SIMULATION OPTIMIZATION SYSTEMS
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GENERALIZED LARGE CHANGE SENSITIVITY ANALYSIS IN LINEAR SYSTEMS

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Summary

In this paper, we investigate large change sensitivity problems and formulate a comprehensive set of formulas associated with matrix and vector operations. They can be directly applied to linear systems. Different formulas are compared. Relevant expressions appropriate for computational purposes are provided. These formulas will be very powerful, especially when the variables exist only in some local areas of the linear system. In an optimization procedure, the calculation of large change effects can be reduced from solving an n x n system to solving an r x r system, where r is the rank of the deviation matrix of the linear system.

Large change sensitivity problems have been approached in many ways, e.g., [1-6]. We feel our method has advantages in efficiency, flexibility and simplicity. Results of computer implementation show significant reduction in computational time and an easy procedure for the sensitivity analysis. Also, as a special case, a series of first-order sensitivity expressions are obtained without using Tellegen's theorem. All these features are discussed and presented in the paper.

Basic notation, definitions and relations are introduced and used uniformly throughout the paper. Formulas are derived for efficient evaluation of large changes in the inverse of a square matrix and in the solutions of a linear system. Basic large change formulas are also expanded to various expressions suitable for computational purposes due to different formulations of the original problem. We provide operational counts for major formulas to facilitate a deep investigation of the computational effort. Appropriate formulations yielding efficient calculation of large change effects are further discussed. Numerical examples are given. A comprehensive set of expressions designed for computation are provided through tables which summarize and complete the various cases of our analysis. These tables can be used as a tool for computer-aided designers to yield efficient design procedures.

Let $\mathbf{A}$ be an nxn matrix containing the variables $\boldsymbol{\phi}$ which exist only in rows $i_1, i_2, \ldots, i_{nI}$ and columns $j_1, j_2, \ldots, j_{nJ}$.

Let $\mathbf{U}_I$ and $\mathbf{U}_J$ be matrices whose columns are unit vectors $\mathbf{u}_{i_1}, \mathbf{u}_{i_2} \ldots, \mathbf{u}_{i_{nI}}$ and $\mathbf{u}_{j1}, \mathbf{u}_{j2}, \ldots, \mathbf{u}_{j_{nJ}}$, respectively, where unit vector $\mathbf{u}_i$, $1 \leq i \leq n$, is defined as an n-dimensional unit-vector with 1 in its ith row and zeros everywhere else.

The intersection elements of rows $i_1, i_2, \ldots, i_{nI}$ and columns $j_1, j_2, \ldots, j_{nJ}$ in $\mathbf{A}$ constitute an n_I by n_J submatrix denoted by $\mathbf{A}_{IJ}$. Let $\Delta\mathbf{A}$ represent the large changes of $\mathbf{A}$ due to changes in $\boldsymbol{\phi}$. We define

$$\Delta(\mathbf{A}^{-1}) \triangleq (\mathbf{A} + \Delta\mathbf{A})^{-1} - \mathbf{A}^{-1}. \tag{1}$$

It should be noted that $\Delta\mathbf{A}$ can always be expressed in the form of

$$\Delta\mathbf{A} = \mathbf{V}\,\mathbf{D}\,\mathbf{W}^{\mathrm{T}}, \tag{2}$$

where $\mathbf{V}$, $\mathbf{D}$ and $\mathbf{W}$ are matrices of order of $n \times r_1$, $r_1 \times r_2$ and $n \times r_2$, respectively. We have great freedom to choose $\mathbf{V}$, $\mathbf{D}$ and $\mathbf{W}$ for the realization of (2). For example, a simple and useful realization is

$$\Delta\mathbf{A} = \mathbf{U}_I\,\Delta\mathbf{A}_{IJ}\,\mathbf{U}_J^{\mathrm{T}}. \tag{3}$$

We can fully exploit the freedom of selecting **V**, **D** and **W** so that various efficient computational schemes are constructed. This is discussed in detail in our paper.

