SIMULATION OPTIMIZATION SYSTEMS
Research Laboratory

COLLECTED PROBLEMS IN COMPUTATIONAL METHODS, DESIGN AND OPTIMIZATION

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SOS-83-16-N

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IMPORTANT NOTE

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Question 1 Develop an algorithm to efficiently calculate the value of

$$\frac{a_0 + a_2 s^2 + a_4 s^4 + \dots + a_n s^n}{b_1 s + b_3 s^3 + \dots + b_m s^m}$$

given m, n, the coefficients and s. Test m and n. State the number of multiplications and divisions and the number of additions and subtractions.

Question 2 Develop an algorithm to efficiently calculate the value of

$$Z_0 \frac{Z_L + jZ_0 \tan\theta}{Z_0 + jZ_L \tan\theta}$$

given real Z_0 , $0 \leq \theta \leq \pi$ and complex Z_L. Avoid $\theta = \frac{\pi}{2}$. State the of multiplications and divisions, the number of additions and subtractions and the number of calls to a trigonometric function evaluation routine.

Question 3 Develop an algorithm to efficiently calculate the value of

$$a \sinh x + b \tanh x$$

given a, b and e^x. State the number of multiplications and divisions, the number of additions and subtractions and the number of calls to function subprograms.

Question 4 State Horner's rule for polynomial evaluation. Explain its advantages compared with the direct method of evaluating a polynomial.

Question 5 Develop an algorithm to calculate as efficiently as possible the value of

$$a_1 \sin \theta + a_3 \sin 3\theta + a_5 \sin 5\theta$$

given a_1, a_3, a_5 and θ. State the number of multiplications and divisions, the number of additions and subtractions and the number of calls to a trigonometric function evaluation routine.

Question 6 Write an efficient algorithm for converting binary numbers to decimal numbers. Test it on the numbers 1101, 10111 and 1010101.

Question 7 Write and test on 44 an efficient algorithm for converting decimal numbers to binary numbers.

Question 8 Write an algorithm to efficiently evaluate $\underset{\sim}{\nabla} F$ and $\partial F/\partial s$ where

$$F(\underset{\sim}{\phi}, s) = \sum_{i=0}^{n} a_i s^i$$

and

$$\underset{\sim}{\phi} = \begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix}, \quad \underset{\sim}{\nabla} F = \begin{bmatrix} \partial F/\partial a_0 \\ \partial F/\partial a_1 \\ \cdot \\ \cdot \\ \cdot \\ \partial F/\partial a_n \end{bmatrix}.$$

Question 9 Write an algorithm to efficiently calculate the value of the objective function

$$U(\underset{\sim}{\phi}) = \sum_{i=1}^{n} (F(\underset{\sim}{\phi},t_i) - S(t_i))^2$$

and the gradient vector $\underset{\sim}{\nabla} U(\underset{\sim}{\phi})$ m times for different $\underset{\sim}{\phi}$, where

$$S(t) = \frac{3}{20} e^{-t} + \frac{1}{52} e^{-5t} - \frac{1}{65} e^{-2t} (3\sin 2t + 11\cos 2t)$$

is the specified function of time t (system response)

$$F(\underset{\sim}{\phi},t) = \frac{c}{\beta} e^{-\alpha t} \sin \beta t$$

is the approximating function of time (model response),

$$\underset{\sim}{\phi} \triangleq \begin{bmatrix} \alpha \\ \beta \\ c \end{bmatrix} \text{ and } \underset{\sim}{\nabla} \triangleq \begin{bmatrix} \frac{\partial}{\partial\phi_1} \\ \frac{\partial}{\partial\phi_2} \\ \frac{\partial}{\partial\phi_3} \end{bmatrix} .$$

Question 10 Write an algorithm to efficiently calculate the frequency response $V_2(j\omega)/V_1(j\omega)$ for the circuit of Fig. 1. Use the algorithm to calculate the response when $L_1 = L_2 = 2H$, $C_1 = C_2 = 0.5F$, and $\omega = 2$ rad/s.

Question 11 Write an algorithm to efficiently evaluate $\underset{\sim}{\nabla} F$ where

$$F(\underset{\sim}{\phi},s) = \frac{\sum_{i=0}^{n} a_i s^i}{\sum_{i=0}^{m} b_i s^i}$$

and $\underset{\sim}{\phi} = [a_0 \; a_1 \; \dots \; a_n \; b_0 \; b_1 \; \dots \; b_m]^T$.

Question 12 Write an algorithm to efficiently evaluate $\underset{\sim}{\nabla}T$ where

$$T(\underset{\sim}{\phi},s) = \frac{1}{R_1R_2C_1C_2s^2 + (R_1C_1 + R_1C_2 + R_2C_2)s + 1}$$

and

$$\underset{\sim}{\phi} = \begin{bmatrix} R_1 \\ C_1 \\ R_2 \\ C_2 \end{bmatrix} .$$

$T(s) = V_2(s)/V_1(s)$ for the circuit of Fig. 2.

Question 13 Show how the errors propagate in the calculation of

(a) $\frac{a}{b - cd}$,

(b) $\frac{a}{b(c - d)}$,

(c) $\frac{xy}{u - v}$.

What is the relative error? Assuming all results are subject to the same roundoff errors, develop an expression yielding the maximum possible error.

Question 14 Derive an expression for the relative error in the computation of x/y. Neglect terms involving products of errors.

Question 15 Calculate and state the maximum number of multiplications and divisions in the efficient solution for $\underset{\sim}{x}$ of the linear system

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x \\ x \\ x \\ x \end{bmatrix} ,$$

where $\underset{\sim}{x} \overset{\Delta}{=} [x_1 \; x_2 \; x_3 \; x_4]^T$.

Question 16 Write an efficient Fortran program to calculate all the branch voltages and currents in the resistive ladder network of Fig. 3, allowing up to 100 resistors. Essential data: V_g, R_1, R_2, ..., R_n.

Let $n = 8$, $R_1 = R_3 = R_5 = R_7 = 3\Omega$, $R_2 = R_4 = R_6 = R_8 = 1\Omega$. Calculate the voltages and currents for $V_g = 1V$ using the program written.

Question 17 Write a program to calculate the input resistance of the circuit of Question 16. Use the program written to calculate the input resistance for the numerical example in Question 16.

Question 18 Write an efficient Fortran program using LU factorization to calculate and print out all the branch voltages and currents of the resistive ladder network of Fig. 4, allowing up to 99 resistors. Take account of symmetry and the tridiagonal nature of the admittance matrix. Essential data: V_g, R_1, R_2, ..., R_n.

Let $n = 7$, $R_2 = R_4 = R_6 = 1/3\Omega$, $R_1 = R_3 = R_5 = R_7 = 1\Omega$. Calculate the voltages and currents for $V_g = 1V$ using the program.

Question 19 Write a program to calculate the input conductance of the circuit of Question 18. Use the program written to calculate the input conductance for the numerical example in Question 18.

Question 20 Consider the ladder network of Fig. 5.

(a) Showing clearly all major steps, calculate the node voltages by
  (i) matrix inversion,
  (ii) LU factorization.

(b) What is the computational effort involved in (a)?

(c) Set the right-hand source to zero and recalculate the node voltages. In general, what would the computational effort be for different excitations?

Question 21 Is the inverse of a tridiagonal matrix (in general) sparse, dense or tridiagonal? Justify your answer by a physically meaningful example.

Question 22 Define the term "relaxation method".

Question 23 State the Gauss-Seidel iterative formula for the solution of the linear system $\underset{\sim}{A}\ \underset{\sim}{x} = \underset{\sim}{b}$, defining precisely any new symbols introduced.

Question 24 Factorize the following matrix into LU form utilizing available storage locations as much as possible:

$$\begin{bmatrix} 5 & -1 & 0 & 0 \\ -1 & 6 & -1 & 0 \\ 0 & -1 & 6 & -1 \\ 0 & 0 & -1 & 5 \end{bmatrix}$$

<u>Question 25</u> Consider the resistive network in Fig. 6. Take $G_1 = 2$ and $G_3 = 1$ mho. Showing clearly all major steps, calculate the node voltages by LU factorization.

<u>Question 26</u> Apply the Gauss-Seidel (relaxation) method to the circuit of Question 25. Take the initial node voltages as zero and use two iterations. Repeat with an overrelaxation factor of 1.5.

<u>Question 27</u> Consider the resistive network shown in Fig. 7. Take $G_1 = G_3 = G_5 = 1$ mho and $R_2 = R_4 = 0.5$ ohm. Apply the Gauss-Seidel (relaxation) method to this network. Take the initial node voltages as zero and use two iterations. Repeat with an overrelaxation factor of 1.5.

<u>Question 28</u> Consider the resistive network shown in Fig. 8. Let $G_1 = 1$ and $G_2 = 2$. Showing clearly all major steps, apply two iterations of the Gauss-Seidel relaxation method starting with $v_1 = 1$, $v_2 = 0.5$, $v_3 = 0$. <u>Continue</u> the solution process with two iterations using an overrelaxation factor of 1.75. Expressing the nodal equations as error functions, calculate the Euclidean norm of the errors for each iteration.

Question 29 Consider the circuit shown in Fig. 9, which is operating in the sinusoidal steady state. Find V_3/V_1 for this circuit at $\omega = 2$ rad/s in the following ways, comparing the effort required. Take $R_1 = R_2 = R_3 = 2\Omega$, $C_1 = C_2 = C_3 = 1F$. Show clearly all the steps in your calculations.

(a) From an analytical expression of $V_3(s)/V_1(s)$ derived by the Gauss elimination method.

(b) By actual numerical inversion of the nodal admittance matrix.

(c) By LU factorization of the nodal admittance matrix.

(d) By assuming V_3 and working backwards.

(e) By ABCD or chain matrix analysis.

Question 30 Consider the circuit shown in Fig. 10, which is operating in the sinusoidal steady state. Find V_3/V_1 for this network at $\omega = 1$ rad/s in the following ways. Take $R_1 = R_2 = R_3 = 1\Omega$, $C_1 = C_2 = C_3 = 2F$. Show clearly all the steps in your calculations.

(a) From an analytical expression of $V_3(s)/V_1(s)$. Use the Gauss elimination method.

(b) By actual numerical inversion of the nodal admittance matrix.

(c) By LU factorization of the nodal admittance matrix.

(d) By network reduction.

(e) By assuming a value for V_3 and working back through the ladder.

Question 31 Apply the Gauss-Seidel (relaxation) method to the circuit of Question 30. Take the initial node voltages to be zero and use two iterations. Repeat over with an overrelaxation factor of 1.5.

Question 32 Calculate and plot the reflection coefficient of the circuit shown in Fig. 11, where $C_1 = 1.0F$, $C_2 = 0.125F$, $L = 2.0H$, $0 \leq \omega \leq 4$ rad/s.

Question 33 Consider the iterative scheme

$$\underset{\sim}{y}^{i+1} = \underset{\sim}{A}^{i}\underset{\sim}{y}^{i}, \quad i = 1, 2, \ldots, n$$

where the $\underset{\sim}{y}$ vectors are of dimension 2 and the $\underset{\sim}{A}$ matrices are 2 x 2 with known values. Given the terminating conditions

$$y_1^{n+1} = 1,$$

$$y_1^1 = c^1 y_2^1,$$

where c^1 is known, derive an analogous iterative scheme culminating in the evaluation of $\underset{\sim}{y}^1$.

Question 34 Consider the iterative scheme described in Question 33. Given the terminating condition

$$y_1^1 = c^1 y_2^1 ,$$

where c^1 is known, develop a computational scheme to evaluate

$$c^n = y_1^n / y_2^n.$$

Question 35 Assume that each matrix $\underset{\sim}{A}^i$ in Question 33 is a function of a single variable x_i. Derive from first principles an approach to calculating $\partial y_1^1 / \partial \underset{\sim}{x}$, where $\underset{\sim}{x}$ is a column vector containing the x_i, $i = 1, 2, \ldots, n$.

Question 36 Consider the system described by the iterative schemes

$$\underset{\sim}{y}^{i+1} = \underset{\sim}{A}^{i}\underset{\sim}{y}^{i}, \quad i = 1, 2, \ldots, n, \; i \neq j.$$

$$\underset{\sim}{z}^{i+1} = \underset{\sim}{B}^{i}\underset{\sim}{z}^{i}, \quad i = 1, 2, \ldots, m,$$

the equation

$$\underset{\sim}{C} \begin{bmatrix} y_1^{j} \\ y_1^{j+1} \\ z_1^{m+1} \end{bmatrix} = \begin{bmatrix} -y_2^{j} \\ y_2^{j+1} \\ -z_2^{m+1} \end{bmatrix},$$

the terminating conditions

$$z_1^1 = z_2^1,$$
$$y_1^1 = y_2^1,$$
$$y_1^{n+1} = 1,$$

where the $\underset{\sim}{y}$ and $\underset{\sim}{z}$ vectors are of dimension 2 and the $\underset{\sim}{A}$ and $\underset{\sim}{B}$ matrices are 2 x 2 with known values and $\underset{\sim}{C}$ is a given 3 x 3 matrix.

Carefully describe and explain an algorithm for evaluating y_2^{n+1} efficiently.

Question 37 Use the multi-dimensional Taylor series expansion to show that a turning point of a convex differentiable function is a global minimum. Justify all assumptions.

Question 38 Given a differentiable function f of many variables $\underset{\sim}{x}$ and a corresponding direction vector $\underset{\sim}{s}$,

$$\lim_{\lambda \to 0^+} \frac{f(\underset{\sim}{x} + \lambda \underset{\sim}{s}) - f(\underset{\sim}{x})}{\lambda} = \,.\,.\,.\,.\,.\,.\,.\,.\,.\, \text{(please state) ?}$$

Explain in a few words the meaning of the above expression.

Question 39 Use the method of Lagrange multipliers to prove that the greatest first-order change in a function of many variables occurs, for a given step size, in the direction of the gradient vector w.r.t. the variables.

Question 40 Use the method of Lagrange multipliers to minimize w.r.t. ϕ_1 and ϕ_2 the function

$$U = \phi_1^2 + \phi_2^2$$

subject to

$$\phi_1 + \phi_2 = 1$$

Sketch a diagram to illustrate the problem and its solution w.r.t. ϕ_1 and ϕ_2. Verify your answer by substituting the constraint into the function.

Question 41 If $\underset{\sim}{g}(\underset{\sim}{\phi})$ is concave, verify that $\underset{\sim}{g}(\underset{\sim}{\phi}) \geq \underset{\sim}{0}$ describes a convex feasible region.

Question 42 Under what conditions could equality constraints be included in convex programming?

Question 43 Comment on each of the following concepts independently.

(a) The minimum of $(\phi - a)^2$ and the maximum of $b - (\phi - a)^2$, where a and b are constants.

(b) The minimum of U, where

$$U = \begin{cases} -2\phi + 2, & \phi \leq 1 \\ \phi - 1, & \phi \geq 1 \end{cases}$$

and the minimum of U subject to $0 \leq \phi \leq 3$.

(c) The minimum of $a\phi^2 + b$ contrasted with the minimum of $a\phi^2 + b$ subject to $\phi \geq 0$, where a, b are constants.

(d) The number of equality constraints in a nonlinear program will generally be less than the number of independent variables.

Question 44 Find suitable transformations for the following constraints so that we can use an unconstrained optimization algorithm.

(a) $0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq \phi_i \leq \ldots \leq \phi_k$.

(b) $0 < \ell \leq \phi_2/\phi_1 \leq u$, $\phi_1 > 0$, $\phi_2 > 0$.

Question 45 Write the following constraints in the form $g_i(\underset{\sim}{\phi}) \geq 0$, i = 1, 2, ..., m.

(a) $\ell_i \leq \phi_i \leq u_i$, i = 1, 2, ..., k.

(b) $a \leq \phi_i/\phi_{i+1} \leq b$, i = 1, 2, ..., k-1.

(c) $1 \leq \phi_1 \leq \phi_2 \leq \ldots \leq \phi_k \leq 3$.

(d) $h_i(\underset{\sim}{\phi}) = 0$, i = 1, 2, ..., s.

Question 46 Discuss the scaling effects of the transformation $\phi_i = \exp \phi_i'$.

Question 47 Use an appropriate transformation to create the minimization of an unconstrained objective function for the problems

(a) $\underset{\phi}{\text{minimize}}\ U = b\phi + c$ subject to $\phi \geq 0$ with $b > 0$.

(b) $\underset{\underset{\sim}{\phi}}{\text{minimize}}\ U = a_1\phi_1^2 + a_2\phi_2^2$ subject to $1 \leq \phi_i \leq 2$, $i = 1, 2$ with a_1, $a_2 > 0$.

Question 48 Derive the gradient vector of $U(\underset{\sim}{\phi})$ w.r.t. $\underset{\sim}{\phi}$ for the objective functions

$$U = \int_{\psi_\ell}^{\psi_u} | e(\underset{\sim}{\phi},\psi) |^p \, d\psi$$

and

$$U = \sum_{i=1}^{n} | e_i(\underset{\sim}{\phi}) |^p,$$

where the appropriate error functions are complex.

