INTERACTIVE OPTIMIZATION OF MULTI-COUPLED CAVITY MICROWAVE FILTERS

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Abstract

This technical report proposes algorithms and a structure for an interactive computer program for design optimization of multi-coupled cavity filters. User-oriented interactive features are emphasized. In conjunction with the nonlinear minimax method, the cubic interpolation technique is introduced to improve computational efficiency. A detailed description of the incorporation of the cubic interpolation technique with the optimization procedure is presented.

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I. INTRODUCTION

The superior electrical performance and favorable physical size of multi- coupled cavity narrow-bandpass microwave filters entitle them to be prominent components of high-quality satellite communication systems [1]. Efficient approaches to the simulation and sensitivity evaluation of such filters have been developed and exhaustively discussed in a report by Bandler, Chen and Daijavad [2], where various formulas and tables are presented in detail.

One immediate application of this theory is found in the filter design optimization problem. Conventionally, a nominal design is achieved by the two major stages of approximation and realization. For the approximation, a filter type suitable for the given specifications is chosen and the filter function is calculated following one of the synthesis procedures, which vary with different filter types. Then, for realization, the coupling matrix is obtained after certain matrix orthogonalization and, to suit the desired physical structure, similarity transformations. See, for example, Atia, Williams et al. [1,3,4]. Modern communication systems demand a greater variety and more stringent requirements on microwave filters and, as special filter characteristics, such as asymmetric, nonminimum-phase, etc., gain increasing attention, the conventional approach could become tedious or even inapplicable.

Employing modern computer-aided design techniques, directly taking the engineering specifications as optimization objectives and the practical network parameters as optimization variables, our approach offers a unified and user-oriented treatment of multi-coupled cavity filter design. Various network structures and specifications, including group delay specifications, are accommodated in an elegant manner. Powerful gradient-based optimization techniques are integrated with efficient simulation and sensitivity evaluation.

II. FORMULATION OF THE PROBLEM

Introduction

A thorough treatment of the general formulation of the design optimization problem has been provided by Bandler [5]. Here, to clarify the ensuing presentations, a brief review of some principles is appropriate.

We define the error function as

$$e(\boldsymbol{\phi},\omega) = w(\omega)\,[F(\boldsymbol{\phi},\omega) - S(\omega)]\,, \tag{1}$$

where ω denotes the frequency variable and the vector

$$\boldsymbol{\phi} = [\phi_1, \phi_2, \ldots \phi_n]^T \tag{2}$$

contains the design variables of the network under consideration. $F(\boldsymbol{\phi},\omega)$ is the network response of interest and $S(\omega)$ the prescribed specification. The weighting function $w(\omega)$ is defined as

$w(\omega) > 0$ corresponds to an upper specification,

$w(\omega) < 0$ corresponds to a lower specification.

It follows that the specification is satisfied when $e(\boldsymbol{\phi},\omega) \leq 0$, or violated when $e(\boldsymbol{\phi},\omega) > 0$. Therefore, the objective of optimization becomes essentially

$$\underset{\boldsymbol{\phi}}{\text{Minimize}}\ e(\boldsymbol{\phi},\omega). \tag{3}$$

We denote the first-order gradients of the error function by

$$e'_{\boldsymbol{\phi}} \triangleq \frac{\partial e}{\partial \boldsymbol{\phi}} \tag{4}$$

and

$$e'_{\omega} \triangleq \frac{\partial e}{\partial \omega}\ . \tag{5}$$

Discretization w.r.t. ω

In practice, the frequency band of interest is divided into a number of subintervals. Within each subinterval, the specification and the weighting function are kept constant and the error function (1) is discretized in frequency, that is, we evaluate the error function at a finite number of sample points, resulting in a set of discrete error functions, as

$$e_{ij}(\boldsymbol{\phi}) = w_j\,[F(\boldsymbol{\phi}, \omega_{ij}) - S_j], \quad i=1,2,\ldots,m_j, \quad j=1,2,\ldots,k \tag{6}$$

where m_j is the number of sample points of the jth interval and k is the total number of subintervals.

Relative Responses

Sometimes, as in the case of group delay, we are interested in the variations of the response within a certain frequency band. Taking the minimum of the response over the set of sample points

$$F_{min}(\boldsymbol{\phi}) = \min_{1 \le i \le m} F(\boldsymbol{\phi},\omega_i) \tag{7}$$

as a reference point, we can define the deviation of $F(\boldsymbol{\phi},\omega_i)$ as

$$\hat{F}(\boldsymbol{\phi},\omega_i) = F(\boldsymbol{\phi},\omega_i) - F_{min}(\boldsymbol{\phi})\,. \tag{8}$$

