STATISTICAL DESIGN VIA

EFFICIENT QUADRATIC MODELING

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Abstract An extremely efficient method for quadratic approximation of circuit response functions is presented. Using a suitable pattern of base points, we are able to obtain all the coefficients of the quadratic model using simple analytical formulas. Consequently, major obstacles for quadratic approximation in the case of high dimensionality, namely, the requirement for prohibitive storage and computational effort, are in effect eliminated. For example, a circuit with 100 variables requires only 400 multiplications to build the quadratic model we propose. We demonstrate the accuracy and efficiency of the quadratic modeling approach in statistical circuit design.

SUMMARY

Quadratic approximation has been used as a powerful tool to reduce the number of circuit simulations in statistical design [1-4]. The determination of a quadratic model itself for a large number of variables, however, is generally expensive. For example, a circuit with 20 variables needs 231 circuit simulations and the solution of a 231x231 linear system of equations in order to uniquely identify a quadratic model.

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Recently, Biernacki and Styblinski [3] introduced a maximally flat interpolation scheme. Their method allows the number of base points required for an accurate model to be much less than needed for a full quadratic approximation. Following their approach, we derive a set of formulas drastically simplifying the computation of the quadratic model. Our method takes advantage of a particular fixed pattern of base points. All coefficients of the quadratic model can be obtained explicitly using simple analytical formulas. While retaining the advantage of [3], i.e., using a small number of base points, our method offers additional and much more significant savings of computer time and storage. A low pass filter example is used to illustrate the efficiency of our method.

Quadratic Approximations

A quadratic polynomial to be used to approximate a given function $f(\mathbf{x})$, $\mathbf{x} = [x_1 \ x_2 \ \ldots \ x_n]^T$, can be written as

$$q(\mathbf{x}) = a_0 + \sum_{i=1}^{n} a_i (x_i - x_i^0) + \sum_{\substack{i,j=1 \\ i \ge j}}^{n} a_{ij} (x_i - x_i^0)(x_j - x_j^0), \tag{1}$$

where $\mathbf{x}^0$ is a reference point. Determining the approximating quadratic function is equivalent to determining all unknown coefficients in (1). Suppose that m ($m > n + 1$) base points, $\mathbf{x}^k$, $k = 0, 1, \cdots, m-1$, are to be used to construct the quadratic approximation. Using the function values at these points, $f_k = f(\mathbf{x}^k)$, we set up a system of linear equations

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}, \tag{2}$$

where the coefficient vectors $\mathbf{a}$ and $\mathbf{v}$ are arranged to have the following orders: $\mathbf{a} = [a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_n]^T$ and $\mathbf{v} = [a_{11} \quad a_{22} \quad \ldots \quad a_{nn}$

$a_{12}\ \ a_{13}\ \ \ldots\ \ a_{n-1\ n}]^T$. The vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ contain function values f_k with appropriate dimensions. The notation used here is similar to that of [3], but the function $q(\mathbf{x})$ has been defined w.r.t. a reference point $\mathbf{x}^0$ rather than w.r.t. the origin. Following [3], the model coefficients can be solved as

$$\mathbf{v} = \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}\mathbf{e} \tag{3}$$

and

$$\mathbf{a} = \mathbf{Q}_{11}^{-1}\mathbf{f}_1 - \mathbf{Q}_{11}^{-1}\mathbf{Q}_{12}\mathbf{v}, \tag{4}$$

where

$$\mathbf{C} = \mathbf{Q}_{22} - \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12} \tag{5}$$

and

$$\mathbf{e} = \mathbf{f}_2 - \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{f}_1. \tag{6}$$

When $m < (n+2)(n+1)/2$, equation (3) is a minimal norm $||\mathbf{v}||$ solution of an underdetermined system and $\mathbf{v}$ and $\mathbf{a}$ give a maximally flat quadratic approximate to $f(\mathbf{x})$.