Question 49 For the linear function (a polynomial is a special case)

$$F(\underset{\sim}{\phi},\psi) = \sum_{i=1}^{k} \phi_i f_i(\psi),$$

(a) Formulate the discrete minimax approximation of $S(\psi)$ by $F(\underset{\sim}{\phi},\psi)$ as a linear programming problem, assuming $\underset{\sim}{\phi}$ to be unconstrained.

(b) Assuming an objective function of the form of

$$U = \sum_{i=1}^{n} [e_1(\underset{\sim}{\phi})]^p$$

derive the gradient vector of U and the Hessian matrix w.r.t. $\underset{\sim}{\phi}$.

Question 50 Derive and compare the Newton methods for (a) minimization of a nonlinear differentiable objective function of many variables (as required in design), and (b) solving systems of nonlinear simultaneous equations (as required in nonlinear d.c. network analysis). Sketch carefully each process for a single nonlinear function of a single variable indicating the various iterations. Under what conditions would you expect divergence from the solution?

Question 51 Derive from first principles Newton's method for function minimization w.r.t. many variables. Define all symbols introduced. Under what conditions would you expect convergence to a minimum? Prove that the direction of search is downhill if the Hessian matrix is positive definite. Sketch diagrams w.r.t. one variable showing

(a) convergence to a minimum,

(b) convergence to a maximum, and

(c) oscillatory behaviour.

Describe and explain the "damped" Newton method.

Question 52 Derive carefully from first principles a numerical approach to finding the gradient vector $\partial f/\partial \underset{\sim}{x}$ subject to the system of equations $\underset{\sim}{h}(\underset{\sim}{x},\underset{\sim}{y}) = \underset{\sim}{0}$ given values for $\underset{\sim}{x}$, where $f \equiv f(\underset{\sim}{y}(\underset{\sim}{x}), \underset{\sim}{x})$ is a scalar function and where the vector $\underset{\sim}{h}$ is nonlinear both in $\underset{\sim}{x}$ and in $\underset{\sim}{y}$. Assume that $\underset{\sim}{h}$ and $\underset{\sim}{y}$ have the same dimensions and that the Jacobian of $\underset{\sim}{h}$ w.r.t. $\underset{\sim}{y}$ is nonsingular. Define all symbols used, and exhibit the structure of all matrices employed. Summarize the main steps of the computational procedure you would employ to solve a large problem.

Question 53 Define the term "positive definite" as it relates to a square symmetric matrix.

Question 54 Provide and discuss a link between the Hessian matrix of a differentiable function $U(\underset{\sim}{\phi})$, where $\underset{\sim}{\phi}$ is a k-vector, with the Jacobian matrix of $\underset{\sim}{f}(\underset{\sim}{\phi})$, where $\underset{\sim}{f}$ is a k-vector of functions of $\underset{\sim}{\phi}$.

Question 55 For the resistor-diode network shown in Fig. 12, illustrate with the aid of an i-v diagram an iterative method of finding v at d.c. State Newton's method for solving this problem and derive the network model corresponding to the situation at the jth iteration. What is the significance of this model?

Question 56 We wish to calculate $\partial f/\partial \underset{\sim}{x}$ subject to $\underset{\sim}{h}(\underset{\sim}{x},\underset{\sim}{y}) = \underset{\sim}{0}$ where $f \equiv f(\underset{\sim}{y}(\underset{\sim}{x}), \underset{\sim}{x})$ given values for $\underset{\sim}{x}$.

Explain fully the formula

$$\left.\frac{\partial f}{\partial \underset{\sim}{x}}\right|_{\underset{\sim}{h}=\underset{\sim}{0}} = - \frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{x}} \hat{\underset{\sim}{y}} + \frac{\partial f}{\partial \underset{\sim}{x}} ,$$

where $\hat{\underset{\sim}{y}}$ is the solution to

$$\left(\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{y}}\right) \hat{\underset{\sim}{y}} = \frac{\partial f}{\partial \underset{\sim}{y}} .$$

Describe the computational and analytical effort required in any given problem.

Let

$$4x_1^2y_1^2 - 3\, y_2 - 2 = 0,$$

$$-x_1y_1 + 2x_2^2\, y_1y_2 - 3 = 0,$$

$$f = y_1^2 + y_2^2 + 2x_2.$$

Set up all the matrices and vectors required both for the solution of the nonlinear equations and also for the evaluation of $\partial f/\partial \underset{\sim}{x}$ s.t. $\underset{\sim}{h} = \underset{\sim}{0}$.

Question 57 Write down and define the first three terms of the multidimensional Taylor series expansion of a scalar function U of many variables $\underset{\sim}{\phi}$, defining any expressions used appropriately.

Question 58 Show that a step in the negative gradient direction reduces the function (neglecting second and higher-order terms) unless the gradient vector is zero.

Question 59 Derive a formula to approximately calculate all first partial derivatives of a function of k variables by perturbation, using 2k function evaluations.

Question 60 What are the implication of a positive-semidefinite Hessian matrix in minimization problems?

Question 61 Derive Newton's method for function minimization. Explain under what conditions you would expect convergence. Sketch the algorithm for a function of one variable showing

(i) a convergent process, and

(ii) a divergent process.

Question 62 Write down a quadratic function of many variables and express its gradient vector and Hessian matrix in terms of constants involved in the function.

Question 63 Write down an objective function which can be minimized in an effort to solve the system of nonlinear equations $\underset{\sim}{f} = \underset{\sim}{0}$. Differentiate it w.r.t. the variables and express the gradient vector in compact form.

Question 64 What is the implication of a negative first-order term in the multidimensional Taylor expansion of a differentiable function of many variables? Sketch your answer w.r.t. a function of two variables.

Question 65 State the principle behind the steepest descent approach to minimizing functions and sketch carefully the path taken on a contour diagram w.r.t. two variables.

Question 66 Write a simple Fortran program to implement steepest descent in the minimization of a scalar differentiable function of many variables and test it on suitable examples.

Question 67 Write a simple program to implement the one-at-a-time method of direct search for the minimization without derivatives of a function of many variables and test it on suitable examples.

Question 68 Describe the pattern search algorithm. Illustrate it on two-dimensional sketches of contours of a function to be minimized, noting exploratory moves, pattern moves and base points. Discuss any advantages enjoyed by this search method.

Question 69 Contrast the method of steepest descent with the method of changing one variable at a time to minimize an unconstrained function. Provide algorithms for both methods.

Question 70 Describe pitfalls in attempting the solution of constrained optimization problems using the algorithms of Question 69.

Question 71 Explain the concept norm. Give examples in (a) the continuous, (b) the discrete, approximation of a specified function of an independent variable by an appropriate function of many variables on a given interval of the independent variable. Use diagrams to illustrate your answer.

Question 72 For an electrical circuit design problem with upper and lower response specifications, explain the role of relative differences in the weighting factor(s) in the error functions. Distinguish the cases of specifications violated and specifications satisfied.

Question 73 Sketch contour and vector diagrams relating to constrained optimization problems illustrating the application of Kuhn-Tucker (KT) necessary conditions and showing

(a) Points satisfying the KT conditions for minimization.

(b) Points satisfying the KT conditions for maximization.

(c) Points not satisfying the KT conditions for either maximization or minimization.

Question 74

(a) What is a convex function?

(b) What is a convex region?

(c) How are these concepts related to a nonlinear optimization problem?

Question 75 Discuss the necessary conditions for an unconstrained optimum of a differentiable function. Derive them from

(a) Conditions for a minimax optimum.

(b) Conditions for a constrained minimum.

Question 76 Sketch contours and vector diagrams to illustrate the application of the Kuhn-Tucker (KT) conditions for a point satisfying the KT conditions for maximization of a constrained function.

Question 77 Sketch curves of $|x - x^0|^p$ against x for p = 0.5, 1, 2, 4 and ∞. Discuss the differentiability and convexity of these curves.

Question 78 Sketch in two dimensions the unit spheres centered at $\underset{\sim}{x}^0$ defined by

$$||\underset{\sim}{x} - \underset{\sim}{x}^0||_p \leq 1$$

for p = 1, 2, 4 and ∞. Comment on the convexity of these regions and the corresponding one for p = 0.5.

Question 79 Derive the necessary conditions (NC) for a minimax optimum for a set of nonlinear differentiable functions from the Kuhn-Tucker conditions (necessary conditions for a constrained minimum). Illustrate the results for the special cases of

(a) a single function satisfying NC,

(b) two active functions satisfying NC,

(c) three active functions satisfying NC,

(d) two active functions not satisfying NC.

Question 80 Draw a diagram for violated specifications that would illustrate the situation of multiple optimization of the frequency response and time response of an electrical circuit. Write down error functions in a form suitable for minimax optimization.

Question 81 Set up as a minimization problem the solution of the complex nodal equations of a linear analog circuit, required simultaneously for a number of frequencies. Identify clearly and compactly the objective function, the variables and any necessary gradient vectors required by the optimization program.

Question 82 Consider the problem of minimizing

$$U = \phi_3(\phi_1 + \phi_2)^2$$

subject to

$$g_1 = \phi_1 - \phi_2^2 \geq 0, \; g_2 = \phi_2 \geq 0, \; h = (\phi_1 + \phi_2)\phi_3 - 1 = 0.$$

Is this a convex programming problem? Formulate it for solution by the sequential unconstrained minimization method. Starting with a feasible point, show how the constrained minimum is approached as the parameter r → 0. Draw a contour sketch to illustrate the process. Are the conditions for a constrained minimum satisfied?

Question 83 Apply the Fletcher-Powell-Davidon updating formula to the minimization of

$$\phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + 2\phi_1 + 1$$

w.r.t. ϕ_1 and ϕ_2 starting at $\phi_1 = 0$, $\phi_2 = 0$, showing all steps explicitly and commenting on the results obtained.

Question 84 Apply the conjugate gradient algorithm for minimizing a differentiable function of many variables to the minimization of

$$\phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + 2\phi_1 + 1$$

w.r.t. ϕ_1 and ϕ_2 starting at $\phi_1 = 0$, $\phi_2 = 0$, showing all steps explicitly and commenting on the results obtained.

Question 85 Apply the conjugate gradient algorithm for minimizing a differentiable function of many variables to the following data.

$$\text{Point:} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 2 \end{bmatrix}, \begin{bmatrix} 8.4 \\ 2.45 \end{bmatrix}, \ldots$$

$$\text{Gradient:} \quad \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}, \ldots$$

Sketch contours of a reasonable function that might have produced these numbers and plot the path taken by the algorithm.

Question 86 Consider the linear programming problem

$$\text{minimize } \phi_1 + 0.5\,\phi_2 - 1$$

w.r.t. ϕ_1, ϕ_2 subject to $\phi_1 \geqslant 0$, $\phi_2 \geqslant 0$, $\phi_1 + \phi_2 \geqslant 1$. Starting at $\phi_1 = 2$, $\phi_2 = 0$, solve this analytically by steepest descent. Show how two one-dimensional searches yield the exact solution. Verify that the

Kuhn-Tucker relations (the necessary conditions for an optimum) are satisfied only at the solution.

Question 87 Minimize w.r.t. $\underset{\sim}{\phi}$

$$U = \phi_1^2 + 4\phi_2^2$$

subject to

$$\phi_1 + 2\phi_2 - 1 = 0$$

The function has a minimum value of 0.5 at $\phi_1 = 0.5$, $\phi_2 = 0.25$. Suggested starting point: $\phi_1 = \phi_2 = 1$.

[Source: Fletcher (1970). See also Charalambous (1973).]

Question 88 Sketch contours of the function

$$V = \max[U,\ U + \alpha h,\ U - \alpha h]$$

w.r.t. $\underset{\sim}{\phi}$ for $U = \phi_1^2 + 4\phi_2^2$ and $h = \phi_1 + 2\phi_2 - 1$ in the vicinity of the solution stated in Question 87 for $\alpha = 0.1$, 1.0 and 100, taking care to indicate points of discontinuous derivatives.

[Source: Bandler and Charalambous (1974).]

Question 89 Minimize w.r.t. $\underset{\sim}{\phi}$

$$f = -\phi_1 \phi_2 \phi_3$$

subject to

$$\phi_1 \geq 0,\ i = 1, 2, 3,$$

$$20 - \phi_1 \geq 0,\ 11 - \phi_2 \geq 0,\ 42 - \phi_3 \geq 0,$$

$$72 - \phi_1 - 2\phi_2 - 2\phi_3 \geq 0.$$

The function has a minimum of -3300 at $\phi_1 = 20$, $\phi_2 = 11$, $\phi_3 = 15$. This problem is referred to as the Post Office Parcel problem.

[Source: Rosenbrock (1960). See also Bandler and Charalambous (1974).]

<u>Question 90</u> Minimize w.r.t. $\underset{\sim}{\phi}$

$$f = \phi_1^2 + \phi_2^2 + 2\phi_3^2 + \phi_4^2 - 5\phi_1 - 5\phi_2 - 21\phi_3 + 7\phi_4$$

subject to

$$-\phi_1^2 - \phi_2^2 - \phi_3^2 - \phi_4^2 - \phi_1 + \phi_2 - \phi_3 + \phi_4 + 8 \geq 0,$$
$$-\phi_1^2 - 2\phi_2^2 - \phi_3^2 - 2\phi_4^2 + \phi_1 + \phi_4 + 10 \geq 0,$$
$$-2\phi_1^2 - \phi_2^2 - \phi_3^2 - 2\phi_1 + \phi_2 + \phi_4 + 5 \geq 0.$$

The function has a minimum of -44 at $\phi_1 = 0$, $\phi_2 = 1$, $\phi_3 = 2$, $\phi_4 = -1$. Suggested starting point: $\phi_1 = 0$, $\phi_2 = 0$, $\phi_3 = 0$, $\phi_4 = 0$. This problem is referred to as the Rosen-Suzuki problem.

[Source: Rosen and Suzuki (1965). See also Kowalik and Osborne (1968).]

<u>Question 91</u> Minimize w.r.t. $\underset{\sim}{\phi}$

$$f = 9 - 8\phi_1 - 6\phi_2 - 4\phi_3 + 2\phi_1^2 + 2\phi_2^2 + \phi_3^2 + 2\phi_1\phi_2 + 2\phi_1\phi_3$$

subject to

$$\phi_i \geq 0, \; i = 1, 2, 3,$$
$$3 - \phi_1 - \phi_2 - 2\phi_3 \geq 0.$$

The function has a minimum of 1/9 at $\phi_1 = 4/3$, $\phi_2 = 7/9$, $\phi_3 = 4/9$. Suggested starting points: (a) $\phi_1 = 1$, $\phi_2 = 2$, $\phi_3 = 1$; (b) $\phi_1 = \phi_2 = \phi_3 = 1$; (c) $\phi_1 = \phi_2 = \phi_3 = 0.5$; (d) $\phi_1 = \phi_2 = \phi_3 = 0.1$. This problem is referred to as the Beale problem.

[Source: Beale (1967). See also Kowalik and Osborne (1968).]

<u>Question 92</u> Minimize w.r.t. $\underset{\sim}{\phi}$ the maximum of

$$f_1 = \phi_1^4 + \phi_2^2,$$
$$f_2 = (2-\phi_1)^2 + (2-\phi_2)^2,$$
$$f_3 = 2\exp(-\phi_1+\phi_2).$$

The minimax solution occurs at $\phi_1 = \phi_2 = 1$, where $f_1 = f_2 = f_3 = 2$.

Suggested starting point: $\phi_1 = \phi_2 = 2$.

[Source: Charalambous (1973).]

Question 93 Minimize w.r.t. $\underset{\sim}{\phi}$ the maximum of

$$f_1 = \phi_1^2 + \phi_2^4,$$
$$f_2 = (2-\phi_1)^2 + (2-\phi_2)^2,$$
$$f_3 = 2\exp(-\phi_1+\phi_2).$$

The minimax solution occurs at

$$\phi_1 = 1.13904, \ \phi_2 = 0.89956,$$

where

$$f_1 = f_2 = 1.95222,$$
$$f_3 = 1.57408.$$

Suggested starting point: $\phi_1 = \phi_2 = 2$.

[Source: Charalambous (1973).]

Question 94

(a) Formulate the design of a notch filter in terms of inequality constraints, given the following requirements. The attentuation should not exceed A_1 dB over the range 0 to ω_1, and A_2 dB over the range ω_2 to ω_3, with $0 < \omega_1 < \omega_2 < \omega_3$. At ω_0, where $\omega_1 < \omega_0 < \omega_2$, the attenuation must exceed A_0 dB.

(b) Describe very briefly and illustrate the Sequential Unconstrained Minimization Technique (Fiacco-McCormick method) for constrained optimization.