The error function here is of the form

$$e(\boldsymbol{\phi},\omega_i) = w[\hat{F}(\boldsymbol{\phi},\omega_i) - S] \tag{9}$$

with $w > 0$, implying an upper specification, since it is obvious that $\hat{F} \ge 0$. Alternatively, by designating the average of $F(\boldsymbol{\phi},\omega_i)$ given by

$$\bar{F}(\boldsymbol{\phi}) = \frac{1}{m} \sum_{i=1}^{m} F(\boldsymbol{\phi},\omega_i) \tag{10}$$

as the reference point, the variations in F can be measured by

$$\hat{F}(\boldsymbol{\phi},\omega_i) = F(\boldsymbol{\phi},\omega_i) - \bar{F}(\boldsymbol{\phi})\,. \tag{11}$$

One advantage of definition (11) is that the functions $\hat{F}$ so defined have continuous derivatives w.r.t. the optimization variables $\boldsymbol{\phi}$. The corresponding error function can be constructed either by

$$e_{\ell i}(\boldsymbol{\phi}) = w_\ell\,[\hat{F}(\boldsymbol{\phi},\omega_i) - S], \; w_\ell < 0, \; i=1,2,\ldots,m\,,$$

$$e_{ui}(\boldsymbol{\phi}) = w_u\,[\hat{F}(\boldsymbol{\phi},\omega_i) - S], \; w_u > 0, \; i=1,2,\ldots,m\,, \tag{12}$$

where both upper and lower specifications are applied, since $\hat{F}$ has arbitrary sign, or, by

$$e_i(\boldsymbol{\phi}) = w\{|\hat{F}(\boldsymbol{\phi},\omega_i)| - S\}, \ w > 0, \ i=1,2,\ldots,m . \tag{13}$$

By definition, it is clear that the derivatives of the error functions are given in terms of the derivatives of the corresponding response, which are readily obtained utilizing the results we have reported previously [2].

Optimization Technique

Since the microwave filter design problem we are concerned with is known to be a highly nonlinear one, a powerful and robust optimization package is essential for a successful software system. Among the various techniques available, the minimax approach is the most preferred in this area. The theory and general description of the minimax algorithm employed in our computer program are available elsewhere [6,7].

III. CUBIC INTERPOLATION TECHNIQUE

General Considerations

As required for numerical optimization, the error function needs to be discretized in the frequency domain. This could be done, in the simplest way, by taking a set of fixed sample points, which is adequate for a well-behaved error function. As the behaviour of the error function degenerates, however, the sample points may have to be densely spaced and, consequently, a large number of discretized error functions may have to be handled by the optimization procedure. For the filter problem considered here, the presence of sharp ripples of the response, especially near the band edges, worsens the situation. As illustrated in Fig. 1, the solution obtained using fixed sample points may not be optimal in the continuous minimax sense because some error peaks are missed. As a result, a distinguishing feature of the minimax method, namely its potentially equal-ripple solution, is degraded. To overcome the difficulty, we adapt the cubic interpolation technique to locate the maxima of the error function (referred to as the error maxima henceforth) w.r.t. frequency.

Cubic Interpolation Formula

As a well-known fact, a maximum of a continuous differentiable function $e(\omega)$ is characterized by $\partial e/\partial\omega = 0$ and $\partial^2 e/\partial\omega^2 < 0$. This implies a change in the sign of $\partial e/\partial\omega$ and, in the neighbourhood of the maximum, $\partial e/\partial\omega$ decreases as frequency increases. It follows that if there exist two points $\omega_1 < \omega_2$ such that

$$e'_{\omega_1} > 0 \quad \text{and} \quad e'_{\omega_2} < 0$$

at least one maximum of $e(\omega)$ lies between ω_1 and ω_2. If ω_1 and ω_2 are close enough to exclude the existence of multiple maxima, the location of the detected maximum can be estimated by the cubic interpolation formula [8]

$$\omega_{max} = \omega_2 - \frac{(\omega_2 - \omega_1)[x - y - e'_{\omega_2}]}{e'_{\omega_1} - e'_{\omega_2} + 2x}, \tag{14}$$

where

$$y = -e'_{\omega_1} - e'_{\omega_2} + 3\,\frac{e(\omega_2) - e(\omega_1)}{\omega_2 - \omega_1} \tag{15}$$

and

$$x = [y^2 - e'_{\omega_1} e'_{\omega_2}]^{1/2}. \tag{16}$$

The cubic interpolation formulas (14)-(16) have been used by Fletcher and Powell [8] in their famous paper of 1963, where the origin of the formulas was said to be a report by Davidon in 1959 [9]. For convenient reference, we also give a proof in the Appendix.

The Search Procedure

The search procedure for a maximum can be performed within a subinterval, where the error function corresponding to a constant specification is continuous, starting from the lower frequency edge with a given step-length. As the searching step-length decreases, the accuracy of maximum location increases, but so does the computational effort. On the other hand, a maximum could be missed if the step-length is too large. Fortunately, some helpful preliminary knowledge, such as the general distribution of poles and zeros, is usually available.

