EFFICIENT QUADRATIC APPROXIMATION FOR STATISTICAL DESIGN

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Abstract A highly efficient approach to quadratic approximation of circuit responses is presented. Because it uses a fixed pattern of base points, this approach requires extremely small amounts of CPU time and storage space. Using this approach, a major obstacle for the traditional quadratic approximation to deal with large problems, namely, the prohibitive requirement for storage and computational effort, is effectively eliminated. The accuracy and efficiency of this quadratic approximation approach are strongly demonstrated by results of two statistical circuit design examples.

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I. INTRODUCTION

In order to make existing statistical circuit design methods more practically usable, many approaches have been devised to reduce very costly computational effort by approximating acceptable regions or circuit responses. Quadratic approximation has proven suitable and successful [1-4]. However, the determination of a quadratic model itself for a problem with a larger number of variables may be too expensive.

For a circuit with 50 elements, the number of coefficients in the quadratic model is 1326. The calculation of the coefficients in a traditional manner involves 1326 circuit simulations and solving a linear system with 1326 equations. Besides all coefficients, the matrix of the linear system requires storage of a 1326 by 1326 array. Determining a quadratic approximation to the response of this circuit creates quite a large problem in the terms of computer time and storage, although the circuit itself may be of a moderate scale. Therefore, for large scale problems the traditional approaches that aim to obtain unique quadratic models do not effectively reduce computational costs.

Biernacki and Styblinski [4] introduced the concept of the maximally flat interpolation and presented an updating algorithm. The most significant property of their approach is that the method allows the number of actual circuit simulations required for an accurate model to be much less than that needed for a full unique quadratic approximation. However, the computational requirement of the method, especially storage space, is still high.

In this paper we substantially enhance the maximally flat quadratic interpolation. Our approach makes use of a fixed pattern of points at which simulation is performed, resulting in very low computational requirements for both CPU time and storage. The basic concept is reviewed in Section II. Our new approach is described in Section III. Two examples of circuit statistical design are given in Section IV. Finally, Section V contains the conclusions.

II. THE BASIC CONCEPT OF THE MAXIMALLY FLAT QUADRATIC APPROXIMATION

A quadratic model in polynomial form to be used to approximate a given function f(x), $\mathbf{x} = [x_1\ x_2\ .\ .\ .\ x_n]^T$, can be written as

$$q(\mathbf{x}) = a_0 + \sum_{i=1}^{n} a_i(x_i - r_i) + \sum_{\substack{j,i=1 \\ j \geq i}}^{n} a_{ij}(x_i - r_i)(x_j - r_j), \tag{1}$$

where $\mathbf{r} = [r_1\ r_2\ .\ .\ .\ r_n]^T$ is a known reference point. The form of the quadratic function used is similar to that of [4]. However, q(x) is defined here w.r.t. the reference point **r** rather than w.r.t. the origin. Note that the subscript notation is such that each coefficient can be easily identified with its corresponding x term, e.g., a_{ij} is the coefficient of $(x_i - r_i)(x_j - r_j)$. Determining a quadratic model is equivalent to determining all its coefficients, which are now unknowns in (1).

Suppose that m ($m > n + 1$) evaluations of f(x) are performed at some points $\mathbf{x}^i$, i = 1, 2, . . ., m. These points are called the base points. Using $f(\mathbf{x}^i)$, we set up a system of linear equations

$$\begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}, \tag{2}$$

where **a** and **v** are arranged to have the following orders: $\mathbf{a} = [a_0\ a_1\ a_2\ .\ .\ .\ a_n]^T$ and $\mathbf{v} = [a_{11}\ a_{22}\ .\ .\ .\ a_{nn}\ a_{12}\ a_{13}\ .\ .\ .\ a_{n-1,n}]^T$, respectively. The vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ are of dimensions $(n + 1)$ and $(m - (n + 1))$, respectively. They contain function values $f(\mathbf{x}^i)$. The matrix $\mathbf{Q}_{ij}$, i, j = 1, 2, is determined by the coordinates of the base points and of **r**.

Similarly to [4], the reduced system with variables **v** is obtained as

$$\mathbf{C}\ \mathbf{v} = \mathbf{e}, \tag{3}$$

where

$$\mathbf{C} = \mathbf{Q}_{22} - \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12} \tag{4}$$

and

$$\mathbf{e} = \mathbf{f}_2 - \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{f}_1. \tag{5}$$

If $m < (n + 1)(n + 2)/2$, the above system is under-determined.

When the least squares constraint is applied to $\mathbf{v}$, the unique solution to (3) can be found as

$$\mathbf{v} = \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}\mathbf{e} \tag{6}$$

and $\mathbf{a}$ is readily obtained as

$$\mathbf{a} = \mathbf{Q}_{11}^{-1}\mathbf{f}_1 - \mathbf{Q}_{11}^{-1}\mathbf{Q}_{12}\mathbf{v}. \tag{7}$$

Then $\mathbf{v}$, called the minimal Euclidean norm solution of (3), and $\mathbf{a}$ give the maximally flat quadratic interpolation in the form of (1) to $f(\mathbf{x})$. The term of the maximally flat quadratic approximation comes from the mechanism of the least squares constraint that forces the second order derivatives to be as small as possible.