P or **p** and **Q** or **q** are used to represent the solutions of the linear systems with coefficient matrix **A** and $\mathbf{A}^T$, respectively.

Subscripts similar to the notation of the R.H.S. of the linear equations are added to **P**, **p**, **Q** and **q** to identify the solutions for different R.H.S. As an example, $\mathbf{p}_b$ and $\mathbf{Q}_W$ are solutions of

$$\mathbf{A}\,\mathbf{p}_b = \mathbf{b} \tag{4}$$

and

$$\mathbf{A}^T\,\mathbf{Q}_W = \mathbf{W}\,, \tag{5}$$

respectively.

We have the following formula to calculate the large change effects of $\mathbf{A}^{-1}$ as

$$\Delta(\mathbf{A}^{-1}) = -\,\mathbf{A}^{-1}\,\mathbf{V}\,\mathbf{D}\,(\mathbf{1} + \mathbf{W}^T\,\mathbf{A}^{-1}\,\mathbf{V}\,\mathbf{D})^{-1}\,\mathbf{W}^T\,\mathbf{A}^{-1}\,. \tag{6}$$

Notice that **D** is a matrix of dimension $r_1 \times r_2$. The inversion of $(\mathbf{1} + \mathbf{W}^T\,\mathbf{A}^{-1}\,\mathbf{V}\,\mathbf{D})$ in (6) is the order of r_2. A dual formula is also available which results in an $r_1 \times r_1$ matrix inversion. Therefore, if we already know $\mathbf{A}^{-1}$, the large change effects of $\mathbf{A}^{-1}$ due to changes in elements of **A** may be evaluated by inverting an $r_1 \times r_1$ or $r_2 \times r_2$ matrix and performing appropriate matrix multiplications. To obtain $(\mathbf{A} + \Delta\mathbf{A})^{-1}$, we simply add $\Delta(\mathbf{A}^{-1})$ to $\mathbf{A}^{-1}$.

We can always choose **V**, **D**, and **W** such that (2) holds and $r_1 \leq n$ and $r_2 \leq n$. Especially, when rank $\Delta\mathbf{A} < n$, we can have $r_1 < n$ and/or $r_2 < n$. Thus, instead of directly inverting an $n \times n$ system, we can obtain the $(\mathbf{A} + \Delta\mathbf{A})^{-1}$ matrix by solving a smaller system of $r_1 \times r_1$ or $r_2 \times r_2$. This is one of the major attractions of these formulas. For example, the simplest way to choose **V**, **D** and **W** is to use $\mathbf{U}_I$, $\Delta\mathbf{A}_{IJ}$ and $\mathbf{U}_J$, yielding an $n_I \times n_I$ or $n_J \times n_J$ smaller system. Further discussion on the formulation of **V**, **D** and **W** as well as computational aspects of large change formulas associated with $\mathbf{A}^{-1}$ will be presented in the paper.

As a special case, the well-known Householder formulas [7-8] can be obtained from our formula. It should be noted that, mathematically, the Householder formulas require **D** to be a nonsingular square matrix. In our formulas **D** can be singular or even rectangular, thus more freedom can be exploited in the formulation of **D**. Computationally, compared to Householder formulas, using our formula, one can either avoid matrix inversion of **D** or perform fewer matrix multiplications.

Other special cases of our formula are also discussed, including those originated by Sherman and Morrison [9] and frequently used in large change analysis, fault location and tolerance assignment, e.g., [10].