It is clear that the positions of error-maxima are functions of the optimization variables. During the optimization process, they shift along the frequency axis, and even the total number of error maxima could be different when we are far from the solution. While we try to track error maxima, the shift of sample points could cause discontinuity in the gradients of the corresponding error functions w.r.t. the optimization variables. To smooth this effect and to stabilize the algorithm, an appropriate number of sample points, e.g. 2-3 times the order of the filter, should be taken.

For a subinterval, a search step-length is chosen, and the total number of search points for error maxima is given by

$$\text{Integer}\left[\frac{\text{the width of the subinterval}}{\text{search step} - \text{length}}\right] + 1 \ . \tag{17}$$

The concepts of search points and sample points should not be confused. At the search points the function values and first-order derivatives w.r.t. frequency of $e(\boldsymbol{\phi},\omega)$ are computed in order to detect and locate the error maxima. At the optimally selected sample points, the function values of $e(\boldsymbol{\phi},\omega)$ as well as their gradients w.r.t. $\boldsymbol{\phi}$ are computed to construct discretized error functions to be handled by the optimization package. The number of search points is usually larger than that of sample points. Computational efficiency can be improved if these points are manipulated appropriately. A practical algorithm is illustrated in Section V.

IV. AN INTERACTIVE MAIN PROGRAM

We now proceed to discuss the organization and structure of an interactive main program. The user is relieved of the tedious task of writing a calling segment or data files, at the same time he is allowed the maximum freedom to define the system and design goals. Moreover, after a design is obtained, the user has the choice of simulating the resulting design and, if the outcome is not satisfactory, to modify the specifications or redefine the system in order to restart the optimization procedure.

A step-by-step description of such a main program follows.

<u>Step 1</u> Initially set up or subsequently modify the coupling matrix:

- its order,
- the nonzero elements (the desired and physically available couplings),
- the indices of the variable couplings,
- the symmetry pattern.

<u>Comment</u> Considerable computational effort can be saved if the coupling matrix is symmetrical w.r.t. both its diagonal and anti-diagonal as described in [2]. The symmetry pattern also determines the sensitivity formulas to be used.

<u>Step 2</u> Initialize or modify the remaining network parameters:

- the center frequency and the bandwidth parameter,
- the load and the source impedances,
- indicate whether the input- and output-couplings are variables.

<u>Comment</u> It has been verified that the bandwidth parameter acts as a scaling factor and hence it should not be taken as an independent variable.

<u>Step 3</u> If a pre-design simulation is desired, go to Step 8.

<u>Step 4</u> Define or update the subintervals and the specifications:

- type of response: the reflection coefficient,
  the transducer loss,
  with or without the group delay,
- the number of subintervals to be defined or updated,
- for each such subinterval:

(i) two frequency edges,

(ii) the number of sample points and their initial positions,

(iii) whether there is to be an error maxima search or specified fixed points; if a search for error maxima is required, the search step- length,

(iv) a constant specification and a weighting factor associated with the subinterval.

Comment For lossless designs, the reflection coefficient and the transducer loss are interchangeable but have different weighting effects, as discussed in the last part of this section.

Step 5 Define the linear constraints on the variables, consistent with the optimization package being used.

Step 6 Define the control parameters of the optimization package, such as

- maximum number of function evaluations,
- initial optimization step-length,
- accuracy requirement of the solution,
- printout control.

Step 7 Call the optimization package.

Return from the optimization package, go to Step 9.

Step 8 Simulation. The user defines:

- the frequency band in which the filter is to be simulated, i.e., the lower and upper edges and the number of frequency points,
- the type of response to be simulated.

The simulation subroutine is called; the results are printed and plotted.

Step 9 User's options:

- redefine the coupling matrix → go to Step 1.
- redefine the terminations → go to Step 2.
- modify the specifications → go to Step 4.
- modify the constraints → go to Step 5.
- restart the optimization → go to Step 6.

- simulation → go to Step 8.
- stop.

Comments on Scaling

The different weighting effects between specifying the reflection coefficient and specifying the transducer loss will be briefly discussed. Suppose we let $w(\omega) = 1$. Then, for an equal-ripple solution, any violation or satisfaction of the specification will be uniform for both the passband and the stopband. First, we consider the reflection coefficient. A deviation from the specified value, say, of 0.01 is negligible for the passband but significant for the stopband. In the case of transducer loss, the opposite situation occurs, where a difference of 0.5 dB is insignificant for the stopband but critical for the passband. This becomes clearer when we relate the reflection coefficient $|\rho|$ to the transducer loss Λ by

$$\Lambda = -10 \log_{10} (1 - |\rho|^2) . \tag{18}$$

From (18), we relate their first-order changes, denoted by $\delta\Lambda$ and $\delta|\rho|$, by

$$\delta\Lambda = \frac{20}{\ell n 10} \left(\frac{|\rho|}{1 - |\rho|^2} \right) \delta|\rho| . \tag{19}$$

For a typical passband specification of $|\rho| = 0.1$, we have

$$\delta\Lambda \approx 0.88\, \delta|\rho| . \tag{20}$$

For a stopband specification of $|\rho| = 0.995$ (i.e., $\Lambda = 20$ dB), it becomes

$$\delta\Lambda \approx 87\, \delta|\rho| . \tag{21}$$

The interpretation here is that the passband (or the stopband) is much more emphasized by specifying $|\rho|$ (or Λ). Such an effect becomes more pronounced when tighter specifications are used. The situation can be balanced by defining an appropriate weighting function $w(\omega)$.

