THE COMPLEX ADJOINT APPROACHES TO NETWORK SENSITIVITIES

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Abstract

We present a comprehensive comparison between the widely used Lagrange multiplier and Tellegen's theorem approaches to sensitivity calculations in electrical networks. The two approaches are described on a unified basis using the conjugate notation. Different aspects of comparison can thereby be investigated. The linear electronic circuit analysis case is seen to be a special case. Analytical and numerical results are shown for 2-bus and 6-bus power systems.

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I. INTRODUCTION

Sensitivity calculations are performed routinely in electrical network analysis and design to supply first-order changes and gradients of functions of interest w.r.t. practically defined control or design variables.

Two approaches, namely the Lagrange multiplier approach [1,2] and Tellegen's theorem approach [3,4], are intensively used for sensitivity calculations in both electronic and power networks. Methods based on the two approaches have been described and applied [1-4] on an individual basis. A complex formulation of the Lagrange multiplier approach has been described in [5].

The material presented in this paper aims at investigating relationships between the two approaches. This investigation is accomplished by employing common bases of description and analysis through which the required aspects of comparison can be clearly stated.

We state the notation used and the basic formulation in Section II. In Sections III and IV, we describe, on a unified basis, the application of the Lagrange multiplier and the Tellegen's theorem approaches to sensitivity analysis of electrical networks. A comprehensive discussion of some aspects of comparison is then presented in Section V and applications to a 2-bus and a 6-bus sample power systems are presented in Section VI.

II. BASIC FORMULATION

We denote by f a single valued continuous complex function of $2n_x$ system complex state variables $(\underset{\sim}{x}, \underset{\sim}{x}^*)$ and $2n_u$ complex control variables $(\underset{\sim}{u}, \underset{\sim}{u}^*)$ arranged as column vectors. We also denote by $\underset{\sim}{h}$ a set of n_x

complex equality constraints relating $(\underset{\sim}{x}, \underset{\sim}{x}^*)$ to $(\underset{\sim}{u}, \underset{\sim}{u}^*)$.

Using the conjugate notation [4-6], the first-order change of f is written as

$$\delta f = \underset{\sim}{f}_x^T \, \delta\underset{\sim}{x} + \bar{\underset{\sim}{f}}_x^T \, \delta\underset{\sim}{x}^* + \underset{\sim}{f}_u^T \, \delta\underset{\sim}{u} + \bar{\underset{\sim}{f}}_u^T \, \delta\underset{\sim}{u}^*, \tag{1}$$

where δ denotes first-order change, T denotes transposition and $\underset{\sim}{f}_x$, $\bar{\underset{\sim}{f}}_x$, $\underset{\sim}{f}_u$ and $\bar{\underset{\sim}{f}}_u$ denote the formal derivatives [5] $\partial f/\partial\underset{\sim}{x}$, $\partial f/\partial\underset{\sim}{x}^*$, $\partial f/\partial\underset{\sim}{u}$ and $\partial f/\partial\underset{\sim}{u}^*$, respectively. Also, the first-order change of $\underset{\sim}{h}$ is written as

$$\delta\underset{\sim}{h} = \underset{\sim}{H}_x \, \delta\underset{\sim}{x} + \bar{\underset{\sim}{H}}_x \, \delta\underset{\sim}{x}^* + \underset{\sim}{H}_u \, \delta\underset{\sim}{u} + \bar{\underset{\sim}{H}}_u \, \delta\underset{\sim}{u}^* = \underset{\sim}{0}, \tag{2}$$

where $\underset{\sim}{H}_x$, $\bar{\underset{\sim}{H}}_x$, $\underset{\sim}{H}_u$ and $\bar{\underset{\sim}{H}}_u$ stand for $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{x})^T$, $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{x}^*)^T$, $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{u})^T$ and $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{u}^*)^T$, respectively.

When dealing with electrical networks, $\underset{\sim}{x}$, $\underset{\sim}{x}^*$, $\underset{\sim}{u}$ and $\underset{\sim}{u}^*$ may be classified [4] into 2-component subvectors $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$, respectively, associated with different element (branch) types, b denoting the bth branch. The elements of $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ may constitute complex conjugate pairs of network variables. In general, $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ constitute node branch variables $\underset{\sim}{x}_m$ and $\underset{\sim}{u}_m$ and line branch variables $\underset{\sim}{x}_t$ and $\underset{\sim}{u}_t$. For example, $\underset{\sim}{x}_m$ may represent node voltages in a typical linear electronic network. In this case the components of $\underset{\sim}{x}_m$ are V_m and V_m^*.

In power networks $\underset{\sim}{x}_m$ and $\underset{\sim}{u}_m$ are further classified [4,6] into vectors associated with load $(\underset{\sim}{x}_\ell, \underset{\sim}{u}_\ell)$, generator $(\underset{\sim}{x}_g, \underset{\sim}{u}_g)$ and slack generator $(\underset{\sim}{x}_n, \underset{\sim}{u}_n)$ branches. An element of $\underset{\sim}{x}_\ell$, $\underset{\sim}{x}_g$, $\underset{\sim}{x}_n$ and $\underset{\sim}{x}_t$ may be defined as V_ℓ, $Q_g + j\delta_g$, $P_n + jQ_n$ and I_t, respectively, where P_m and Q_m are the real and reactive powers associated with bus m and $V_m = V_m \angle\delta_m$. A corresponding element of $\underset{\sim}{u}_\ell$, $\underset{\sim}{u}_g$, $\underset{\sim}{u}_n$ and $\underset{\sim}{u}_t$ is defined, respectively, as $P_\ell + jQ_\ell$, $P_g + j|V_g|$, V_n and Y_t.

