LARGE CHANGE SENSITIVITY ANALYSIS IN LINEAR SYSTEMS USING GENERALIZED HOUSEHOLDER FORMULAS

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Abstract

This paper investigates multiparameter large change sensitivity problems in linear systems by a set of generalized Householder formulas. The newly developed Rectangular Formulas can accommodate large, small and zero parameter changes directly by avoiding a critical matrix inversion as compared to the traditional Square Formulas. Possible determination of a minimum order reduced system, whose solution procedure constitutes the major work in large change evaluation is discussed. Applications to linear systems are considered for the original and adjoint systems w.r.t. single as well as multiple input-output cases. This approach makes it possible to use large change analysis algorithms even if many parameters are changed.

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I. INTRODUCTION

In computer aided circuit design, it is often required to calculate network responses after a certain set of parameters are changed. This problem, referred to as large change sensitivity analysis, has been studied by many people. Fidler [1] and Singhal et al. [2], considered single and multiple parameter changes, respectively, and developed methods to calculate the response function as a multilinear form in variable parameters. Another method is to formulate a reduced system, whose solutions are then used to update the responses. This method has been treated from different angles, e.g., the current source substitution approach of Leung and Spence [3], the adjoint network approach of Temes et al. [4], the Householder formula approach [3,5], the scattering matrix approach of Haley [6-7] and the matrix partitioning approach of Vlach and Singhal [8]. Hajj has derived and summarized a set of algorithms where finite, infinite and zero parameter changes are all permitted and sparsity is exploited [5]. A recent overview of this area is given by Haley and Current who presented general approaches encompassing most of the previous methods [7].

As already noticed [3,5], large change analysis algorithms will lose efficiency when too many parameters are changed. This is mainly because the algorithms involve the solution of a reduced system of order n_Φ, the number of variables. However, cases exist where this system is larger than needed. Also, in a Monte-Carlo analysis or in an optimization procedure, it is possible that some variables change slightly while others change substantially. In this case, the small parameter changes may cause ill-conditioning in a non-iterative method [3-5] and the large parameter changes may affect the convergence rate in an iterative method [5].

In this paper, we present a set of generalized Householder formulas which is capable of handling complicated cases encountered in practice. The problem of determining a minimum reduced system is investigated. Different aspects of the basic set of formulas are discussed in terms of duality property and operational count. Applications to general linear systems are considered for original and adjoint responses with single and multiple input and output situations. Also, as a special case, a series of first-order sensitivity expressions are obtained without reference to Tellegen's theorem. Numerical examples are given for a general system of linear equations and for an arbitrary 10 node electrical circuit.

II. A SET OF GENERALIZED HOUSEHOLDER FORMULAS

Let the linear system be characterized by an n x n matrix $\mathbf{A}$. Suppose the parameters $\boldsymbol{\phi}$ of the system are changed by $\Delta\boldsymbol{\phi}$. The system matrix $\mathbf{A}$ will then be affected by $\Delta\mathbf{A}$. We can express

$$\Delta\mathbf{A} = \mathbf{V}\,\mathbf{D}\,\mathbf{W}^T, \tag{1}$$

where $\mathbf{V}$, $\mathbf{D}$ and $\mathbf{W}$ are $n \times r_1$, $r_1 \times r_2$ and $n \times r_2$ matrices, respectively. For a network example, $\mathbf{D}$ can be an $n_\phi \times n_\phi$ diagonal matrix containing variables and $\mathbf{V}$ and $\mathbf{W}$ are $n \times n_\phi$ matrices containing +1 and −1 [5,8].

The effect of $\Delta\boldsymbol{\phi}$ in the response matrix $\mathbf{A}^{-1}$ is defined as

$$\Delta(\mathbf{A}^{-1}) \triangleq (\mathbf{A} + \Delta\mathbf{A})^{-1} - \mathbf{A}^{-1}\,. \tag{2}$$

For the calculation of $\Delta(\mathbf{A}^{-1})$, commonly suggested is the Householder formula [9], which can be represented by

$$\Delta(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}\,\mathbf{V}(\mathbf{D}^{-1} + \mathbf{W}^T\,\mathbf{A}^{-1}\,\mathbf{V})^{-1}\,\mathbf{W}^T\,\mathbf{A}^{-1}\,. \tag{3}$$

In this formula, $\mathbf{D}$ is required to be a square and non-singular matrix. Even if this can be satisfied, ill-conditioning may still happen when $\mathbf{D}$ is inverted. In fact, cases exist where $\mathbf{D}$ is simply not invertible and additional measures such as the partitioning procedures developed by Hajj [5], Vlach and Singhal [8] must be applied. Another formula by Householder is [10]

$$\Delta(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}\,\mathbf{V}\mathbf{D}(\mathbf{D} + \mathbf{D}\mathbf{W}^T\,\mathbf{A}^{-1}\,\mathbf{V}\mathbf{D})^{-1}\,\mathbf{D}\mathbf{W}^T\,\mathbf{A}^{-1}\,. \tag{4}$$

This formula avoids actually performing the inversion of $\mathbf{D}$. But it still has the same limitation as that of (3).

