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Abstract The usual assumptions for the tolerance problem in the frequency domain are that the worst cases occur at boundary points of a tolerance region, and that the acceptable region is simply connected. These assumptions are investigated and conditions for validity are given for the class of networks which have bilinear dependence on the parameter of interest.

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I INTRODUCTION

Large change sensitivities and worst-case tolerance problems dealing with linear networks in the frequency domain have attracted much attention recently [1]-[5]. The workers in these areas usually assume that the worst cases occur at the vertices or the surfaces of the tolerance region and that the acceptable region is simply connected. Although the assumptions may be true if the tolerances are small certain conditions have to be met.

The purpose of this paper is to justify these assumptions and state the conditions for the assumptions to be valid. We are interested in the effect of variation of a single parameter on the overall network function. We shall be concerned with the class of networks for which the network function can be expressed as a bilinear function of the parameter of interest [6]-[8].

II THE BIQUADRATIC FUNCTION

General Properties

Consider the biquadratic function

$$F(\phi) = \frac{N(\phi)}{M(\phi)} = \frac{c\phi^2 + 2d\phi + e}{\phi^2 + 2a\phi + b} . \tag{1}$$

The first derivative of $F(\phi)$ is

$$F'(\phi) = 2\,\frac{(c\phi+d)M(\phi) - (\phi+a)N(\phi)}{M^2(\phi)} . \tag{2}$$

It may be noted that the numerator of (2) is a quadratic function of ϕ which implies that the derivative has at most two changes of sign for finite values of ϕ. Furthermore, the function value approaches the value of c as $\phi \to \pm \infty$.

Take any two points ϕ^r and ϕ^s and let $\Delta\phi = \phi^s - \phi^r$. $F(\phi^s)$ may be expressed in terms of ϕ^r, $\Delta\phi$ and the coefficients of $N(\phi)$ and $M(\phi)$ as follows:

$$F(\phi^s) = \frac{N(\phi^s)}{M(\phi^s)} = \frac{N(\phi^r) + 2\Delta\phi(c\phi^r + d) + c\Delta\phi^2}{M(\phi^r) + 2\Delta\phi(\phi^r + a) + \Delta\phi^2} \quad . \tag{3}$$

The large change sensitivity

$$\frac{\Delta F}{\Delta\phi} \triangleq \frac{F(\phi^s) - F(\phi^r)}{\phi^s - \phi^r} \tag{4}$$

may be related to the first differential sensitivity $F'(\phi^r)$. We have

$$F(\phi^s) - F(\phi^r) = \frac{2\Delta\phi\{(c\phi^r + d)M(\phi^r) - (\phi^r + a)N(\phi^r)\} - \Delta\phi^2\{N(\phi^r) - cM(\phi^r)\}}{M(\phi^r)M(\phi^s)}$$

$$= \Delta\phi\, F'(\phi^r)\frac{M(\phi^r)}{M(\phi^s)} - \Delta\phi^2 \frac{(F(\phi^r) - c)}{M(\phi^s)}$$

therefore

$$M(\phi^s)\frac{\Delta F}{\Delta\phi} = F'(\phi^r)M(\phi^r) - \Delta\phi(F(\phi^r) - c) \quad . \tag{5}$$

Given a fixed value ϕ^r, we can find uniquely one other point ϕ^s such that $F(\phi^s) = F(\phi^r)$, except when the function $F(\phi^r) = c$, $F'(\phi^r) = 0$, or $M(\phi^r) = 0$. The point ϕ^s is given, using (5) with $\Delta F = 0$, by

$$\phi^s = \phi^r + \frac{F'(\phi^r)M(\phi^r)}{F(\phi^r) - c} \quad . \tag{6}$$

For the case $F'(\phi^r) = 0$, the point ϕ^r is either at the maximum or at the minimum of the function. There is only one finite point ϕ^c such that $F(\phi^c) = c$. The other points with the same value can only be at infinity. See, for example, Fig. 1.

Assumptions

In the following discussion, we shall assume that $M(\phi)$ does not change sign on $[\phi^r, \phi^s]$. We shall also exclude points where $M(\phi) = 0$ since the derivative of $F(\phi)$ is not defined at such points.

