marcinkiewicz)</u>. Let T be a sublinear operator of weak type (p_0,q_0) and (p_1,q_1). Furthermore suppose $1 \le p_i \le q_i \le \infty$ for $i = 0, 1$ and $q_0 \ne q_1$. If $0 < \Theta < 1$ and $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $\frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$, then T is of strong type (p,q), with norm $O(1/t)$ and $O(1/(1-t))$ as $t \to 0$ and 1 respectively.

To extend these theorems to larger classes of spaces we need the following concepts.

Let $\mathcal{A}$ be a linear Hausdorff space and A_0 and A_1 be two Banach subspaces of $\mathcal{A}$ such that the injections of the $\hat{A}_i$ $(i=0,1)$ are continuous. We call such a pair (A_0,A_1) an interpolation pair and define the intersection $A_0 \cap A_1$ and the algebraic sum $A_0 + A_1$ in the usual manner.

<u>Theorem</u>. The spaces $A_0 \cap A_1$ and $A_0 + A_1$ are Banach spaces under the norms

$$||f||_{A_0\cap A_1} = \max (||f||_{A_0}, ||f||_{A_1}) ,$$

$$||f||_{A_0+A_1} = \inf_{\substack{f=f_0+f_1 \\ f_i\in A_i}} (||f_0||_{A_0} + ||f_1||_{A_1}).$$

Furthermore $A_0 \cap A_1 \subseteq A_i \subseteq A_0 + A_1$ $(i=0,1)$.

<u>Definition</u>. For $f \in A_0 + A_1$ set

$$K(t,f;\, A_0,A_1) = \inf_{\substack{f=f_0+f_1 \\ f_i\in A_i}} (||f_0||_{A_0} + t||f_1||_{A_1}),\ t > 0.$$

<u>Definition</u>. The space $(A_0,A_1)_{\theta,q}$ is defined to be the space of all $f \in A_0 + A_1$ such that

$$||f||_{(A_0,A_1)_{\theta,q}} = \begin{cases} \left(\int_0^\infty [t^{-\theta} K(t,f)]^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < q < \infty, \\ & 0 < \theta < 1 \\ \operatorname{ess\,sup}_{0<t<\infty} t^{-\theta}K(t,f) & , 0 < \theta < 1 ,\ q = \infty \end{cases}$$

is finite.

The spaces so constructed are of interest because of the following result.

<u>Theorem</u>. Let $T: A_0 \to B_0$ and $T: A_1 \to B_1$. Then

$$T: (A_0,A_1)_{\theta,q} \to (B_0,B_1)_{\theta,q}$$

for $0 < \theta < 1$, $0 < q \le \infty$.

<u>Theorem (Reiteration)</u>. Let $A_{\theta_1} = (A_0,A_1)_{\theta_1,q_1}$ and $A_{\theta_2} = (A_0,A_1)_{\theta_2,q_2}$, $0 \le \theta_1 < \theta_2 \le 1$, $0 < q_1, q_2 \le \infty$. Write $\theta = (1-\theta')\theta_1 + \theta'\theta_2$, $0 < \theta' < 1$, and let $0 < q \le \infty$. Then $(A_{\theta_1},A_{\theta_2})_{\theta',q} = (A_0,A_1)_{\theta,q}$.

Since $(L^{p_0},L^{p_1})_{\theta,q} = L(p,q)$, $0 < \theta < 1$, $0 < q < \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, we obtain the Riesz-Thorin theorem as a special case of the interpolation theorem above.

The last interpolation theorem we mention is a well-known result of Stein and Weiss [70, p. 205] and allows us to interpolate among families of operators which vary in a sufficiently smooth manner.