According to the formulation of $\mathbf{D}$, we refer to (3) as Square with Inversion Formula (SIF) and (4) as Square without Inversion Formula (SF).

To alleviate the limitations, corresponding formulas can be derived as [11]

$$\Delta(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}\,\mathbf{V}\mathbf{D}(\mathbf{1} + \mathbf{W}^{T}\,\mathbf{A}^{-1}\,\mathbf{V}\mathbf{D})^{-1}\,\mathbf{W}^{T}\,\mathbf{A}^{-1} \tag{5}$$

and

$$\Delta(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}\,\mathbf{V}(\mathbf{1} + \mathbf{D}\mathbf{W}^{T}\,\mathbf{A}^{-1}\,\mathbf{V})^{-1}\,\mathbf{D}\mathbf{W}^{T}\,\mathbf{A}^{-1}\,. \tag{6}$$

These two formulas permit **D** to be singular or even rectangular. Thus, more freedom can be exploited using different formulations of **D** and ill-conditioning can be avoided.

The reduced systems in (5) and (6) are the order of r_2 and r_1, respectively, where r_1 is the number of rows of **D** and r_2 is the number of columns of **D**. Therefore, (5) may be preferred if $r_1 > r_2$, otherwise (6) should be used. It is reasonable to refer to (5) as Vertical Rectangular Formula (VRF) and (6) as Horizontal Rectangular Formula (HRF), respectively, reflecting the form of **D**.

The case of a rectangular **D** may occur, e.g., when we construct a minimum order reduced system involving variables that are active element parameters, and when large change algorithms are applied to algebraic linear systems other than electrical networks [11]. In those cases, the rectangular **D** may be used in VRF and HRF without modification leaving **V** and **W** free of values $\Delta\boldsymbol{\phi}$. Hence, **V** and **W** need to be preprocessed only once.

It should be noted that mathematically, the Square Formulas are special cases of the Rectangular ones. Computationally, the latter have good stability.

III. PROPERTIES OF THE SET OF GENERALIZED HOUSEHOLDER FORMULAS

Duality Property

The HRF and the VRF can be considered as dual to each other. If we apply the following interchanges

$$\mathbf{A} \leftrightarrow \mathbf{A}^{T}\,, \tag{7}$$

$$\mathbf{D} \leftrightarrow \mathbf{D}^{T} \tag{8}$$

and

$$\mathbf{V} \leftrightarrow \mathbf{W}\,, \tag{9}$$

then the two formulas, i.e. (5) and (6), are completely interchanged.

This duality property can be employed to save our analytical effort by half. Unless otherwise stated, we will focus on the Vertical Formula in the ensuing sections. Results for the Horizontal ones can be similarly obtained.

The Minimum Order of the Reduced System

Using the scattering theory approach, Haley has found that the order of the reduced system can be as low as rank ($\Delta \mathbf{A}$). Using our approach of only simple matrix manipulations one can also verify that [11]

$$\min_{(\mathbf{V}, \mathbf{D}, \mathbf{W})} r_1 = \min_{(\mathbf{V}, \mathbf{D}, \mathbf{W})} r_2 = \operatorname{rank}(\Delta \mathbf{A}) . \tag{10}$$

This equation yields the conclusion that, for evaluating large change effects involving Householder formulas, the minimum order of the reduced system is equal to the rank of $\Delta \mathbf{A}$.

Consider the circuit of Fig. 1 in which 7 parameters are changed from their nominal values. By the methods of [3-5,8], the reduced system is 7 x 7. However, the rank of the nodal admittance deviation matrix is 4. Thus, an even smaller system of size 4 x 4 is sufficient for this problem.

Operational Count

Consider the computation of $\Delta(\mathbf{A}^{-1})$. Suppose $r_1 + r_2 < n$ and the matrix $\mathbf{A}$ has already been LU factorized. Usually, **V**, **D** and **W** are formulated such that **D** contains variables and **V** and **W** indicate the positions of the variables and are constant. Preparatory calculations involving **V** and **W** are performed only once for each set of variables. Table I gives operational counts (number of operations, i.e., multiplications or divisions) for the set of generalized Householder formulas. As shown in the table, the computational stability of the HRF and the VRF is achieved at the cost of one more matrix multiplication, as compared with the SIF. It should be noticed that these operation counts are for arbitrary algebraic linear equations. When linear circuits are concerned, the operational count is reduced as discussed in Section IV.