Our basic formulas are expanded according to different formulations of the original problem. For example, the large change evaluation in a multi-measurement, multi-excitation system can be represented by

$$\Delta(\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C}) \quad ,$$

where the columns of the nxm′ matrix **B** select appropriate measurements and the columns of the nxn′ matrix **C** contain different excitation vectors. According to the values of r_1, r_2 and the number of measurements and excitations we can choose appropriate approaches such that the computational effort is reduced. For example, we can use

$$\Delta(\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C}) = -\, \mathbf{S}^T \mathbf{Q}_W{}^T \mathbf{C}\,, \qquad (7)$$

where **S** is the solution of the $r_2 \times r_2$ linear system

$$(\mathbf{1} + \mathbf{Q}_W{}^T \mathbf{V}\,\mathbf{D})^T \mathbf{S} = \mathbf{RHS} \qquad (8)$$

with the right-hand sides as

$$\mathbf{RHS} = (\mathbf{Q}_B{}^T \mathbf{V}\,\mathbf{D})^T . \qquad (9)$$

This approach requires $m' + r_2$ FBS in the $n \times n$ linear system for $\mathbf{Q}_B$ and $\mathbf{Q}_W$, one LU factorization and m′ FBS in the $r_2 \times r_2$ linear system of (8).

As another example, consider

$$\Delta(\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C}) = -\, \mathbf{B}^T \mathbf{P}_V \mathbf{D}\,\mathbf{S}\,, \qquad (10)$$

where **S** is the solution of the $r_2 \times r_2$ linear system

$$(1 + \mathbf{W}^T \mathbf{P}_V \mathbf{D}) \mathbf{S} = \mathbf{RHS} \quad (11)$$

with the right-hand sides as

$$\mathbf{RHS} = \mathbf{W}^T \mathbf{P}_C . \quad (12)$$

This approach requires $r_1 + n'$ FBS in the $n \times n$ linear system for $\mathbf{P}_V$ and $\mathbf{P}_C$, one LU factorization and n' FBS in the $r_2 \times r_2$ linear system of (11).

Comparing the major computational effort required by the approaches of (7) and (10), we find that (7) can be recommended if $m' < n'$ and $r_2 + m' < r_1 + n'$, otherwise, if $m' > n'$ and $r_2 + m' > r_1 + n'$, (10) may be used.

In the paper, we present an exhaustive search for all possible cases of $\Delta(\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C})$ distinguished by the number of FBS in the $n \times n$ and $r_1 \times r_1$ or $r_2 \times r_2$ systems. Suitable expressions are formulated for efficient calculations of $\Delta(\mathbf{B}^T \mathbf{A}^{-1} \mathbf{C})$. Different cases are classified such that the number of FBS in the $n \times n$ system equals the minimum of $m' + r_2$, $n' + r_1$ and $r_1 + r_2$ and the number of FBS in the $r_1 \times r_1$ or $r_2 \times r_2$ smaller system equals the minimum of r_1, r_2, m' and n'.

Also included are formulas suitable for computations of large change effects of $\mathbf{A}^{-1}$ associated with vectors $\mathbf{b}$ and $\mathbf{c}$.

Operational counts [11] for major formulas are presented for deep investigation of computational effort. For example, to calculate $\Delta(\mathbf{A}^{-1} \mathbf{c})$ using (6) we will need $r_1 n (n + r_2) + n r_2 + r_2^3/3 - r_2/3 + r_2^2 + r_1 n + r_1 r_2 (r_2 + 1)$ operations.

If we can express $\Delta\mathbf{A}$ as $\mathbf{V}\,\mathbf{D}\,\mathbf{W}^T$ such that r_1 and r_2 are very small, using large change formulas can save considerable computation as compared with direct methods which calculate everything all over again.

If r_1 and r_2 are not small enough, the above computational counts of large change formulas will even exceed those of the direct methods. However, this computational count can be further reduced significantly if variable parameters are changed many times. For example, in an optimization procedure, variables will usually be updated many times before the optimum is reached. Another example is centering and yield estimation, where Monte-

Carlo procedures are often involved to choose random values repeatedly for variable parameters. When we apply large change formulas in these cases, we can choose **V**, **D** and **W** such that **D** contains variables and **V** and **W** are constant. Thus all the FBS in the n x n system represented by **A**, are calculated only once and can be used for all subsequent changes of variables. The operational count becomes $r_2^3/3 - r_2/3 + r_2^2 + r_1 n + r_1 r_2 (r_2 + 1)$ for $\Delta(\mathbf{A}^{-1}\mathbf{c})$ using formulas corresponding to (6), for large changes of the second time and on.

