MFNC - A FORTRAN PACKAGE FOR MINIMIZATION WITH GENERAL CONSTRAINTS

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Abstract

MFNC is a package of subroutines for minimization of a nonlinear objective function subject to nonlinear constraints. It is an extension and modification of a set of subroutines of the Harwell Subroutine Library (subroutines VF02AD, VF02BD, VF02CD, VE02A, LA02A, MB01C, FM02AS). First derivatives of all functions with respect to all variables are assumed to be available. The solution is found by an iteration that minimizes a quadratic approximation of the objective function subject to linearized constraints. The method was presented by Han and Powell. The package and documentation have been developed for the CDC 170/730 system with the NOS 1.4 level 552 operating system and the Fortran Extended (FTN) version 4.8 compiler.

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I. INTRODUCTION

The package of Harwell subroutines (with the main subroutine VF02AD) for minimization with nonlinear constraints [1,2,3] has been modified and extended to provide a uniform printed output of input parameters as well as intermediate and final results of optimization. The modifications include:

(1) conversion to single precision,

(2) replacement of the subroutine MB01B by MB01C, which supersedes MB01B and removes the restriction on the matrix order,

(3) adjustments in the subroutines LA02A and VE02A required by MB01C,

(4) standardization of the source code.

The extensions, in the form of additional subroutines, contain:

(1) more flexible and more detailed printed output generated by the package,

(2) numerical verification of partial derivatives,

(3) replacement of the "reverse communication" by the separate user-defined subroutine that evaluates the functions and their first-order derivatives.

Consequently, the calling sequences have been changed appropriately, however, the original call to the subroutine VF02A (in single precision) has been preserved with slight modifications only.

The whole package is written in Fortran IV for the CDC 170/730 system. At McMaster University it is available in the form of a library of binary relocatable subroutines which is linked with the user's program by an appropriate call to the main subroutine of the package. The name of the library is LIBRMFN. The library is available as a group indirect file under the charge RJWBAND. The general sequence of NOS commands to use the package can be as follows:

/GET(LIBRMFN/GR) - fetch the library,

/LIBRARY(LIBRMFN) - indicate library to the loader.

The user's program should be composed (at least) of:

- the main segment that prepares arguments and calls the main subroutine of the package,
- the subroutine which evaluates the objective and constraint functions and their partial derivatives at points determined by the package; the name of this subroutine can be arbitrary because it is transferred to the package as one of the arguments.

This document includes the user's manual of the MFNC package presented together with illustrative examples. A Fortran listing of the package is found in [4].

II. GENERAL DESCRIPTION

The purpose of the package is to minimize the objective function $F(\underset{\sim}{x})$ of n variables, $\underset{\sim}{x} = [x_1 \dots x_n]^T$, subject to the general equality and inequality constraints

$$f_j(\underset{\sim}{x}) = 0, \qquad j=1, \dots, \ell_{eq},$$
$$f_j(\underset{\sim}{x}) \geq 0, \qquad j=\ell_{eq}+1, \dots, \ell,$$

where the objective and the constraint functions are differentiable and their first-order derivatives are available.

The algorithm used in the package is Powell's [1,5] variable metric method for constrained optimization which is based on the results of Han [2]. In the kth iteration the search direction $\underset{\sim}{h}^k$ is determined as the solution of the linearly constrained quadratic minimization subproblem

$$\underset{\underset{\sim}{h}^k}{\text{Minimize}}\ \tilde{F}(\underset{\sim}{x}^{k-1}, \underset{\sim}{h}^k) = F(\underset{\sim}{x}^{k-1}) + \underset{\sim}{h}^{k^T} \underset{\sim}{F}'(\underset{\sim}{x}^{k-1}) + 0.5\ \underset{\sim}{h}^{k^T} \underset{\sim}{B}^k \underset{\sim}{h}^k$$

subject to the constraints

$$\underset{\sim}{h}^{k^T} \underset{\sim}{f}_j'(\underset{\sim}{x}^{k-1}) + \alpha^k f_j(\underset{\sim}{x}^{k-1}) = 0\ , \qquad j=1,\ldots,\ell_{eq}\ ,$$

$$\underset{\sim}{h}^{k^T} \underset{\sim}{f}_j'(\underset{\sim}{x}^{k-1}) + \alpha_j^k f_j(\underset{\sim}{x}^{k-1}) \geq 0\ , \qquad j=\ell_{eq}+1,\ldots,\ell\ ,$$

$$0 \leq \alpha^k \leq 1\ ,$$

where $\underset{\sim}{F}'(\underset{\sim}{x})$ and $\underset{\sim}{f}_j'(\underset{\sim}{x})$, $j=1,\ldots,\ell$, are the gradient vectors of the objective and constraint functions, respectively, $\underset{\sim}{B}^k$ is a positive definite square matrix of dimension n containing second-order derivative information which is updated in consecutive iterations according to the BFGS formula (initially the matrix is set to the unit matrix, $\underset{\sim}{B}^0 = \underset{\sim}{1}$), and α^k is an additional variable introduced in order to allow infeasibility in linearized constraints, while α_j^k, $j=\ell_{eq}+1,\ldots,\ell$, are defined as

$$\alpha_j^k = \begin{cases} 1, \text{ if } f_j(\underset{\sim}{x}^{k-1}) > 0\ , \\ \alpha^k, \text{ if } f_j(\underset{\sim}{x}^{k-1}) \leq 0\ . \end{cases}$$

Usually the solution of the quadratic subproblem results in $\alpha^k = 1$. If the only feasible solution corresponds to $\alpha^k = 0$ and $\underset{\sim}{h}^k = \underset{\sim}{0}$, the algorithm terminates and it is assumed that the constraints are inconsistent. Positive values of α^k are used in a subsequent one-dimensional search of the consecutive approximations $\underset{\sim}{x}^k$ of the solution

$$x^k = x^{k-1} + \beta^k h^k ,$$

where β^k is a positive multiplier, $0 < \beta^k \leq 1$, which is chosen in such a way that

$$\bar{F}(x^{k-1} + \beta^k h^k, \mu^k) < \bar{F}(x^{k-1}, \mu^k) ,$$

where

$$\bar{F}(x, \mu) = F(x) + c(x, \mu)$$

and

$$c(x, \mu) = \sum_{1 \leq j \leq \ell_{eq}} \mu_j |f_j(x)| + \sum_{\ell_{eq} < j \leq \ell} \mu_j |\min(0, f_j(x))| .$$

$c(x, \mu)$ is equal to zero when all the constraints are satisfied, and is positive otherwise. The vectors μ^k depend on the Lagrangian multipliers λ^k (determined at the solution h^k of the quadratic subproblem) in the following way:

$$\mu_j^1 = |\lambda_j^1|, \ j=1,\dots,\ell,$$
$$\mu_j^k = \max(|\lambda_j^k|, 0.5\ (\mu_j^{k-1} + |\lambda_j^k|)), \ j=1,\dots,\ell, \ k=2,3,\dots .$$

The multiplier β^k is determined iteratively (line search) starting with the value $\beta_1^k = 1$. In each step i of the search

$$\beta_{i+1}^k = \max(0.1\ \beta_i^k, \bar{\beta}_i^k)$$

where $\bar{\beta}_i^k$ is the value that minimizes the quadratic approximation of the function

$$\bar{F}(x^{k-1} + \beta_i^k h^k, \mu^k) .$$

The value β^k is equal to the first β_i^k that satisfies the condition

$$\bar{F}(\underset{\sim}{x}^{k-1} + \beta_i^k \underset{\sim}{h}^k, \underset{\sim}{\mu}^k) \le \bar{F}(\underset{\sim}{x}^{k-1}, \underset{\sim}{\mu}^k) + 0.1\ \beta_i^k(\underset{\sim}{h}^{k^T} \underset{\sim}{F}'(\underset{\sim}{x}^{k-1}) - \alpha^k\ c(\underset{\sim}{x}^{k-1}, \mu^k)).$$

Usually the condition is satisfied in the first step of the line search and $\beta^k = \beta_1^k = 1$. However, when the starting point is far from the solution, more line search steps can be required. If the number of required line search steps is greater than 5 it is assumed that the gradient vectors are incorrect and the algorithm terminates.

The algorithm terminates when any one of the following conditions is satisfied:

(1) the required accuracy is obtained

$$|\underset{\sim}{h}^{k^T} \underset{\sim}{F}'(\underset{\sim}{x}^{k-1})| + \sum_{1 \le j \le \ell} |\lambda_j^k f_j(\underset{\sim}{x}^{k-1})| \le \varepsilon ,$$

where ε is defined by the user (argument EPS),

(2) an uphill search direction is obtained, which can only be due to rounding errors; the required accuracy cannot be obtained in this case,

(3) the number of function evaluations exceeds the limit defined by the user (argument MAXF),

(4) the line search procedure requires more than 5 steps, which is usually due to incorrect derivatives but can also occur when the required accuracy cannot be achieved and the function values are dominated by rounding errors,

(5) a vector of variables that satisfy the constraints cannot be determined, which is usually due to inconsistent constraints but can also occur when constraint function derivatives are incorrect,

(6) the changes of the values of variables are restricted by an artificial bound (with default value 10^6) which is usually due to an unbounded solution but may also occur when the problem is badly scaled.