IV. COMPUTATIONS OF ORIGINAL AND ADJOINT LINEAR SYSTEM RESPONSES CORRESPONDING TO DIFFERENT NUMBERS OF INPUTS AND OUTPUTS

In this section, we examine the computations of large change sensitivities in different input and output cases. The VRF is applied. All results of Forward and Backward Substitutions (FBS) involving $\mathbf{A}$ are calculated in the preparatory step and are represented by $\mathbf{P}$ and $\mathbf{p}$ for the original system (coefficient matrix $\mathbf{A}$) and by $\mathbf{Q}$ and $\mathbf{q}$ for the adjoint system (coefficient matrix $\mathbf{A}^T$). To distinguish these solutions for different R.H.S., we use the characters, similar to the R.H.S., as subscript. For example, $\mathbf{P}_V$ is the solution of

$$\mathbf{A}\mathbf{P}_V = \mathbf{V} \tag{11}$$

and $\mathbf{q}_b$ is the solution of

$$\mathbf{A}^T \mathbf{q}_b = \mathbf{b} \,. \tag{12}$$

<u>Case 1: Response Matrix $\mathbf{A}^{-1}$</u>

$$\Delta(\mathbf{A}^{-1}) = -\,\mathbf{P}_V\, \mathbf{D}\mathbf{S}\mathbf{Q}_W^{\,T} , \tag{13}$$

where $\mathbf{S}$ is the inverse of $(\mathbf{1} + \mathbf{W}^T \mathbf{P}_V\, \mathbf{D})$.

<u>Case 2: System Responses for a Single Excitation Vector $\mathbf{c}$</u>

Suppose the response vector corresponding to excitation $\mathbf{c}$ is $\mathbf{x} = [x_1\ x_2\ \ldots\ x_n]^T$, i.e.,

$$\mathbf{A}\mathbf{x} = \mathbf{c} \,. \tag{14}$$

We have

$$\Delta\mathbf{x} = \Delta(\mathbf{A}^{-1}\,\mathbf{c})$$

$$= -\,\mathbf{P}_V\, \mathbf{D}\mathbf{s} , \tag{15}$$

where $\mathbf{s}$ is the solution of

$$(\mathbf{1} + \mathbf{W}^T\, \mathbf{P}_V \mathbf{D})\mathbf{s} = \mathbf{W}^T\, \mathbf{x} \,. \tag{16}$$

Case 3: Adjoint Responses for a Single Excitation Vector **b**

Suppose the adjoint response vector corresponding to excitation **b** is $\mathbf{y} = [y_1\ y_2 \ldots y_n]^T$, i.e.,

$$\mathbf{A}^T \mathbf{y} = \mathbf{b} . \tag{17}$$

We have

$$\begin{aligned} \Delta \mathbf{y}^T &= \Delta(\mathbf{b}^T \mathbf{A}^{-1}) \\ &= -\mathbf{s}'^T \mathbf{Q}_W^T , \end{aligned} \tag{18}$$

where $\mathbf{s}'$ is the solution of

$$(\mathbf{1} + \mathbf{Q}_W{}^T \mathbf{VD})^T \mathbf{s}' = \mathbf{D}^T \mathbf{V}^T \mathbf{q}_b . \tag{19}$$

Case 4: Response of Single-Input and Single-Output (SISO) System

If we use vector **b** to select the desired output from response vector **x**, then

$$\begin{aligned} \Delta(\mathbf{b}^T \mathbf{x}) &= \Delta(\mathbf{b}^T \mathbf{A}^{-1} \mathbf{c}) \\ &= -\mathbf{b}_1^T \mathbf{D} \mathbf{s} , \end{aligned} \tag{20}$$

where **s** is defined in (16) and $\mathbf{b}_1$ equals $\mathbf{P}_V{}^T \mathbf{b}$ and is obtained in the preparatory step.

Case 5: Responses of Multi-Input and Multi-Output (MIMO) System

Suppose **C** is an n x n′ matrix whose columns represent different excitation vectors and **B** is an n x m′ matrix whose columns select the desired output measurements. Then the n′ - input m′ - output case can be expressed, formally, by $\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C}$. Thus

$$\Delta(\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C}) = -\mathbf{B}^T \mathbf{A}^{-1} \mathbf{VD}(\mathbf{1} + \mathbf{W}^T \mathbf{A}^{-1} \mathbf{VD})^{-1} \mathbf{W}^T \mathbf{A}^{-1} \mathbf{C} . \tag{21}$$

We notice that the term $\mathbf{B}^T\mathbf{A}^{-1}\mathbf{V}$ can be computed either as $\mathbf{B}^T \mathbf{P}_V$ or $\mathbf{Q}_B{}^T\mathbf{V}$ with a difference of operational count as $n^2 (r_1 - m')$. Therefore, comparing r_1 and m', we can calculate $(\mathbf{B}^T \mathbf{A}^{-1} \mathbf{V})$ as

$$\mathbf{B}^T \mathbf{A}^{-1} \mathbf{V} = \begin{cases} \mathbf{B}^T \mathbf{P}_V , & \text{if } r_1 \le m' \qquad (22a) \\ \mathbf{Q}_B^T \mathbf{V} , & \text{if } r_1 > m'. \qquad (22b) \end{cases}$$

Similarly,

$$\mathbf{W}^T \mathbf{A}^{-1} \mathbf{C} = \begin{cases} \mathbf{W}^T \mathbf{P}_C, & \text{if } r_2 > n' \quad (23a) \\ \mathbf{Q}_W^T \mathbf{C}, & \text{if } r_2 \leq n'. \quad (23b) \end{cases}$$