III. APPROACH USING A FIXED PATTERN OF BASE POINTS

In the original scheme of [4] all base points are randomly selected. This type of selection allows certain freedom. However, matrices $\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12}$ and $\mathbf{C}$ have to be stored in two arrays with dimensions $(n + 1) \times n(n + 1)/2$ and $(m - (n + 1)) \times n(n + 1)/2$, respectively. For a circuit with 50 variables and an m equal to 500, the required storage consists of two arrays of 51 by 1275 and 449 by 1275. Meanwhile, some fairly complicated calculations are required, as shown in (4)-(7) where matrix multiplication and matrix inversion, or equivalent calculations, are involved. Here, we shall propose a new approach which is based on a fixed pattern of base points. The regularity of the pattern will greatly reduce storage and simplify the calculation of coefficients.

In our approach, only m ($n + 1 < m \leq 2n + 1$) base points are used. The reference point $\mathbf{r}$ is selected as the first base point $\mathbf{x}^1$. The next n base points are determined by perturbing one variable at a time around $\mathbf{r}$, i.e.,

$$\mathbf{x}^{i+1} = \mathbf{r} + [0 \ldots 0 \; \beta_i \; 0 \ldots 0]^T, \quad i = 1, 2, \ldots, n, \tag{8}$$

where β_i is a predetermined perturbation. It can be shown that the first $(n + 1)$ base points lead to very simple forms of matrices $\mathbf{Q}_{11}^{-1}$ and $\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12}$. They are

$$\mathbf{Q}_{11}^{-1} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ -1/\beta_1 & 1/\beta_1 & & & \\ \vdots & & \ddots & & \mathbf{0} \\ \vdots & \mathbf{0} & & \ddots & \\ -1/\beta_n & & & & 1/\beta_n \end{bmatrix} \tag{9}$$

and

$$\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12} = \left[\begin{array}{cccc|c} 0 & \cdots & & 0 & \\ \beta_1 & & & & \\ & \ddots & \mathbf{0} & & \mathbf{0} \\ & \mathbf{0} & \ddots & & \\ & & & \beta_n & \end{array}\right]. \tag{10}$$

Because of this simple pattern they need not be stored in matrix form.

After the first $(n + 1)$ base points, the remaining $(m - (n + 1))$ points follow to provide the second order information on the function. Similar to the base points defined in (8), the consecutive base points are selected by also perturbing one variable at a time. For simplicity, these base points are determined by perturbing $\mathbf{r}$ consecutively, that is

$$\mathbf{x}^{n+1+i} = \mathbf{r} + [0\ 0\ \ldots\ 0\ \gamma_i\ 0\ \ldots\ 0]^T, \quad i = 1, 2, \ldots, k, \tag{11}$$

where γ_i is another perturbation of r_i, which must not equal β_i, and k equals $(m - (n + 1))$. Under this arrangement matrices $\mathbf{Q}_{21}$ and $\mathbf{Q}_{22}$ have very regular structures. Substituting $\mathbf{Q}_{21}$ and $\mathbf{Q}_{22}$ into (4), the matrix $\mathbf{C}$ takes a concise analytical form

$$\mathbf{C} = \left[\begin{array}{ccccc|c} (\gamma_1-\beta_1)\gamma_1 & & & & \mathbf{0} & \\ & \ddots & & & & \\ & & (\gamma_i-\beta_i)\gamma_i & & & \mathbf{0} \\ & & & \ddots & & \\ \mathbf{0} & & & & (\gamma_k-\beta_k)\gamma_k & \end{array}\right] \tag{12}$$

and the vector $\mathbf{e}$ can be expressed by

$$
\mathbf{e} = \mathbf{f}_2 - \left[\begin{array}{cccccc|c}
1-\gamma_1/\beta_1 & \gamma_1/\beta_1 & & & & & \\
1-\gamma_2/\beta_2 & & \gamma_2/\beta_2 & & \mathbf{0} & & \\
\cdot & & & \cdot & & & \\
\cdot & & & & \cdot & & \\
\cdot & & & & & & \\
1-\gamma_i/\beta_i & & & \gamma_i/\beta_i & & & \mathbf{0} \\
\cdot & & & & \cdot & & \\
\cdot & & & & & \cdot & \\
\cdot & & \mathbf{0} & & & \cdot & \\
1-\gamma_k/\beta_k & & & & & \gamma_k/\beta_k &
\end{array}\right] \mathbf{f}_1. \tag{13}
$$