Moreover, the user can terminate the iterative procedure and cause the return from the package by setting one of parameters during evaluation of functions and their first-order derivatives (see argument FCD).

III. STRUCTURE OF THE PACKAGE

There are 3 different entries to the package and 3 corresponding "main" (or interfacing) subroutines:

1. subroutine MFNC1A - standard entry which provides uniform printing of input parameters as well as intermediate and final results,
2. subroutine MFNC2A - basic entry which does not provide any form of printed output (it is the user's responsibility to organize printing of data and results in this case),
3. subroutine VF02A - original entry, as defined in VF02AD subroutine specification [3].

Block diagrams of the package, corresponding to entries 1, 2 and 3 are shown in Fig. 1, 2 and 3, respectively. It can be observed that the PRINTOUT package of subroutines is used only when entry 1 (subroutine MFNC1A) is called, and that the subroutine MFN00Q (Fig. 1), which is for printing the values of the functions and their first-order derivatives, is replaced by the dummy subroutine MFN00Z (Fig. 2) when entry 2 is used.

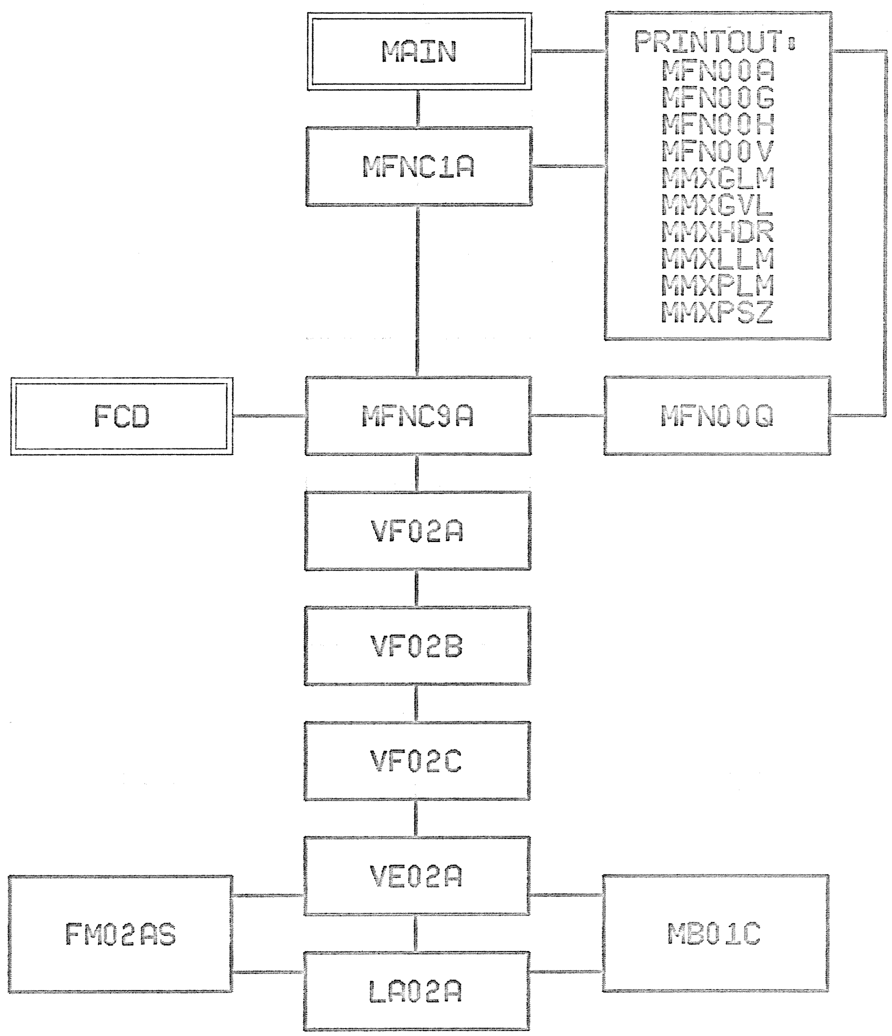

Fig. 1 Structure of the MFNC package corresponding to the standard entry (subroutine MFNC1A).

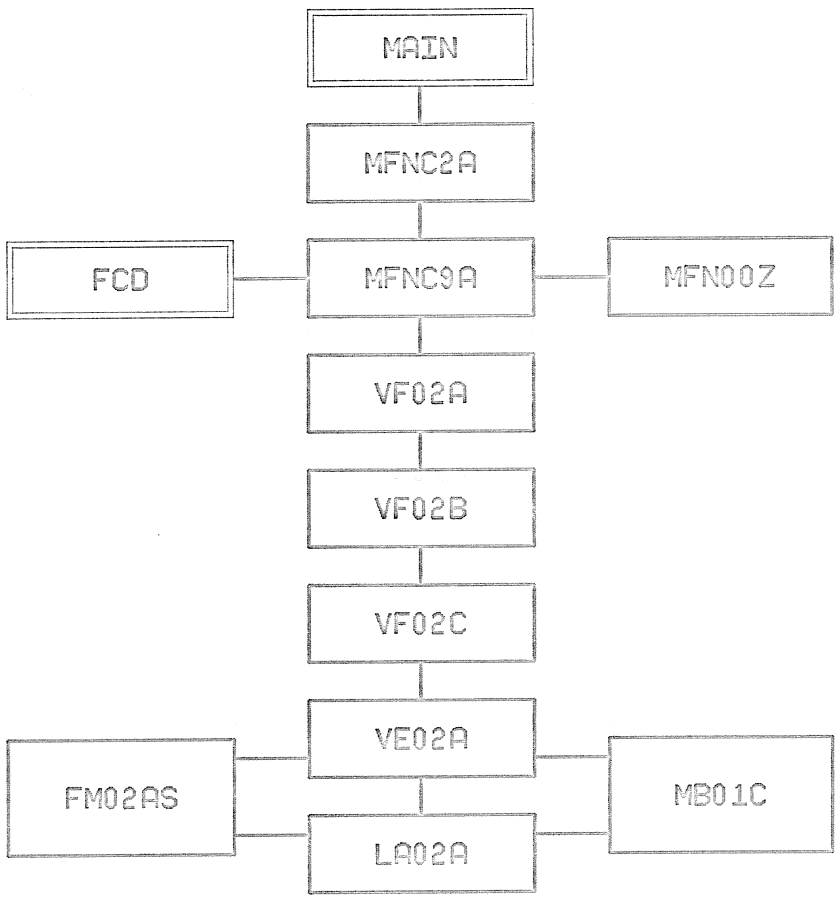

Fig. 2 Structure of the MFNC package corresponding to the basic entry (subroutine MFNC2A).

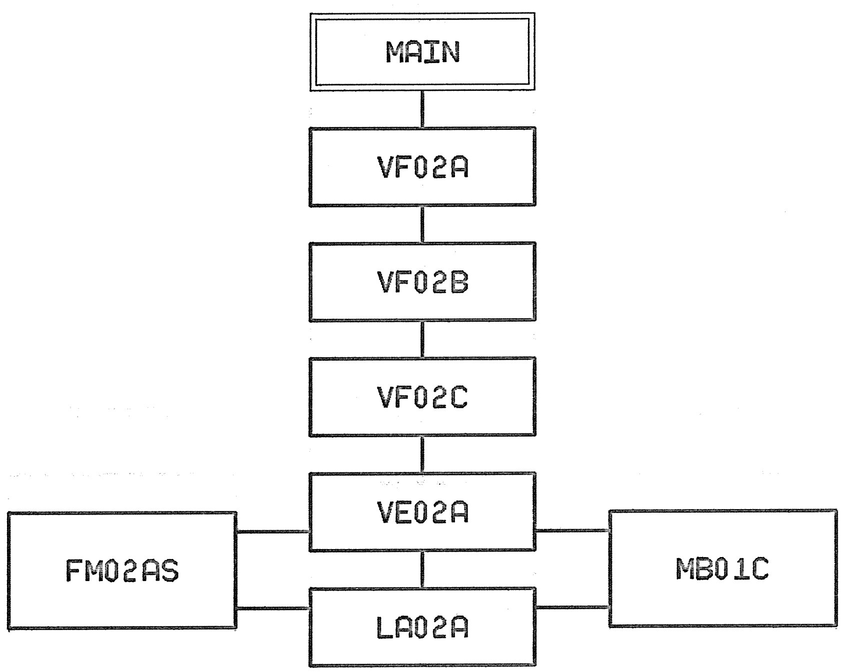

Fig. 3 Structure of the MFNC package corresponding to the original entry (subroutine VF02A).

The common part of the package is composed of subroutines VF02A, VF02B, VF02C, VE02A, LA02A, MB01C and FM02AS. VF02A subdivides the workspace (defined by the user) into a set of vectors and matrices used by the remaining subroutines and checks formal correctness of some parameters. VF02B controls the minimization procedure, implements the line search, calls VF02C for solving quadratic subproblems, updates the approximation of the Hessian matrix, and checks the convergence of the algorithm. VF02C determines linear approximations of the constraint functions, calls VE02A to solve linearly constrained quadratic minimization, and calculates Lagrangian multipliers; it also checks whether the required feasibility conditions hold for the solution returned by VE02A. VE02A finds a minimum of the quadratic function subject to linear equality and inequality constraints using the method of Fletcher. The method requires a feasible initial point, and this is obtained by calling LA02A. MB01C is used for the matrix inversion, and FM01AS for the evaluation of the inner product of two real vectors.