In general, we write

$$\{\underset{\sim}{x}, \underset{\sim}{x}^*\} = \{\underset{\sim}{x}_b\} = \{\underset{\sim}{x}_m, \underset{\sim}{x}_t\} \tag{3}$$

and

$$\{\underset{\sim}{u}, \underset{\sim}{u}^*\} = \{\underset{\sim}{u}_b\} = \{\underset{\sim}{u}_m, \underset{\sim}{u}_t\}. \tag{4}$$

In this formulation, we have assumed that the number of state or control variables defined is $2n_B$, n_B denoting the number of branches in the network. This assumption is made to simplify the comparison between the Lagrange multiplier and Tellegen's theorem approaches performed in the following sections. Both approaches can, however, be applied [2,5] for a general number of state variables.

III. LAGRANGE MULTIPLIER APPROACH

In this approach, we use (2) and its complex conjugate to write the first-order change δf of (1) in the form [5]

$$\delta f = (\underset{\sim}{f}_u - \underset{\sim}{H}_u^T \underset{\sim}{\lambda} - \overline{\underset{\sim}{H}}_u^{*T} \overline{\underset{\sim}{\lambda}})^T \delta\underset{\sim}{u} + (\overline{\underset{\sim}{f}}_u - \overline{\underset{\sim}{H}}_u^T \underset{\sim}{\lambda} - \underset{\sim}{H}_u^{*T} \overline{\underset{\sim}{\lambda}})^T \delta\underset{\sim}{u}^*, \tag{5}$$

where $\underset{\sim}{\lambda}$ and $\overline{\underset{\sim}{\lambda}}$ are vectors of Lagrange multipliers obtained by solving the adjoint equations

$$\underset{\sim}{H}_x^T \underset{\sim}{\lambda} + \overline{\underset{\sim}{H}}_x^{*T} \overline{\underset{\sim}{\lambda}} = \underset{\sim}{f}_x, \tag{6a}$$

$$\overline{\underset{\sim}{H}}_x^T \underset{\sim}{\lambda} + \underset{\sim}{H}_x^{*T} \overline{\underset{\sim}{\lambda}} = \overline{\underset{\sim}{f}}_x. \tag{6b}$$

Hence, from (5)

$$\frac{df}{d\underset{\sim}{u}} = \underset{\sim}{f}_u - \underset{\sim}{H}_u^T \underset{\sim}{\lambda} - \overline{\underset{\sim}{H}}_u^{*T} \overline{\underset{\sim}{\lambda}}, \tag{7a}$$

$$\frac{df}{d\underset{\sim}{u}^*} = \overline{\underset{\sim}{f}}_u - \overline{\underset{\sim}{H}}_u^T \underset{\sim}{\lambda} - \underset{\sim}{H}_u^{*T} \overline{\underset{\sim}{\lambda}}. \tag{7b}$$

In practice, we solve the $2n_x$ complex adjoint equations (6) for the Lagrange multipliers $\underset{\sim}{\lambda}$ and $\overline{\underset{\sim}{\lambda}}$ which are then substituted into (7) to obtain the required total formal derivatives of f w.r.t. control variables.

For use later, we now describe the approach in a slightly different way. We employ the classifications of (3) and (4) to define the change of an element-local Lagrangian term as

$$\delta L_b \triangleq (\underset{\sim}{\lambda}^T [\underset{\sim}{h}_{bx} \quad \overline{\underset{\sim}{h}}_{bx}] + \overline{\underset{\sim}{\lambda}}^T [\overline{\underset{\sim}{h}}^*_{bx} \quad \underset{\sim}{h}^*_{bx}]) \, \delta\underset{\sim}{x}_b + (\underset{\sim}{\lambda}^T [\underset{\sim}{h}_{bu} \quad \overline{\underset{\sim}{h}}_{bu}] + \overline{\underset{\sim}{\lambda}}^T [\overline{\underset{\sim}{h}}^*_{bu} \quad \underset{\sim}{h}^*_{bu}]) \, \delta\underset{\sim}{u}_b, \qquad (8)$$

where

$$\underset{\sim}{H}_x \triangleq [\underset{\sim}{h}_{1x} \; \cdots \; \underset{\sim}{h}_{n_B x}], \qquad (9a)$$

$$\overline{\underset{\sim}{H}}_x \triangleq [\overline{\underset{\sim}{h}}_{1x} \; \cdots \; \overline{\underset{\sim}{h}}_{n_B x}] \qquad (9b)$$

and

$$\underset{\sim}{H}_u \triangleq [\underset{\sim}{h}_{1u} \; \cdots \; \underset{\sim}{h}_{n_B u}], \qquad (10a)$$

$$\overline{\underset{\sim}{H}}_u \triangleq [\overline{\underset{\sim}{h}}_{1u} \; \cdots \; \overline{\underset{\sim}{h}}_{n_B u}], \qquad (10b)$$

$\underset{\sim}{h}_{bx}$ and $\underset{\sim}{h}_{bu}$ being n_B vectors.