Substituting (12) and (13) into (6), the coefficients are determined by

$$a_{ii} = \{[f(x^{n+1+i}) - f(x^1)]/\gamma_i - [f(x^{i+1}) - f(x^1)]/\beta_i\}/(\gamma_i - \beta_i),\ i = 1, 2, \ldots, k, \tag{14a}$$

$$a_{ii} = 0, \quad i = k + 1, \ldots, n, \tag{14b}$$

and

$$a_{ij} = 0, \ \ i \neq j, \ \ i, j = 1, 2, \ldots, n. \tag{14c}$$

The coefficients a_0 and a_i are easily obtained as

$$a_0 = f(x^1), \tag{15a}$$

and

$$a_i = [f(x^{i+1}) - f(x^1)]/\beta_i - \beta_i a_{ii}, \quad i = 1, 2, \ldots, n. \tag{15b}$$

In our approach storage is reduced to a minimum. Only the $(m - 1)$ perturbations and the m function values need be stored. Our approach does not require any matrix manipulations. All calculations are simplified to (14) and (15). The operational count to calculate all coefficients using this pattern can be merely $(n + 3k)$. This varies linearly with the number of variables. For a circuit with 50 elements and m chosen to be 101, only 200 multiplications are needed.

The maximally flat quadratic interpolation, using a fixed pattern of base points defined here, has an interesting property. All the coefficients of the mixed terms, a_{ij} for $i \neq j$, are conveniently forced to be zero because no related information can be extracted from the fixed pattern. Any of the a_{ii}'s in (14a), $i \leq k$, can be nonzero because double perturbations are made along a straight line parallel to the ith axis. If a third perturbation is made along the same line, it can be shown that the **C** matrix will not have full row rank, and, therefore, the third perturbation does not provide any extra useful information for the quadratic model.

IV. EXAMPLES

A. Design of an 11 Element Low-Pass Filter

We have tested our method within the framework of a circuit design program, which uses a general-purpose simulator and the generalized ℓ_p centering approach [5] to perform statistical design. For the quadratic approximation, the individual circuit responses are chosen as the functions to be approximated. The reference point is defined as the nominal point. At each iteration of optimization, a set of quadratic models is built. The models are evaluated for all outcomes, namely, the statistically sampled circuits. The objective function is calculated from the resulting approximate error functions.

A low-pass ladder filter with 11 elements [6,7], shown in Fig.1, was used as an example. The upper specification was 0.32 dB at (0.02, 0.04, ..., 1 Hz), and the lower specification 52dB at 1.3 Hz, on the insertion loss. A tolerance of 1.5% was assumed for all elements. Outcomes were uniformly distributed between tolerance extremes. The starting point was the result of a synthesis procedure [6].

Results and comparisons are given in Table I. Two approaches to calculate circuit responses, namely actual circuit simulation and our approximation method, were utilized in the two designs. Each design consisted of two successive centering processes shown as phase 1 and phase 2 in Table I. Two phases of design with actual simulation took approximately 96.2 and 62.4 minutes on the VAX 8600 and required 48000 and 31200 circuit simulations, respectively. Achieved yields were 61.7% and 63.7%, respectively. Two phases of design with our approximation method used only 2.5 and 3.9 minutes on the VAX 8600 to reach 70.2% and 79.7% yields, respectively. Only 529 and 828 actual circuit simulations were required. For this example, our method not only presented greatly reduced computational effort as compared with the actual simulation approach, but also reached a higher final yield.

B. Design of a 5 Channel Microwave Multiplexer

This example is a 5-channel 12 GHz contiguous band microwave multiplexer consisting of multi-cavity filters distributed along a waveguide manifold [8]. Fig. 2 illustrates the equivalent circuit of the multiplexer. Tuning is essential and expensive for multiplexers to satisfy the ultimate specifications. The goal of this design is to ease the tuning process.

In order to take the appropriate tolerances into account, specifications were chosen to be 10dB for the common port return loss and for the individual channel stopband insertion losses, resulting in 124 nonlinear constraint functions. Design variables included 60 couplings, 10 input and output transformer ratios and 5 waveguide spacings. Tolerances of 5% were assumed for the spacings, and tolerances of 0.5% for the remaining variables. The starting point was the solution of the conventional minimax nominal design w.r.t. specifications of 20dB. The corresponding responses are shown in Fig. 3. The estimated yield w.r.t. specifications of 10dB at this point was 75%.

Yield optimization was carried out on the CRAY X-MP/22 using the generalized ℓ_p centering algorithm [5], our approximation scheme and utilizing a multiplexer simulation program [8]. The process consisted of 4 phases as shown in Table II. At the beginning of each phase, a set of quadratic models corresponding to 124 responses was constructed. These models were used for all outcomes in the phase. This is different from the first example where the quadratic models were rebuilt at each optimization iteration.