(c) Set up a suitable objective function for the optimization of the notch filter of (a).

Question 95 Write down explicitly the generalized least pth objective function comprising real functions f_i (not necessarily positive) of $\underset{\sim}{\phi}$, level ξ, maximum M, multipliers u_i and any other necessary symbols. Ensure that $M > 0$, $M = 0$ and $M < 0$ are included in your description.

Question 96 Derive the gradient vector of the generalized least pth objective of Question 95 and discuss its features.

Question 97 Derive necessary conditions for a minimax optimum from the gradient vector of the least pth objective of Question 96, where the f_i are assumed differentiable functions of $\underset{\sim}{\phi}$.

Question 98 Fit $f = \phi_1\psi + \phi_2$ to $S(\psi)$, where $\psi_1 = 1$, $\psi_2 = 2$, $\psi_3 = 3$, $\psi_4 = 4$, $S(\psi_1) = 1$, $S(\psi_2) = 1$, $S(\psi_3) = 1.5$, $S(\psi_4) = 1$, using a program for least pth approximation. Consider $p = 1$, 2 and ∞ with uniform weighting to all errors.

Question 99 Solve analytically the problems described in Question 98 invoking optimality conditions.

Question 100 Consider the functions e_1 and e_2 of one variable ϕ shown in Fig. 13. Explain the implications of least pth approximation with $p = 1$ and 2, minimax approximation and simultaneous minimization of $|e_1|$ and $|e_2|$ w.r.t. ϕ.

Question 101 Consider the functions f_1 and f_2 of one variable ϕ shown in Fig. 14. Explain the implications of generalized least pth optimization of f_1 and f_2 w.r.t. ϕ for $p > 0$.

Question 102 Consider the two functions of one variable

$$e_1 = -\phi + 4$$

$$e_2 = \phi/3$$

Expose and explain the distinctive features and implications of

(a) the least pth approximation with $p = 1$ and $p = 2$ of $|e_1|$ and $|e_2|$ w.r.t. ϕ,

(b) the minimax optimization of $|e_1|$ and $|e_2|$ w.r.t. ϕ,

(c) the simultaneous minimization of $|e_1|$ and $|e_2|$ w.r.t. ϕ.

Question 103 Consider a transfer function of a filter as

$$H(j\omega) = \frac{1}{(j\omega - \alpha_1)(j\omega - \alpha_2)(j\omega - \alpha_3)}$$

All α_i are real variables which are adjusted to satisfy given specifications for the filter gain and $j = \sqrt{-1}$. Filter gain $G(\omega)$ is defined by

$$G(\omega) = - 20 \log|H(j\omega)|$$

and specifications $S(\omega)$ are

$$S(\omega) \leq 1 \text{ dB for } 0 \leq \omega \leq 1$$

$$S(\omega) \geq 40 \text{ dB for } \omega \geq 5$$

Formulate the optimization problem in a form suitable for programming with specific relevance to an available package you are familiar with.

Question 104 Suppose that the following table has been derived from impedance measurements at four frequencies.

frequency (rad/s)	real part (Ω)	imaginary part (Ω)
1	1.9	1.6
2	2.1	2.9
3	4.5	2.0
4	2.0	6.0

Obtain a uniformly weighted least pth approximation based on real approximating functions for (a) $p = 1$, (b) $p = 2$, and (c) $p = \infty$, for a proposed series RL circuit model with resistance R and inductance L as independent unknowns. Consider error functions of the form $|R - S_R|$, $|L - S_L|$. Comment on the data in the table and on your solutions.

Question 105 Set up as a nonlinear program the problem of least pth optimization with $p = 1$ given by

$$\min_{\underset{\sim}{\phi}} \sum_{i=1}^{n} | e_i(\underset{\sim}{\phi}) | ,$$

where the e_i are real functions of $\underset{\sim}{\phi}$. State necessary conditions for optimality of the problem and discuss them. Apply these ideas to

(a) $\min_{\phi} |\phi - 1| + |\phi|$,

(b) $\min_{\phi_1,\phi_2} |\phi_1 + \phi_2 - 1| + |\phi_1| + |\phi_2|$.

Question 106 Optimize the LC lowpass filter shown in Fig. 15. Write all necessary subprograms to calculate the response and its sensitivities. Verify your results with an available analysis program.

Specifications

Frequency Range (rad/s)	Insertion Loss (dB)
0 - 1	< 1.5
> 2.5	> 25

Question 107 Consider the following specification for a transient response of a linear system:

$$S(t) = \begin{cases} 5t, & 0 \le t \le 0.2 \\ -1.25t + 1.25, & 0.2 \le t \le 1 \\ 0, & t \ge 1 \end{cases}$$

Optimize the impulse response of the LC circuit of Question 106 to fit this specification in the least squares sense.

Question 108 Consider the linear circuit shown in Fig. 16, which is assumed to be in the sinusoidal steady state. Let R = 2Ω, C = 1F, ω = 2 rad/s.

(a) Obtain by direct differentiation simplified formulas for $\frac{\partial V_R}{\partial C}$, $\frac{\partial V_R}{\partial R}$ and $\frac{\partial V_R}{\partial \omega}$.

(b) Obtain the formulas of (a) by the adjoint network method from first principles.

Question 109 Consider the linear circuit shown in Fig. 17, which is assumed to be in the sinusoidal steady state. Let $V_g = 1V$, $R_g = 0.5\Omega$, $C = 2$ F, $R = 1\Omega$, $\omega = 10$ rad/s.

Use the adjoint network approach to evaluate $\partial V_R/\partial C$, $\partial V_R/\partial R$ and $\partial V_R/\partial\omega$. Estimate the change in V_R when both C and R decrease by 5% using these partial derivatives and compare with the exact change. How would you conduct a worst-case tolerance analysis, in general?

Question 110 Consider the circuit shown in Fig. 18, which is assumed to be in the sinusoidal steady state.

Derive from first principles the adjoint network and sensitivity expressions for all the elements of the circuit. Derive the adjoint excitations appropriate for calculating the first-order sensitivities of V_{C_2} w.r.t. all the parameters.

Question 111 Derive the first-order sensitivity expression

$$- \underset{\sim}{V}^T \Delta\underset{\sim}{Y}^T \hat{\underset{\sim}{V}}$$

for linear time-invariant networks in the frequency domain, where $\underset{\sim}{Y}$ is the s.c. admittance matrix of an element, $\underset{\sim}{V}$ the voltage vector in the original network and $\hat{\underset{\sim}{V}}$ the corresponding vector in the adjoint network of the element under consideration.

Question 112 Derive from first principles an approach to finding $\partial y_i/\partial\underset{\sim}{x}$, where $\underset{\sim}{A}\,\underset{\sim}{y} = \underset{\sim}{b}$ is a linear system in $\underset{\sim}{y}$, $\underset{\sim}{A}$ is a square matrix whose coefficients are nonlinear functions of $\underset{\sim}{x}$, the term y_i is the ith component of the column vector $\underset{\sim}{y}$ and $\partial y_i/\partial\underset{\sim}{x}$ represents a column vector containing partial derivatives of y_i w.r.t. corresponding elements of the column vector $\underset{\sim}{x}$. Discuss the computational effort involved.

Question 113 Derive from first principles an approach to finding $\partial V_i / \partial \omega$, where ω is frequency, V_i is an ith nodal voltage in the nodal equation of a linear, time-invariant circuit in the frequency domain, namely,

$$\underset{\sim}{Y}\,\underset{\sim}{V} = \underset{\sim}{I},$$

assuming $\underset{\sim}{I}$ is independent of ω.

Question 114 Consider the system of complex linear equations

$$\underset{\sim}{Y}\,\underset{\sim}{V} = \underset{\sim}{I}\ ,$$

where $\underset{\sim}{Y}$ is a square nodal admittance matrix of constant, complex coefficients, and $\underset{\sim}{I}$ is a specified excitation vector. Set up the appropriate objective function for the least squares solution of this system of equations and derive the gradient vector w.r.t. the real and imaginary parts of the components of $\underset{\sim}{V}$.

Question 115 Derive an approach to calculating $\partial \underset{\sim}{y} / \partial x_i$, where $\underset{\sim}{A}\,\underset{\sim}{y} = \underset{\sim}{b}$ is a linear system in $\underset{\sim}{y}$, $\underset{\sim}{A}$ is a square matrix whose coefficients are nonlinear functions of $\underset{\sim}{x}$ and x_i is the ith component of $\underset{\sim}{x}$. Discuss the computational effort involved.

Question 116 Derive from first principles an approach to calculating

$$\frac{\partial^2 y_i}{\partial x_j \partial x_k}$$

for the system described in Question 112, where x_j and x_k are elements of the vector $\underset{\sim}{x}$.

Question 117 Derive from first principles an approach to finding $\partial\lambda/\partial\underset{\sim}{x}$, where λ is an eigenvalue of the square matrix $\underset{\sim}{A}$ whose coefficients are (in general) nonlinear functions of $\underset{\sim}{x}$, i.e.,

$$\underset{\sim}{A}\underset{\sim}{y} = \lambda\underset{\sim}{y}.$$

The expression $\partial\lambda/\partial\underset{\sim}{x}$ is a column vector containing all first partial derivatives of λ w.r.t. corresponding elements of the column vector $\underset{\sim}{x}$. Discuss the computational effort involved. Give interpretations of any new symbols introduced. [Hint: λ is also an eigenvalue of $\underset{\sim}{A}^T$.]

Question 118 Derive an approach to calculating

$$\frac{\partial^2\lambda}{\partial x_j \partial x_k}$$

for the system described in Question 117, where x_j and x_k are elements of the vector $\underset{\sim}{x}$.

Question 119 Consider the quadratic approximation to a response function given by

$$f(\underset{\sim}{\phi},\psi) = \frac{1}{2} [\underset{\sim}{\phi}^T \ \psi] \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{a} \\ \underset{\sim}{a}^T & a \end{bmatrix} \begin{bmatrix} \underset{\sim}{\phi} \\ \psi \end{bmatrix} + [\underset{\sim}{\phi}^T \ \psi] \begin{bmatrix} \underset{\sim}{b} \\ b \end{bmatrix} + c ,$$

where $\underset{\sim}{A}$ is a symmetric square matrix of the dimensions of the column vector $\underset{\sim}{\phi}$; $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are column vectors of constants of the same dimension as $\underset{\sim}{\phi}$; and a, b and c are constants. Develop a compact expression for $f(\underset{\sim}{\phi},\psi)$ subjected to the condition

$$\frac{\partial f}{\partial \psi} = 0 .$$

Question 120 Develop from first principles a computationally attractive method of obtaining the Thevenin equivalent of an arbitrary linear, time-invariant circuit in the frequency domain using only one analysis of a suitable circuit. [Hint: Show that this circuit is the adjoint of the given circuit and derive the appropriate terminations and all necessary formulas.]

Question 121 Derive from first principles the sensitivity expression and adjoint element corresponding to a voltage controlled current source. Draw circuit diagrams to fully illustrate your results.

Question 122 Derive from first principles the first-order sensitivity expressions relating to:

(a) a voltage controlled voltage source,

(b) a current controlled voltage source,

(c) an open-circuited uniformly distributed line,

(d) a uniform RC line.

Question 123 Derive from first principles the adjoint element equation and sensitivity expression for a two-port characterized by

$$\begin{bmatrix} V_p \\ I_p \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_q \\ -I_q \end{bmatrix}$$

Apply the result to the element shown in Fig. 19.

Question 124 Verify that the adjoint network may be characterized by the hybrid matrix description

$$\begin{bmatrix} \hat{\underset{\sim}{I}}_a \\ \hat{\underset{\sim}{V}}_b \end{bmatrix} = \begin{bmatrix} \underset{\sim}{Y}^T & -\underset{\sim}{M}^T \\ -\underset{\sim}{A}^T & \underset{\sim}{Z}^T \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_a \\ \hat{\underset{\sim}{I}}_b \end{bmatrix} ,$$

where the corresponding description for the original network is

$$\begin{bmatrix} \underset{\sim}{I}_a \\ \underset{\sim}{V}_b \end{bmatrix} = \begin{bmatrix} \underset{\sim}{Y} & \underset{\sim}{A} \\ \underset{\sim}{M} & \underset{\sim}{Z} \end{bmatrix} \begin{bmatrix} \underset{\sim}{V}_a \\ \underset{\sim}{I}_b \end{bmatrix} .$$

Question 125 Verify that, for a network excited by a set of independent voltages J_V and a set of independent currents J_I,

$$\underset{\sim}{G} = \sum_{i \varepsilon J_V} \hat{V}_i \, \underset{\sim}{\nabla} I_i - \sum_{i \varepsilon J_I} \hat{I}_i \, \underset{\sim}{\nabla} V_i ,$$

where

$$\underset{\sim}{\nabla} \overset{\Delta}{=} \begin{bmatrix} \frac{\partial}{\partial \phi_1} \\ \frac{\partial}{\partial \phi_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial}{\partial \phi_k} \end{bmatrix}$$

implies differentiation w.r.t. k parameters $\phi_1, \phi_2, \ldots, \phi_k$ and $\underset{\sim}{G}$ is a vector of corresponding sensitivity expressions associated with elements of the network. The remaining variables V_i, I_i, $\hat{V}_i$ and $\hat{I}_i$ are associated with excitations and responses in the original network and adjoint network as implied by Fig. 20.

Question 126 Consider the linear circuit shown in Fig. 21 excited by a unit step u(t). Obtain $\partial v/\partial R$ and $\partial v/\partial C$ using the adjoint network method and verify the resulting formulas by directly differentiating v(t).

Question 127 Evaluate at 0.5 rad/s the partial derivatives of the input impedance (see Fig. 22) w.r.t. the inductors and capacitors of the filter of Question 32.

Question 128 Consider the circuit of Question 106 at $\omega = 1$ rad/s. Let $L_1 = L_2 = 2$H, C = 1F. Obtain the partial derivative values of the insertion loss in dB of the filter between the terminating resistors with respect to L_1, C and L_2 using the adjoint network method. If L_1 changes by +5%, L_2 by -5% and C by +10%, estimate the change in insertion loss at $\omega = 1$ rad/s. Check your results by calculating the change in loss directly and explain any discrepancies.

Question 129 Derive from first principles an approach to finding the exact large change Δy_i due to the large change Δa_{jj}, where $\underset{\sim}{A}\, \underset{\sim}{y} = \underset{\sim}{b}$ is a linear system in $\underset{\sim}{y}$, $\underset{\sim}{A}$ is a square matrix, the term y_i is the ith component of the column vector $\underset{\sim}{y}$ and a_{jj} represents the jth diagonal element of $\underset{\sim}{A}$. Discuss the computational effort involved. [Hint: First find Δy_j.]

Question 130 Consider the resistive network of Fig. 23.

(a) Calculate the node voltages by LU factorization of the nodal admittance matrix showing all major steps. Verify that

$$\underset{\sim}{L} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 11/3 & 0 \\ 0 & -2 & 21/11 \end{bmatrix},$$

$$\underset{\sim}{U} = \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1 & -6/11 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Draw the adjoint circuit appropriately excited with a unit current for finding the first-order sensitivities of the voltage V across G_3.

(c) Calculate the node voltages of the adjoint circuit using the LU factors already obtained above.

(d) Calculate $\underset{\sim}{\nabla}V$, where

$$\underset{\sim}{\nabla} = \begin{bmatrix} \partial/\partial G_1 \\ \partial/\partial R_2 \\ \partial/\partial G_3 \\ \partial/\partial R_4 \\ \partial/\partial G_5 \end{bmatrix}$$

using sensitivity formulas shown in the table.

Element	Branch Equation		Sensitivity	Parameters
	Original	Adjoint		
Resistor	$V = RI$	$\hat{V} = R\hat{I}$	$I\hat{I}$	R
	$I = GV$	$\hat{I} = G\hat{V}$	$-V\hat{V}$	G

Question 131 Consider the resistive network of Question 27.

(a) Calculate the LU factors of the nodal admittance matrix.

(b) Calculate using Tellegen's theorem (unperturbed) the Thevenin equivalent of the network as seen by the element G_3. Proceed as follows. You need

(i) the open circuit voltage V_{TH} seen by G_3,

(ii) the impedance Z_{TH} seen by G_3 with $I_g = 0$.

Prove that one adjoint network analysis can be used for both quantities, draw the appropriate excited adjoint network, and solve it using the LU factors of (a).

(c) Calculate using your Thevenin equivalent the change in voltage across G_3 when G_3 increases from 1 mho to 2 mho. Now represent this change by an independent current source applied across G_3.

(d) Hence, find the voltage across G_5 due to the specified change in G_3 using the LU factors obtained in (a).

(e) Check by a direct method that your result in (d) is correct.