As an example, Table I shows the operational counts of calculating $\Delta(\mathbf{A}^{-1}\mathbf{c})$ by using the direct method and by using our large change formulas. The direct method refers to the method of calculating the system all over again after a large change. In the paper, we present the operational counts for $\Delta(\mathbf{A}^{-1})$ and $\Delta(\mathbf{A}^{-1}\mathbf{c})$ by different approaches. Computer programs calculating $\Delta(\mathbf{A}^{-1})$ and $\Delta(\mathbf{A}^{-1}\mathbf{c})$ by our large change formulas have been developed which confirm the corresponding operational counts.

From the previous discussion, we find that the calculation of large change effects with $\mathbf{A}^{-1}$ involved is essentially reduced to solving a smaller system of size r_1 by r_1 or r_2 by r_2, where r_1 and r_2 are the numbers of rows and columns in **D**, respectively. If $r_1 \geq r_2$, we use (6) for an $r_2 \times r_2$ smaller system, otherwise use the dual formula yielding an $r_1 \times r_1$ system. Different formulations of **V**, **W** and **D** can lead to different computational cost which depends heavily upon the value of r_1 (if $r_1 < r_2$) or r_2 (if $r_1 > r_2$). Thus, it is always desirable to have the value of either r_1 or r_2 as low as possible. For example, the formulation of (3) gives $r_1 = n_I$ and $r_2 = n_J$. Therefore, $r_1 \leq n$ and $r_2 \leq n$. So, we are assured that this formulation results in solving a system no greater than the original n x n system.

Usually, r_1 or r_2 can be further reduced. We have,

$$\min_{(\mathbf{V},\mathbf{D},\mathbf{W})} r_1 = \min_{(\mathbf{V},\mathbf{D},\mathbf{W})} r_2 = r\ , \qquad (13)$$

where r is the rank of $\Delta\mathbf{A}$. The above equation gives the conclusion that for evaluating large changes involving $\mathbf{A}^{-1}$, the minimum size of the system we need to solve is r by r.

However, in practical problems, e.g., in computer-aided circuit design, using the branch and nodal information and the topological relation of the variable elements, the physical properties of the problem can lead to a formulation of **V**, **W** and **D** such that the minimum size system is achieved and no additional computations are introduced. This will result in an algorithm which further reduces computation as compared with [2].

In the following numerical example, we consider a 10x10 system

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}\ , \tag{14}$$

where the intersection elements of **A** at rows 2, 5 and 9 and columns 3 and 6 are constantly changed. **V**, **D** and **W** are formulated by (3). Numerical solutions as well as intermediate results are shown in Fig. 1. Fig. 2 shows the efficiency of this method compared with the direct method, which is to solve the n by n (here 10 by 10) system repeatedly. For example, if the variables change 5 times, then the computational cost can be reduced by 60%, if they change 20 times, then 80%. Notice that in Fig. 2, the horizontal axis represents the number of simulations which include the initial system simulation and subsequent simulations when system parameters are changed.

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TABLE I

OPERATIONAL COUNTS FOR $\Delta(\mathbf{A}^{-1}\mathbf{c})$

n, r_1, r_2	By Direct Method	By Formula (6)	
		Variables Change for the First Time	Variables Change for the Second Time and On
10,1,1	430	133	13
10,3,2	430	434	54
10,5,3	430	807	127
10,4,4	430	756	156
10,6,4	430	1096	216
10,6,5	430	1255	305
10,6,6	430	1438	418
10,7,6	430	1650	470
4,3,2	36	116	36
7,2,2	161	172	32