Four phases took totally 69.5 seconds on the CRAY X-MP/22 to reach a 90% estimated yield. This approach allowed us to handle this large optimization problem (with 75 toleranced variables, 124 constraints and up to 200 statistically perturbed circuits) in acceptable CPU time. At the solution, the return losses of 3000 outcomes and of satisfactory outcomes are contained within the envelopes in Figs. 4 and 5, respectively.

V. CONCLUSIONS

In this paper we have presented a highly efficient quadratic approximation technique. The new approach takes advantage of the maximally flat interpolation and of a fixed pattern of base points, thus substantially reducing the computational effort. A set of extremely simple formulas to calculate model coefficients has been derived. The elegance of this approach is its conciseness and applicability. The very strong impact of our approach on the feasibility of statistical design of larger circuits should not be underestimated. From the results of two examples of statistical design, very high efficiency and the capability of handling large problems have been proven. It should also be noted that our approach is suitable for a large variety of applications where a large number of expensive simulations is involved.

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REFERENCES

[1] J.W. Bandler and H.L. Abdel-Malek, "Optimal centering, tolerancing, and yield determination via updated approximations and cuts", *IEEE Trans. Circuits and Systems*, vol. CAS-25, 1978, pp. 853-871.

[2] H.L. Abdel-Malek and J.W. Bandler, "Yield optimization for arbitrary statistical distributions: Part I-theory", *IEEE Trans. Circuits and Systems*, vol. CAS-27, 1980, pp. 245-253.

[3] D.E. Hocevar, M.R. Lightner and T.N. Trick, "Monte Carlo based yield maximization with quadratic model", *Proc. IEEE Int. Symp. Circuits and Systems* (Newport Beach, CA), 1983, pp. 550-553.

[4] R.M. Biernacki and M.A. Styblinski, "Statistical circuit design with a dynamic constraint approximation scheme", *Proc. IEEE Int. Symp. Circuits and Systems* (San Jose, CA), 1986, pp. 976-979.

[5] J.W. Bandler and S.H. Chen, "Circuit optimization: the state of the art", *IEEE Trans. Microwave Theory Tech.*, vol. MTT-36, 1988, pp. 424-443.

[6] K. Singhal and J.F. Pinel, "Statistical design centering and tolerancing using parametric sampling", *IEEE Trans. Circuits and Systems*, vol. CAS-28, 1981, pp. 692-701.

[7] E. Wehrhahn and R. Spence, "The performance of some design centering methods", *Proc. IEEE Int. Symp. Circuits and Systems* (Montreal, Canada), 1984, pp. 1424-1438.

[8] J.W. Bandler, S. Daijavad and Q.J. Zhang, "Exact simulation and sensitivity analysis of multiplexing networks," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-34, 1986, pp. 93-102.

TABLE I

COMPARISON OF STATISTICAL DESIGN OF A LOW-PASS FILTER WITH AND WITHOUT QUADRATIC APPROXIMATION

Component x_i	Nominal Design x^0	Actual Circuit Simulation		Maximally Flat Approximation	
		Phase 1 x^1	Phase 2 x^2	Phase 1 x^3	Phase 2 x^4
x_1	0.22510	0.22572	0.22512	0.22266	0.21669
x_2	0.24940	0.24903	0.24944	0.25045	0.25131
x_3	0.25230	0.25269	0.25276	0.25268	0.25083
x_4	0.24940	0.24908	0.24882	0.25028	0.24067
x_5	0.22510	0.22568	0.22594	0.22335	0.22120
x_6	0.21490	0.21589	0.21658	0.22163	0.23347
x_7	0.36360	0.36313	0.36275	0.36291	0.37008
x_8	0.37610	0.37625	0.37698	0.37938	0.37217
x_9	0.37610	0.37633	0.37561	0.37156	0.38529
x_{10}	0.36360	0.36313	0.36305	0.36226	0.37232
x_{11}	0.21490	0.21587	0.21674	0.22168	0.21893
Yield*	54.0%	61.7%	63.7%	70.2%	79.7%
Yield**	54.0%			74.0%	84.5%
No. of Outcomes Used for Design		200	200	200	200
Starting Point		x^0	x^1	x^0	x^3
No. of Simulations		48000	31200	529	828
No. of Iterations		9	7	10	19
CPU Time(VAX 8600)		96.2min.+	62.4min.+	2.5min.	3.9min.
CPU Time(MircoVAX)		481min.	312min.	12.3min.	19.5min.+

CPU times do not include yield estimation based on actual simulation.

* The yield is estimated by actual simulation and using 1000 samples.
** The yield is estimated by maximally flat approximation and the actual outcomes used for design.
\+ The CPU time is approximately given by assuming that the speed ratio of VAX 8600 to MicroVAX is 5.