Question 132 Draw the adjoint network for the active circuit shown in Fig. 24, which is assumed to be in the sinusoidal steady state. Include excitations appropriate to calculating the sensitivities of $V_2(j\omega)$ w.r.t. all parameters, clearly identifying zero and nonzero excitations. Develop an expression for the gradient vector of the following objective function to be minimized:

$$U = \sum_{i=1}^{n} (G(\omega_i) - S(\omega_i))^2 ,$$

where

$$G(\omega) = \left| \frac{V_2(j\omega)}{V_0(j\omega)} \right|^2$$

and S(ω) is a given specification.

Element	Equation		Sensitivity Parameters	
	Original	Adjoint		
Resistor	$V = RI$	$\hat{V} = R\hat{I}$	$I\hat{I}$	R
Capacitor	$I = j\omega CV$	$\hat{I} = j\omega C\hat{V}$	$-j\omega V\hat{V}$	C
Voltage Controlled Voltage Source	$\begin{matrix} I_1 \\ V_2 \end{matrix} = \begin{matrix} 0 & 0 \\ \mu & 0 \end{matrix} \begin{matrix} V_1 \\ I_2 \end{matrix}$	$\begin{matrix} \hat{I}_1 \\ \hat{V}_2 \end{matrix} = \begin{matrix} 0 & -\mu \\ 0 & 0 \end{matrix} \begin{matrix} \hat{V}_1 \\ \hat{I}_2 \end{matrix}$	$V_1\hat{I}_2$	μ

<u>Question 133</u> Consider the circuit of Question 29, which is assumed to be in the sinusoidal steady state.

Let $V_1 = 1V$, $\omega = 2$ rad/s, $R_1 = R_2 = R_3 = 2\Omega$, $C_1 = C_2 = C_3 = 1F$.

(a) Write down the nodal equations for the circuit, using the component values and frequency indicated.

(b) Apply Gauss-Seidel (relaxation) method to find the node voltages, assuming the initial node voltages to be zero. Use two iterations. Repeat with an overrelaxation factor of 1.5.

(c) Factorize the nodal admittance matrix into upper and lower triangular form.

(d) Calculate $\partial V_3/\partial C_2$ and $\partial V_3/\partial R_1$ by the adjoint network method using the above LU factorization results in conjunction with the nodal admittance matrix of the adjoint circuit.

(e) Estimate ΔV_3 (the total change in V_3) when C_2 changes by +3% and R_1

by -5%. Use $\Delta V_3 \simeq \frac{\partial V_3}{\partial C_2} \Delta C_2 + \frac{\partial V_3}{\partial R_1} \Delta R_1$. Check the results by direct perturbation.

Question 134 Compare the computational effort in the ABCD or chain matrix analysis of a network and an efficient method based on a tridiagonal nodal admittance matrix.

Question 135 Discuss carefully the computational effort required in general for each approach used in Question 133.

Question 136 Write an efficient Fortran program using LU factorization in conjunction with Newton's method for solving nonlinear equations to find the node voltages of the resistor-diode network shown in Fig. 25 [Source: Chua and Lin (1975)], where

$$i_d = I_S(e^{\lambda v_d} - 1) ,$$

$$I_S = 10^{-12} \text{ mA} ,$$

$$\lambda = 1/V = 1/0.026 \text{ V}^{-1} ,$$

$$E = 10 \text{ V} ,$$

$$R_1 = R_2 = 1 \text{ k}\Omega .$$

Use the results to calculate

$$\begin{bmatrix} \frac{\partial v_3}{\partial R_1} \\ \\ \frac{\partial v_3}{\partial R_2} \end{bmatrix}$$

subject to satisfying the nonlinear equations.

By running the program again with small perturbations in R_1 and R_2, check these derivatives. Solve the equations for a number of starting points and comment on the results. Also use

$$v_1 = 5.75 \quad v_2 = 0.75 \quad v_3 = 5.0$$

as a test starting point.

Question 137 What is the companion network method of solving nonlinear networks? How does it take advantage of existing linear network simulation methods? Provide an illustrative example.

Question 138 Consider the resistor-diode network shown in Question 136. Draw the corresponding companion network at the jth iteration for its d.c. solution. Write down the nodal equations at this iteration.

Question 139 Consider the resistor-diode network shown in Question 136. Develop the system of linear equations derived from the nodal equations at the jth iteration for solution by the Newton method. Write down explicitly the Jacobian at the jth iteration.

Question 140 Consider the nonlinear circuit shown in Fig. 26, where $i_a = 2v_a^3$, $i_b = v_b^3 + 10v_b$.

(a) Express the nodal equations in the linearized form required at the jth iteration of the Newton algorithm.

(b) Apply two iterations of the Newton method, starting at $v_1 = 2$, $v_2 = 1$.

(c) Draw the companion network at the jth iteration and state the corresponding nodal equations.

(d) Continue with two iterations of the companion network method.

[Source: Chua and Lin (1975).]

Question 141 Consider least pth optimization with both upper and lower response specifications, where the specifications might be violated or satisfied. Discuss in as much detail as possible the role of the value of p and the effects of different weightings on the solution.

Question 142 Show, using the generalized least pth objective, that if specifications cannot be satisfied with a given value of $p \geq 1$, then they cannot be satisfied for any other value, e.g., $p = \infty$.

Question 143 Set up and discuss a suitable least pth objective function for approximate minimization of

$$\max_{i \in I} f_i(\underset{\sim}{\phi})$$

where $\underset{\sim}{\phi}$ contains the adjustable parameters and I denotes an index set relating to the differentiable nonlinear functions f_i, which are not necessarily positive.

Question 144 Relate the problem formulation of Question 143 to filter design, taking care to discuss upper and lower response specifications, errors and weighting functions.

Question 145 Derive the Golden Section search method for functions of one variable from first principles. Explain all the concepts involved.

Under what conditions would you expect a global solution?

Question 146 Apply 3 iterations of the Golden Section search method to the function of one variable given shown in Fig. 27. Show clearly all steps and label the diagrams appropriately. Fit a quadratic function to 3 points corresponding to the lowest function values observed and find its minimum. Estimate function values and points from the graph.

Question 147 Starting with the interval [0,6], apply 4 iterations of the Golden Section search method to the minimization w.r.t. ϕ of a function described by

$$
\begin{aligned}
U &= -\phi + 5 & \phi &\leqslant 1 \\
U &= 0.5(\phi - 3)^2 + 1 & 1 < \phi &\leqslant 4 \\
U &= 3 - (\phi - 6)^2/3 & \phi &> 4
\end{aligned}
$$

What is the solution obtained? By how much has the interval of uncertainty been reduced?

Question 148 Devise an algorithm for finding the extrema of a well-behaved multimodal function of one variable.

Question 149 Discuss mathematically and physically the concept of steepest descent for $\max_{1 \leq i \leq n} f_i(\underset{\sim}{\phi})$, where the $f_i(\underset{\sim}{\phi})$ are n real, nonlinear, differentiable functions of $\underset{\sim}{\phi}$.

Question 150 Suppose we have to minimize

$$(a) \quad U = \Big(\sum_{\omega_i \in \Omega_d} |L(\omega_i) - S(\omega_i)|^p \Big)^{1/p}, \qquad p > 1.$$

$$\text{(b)} \quad U = \sum_{\omega_i \varepsilon \Omega_d} [L(\omega_i) - S(\omega_i)]^p \quad , \qquad p \text{ even} > 0.$$

where the $L(\omega_i)$ is the insertion loss in dB of a filter between R_g and R_L, $S(\omega_i)$ is the desired insertion loss between R_g and R_L and Ω_d is a set of discrete frequencies ω_i. Obtain expressions relating ∇U to $\underset{\sim}{G}(j\omega_i)$, where the elements of $\underset{\sim}{G}$ are appropriate adjoint sensitivity expressions. Assume convenient values for the excitations of the original and adjoint networks.

Question 151 The complex impedance of a body has been measured at a set of frequencies. A linear circuit model to represent this impedance is proposed. Explain the steps you would take to optimize the model, assuming you were to use an available unconstrained optimization program requiring first derivatives.

Question 152 Describe the aims of the project you are carrying out for this course. Explain in detail the steps you are taking to meet these aims. What results have you obtained thus far and are they what you expected?

Question 153 Consider the circuit shown in Fig. 28, which is a linear time-invariant network with parameters $\underset{\sim}{\phi}$. It is desired to obtain the best impedance match between the complex, frequency-dependent load Z_L and the constant source resistance R_g.

Formulate a least squares objective function U of the parameter vector $\underset{\sim}{\phi}$, the optimum of which represents a good match over a band of frequencies Ω. Explain carefully and in detail how the adjoint network method may be used to calculate the gradient vector $\underset{\sim}{\nabla} U(\underset{\sim}{\phi})$.

Question 154 Consider the voltage divider shown in Fig. 29. The specifications are as follows.

$$0.46 \leq \frac{R_2}{R_1+R_2} \leq 0.53 ,$$

$$1.85 \leq R_1+R_2 \leq 2.15 .$$

Assuming $R_1 \geq 0$, $R_2 \geq 0$, derive the worst vertices of a tolerance region for independent tolerance assignment on these two components.

[Reference: Karafin, BSTJ, vol. 50, 1971, pp. 1225-1242.]

Question 155 Consider the problem defined in Question 154. Optimize the tolerances ε_1 and ε_2 on R_1 and R_2 given the cost function

$$C = \frac{R_1^0}{\varepsilon_1} + \frac{R_2^0}{\varepsilon_2}$$

assuming an environmental (uncontrollable) parameter T common to both resistors such that

$$R_1 = (R_1^0 + \mu_1\varepsilon_1)\ (T^0 + \mu_t\varepsilon_t) ,$$

$$R_2 = (R_2^0 + \mu_2\varepsilon_2)\ (T^0 + \mu_t\varepsilon_t) ,$$

where

$$-1 \leq \mu_1, \mu_2, \mu_t \leq 1 ,$$

$$T^0 = 1, \ \varepsilon_t = 0.05 .$$

[The independent designable variables include R_1^0, R_2^0, ε_1 and ε_2.]

Question 156 Consider the problem defined in Question 154. Optimize the tolerance ε_1 on R_1 given the cost function

$$C = \frac{R_1^0}{\varepsilon_1}$$

assuming that R_2 is tunable by ±10% of its nominal value. [The independent designable variables include R_1^0, ε_1 and R_2^0.]

Question 157 Consider the voltage divider shown in Fig. 30 with a nonideal source and load.

It is desired to maintain

$$0.47 \leq V \leq 0.53 \ ,$$

$$1.85 \leq R \leq 2.15 \ ,$$

for all possible

$$R_g \leq 0.01 \ ,$$

$$R_L \geq 100 \ ,$$

with

$$R_1^0 = R_2^0 \ ,$$

$$\varepsilon_1 = \varepsilon_2 \ ,$$

and maximum tolerances. Find the optimal values for R_1^0, R_2^0, ε_1 and ε_2.

Question 158 Consider the voltage divider shown in Question 154. Formulate as precisely as possible the functions involved (objective and constraints) and their first partial derivatives required to optimize the tolerances on R_1 and R_2, allowing the nominal point to move, subject to lower and upper limits on the transer function and input resistance. Assume a worst-case solution is desired, and suggest cost functions.

Question 159 Consider the voltage divider shown in Question 154. Deriving all formulas from first principles, use the adjoint network method to calculate $\partial T/\partial R_1$ and $\partial T/\partial R_2$ given:

$$T \triangleq \frac{V_2}{V_1}, \quad R_1 = 1.1\ \Omega,\ R_2 = 0.9\ \Omega.$$

Show both original and adjoint networks appropriately excited and verify your result by direct differentiation.

Question 160 Consider the voltage divider of Question 154 expressed as a minimax problem. Determine suitable active functions when

$$R_1 = 1.01$$

$$R_2 = 1.14$$

and calculate the steepest descent direction from first principles. Assume that if $|M - f_i| \leqslant 0.01$ for any f_i, then the corresponding f_i is active, where $M \triangleq \max_i f_i$. Show all steps in your calculations.

Question 161 Consider an acceptable region given by

$$2 + 2\phi_1 - \phi_2 \geq 0,$$

$$143 - 11\phi_1 - 13\phi_2 \geq 0,$$

$$-60 + 4\phi_1 + 15\phi_2 \geq 0.$$

Determine optimally centered, optimally toleranced solutions using the following cost functions:

(a) $\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2}$,

(b) $\log_e \dfrac{\phi_1^0}{\varepsilon_1} + \log_e \dfrac{\phi_2^0}{\varepsilon_2}$,

where ε_1 and ε_2 are tolerances and ϕ_1^0 and ϕ_2^0 are nominal values.

Formulate the problem as a nonlinear programming problem and give expressions for derivatives.

Question 162 Consider the voltage divider shown in Question 154 subject to the same specifications. Optimize the tolerances ε_1 and ε_2 on R_1 and R_2, respectively, and find the best corresponding nominal values R_1^0 and R_2^0, using the following cost functions:

(a) $C_1 = \frac{R_1^0}{\varepsilon_1} + \frac{R_2^0}{\varepsilon_2}$,

(b) $C_2 = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}$.

[Source: Karafin, BSTJ, vol. 50, 1971, pp. 1225-1242.]

Question 163 Find the number of state variables and indicate a possible choice of these states for the circuit shown in Fig. 31.

Question 164 The circuit shown in Fig. 32 has the state equations

$$C_D \frac{dv_D}{dt} = -I_S(e^{\lambda v_D}-1)+(E_1-E_2-v_D)/R_1+(v_0-E_2-v_D)/R_2$$

$$C_0 \frac{dv_0}{dt} = (E_2+v_D-v_0)/R_2$$

The parameters are

$R_1 = R_2 = 1\ \mathrm{k}\Omega$

$I_D = I_S(e^{\lambda v_D}-1)$, $\lambda = 40\ \mathrm{V}^{-1}$, $I_S = 10^{-10}\ \mathrm{A}$

$C_1 = 1\ \mu\mathrm{F}$, $C_2 = 10\ \mathrm{pF}$

$E_2 = 1\ \mathrm{V}$

Perform two steps of fourth-order Runge-Kutta integration starting at $t=0$, $v_D(0) = v_0(0) = 0$ and using a time step of 10 ns.

[Source: Chua and Lin (1975).]

Question 165 Describe briefly the principle behind the Runge-Kutta algorithms for solving a differential equation with a given initial value. Consider the following initial value problem

$$\dot{x} = (\cos x) + t \qquad x_0 = 1,\ t \in [0,\ 0.3]$$

A solution is required for a step-size of 0.1.

(a) Use Heun's algorithm.

(b) Use the fourth-order Runge-Kutta method.

Question 166 Approximate in a uniformly weighted minimax sense

$$f(x) = x^2$$

by

$$F(x) = a_1 x + a_2 \exp(x)$$

on the interval [0,2].

[Source: Curtis and Powell (1965). See also Popovic, Bandler and Charalambous (1974).]

Question 167 Approximate in a uniformly weighted minimax sense

$$f(x) = \frac{[(8x - 1)^2 + 1]^{0.5} \tan^{-1}(8x)}{8x}$$

by

$$F(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2}$$

on the interval [-1,1].

[Reference: Popovic, Bandler and Charalambous (1974).]

Question 168 Consider a lumped-element LC transformer (Fig. 33) to match a 1 ohm load to a 3 ohm generator over the range 0.5 - 1.179 rad/s. A minimax approximation should be carried out on the modulus of the reflection coefficient using all six reactive components as variables. The solution is

$$L_1 = 1.041,$$
$$C_2 = 0.979,$$
$$L_3 = 2.341,$$
$$C_4 = 0.781,$$
$$L_5 = 2.937,$$
$$C_6 = 0.347,$$

at which max $|\rho| = 0.075820$. Use 21 uniformly spaced sample points in the band. Suggested starting point:

$$L_1 = C_2 = L_3 = C_4 = L_5 = C_6 = 1.$$

[Source: Hatley (1967). See also Srinivasan (1973). See Example 4 of Report SOS-78-14-U for hints in setting up the subprograms.]

Question 169 Consider the RC active equalizer shown in Fig. 34. The specified linear gain response in dB over the band 1 MHz to 2 MHz is given by $G = 5 + 5f$, where f is in MHz. Find optimal solutions using least pth approximation with $p = 2, 4, 8, \ldots, \infty$ taking as variables C_1, C_2, R_1 and R_2. Twenty-one uniformly distributed sample points are suggested with starting values

$$C_1 = C_2 = R_1 = R_2 = 1$$

and

$$C_1 = C_2 = R_1 = R_2 = 0.5.$$

Comment on the results.

Reconsider the problem using only C_1 and R_1 as variables.

[Source: Temes and Zai (1969).]