Also, at least one of (22a) and (23b) should be used in order to yield either $\mathbf{P}_V$ or $\mathbf{Q}_W$ which is required in calculating

$$(\mathbf{1} + \mathbf{W}^T \mathbf{A}^{-1} \mathbf{VD}) = (\mathbf{1} + \mathbf{Q}_W{}^T \mathbf{VD})$$
$$= (\mathbf{1} + \mathbf{W}^T \mathbf{P}_V \mathbf{D}) . \quad (24)$$

Hence, according to the values of r_1, r_2, m' and n', we can choose appropriate formulations. For example, when $m' < n'$ and $m' < r_2$, we use

$$\Delta(\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C}) = -\mathbf{S}^T \mathbf{Q}_W{}^T \mathbf{C}, \quad (25)$$

where $\mathbf{S}$ is the solution to

$$(\mathbf{1} + \mathbf{Q}_W{}^T \mathbf{VD})^T \mathbf{S} = (\mathbf{Q}_B{}^T \mathbf{VD})^T . \quad (26)$$

This approach requires $m' + r_2$ FBS in the adjoint system for $\mathbf{Q}_B$ and $\mathbf{Q}_W$ as preparatory calculations, one LU factorization and m' FBS in the reduced system of (26).

Expressions for Different Cases of Large Change Evaluation

In Table II, we summarize the various cases of the above discussion. Different situations of the MIMO case are distinguished so that the number of FBS in the n x n system equals the minimum of $m' + r_2$, $n' + r_1$ and $r_1 + r_2$ and the number of FBS in the reduced system equals the minimum of r_1, r_2, m' and n', as shown in Table III. This minimum FBS criterion can be used as a guide to select appropriate expressions for the calculation of $\Delta(\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C})$.

When the number of FBS exceeds the order of the system, a matrix inversion may be directly performed.

Computational Cost Consideration

In Section III, the operational count has been discussed for a general linear system of equations. However, when an electric circuit is concerned, the cost is much less. We consider the SISO network as an example. Suppose the reduced system is of order r. In the preparatory step, we calculate $\mathbf{P}_V$ whose operational count is rn^2 and $\mathbf{P}^T{}_V\,\mathbf{b}$, $\mathbf{W}^T\,\mathbf{P}_V$ and $\mathbf{W}^T\,\mathbf{x}$ which are simply element selections and additions. Then, for each set of parameter changes, we formulate and solve the reduced system by at worst $4r^3/3 - r/3 + r^2$ operations. The operational count for updating the output is r for the SIF and $r + r^2$ for the HRF and the VRF.

Special Case: First-Order Sensitivity

As a special case of large change sensitivity analysis, small change sensitivity computations can be deduced from our large change formulas without reference to Tellegen's theorem. Table IV gives examples of such first-order sensitivities w.r.t. components of a matrix. Table V lists formulas w.r.t. variables. These results are obtained by putting $\Delta\phi$ into the denominator of large change formulas and then letting the parameter change $\Delta\phi$ approach zero. The formulas in Tables IV and V are consistent with the existing ones derived using other approaches, e.g. [12].

V. EXAMPLES

Example 1: A System of Linear Equations With Rectangular **D**

Consider a 10 x 10 system of linear equations with coefficient matrix as **A**. Supose the intersection elements of rows 2,5,9 and columns 3 and 6 are constantly changed. We formulate **V**, **D** and **W** such that

$$\mathbf{V} = [\,\mathbf{u}_2\ \ \mathbf{u}_5\ \ \mathbf{u}_9\,]\,, \tag{27}$$

$$\mathbf{W} = [\,\mathbf{u}_3\ \ \mathbf{u}_6] \tag{28}$$

and

$$\mathbf{D} = \begin{bmatrix} \Delta A_{23} & \Delta A_{26} \\ \Delta A_{53} & \Delta A_{56} \\ \Delta A_{93} & \Delta A_{96} \end{bmatrix} , \tag{29}$$

where $\mathbf{u}_i$, $i=2,3,5,6,9$, is a unit 10-vector containing 1 in the ith row and zeros everywhere else. In this way, no additional effort is involved when applying the VRF and HRF. If we use the Square Formulas, elementary transformations must be employed in order to obtain a square matrix **D**.

Numerical solutions as well as intermediate results are shown in Fig. 2.