<u>Definition</u>. Suppose that to each z in the strip $S = \{z \in \mathbb{C}: 0 \le \text{Re } z \le 1\}$ there corresponds a linear operator T_z on the space of simple functions in $L^1(X,d\mu)$ into

measurable functions on $(Y,d\nu)$ in such a manner that $(T_z f)g \in L^1(Y,d\nu)$ for every simple $f \in L^1(X)$ and simple $g \in L^1(Y)$. The family $\{T_z\}$ is analytic (or admissible) if the mapping

$$z \to \int_Y (T_z f)\, g\, d\nu$$

is analytic in the interior of S, continuous on S and there exists a constant $a < \pi$ such that

$$e^{-a|y|} \log \left|\int_Y (T_z f)\, g\, d\nu\right|,$$

$z = x + iy$ is uniformly bounded above in the strip S.

Theorem. Suppose $(T_z)_{z \in S}$, $S = \{z \in \mathbb{C}: \mathrm{Re}\,(z) \in [0,1]\}$, is an analytic family of linear operators satisying

$$||T_{iy} f||_{q_0} \leq K_0(y)\, ||f||_{p_0}$$

and

$$||T_{1+iy} f||_{q_1} \leq K_1(y)\, ||f||_{p_1}$$

for all simple $f \in L^1(X,\mu)$, where $1 \leq p_j, q_j \leq \infty$, $K_j(y)$, $j = 0,1$, are independent of f and satisfy

$$\sup_{y \in \mathbb{R}} e^{-b|y|} \log K_j(y) < \infty$$

for some $b < \pi$. Then, if $0 \leq \theta \leq 1$, there exists K_θ such that

$$||T_\theta f||_q \leq K_\theta ||f||_p$$

for all simple functions f where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

For a detailed discussion of interpolation theory see [7], [10], [46], and [70].

A 4. The A_p-condition.

If $1 \le p \le \infty$, we say that a non-negative function w on $\mathbb{R}$ satisfies the A_p-condition (i.e. $w \in A_p$) if, for every interval I,

(i) $$\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-1/(p-1)}\,dx\right)^{p-1} \le K < \infty, \; 1 < p < \infty,$$

(ii) $$\left(\frac{1}{|I|}\int_I w(x)\,dx\right) \le K \operatorname*{ess\,inf}_{y\in I} w(y)\;, \; p = 1,$$

(iii) There exist constants $K, \delta > 0$ such that for any interval I and any measurable $E \subseteq I$,

$$\frac{|E|_w}{|I|_w} \le K\left(\frac{|E|}{|I|}\right)^{\delta}\;, \; p = \infty.$$

Here $|A|$ denotes the measure of A and $|A|_w$ denotes the w-measure of A i.e. $|A|_w = \int_A w(x)\,dx$.

As explained in Chapter 2 Muckenhoupt [55] showed that the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|}\int_I |f(t)|\,dt$$

is a bounded operator from $L^p(\mathbb{R}, w(x)\,dx)$ into $L^p(\mathbb{R}, w(x)\,dx)$,

$1 < p < \infty$ if and only if $w \in A_p$ while Hunt, Muckenhoupt and Wheeden [39] proved the same result with M replaced by the Hilbert transform

$$Hf(x) = \frac{\text{p.v.}}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

The basic properties of the A_p-classes are well known. We give the following summary for reference (see [12], [39], [55], [56] for proofs):

(1) $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$, $1 < p < \infty$,

(2) if $w \in A_p$ then $w \in A_q$, $1 \leq p \leq q \leq \infty$,

(3) if $w \in A_p$, $p > 1$, then there exists an $r = r(p,w)$ such that $1 < r < p$ and $w \in A_r$,

(4) $w \in A_\infty$ if and only if $w \in A_p$, some $1 \leq p < \infty$, (that is $A_\infty = \bigcup_{1 \leq p < \infty} A_p$),

(5) if $w \in A_p$ then $w > 0$ almost everywhere and $w^q \in L^1_{loc}$ for some $q > 1$,

(6) if $w \in A_p$ then $|\bar{I}|_w \leq C|I|_w$ for all intervals I where $\bar{I}$ is the interval with the same center as I and twice the length,

(7) if $w \in A_p$, $1 \leq p \leq \infty$, then $\log w \in$ BMO (BMO is the space of functions of bounded mean oscillation discussed in Chapter 4). If $f \in$ BMO there exists a $C = C(f) > 0$ such that $\exp(f/C) \in A_2$.

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