Question 170 Consider the problem of finding a second-order model of a fourth-order system, when the input to the system is an impulse, in the minimax sense. The transfer function of the system is

$$G(s) = \frac{(s+4)}{(s+1)(s^2+4s+8)(s+5)}$$

and of the model is

$$H(s) = \frac{\phi_3}{(s+\phi_1)^2+\phi_2^2} \quad .$$

The problem is, therefore, equivalent to making the function

$$F(\underset{\sim}{\phi},t) = \frac{\phi_3}{\phi_2} \exp(-\phi_1 t) \sin \phi_2 t$$

best approximate

$$S(t) = \frac{3}{20} \exp(-t) + \frac{1}{52} \exp(-5t) - \frac{\exp(-2t)}{65} (3\sin 2t + 11\cos 2t)$$

in the minimax sense. The problem may be discretized in the time interval 0 to 10 seconds and the function to be minimized is

$$\max_{i \varepsilon I} |e_i(\underset{\sim}{\phi})| \ , \quad I = \{1, 2, \ldots, 51\} \ ,$$

where

$$e_i(\underset{\sim}{\phi}) = F(\underset{\sim}{\phi},t_i) - S(t_i) \ .$$

The solution is

$$\phi_1 = 0.68442,$$

$$\phi_2 = \pm\ 0.95409,$$

$$\phi_3 = 0.12286,$$

and the maximum error is 7.9471×10^{-3}. Suggested starting point: $\phi_1 = \phi_2 = \phi_3 = 1$.

[See, for example, Bandler (1977).]

Question 171 Develop a program to calculate and plot insertion loss of the circuit shown in Fig. 35 (elliptic low-pass filter).

Data for the circuit is

$C_1 = 0.89318$ F

$C_2 = 0.1022$ F

$C_3 = 1.57677$ F $\qquad L_2 = 1.26033$ H

$C_4 = 0.29139$ F

$C_5 = 0.74177$ F $\qquad L_4 = 1.03950$ H

$0 \leqslant \omega \leqslant 4$ rad/s.

What specifications does the circuit meet? Suggest ways of meeting these specifications by optimization assuming the solution was not known.

Question 172 Consider the LC filter of Question 106. The minimax solution corresponding to the specifications of Question 106, taking the passband sample points as 0.45, 0.5, 0.55, 1.0 and the stopband as 2.5, is

$$L_1 = L_2 = 1.6280, \ C = 1.0897.$$

Using appropriate optimization programs verify the worst-case tolerance solutions shown in the following table for the objective

$$\frac{L_1^0}{\varepsilon_1} + \frac{L_2^0}{\varepsilon_2} + \frac{C^0}{\varepsilon_C} .$$

	Continuous Solution		Discrete Solution		
Parameters	Fixed Nominal	Variable Nominal	from [1,2,5,10,15]%		
ε_1/L_1^0	3.5%	9.9%	5%	10%	10%
ε_C/C^0	3.2%	7.6%	10%	5%	10%
ε_2/L_2^0	3.5%	9.9%	10%	10%	5%
L_1^0	1.628	1.999		1.999	
C^0	1.090	0.906		0.906	
L_2^0	1.628	1.999		1.999	

[Source: Bandler, Liu and Chen (1975).]

<u>Question 173</u> For the circuit of Question 172 verify numerically that the active worst-case vertices of the tolerance region are identified as in the table shown.

Vertex	Frequency
6	0.45, 0.50, 0.55
8	1.0
1	2.5

[Source: Bandler, Liu and Tromp (1976).]

Question 174 Consider the 10:1 impedance ratio, lossless two-section transmission-line transformer shown in Fig. 36. The lengths of the sections are ℓ_1 and ℓ_2. The corresponding characteristic impedances are Z_1 and Z_2. Minimize the maximum of the modulus of the reflection coefficient ρ over 100 percent relative bandwidth w.r.t. lengths and/or characteristic impedances. The known quarter-wave solution is given by

$$\ell_1 = \ell_2 = \ell_q \text{ (the quarter wavelength at centre frequency)},$$

$$Z_1 = 2.2361,$$

$$Z_2 = 4.4721,$$

where ℓ_q = 7.49481 cm for 1 GHz centre. The corresponding max $|\rho|$ = 0.42857.

Use 11 uniformly distributed (normalized frequency) sample points, namely 0.5, 0.6, ..., 1.5. Seven suggested starting points and problems are tabulated, namely, a, b, ..., g.

	Problem starting points						
Parameters	a	b	c	d	e	f	g
ℓ_1/ℓ_q		fixed (optimal)			0.8	1.2	1.2
Z_1	1.0	3.5	1.0	3.5	*	3.5	3.5
ℓ_2/ℓ_q		fixed (optimal)			1.2	*	0.8
Z_2	3.0	3.0	6.0	6.0	*	*	3.0

* Parameter is fixed at optimal value.

A suggested specification, if appropriate to the method, is $|\rho| \leq 0.5$. A variation to the problem is to minimize the maximum of $0.5 |\rho|^2$. Suggested termination criterion: max $|\rho|$ within 0.01 percent of the optimal value.

[Source: Bandler and Macdonald (1969).]

Question 175 Consider the problem described in Question 174. Using a computer plotting routine plot the contours

$$\{\max |\rho|\} = \{0.45, 0.50, 0.55, 0.60, 0.65, 0.70, 0.75, 0.80\}$$

for the following situations:

(a) $1 \leq Z_1 \leq 3.5$, $3 \leq Z_2 \leq 6$,

(b) $0.8 \leq \ell_1/\ell_q$, $\ell_2/\ell_q \leq 1.2$,

(c) $0.8 \leq \ell_1/\ell_q \leq 1.2$, $1 \leq Z_1 \leq 3.5$.

Parameters not specified are held fixed at optimal values.

[Source: Bandler and Macdonald (1969).]

Question 176 Consider the problems described in Questions 174 and 175. Use a computer plotting routine to plot contours of a generalized least pth objective function for $p = 1, 2, 10, \infty$, taking $|\rho|$ as the approximating function and 0.5 as the upper specification.

[Source: Bandler and Charalambous (1972).]

Question 177 Consider the same circuits, terminations and specifications as in Question 174. Let ε_1 and ε_2 be the tolerances on Z_1 and Z_2, respectively. Starting at the known minimax solution with $\varepsilon_1 = 0.2$ and $\varepsilon_2 = 0.4$ minimize w.r.t. Z_1^0, Z_2^0, ε_1 and ε_2

(a) $$C_1 = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2},$$

(b) $$C_2 = \frac{Z_1^0}{\varepsilon_1} + \frac{Z_2^0}{\varepsilon_2},$$

for a worst-case design (yield = 100%).

[Source: Bandler, Liu and Chen (1975). See also Abdel-Malek (1977).]

Question 178 Consider the same circuit and terminations as in Question 174 but with three sections. The known quarter-wave solution is given by (see Question 174 for definition and value of ℓ_q)

$$\ell_1 = \ell_2 = \ell_3 = \ell_q,$$

$$Z_1 = 1.63471,$$

$$Z_2 = 3.16228,$$

$$Z_3 = 6.11729.$$

The corresponding max $|\rho| = 0.19729$.

Use the 11 (normalized frequency) sample points 0.5, 0.6, 0.7, 0.77, 0.9, 1.0, 1.1, 1.23, 1.3, 1.4, 1.5. Three suggested starting points are tabulated, namely, a, b and c.

Parameters	Problem starting points		
	a	b	c
ℓ_1/ℓ_q	*	**	0.8
Z_1	1.0	1.0	1.5
ℓ_2/ℓ_q	*	**	1.2
Z_2	**	**	3.0
ℓ_3/ℓ_q	*	**	0.8
Z_3	10.0	10.0	6.0

* Parameter is fixed at optimal value.
** Parameter varies, starting at optimal value.

A variation to the problem is to minimize the maximum of $0.5\ |\rho|^2$. Suggested termination criterion: max $|\rho|$ agrees with the optimal value to 5 significant figures.

[Source: Bandler and Macdonald (1969).]

Question 179 Design a recursive digital lowpass filter of the cascade form to best approximate a magnitude response of 1 in the passband, normalized frequency ψ of 0-0.09, and 0 in the stopband above $\psi = 0.11$. Take the transfer function as

$$H(z) = A \prod_{k=1}^{K} \frac{1+a_k z^{-1}+b_k z^{-2}}{1+c_k z^{-1}+d_k z^{-2}} ,$$

where K is the number of second-order sections,

$$z = \exp(j\psi\pi) ,$$

$$\psi = \frac{2f}{f_s} ,$$

f is frequency and f_s is the sampling frequency.

Analytical derivatives w.r.t. the coefficients a_k, b_k, c_k and d_k are readily derived.

Suggested sample points ψ are

0.0	to 0.8	in steps of 0.01,
0.0801	to 0.09	in steps of 0.00045,
0.11	to 0.2	in steps of 0.01,
0.3	to 1.0	in steps of 0.1.

Use one section and a starting point of

$$a_1 = 0,$$

$$b_1 = 0,$$

$$c_1 = 0,$$

$$d_1 = -0.25,$$

$$A = 0.1,$$

for least pth approximation with p = 2, 10, 100, 1000, 10000 and minimax approximation, each optimization starting at the solution to the previous one.

[See Bandler and Bardakjian (1973).]

Question 180 Grow a second section at the solution to Question 179 and reoptimize appropriately.

[See Bandler and Bardakjian (1973).]

Question 181 Optimize the coefficients of a recursive digital lowpass filter of the cascade form (see Question 179) to meet the following specifications:

$$0.9 \leq |H| \leq 1.1 \text{ in the passband,}$$

$$|H| \leq 0.1 \text{ in the stopband,}$$

where the passband sample points ψ are

0.0 to 0.18 in steps of 0.02,

and the stopband sample points ψ are

0.24,

0.3 to 1.0 in steps of 0.1.

Begin optimizing with one section starting at

$$a_1 = 0 ,$$
$$b_1 = 1 ,$$
$$c_1 = -1 ,$$
$$d_1 = 0.5 ,$$
$$A = 0.1 ,$$

for least pth approximation with $p = 2, 10, 1000, 10000$ and minimax approximation, each optimization starting at the solution to the previous one.

[See Bandler and Bardakjian (1973).]

Question 182 Grow a second section at the solution to Question 181 and reoptimize appropriately.

[See Bandler and Bardakjian (1973).]

Question 183 For the five-section, lossless, transmission-line filter shown in Fig. 37, the following objectives provide two distinct problems, each of which is subjected to a passband insertion loss of no more than 0.01 dB over the band 0 - 1 GHz.

(a) Maximize the stopband loss at 5 GHz.

(b) Maximize the minimum stopband loss over the range 2.5 - 10 GHz.

The characteristic impedances are to be fixed at the values

$$Z_1 = Z_3 = Z_5 = 0.2$$
$$Z_2 = Z_4 = 5$$

and the section lengths (normalized to ℓ_q as the quarter-wavelength at 1 GHz) as variables. Suggested sample points are: 21 uniformly distributed in the passband, 16 for the stopband in problem (b). A suggested starting point is

$$\ell_1/\ell_q = \ell_5/\ell_q = 0.07,$$
$$\ell_3/\ell_q = 0.15,$$
$$\ell_2/\ell_q = \ell_4/\ell_q = 0.15.$$

[Source for Problem (a): Brancher, Maffioli and Premoli (1970). See also Bandler and Charalambous (1972).]

Question 184 Solve Question 183(a) with normalized lengths fixed at 0.2 and impedances variable.

[See Levy (1965).]

Question 185 Consider the circuit of Question 183. Let the passband be 0 - 1 GHz. Consider a single stopband frequency of 3 GHz. The attenuation in the passband should not exceed 0.4 dB, while the attenuation at 3 GHz should be as high as possible, subject to the following constraints:

$$\ell_i = \ell_q, \; 0.5 \le Z_i \le 2.0, \; i = 1, 2, \ldots, 5 ,$$

where

$$\ell_q = 2.5 \text{ cm (quarterwave at 3 GHz)}.$$

It is suggested that 21 uniformly spaced frequencies are chosen in the passband.

[See Srinivasan (1973) and Carlin (1971).]

Question 186 Reoptimize the example of Question 185 subject to the constraints

$$\left.\begin{array}{l} 0 \le \ell_i/\ell_q \le 2 , \\ 0.4416 \le Z_i \le 4.419 , \end{array}\right. \quad i = 1, 2, \ldots, 5$$

$$0 \leq \sum_{i=1}^{5} \ell_i/\ell_q \leq 5 ,$$

where lengths ℓ_i and impedances Z_i are allowed to vary.

[See Srinivasan and Bandler (1975).]

<u>Question 187</u> Consider a third-order lumped-distributed-active lowpass filter as shown in Fig. 38. The passband is 0 - 0.7 rad/s, the stopband 1.415 - ∞ rad/s. Three design problems are to be solved for minimax results.

(a) An attenuation and ripple in the passband of less than 1 dB, with the attenuation in the stopband at least 30 dB (second amplifier removed).

(b) An attenuation and ripple of 1 dB in the passband with the best stopband response.

(c) A minimum attenuation and ripple in the passband subject to at least 30 dB attenuation in the stopband.

The nodal equations for the circuit are

$$\begin{bmatrix} y_{22}+j\omega C_1 & -(y_{22}+y_{12}) & 0 \\ -(y_{22}+y_{12}+\frac{A}{R_0}) & y_{11}+y_{22}+y_{12}+y_{21}+\frac{1}{R_0} & 0 \\ -\frac{A}{R_1} & 0 & \frac{1}{R_1}+j\omega C_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -y_{12}V_S \\ (y_{11}+y_{12})V_S \\ 0 \end{bmatrix}$$

where y_{11}, y_{12}, y_{21} and y_{22} are the y parameters of the uniform distributed RC line given by

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = Y \begin{bmatrix} \coth\theta & -\operatorname{csch}\theta \\ -\operatorname{csch}\theta & \coth\theta \end{bmatrix}$$

where $Y = \sqrt{\frac{sC}{R}}$ and $\theta = \sqrt{sRC}$.

Suggested passband sample points are

{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.65, 0.7} rad/s.

Suggested stopband sample points are

{1.415, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3.0} rad/s.

Let C_2R_1 be one variable with C_2 fixed at 2.62. Variables to be used for problem (a) are A, R, C, R_0, R_1 and C_1. For problems (b) and (c) the variables are A, C, R_1 and C_1 with $R_0 = 1$ and $R = 17.786$. It is suggested that the transformation

$$\phi_i = \exp \phi_i'$$

is used so that the variables ϕ_i' are unconstrained while the ϕ_i are positive.

[Source: Charalambous (1974).]

Question 188 A seven-section, cascaded, lossless, transmission-line filter with frequency-dependent terminations is depicted in Fig. 39. The frequency dependence of the terminations is given by

$$R_g = R_L = 377/\sqrt{1-(f_c/f)^2} ,$$

where

$$f_c = 2.077 \text{ GHz}.$$

The section lengths are to be kept fixed at 1.5 cm. The problem is to optimize the 7 characteristic impedances such that a passband specification of 0.4 dB insertion loss is met in the range 2.16 to 3 GHz while the loss at 5 GHz is maximized. Suggested passband sample points are 22 uniformly spaced frequencies including band edges.

[Reference: Bandler, Srinivasan and Charalambous (1972).]

Question 189 Consider the active filter shown in Fig. 40. Let $R_g = 50$ Ω, R = 75 Ω. Take a model of the amplifier as

$$A(s) = \frac{A_0 \omega_a}{s+\omega_a} ,$$

where s is the complex frequency variable, A_0 is the d.c. gain and $\omega_a = 12\pi$ rad/s. Use the equivalent circuit shown in Fig. 41 for the purpose of nodal analysis.

The ideal transfer function, i.e., for $A_0 \to \infty$ and $R_3 \to \infty$ is

$$\frac{V_2}{V_g} = - G_1 \frac{sC_1}{s^2C_1C_2+sG_2(C_1+C_2)+G_2(G_4+G_1)}$$

and the nodal equations for the nonideal filter are

$$\begin{bmatrix} G_1+G_g & 0 & -G_1 & 0 \\ 0 & G_2+G_3+sC_2+A_2G_3 & -sC_2 & -G_2+A_1A_2G_3 \\ -G_1 & -sC_2 & G_1+G_4+sC_1+sC_2 & -sC_1 \\ 0 & -G_2 & -sC_1 & G_2+sC_1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} G_gV_g \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

Let $F = |V_2/V_g|$. The specifications are w.r.t. frequency f:

$F \leq 1/\sqrt{2}$ for $f \leq 90$ Hz,

$F \leq 1.1$ for $90 \leq f \leq 110$ Hz,

$F \leq 1/\sqrt{2}$ for $f \geq 110$ Hz,

$F \geq 1/\sqrt{2}$ for $92 \leq f \leq 108$ Hz,

$F \geq 1$ for $f = 100$ Hz.

Find an optimum solution in the minimax sense for components R_1, C_1, C_2 and R_4, given

$$A_0 = 2 \times 10^5 ,$$
$$R_2 = 2.65 \times 10^4 \ \Omega ,$$
$$C_1 = C_2 = C.$$

Question 190 Describe in detail and explain all the information to be supplied by a user to run the optimization package you are currently using or are familiar with.