The main segment MAIN and the subroutine FCD for the evaluation of functions and their first-order derivatives must be supplied by the user.

When the standard entry (Fig. 1) is used, the subroutine MFNC1A and the set of subroutines PRINTOUT provide printed output containing principal input parameters of the problem to be solved, and the solution obtained by the package. Moreover, the subroutine MFN00Q outputs the values of functions and their derivatives according to the argument IPR in the call statement of MFNC1A.

For the standard entry (Fig. 1) and the basic entry (Fig. 2) the subroutine MFNC9A checks the formal correctness of input parameters, calls the user-defined subroutine FCD for the evaluation of functions and their derivatives and sets the output parameters to the values corresponding to the solution found by the package.

IV. LIST OF ARGUMENTS

Standard entry (subroutine MFNC1A)

The subroutine call is

```
CALL MFNC1A (FCD,N,L,LEQ,X,EPS,MAXF,W,IW,ICH,IPR,IFLAG)
```

The arguments are as follows.

FCD is the name of a subroutine supplied by the user. It must have the form

```
SUBROUTINE FCD(N,L,X,F,G,C,D,K)
DIMENSION X(N),G(N),C(L),D(K,L)
```

and it must calculate the values of the objective function F, its gradient G, the constraint functions $f_i(\underset{\sim}{x})$ and their derivatives $\partial f_i(\underset{\sim}{x})/\partial x_j$ at the point $\underset{\sim}{x}$ corresponding to X(1),X(2),...,X(N), and store the values in the following way:

G(J) = $\partial F(\underset{\sim}{x})/\partial x_J$, J=1,...,N,

C(I) = $f_I(\underset{\sim}{x})$, I=1,...,L,

D(J,I) = $\partial f_I(\underset{\sim}{x})/\partial x_J$, I=1,...,L, J=1,...,N.

Note: The name FCD can be arbitrary (user's choice) and must appear in an EXTERNAL statement in the segment calling MFNC1A.

The user can terminate the iterative procedure and force the return from the package by setting to zero (in the subroutine FCD) the variable MARK in the common area MFN000

```
COMMON /MFN000/ MARK
```

(on entry to the package MARK is set to 1).

N is an INTEGER argument which must be set to n, the number of optimization parameters. Its value must be positive and it is not changed by the package.

L is an INTEGER argument which must be set to ℓ, the total number of equality and inequality constraints. Its value must be positive or zero and it is not changed by the package.

LEQ is an INTEGER argument which must be set to ℓ_{eq}, the number of equality constraints. Its value must be positive or zero and not greater than L, and not greater than N. Its value is not changed by the package.

X is a REAL array of the length at least N, which on entry must be set to the initial approximation of the solution, $X(I)=x_I^0$, I=1,...,N. On exit, X contains the best solution found by the package.

EPS is a REAL variable which on entry must be set to the required accuracy of the solution. The iteration terminates when the objective function is predicted to be within EPS of its final value and allowance is made for any constraint violation. If EPS is chosen too small, the iteration terminates when no better estimation of the solution can be obtained because of rounding errors.

MAXF is an INTEGER variable which must be set to an upper bound on the number of calls to FCD (i.e., the maximum number of functions evaluations). On exit, MAXF contains the number of calls to FCD performed by the package.

W is a REAL array which is used as workspace. Its length is given by IW. On exit, the first L+1 elements of W contain the function values at the solution, i.e., $W(1)=F(\underset{\sim}{x})$ and $W(I+1)=f_I(\underset{\sim}{x})$, I=1,...,L.

IW is an INTEGER argument which must be set to the length of W. Its value must be at least

IWR = 19+5*N*N+24*N+6*L+N*L+max(L,3*N+3).

The values of IWR for a set of initial values of arguments L and N are given in Table 1.

ICH is an INTEGER argument which must be set to the unit number (or channel number) that is to be used for the printed output generated by the package. Usually it is the unit number of the file OUTPUT. If ICH is less than or equal to zero, no printed output will be generated by the package. The value of ICH is not changed by the package.

IPR is an INTEGER argument which controls the printed output generated by the package. It must be set by the user and is not changed by the package. The absolute value of IPR, as a decimal number, is "logically" composed of 4 fields

$$|IPR| = pqrs$$

where q, r and s are the least significant one-digit fields, and p is the remaining part of the number. If q is not equal to zero (i.e. q=1,...,9) then the first q evaluations of functions (i.e., the first q calls to FCD) are reported in the printed output. Further, if p is not equal to zero then every

TABLE I

MINIMUM WORKSPACE FOR THE MFNC PACKAGE

L: \ N:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	61	104	157	220	293	376	469	572	685	808	941	1084	1237	1400	1573	1756	1949	2152	2365	2588
2	68	112	166	230	304	388	482	586	700	824	958	1102	1256	1420	1594	1778	1972	2176	2390	2614
3	75	120	175	240	315	400	495	600	715	840	975	1120	1275	1440	1615	1800	1995	2200	2415	2640
4	82	128	184	250	326	412	508	614	730	856	992	1138	1294	1460	1636	1822	2018	2224	2440	2666
5	89	136	193	260	337	424	521	628	745	872	1009	1156	1313	1480	1657	1844	2041	2248	2465	2692
6	96	144	202	270	348	436	534	642	760	888	1026	1174	1332	1500	1678	1866	2064	2272	2490	2718
7	104	152	211	280	359	448	547	656	775	904	1043	1192	1351	1520	1699	1888	2087	2296	2515	2744
8	112	160	220	290	370	460	560	670	790	920	1060	1210	1370	1540	1720	1910	2110	2320	2540	2770
9	120	168	229	300	381	472	573	684	805	936	1077	1228	1389	1560	1741	1932	2133	2344	2565	2796
10	128	177	238	310	392	484	586	698	820	952	1094	1246	1408	1580	1762	1954	2156	2368	2590	2822
11	136	186	247	320	403	496	599	712	835	968	1111	1264	1427	1600	1783	1976	2179	2392	2615	2848
12	144	195	256	330	414	508	612	726	850	984	1128	1282	1446	1620	1804	1998	2202	2416	2640	2874
13	152	204	266	340	425	520	625	740	865	1000	1145	1300	1465	1640	1825	2020	2225	2440	2665	2900
14	160	213	276	350	436	532	638	754	880	1016	1162	1318	1484	1660	1846	2042	2248	2464	2690	2926
15	168	222	286	360	447	544	651	768	895	1032	1179	1336	1503	1680	1867	2064	2271	2488	2715	2952
16	176	231	296	371	458	556	664	782	910	1048	1196	1354	1522	1700	1888	2086	2294	2512	2740	2978
17	184	240	306	382	469	568	677	796	925	1064	1213	1372	1541	1720	1909	2108	2317	2536	2765	3004
18	192	249	316	393	480	580	690	810	940	1080	1230	1390	1560	1740	1930	2130	2340	2560	2790	3030
19	200	258	326	404	492	592	703	824	955	1096	1247	1408	1579	1760	1951	2152	2363	2584	2815	3056
20	208	267	336	415	504	604	716	838	970	1112	1264	1426	1598	1780	1972	2174	2386	2608	2840	3082

TABLE I

MINIMUM WORKSPACE FOR THE MFNC PACKAGE

L: \ N:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
21	216	276	346	426	516	616	729	852	985	1128	1281	1444	1617	1800	1993	2196	2409	2632	2865	3108
22	224	285	356	437	528	629	742	866	1000	1144	1298	1462	1636	1820	2014	2218	2432	2656	2890	3134
23	232	294	366	448	540	642	755	880	1015	1160	1315	1480	1655	1840	2035	2240	2455	2680	2915	3160
24	240	303	376	459	552	655	768	894	1030	1176	1332	1498	1674	1860	2056	2262	2478	2704	2940	3186
25	248	312	386	470	564	668	782	908	1045	1192	1349	1516	1693	1880	2077	2284	2501	2728	2965	3212
26	256	321	396	481	576	681	796	922	1060	1208	1366	1534	1712	1900	2098	2306	2524	2752	2990	3238
27	264	330	406	492	588	694	810	936	1075	1224	1383	1552	1731	1920	2119	2328	2547	2776	3015	3264
28	272	339	416	503	600	707	824	951	1090	1240	1400	1570	1750	1940	2140	2350	2570	2800	3040	3290
29	280	348	426	514	612	720	838	966	1105	1256	1417	1588	1769	1960	2161	2372	2593	2824	3065	3316
30	288	357	436	525	624	733	852	981	1120	1272	1434	1606	1788	1980	2182	2394	2616	2848	3090	3342
31	296	366	446	536	636	746	866	996	1136	1288	1451	1624	1807	2000	2203	2416	2639	2872	3115	3368
32	304	375	456	547	648	759	880	1011	1152	1304	1468	1642	1826	2020	2224	2438	2662	2896	3140	3394
33	312	384	466	558	660	772	894	1026	1168	1320	1485	1660	1845	2040	2245	2460	2685	2920	3165	3420
34	320	393	476	569	672	785	908	1041	1184	1337	1502	1678	1864	2060	2266	2482	2708	2944	3190	3446
35	328	402	486	580	684	798	922	1056	1200	1354	1519	1696	1883	2080	2287	2504	2731	2968	3215	3472
36	336	411	496	591	696	811	936	1071	1216	1371	1536	1714	1902	2100	2308	2526	2754	2992	3240	3498
37	344	420	506	602	708	824	950	1086	1232	1388	1554	1732	1921	2120	2329	2548	2777	3016	3265	3524
38	352	429	516	613	720	837	964	1101	1248	1405	1572	1750	1940	2140	2350	2570	2800	3040	3290	3550
39	360	438	526	624	732	850	978	1116	1264	1422	1590	1768	1959	2160	2371	2592	2823	3064	3315	3576
40	368	447	536	635	744	863	992	1131	1280	1439	1608	1787	1978	2180	2392	2614	2846	3088	3340	3602