We also define

$$\delta L \triangleq \sum_b \delta L_b, \qquad (11)$$

hence, from (2) and (8)

$$\delta L = 0. \qquad (12)$$

Using (8), (12) and

$$\delta f = \sum_b ([f_{bx} \quad \overline{f}_{bx}] \, \delta\underset{\sim}{x}_b + [f_{bu} \quad \overline{f}_{bu}] \, \delta\underset{\sim}{u}_b) \qquad (13)$$

we may write, from (11)

$$\delta L = \delta f - \sum_b \{([f_{bx} \quad \bar{f}_{bx}] - \underset{\sim}{\lambda}^T [\underset{\sim}{h}_{bx} \quad \bar{\underset{\sim}{h}}_{bx}] - \bar{\underset{\sim}{\lambda}}^T [\bar{\underset{\sim}{h}}^*_{bx} \quad \underset{\sim}{h}^*_{bx}])\, \delta \underset{\sim}{x}_b +$$

$$([f_{bu} \quad \bar{f}_{bu}] - \underset{\sim}{\lambda}^T [\underset{\sim}{h}_{bu} \quad \bar{\underset{\sim}{h}}_{bu}] - \bar{\underset{\sim}{\lambda}}^T [\bar{\underset{\sim}{h}}^*_{bu} \quad \underset{\sim}{h}^*_{bu}])\, \delta \underset{\sim}{u}_b\} \,. \tag{14}$$

Observe that when $\underset{\sim}{\lambda}$ and $\bar{\underset{\sim}{\lambda}}$ of (14) satisfy (6), namely

$$\underset{\sim}{h}^T_{bx} \underset{\sim}{\lambda} + \bar{\underset{\sim}{h}}^{*T}_{bx} \bar{\underset{\sim}{\lambda}} = f_{bx}, \tag{15a}$$

$$\bar{\underset{\sim}{h}}^T_{bx} \underset{\sim}{\lambda} + \underset{\sim}{h}^{*T}_{bx} \bar{\underset{\sim}{\lambda}} = \bar{f}_{bx}, \tag{15b}$$

then (14) reduces to

$$\delta L = \delta f - \sum_b \left([f_{bu} \quad \bar{f}_{bu}]^T - \begin{bmatrix} \underset{\sim}{h}^T_{bu} \\ \bar{\underset{\sim}{h}}^T_{bu} \end{bmatrix} \underset{\sim}{\lambda} - \begin{bmatrix} \bar{\underset{\sim}{h}}^{*T}_{bu} \\ \underset{\sim}{h}^{*T}_{bu} \end{bmatrix} \bar{\underset{\sim}{\lambda}}\right)^T \delta \underset{\sim}{u}_b, \tag{16}$$

hence, from (12)

$$\delta f = \sum_b \left([f_{bu} \quad \bar{f}_{bu}]^T - \begin{bmatrix} \underset{\sim}{h}^T_{bu} \\ \bar{\underset{\sim}{h}}^T_{bu} \end{bmatrix} \underset{\sim}{\lambda} - \begin{bmatrix} \bar{\underset{\sim}{h}}^{*T}_{bu} \\ \underset{\sim}{h}^{*T}_{bu} \end{bmatrix} \bar{\underset{\sim}{\lambda}}\right)^T \delta \underset{\sim}{u}_b, \tag{17}$$

so that

$$\frac{df}{d\underset{\sim}{u}_b} = \begin{bmatrix} f_{bu} \\ \bar{f}_{bu} \end{bmatrix} - \begin{bmatrix} \underset{\sim}{h}^T_{bu} \\ \bar{\underset{\sim}{h}}^T_{bu} \end{bmatrix} \underset{\sim}{\lambda} - \begin{bmatrix} \bar{\underset{\sim}{h}}^{*T}_{bu} \\ \underset{\sim}{h}^{*T}_{bu} \end{bmatrix} \bar{\underset{\sim}{\lambda}}, \tag{18}$$

which is a form of (7).

IV. TELLEGEN'S THEOREM APPROACH

In this approach, the application of Tellegen's theorem [4] results in the identity

$$\delta T = 0, \tag{19}$$

where

$$\delta T \triangleq \sum_b \delta T_b, \tag{20}$$

and the element-local Tellegen term δT_b is defined as

$$\delta T_b \triangleq \hat{\eta}_{bx}^T \, \delta x_b + \hat{\eta}_{bu}^T \, \delta u_b \tag{21}$$

and the 2-component vectors $\hat{\eta}_{bx}$ and $\hat{\eta}_{bu}$ are linear functions of the formulated adjoint network current variables $\hat{I}_b$ and voltage variables $\hat{V}_b$ and their complex conjugate. Hence, the $\hat{\eta}_{bx}$ and $\hat{\eta}_{bu}$ are related through Kirchhoff's current and voltage laws formulating a set of network equations. Using (13) and (21), we may write, from (20)

$$\delta T = \delta f - \sum_b \{([f_{bx} \quad \bar{f}_{bx}] - \hat{\eta}_{bx}^T) \, \delta x_b + ([f_{bu} \quad \bar{f}_{bu}] - \hat{\eta}_{bu}^T) \, \delta u_b\}. \tag{22}$$

The adjoint network is defined by setting

$$\hat{\eta}_{bx} = \begin{bmatrix} f_{bx} \\ \bar{f}_{bx} \end{bmatrix}, \tag{23}$$

hence (22) reduces to

$$\delta T = \delta f - \sum_b \left(\begin{bmatrix} f_{bu} \\ \bar{f}_{bu} \end{bmatrix} - \hat{\eta}_{bu} \right)^T \delta u_b, \tag{24}$$

from which

$$\frac{df}{du_b} = \begin{bmatrix} f_{bu} \\ \bar{f}_{bu} \end{bmatrix} - \hat{\eta}_{bu} \, . \tag{25}$$

In practice, we formulate the adjoint network using (23) and solve the 2n adjoint network equations to get $\hat{\eta}_{bu}$, which are then substituted into (21) to obtain the required total formal derivatives of f w.r.t. complex control variables.