Question 191 Describe all necessary steps required to access the optimization package described in Question 190 to execute an optimization problem in conjunction with user-supplied programs.

Question 192 What is the effect on the number of function evaluations or iterations of changing starting points in the minimization problems you have tested using the package of Question 190.

Question 193 Each student should familiarize himself with the optimization package under study by running the examples in the user's manual. Run each example from starting points different to the ones given and compare the results with those in the manual.

Question 194 For the resistive network of Question 27, solve the nodal equations by an unconstrained minimization package. Take $G_1 = G_3 = G_5 =$ 1 mho, $R_2 = R_4 = 0.5$ ohm. Write all necessary subprograms.

Question 195 For the voltage divider of Question 154, the specifications

$$0.46 \leqslant \frac{R_2}{R_1 + R_2} \leqslant 0.53$$

$$1.85 \leqslant R_1 + R_2 \leqslant 2.15$$

must be met in the minimax sense using an available package. Write all necessary subprograms.

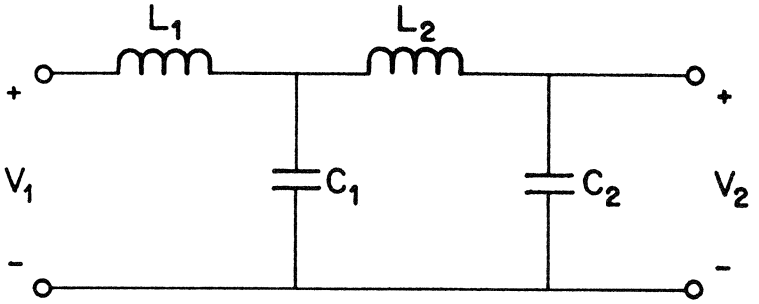


Fig. 1 LC ladder network (Question 10).

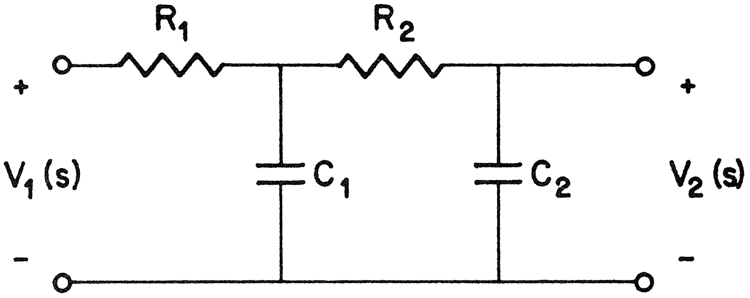


Fig. 2 RC ladder network (Question 12).

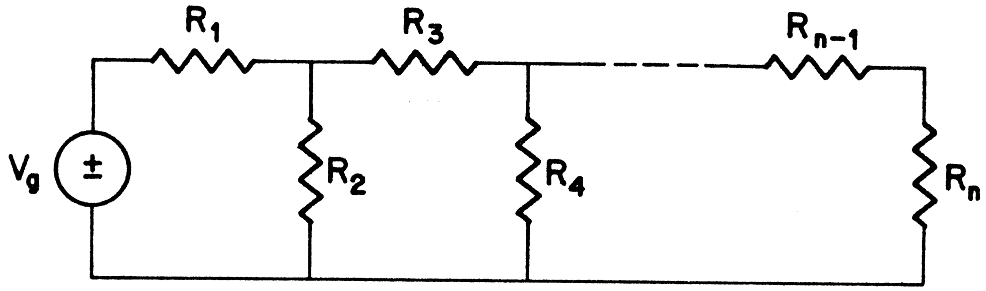


Fig. 3 Resistive ladder network (Question 16).

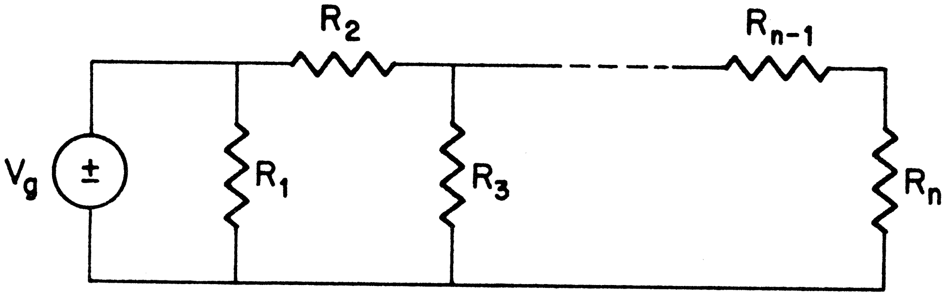


Fig. 4 Resistive ladder network (Question 18).

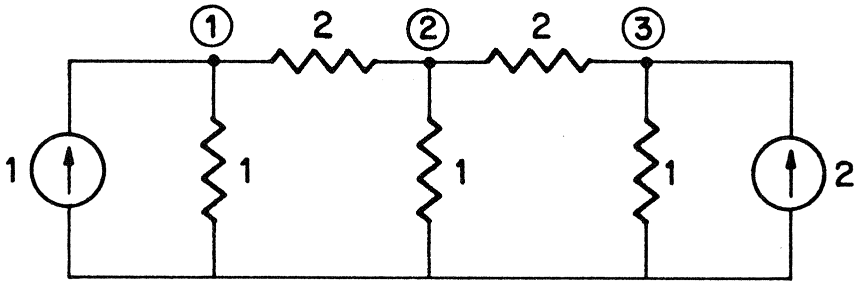


Fig. 5 Three-node resistive ladder network (Question 20).

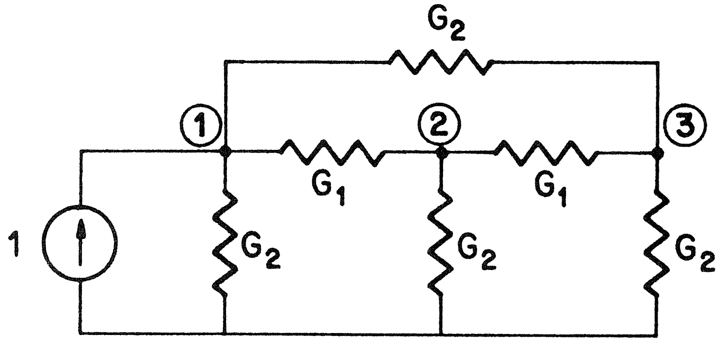


Fig. 6 Three-node resistive ladder network (Question 25).

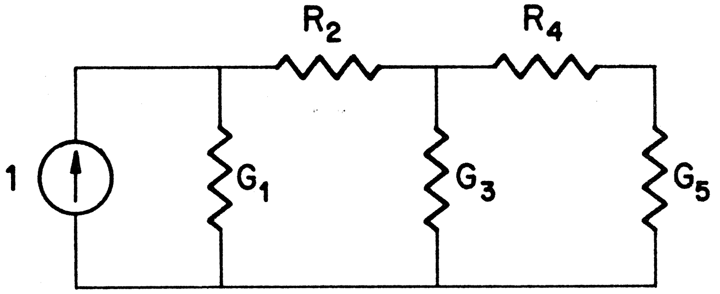


Fig. 7 Resistive ladder network (Question 27).

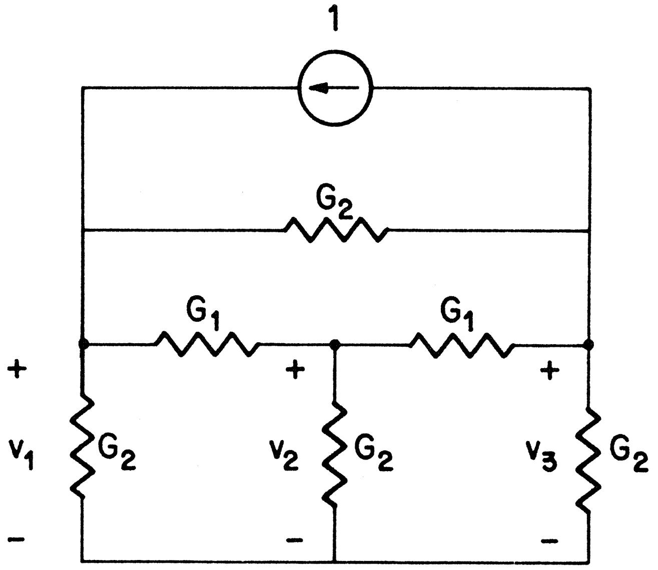


Fig. 8 Resistive network (Question 28).

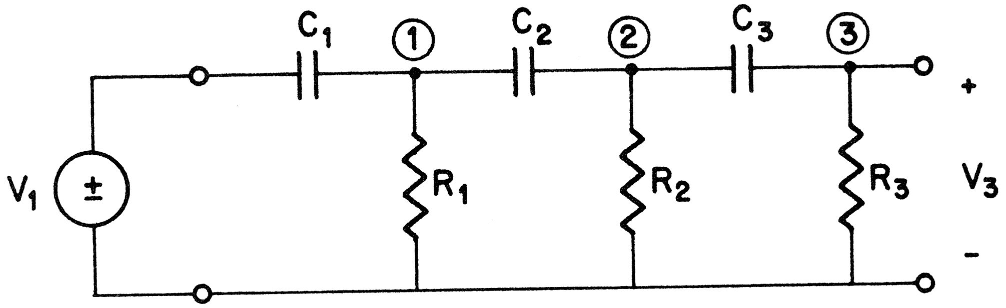


Fig. 9 CR ladder network (Question 29).

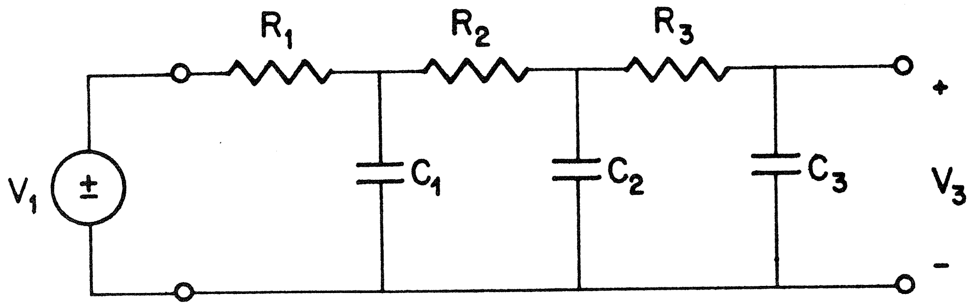


Fig. 10 RC ladder network (Question 30).

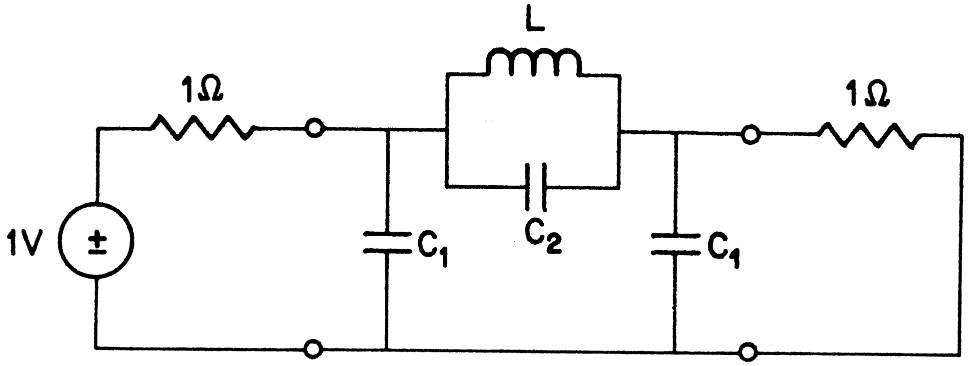


Fig. 11 LC filter network (Question 32).

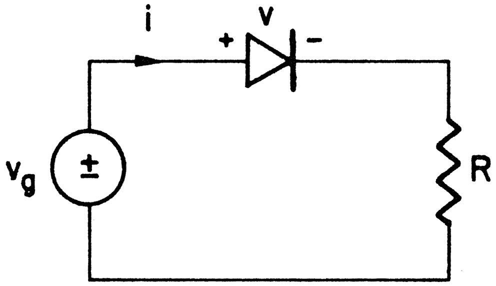


Fig. 12 Resistor-diode network (Question 55).

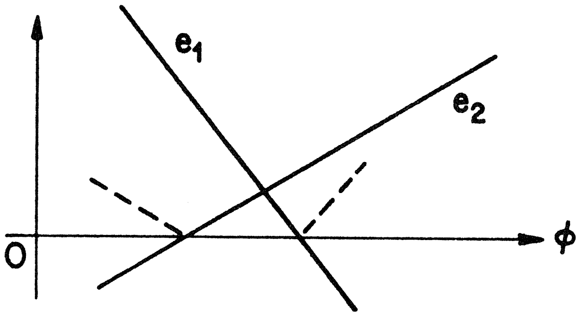


Fig. 13 Error functions (Question 100).

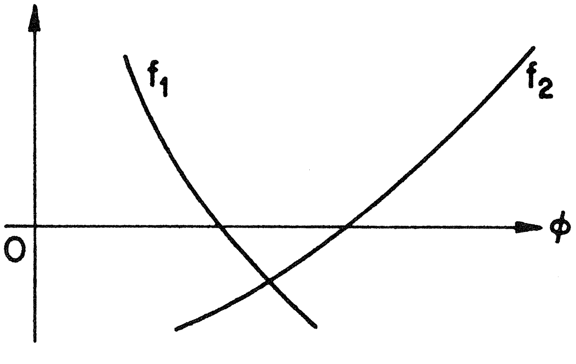


Fig. 14 Two functions of one variable (Question 101).

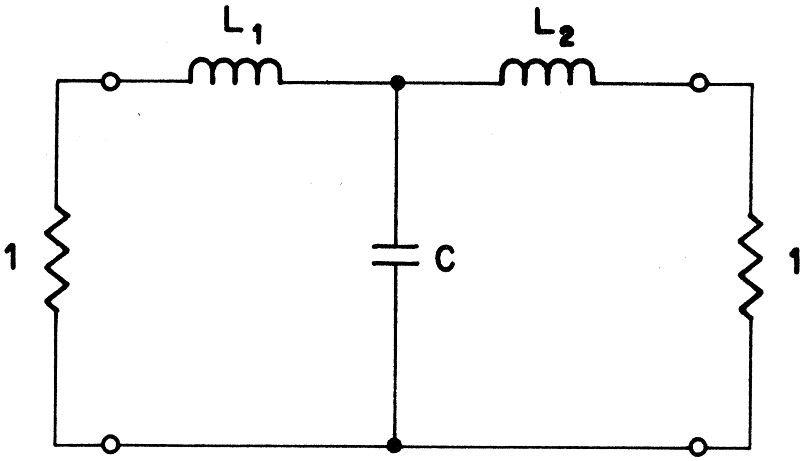


Fig. 15 . LC lowpass filter (Question 106).

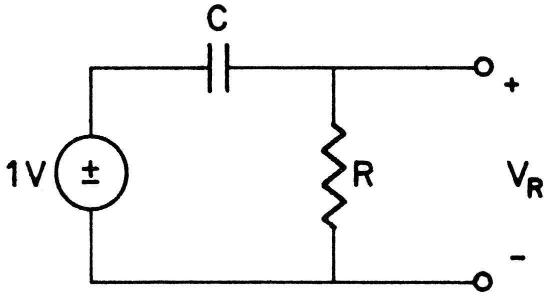


Fig. 16 RC circuit (Question 108).

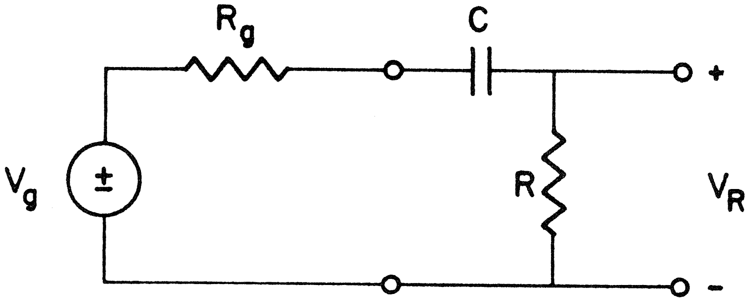


Fig. 17 RC circuit (Question 109).