Example 2: An Electrical Network with Its Minimum Order System Achieved

The 10-node circuit of Fig. 1 is solved using the generalized Householder formulas with simultaneous changes of 7 variable components. The minimum order of the reduced system is 4, which is achieved by formulating **V**, **D** and **W** as

$$\mathbf{V} = \mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

and

$$\mathbf{D} = \begin{bmatrix} \Delta\phi_1+\Delta\phi_2+\Delta\phi_4 & -\Delta\phi_4 & -\Delta\phi_2 & 0 \\ -\Delta\phi_4 & \Delta\phi_4+\Delta\phi_5+\Delta\phi_6 & -\Delta\phi_5 & 0 \\ -\Delta\phi_2 & -\Delta\phi_5 & \Delta\phi_2+\Delta\phi_3+\Delta\phi_5 & 0 \\ 0 & 0 & 0 & \Delta\phi_7 \end{bmatrix} . \tag{31}$$

The -1's in **V** and **W** correspond to reference nodes associated with the variables. If a loop or several connected loops are formulated by the variable branches, a common reference node is appointed for all the variables contributed to the loop or loops. For example, node 9 in Fig. 1 is chosen as the common reference node for variables ϕ_1, ϕ_2, ..., ϕ_6 and -1 appears in the 9th row of **V** accordingly. The 1's in **V** and **W** correspond to the non-reference nodes associated with the variable branches, e.g., nodes 3, 4, and 8 in Fig. 1. With respect to each reference node, a submatrix is formulated using $\Delta\phi$ in just the same way as if a nodal admittance matrix is formulated using ϕ w.r.t. a ground node. **D** is a block diagonal matrix containing those submatrices.

The changes of variables range from 0.00001 to 90. Zero changes are also included as shown in Table VI. These simultaneous small, large and zero changes are handled directly by the VRF [11]. For the two extreme cases of $\Delta\phi$, the SIF can handle $\Delta\phi\rightarrow\infty$ while the VRF and HRF accommodate $\Delta\phi\rightarrow 0$. In a Monte-Carlo analysis, network optimization, identification and tuning, various unpredictable patterns of $\Delta\phi\rightarrow 0$ in multiparameter changes may be possible while $\Delta\phi\rightarrow\infty$ is often limited by, e.g., tolerances and tuning ranges or by step size constraints. For 100 sets of variable changes of ϕ_1 to ϕ_7, the operational count for our method using SIF, VRF, the existing method of [3-5,8] and the direct method are in the order of 8430, 10030, 27730 and 43430, respectively.

VI. CONCLUSIONS

We have presented a multiparameter large change sensitivity analysis approach for a general system involving solutions of linear equations. Particular attention has been devoted to the formulation and order of the reduced system, which in turn affects the stability and efficiency of the system response evaluation. The mathematical essence of the generalized Householder formulas also provides basic links with other approaches, indicating their theoretical equivalence. However, our extended formulas accommodate more cases of various formulations of the reduced system which the traditional methods cannot handle directly.

For a general circuit with arbitrary distribution of variable components, proper formulations of **V**, **D** and **W** are possible to ensure the large change calculation to be performed via a minimum order reduced system. Thus, under certain circumstances, large change algorithms are still feasible even if many system parameters are changed. These circumstances may be, for example, a case where loops are constructed by branches containing variables. It is also possible that a general formulation of **V**, **D** and **W**, together with the set of Householder formulas, can be embedded into the different iterative and non-iterative methods of Hajj [5] to yield various powerful design procedures.

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TABLE I

OPERATIONAL COUNT FOR THE GENERALIZED HOUSEHOLDER FORMULAS

Cases	Square with Inversion Formula (SIF)	Vertical Rectangular Formula (VRF)	Horizontal Rectangular Formula (HRF)	Square without Inversion Formula (SF)
<u>Case 1</u>				
$r_1 \neq r_2$				
preparatory calculation	–	C_P	C_P	–
calculation for each set of parameter changes	–	C_2	C_1	–
<u>Case 2</u>				
$r_1 = r_2 = r$				
preparatory calculation	C_P	C_P	C_P	C_P
calculation for each set of parameter changes	$2C_A + C_B$	$3C_A + C_B$	$3C_A + C_B$	$5C_A + C_B$

$C_P = n^2(r_1 + r_2) + nr_1r_2$

$C_1 = r_1(2r_1r_2 + r_1{}^2 + r_2n + n^2)$, $C_2 = r_2(2r_1r_2 + r_2{}^2 + r_1n + n^2)$