V. CONSTRUCTION AND EVALUATION ON FUNCTIONS AND GRADIENTS

The subroutine described in this section works interactively with the optimization package. At each iteration of the optimization procedure it is called with a new set of design variables and returns with the updated functions and their gradients. Different types of specifications and the search for error maxima are handled here. Hence, this segment is very important and requires delicate treatment. The necessary information such as the network structure, constant parameters, definition of subintervals and specifications should be provided from the main program through common data. The calling segment is the optimization package.

<u>Step 1</u> Update the variable parameters (couplings, etc.).

<u>Step 2</u> Initialize the subinterval index, $j \leftarrow 1$.

<u>Step 3</u> If the jth subinterval is associated with a group delay specification, go to Step 13. If the jth subinterval is specified with fixed sample points, go to Step 12.

<u>Step 4</u> Set ω_ℓ and ω_u equal to the lower and upper edges of the jth subinterval, respectively; set $\delta\omega$ equal to the user-defined search step-length. Initialize $\omega_2 = \omega_\ell$, ω_2 being the current search point. Initialize step counter $k \leftarrow 1$.

<u>Step 5</u> If ω_2 is within 0.5 $\delta\omega$ of one of the sample points, set ω_2 to that sample point and set MODE $\leftarrow 2$; otherwise set MODE $\leftarrow 1$.

<u>Comment</u> Here MODE is an indicator. When MODE = 2 it means that ω_2 is both a search point and a sample point, hence e, e'_ω and e'_ϕ need to be computed at ω_2. MODE = 1 means ω_2 is only a search point, thus e'_ϕ is not needed. MODE = 3, as used later, means ω_2 is only a sample point, thus e'_ω is not needed.

<u>Step 6</u> Call the simulation subroutine.

<u>Step 7</u> If this is the first search point ($\omega_2 = \omega_\ell$), go to Step 9.

<u>Comment</u> In order to perform cubic interpolation, we need two consecutive search points.

<u>Step 8</u> If $e'_{\omega 1} > 0$ and $e'_{\omega 2} < 0$, call the cubic interpolation subroutine to locate the error maximum between ω_2 and ω_1. Denote the result by ω_m. Replace the sample point closest to ω_m by ω_m, call the simulation subroutine with MODE = 3.

<u>Step 9</u> Set $\omega_1 \leftarrow \omega_2$, $e(\boldsymbol{\phi},\omega_1) \leftarrow e(\boldsymbol{\phi},\omega_2)$ and $e'_{\omega 1} \leftarrow e'_{\omega 2}$.

Comment This step stores the useful information of the current point before proceeding to the next point.

Step 10 If $\omega_2 = \omega_u$, go to Step 14.

Step 11 Set $\omega_2 \leftarrow \omega_\ell + k\,\delta\omega$. Set $k \leftarrow k+1$. If $\omega_2 > \omega_u$, set $\omega_2 \leftarrow \omega_u$. Go to Step 5.

Step 12 Call the simulation subroutine with MODE = 3 at each and every sample point associated with the jth subinterval. Go to Step 14.

Comment This step processes a subinterval specified with fixed sample points.

Step 13 Call the group delay simulation subroutine at each and every sample point associated with the jth subinterval. Construct the residual functions and their gradients using formulas such as (7)-(9).

Comment This step processes a subinterval associated with a group delay specification.

Step 14 Return if the last subinterval has been processed.

Step 15 Set $j \leftarrow j+1$. Go to Step 3.

The procedure described here is also illustrated in Fig. 2.

VI. CONCLUSION

Algorithms and the structure for an interactive computer program for design optimization of multi-coupled cavity filters have been proposed. A user-oriented interactive main program is described in detail. A subroutine efficiently incorporating the cubic interpolation technique with the evaluation of objective functions and gradients is illustrated step by step. Detailed and explicit formulas for computation of various filter responses sensitivities used in the proposed computer program can be found in [2]. Two individual subroutines, one for the simulation of gain responses and the other for the group delay, need to be constructed because they require different adjoint analyses. A computer software package implementing our approach has been successfully tested. Selected numerical examples will be presented in separate reports.

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REFERENCES

[1] A.E. Atia and A.E. Williams, "New types of waveguide bandpass filters for satellite transponders", COMSAT Technical Review, vol. 1, 1971, pp. 21-43.