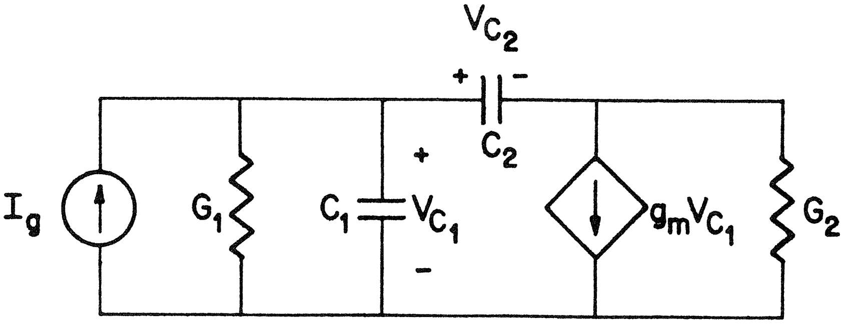
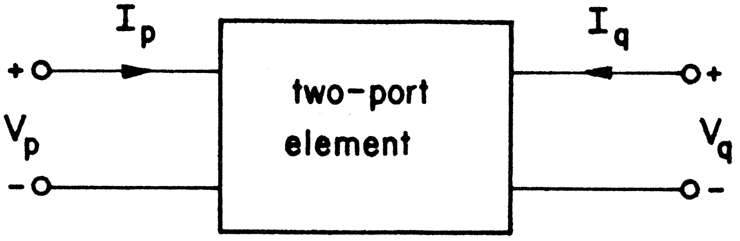


Fig. 18 Active circuit (Question 110).



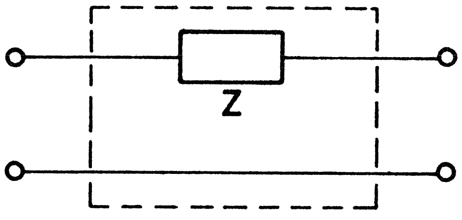


Fig. 19 Example of two-port (Question 123).

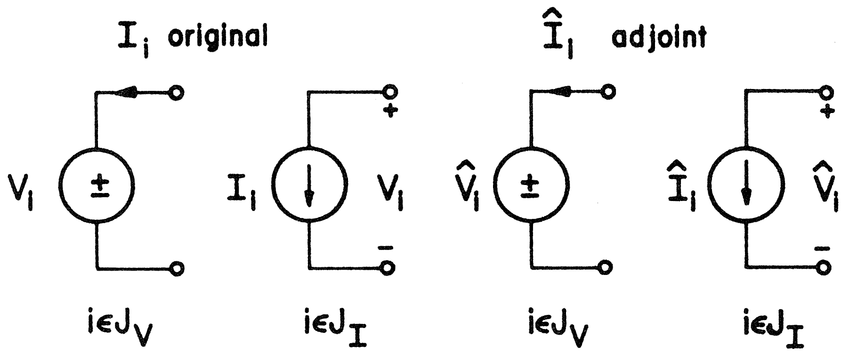


Fig. 20 Excitations and responses in the original and adjoint networks (Question 125).

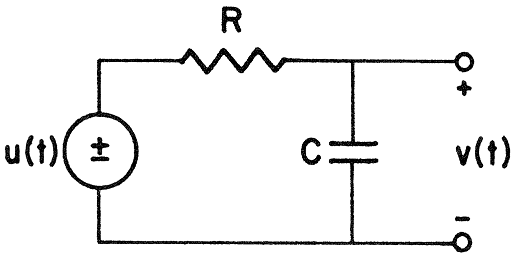


Fig. 21 RC circuit (Question 126).

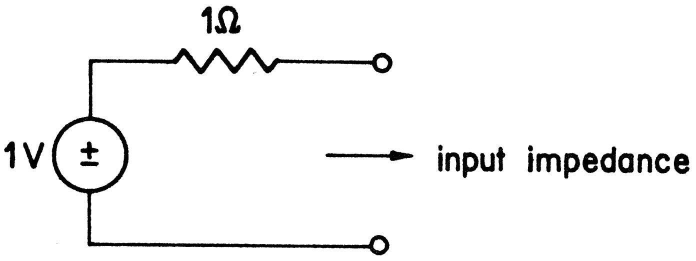


Fig. 22 Source for input impedance calculation (Question 127).

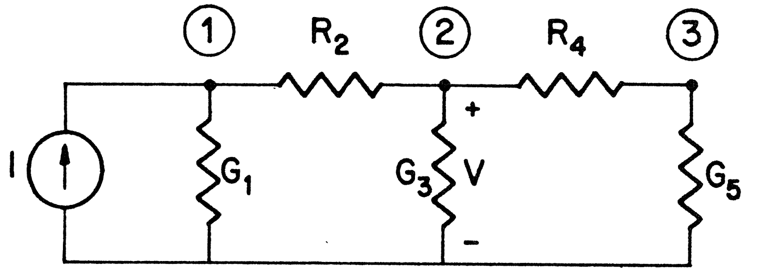


Fig. 23 Three-node resistive network (Question 130).

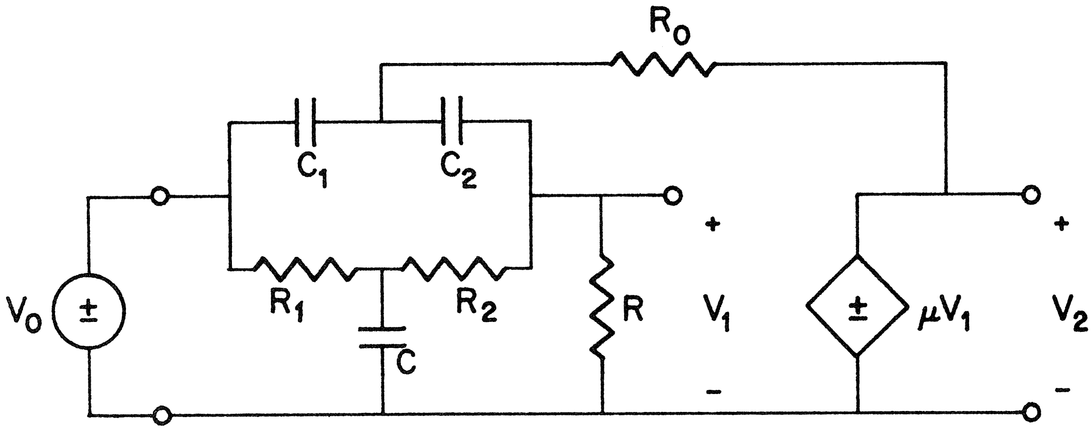


Fig. 24 Active circuit example (Question 132).

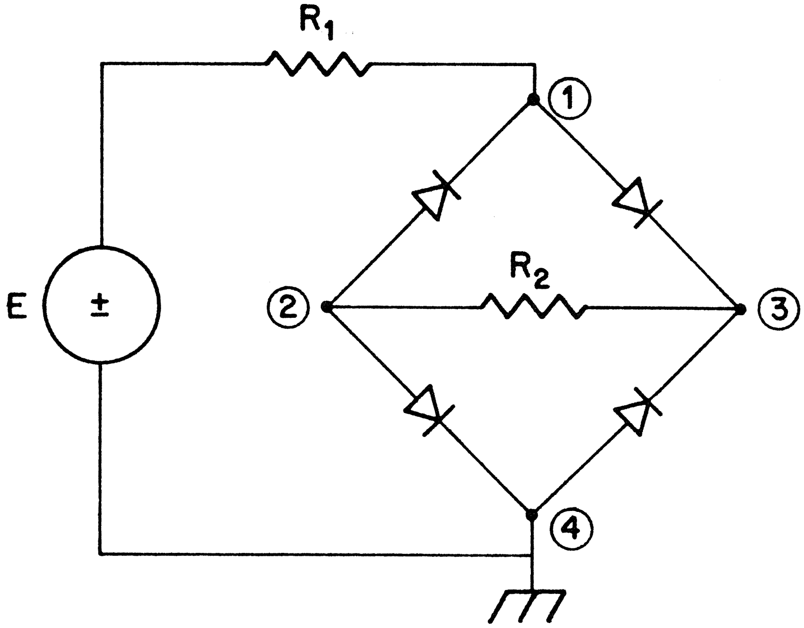


Fig. 25 Resistor-diode network (Question 136).

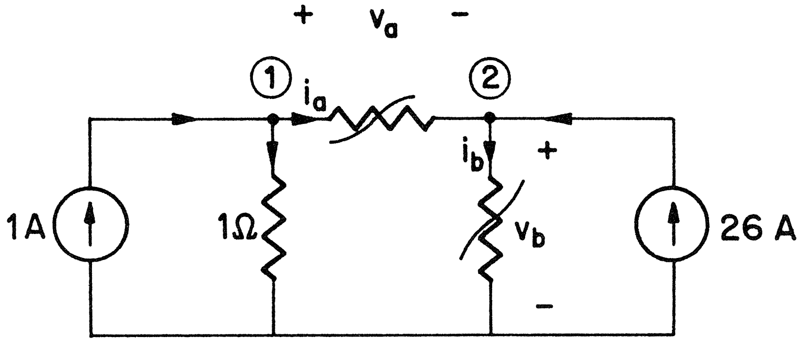


Fig. 26 Nonlinear circuit example (Question 140).

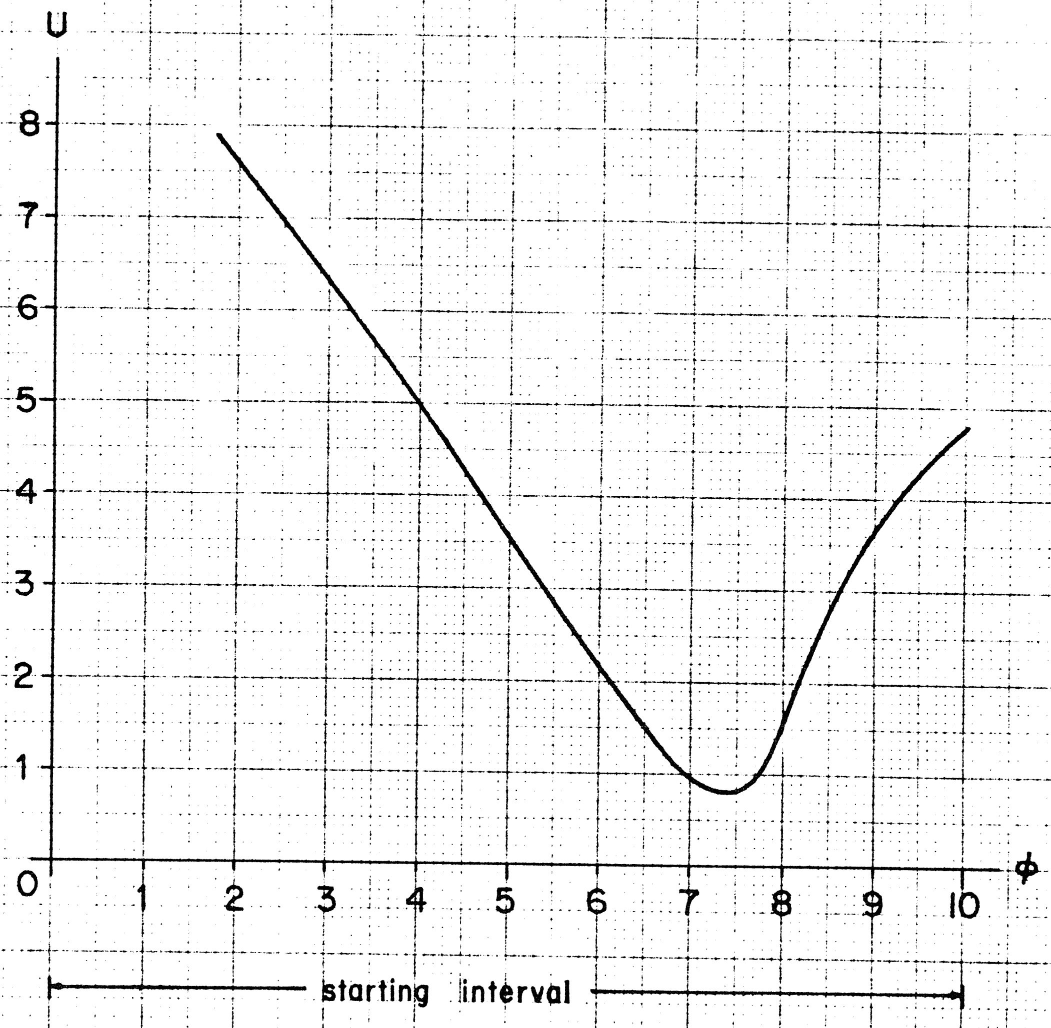


Fig. 27 Function of one variable (Question 146).

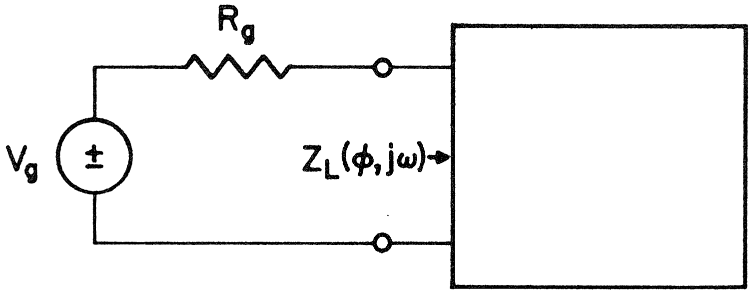


Fig. 28 Impedance matching example (Question 153).

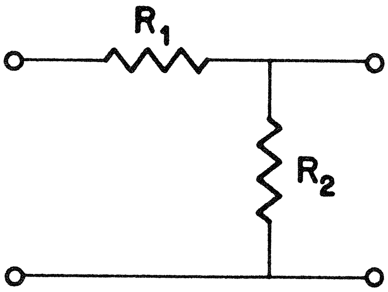


Fig. 29 Voltage divider circuit (Question 154).

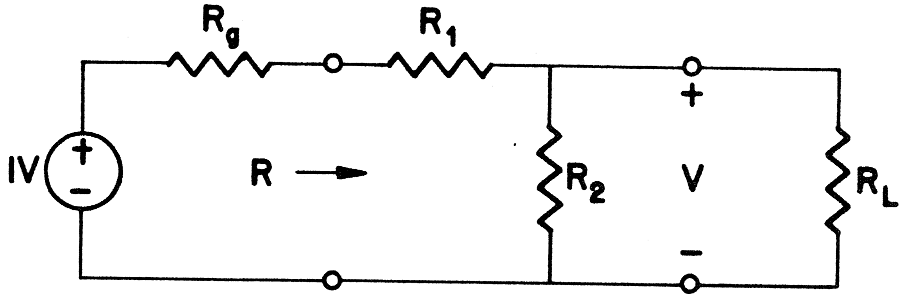


Fig. 30 Nonideal voltage divider circuit (Question 157).

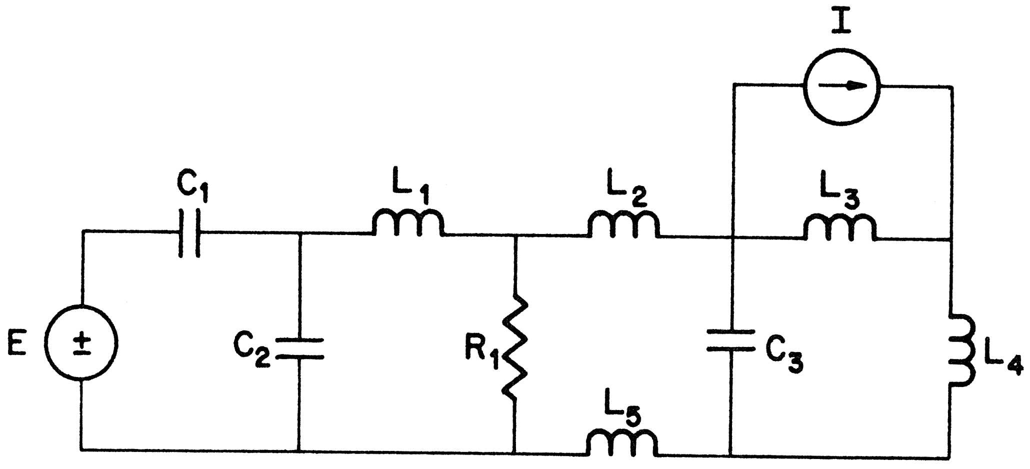


Fig. 31 Arbitrary network (Question 163).

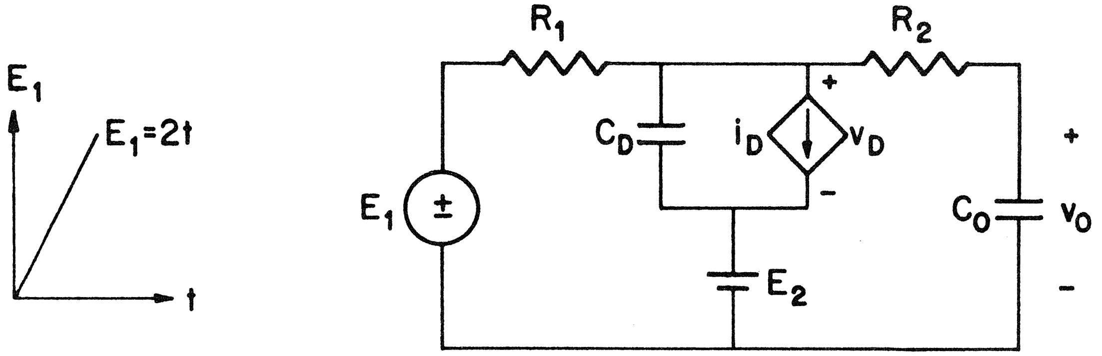


Fig. 32 Time domain circuit example (Question 164).