TABLE II

STATISTICAL DESIGN OF A 5-CHANNEL MULTIPLEXER

USING QUADRATIC APPROXIMATION

	Phase 1	Phase 2	Phase 3	Phase 4
Initial *	75.0%	(81.0%)	(84.3%)	(90.0%)
Yield **	56.3%	69.0%	69.3%	92.0%
Final *	81.0%	84.3%	90.0%	90.3%
Yield **	77.3%	77.3%	91.3%	94.0%
No. of Outcomes Used for Design	50	100	150	200
No. of Iterations	4	6	6	4
CPU Time (CRAY X-MP/22)	16.5s	17.6s	17.8s	17.6s

CPU times do not include yield estimation based on actual simulation.

All yields are estimated using 300 samples.
* The yield is estimated by actual simulation.
** The yield is estimated by maximally flat approximation.

Figure Captions

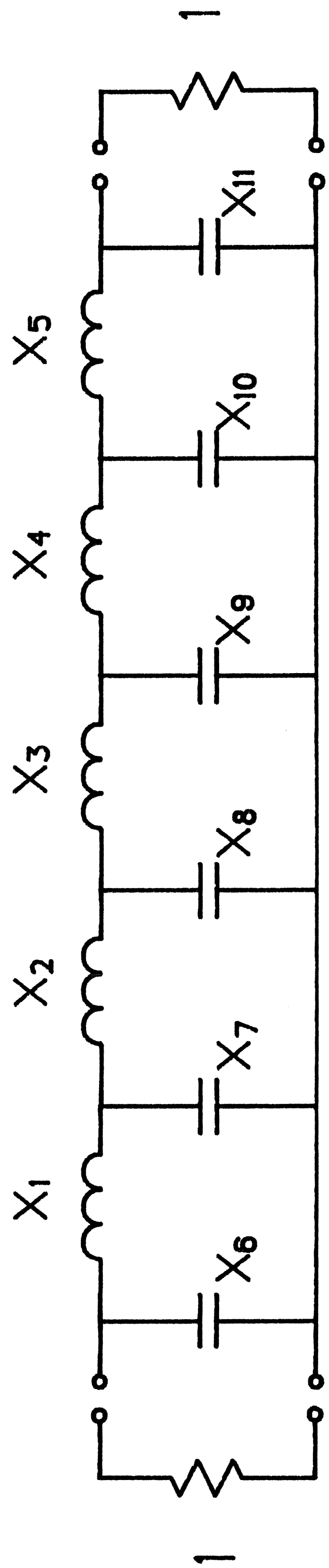
Fig 1. The LC low-pass filter [6,7].

Fig 2. The equivalent circuit of a 5-channel multiplexer. In the *i*th channel, V_i is the output, Z_i denotes the multi-coupled cavity filter, I_i symbolizes the impedance inverter, n_1^i and n_2^i are the input and output transformer ratios, Y_c^i and Y_a^i represent the non-ideal series junction susceptances, and ℓ^i stands for the waveguide spacing.

Fig 3. Optimized return and insertion loss vs. frequency for the 5-channel multiplexer.

Fig 4. The envelope containing return loss responses of all 3000 Monte-Carlo samples.

Fig 5. The envelope containing return loss responses of acceptable circuits among 3000 Monte-Carlo samples.