[2] J.W. Bandler, S.H. Chen and S. Daijavad, "Efficient approaches to the simulation of narrow-band multi-cavity filters", Department of Electrical and Computer Engineering, McMaster University, Hamilton, Canada, Report SOS-84-9-R, 1984.

[3] A.E. Atia and A.E. Williams, "Narrow-bandpass waveguide filters", IEEE Trans. Microwave Theory Tech., vol. MTT-20, 1972, pp. 258-265.

[4] A.E. Atia, A.E. Williams and R.W. Newcomb, "Narrow-band multiple- coupled cavity synthesis", IEEE Trans. Circuits and Systems, vol. CAS-21, 1974, pp. 649-655.

[5] J.W. Bandler, "Computer-aided circuit optimization", in Modern Filter Theory and Design, G.C. Temes and S.K. Mitra, Eds., New York - Interscience, 1973, pp. 211-271.

[6] J. Hald and K. Madsen, "Combined LP and quasi-Newton methods for minimax optimization", Mathematical Programming, vol. 20, 1981, pp. 49-62.

[7] J.W. Bandler and W.M. Zuberek, "MMLC - a Fortran package for linearly constrained minimax optimization", Department of Electrical and Computer Engineering, McMaster University, Hamilton, Canada, Report SOS-82-5-U2, 1983.

[8] R. Fletcher and M.J.D. Powell, "A rapidly convergent descent method for minimization", Computer J., vol. 6, 1963, pp. 163-168.

[9] W.C. Davidon, "Variable metric method for minimization", AEC Research Development Report ANL-5990 (Rev.), 1959.

[10] R.J. Cameron, "Dual-mode realisations for asymmetric filter characteristics", ESA Journal, vol. 6, 1982, pp. 339-356.

APPENDIX

PROOF OF CUBIC INTERPOLATION FORMULA

We want to estimate the location of a maximum of the function $e(\omega)$, namely ω_3, using computed function values and first-order derivatives w.r.t. ω at two search points ω_1 and ω_2, as shown in Fig. 3. We define

$$e_i \triangleq e(\omega_i) \ , \tag{A1}$$

$$e_i' \triangleq \left.\frac{\partial e}{\partial \omega}\right|_{\omega_i} \ , \tag{A2}$$

$$e_i'' \triangleq \left.\frac{\partial^2 e}{\partial \omega^2}\right|_{\omega_i} \ , \tag{A3}$$

and

$$e_i''' \triangleq \left.\frac{\partial^3 e}{\partial \omega^3}\right|_{\omega_i} \ , \qquad i = 1, 2, 3. \tag{A4}$$

The values of ω_1, ω_2, e_1, e_2, e_1' and e_2' are known. Also we know that $\omega_2 > \omega_1$, $e_1' \geq 0$ and $e_2' \leq 0$. Approximating $e(\omega)$ by Taylor's expansion to the cubic terms about ω_3, bearing in mind that $e_3' = 0$, we have

$$e_1 \approx e_3 + \frac{1}{2} e_3''(\omega_1 - \omega_3)^2 + \frac{1}{6} e_3'''(\omega_1 - \omega_3)^3 \ , \tag{A5}$$

$$e_2 \approx e_3 + \frac{1}{2} e_3''(\omega_2 - \omega_3)^2 + \frac{1}{6} e_3'''(\omega_2 - \omega_3)^3 \ , \tag{A6}$$

$$e_1' \approx e_3''(\omega_1 - \omega_3) + \frac{1}{2} e_3'''(\omega_1 - \omega_3)^2 \tag{A7}$$

and

$$e_2' \approx e_3''(\omega_2 - \omega_3) + \frac{1}{2} e_3'''(\omega_2 - \omega_3)^2 \ . \tag{A8}$$

From these 4 approximate equations the 4 unknowns, namely ω_3, e_3. e_3'' and e_3''', can be solved. Among them only ω_3 is of interest here. For brevity, we use $=$ instead of $\approx$. Subtract (A5) from (A6),

$$e_2 - e_1 = (\omega_2 - \omega_1)\left[\frac{1}{2} e_3'' (\omega_2 + \omega_1 - 2\omega_3) \right.$$

$$\left. + \frac{1}{6} e_3''' (3\omega_3^2 - 3\omega_3\omega_2 - 3\omega_3\omega_1 + \omega_2^2 + \omega_1^2 + \omega_1\omega_2) \right] . \qquad \text{(A9)}$$

Add (A7) to (A8),

$$e_2' + e_1' = e_3''(\omega_2 + \omega_1 - 2\omega_3) + \frac{1}{2} e_3''' (2\omega_3^2 - 2\omega_3\omega_2 - 2\omega_3\omega_1 + \omega_2^2 + \omega_1^2) . \qquad \text{(A10)}$$

Utilizing (A10) and (A11), we define two auxiliary parameters y and x as

$$y \triangleq 3\frac{e_2 - e_1}{\omega_2 - \omega_1} - (e_2' + e_1') = \frac{1}{2}\left[e_3''(\omega_2 + \omega_1 - 2\omega_3) + e_3'''(\omega_2 - \omega_3)(\omega_1 - \omega_3) \right] \qquad \text{(A11)}$$

and

$$x \triangleq (y^2 - e_1' e_2')^{1/2} = \left[\frac{1}{4} (\omega_2 - \omega_1)^2 (e_3'')^2 \right]^{1/2} = -\frac{1}{2} (\omega_2 - \omega_1) e_3'' , \qquad \text{(A12)}$$

the minus sign on the right-hand-side of (A12) is due to the fact that $e_3'' < 0$ and $\omega_2 > \omega_1$.