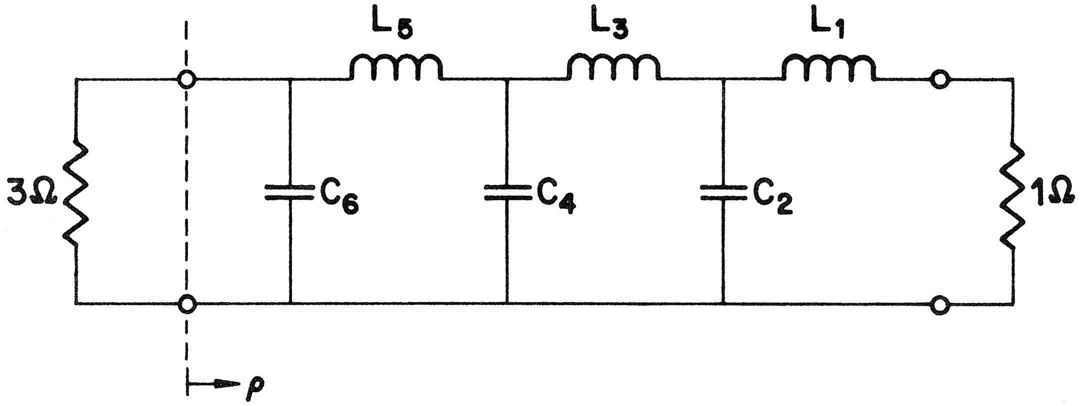


Fig. 33 Lumped element LC transformer (Question 168).

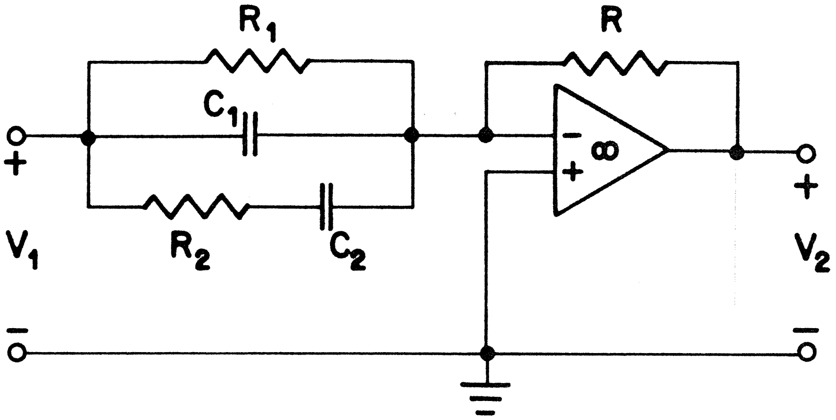


Fig. 34 RC active equalizer example (Question 169).

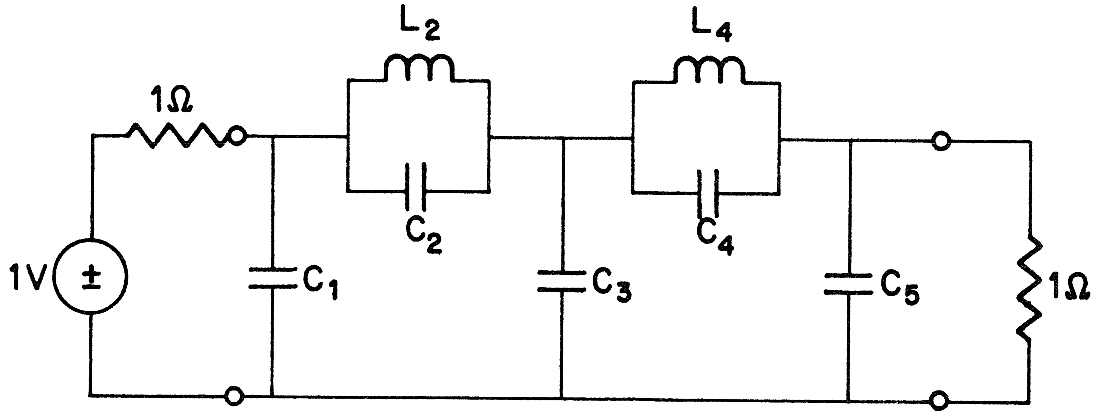


Fig. 35 Elliptic low-pass filter (Question 171).

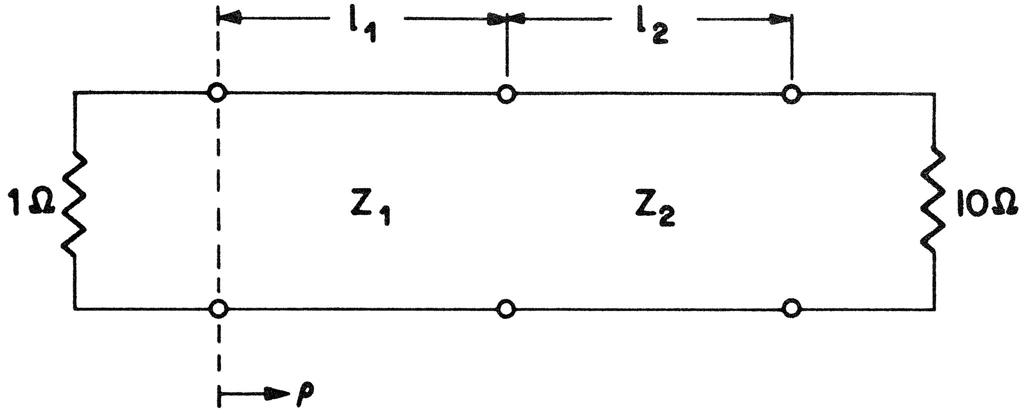


Fig. 36 Two-section transmission-line transformer example (Question 174).

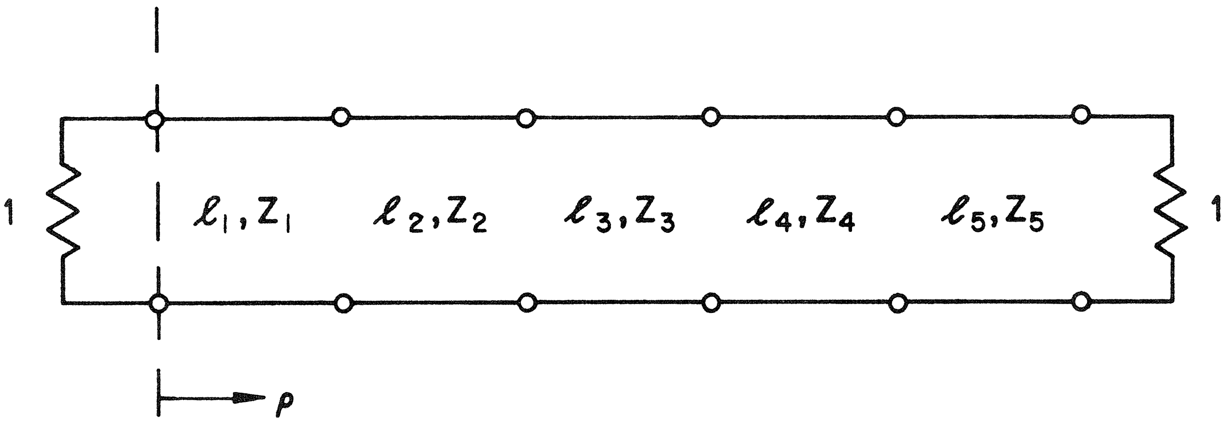


Fig. 37 Five-section transmission-line filter (Question 183).

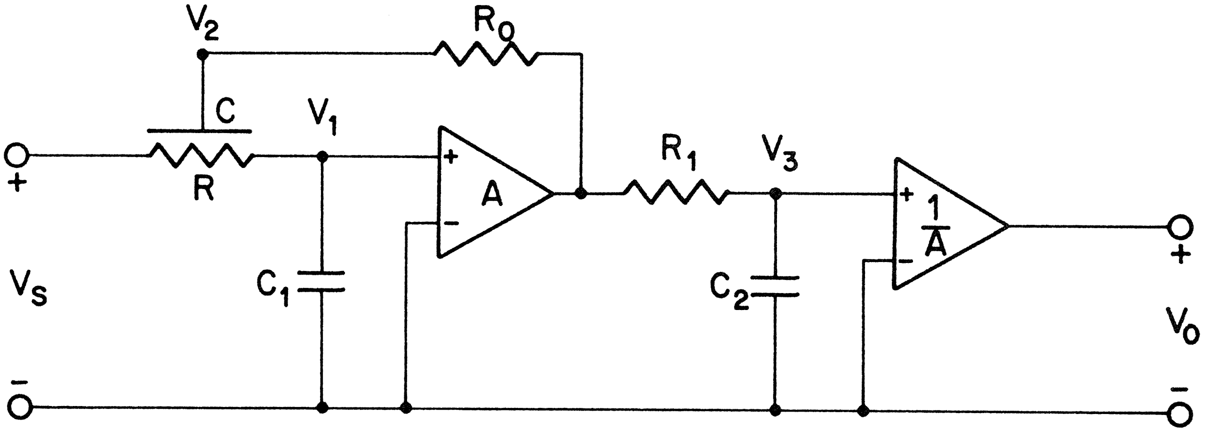


Fig. 38 Third-order lumped-distributed-active lowpass filter. (Question 187).

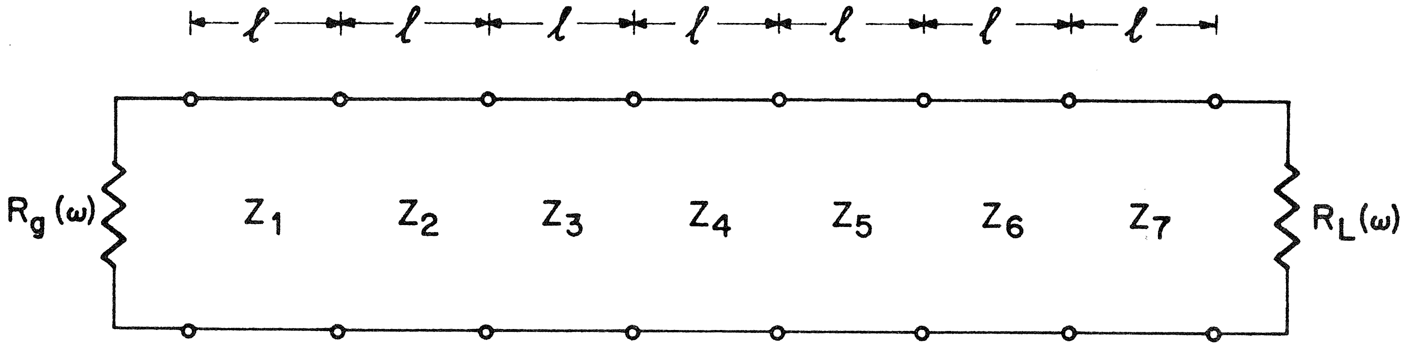


Fig. 39 Seven-section, cascaded transmission-line filter (Question 188).

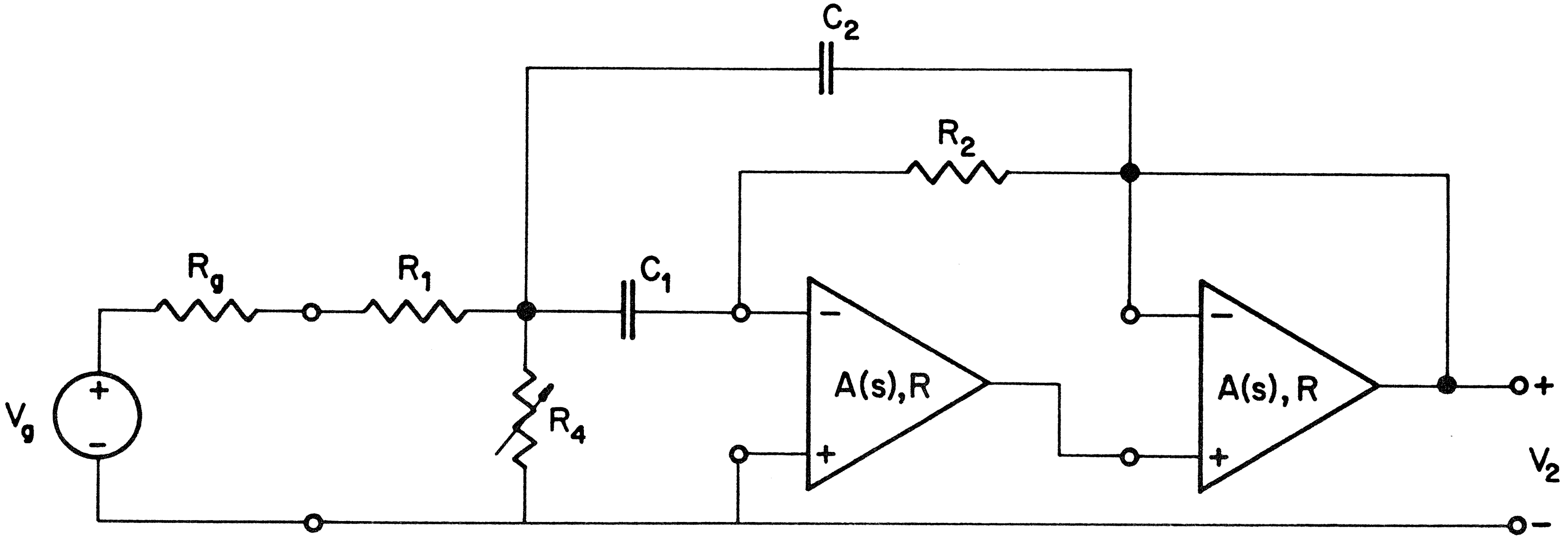


Fig. 40 Active filter (Question 189).

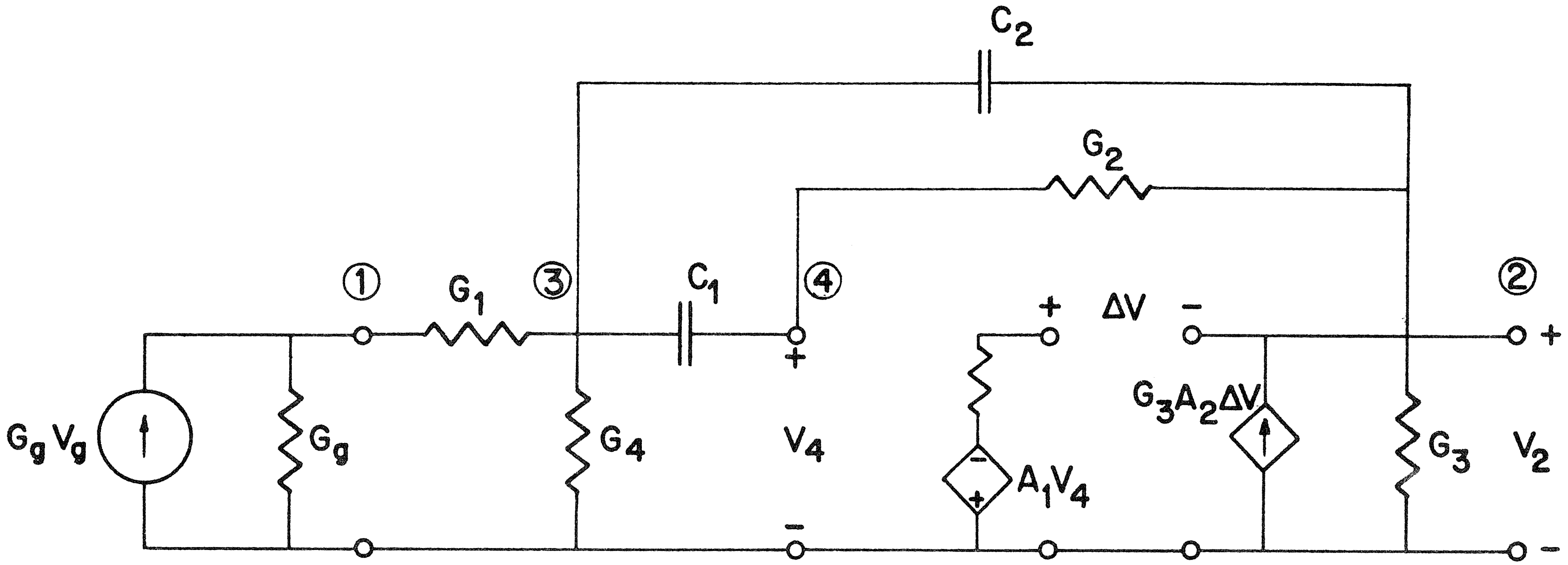


Fig. 41 Equivalent circuit for the active filter of Fig. 40 (Question 189).