$C_A = r^3$, $C_B = rn(r + n)$

TABLE II

FORMULAS FOR THE COMPUTATION OF LARGE CHANGES WHEN $\mathbf{A}^{-1}$ IS INVOLVED AND WHEN $r_1 \geq r_2$

	Identification		Formula	Definition of **S** or **s**
	$\Delta(\mathbf{A}^{-1})$		$-\mathbf{P}_V \mathbf{D} \mathbf{S} \mathbf{Q}_W{}^T$	$\mathbf{H}_1 \mathbf{S} = \mathbf{1}$ or $\mathbf{H}_2 \mathbf{S} = \mathbf{1}$
	$\Delta(\mathbf{b}^T \mathbf{A}^{-1})$		$-\mathbf{s}^T \mathbf{Q}_W{}^T$	$\mathbf{H}_1{}^T \mathbf{s} = \mathbf{D}^T(\mathbf{V}^T \mathbf{q}_b)$
	$\Delta(\mathbf{A}^{-1} \mathbf{c})$		$-\mathbf{P}_V \mathbf{D} \mathbf{s}$	$\mathbf{H}_2 \mathbf{s} = \mathbf{W}^T \mathbf{p}_c$
	$\Delta(\mathbf{b}^T \mathbf{A}^{-1} \mathbf{c})$		$-(\mathbf{b}^T \mathbf{P}_V) \mathbf{D} \mathbf{s}$	$\mathbf{H}_2 \mathbf{s} = \mathbf{W}^T \mathbf{p}_c$
†	$\Delta(\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C})$	(1)	$-\mathbf{S}^T(\mathbf{Q}_W{}^T \mathbf{C})$	$\mathbf{H}_1{}^T \mathbf{S} = \mathbf{D}^T(\mathbf{V}^T \mathbf{Q}_B)$
		(2)	$-(\mathbf{B}^T \mathbf{P}_V)\mathbf{D} \mathbf{S}$	$\mathbf{H}_2 \mathbf{S} = \mathbf{W}^T \mathbf{P}_C$
		(3)	$-(\mathbf{Q}_B{}^T \mathbf{V})\mathbf{D} \mathbf{S}$	$\mathbf{H}_1 \mathbf{S} = \mathbf{Q}_W{}^T \mathbf{C}$
		(4)	$-(\mathbf{B}^T \mathbf{P}_V)\mathbf{D} \mathbf{S}(\mathbf{Q}_W{}^T \mathbf{C})$	$\mathbf{H}_1 \mathbf{S} = \mathbf{1}$ or $\mathbf{H}_2 \mathbf{S} = \mathbf{1}$
		(5)	$-(\mathbf{Q}_B{}^T \mathbf{V})\mathbf{D} \mathbf{S}(\mathbf{Q}_W{}^T \mathbf{C})$	$\mathbf{H}_1 \mathbf{S} = \mathbf{1}$

where $\mathbf{H}_1 = (\mathbf{1} + \mathbf{Q}_W{}^T \mathbf{V} \mathbf{D})$, $\mathbf{H}_2 = (\mathbf{1} + \mathbf{W}^T \mathbf{P}_V \mathbf{D})$

† Table III can be used as a guide to select among (1) to (5) by the minimum FBS criterion.

TABLE III

MAJOR COMPUTATIONAL EFFORT FOR CALCULATING $\Delta(\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C})$

BY FORMULAS IN TABLE II WHERE $r_1 \geq r_2$

Category	Corresponding Case in Table II	The n x n System Represented By **A**	The r_2 x r_2 System Represented By $\mathbf{H}_1$ or $\mathbf{H}_2$
No. of LU Factorizations	(1) - (5)	1	1
No. of FBS	(1)	$m' + r_2$	m'
	(2)	$n' + r_1$	n'
	(3)	$m' + r_2$	n'
	(4)	$r_1 + r_2$	r_2
	(5)	$m' + r_2$	r_2

TABLE IV

EXPRESSIONS APPROPRIATE FOR COMPUTATIONS FOR SENSITIVITIES W.R.T. COMPONENTS OF MATRIX A WHEN $\mathbf{A}^{-1}$ IS INVOLVED

Identification	Sensitivity Expression	
	(a) General	(b) when $\mathbf{A} = \mathbf{A}^T$ and $i \neq j$
$\dfrac{\partial \mathbf{A}^{-1}}{\partial A_{ij}}$	$-\mathbf{p}_{ui}\,\mathbf{q}_{uj}^T$	$-(\mathbf{p}_{ui}\,\mathbf{p}_{uj}^T + \mathbf{p}_{uj}\,\mathbf{p}_{ui}^T)$
$\dfrac{\partial(\mathbf{b}^T\mathbf{A}^{-1}\mathbf{c})}{\partial \mathbf{A}}$	$-\mathbf{q}_b\,\mathbf{p}_c^T$	$-(\mathbf{p}_b\,\mathbf{p}_c^T + \mathbf{p}_c\,\mathbf{p}_b^T)$
$\dfrac{\partial(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{C})}{\partial A_{ij}}$	$-\mathbf{B}^T\mathbf{p}_{ui}\,\mathbf{q}_{uj}^T\mathbf{C}$	$-\mathbf{B}^T(\mathbf{p}_{ui}\,\mathbf{p}_{uj}^T + \mathbf{p}_{uj}\,\mathbf{p}_{ui}^T)\mathbf{C}$
$\dfrac{\partial[\mathbf{B}^T\mathbf{A}^{-1}\mathbf{C}]_{\ell k}}{\partial \mathbf{A}}$ †	$-\mathbf{q}_b\,\mathbf{p}_c^T$	$-(\mathbf{p}_b\,\mathbf{p}_c^T + \mathbf{p}_c\,\mathbf{p}_b^T)$††

$\mathbf{u}_i$ ($\mathbf{u}_j$) is a unit n-vector containing **1** at the ith (jth) row and zeros everywhere else.

† where $[*]_{\ell k}$ is the (ℓ,k) th element of matrix *.

†† where **b** is the ℓth column of **B** and **c** is the kth column of **C**. Both **b** and **c** are used as the R.H.S. of the system involving **A** for original solutions $\mathbf{p}_b$, $\mathbf{p}_c$ and adjoint solution $\mathbf{q}_b$.