V. ANALOGY AND COMPARISON

In the last two sections, we have described both the complex Lagrange multiplier and Tellegen's theorem approaches to sensitivity calculations in electrical networks. In this section, we investigate the analogous features of the two approaches and state a general comparison between them.

Both approaches have been applied to both real and complex functions [6]. The application of the Lagrangian approach to complex functions in power networks implies the direct solution of $2n_B \times 2n_B$ complex equations of (6) or (15). For real functions $\underset{\sim}{f}_x = \underset{\sim}{f}_x^*$ and it is sufficient to solve either (6a) or (6b) where $\underset{\sim}{\lambda} = \underset{\sim}{\lambda}^*$. On the other hand, the application of the Tellegen theorem approach to real function sensitivities implies a consistency analysis [6] which depends on the particular network being analyzed. In order to attain a reasonable comparison between the two approaches, only the case of real functions will be considered. The equations derived are general and the case of complex function sensitivities can be analyzed in a similar, straightforward manner.

We remark on the resemblance between the element-local Lagrangian term δL_b of (8) and the element-local Tellegen term δT_b of (21). We also remark on the resemblance between equation (12) formed to satisfy (2) and equation (19) formed by applying Tellegen's theorem. The δf of (14) and (22) is expressed solely in terms of the control variables via defining, respectively, the adjoint systems (15) and (23). The solution of the adjoint network is then used to obtain the total derivatives $df/d\underset{\sim}{u}_b$ from (18) and (25), respectively.

In the complex Lagrange multiplier approach, the adjoint system of equations to be solved for the adjoint variables (Lagrange multipliers) $\underset{\sim}{\lambda}$ and $\underset{\sim}{\bar{\lambda}}$ constitutes a $2n_B$ x $2n_B$ complex matrix of coefficients. In general, when other state variables are defined [2], the order of the matrix of coefficients is determined by the total number of state variables defined. On the other hand, the adjoint system of equations in the Tellegen's theorem approach represents a set of network equations and constitutes only a 2n x 2n real matrix of coefficients, n denoting the number of nodes (or buses) in the original network.

The compactness of the adjoint system formulation in the Tellegen's theorem approach is afforded in essence by realizing, when formulating the adjoint equations, Kirchhoff's relations between the different adjoint variables which constitute a fictitious electrical network.

Assuming that the effort required is divided into formulation and solution parts of the adjoint system, we immediately see that the Tellegen's theorem approach sweeps the major effort into the formulation part and results in only 2n real adjoint equations to be solved. In contrast, the Lagrange multiplier approach requires almost nothing to formulate the adjoint system which then constitutes n_x adjoint equations to be solved.

VI. APPLICATIONS

In this section, we illustrate the implementation of the sensitivity approaches described in the previous sections by analytical and numerical results for a 2-bus and a 6-bus sample power systems.

Application to a 2-Bus Power System

This simple system is mainly used to clarify analytical aspects of the sensitivity approaches. The system consists of a load bus ($m=\ell=1$), the slack bus ($n=2$) and three transmission branches ($t=3,4,5$). The single line diagram for this sample system is shown in Fig. 1. The system complex state variables $(\underset{\sim}{x},\underset{\sim}{x}^*)$ of Section II are

$$\underset{\sim}{x} = \begin{bmatrix} V_1 \\ S_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{x}^* = \begin{bmatrix} V_1^* \\ S_2^* \\ I_3^* \\ I_4^* \\ I_5^* \end{bmatrix} ,$$

where $n_x=5$ for this example. Similarly, the system complex control variables $(\underset{\sim}{u},\underset{\sim}{u}^*)$ are

$$\underset{\sim}{u} = \begin{bmatrix} S_1 \\ V_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{u}^* = \begin{bmatrix} S_1^* \\ V_2^* \\ Y_3^* \\ Y_4^* \\ Y_5^* \end{bmatrix} .$$

The vector $\underset{\sim}{h}$ of n_x complex equality constraints relating $(\underset{\sim}{x},\underset{\sim}{x}^*)$ to $(\underset{\sim}{u},\underset{\sim}{u}^*)$ is given, for example, by

$$\underset{\sim}{h} = \begin{bmatrix} S_1^* + V_1^*(Y_3 + Y_5)V_1 - V_1^* Y_5 V_2 \\ S_2^* + V_2^*(Y_4 + Y_5)V_2 - V_2^* Y_5 V_1 \\ I_3 - Y_3 V_1 \\ I_4 - Y_4 V_2 \\ I_5 - Y_5 V_1 + Y_5 V_2 \end{bmatrix} = \underset{\sim}{0} ,$$

hence, the first-order formal partial derivative matrices $\underset{\sim}{H}_x$, $\underset{\sim}{\bar{H}}_x$, $\underset{\sim}{H}_u$ and $\underset{\sim}{\bar{H}}_u$ of (2) are given by