1
X1
X2
X3
X4
X5
X6
X7
X8
X9
X10
X11
1

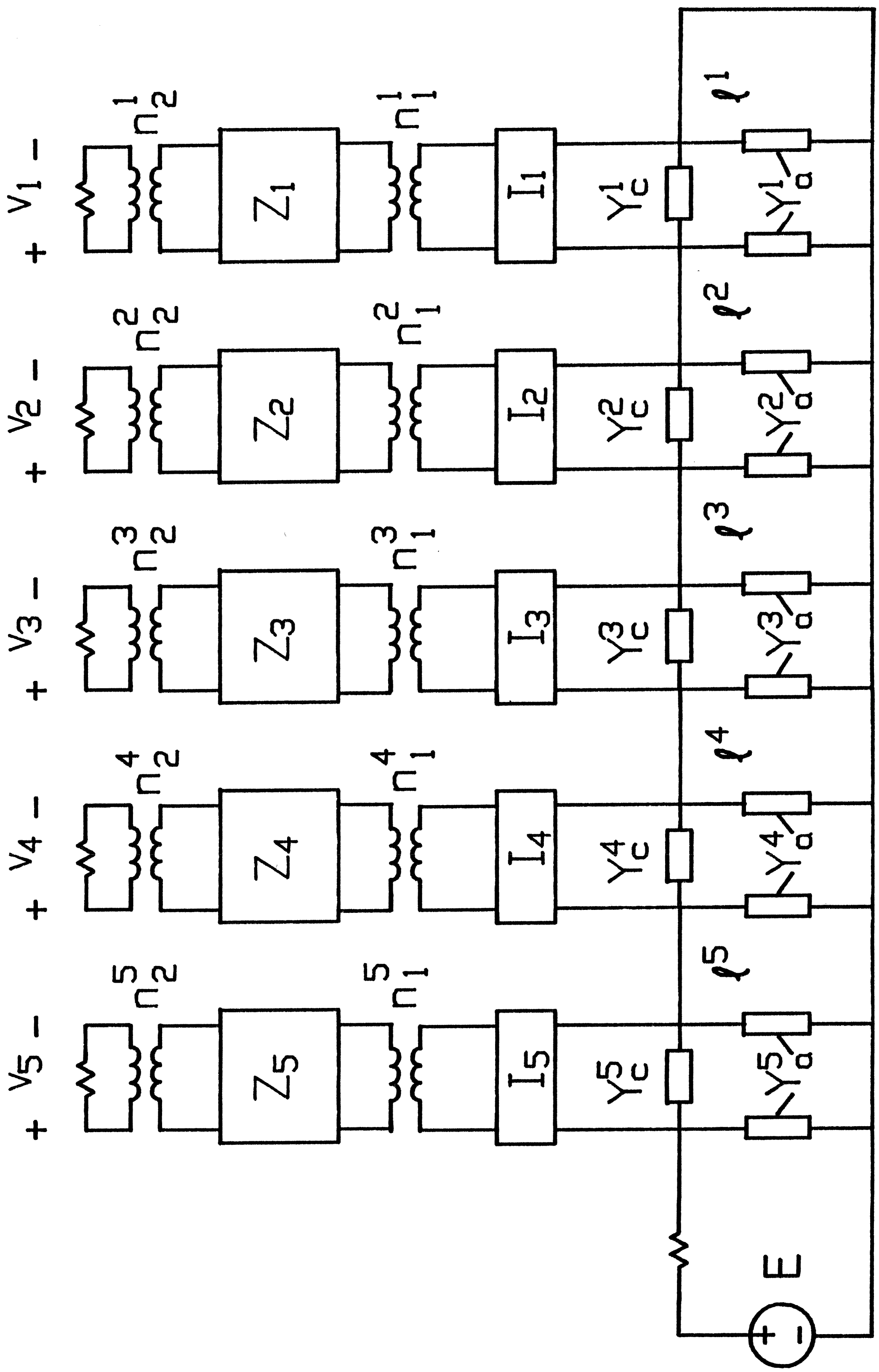

V1 V2 V3 V4 V5
Z1 Z2 Z3 Z4 Z5
I1 I2 I3 I4 I5
E

Fig. 3