<u>Direct Computation of Matrix **C**</u>

Here we propose that the first n+1 base points are selected by perturbing one variable at a time around the reference point $\mathbf{x}^0$, i.e.,

$$\mathbf{x}^i = \mathbf{x}^0 + [0\ 0\ \ldots\ 0\ \beta_i\ 0\ \ldots\ 0]^T, \quad i = 1, 2, \ldots, n, \tag{7}$$

where β_i is a certain length of perturbation. Matrix $\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12}$ is now simply

$$\mathbf{Q}_{11}^{-1}\mathbf{Q}_{12} = \left[\begin{array}{ccc|c} 0 & \ldots & 0 & \\ \beta_1 & & \mathbf{0} & \\ & \ddots & & \mathbf{0} \\ \mathbf{0} & & \beta_n & \end{array}\right] \tag{8}$$

and need not be stored. After these first (n + 1) base points, a sequence of other base points follows with no particular restriction about their locations. Each new base point results in the addition of a new row to **C**. Suppose $\mathbf{x}^k$ (k > n) is added as a new point. The components of the (k − n)th row of **C** are simply either

$$c_{k-n,(ii)} = (x_i^k - x_i^0)^2 - \beta_i(x_i^k - x_i^0) \tag{9a}$$

or

$$c_{k-n,(ij)} = (x_i^k - x_i^0)(x_j^k - x_j^0), \quad \text{for} \quad i \neq j, \tag{9b}$$

where the arrangement of the index pair (ij) in the new row of **C** corresponds to that of the a_{ij}'s in vector **v**, i.e., in the order (1 1), (2 2), ..., (n n), (1 2), (1 3), ..., (n−1 n). Using these formulas, some involved matrix-vector multiplications, required by the original approach of [3], are avoided.

Updating Model Coefficients upon Updating of Base Points

To efficiently calculate **v** after adding a new row in **C**, Biernacki and Styblinski [3] used a scheme to update the UL factors of $(\mathbf{CC}^T)^{-1}$. Their scheme does not accommodate deletion of old base points. We have derived a set of formulas to update matrix $(\mathbf{CC}^T)^{-1}$, allowing the freedom of adding or dropping any base point. Denote m′ as the number of base points created after the first n+1 ones. Suppose m′ equals half of the total number of second order coefficients. The operational count to obtain **v** for adding and deleting a base point is in the order of $6m'^2$ and $3m'^2$, respectively. When excluding the case of deleting, we use the original method of [3] to update the UL factors of $(\mathbf{CC}^T)^{-1}$. The operational count is then only in the order of $3m'^2$.

Direct Computation of Model Coefficients for Fixed Base Point Pattern

In our approach, m ($m \leq 2n + 1$) base points are used. The first n+1 ones are defined in (7) and the rest m′ ($m' = m - (n + 1)$) points are also selected deterministically. These m′ base points are selected by perturbing one variable at a time and, for the sake of simplicity, the variables are perturbed consecutively, that is

$$\mathbf{x}^{n+i} = \mathbf{x}^0 + [0\ 0 \ldots 0\ \gamma_i\ 0 \ldots\ 0]^T, \quad i = 1, 2, \ldots, m'. \qquad (10)$$

γ_i is the length of the perturbation, which must not equal β_i. It can be seen that these m′ base points are generated by perturbing the first m′ elements of $\mathbf{x}^0$ again. Under this arrangement, the matrix **C** takes the analytical form

$$\mathbf{C} = \left[\begin{array}{ccc|c} (\gamma_1-\beta_1)\gamma_1 & & \mathbf{0} & \\ & \cdot & & \\ & (\gamma_i-\beta_i)\gamma_i & & \mathbf{0} \\ & & \cdot & \\ \mathbf{0} & & (\gamma_{m'}-\beta_{m'})\gamma_{m'} & \end{array}\right] \qquad (11)$$