$$\underset{\sim}{H}_x = \begin{bmatrix} V_1^*(Y_3 + Y_5) & 0 & 0 & 0 & 0 \\ -V_2^* Y_5 & 0 & 0 & 0 & 0 \\ -Y_3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -Y_5 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\bar{\underset{\sim}{H}}_x = \begin{bmatrix} [V_1(Y_3 + Y_5) - Y_5 V_2] & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\underset{\sim}{H}_u = \begin{bmatrix} 0 & -V_1^* Y_5 & V_1^* V_1 & 0 & V_1^*(V_1 - V_2) \\ 0 & V_2^*(Y_4 + Y_5) & 0 & V_2^* V_2 & V_2^*(V_2 - V_1) \\ 0 & 0 & -V_1 & 0 & 0 \\ 0 & -Y_4 & 0 & -V_2 & 0 \\ 0 & Y_5 & 0 & 0 & (V_2 - V_1) \end{bmatrix}$$

and

$$\bar{\underset{\sim}{H}}_u = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & [(Y_4 + Y_5)V_2 - Y_5 V_1] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Note that $\underset{\sim}{H}_x^* \neq \bar{\underset{\sim}{H}}_x$ and $\underset{\sim}{H}_u^* \neq \bar{\underset{\sim}{H}}_u$ since $\underset{\sim}{h} = \underset{\sim}{0}$ represents general complex equality constraints. Note also that the quantities $Y_5(V_1 - V_2)$ and

$Y_3 V_1$ in terms of the state V_1 and controls V_2, Y_3 and Y_5 could be replaced in the vector $\underset{\sim}{h}$ by the states I_5 and I_3, respectively.

Now, consider the real function

$$f = |V_1|^2 = V_1 V_1^* \quad , \tag{26}$$

which represents the square of the load bus voltage magnitude. Hence, the first-order formal partial derivatives $\underset{\sim}{f}_x$, $\underset{\sim}{\bar{f}}_x$, $\underset{\sim}{f}_u$ and $\underset{\sim}{\bar{f}}_u$ of (1) are given by

$$\underset{\sim}{f}_x = \begin{bmatrix} V_1^* \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \underset{\sim}{\bar{f}}_x = \begin{bmatrix} V_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\underset{\sim}{f}_u = \underset{\sim}{\bar{f}}_u = \underset{\sim}{0}.$$

Note that for the <u>real</u> function f, $\underset{\sim}{f}_x^* = \underset{\sim}{\bar{f}}_x$ and $\underset{\sim}{f}_u^* = \underset{\sim}{\bar{f}}_u$.

The matrices $\underset{\sim}{H}_x$, $\underset{\sim}{\bar{H}}_x$, $\underset{\sim}{H}_u$ and $\underset{\sim}{\bar{H}}_u$ can easily be partitioned into the element vectors $\underset{\sim}{h}_{bx}$, $\underset{\sim}{\bar{h}}_{bx}$, $\underset{\sim}{h}_{bu}$, $\underset{\sim}{\bar{h}}_{bu}$, respectively, as given by (9) and (10). For example,

$$\underset{\sim}{h}_{1x} = \begin{bmatrix} V_1^*(Y_3 + Y_5) \\ -V_2^* Y_5 \\ -Y_3 \\ 0 \\ -Y_5 \end{bmatrix} \quad , \quad \underset{\sim}{h}_{3u} = \begin{bmatrix} V_1^* V_1 \\ 0 \\ -V_1 \\ 0 \\ 0 \end{bmatrix} .$$

Similarly, f_{bx}, $\bar{f}_{bx}$, f_{bu} and $\bar{f}_{bu}$ are simply the corresponding elements of $\underset{\sim}{f}_x$, $\underset{\sim}{\bar{f}}_x$, $\underset{\sim}{f}_u$ and $\underset{\sim}{\bar{f}}_u$ calculated before. Therefore, the Lagrange multipliers $\underset{\sim}{\lambda}$ and $\underset{\sim}{\bar{\lambda}}$ can be calculated from (6) or (15) which constitute

10 equations each ($2n_x = 10$). The total derivatives can subsequently be evaluated from (7) or (18) as described in the paper.

In the Tellegen's theorem approach, the transformed adjoint network vectors $\hat{\underset{\sim}{\eta}}_{bx}$ and $\hat{\underset{\sim}{\eta}}_{bu}$ are given [6] for various elements by

$$\hat{\underset{\sim}{\eta}}_{1x} = \begin{bmatrix} \hat{I}_1 + \hat{V}_1^* I_1^*/V_1 \\ \hat{I}_1^* + \hat{V}_1 I_1/V_1^* \end{bmatrix}, \quad \hat{\underset{\sim}{\eta}}_{1u} = \begin{bmatrix} -\hat{V}_1^*/V_1 \\ -\hat{V}_1/V_1^* \end{bmatrix}$$

$$\hat{\underset{\sim}{\eta}}_{2x} = \begin{bmatrix} -\hat{V}_2^*/V_2 \\ -\hat{V}_2/V_2^* \end{bmatrix}, \quad \hat{\underset{\sim}{\eta}}_{2u} = \begin{bmatrix} \hat{I}_2 + I_2^* \hat{V}_2^*/V_2 \\ \hat{I}_2^* + I_2 \hat{V}_2/V_2^* \end{bmatrix}$$

$$\hat{\underset{\sim}{\eta}}_{3x} = \begin{bmatrix} \hat{I}_3/Y_3 - \hat{V}_3 \\ \hat{I}_3^*/Y_3^* - \hat{V}_3^* \end{bmatrix}, \quad \hat{\underset{\sim}{\eta}}_{3u} = \begin{bmatrix} V_3 \hat{I}_3/Y_3 \\ V_3^* \hat{I}_3^*/Y_3^* \end{bmatrix}$$