III LEMMAS AND THEOREMS

Lemma 1 $F(\phi^r+\lambda(\phi^s-\phi^r)) > \min[F(\phi^r),F(\phi^s)]$ for all λ satisfying $0 < \lambda < 1$ provided that

$$\frac{\Delta F}{\Delta\phi} \cdot \left.\frac{dF}{d\phi}\right|_{\phi=\check{\phi}} > 0 \tag{7}$$

where $\frac{\Delta F}{\Delta\phi}$ is given in (4), $\check{\phi}$ is ϕ^r or ϕ^s whichever corresponds to the lower function value.

Fig. 2 illustrates this lemma.

Proof The case $F(\phi^s) > F(\phi^r)$ will be considered first. From equation (5), we have

$$M(\phi)\,\frac{F(\phi)-F(\phi^r)}{\lambda\Delta\phi} = F'(\phi^r)M(\phi^r)-\lambda\Delta\phi(F(\phi^r) - c) \tag{8}$$

where

$$\phi = \phi^r + \lambda(\phi^s-\phi^r) \ , \ 0 < \lambda < 1 \quad . \tag{9}$$

If condition (7) is satisfied, $F'(\phi^r) = \left.\frac{dF}{d\phi}\right|_{\phi=\phi^r} > 0$

then

$$\frac{1}{M(\phi^s)} \left[F'(\phi^r)M(\phi^r) - \Delta\phi(F(\phi^r) - c)\right] > 0$$

implies, since $M(\phi)$ must not change sign, that

$$\frac{1}{M(\phi)} \left[F'(\phi^r)M(\phi^r) - \lambda\Delta\phi(F(\phi^r) - c)\right] > 0 .$$

Therefore,

$$F(\phi) - F(\phi^r) > 0. \tag{10}$$

Similarly, for the case when $F(\phi^r) > F(\phi^s)$, it is required from (7) that $F'(\phi^s) = \left.\frac{dF}{d\phi}\right|_{\phi=\phi^s} < 0$. The equations corresponding to (5) and (8) are, respectively,

$$M(\phi^r) \frac{F(\phi^s)-F(\phi^r)}{\Delta\phi} = F'(\phi^s)M(\phi^s) + \Delta\phi(F(\phi^s)-c) \tag{11}$$

and

$$M(\phi) \frac{F(\phi^s)-F(\phi)}{(1-\lambda)\Delta\phi} = F'(\phi^s)M(\phi^s) + (1-\lambda)\Delta\phi(F(\phi^s)-c) . \tag{12}$$

Since $\frac{\Delta F}{\Delta\phi} < 0$

$$\frac{1}{M(\phi^r)} \left[F'(\phi^s)M(\phi^s)+\Delta\phi(F(\phi^s)-c)\right] < 0$$

implies, since $M(\phi)$ must not change sign, that

$$\frac{1}{M(\phi)} \left[F'(\phi^s)M(\phi^s) + (1-\lambda)\Delta\phi(F(\phi^s) - c)\right] < 0$$

and hence that

$$F(\phi) - F(\phi^s) > 0 \quad . \tag{13}$$

Inequalities (10) and (13) are true for all $0 < \lambda < 1$, hence the lemma is proved.

<u>Corollary</u> $F(\phi^r + \lambda(\phi^s-\phi^r)) < \max[F(\phi^r), F(\phi^s)]$, where $0 < \lambda < 1$, provided that

$$\frac{\Delta F}{\Delta\phi} \cdot \frac{dF}{d\phi}\bigg|_{\phi=\hat{\phi}} > 0 \tag{14}$$

where $\hat{\phi}$ is ϕ^r or ϕ^s whichever corresponds to the higher function value.

The corollary may be proved by defining a new function $G(\phi) = -F(\phi)$ and applying Lemma 1. See Fig. 3 for an illustration. Fig. 4 shows an example where both the lemma and its corollary apply.

<u>Lemma 2</u> The function $F(\phi)$ is pseudoconcave [9] on the interval $[\phi^r, \phi^s]$ except where $M(\phi) = 0$ if the conditions of Lemma 1 are satisfied.