and the vector **e** can be expressed as

$$\mathbf{e} = \mathbf{f}_2 - \left[\begin{array}{cccccc|c} 1-\gamma_1/\beta_1 & \gamma_1/\beta_1 & & & & \mathbf{0} & \\ 1-\gamma_2/\beta_2 & & \gamma_2/\beta_2 & & & & \\ \cdot & & & \cdot & & & \\ 1-\gamma_i/\beta_i & & & \gamma_i/\beta_i & & & \mathbf{0} \\ \cdot & & & & \cdot & & \\ 1-\gamma_{m'}/\beta_{m'} & \mathbf{0} & & & & \gamma_{m'}/\beta_{m'} & \end{array}\right] \mathbf{f}_1 . \qquad (12)$$

From (3), (11) and (12) the coefficients are determined by

$$a_{ii} = [(f_{n+i}-f_0)/\gamma_i - (f_i-f_0)/\beta_i]/(\gamma_i-\beta_i), \quad i = 1,2,\ldots,m', \qquad (13a)$$

$$a_{ii} = 0, \quad i = m'+1, \ldots, n, \qquad (13b)$$

and

$$a_{ij} = 0, \quad i \neq j,\ i,j = 1, 2, \ldots, n. \qquad (13c)$$

Coefficients a_0 and a_i are easily obtained as,

$$a_0 = f_0, \qquad (14a)$$

$$a_i = (f_i - f_0)/\beta_i - \beta_i a_{ii}, \quad i = 1, 2, \ldots, n. \qquad (14b)$$

The operational count to calculate all coefficients by using this pattern can be merely n+3m′. In the above approach all the coefficients of mixed terms, i.e., a_{ij}, $i \neq j$, are conveniently forced to be zero because no related information can be extracted from the fixed base point pattern. Any of the a_{ii}'s in (13a), i.e., $i \leq m'$, can be nonzero because double perturbations are made along a straight line parallel to the ith axis. If a third perturbation is made along the same straight line, it can be seen that the **C** matrix will not have a full row rank, and, therefore, the third perturbation contains no useful information for the quadratic interpolation.

Flexible Arrangement of Nonzero Second Order Coefficients

If some of the mixed term coefficients a_{ij}, $i \neq j$, are dominant, the direct use of the above approach may result in a model with poorer accuracy. A transformation of the base points is introduced to enhance our approach.

Suppose that we have a set of base points, $\mathbf{y}^k$, k = 0, 1, ..., m, defined similarly to (7) and (10). Then a quadratic function can be determined in the Y-space, which is written in a compact form

$$q(\mathbf{y}) = b_0 + \mathbf{b}^T(\mathbf{y}-\mathbf{y}^0) + 1/2\ (\mathbf{y}-\mathbf{y}^0)^T\mathbf{B}(\mathbf{y}-\mathbf{y}^0), \tag{15}$$

where **b** consists of the coefficients of the first order terms, and **B** is a diagonal matrix with the diagonal elements equal to $2b_{ii}$, where b_{ii}'s are the coefficients of the second order terms. A transformation matrix, **P**, can be introduced such that

$$\mathbf{P}[\Delta\mathbf{x}^1\ \ \Delta\mathbf{x}^2\ \ldots\ \Delta\mathbf{x}^m] = [\Delta\mathbf{y}^1\ \ \Delta\mathbf{y}^2\ \ldots\ \Delta\mathbf{y}^m]. \tag{16}$$

Notice that the reference point is not changed, i.e., $\mathbf{x}^0 = \mathbf{y}^0$. Thereafter, q(**x**) is obtained by substituting $\mathbf{P}(\mathbf{x}-\mathbf{x}^0) = (\mathbf{y}-\mathbf{y}^0)$ in (15),

$$q(\mathbf{x}) = a_0 + \mathbf{a}^T(\mathbf{x}-\mathbf{x}^0) + 1/2(\mathbf{x}-\mathbf{x}^0)^T\mathbf{A}(\mathbf{x}-\mathbf{x}^0), \tag{17}$$

where $a_0 = b_0$, $\mathbf{a} = \mathbf{P}^T\mathbf{b}$ and $\mathbf{A} = \mathbf{P}^T\mathbf{B}\mathbf{P}$. It follows from the above that **A** may not be a diagonal matrix, i.e., some coefficients of mixed terms are nonzero, if **P** is properly chosen. As an example, consider the following matrix as $\mathbf{P}^{-1}$,