$$\hat{\underset{\sim}{\eta}}_{4x} = \begin{bmatrix} \hat{I}_4/Y_4 - \hat{V}_4 \\ \hat{I}_4^*/Y_4^* - \hat{V}_4^* \end{bmatrix}, \quad \hat{\underset{\sim}{\eta}}_{4u} = \begin{bmatrix} V_4 \hat{I}_4/Y_4 \\ V_4^* \hat{I}_4^*/Y_4^* \end{bmatrix}$$

$$\hat{\underset{\sim}{\eta}}_{5x} = \begin{bmatrix} \hat{I}_5/Y_5 - \hat{V}_5 \\ \hat{I}_5^*/Y_5^* - \hat{V}_5^* \end{bmatrix}, \quad \hat{\underset{\sim}{\eta}}_{5u} = \begin{bmatrix} V_5 \hat{I}_5/Y_5 \\ V_5^* \hat{I}_5^*/Y_5^* \end{bmatrix}$$

The adjoint network is described by (23), namely

$$\begin{bmatrix} \hat{I}_1 + \hat{V}_1^* I_1^*/V_1 \\ \hat{I}_1^* + \hat{V}_1 I_1/V_1^* \\ -\hat{V}_2^*/V_2 \\ -\hat{V}_2/V_2^* \\ \hat{I}_3/Y_3 - \hat{V}_3 \\ \hat{I}_3^*/Y_3^* - \hat{V}_3^* \\ \hat{I}_4/Y_4 - \hat{V}_4 \\ \hat{I}_4^*/Y_4^* - \hat{V}_4^* \\ \hat{I}_5/Y_5 - \hat{V}_5 \\ \hat{I}_5^*/Y_5^* - \hat{V}_5^* \end{bmatrix} = \begin{bmatrix} V_1^* \\ V_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that the adjoint transmission elements are modelled as $\hat{I}_t = Y_t \hat{V}_t$, $t = 3, 4, 5$ as in the original networks.

The solution of the adjoint network which constitutes four equations ($2n = 4$) provides the unknown adjoint network currents and voltages. The total derivatives can subsequently be evaluated from (25) using the values of f_{bu}, $\bar{f}_{bu}$ and $\hat{\tilde{n}}_{bu}$ calculated before.

Application to a 6-Bus Power System

Consider the 6-bus power system [6] shown in Fig. 2. The system consists of three load branches ($\ell = 1, 2, 3$), two generator branches ($g = 4, 5$) and the slack generator branch ($n = 6$) as well as eight transmission branches ($t = 7, 8, ..., 14$). Therefore, for this system, $n = 6$ and $n_B = 14$. Tables I and II show, respectively, the bus and line data of the system using the notation described in Section II. The power flow solution associated with the base-case data of Tables I and II is shown in Table III.

The element state and control variables $\tilde{x}_b$ and $\tilde{u}_b$ of (3) and (4), respectively, are given as follows [4,6].

The element variables for a load are defined as

$$\bar{\tilde{z}}_\ell = \begin{bmatrix} \tilde{x}_\ell \\ \hline \tilde{u}_\ell \end{bmatrix} \triangleq \begin{bmatrix} |V_\ell| \\ \delta_\ell \\ \hline P_\ell \\ Q_\ell \end{bmatrix} = \begin{bmatrix} (V_\ell \, V_\ell^*)^{1/2} \\ \tan^{-1}[j(V_\ell^* - V_\ell)/(V_\ell + V_\ell^*)] \\ \hline (V_\ell \, I_\ell^* - V_\ell^* \, I_\ell)/2 \\ j(V_\ell^* \, I_\ell - V_\ell \, I_\ell^*)/2 \end{bmatrix} \tag{27}$$

or, for example, as

$$\underset{\sim}{z}_\ell \overset{\Delta}{=} \begin{bmatrix} V_\ell \\ V^*_\ell \\ S_\ell \\ S^*_\ell \end{bmatrix} = \begin{bmatrix} V_\ell \\ V^*_\ell \\ V_\ell\ I^*_\ell \\ V^*_\ell\ I_\ell \end{bmatrix} . \tag{28}$$

The element variables for a generator are defined as

$$\bar{\underset{\sim}{z}}_g \overset{\Delta}{=} \begin{bmatrix} \underset{\sim}{x}_g \\ \hline \underset{\sim}{u}_g \end{bmatrix} \overset{\Delta}{=} \begin{bmatrix} \delta_g \\ Q_g \\ \hline |V_g| \\ P_g \end{bmatrix} = \begin{bmatrix} \tan^{-1}[j(V^*_g - V_g)/(V_g + V^*_g)] \\ j(V^*_g\ I_g - V_g\ I^*_g)/2 \\ \hline (V_g\ V^*_g)^{1/2} \\ (V_g\ I^*_g + V^*_g\ I_g)/2 \end{bmatrix} \tag{29}$$

or, for example, as

$$\underset{\sim}{z}_g \overset{\Delta}{=} \begin{bmatrix} V_g \\ I_g \\ |V_g|^2 \\ S_g + S^*_g \end{bmatrix} = \begin{bmatrix} V_g \\ I_g \\ V_g\ V^*_g \\ V_g\ I^*_g + V^*_g\ I_g \end{bmatrix} . \tag{30}$$