pth evaluation of functions is reported in the printed output. Consequently, if p=1, the value of q is insignificant because all function evaluations will be reported by the package. Printing of partial derivatives is controlled by the fields r and s. If s is not equal to zero (and is not greater than q) then the values of partial derivatives calculated in the first s calls to FCD are reported in the printed output. If r is not equal to zero (and p is greater than zero) then every (p*r)th evaluation of partial derivatives is reported as well. Moreover, if q is equal to zero and p is not equal to 1 (i.e., when the first call to FCD is not reported by the package), then the "starting point" values of optimization variables $\underset{\sim}{x}^0$ and corresponding function values $\underset{\sim}{f}(\underset{\sim}{x}^0)$ are printed; if, at the same time, s is greater than zero, the values of partial derivatives are included in the "starting point" information. It should be noted that the values of partial derivatives can only be printed for those evaluations for which printing of function values is indicated.

If the value of IPR is negative, the partial derivatives calculated by FCD are verified numerically by comparing values supplied by FCD with the differences of function values in the small environment of the starting point. All partial derivatives which differ from the numerically approximated ones by more than 1% (with respect to the numerical approximation) are reported in the printed output. Partial derivatives of the objective function are indicated by the subscript equal to zero.

IFLAG is an INTEGER variable which on exit contains information about the solution:

- IFLAG = -4 artificial bound reached (usually because of unbounded solution),
- IFLAG = -3 line search requires more than 5 steps (usually because of incorrect derivatives),
- IFLAG = -2 feasible region is empty (usually because of inconsistent constraints),
- IFLAG = -1 incorrect input arguments,
- IFLAG = 0 required accuracy obtained,
- IFLAG = 1 machine accuracy reached,
- IFLAG = 2 limit of function evaluations reached,
- IFLAG = 3 iteration terminated by the user.

Basic entry (subroutine MFNC2A)

The subroutine call is

```
CALL MFNC2A (FCD,N,L,LEQ,X,EPS,MAXF,W,IW,IFLAG)
```

All arguments are the same as for the standard entry. It should be noted, however, that 2 arguments of the standard entry do not exist in this case (arguments ICH and IPR), since no printed output is generated for the basic entry to the package, however, diagnostic messages can be obtained by setting the variable LPR in the common area MFN111

```
COMMON /MFN111/ LPR
```

to the unit number of the output file (LPR has a preset value 0).

Original entry (subroutine VF02A)

The subroutine call is

CALL VF02A (N,L,LEQ,X,F,G,C,D,K,MAXF,EPS,IP,W,IW)

The arguments are described in the documentation of the subroutine VF02AD [3] of the Harwell Subroutine Library, however:

(1) the length IW of the workspace W must be at least

18+5*N*N+23*N+4*L+max(L,3*N+3),

(2) to obtain printed output, the variable LPR in the common area VF02D

COMMON /VF02D/ VLN,LPR

must be set to the unit number of the output file (LPR has a preset value 0); VLN controls the artificial bound and is preset to 10^6.

V. AUXILIARY SUBROUTINES

The package contains several auxiliary subroutines which can be used to change or to set the values of additional parameters controlling the form of the printed output generated by the package. All these subroutines (if used) should be called before the standard entry to the package.

Subroutine MMXHDR

Subroutine MMXHDR defines the title line which is printed within the page header. The title must be a string of up to 80 characters which is stored in consecutive elements of a REAL array, 10 characters in one element.

The subroutine call is

```
CALL MMXHDR(L,T)
```

where L is the number of array elements required for the title, and T is the name of an array or the first element storing the title. If L is equal to zero, no title line is printed by the package.

Subroutine MMXPSZ

Subroutine MMXPSZ defines the "page size", that is the maximum number of lines printed on a page. The preset value is 65.

The subroutine call is

```
CALL MMXPSZ(L)
```

where L is the defined page size. If the value of L is equal to zero, the printed output is generated without page control.

Subroutine MMXPLM

Subroutine MMXPLM defines the limit of printed pages. The preset value of this limit is 10, and it cannot be changed to more than 50.

The subroutine call is

```
CALL MMXPLM (L)
```

where L is the defined limit of pages.

When the limit of pages is reached the further output generated by the package is suppressed except of the results of optimization.

Subroutine MMXLLM

Subroutine MMXLLM defines the limit of printed lines. The preset value of this limit is 750.

The subroutine call is

```
CALL MMXLLM(L)
```

where L is the defined limit of lines.

When the limit of printed lines is reached the further output generated by the package is suppressed except of the results of optimization.

Subroutine MMXGLM

Subroutine MMXGLM defines the bounds on the number of variables and the number of constraint functions when the matrix of partial derivatives is printed by the package (for some problems this matrix can be quite large and it can be reasonable to print the initial part of it only). The preset bound on the number of variables is 10, and on the number of functions is 25.

The subroutine call is

```
CALL MMXGLM(K,L)
```

where K is the defined bound on the number of variables, and L is the defined bound on the number of functions.

Subroutine MMXGVL

Subroutine MMXGVL defines, for the matrix of partial derivatives, the number of columns printed in one line. The preset value is 10, and it corresponds to 120 character line. If the standard form of generated output is to be preserved this number should be defined as 6.

The subroutine call is

```
CALL MMXGVL(K)
```

where K is the defined number of columns per line.

VI. GENERAL INFORMATION

Use of COMMON: COMMON /MFN000/ (see argument FCD),

COMMON /MFN111/ (for basic entry),

COMMON /MMX000/ (for standard entry),

COMMON /VF02D/

COMMON /VF02E/

COMMON /VE02X/

COMMON /LA02B/

COMMON /MB01D/

Workspace: Provided by the user; see arguments W and IW.

Input/output: Output (for standard entry only) as defined by the user; see argument ICH.

Subroutines: VF02A, VF02B, VF02C, VE02A, LA02A, MB01C, FM01AS and:

a) for standard entry: MFNC1A, MFNC9A, MFN00Q, MFN00A, MFN00G, MFN00H, MFN00V, MMXPSZ, MMXPLM, MMXLLM, MMXHDR, MMXGLM, MMXGVL;

b) for basic entry: MFNC2A, MFNC9A, MFN00Z.

Restrictions: $N>0$, $L\geq 0$, $LEQ\geq 0$, $LEQ\leq L$, $LEQ\leq N$, $EPS\geq 0$, $MAXF>0$, $IW\geq IWR$.

VII. EXAMPLES

Example 1 [3]

Minimize

$$F(\underset{\sim}{x}) = x_1^2 + x_2^2 + x_3$$

subject to the constraints

$$x_1 x_2 - x_3 = 0 ,$$

$$x_3 - 1 \geq 0 .$$

For the starting point

$$\underset{\sim}{x}^0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and the required accuracy 10^{-7} the solution is obtained after 12 evaluations of the function (as in [3]).

```
      PROGRAM TRMFN1 (OUTPUT,TAPE6=OUTPUT)
C
C   HARWELL TEST PROGRAM.
C
      DIMENSION X(3),T(3),W(166)
      EXTERNAL FCD
      DATA T/10HTRMFN1 : H,10HHARWELL EXA,10HMPLE.       /
      CALL MMXHDR(3,T)
      X(1)=1.0
      X(2)=2.0
      X(3)=3.0
      N=3
      LEQ=1
      L=2
      MAXF=25
      EPS=1.0E-7
      ICH=6
      IPR=-10
      LW=166
      CALL MFNC1A(FCD,N,L,LEQ,X,EPS,MAXF,W,LW,ICH,IPR,IFLAG)
      STOP
      END
C
C
      SUBROUTINE FCD (N,M,X,F,G,C,D,K)
      DIMENSION X(N),G(N),C(M),D(K,M)
      X1=X(1)
      X2=X(2)
      X3=X(3)
      F=X1*X1+X2*X2+X3
      G(1)=X1+X1
      G(2)=X2+X2
      G(3)=1.0
      C(1)=X1*X2-X3
      D(1,1)=X2
      D(2,1)=X1
      D(3,1)=-1.0
      C(2)=X3-1.0
      D(1,2)=0.0
      D(2,2)=0.0
      D(3,2)=1.0
      RETURN
      END
```

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MINIMIZATION WITH NONLINEAR CONSTRAINTS (MFNC PACKAGE) (V:82.05)

TRMFN1 : HARWELL EXAMPLE.