<u>Proof</u> Consider the case $F(\phi^s) > F(\phi^r)$. The other case follows a similar argument. Let us assume that the function has more than one turning point in the interval. By the nature of the biquadratic function, there are at most two turning points. If we assume that there are two turning points on $[\phi^r, \phi^s]$, there exist two points $\phi^\alpha = \phi^r + \alpha\Delta\phi$ and $\phi^\beta = \phi^r + \beta\Delta\phi$, where $0 < \alpha < \beta < 1$ such that the following inequalities hold:

$$F(\phi^\alpha) > F(\phi^\beta) \tag{15}$$

and

$$F'(\phi^\beta) > 0 \quad . \tag{16}$$

As a direct consequence of Lemma 1 and inequality (16), the following inequalities can be made to hold:

$$F(\phi^s) > F(\phi^\beta) \tag{17}$$

and

$$F(\phi^\beta) > F(\phi^r) \ . \tag{18}$$

Rewriting the function values in terms of $F'(\phi^\beta)$, $F(\phi^\beta)$ and $M(\phi^\beta)$ as in equation (5), we obtain

$$\frac{1}{M(\phi^\alpha)} [F'(\phi^\beta)M(\phi^\beta) + (\beta-\alpha)\Delta\phi(F(\phi^\beta) - c)] < 0 \tag{19}$$

$$\frac{1}{M(\phi^r)} [F'(\phi^\beta)M(\phi^\beta) + \beta\Delta\phi(F(\phi^\beta) - c)] > 0 \tag{20}$$

and

$$\frac{1}{M(\phi^s)} [F'(\phi^\beta)M(\phi^\beta) - (1-\beta)\Delta\phi(F(\phi^\beta) - c)] > 0 \ . \tag{21}$$

Multiply (19) by $M(\phi^\alpha)$, (20) by $M(\phi^r)$ and (21) by $M(\phi^s)$.
Subtracting appropriately, we have

$$\alpha\Delta\phi(F(\phi^\beta) - c) \begin{cases} > 0 \text{ for } M > 0 \\ < 0 \text{ for } M < 0 \end{cases}$$

and

$$-(1-\alpha)\Delta\phi(F(\phi^\beta) - c) \begin{cases} > 0 \text{ for } M > 0 \\ < 0 \text{ for } M < 0 \ . \end{cases}$$

The last two pairs of inequalities are inconsistent, therefore the assumption that there are two turning points on the interval is false. $F(\phi)$, $\phi\varepsilon[\phi^r,\phi^s]$, is unimodal with a positive derivative at ϕ^r.

Given any two points ϕ^a and ϕ^b, such that $F(\phi^b) > F(\phi^a)$, we will consider the following:

(1) $F'(\phi^a) > 0$, then $\phi^b > \phi^a$ because F is an increasing function between ϕ^r and ϕ^a.

(2) $F'(\phi^a) < 0$, then $\phi^b < \phi^a$ because F is a decreasing function between ϕ^a and ϕ^s.

Therefore, in both cases $F(\phi^b) > F(\phi^a)$ implies $F'(\phi^a)(\phi^b-\phi^a) > 0$, which proves the lemma.

Corollary The function $F(\phi)$ is pseudoconvex on the interval $[\phi^r,\phi^s]$ except where $M(\phi) = 0$ if the conditions of the corollary to Lemma 1 are satisfied.

Theorem 1 The $\begin{smallmatrix}\text{minimum}\\ \text{maximum}\end{smallmatrix}$ of $F(\phi)$, $\phi\varepsilon[\phi^r,\phi^s]$, lies on the boundary of the interval if one of the following conditions is satisfied.

$$F'(\phi^r) \gtrless 0 \text{ and } F'(\phi^s) \lessgtr 0 \qquad \begin{pmatrix}22a\\22b\end{pmatrix}$$

$$F'(\phi^r) > 0,\ F'(\phi^s) > 0 \text{ and } F(\phi^r) < F(\phi^s) \qquad (23)$$

or

$$F'(\phi^r) < 0,\ F'(\phi^s) < 0 \text{ and } F(\phi^r) > F(\phi^s) \qquad (24)$$

See, for example, Figs. 2-4.

Proof We will prove the case for the minimum of $F(\phi)$ to be on the boundary of an interval for the conditions of (22a), (23) and (24).

(1) Take $\check{\phi}=\phi^r$, then $F(\phi^s) > F(\phi^r)$ and $\frac{\Delta F}{\Delta\phi} > 0$. Using Lemma 1,

$F(\phi^r+\lambda(\phi^s-\phi^r)) > \min[F(\phi^r),F(\phi^s)]$, $0 < \lambda < 1$, will hold if $F'(\phi^r) > 0$. This is satisfied in (22a) and (23).

(2) Take $\check{\phi}=\phi^s$, than $F(\phi^r) > F(\phi^s)$ and $\frac{\Delta F}{\Delta \phi} < 0$. Using Lemma 1 again, the requirement that $F'(\phi^s) < 0$ will be met in (22a) and (24).