MATRIX [A]										VECTOR [B]
1.0	5.0	5.0	1.0	5.0	2.0	1.0	1.0	7.0	2.0	35.0
2.0	3.0	3.0	7.0	0.0	4.0	3.0	6.0	8.0	3.0	32.0
3.0	0.0	2.0	4.0	2.0	6.0	4.0	4.0	9.0	7.0	16.0
6.0	1.0	2.0	5.0	2.0	3.0	3.0	7.0	3.0	5.0	51.0
8.0	1.0	2.0	2.0	4.0	4.0	6.0	8.0	4.0	8.0	42.0
4.0	1.0	6.0	7.0	3.0	5.0	7.0	3.0	5.0	3.0	19.0
7.0	0.0	6.0	5.0	9.0	4.0	8.0	9.0	2.0	9.0	34.0
2.0	0.0	4.0	2.0	2.0	5.0	3.0	5.0	4.0	3.0	71.0
3.0	2.0	0.0	1.0	5.0	3.0	4.0	2.0	3.0	1.0	36.0
4.0	2.0	4.0	4.0	6.0	2.0	9.0	6.0	1.0	7.0	61.0

SOLUTIONS BEFORE LARGE CHANGE :

VECTOR [X]
-8.89217
39.80097
-3.00067
2.31014
-5.40544
48.42778
-12.11626
-3.61726
-32.93004
16.99799

Fig. 1(a) The original linear system and its solutions. **A** is a 10x10 matrix containing parameters of the system. **b** is the excitation vector. **x** is the solution vector.

MATRIX [V]

0.0	0.0	0.0
1.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	1.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	0.0
0.0	0.0	1.0
0.0	0.0	0.0

MATRIX [W]

0.0	0.0
0.0	0.0
1.0	0.0
0.0	0.0
0.0	0.0
0.0	1.0
0.0	0.0
0.0	0.0
0.0	0.0
0.0	0.0

MATRIX [PV]

-.03684	.19072	-.00936
-.30799	-.26526	.04838
.00454	.09406	-.15865
-.04645	-.23600	.02949
.02608	-.09579	.12002
-.48948	-.43199	.13012
.18919	.22984	.01487
.27060	.13754	.00846
.36321	.32717	-.02238
-.27658	-.20670	-.09789

VECTOR [RHS]

-3.00067

48.42778

Fig. 1(b) Matrices **V**, **W**, $\mathbf{P}_V$ and vector **RHS**, where $\mathbf{P}_V$ is the solution of $\mathbf{A}\,\mathbf{P}_V = \mathbf{V}$ and $\mathbf{RHS} = \mathbf{W}^T\mathbf{x}$.

MATRIX [D]

2.00000	3.00000
4.00000	5.00000
2.00000	3.00000

MATRIX [H]

1.06802	**.00797**
-2.44666	**-2.23801**

VECTOR [S]

-2.66983

-18.72004

SOLUTIONS AFTER LARGE CHANGE :

VECTOR [X]

8.15496

-3.82546

-2.66983

-23.34277

-6.40995

-18.72004

24.40133

27.88727

22.14824

-27.58607

Fig. 1(c) Results corresponding to a particular change of variable parameters represented by **D**. **H** repesents $(\mathbf{1} + \mathbf{W}^T \mathbf{A}^{-1} \mathbf{V} \mathbf{D})$ and **s** is the solution of the smaller system $\mathbf{H}\,\mathbf{s} = \mathbf{W}^T \mathbf{x}$.

VARIABLES CHANGE FOR ANOTHER TIME CAUSING THE CHANGE OF [D].
BUT [V] AND [W] ARE THE SAME AS THOSE WITH THE PREVIOUS [D].

MATRIX [D]

6.00000	7.00000
5.00000	4.00000
3.00000	4.00000

MATRIX [H]

1.02160	-.22657
-4.70646	-3.63383

VECTOR [S]

-4.57788
-7.39775

SOLUTIONS AFTER ANOTHER LARGE CHANGE :

VECTOR [X]

-2.20815
3.56798
-4.57788
-12.47901
-3.16642
-7.39775
15.38395
25.41279
12.05534
-20.00992

Fig. 1(d) Results corresponding to another change of variable parameters. **H** and **s** are similarly defined to those in Fig. 1(c).