INPUT DATA

NUMBER OF VARIABLES (N) 3

NUMBER OF EQUALITY CONSTRAINTS (LEQ). 1

TOTAL NUMBER OF CONSTRAINTS (L) 2

ACCURACY (EPS) 1.000E-07

MAX NUMBER OF FUNCTION EVALUATIONS (MAXF) 25

WORKING SPACE (IW) 166

PRINTOUT CONTROL (IPR) . . . . -10

STARTING POINT. OBJECTIVE FUNCTION : 8.000000000000E+00

	VARIABLES	GRADIENT		CONSTRAINTS
1	1.000000000000E+00	2.0000000000E+00	1	-1.000000000000E+00
2	2.000000000000E+00	4.0000000000E+00	2	2.000000000000E+00
3	3.000000000000E+00	1.0000000000E+00		

VERIFICATION OF PARTIAL DERIVATIVES PERFORMED.

SOLUTION

OBJECTIVE FUNCTION : 2.999999999999E+00

	VARIABLES	GRADIENT		CONSTRAINTS
1	9.999999999993E-01	2.0000000000E+00	1	-4.618527782441E-13
2	1.000000000000E+00	2.0000000000E+00	2	0.
3	1.000000000000E+00	1.0000000000E+00		

TYPE OF SOLUTION (IFLAG) 0

NUMBER OF FUNCTION EVALUATIONS 12

NUMBER OF QUADRATIC ITERATIONS 10

EXECUTION TIME (IN SECONDS) 268

Example 2 [6, Example 3]

This is the problem proposed by Brent [7] as an example in which the continuous analogue of the Newton-Raphson method is not globally convergent. The problem is to solve a system of 2 nonlinear equations

$$4(x_1+x_2) = 0 ,$$
$$(x_1-x_2)((x_1-2)^2 + x_2^2) + 3x_1 + 5x_2 = 0 .$$

More details and some solutions are given in [6]. It can be observed, however, that the solution can be obtained by minimizing the objective function

$$F(\underset{\sim}{x}) = (x_1+x_2)^2$$

subject to the nonlinear constraint

$$(x_1-x_2)((x_1-2)^2 + x_2^2) + 3x_1 + 5x_2 = 0 .$$

The solutions are shown for 4 different starting points $\underset{\sim}{x}^0$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

as in [6].

```
      PROGRAM TRMFN2 (OUTPUT,TAPE1=OUTPUT)
C
C  BRENT EXAMPLE.
C
      DIMENSION X(2),XX(4,2),T(3),W(104)
      EXTERNAL FFF
      DATA T/10HTRMFN2 : B,10HRENT EXAMP,10HLE          /
      DATA XX/2.0,-2.0,2.0,2.0,
     1        2.0,-2.0,0.0,1.0/
      CALL MMXHDR(3,T)
      N=2
      LEQ=1
      L=1
      DO 10 I=1,4
      X(1)=XX(I,1)
      X(2)=XX(I,2)
      EPS=1.E-6
      MAXF=25
      IW=104
      ICH=1
      IPR=-10
      CALL MFNC1A(FFF,N,L,LEQ,X,EPS,MAXF,W,IW,ICH,IPR,IFLAG)
   10 CONTINUE
      STOP
      END
C
C
      SUBROUTINE FFF(N,L,X,F,G,C,D,K)
      DIMENSION X(N),G(N),C(L),D(K,L)
      X1=X(1)
      X2=X(2)
      R1=X1-X2
      R2=(X1-2.0)**2+X2*X2
      R3=X1+X2
      R4=R3+R3
      F=R3*R3
      G(1)=R4
      G(2)=R4
      C(1)=R1*R2+3.0*X1+5.0*X2
      D(1,1)=R2+(R1+R1)*(X1-2.0)+3.0
      D(2,1)=-R2+R1*(X2+X2)+5.0
      RETURN
      END
```

```
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MINIMIZATION WITH NONLINEAR CONSTRAINTS (MFNC PACKAGE)                    (V:82.05)

TRMFN2 : BRENT EXAMPLE


INPUT DATA
----------

  NUMBER OF VARIABLES (N) . . . . . . . . . . . . . . . . . . . . . . . . . .      2

  NUMBER OF EQUALITY CONSTRAINTS (LEQ). . . . . . . . . . . . . . . . . . . .      1

  TOTAL NUMBER OF CONSTRAINTS (L) . . . . . . . . . . . . . . . . . . . . . .      1

  ACCURACY (EPS)  . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1.000E-06

  MAX NUMBER OF FUNCTION EVALUATIONS (MAXF) . . . . . . . . . . . . . . . . .     25

  WORKING SPACE (IW) . . . . . . . . . . . . . . . . . . . . . . . . . . . .    104

  PRINTOUT CONTROL (IPR) . . . . . . . . . . . . . . . . . . . . . . . . . .    -10

  STARTING POINT.                    OBJECTIVE FUNCTION :  1.600000000000E+01

              VARIABLES            GRADIENT                    CONSTRAINTS

       1   2.000000000000E+00   8.0000000000E+00        1   1.600000000000E+01
       2   2.000000000000E+00   8.0000000000E+00

VERIFICATION OF PARTIAL DERIVATIVES PERFORMED.


SOLUTION
--------
                                     OBJECTIVE FUNCTION :  1.186745702620E-08

              VARIABLES            GRADIENT                    CONSTRAINTS

       1  -1.082246213238E-04   2.1787571711E-04        1  -5.405507481288E-04
       2   2.171624798772E-04   2.1787571711E-04

  TYPE OF SOLUTION (IFLAG)  . . . . . . . . . . . . . . . . . . . . . . . . .      0

  NUMBER OF FUNCTION EVALUATIONS  . . . . . . . . . . . . . . . . . . . . . .      9

  NUMBER OF QUADRATIC ITERATIONS  . . . . . . . . . . . . . . . . . . . . . .      8

  EXECUTION TIME (IN SECONDS) . . . . . . . . . . . . . . . . . . . . . . . .   .121
```

```
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MINIMIZATION WITH NONLINEAR CONSTRAINTS (MFNC PACKAGE)                  (V:82.05)

TRMFN2 : BRENT EXAMPLE


INPUT DATA
----------

  NUMBER OF VARIABLES (N) . . . . . . . . . . . . . . . . . . . . . . . . . .     2

  NUMBER OF EQUALITY CONSTRAINTS (LEQ). . . . . . . . . . . . . . . . . . . .     1

  TOTAL NUMBER OF CONSTRAINTS (L) . . . . . . . . . . . . . . . . . . . . . .     1

  ACCURACY (EPS)  . . . . . . . . . . . . . . . . . . . . . . . . . . 1.000E-06

  MAX NUMBER OF FUNCTION EVALUATIONS (MAXF) . . . . . . . . . . . . . . . . .    25

  WORKING SPACE (IW) . . . . . . . . . . . . . . . . . . . . . . . . . . . .   104

  PRINTOUT CONTROL (IPR) . . . . . . . . . . . . . . . . . . . . . . . . . .   -10

  STARTING POINT.                    OBJECTIVE FUNCTION :  1.600000000000E+01

              VARIABLES            GRADIENT                     CONSTRAINTS

       1  -2.000000000000E+00  -8.0000000000E+00         1  -1.600000000000E+01
       2  -2.000000000000E+00  -8.0000000000E+00

VERIFICATION OF PARTIAL DERIVATIVES PERFORMED.


SOLUTION
--------
                                     OBJECTIVE FUNCTION :  4.270746189808E-12

              VARIABLES            GRADIENT                     CONSTRAINTS

       1  -2.117626035708E-05  -4.1331567547E-06         1  -1.291275529750E-04
       2   1.910968197971E-05  -4.1331567547E-06


  TYPE OF SOLUTION (IFLAG)  . . . . . . . . . . . . . . . . . . . . . . . . .     0

  NUMBER OF FUNCTION EVALUATIONS  . . . . . . . . . . . . . . . . . . . . . .     7

  NUMBER OF QUADRATIC ITERATIONS  . . . . . . . . . . . . . . . . . . . . . .     6

  EXECUTION TIME (IN SECONDS) . . . . . . . . . . . . . . . . . . . . . . .  .094
```

```
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MINIMIZATION WITH NONLINEAR CONSTRAINTS (MFNC PACKAGE)                       (V:82.05)

TRMFN2 : BRENT EXAMPLE


INPUT DATA
----------

  NUMBER OF VARIABLES (N) . . . . . . . . . . . . . . . . . . . . . . . . .      2

  NUMBER OF EQUALITY CONSTRAINTS (LEQ). . . . . . . . . . . . . . . . . . .      1

  TOTAL NUMBER OF CONSTRAINTS (L) . . . . . . . . . . . . . . . . . . . . .      1

  ACCURACY (EPS)  . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1.000E-06

  MAX NUMBER OF FUNCTION EVALUATIONS (MAXF) . . . . . . . . . . . . . . . .     25

  WORKING SPACE (IW) . . . . . . . . . . . . . . . . . . . . . . . . . . .    104

  PRINTOUT CONTROL (IPR) . . . . . . . . . . . . . . . . . . . . . . . . .    -10

  STARTING POINT.                      OBJECTIVE FUNCTION :  9.000000000000E+00

               VARIABLES            GRADIENT                    CONSTRAINTS

       1   2.000000000000E+00   6.0000000000E+00        1   1.200000000000E+01
       2   1.000000000000E+00   6.0000000000E+00

VERIFICATION OF PARTIAL DERIVATIVES PERFORMED.


SOLUTION
--------
                                       OBJECTIVE FUNCTION :  2.811748620719E-07

               VARIABLES            GRADIENT                    CONSTRAINTS

       1  -2.157351558771E-04   1.0605184809E-03        1  -7.649821902926E-04
       2   7.459943963171E-04   1.0605184809E-03

  TYPE OF SOLUTION (IFLAG)  . . . . . . . . . . . . . . . . . . . . . . . .      0

  NUMBER OF FUNCTION EVALUATIONS  . . . . . . . . . . . . . . . . . . . . .      6

  NUMBER OF QUADRATIC ITERATIONS  . . . . . . . . . . . . . . . . . . . . .      6

  EXECUTION TIME (IN SECONDS) . . . . . . . . . . . . . . . . . . . . . . .   .087
```

```
DATE : 82/05/19.                      TIME : 14.56.16.                        PAGE :  1
MINIMIZATION WITH NONLINEAR CONSTRAINTS (MFNC PACKAGE)                        (V:82.05)

TRMFN2 : BRENT EXAMPLE


INPUT DATA
----------

  NUMBER OF VARIABLES (N) . . . . . . . . . . . . . . . . . . . . . . . . .          2

  NUMBER OF EQUALITY CONSTRAINTS (LEQ). . . . . . . . . . . . . . . . . . .          1

  TOTAL NUMBER OF CONSTRAINTS (L) . . . . . . . . . . . . . . . . . . . . .          1

  ACCURACY (EPS)  . . . . . . . . . . . . . . . . . . . . . . . . . . . .   1.000E-06

  MAX NUMBER OF FUNCTION EVALUATIONS (MAXF) . . . . . . . . . . . . . . . .         25

  WORKING SPACE (IW) . . . . . . . . . . . . . . . . . . . . . . . . . . .         104

  PRINTOUT CONTROL (IPR) . . . . . . . . . . . . . . . . . . . . . . . . .         -10

  STARTING POINT.                    OBJECTIVE FUNCTION :  4.000000000000E+00

                VARIABLES           GRADIENT                      CONSTRAINTS

       1   2.000000000000E+00   4.0000000000E+00          1   6.000000000000E+00
       2   0.                   4.0000000000E+00

VERIFICATION OF PARTIAL DERIVATIVES PERFORMED.


SOLUTION
--------
                                     OBJECTIVE FUNCTION :  3.474781237047E-10


                VARIABLES           GRADIENT                      CONSTRAINTS

       1   6.499981749980E-06  -3.7281530210E-05          1   2.035830275945E-05
       2  -2.514074685511E-05  -3.7281530210E-05


  TYPE OF SOLUTION (IFLAG)  . . . . . . . . . . . . . . . . . . . . . . . .          0

  NUMBER OF FUNCTION EVALUATIONS  . . . . . . . . . . . . . . . . . . . . .          6

  NUMBER OF QUADRATIC ITERATIONS  . . . . . . . . . . . . . . . . . . . . .          6

  EXECUTION TIME (IN SECONDS) . . . . . . . . . . . . . . . . . . . . . . .       .095
```