(3) Suppose $F(\phi^r) = F(\phi^s)$ and hence $\frac{\Delta F}{\Delta \phi} = 0$. We can find one point ϕ^a such that $F(\phi^a) > F(\phi^r) = F(\phi^s)$. Two subintervals are thus obtained, each of which may be considered under cases (1) and (2) above.

It should be noted that, from Lemma 2, (22a), (23), and (24) imply pseudoconcavity. From its corollary, (22b), (23) and (24) imply pseudoconvexity.

Let us define the upper and lower specifications by S_{ui}, $i \varepsilon I_u$, and $S_{\ell i}$, $i \varepsilon I_\ell$, respectively, where I_u and I_ℓ are disjoint index sets. An acceptable interval I_a may be defined as

$$I_a \triangleq \{\phi | S_{ui} - F_i(\phi) \geq 0,\ i \varepsilon I_u,$$

$$F_j(\phi) - S_{\ell j} \geq 0,\ j \varepsilon I_\ell\} \tag{25}$$

Theorem 2 I_a is convex if the condition (22a), (23) or (24) is satisfied by $F_i(\phi)$, for all $i \varepsilon I_\ell$, and condition (22b), (23) or (24) is satisfied by $F_i(\phi)$, for all $i \varepsilon I_u$.

Proof Consider the case $i \varepsilon I_\ell$ and let

$$I_i \triangleq \{\phi | F_i(\phi) - S_{\ell i} \geq 0\},\ i \varepsilon I_\ell. \tag{26}$$

Take two different points ϕ^r, ϕ^s εI_i. If the condition (22a), (23) or (24) is satisfied, then from Theorem 1

$$F_i(\phi^\lambda) = F_i(\phi^r + \lambda(\phi^s - \phi^r)) > \min[F_i(\phi^r),\ F_i(\phi^s)],$$

$$0 < \lambda < 1$$

thus

$$F_i(\phi^\lambda) - S_{\ell i} > \min[F_i(\phi^r) - S_{\ell i} \quad , \quad F_i(\phi^s) - S_{\ell i}],$$

$$0 < \lambda < 1.$$

Since

$$\phi^r \ , \ \phi^s \ \varepsilon I_i$$

$$F_i(\phi^\lambda) - S_{\ell i} > 0. \tag{27}$$

Therefore,

$$\phi^\lambda = \phi^r + \lambda(\phi^s - \phi^r) \ \varepsilon I_i. \tag{28}$$

Hence I_i, $i\varepsilon I_\ell$, is a convex interval by definition of a convex set. Similarly, for the case $i\varepsilon I_u$, if the condition (22b), (23) or (24) is satisfied, using Theorem 1, we may prove that I_i, $i\varepsilon I_u$, is convex.

The intersection of convex sets is convex, and since by definition $I_a = \bigcap_{\substack{i\varepsilon I_\ell \\ i\varepsilon I_u}} I_i$, I_a is convex.

If any $F(\phi)$ has both upper and lower specifications, the required conditions for a convex acceptable interval are restricted to (23) and (24).

IV THE NETWORK TOLERANCE PROBLEM

We consider a bilinear network function [6]-[8] of the form $(A + \phi B)/(C + \phi D)$ where A, B, C, and D are, in general, complex and frequency dependent. Thus we assume a function of the form

$$F(\phi) = \left| \frac{A + \phi B}{C + \phi D} \right|^2 = \frac{N(\phi)}{M(\phi)} .$$

In this case N, M $\geq$ 0. The coefficients of (1) are related to the bilinear function as follows:

$$a = \frac{C_r D_r + C_i D_i}{|D|^2} , \quad b = \frac{|C|^2}{|D|^2} ,$$

$$c = \frac{|B|^2}{|D|^2} , \; d = \frac{A_r B_r + A_i B_i}{|D|^2} \quad \text{and } e = \frac{|A|^2}{|D|^2} ,$$

where the subscripts i and r denote the imaginary and real parts of the complex number.

Take the example of an LC lowpass filter shown in Fig. 5. We have studied the behaviour of $|\rho|^2$ with respect to the variations of L, C_2 and C_3, respectively. Fig. 6 shows some of the curves for different values of frequency. The three vertical lines on each drawing represent the nominal values and the extreme values of ± 25% relative tolerance. The nominal values for L, C_2 and C_3 are 2, .125 and 1, respectively. $C_1 = C_3$ for reasons of symmetry.