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ & & & & \ldots\ldots & & & & \\ 0 & 0 & 0 & \ldots & 1 & \ldots & -1 & \ldots & 0 \\ & & & & \ldots\ldots & & & & \\ 0 & 0 & 0 & \ldots & 1 & \ldots & 1 & \ldots & 0 \\ & & & & \ldots\ldots & & & & \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 1 \end{bmatrix}. \qquad (18)$$

This matrix rotates two axes such that the rotated pattern will give some information for the coefficients of related mixed terms. The above method can be generalized to deal with the case of more than two axes needed to be rotated.

Example

The statistical design of a ladder low-pass filter with 11 elements [5,6] is used as an example to demonstrate the usefulness of our method. The generalized ℓ_p centering approach to yield enhancement [7] is employed. 23 base points are located in a symmetric pattern, that is

$$\beta_i = -\gamma_i, \quad i = 1, 2, \ldots 11.$$

The details of the computational results and the comparisons are given in Table I. The process of yield maximization consists of two phases. $\mathbf{x}^1$ and $\mathbf{x}^2$, the nominal values of the solutions obtained by using exact circuit simulations, are given for the purpose of comparison. When the maximally flat interpolation approach is adopted, each phase uses a set of quadratic functions, which approximate the circuit constraints, to reach $\mathbf{x}^3$ and $\mathbf{x}^4$, respectively. The maximally flat interpolation approach is also used to estimate yields at $\mathbf{x}^0$, $\mathbf{x}^3$ and $\mathbf{x}^4$.

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TABLE I STATISTICAL DESIGN OF A LOW-PASS FILTER

Component x_i	Nominal Design $\mathbf{x}^0$	Actual Circuit Simulation Phase 1 $\mathbf{x}^1$	Actual Circuit Simulation Phase 2 $\mathbf{x}^2$	Maximally Flat Interpolation Phase 1 $\mathbf{x}^3$	Maximally Flat Interpolation Phase 2 $\mathbf{x}^4$
x_1	0.22510	0.22668	0.22511	0.22616	0.22623
x_2	0.24940	0.24842	0.25050	0.24863	0.24920
x_3	0.25230	0.25317	0.25023	0.25201	0.25186
x_4	0.24940	0.24782	0.24982	0.24728	0.24791
x_5	0.22510	0.22612	0.22715	0.22490	0.22487
x_6	0.21490	0.21648	0.22000	0.21926	0.21916
x_7	0.36360	0.36273	0.35944	0.36206	0.36253
x_8	0.37610	0.37693	0.37624	0.37748	0.37763
x_9	0.37610	0.37548	0.37760	0.37582	0.37599
x_{10}	0.36360	0.36273	0.36003	0.36251	0.36299
x_{11}	0.21490	0.21648	0.22000	0.21982	0.21973
Yield*	75.33%	91.67%	94.00%	89.00%	93.00%
Yield+	77.33%			93.33%	
Yield++				91.33%	96.33%
Yield+++					95.67%
No. of samples used for design		50	100	50	100
Starting point		$\mathbf{x}^0$	$\mathbf{x}^1$	$\mathbf{x}^0$	$\mathbf{x}^3$
Number of simulations		600	1700	23	23
Number of iterations		12	17	7	5

Tolerances: 1% with independent uniform distributions for each element.
Yields: All yield estimates are based on 300 samples.

*: The yield is estimated by using actual simulations.
+, ++, and +++: The yield is estimated by using maximally flat interpolation with the reference point $\mathbf{x}^0$, $\mathbf{x}^3$, or $\mathbf{x}^4$, respectively.