Example 3

This is the Rosen-Suzuki constrained minimization problem [8]. It is to minimize

$$F(\underset{\sim}{x}) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4$$

subject to the constraints

$$- x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 + 8 \geq 0 \ ,$$
$$- x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_1 + x_4 + 10 \geq 0 \ ,$$
$$- 2x_1^2 - x_2^2 - x_3^2 - 2x_1 + x_2 + x_4 + 5 \geq 0 \ .$$

The solution is $\underset{\sim}{x}^* = [0 \ 1 \ 2 \ -1]^T$ with $F(\underset{\sim}{x}^*) = -44$.

Two solutions are shown, which correspond to starting points $\underset{\sim}{x}^0 = [2 \ 2 \ 5 \ 0]^T$ and $\underset{\sim}{x}^0 = \underset{\sim}{0}$, as in [9]. Both the solutions require slightly different numbers of function evaluations. The differences, however, are not significant.

```
      PROGRAM TRMFN3 (OUTPUT,TAPE6=OUTPUT)
C
C   ROSEN-SUZUKI PROBLEM.
C
      DIMENSION X(4),XX(4,2),T(3),W(300)
      EXTERNAL FCD
      DATA T/10HTRMFN3 : R,10HOSEN-SUZUK,10HI PROBLEM /
      DATA XX/2.0,2.0,5.0,0.0,0.0,0.0,0.0,0.0/
      CALL MMXHDR(3,T)
      N=4
      LEQ=0
      L=3
      DO 10 II=1,2
      DO 20 JJ=1,4
   20 X(JJ)=XX(JJ,II)
      MAXF=30
      EPS=1.E-6
      LW=300
      ICH=6
      IPR=-10
      CALL MFNC1A(FCD,N,L,LEQ,X,EPS,MAXF,W,LW,ICH,IPR,IFLAG)
   10 CONTINUE
      STOP
      END
C
C
      SUBROUTINE FCD(N,L,X,F,G,C,D,K)
      DIMENSION X(N),G(N),C(L),D(K,L)
      X1=X(1)
      X2=X(2)
      X3=X(3)
      X4=X(4)
      R1=X1+X1+1.0
      R2=X2+X2-1.0
      R3=X3+X3
      R4=X4+X4-1.0
      F=X1*(X1-5.0)+X2*(X2-5.0)+X3*(R3-21.0)+X4*(X4+7.0)
      G(1)=R1-6.0
      G(2)=R2-4.0
      G(3)=4.0*X3-21.0
      G(4)=X4+X4+7.0
      C(1)=X1*(-X1-1.0)+X2*(1.0-X2)+X3*(-X3-1.0)+X4*(1.0-X4)+8.0
      C(2)=X1*(1.0-X1)-X2*(X2+X2)-X3*X3+X4*(1.0-X4-X4)+10.0
      C(3)=-2.0*X1*(X1+1.0)+X2*(1.0-X2)-X3*X3+X4+5.0
      D(1,1)=-R1
      D(2,1)=-R2
      D(3,1)=-R3-1.0
      D(4,1)=-R4
      D(1,2)=-R1+2.0
      D(2,2)=-4.0*X2
      D(3,2)=-R3
      D(4,2)=1.0-4.0*X4
      D(1,3)=-2.0*R1
      D(2,3)=-R2
      D(3,3)=-R3
      D(4,3)=1.0
      RETURN
      END
```

```
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MINIMIZATION WITH NONLINEAR CONSTRAINTS (MFNC PACKAGE)                  (V:82.05)

TRMFN3 : ROSEN-SUZUKI PROBLEM


INPUT DATA
----------

  NUMBER OF VARIABLES (N) . . . . . . . . . . . . . . . . . . . . . . . . . .    4

  NUMBER OF EQUALITY CONSTRAINTS (LEQ). . . . . . . . . . . . . . . . . . . .    0

  TOTAL NUMBER OF CONSTRAINTS (L) . . . . . . . . . . . . . . . . . . . . . .    3

  ACCURACY (EPS)  . . . . . . . . . . . . . . . . . . . . . . . . . . . .  1.000E-06

  MAX NUMBER OF FUNCTION EVALUATIONS (MAXF) . . . . . . . . . . . . . . . . .   30

  WORKING SPACE (IW)  . . . . . . . . . . . . . . . . . . . . . . . . . . . .  300

  PRINTOUT CONTROL (IPR)  . . . . . . . . . . . . . . . . . . . . . . . . . .  -10

  STARTING POINT.                       OBJECTIVE FUNCTION : -6.700000000000E+01

              VARIABLES            GRADIENT                       CONSTRAINTS

       1   2.000000000000E+00  -1.0000000000E+00          1  -3.000000000000E+01
       2   2.000000000000E+00  -1.0000000000E+00          2  -2.500000000000E+01
       3   5.000000000000E+00  -1.0000000000E+00          3  -3.400000000000E+01
       4   0.                   7.0000000000E+00

VERIFICATION OF PARTIAL DERIVATIVES PERFORMED.


SOLUTION
--------
                                        OBJECTIVE FUNCTION : -4.400000000261E+01

              VARIABLES            GRADIENT                       CONSTRAINTS

       1  -3.280674729311E-07  -5.0000006561E+00          1  -7.651891204831E-10
       2   9.999999248445E-01  -3.0000001503E+00          2   1.000000408976E+00
       3   2.000000256093E+00  -1.2999998976E+01          3  -9.219149887940E-10
       4  -9.999997078416E-01   5.0000005843E+00


  TYPE OF SOLUTION (IFLAG)  . . . . . . . . . . . . . . . . . . . . . . . . .    0

  NUMBER OF FUNCTION EVALUATIONS  . . . . . . . . . . . . . . . . . . . . . .   15

  NUMBER OF QUADRATIC ITERATIONS  . . . . . . . . . . . . . . . . . . . . . .   11

  EXECUTION TIME (IN SECONDS) . . . . . . . . . . . . . . . . . . . . . . . . .466
```

```
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MINIMIZATION WITH NONLINEAR CONSTRAINTS (MFNC PACKAGE)              (V:82.05)

TRMFN3 : ROSEN-SUZUKI PROBLEM


INPUT DATA
----------

 NUMBER OF VARIABLES (N) . . . . . . . . . . . . . . . . . . . . . . . .      4

 NUMBER OF EQUALITY CONSTRAINTS (LEQ). . . . . . . . . . . . . . . . . .      0

 TOTAL NUMBER OF CONSTRAINTS (L) . . . . . . . . . . . . . . . . . . . .      3

 ACCURACY (EPS)  . . . . . . . . . . . . . . . . . . . . . . . . . .  1.000E-06

 MAX NUMBER OF FUNCTION EVALUATIONS (MAXF) . . . . . . . . . . . . . . .     30

 WORKING SPACE (IW) . . . . . . . . . . . . . . . . . . . . . . . . . .    300

 PRINTOUT CONTROL (IPR) . . . . . . . . . . . . . . . . . . . . . . . .    -10

 STARTING POINT.                    OBJECTIVE FUNCTION :  0.

             VARIABLES           GRADIENT                  CONSTRAINTS

      1   0.                  -5.0000000000E+00      1   8.000000000000E+00
      2   0.                  -5.0000000000E+00      2   1.000000000000E+01
      3   0.                  -2.1000000000E+01      3   5.000000000000E+00
      4   0.                   7.0000000000E+00

VERIFICATION OF PARTIAL DERIVATIVES PERFORMED.


SOLUTION
--------
                                    OBJECTIVE FUNCTION : -4.400000000045E+01

             VARIABLES           GRADIENT                  CONSTRAINTS

      1   4.479937861972E-07  -4.9999991040E+00      1  -1.568566139081E-10
      2   1.000000426545E+00  -2.9999991469E+00      2   9.999982845454E-01
      3   1.999999558176E+00  -1.3000001767E+01      3  -1.492992396379E-10
      4  -1.000000444913E+00   4.9999991102E+00

 TYPE OF SOLUTION (IFLAG)  . . . . . . . . . . . . . . . . . . . . . . .      0

 NUMBER OF FUNCTION EVALUATIONS  . . . . . . . . . . . . . . . . . . . .     12

 NUMBER OF QUADRATIC ITERATIONS  . . . . . . . . . . . . . . . . . . . .     10

 EXECUTION TIME (IN SECONDS) . . . . . . . . . . . . . . . . . . . . . .   .390
```