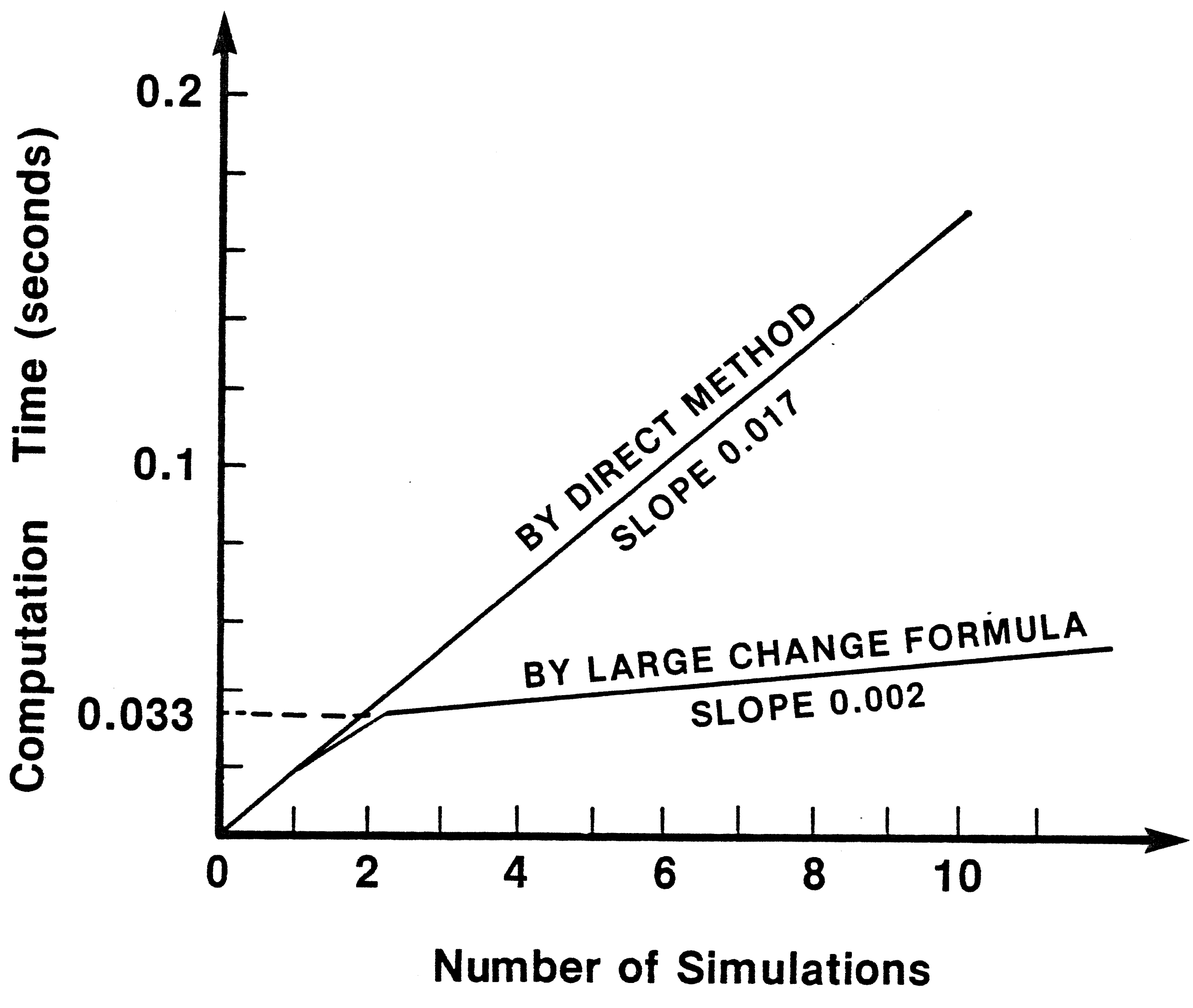

Fig. 2 Computational time for evaluating large change effects of the linear system of the example. (The CDC 170/815 System is used. Compilation time is not included.)