The element variables for the slack generator are defined as

$$\bar{\underset{\sim}{z}}_n = \begin{bmatrix} \underset{\sim}{x}_n \\ \hline \underset{\sim}{u}_n \end{bmatrix} \overset{\Delta}{=} \begin{bmatrix} P_n \\ Q_n \\ \hline |V_n| \\ \delta_n \end{bmatrix} = \begin{bmatrix} (V_n\ I^*_n + V^*_n\ I_n)/2 \\ j(V^*_n\ I_n - V_n\ I^*_n)/2 \\ \hline (V_n\ V^*_n)^{1/2} \\ \tan^{-1}[j(V^*_n - V_n)/(V_n + V^*_n)] \end{bmatrix} \tag{31}$$

or, for example, as

$$\underset{\sim}{\tilde{z}}_n \triangleq \begin{bmatrix} I_n \\ I_n^* \\ V_n \\ V_n^* \end{bmatrix} . \tag{32}$$

For other branches, the element variables are defined according to the element type. The element variables for a transmission element, for example, may be defined as

$$\bar{\underset{\sim}{z}}_t = \begin{bmatrix} \underset{\sim}{x}_t \\ \hline \underset{\sim}{u}_t \end{bmatrix} \triangleq \begin{bmatrix} \mathrm{Re}\{I_t\} \\ \mathrm{Im}\{I_t\} \\ \hline G_t \\ B_t \end{bmatrix} = \begin{bmatrix} (I_t + I_t^*)/2 \\ j(I_t^* - I_t)/2 \\ \hline (I_t/V_t + I_t^*/V_t^*)/2 \\ j(V_t^*/V_t^* - I_t/V_t)/2 \end{bmatrix} \tag{33}$$

or, for example, as

$$\underset{\sim}{\tilde{z}}_t \triangleq \begin{bmatrix} I_t \\ I_t^* \\ Y_t \\ Y_t^* \end{bmatrix} = \begin{bmatrix} I_t \\ I_t^* \\ I_t/V_t \\ I_t^*/V_t^* \end{bmatrix} . \tag{34}$$

Now, consider the function

$$f = \sum_t |I_t|^2 R_t$$

$$= \sum_t \frac{1}{2}(I_t \; I_t^*)\left(\frac{1}{Y_t} + \frac{1}{Y_t^*}\right) , \tag{35}$$

which represents the total transmission losses in the power network.

Using either set of element variables ($\bar{\underset{\sim}{z}}_b$ or $\underset{\sim}{\tilde{z}}_b$) defined in (27)-(34) and following the procedure of Section III for the Lagrange multiplier approach (28 adjoint equations are to be solved) or the procedure of Section IV for Tellegen's theorem approach (12 adjoint equations are to be solved), the reduced gradients of f with respect to various control variables can be evaluated. For the set of element variables $\bar{\underset{\sim}{z}}_b$, these reduced gradients are given in Table IV.

VII. CONCLUSIONS

The two widely used approaches to sensitivity calculations in electrical networks, namely the Lagrange multiplier and Tellegen's theorem approaches have been described and compared. The description has been performed on a unified basis, where we have defined and employed element-local terms in formulating the two approaches so that different aspects of comparison are clearly investigated. The resemblance in formulating the adjoint systems of the two approaches has been discussed.

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TABLE I

BUS DATA FOR 6-BUS POWER SYSTEM

Bus Index, i	Bus Type	P_i (pu)	Q_i (pu)	$\|V_i\|$ (pu)	$\angle\delta_i$
1	load	-2.40	0	-	$\angle -$
2	load	-2.40	0	-	$\angle -$
3	load	-1.60	-0.40	-	$\angle -$
4	generator	-0.30	-	1.02	$\angle -$
5	generator	1.25	-	1.04	$\angle -$
6	slack	-	-	1.04	$\angle 0$

TABLE II

LINE DATA FOR 6-BUS POWER SYSTEM

Branch Index, t	Terminal Buses	Resistance R_t (pu)	Reactance X_t (pu)	Number of Lines
7	1,4	0.05	0.20	1
8	1,5	0.025	0.10	2
9	2,3	0.10	0.40	1
10	2,4	0.10	0.40	1
11	2,5	0.05	0.20	1
12	2,6	0.01875	0.075	4
13	3,4	0.15	0.60	1
14	3,6	0.0375	0.15	2

TABLE III

LOAD FLOW SOLUTION OF 6-BUS POWER SYSTEM

Load Buses		
	$V_1 = 0.9787 \underline{/-0.6602}$	
	$V_2 = 0.9633 \underline{/-0.2978}$	
	$V_3 = 0.9032 \underline{/-0.3036}$	
Generator Buses		
	$Q_4 = 0.7866$,	$\delta_4 = -0.5566$
	$Q_5 = 0.9780$,	$\delta_5 = -0.4740$
Slack Bus		
	$P_6 = 6.1298$,	$Q_6 = 1.3546$

TABLE IV

RESULTS OF 6-BUS EXAMPLE

Line Quantities

Line	Derivatives w.r.t. G_t	Derivatives w.r.t. B_t
1,4	0.016462	0.008741
1,5	0.048977	0.027370
2,3	0.003490	0.002102
2,4	0.084665	0.044962
2,5	0.045468	0.022680
2,6	0.103966	0.060904
3,4	0.089397	0.042758
3,6	0.113314	0.069869

Load Bus Quantities

Bus	Derivatives w.r.t. P_ℓ	Derivatives w.r.t. Q_ℓ
1	-0.453538	-0.020390
2	-0.201703	-0.054098
3	-0.221666	-0.094646

Generator Bus Quantities

Bus	Derivatives w.r.t. $\|V_g\|$	Derivatives w.r.t. P_g
4	-0.373561	-0.375812
5	-0.184047	-0.312838