Subtract (A8) and (A11) from (A12),

$$x - y - e_2' = -(\omega_2 - \omega_3)\left[2e_3'' + \frac{1}{2} e_3'''(\omega_2 + \omega_1 - 2\omega_3) \right] . \qquad \text{(A13)}$$

The task now is to suppress the terms e_3'' and e_3''' from (A13). Subtract (A8) from (A7),

$$e_1' - e_2' = -(\omega_1 - \omega_2)\left[2e_3'' + \frac{1}{2} e_3'''(\omega_2 + \omega_1 - 2\omega_3) \right] . \qquad \text{(A14)}$$

Add 2 x to (A14),

$$e_1' - e_2' + 2x = (\omega_1 - \omega_2)\left[2e_3'' + \frac{1}{2} e_3'''(\omega_2 + \omega_1 - 2\omega_3) \right] . \qquad \text{(A15)}$$

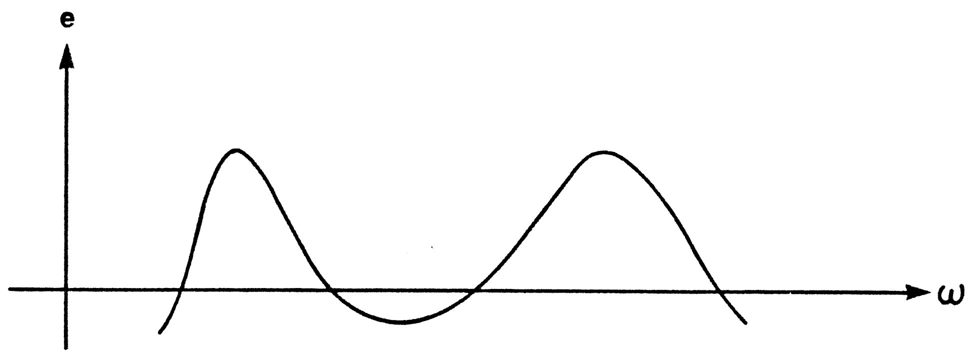
Dividing (A13) by (A15) gives

$$\frac{x - y - e_2'}{e_1' - e_2' + 2x} = -\frac{\omega_2 - \omega_3}{\omega_1 - \omega_2} . \qquad \text{(A16)}$$

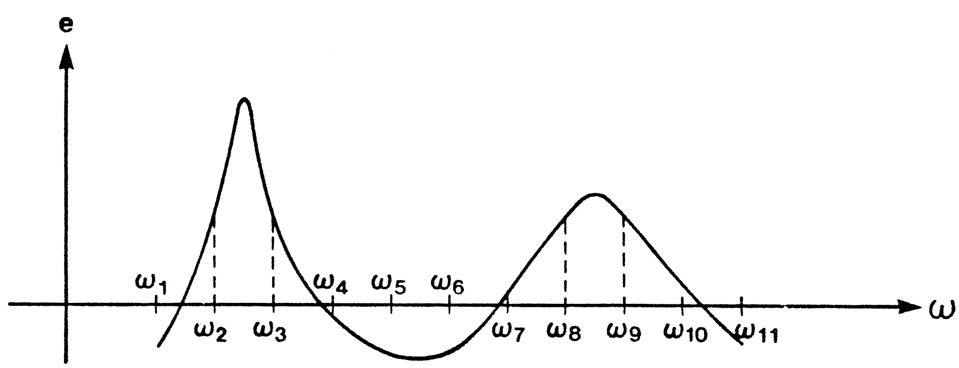
From (A16) ω_3 can be solved to be

$$\omega_3 \approx \omega_2 - \frac{(\omega_2 - \omega_1)(x - y - e_2')}{e_1' - e_2' + 2x} . \qquad \text{(A17)}$$

Equation (A17), together with (A11) and (A12) defining y and x, is the cubic interpolation formula used in this report.



(a) The continuous minimax solution.



(b) The solution obtained by using uniformly spaced sample points ω_i, $i=1,2,\ldots,11$. Notice that

$$\max_{1 \leq i \leq 11} e(\omega_i) = e(\omega_2) = e(\omega_3) = e(\omega_8) = e(\omega_9).$$

Fig. 1 Comparison between a possible solution obtained using fixed sample points and the continuous minimax solution.