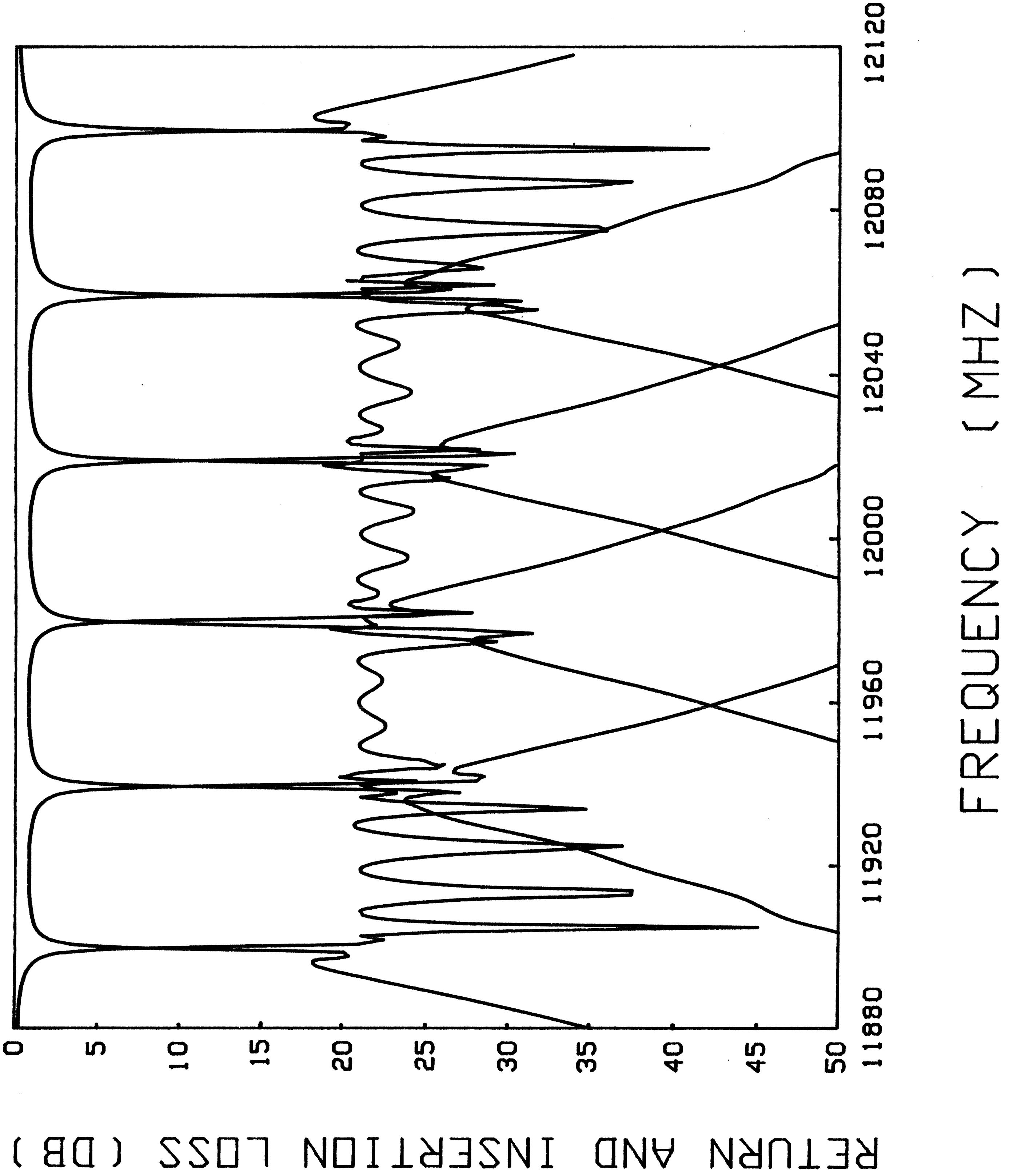

RETURN AND INSERTION LOSS (DB)
0
5
10
15
20
25
30
35
40
45
50
11880
11920
11960
12000
12040
12080
12120
FREQUENCY (MHZ)