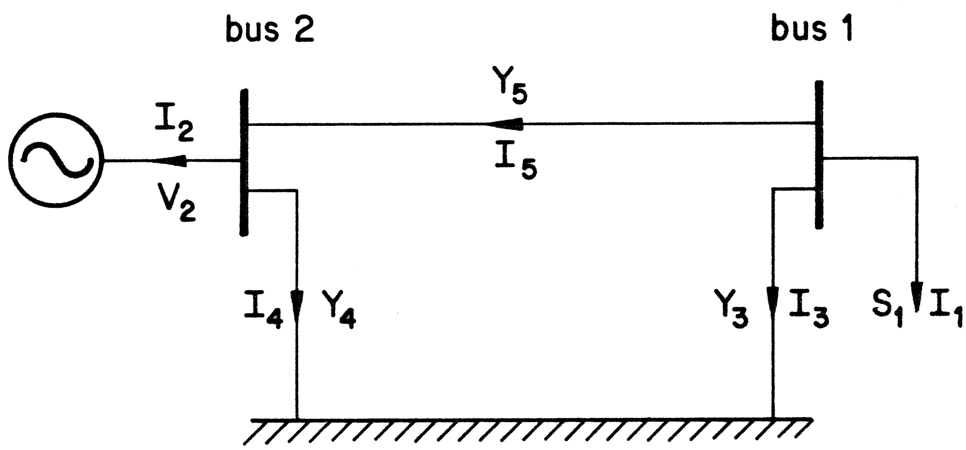

Fig. 1 2-bus power system.

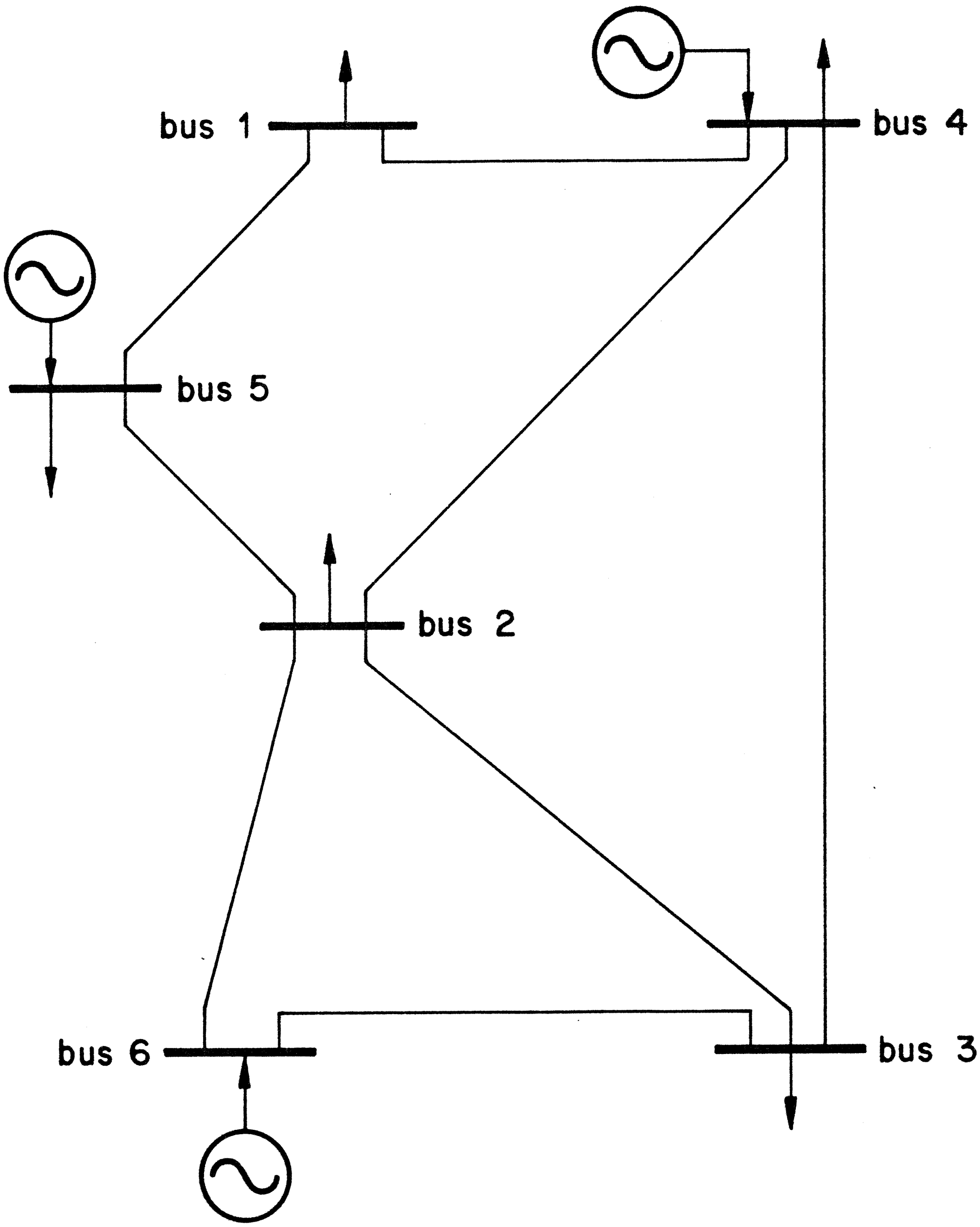

Fig. 2 6-bus sample power system.