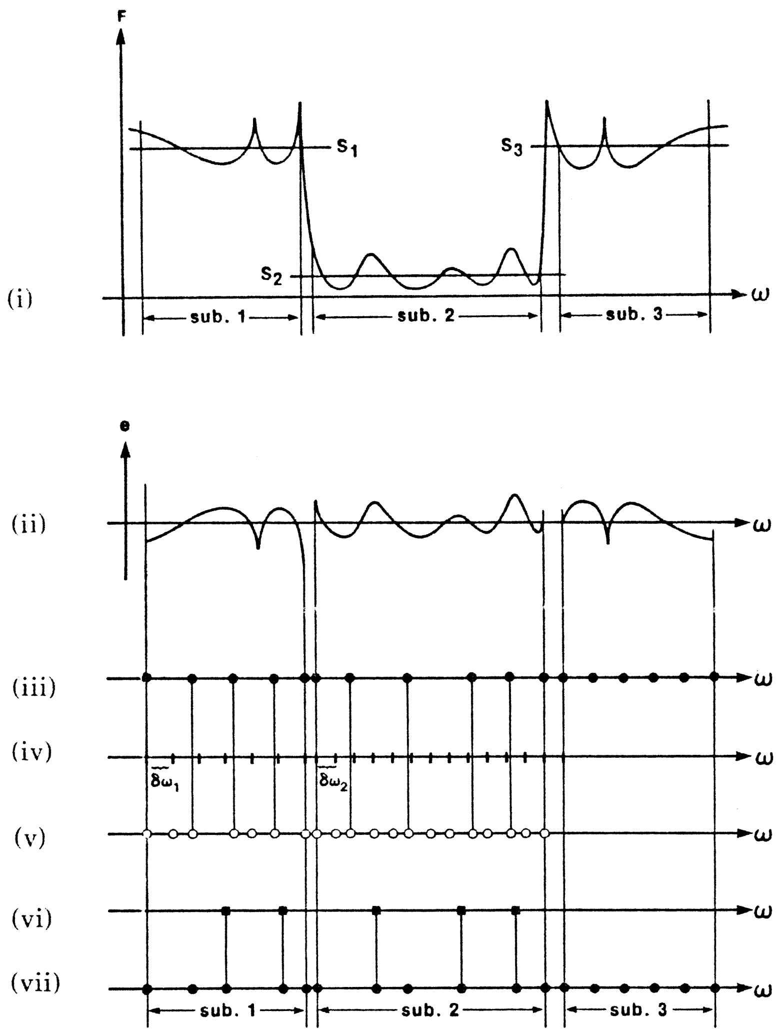


Fig. 2 Illustration of the algorithm of Section V.

(i) Response of the filter. Three subintervals are defined. S_1 and S_3 are lower specifications, S_2 is an upper specification.

(ii) The error functions. Assume unit weighting functions.

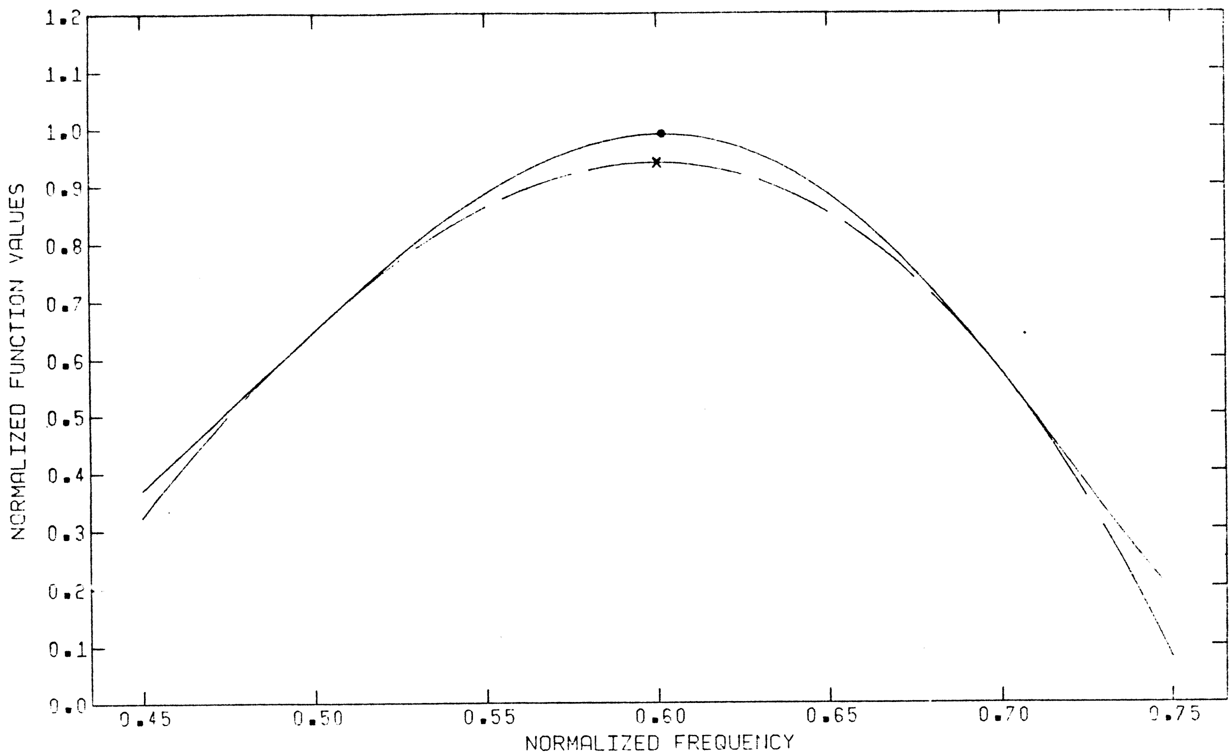
(iii) The initial positions of the sample points.