TABLE V

EXPRESSIONS APPROPRIATE FOR COMPUTATION OF SENSITIVITIES W.R.T. VARIABLE ϕ WHEN $\mathbf{A}^{-1}$ IS INVOLVED

Identification	Sensitivity Expression
$\dfrac{\partial \mathbf{A}^{-1}}{\partial \phi}$	$-\mathbf{P}_{UI} \dfrac{\partial \mathbf{A}_{IJ}}{\partial \phi} \mathbf{Q}_{UJ}^T$
$\dfrac{\partial (\mathbf{b}^T \mathbf{A}^{-1})}{\partial \phi}$	$-(\mathbf{q}_b^T)_I \dfrac{\partial \mathbf{A}_{IJ}}{\partial \phi} \mathbf{Q}_{UJ}^T$
$\dfrac{\partial (\mathbf{A}^{-1} \mathbf{c})}{\partial \phi}$	$-\mathbf{P}_{UI} \dfrac{\partial \mathbf{A}_{IJ}}{\partial \phi} (\mathbf{p}_c)_J$
$\dfrac{\partial (\mathbf{b}^T \mathbf{A}^{-1} \mathbf{c})}{\partial \phi}$	$-(\mathbf{q}_b^T)_I \dfrac{\partial \mathbf{A}_{IJ}}{\partial \phi} (\mathbf{p}_c)_J$
$\dfrac{\partial (\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C})}{\partial \phi}$	$-(*) \dfrac{\partial \mathbf{A}_{IJ}}{\partial \phi} (\dagger)$

I (J) is an index set whose elements indicate the rows (columns) containing the variable ϕ.

$\mathbf{A}_{IJ}$ is a matrix containing the intersection elements of $\mathbf{A}$ in rows i, $i \in I$ and columns j, $j \in J$.

$\mathbf{U}_I$ ($\mathbf{U}_J$) is a matrix whose columns are unit vectors $\mathbf{u}_i$, $i \in I$ ($\mathbf{u}_j$, $j \in J$).

$$(*) = \begin{cases} \mathbf{B}^T \mathbf{P}_{UI} & \text{if } n_I < m' \\ \mathbf{Q}_B^T \mathbf{U}_I & \text{if } n_I \geq m' \end{cases}, \qquad (\dagger) = \begin{cases} \mathbf{Q}_{UJ}^T \mathbf{C} & \text{if } n_J < n' \\ \mathbf{U}_J^T \mathbf{P}_C & \text{if } n_J \geq n' \end{cases}$$

$(\mathbf{q}_b)_I$ and $(\mathbf{p}_c)_J$ are defined as vectors consisting of all ith elements of $\mathbf{q}_b$, $i \in I$, and all jth elements of $\mathbf{p}_c$, $j \in J$, respectively.

TABLE VI

PARAMETER CHANGES FOR EXAMPLE 2

Variable	The First Change $(1/\Omega)$	The Second Change $(1/\Omega)$	The Third Change $(1/\Omega)$
$\Delta\phi_1$	84.0	0.00001	0.2
$\Delta\phi_2$	0.5	0.001	0
$\Delta\phi_3$	0.00001	0.12	3.0
$\Delta\phi_4$	0.02	45.	0
$\Delta\phi_5$	40.	0.00003	0.02
$\Delta\phi_6$	50.	90.	15
$\Delta\phi_7$	0.00002	–2.	0.1

