SIMPLE DERIVATION OF A GENERAL SENSITIVITY FORMULA FOR LOSSLESS TWO-PORTS

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A simple, yet comprehensive proof of an important sensitivity formula for lossless two-ports stated by Orchard, Temes and Cataltepe is presented. Our derivation invokes the principle of conservation of energy and the lossless property of the network under consideration and employs the Cauchy-Riemann equations of complex differentiation. Hence, it bears clear physical interpretation and mathematical elegance.

Consider a doubly terminated lossless two-port as shown in Fig. 1. The kth internal branch is characterized by $V_k = Z_k I_k$, where $Z_k = r_k + jx_k$ and $r_k = 0$ at nominal. The real power associated with the kth branch is given by $P_k = r_k |I_k|^2$, which is equal to zero at nominal. We denote the power in the load by P_2 and define

$$P_1 \triangleq \mathrm{Re}[-I_1^* V_1] = \mathrm{Re}[I_1^*(I_1 R_1 - E)] \ . \tag{1}$$

The conservation of energy of the whole system is implied by

$$P_1 + P_2 + \sum_i P_i = P_1 + P_2 + \sum_i r_i \, |I_i|^2 = 0 \ , \tag{2}$$

where the summation is taken over all the internal branches. Differentiating (2) w.r.t. real parameters r_k and x_k, we have at $r_k = 0$

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$$\frac{\partial P_1}{\partial r_k} + \frac{\partial P_2}{\partial r_k} + |I_k|^2 = 0 \tag{3}$$

and

$$\frac{\partial P_1}{\partial x_k} + \frac{\partial P_2}{\partial x_k} = 0 \ , \tag{4}$$

respectively. Differentiating (1) we have, after simple manipulations,

$$\frac{\partial P_1}{\partial \phi} = \mathrm{Re}\left[(2R_1 I_1^* - E) \frac{\partial I_1}{\partial \phi} \right] = -\mathrm{Re}\left[\rho_1^* E \frac{\partial I_1}{\partial \phi} \right] \ , \tag{5}$$

where ρ_1 is the input reflection coefficient. Utilizing the well-known results, as given by Bandler [1], of $E\,\partial I_1/\partial r_k = -I_k^2$ and $E\,\partial I_1/\partial x_k = -jI_k^2$, we have

$$\frac{\partial P_1}{\partial r_k} = \mathrm{Re}[\rho_1^* I_k^2] \ , \qquad \frac{\partial P_1}{\partial x_k} = \mathrm{Re}[j\rho_1^* I_k^2] \ . \tag{6}$$

The two expressions in (6) can be combined to give

$$\frac{\partial P_1}{\partial r_k} - j \frac{\partial P_1}{\partial x_k} = \rho_1^* I_k^2 \ . \tag{7}$$

From (3), (4) and (7) it follows that

$$\frac{\partial P_2}{\partial r_k} - j \frac{\partial P_2}{\partial x_k} = -[|I_k|^2 + \rho_1^* I_k^2] \ . \tag{8}$$

The complex valued transducer coefficient of the network

$$\theta \triangleq \ell n \frac{E}{2V_2}\sqrt{\frac{R_2}{R_1}} = \alpha + j\beta \tag{9}$$

is analytical in the network parameters wherever it is defined. Differentiating θ w.r.t. the complex variable $Z_k = r_k + jx_k$, we know that the Cauchy-Riemann equations are satisfied as

$$\frac{\partial \alpha}{\partial r_k} = \frac{\partial \beta}{\partial x_k} \ , \quad \frac{\partial \alpha}{\partial x_k} = -\frac{\partial \beta}{\partial r_k} \tag{10}$$

and $\partial\theta/\partial Z_k$ is given by (see, for example, Lang [2])

$$\frac{\partial \theta}{\partial Z_k} = \frac{\partial \alpha}{\partial r_k} - j \frac{\partial \alpha}{\partial x_k} \ . \tag{11}$$

We find that

$$\alpha = \mathrm{Re}\left[\ell n \frac{E}{2V_2}\sqrt{\frac{R_2}{R_1}} \right] = \frac{1}{2}\left[\ell n \frac{E^2}{4R_1} - \ell n \frac{|V_2|^2}{R_2} \right] = \frac{1}{2}\left[\ell n \frac{E^2}{4R_1} - \ell n P_2 \right] \ , \tag{12}$$

which leads to

$$\frac{\partial \alpha}{\partial \phi} = - \frac{1}{2P_2} \frac{\partial P_2}{\partial \phi} . \qquad (13)$$

Combining (8), (11) and (13) we obtain

$$\frac{\partial \theta}{\partial Z_k} = \frac{1}{2P_2} [\,|I_k|^2 + \rho_1^* I_k^2] . \qquad (14)$$

Equation (14) agrees with the basic sensitivity formula stated by Orchard, Temes and Cataltepe [3]. A dual formula for $\partial\theta/\partial Y_k$, Y_k being the admittance of the kth branch, can be easily derived in a very similar manner.

Our result proves that one solution of the original network, due to its lossless property, contains sufficient information for a complete first-order sensitivity analysis of its external behaviour.

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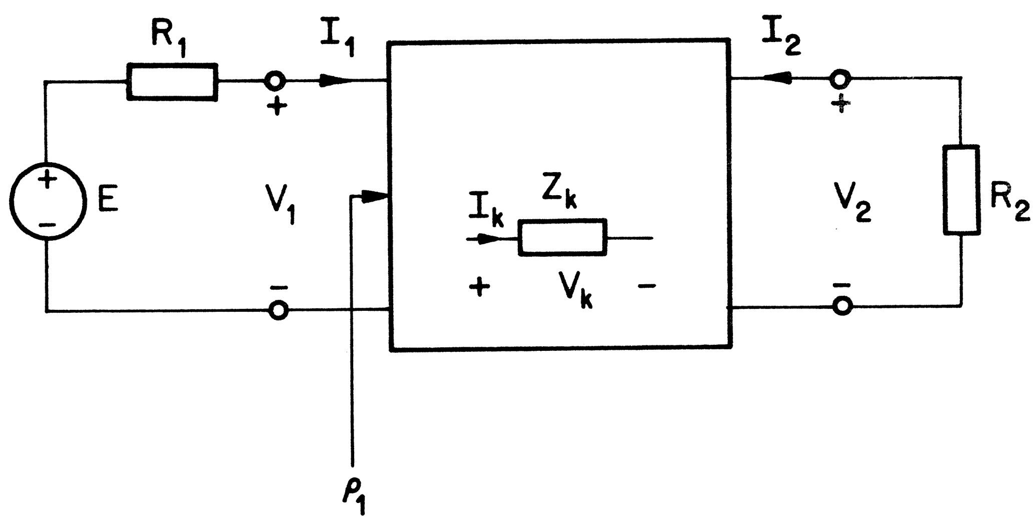

Fig. 1 Doubly terminated lossless two-port.