(iv) The uniformly spaced search points. $\delta\omega_1$ and $\delta\omega_2$ denote the search step-lengths of the 1st and 2nd subintervals, respectively. Fixed sample points are used for the 3rd subinterval.

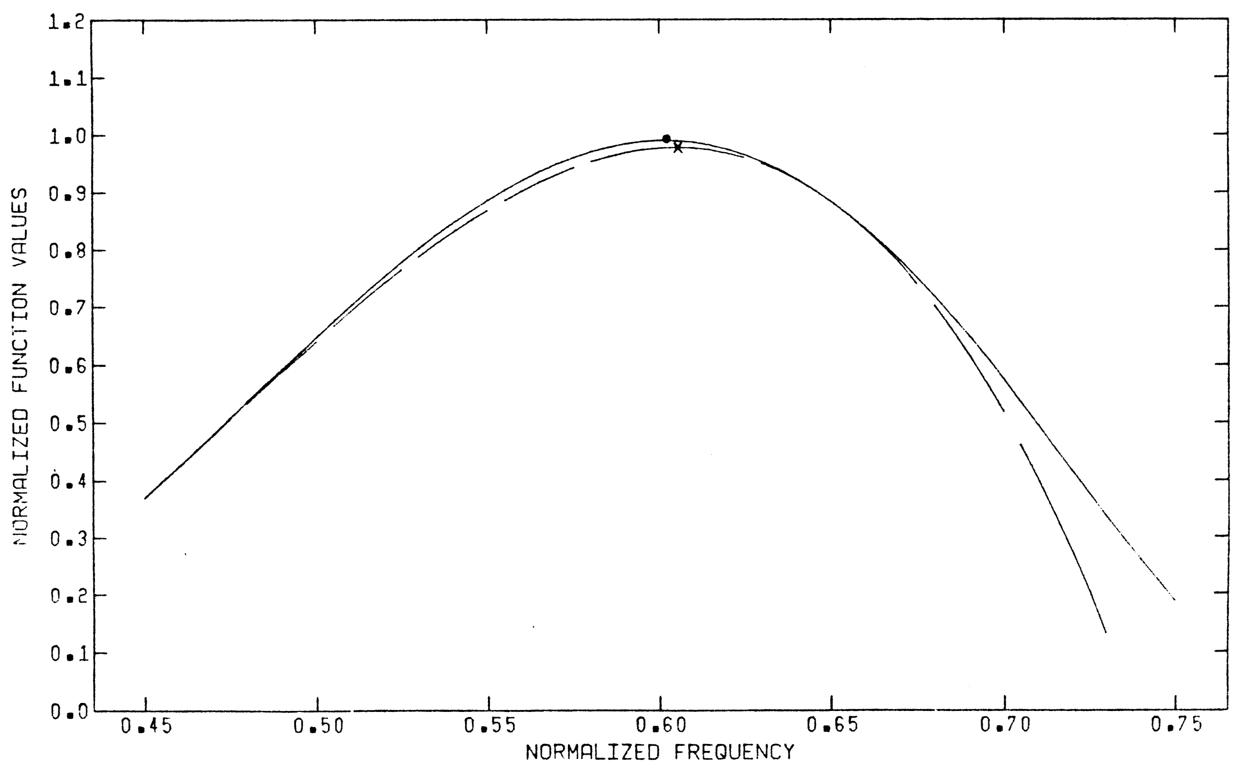
(v) The search points actually used.

(vi) The located maxima of the error functions.

(vii) The optimally selected sample points.



(i) $\omega_1 = 0.5$ and $\omega_2 = 0.7$



(ii) $\omega_1 = 0.45$ and $\omega_2 = 0.65$

Fig. 3 Two examples of locating a maximum by cubic interpolation using two search points ω_1 and ω_2 . The frequency and function values are normalized. The solid line shows a 6-th order filter function and the broken line shows the interpolating function. The exact maximum is indicated by "·" and the located maximum by "x".