Example 4 [10, Example 5.5]

This is the Colville's test problem 2 [11]. It is to minimize the objective function

$$F(\underset{\sim}{x}) = - \sum_{1\leq i\leq 10} b_i \, x_{5+i} + \sum_{1\leq i\leq 5} \sum_{1\leq j\leq 5} c_{ij} x_i x_j + 2 \sum_{1\leq j\leq 5} d_j x_j^3$$

subject to the constraints

$$x_i \geq 0 \,, \qquad i=1,\ldots,15,$$

$$\sum_{1\leq i\leq 10} a_{ij} x_{5+i} \leq e_j + 2 \sum_{1\leq i\leq 5} c_{ij} x_i + 3 d_j x_j^2, \; j=1,\ldots,5,$$

where a_{ij}, b_i, c_{ij}, d_j, e_j are as follows:

	a_{ij}	j = 1	2	3	4	5	b_i
	1	-16	2	0	1	0	-40
	2	0	-2	0	0.4	2	-2
	3	-3.5	0	2	0	0	-.25
	4	0	-2	0	-4	-1	-4
	5	0	-9	-2	1	-2.8	-4
i	6	2	0	-4	0	0	-1
	7	-1	-1	-1	-1	-1	-40
	8	-1	-2	-3	-2	-1	-60
	9	1	2	3	4	5	5
	10	1	1	1	1	1	1

c_{ij}		j = 1	2	3	4	5
	1	30	-20	-10	32	-10
	2	-20	39	-6	-31	32
i	3	-10	-6	10	-6	-10
	4	32	-31	-6	39	-20
	5	-10	32	-10	-20	30
d_j		4	8	10	6	2
e_j		-15	-27	-36	-18	-12

The solution is $F(\underset{\sim}{x}^*) = 32.34868$, and it is obtained for the starting point $\underset{\sim}{x}^0$ where $x_i = 0.0001$, $i \neq 12$, and $x_{12} = 60.0$ (as in [10]), and for the accuracy 10^{-6} after 16 iterations (as in [1]).

```
      PROGRAM TRMFN4 (OUTPUT,TAPE6=OUTPUT)
C
C   COLVILLE TEST PROBLEM 2.
C
      DIMENSION X(15),T(4),W(2000)
      EXTERNAL FCD
      DATA T/10HTRMFN4 : C,10HOLVILLE TE,10HST PROBLEM,2H 2/
      CALL MMXHDR(4,T)
      DO 10 I=1,15
   10 X(I)=0.0001
      X(12)=60.0
      N=15
      LEQ=0
      L=20
      MAXF=50
      EPS=1.0E-6
      ICH=6
      IPR=0
      LW=2000
      CALL MFNC1A(FCD,N,L,LEQ,X,EPS,MAXF,W,LW,ICH,IPR,IFLAG)
      STOP
      END
C
C
      SUBROUTINE FCD(N,L,X,F,G,C,D,KK)
      DIMENSION X(N),G(N),C(L),D(KK,L)
      DIMENSION A(10,5),B(10),C1(5,5),D1(5),E(5)
      DATA A/-16.0,0.0,-3.5,0.0,0.0,2.0,-1.0,-1.0,1.0,1.0,
     +       2.0,-2.0,0.0,-2.0,-9.0,0.0,-1.0,-2.0,2.0,1.0,
     +       0.0,0.0,2.0,0.0,-2.0,-4.0,-1.0,-3.0,3.0,1.0,
     +       1.0,0.4,0.0,-4.0,1.0,0.0,-1.0,-2.0,4.0,1.0,
     +       0.0,2.0,0.0,-1.0,-2.8,0.0,-1.0,-1.0,5.0,1.0/
      DATA B/-40.0,-2.0,-0.25,-4.0,-4.0,-1.0,-40.0,-60.0,5.0,1.0/
      DATA C1/30.0,-20.0,-10.0,32.0,-10.0,
     +        -20.0,39.0,-6.0,-31.0,32.0,
     +        -10.0,-6.0,10.0,-6.0,-10.0,
     +        32.0,-31.0,-6.0,39.0,-20.0,
     +        -10.0,32.0,-10.0,-20.0,30.0/
      DATA D1/4.0,8.0,10.0,6.0,2.0/
      DATA E/-15.0,-27.0,-36.0,-18.0,-12.0/
C
C     OBJECTIVE FUNCTION F
C
      F=0.0
      DO 10 I=1,10
      J=5+I
   10 F=F-B(I)*X(J)
      DO 30 I=1,5
      DO 20 J=1,5
   20 F=F+C1(I,J)*X(I)*X(J)
   30 F=F+2.0*D1(I)*X(I)**3
C
C     GRADIENT G OF THE OBJECTIVE FUNCTION
C
      DO 40 I=1,10
      J=5+I
   40 G(J)=-B(I)
      DO 60 I=1,5
      G(I)=6.0*D1(I)*X(I)*X(I)
      DO 60 J=1,5
   60 G(I)=G(I)+X(J)*(C1(I,J)+C1(J,I))
C
C     CONSTRAINTS C
C
      DO 70 I=1,15
```

```
   70 C(I)=X(I)
      DO 90 J=1,5
      K=J+15
      C(K)=E(J)+3.0*D1(J)*X(J)*X(J)
      DO 80 I=1,5
   80 C(K)=C(K)+2.0*C1(I,J)*X(I)
      DO 85 II=1,10
      JJ=5+II
   85 C(K)=C(K)-A(II,J)*X(JJ)
   90 CONTINUE
C
C     DERIVATIVES D OF THE CONSTRAINTS
C
      DO 100 I=1,15
  100 D(I,I)=1.0
      DO 110 I=1,15
      DO 110 J=1,15
      IF(I.NE.J) D(I,J)=0.0
  110 CONTINUE
      DO 140 J=1,5
      K=15+J
      DO 120 I=1,5
      D(I,K)=2.0*C1(I,J)
      IF(I.EQ.J) D(I,K)=D(I,K)+6.0*D1(J)*X(J)
  120 CONTINUE
      DO 130 I=1,10
      II=5+I
  130 D(II,K)=-A(I,J)
  140 CONTINUE
      RETURN
      END
```

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MINIMIZATION WITH NONLINEAR CONSTRAINTS (MFNC PACKAGE) (V:82.05)

TRMFN4 : COLVILLE TEST PROBLEM 2

INPUT DATA

NUMBER OF VARIABLES (N) 15

NUMBER OF EQUALITY CONSTRAINTS (LEQ). 0

TOTAL NUMBER OF CONSTRAINTS (L) 20

ACCURACY (EPS) 1.000E-06

MAX NUMBER OF FUNCTION EVALUATIONS (MAXF) 50

WORKING SPACE (IW) 2000

PRINTOUT CONTROL (IPR) 0

STARTING POINT. OBJECTIVE FUNCTION : 2.400010525500E+03

	VARIABLES	GRADIENT		CONSTRAINTS
1	1.000000000000E-04	4.4002400000E-03	1	1.000000000000E-04
2	1.000000000000E-04	2.8004800000E-03	2	1.000000000000E-04
3	1.000000000000E-04	-4.3994000000E-03	3	1.000000000000E-04
4	1.000000000000E-04	2.8003600000E-03	4	1.000000000000E-04
5	1.000000000000E-04	4.4001200000E-03	5	1.000000000000E-04
6	1.000000000000E-04	4.0000000000E+01	6	1.000000000000E-04
7	1.000000000000E-04	2.0000000000E+00	7	1.000000000000E-04
8	1.000000000000E-04	2.5000000000E-01	8	1.000000000000E-04
9	1.000000000000E-04	4.0000000000E+00	9	1.000000000000E-04
10	1.000000000000E-04	4.0000000000E+00	10	1.000000000000E-04
11	1.000000000000E-04	1.0000000000E+00	11	1.000000000000E-04
12	6.000000000000E+01	4.0000000000E+01	12	6.000000000000E+01
13	1.000000000000E-04	6.0000000000E+01	13	1.000000000000E-04
14	1.000000000000E-04	-5.0000000000E+00	14	1.000000000000E-04
15	1.000000000000E-04	-1.0000000000E+00	15	1.000000000000E-04
			16	4.500605012000E+01
			17	3.300380024000E+01
			18	2.399590030000E+01
			19	4.200266018000E+01
			20	4.800408006000E+01