Fig. 4

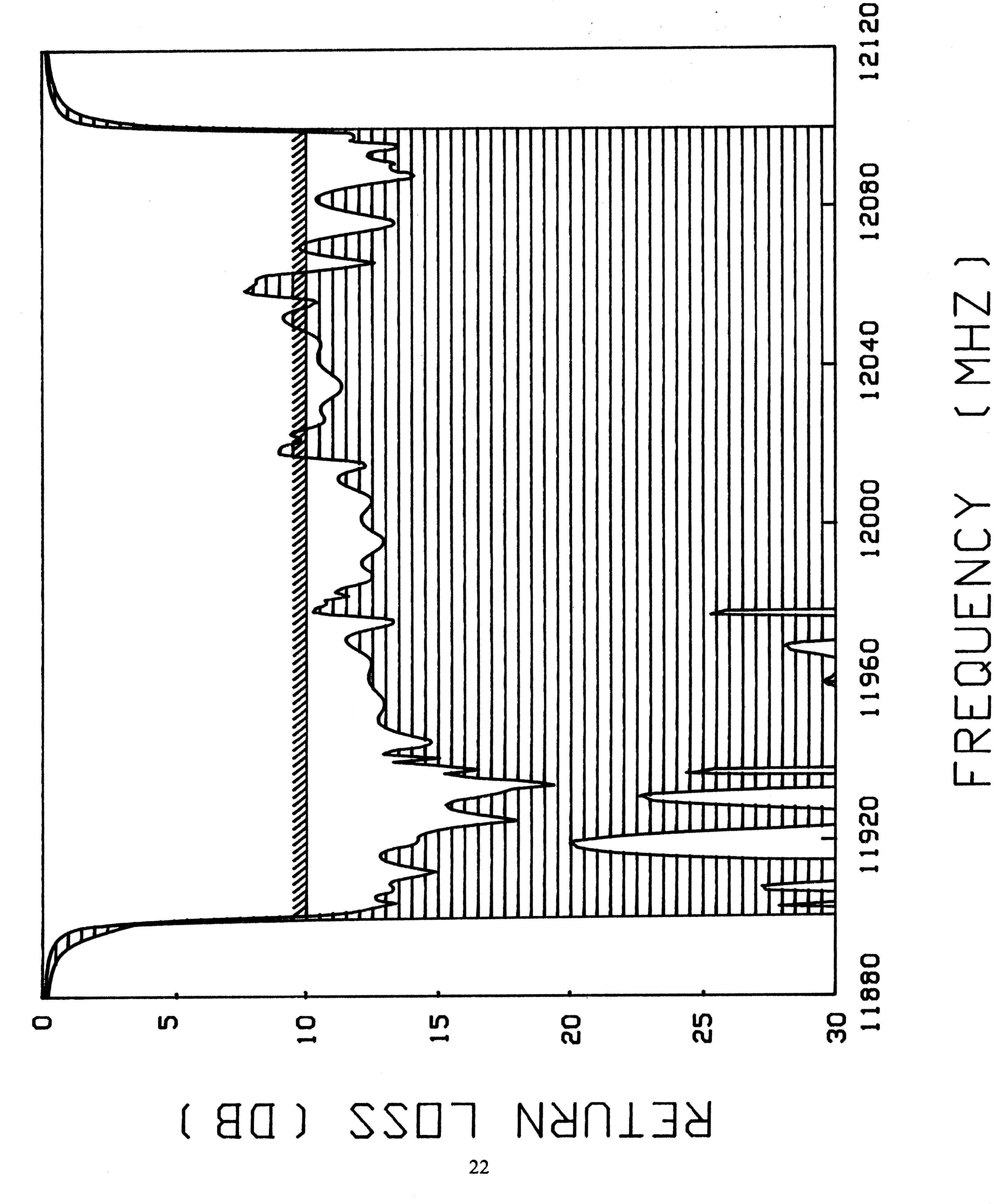
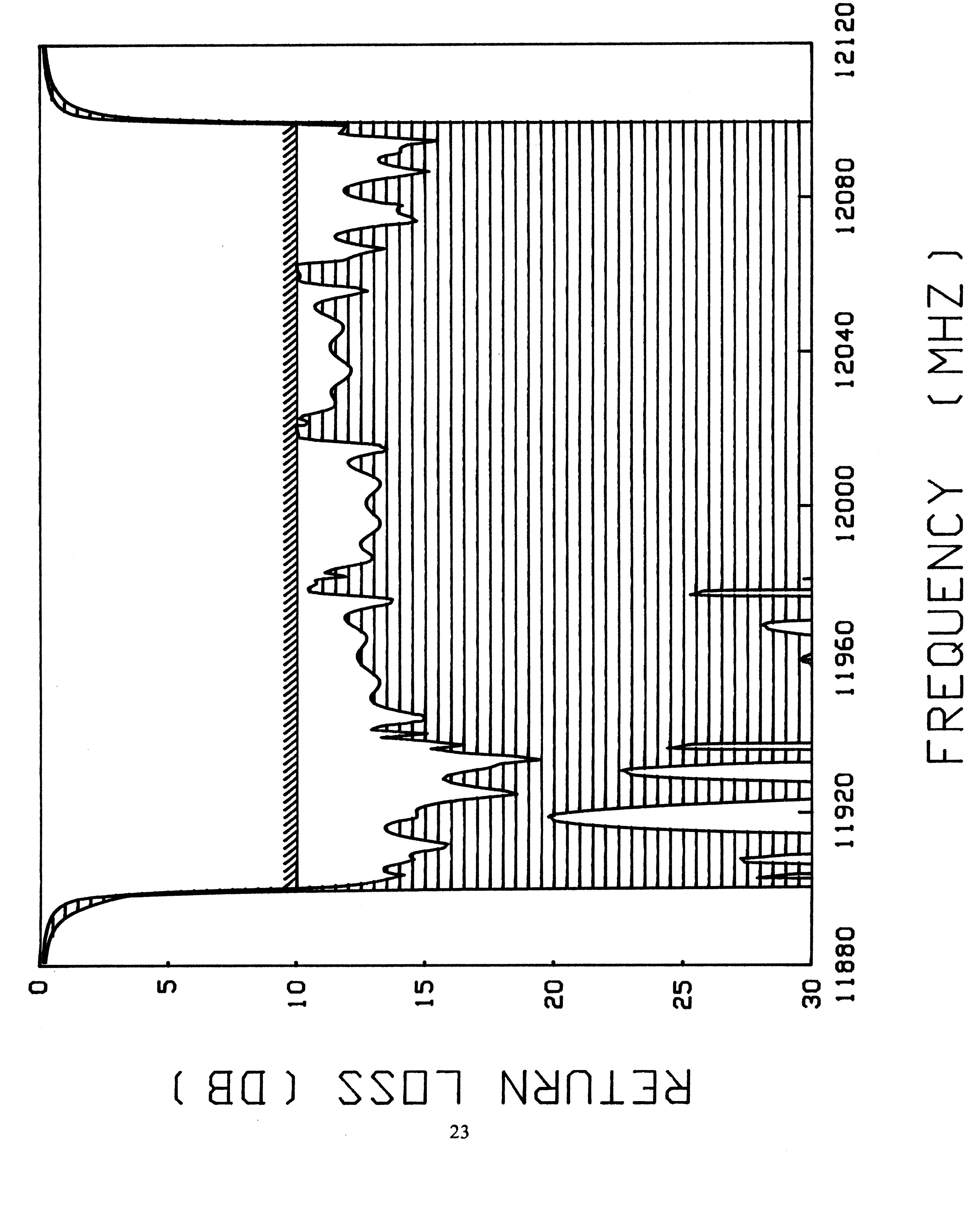

RETURN LOSS (DB)
0
5
10
15
20
25
30
FREQUENCY (MHZ)
11880
11920
11960
12000
12040
12080
12120

Fig. 5



RETURN LOSS (DB)
0
5
10
15
20
25
30
FREQUENCY (MHZ)
11880
11920
11960
12000
12040
12080
12120