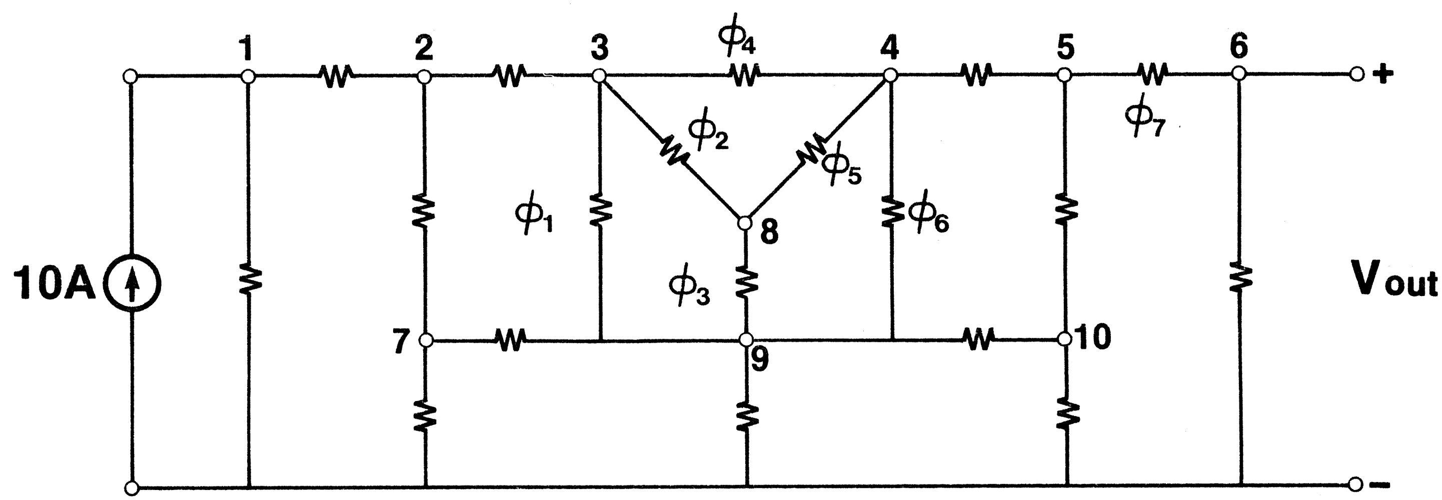

Fig. 1 An arbitrary 10 node network with 7 variable parameters. All element values are assumed as 1. Variables ϕ_1, ϕ_2, ..., ϕ_7 are conductances of the associated components.

MATRIX [A]										VECTOR [B]
1.0	5.0	5.0	1.0	5.0	2.0	1.0	1.0	7.0	2.0	35.0
2.0	3.0	3.0	7.0	0.0	4.0	3.0	6.0	8.0	3.0	32.0
3.0	0.0	2.0	4.0	2.0	6.0	4.0	4.0	9.0	7.0	16.0
6.0	1.0	2.0	5.0	2.0	3.0	3.0	7.0	3.0	5.0	51.0
8.0	1.0	2.0	2.0	4.0	4.0	6.0	8.0	4.0	8.0	42.0
4.0	1.0	6.0	7.0	3.0	5.0	7.0	3.0	5.0	3.0	19.0
7.0	0.0	6.0	5.0	9.0	4.0	8.0	9.0	2.0	9.0	34.0
2.0	0.0	4.0	2.0	2.0	5.0	3.0	5.0	4.0	3.0	71.0
3.0	2.0	0.0	1.0	5.0	3.0	4.0	2.0	3.0	1.0	36.0
4.0	2.0	4.0	4.0	6.0	2.0	9.0	6.0	1.0	7.0	61.0

SOLUTION BEFORE ANY CHANGE :

VECTOR [X]
-8.89217
39.80097
-3.00067
2.31014
-5.40544
48.42778
-12.11626
-3.61726
-32.93004
16.99799

Fig. 2(a) The original linear system and its solutions. **A** is a 10x10 matrix containing parameters of the system. **b** is the excitation vector. **x** is the solution vector.

MATRIX [V]

0.0	0.0	0.0
1.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	1.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	1.0
0.0	0.0	0.0

MATRIX [W]

0.0	0.0
0.0	0.0
1.0	0.0
0.0	0.0
0.0	0.0
0.0	1.0
0.0	0.0
0.0	0.0
0.0	0.0
0.0	0.0

MATRIX [PV]

-.03684	.19072	-.00936
-.30799	-.26526	.04838
.00454	.09406	-.15865
-.04645	-.23600	.02949
.02608	-.09579	.12002
-.48948	-.43199	.13012
.18919	.22984	.01487
.27060	.13754	.00846
.36321	.32717	-.02238
-.27658	-.20670	-.09789

VECTOR [RHS]

-3.00067
48.42778

Fig. 2(b) Matrices **V**, **W**, $\mathbf{P}_V$ and vector **RHS**, where $\mathbf{P}_V$ is the solution of $\mathbf{A}\,\mathbf{P}_V = \mathbf{V}$ and $\mathbf{RHS} = \mathbf{W}^T \mathbf{x}$.

MATRIX [D]

2.00000	3.00000
4.00000	5.00000
2.00000	3.00000

MATRIX [H]

1.06802	.00797
-2.44666	-2.23801

VECTOR [S]

-2.66983

-18.72004

SOLUTION AFTER THE FIRST LARGE CHANGE :

VECTOR [X]

8.15496

-3.82546

-2.66983

-23.34277

-6.40995

-18.72004

24.40133

27.88727

22.14824

-27.58607

Fig. 2(c) Results corresponding to the first change of variable parameters represented by **D**. **H** represents $(\mathbf{1} + \mathbf{W}^T \mathbf{A}^{-1} \mathbf{VD})$ and **s** is the solution of the reduced system $\mathbf{H}\,\mathbf{s} = \mathbf{W}^T \mathbf{x}$.

```
VARIABLES CHANGE AGAIN CAUSING A CHANGE OF [D].

[V] AND [W] REMAIN UNCHANGED.
```

MATRIX [D]

6.00000	7.00000
5.00000	4.00000
3.00000	4.00000

MATRIX [H]

1.02160	-.22657
-4.70646	-3.63383

VECTOR [S]

-4.57788

-7.39775

SOLUTION AFTER THE SECOND LARGE CHANGE :

VECTOR [X]

-2.20815

3.56798

-4.57788

-12.47901

-3.16642

-7.39775

15.58395

25.41279

12.05534

-20.00992

Fig. 2(d) Results corresponding to the second change of variable parameters. **H** and **s** are similarly defined to those in Fig. 2(c).