SOLUTION

OBJECTIVE FUNCTION : 3.234867906597E+01

	VARIABLES	GRADIENT		CONSTRAINTS
1	2.999918085990E-01	2.1753848156E+01	1	2.999918085990E-01
2	3.334635341015E-01	2.3265889599E+00	2	3.334635341015E-01
3	3.999890835345E-01	-2.0212949897E+00	3	3.999890835345E-01
4	4.283149306028E-01	2.4779014835E+01	4	4.283149306028E-01
5	2.239687464390E-01	4.2495198946E+00	5	2.239687464390E-01
6	1.399403725291E-16	4.0000000000E+01	6	1.399403725291E-16
7	1.494513241190E-14	2.0000000000E+00	7	1.494513241190E-14

TRMFN4 : COLVILLE TEST PROBLEM 2

8	5.174131652324E+00	2.5000000000E-01	8	5.174131652324E+00
9	0.	4.0000000000E+00	9	0.
10	3.061114284129E+00	4.0000000000E+00	10	3.061114284129E+00
11	1.183971715885E+01	1.0000000000E+00	11	1.183971715885E+01
12	0.	4.0000000000E+01	12	0.
13	0.	6.0000000000E+01	13	0.
14	1.039335785344E-01	-5.0000000000E+00	14	1.039335785344E-01
15	4.635547259249E-15	-1.0000000000E+00	15	4.635547259249E-15
			16	2.061274154105E-08
			17	7.416374783705E-08
			18	1.652826842332E-07
			19	8.141103620909E-10
			20	1.229539830694E-09

TYPE OF SOLUTION (IFLAG) 0

NUMBER OF FUNCTION EVALUATIONS 16

NUMBER OF QUADRATIC ITERATIONS 16

EXECUTION TIME (IN SECONDS) 12.129

<u>Example 5</u> [12, Example 5]

The problem is to determine an optimally centered point $\underset{\sim}{x}^* = [x_1^* \; x_2^*]^T$ that maximizes the relative tolerance r in the region R_c defined by the inequalities

$$2 + 2x_1 - x_2 \geq 0 \; ,$$
$$143 - 11x_1 - 13x_2 \geq 0 \; ,$$
$$-60 + 4x_1 + 15x_2 \geq 0 \; ,$$

i.e., to find a point $\underset{\sim}{x}^*$ and a tolerance r such that the tolerance region R_ε

$$R_\varepsilon = \{ \underset{\sim}{x} \mid (1-r) \; x_i^* \leq x_i \leq (1+r) x_i^*, \; i=1,2 \}$$

is in the constraint region R_c and is as large as possible.

It can be shown [13] that if the constraint region R_c is one-dimensionally convex (and it is in this case) then it is sufficient that all vertices of R_ε belong to R_c to guarantee that the whole tolerance region R_ε is in the constraint region R_c.

It is convenient to assume that the tolerance r is an additional optimization variable (say x_3) and then the vertices of the tolerance region R_ε are described by the nonlinear expressions

$$[(1 \pm x_3) \; x_1^* \;\; (1 \pm x_3) \; x_2^*]^T \; .$$

Since x_3 is to be maximized, the objective function can take the form

$$F(\underset{\sim}{x}) = -x_3$$

and it is to be minimized subject to the constraints

$$2 + 2(1 \pm x_3)x_1 - (1 \pm x_3)x_2 \geq 0 ,$$

$$143 - 11(1 \pm x_3)x_1 - 13(1 \pm x_3)x_2 \geq 0 ,$$

$$-60 + 4(1 \pm x_3)x_1 + 15(1 \pm x_3)x_2 \geq 0 ,$$

$$x_3 \geq 0 .$$

It should be observed that due to $x_3 \geq 0$ the first 3 constraints (and, in fact, 12 constraints) can be simplified to the form

$$2 + 2(1-x_3)x_1 - (1+x_3)x_2 \geq 0 ,$$

$$143 - 11(1+x_3)x_1 - 13(1+x_3)x_2 \geq 0 ,$$

$$-60 + 4(1-x_3)x_1 + 15(1-x_3)x_2 \geq 0 .$$

The solution is shown for the starting point $\underset{\sim}{x}^0 = \underset{\sim}{0}$. The resulting relative tolerance r is equal to 0.3414 or 34.1% (as in [12]).

```
      PROGRAM TRMFN5 (OUTPUT,TAPE6=OUTPUT)
C
C   TOLERANCING EXAMPLE.
C
      DIMENSION X(3),T(3),W(184)
      EXTERNAL FCD
      DATA T/10HTRMFN5 : T,10HOLERANCING,10H EXAMPLE  /
      CALL MMXHDR(3,T)
      X(1)=0.0
      X(2)=0.0
      X(3)=0.0
      N=3
      LEQ=0
      L=4
      MAXF=25
      EPS=1.0E-6
      ICH=6
      IPR=-10
      LW=184
      CALL MFNC1A(FCD,N,L,LEQ,X,EPS,MAXF,W,LW,ICH,IPR,IFLAG)
      STOP
      END
C
C
      SUBROUTINE FCD(N,L,X,F,G,C,D,K)
      DIMENSION X(N),G(N),C(L),D(K,L)
      X1=X(1)
      X2=X(2)
      X3=X(3)
      R1=1.0-X3
      R2=1.0+X3
      F=-X3
      G(1)=0.0
      G(2)=0.0
      G(3)=-1.0
      C(1)=2.0*(1.0+X1*R1)-X2*R2
      D(1,1)=R1+R1
      D(2,1)=-R2
      D(3,1)=-X1-X1-X2
      C(2)=143.0-R2*(11.0*X1+13.0*X2)
      D(1,2)=-11.0*R2
      D(2,2)=-13.0*R2
      D(3,2)=-11.0*X1-13.0*X2
      C(3)=-60.0+R1*(4.0*X1+15.0*X2)
      D(1,3)=4.0*R1
      D(2,3)=15.0*R1
      D(3,3)=-4.0*X1-15.0*X2
      C(4)=X3
      D(1,4)=0.0
      D(2,4)=0.0
      D(3,4)=1.0
      RETURN
      END
```

```
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MINIMIZATION WITH NONLINEAR CONSTRAINTS (MFNC PACKAGE)                   (V:82.05)

TRMFN5 : TOLERANCING EXAMPLE


INPUT DATA
----------

  NUMBER OF VARIABLES (N) . . . . . . . . . . . . . . . . . . . . . . . . .      3

  NUMBER OF EQUALITY CONSTRAINTS (LEQ). . . . . . . . . . . . . . . . . . .      0

  TOTAL NUMBER OF CONSTRAINTS (L) . . . . . . . . . . . . . . . . . . . . .      4

  ACCURACY (EPS)  . . . . . . . . . . . . . . . . . . . . . . . . . . . 1.000E-06

  MAX NUMBER OF FUNCTION EVALUATIONS (MAXF) . . . . . . . . . . . . . . . .     25

  WORKING SPACE (IW) . . . . . . . . . . . . . . . . . . . . . . . . . . .    184

  PRINTOUT CONTROL (IPR) . . . . . . . . . . . . . . . . . . . . . . . . .    -10

  STARTING POINT.                        OBJECTIVE FUNCTION :  0.

                 VARIABLES           GRADIENT                      CONSTRAINTS

        1   0.                     0.                     1   2.000000000000E+00
        2   0.                     0.                     2   1.430000000000E+02
        3   0.                    -1.0000000000E+00       3  -6.000000000000E+01
                                                          4   0.

VERIFICATION OF PARTIAL DERIVATIVES PERFORMED.


SOLUTION
--------

                                         OBJECTIVE FUNCTION : -3.414065195318E-01


                 VARIABLES           GRADIENT                      CONSTRAINTS

        1   3.670138928676E+00   0.                       1  -3.183515673300E-10
        2   5.094845628439E+00   0.                       2   2.412889443804E-09
        3   3.414065195318E-01  -1.0000000000E+00         3   6.580421541003E-09
                                                          4   3.414065195318E-01

  TYPE OF SOLUTION (IFLAG)  . . . . . . . . . . . . . . . . . . . . . . . .      0

  NUMBER OF FUNCTION EVALUATIONS  . . . . . . . . . . . . . . . . . . . . .     12

  NUMBER OF QUADRATIC ITERATIONS  . . . . . . . . . . . . . . . . . . . . .     11

  EXECUTION TIME (IN SECONDS) . . . . . . . . . . . . . . . . . . . . . . .   .376
```

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