The curves for L and C_2 have two turning points each. For example, at $\omega = 1$,

$$|\rho(L)|^2 = \frac{81L^2 - 144L + 64}{82L^2 - 160L + 128} .$$

The turning points are at L = .889 and L = 8.0 corresponding to the minimum of $|\rho|^2 = 0$ and the maximum of $|\rho|^2 = 1$, respectively. Setting $|\rho|^2 = \frac{81}{82} = c$, we can solve for one unique point L = 4.44 at which the curve is divided into two parts: $|\rho|^2 \geq .988$ for $L \geq 4.44$ and $|\rho|^2 \leq .988$ for $L \leq 4.44$. The maximum and minimum function values occur separately in the two parts. The derivatives at the boundary of the tolerance region are both positive, indicating that the curve is monotonic in the region (both pseudoconvex and pseudoconcave).

For parameter C_2 at $\omega=1$

$$|\rho(C_2)|^2 = \frac{4C_2^2 + 4C_2 + 1}{8C_2^2 + 2} .$$

The maximum and minimum occur at values of .5 and -.5. At $C_2 = 0$, the curve is again divided into two parts for $|\rho|^2 \geq .5$ and $|\rho|^2 \leq .5$ for positive or negative C_2 values, respectively.

The curves for C_3 have only one turning point. The biquadratic function is of the form

$$|\rho(C_3)|^2 = \frac{C_3^2 + 2aC_3 + e}{C_3^2 + 2aC_3 + b} .$$

The minimum occurs at $C_3 = -a$. The curves are pseudoconvex on $(-\infty,\infty)$ for frequencies in both the passband and stopband. For the worst case at stop-band frequencies to occur at the boundary of an interval, it is required that the curves corresponding to these frequencies also be pseudoconcave on the interval, i.e., the curves should be monotonic on the interval.

V CONCLUSIONS

The present work deals with a one-dimensional case. Conditions for the worst-case to occur at the boundary of an interval are given. The conditions may be used at least to partially justify the usual assumptions for the tolerance problem. The analysis presented here is exact unlike an approximation procedure which makes use of the first-and second-order sensitivities at the nominal point. Bandler [10] has already related a one-dimensional convexity assumption for the acceptable interval to that of the k-dimensional case. It was proved that only vertices of the tolerance region need be tested for the worst-case problem if the one-dimensional assumption holds everywhere. Thus, Theorem 1 in the present paper involves necessary conditions for the vertices of a k-dimensional region.

REFERENCES

[1] S.L. Hakimi and J.B. Cruz, Jr., "Measure of sensitivity for linear systems with large multiple parameter variations", IRE WESCON Conv. Rec., Aug. 1960, part 2, pp. 109-115.

[2] R. Spence, P. Villalaz, P. Goddard, "The efficient prediction of tolerance regions, with application to circuit design and device modelling", Proc. IEEE Int. Symp. Circuit Theory (London, England, Sept. 1971), pp. 107-108.

[3] E.M. Butler, "Realistic design using large-change sensitivities and performance contours", IEEE Trans. Circuit Theory, vol. CT-18, Jan. 1971, pp. 58-66.

[4] J.W. Bandler and P.C. Liu, "Automated network design with optimal tolerances", IEEE Trans. Circuits and Systems, vol. CAS-21, March 1974, pp. 219-222.

[5] G.C. Temes and H.J. Orchard, "First-order sensitivity and worst-case analysis of doubly terminated reactance two-ports", IEEE Trans. Circuit Theory, vol. CT-20, Sept. 1973, pp. 567-571.

[6] S.R. Parker, E. Peskin and P.M. Chirlian, "Application of a bilinear theorem to network sensitivity", IEEE Trans. Circuit Theory, vol. CT-12, Sept. 1965, pp. 448-450.

[7] E.V. Sorensen, "General relations governing the exact sensitivity of linear networks", Proc. IEE, vol. 114, Sept. 1967, pp. 1209-1212.

[8] J.K. Fidler and C. Nightingale, "Differential-incremental-sensitivity relationships", Elect. Lett., vol. 8, Dec. 1972, pp. 626-627.

[9] O.L. Mangasarian, Nonlinear Programming. New York, McGraw-Hill, 1969.

[10] J.W. Bandler, "Optimization of design tolerances using nonlinear programming", Proc. 6th Princeton Conf. on Information Sciences and Systems (Princeton, N.J., March 1972), pp. 655-659. To be published in J. Optimization Theory and Applications, vol. 14, July 1974.

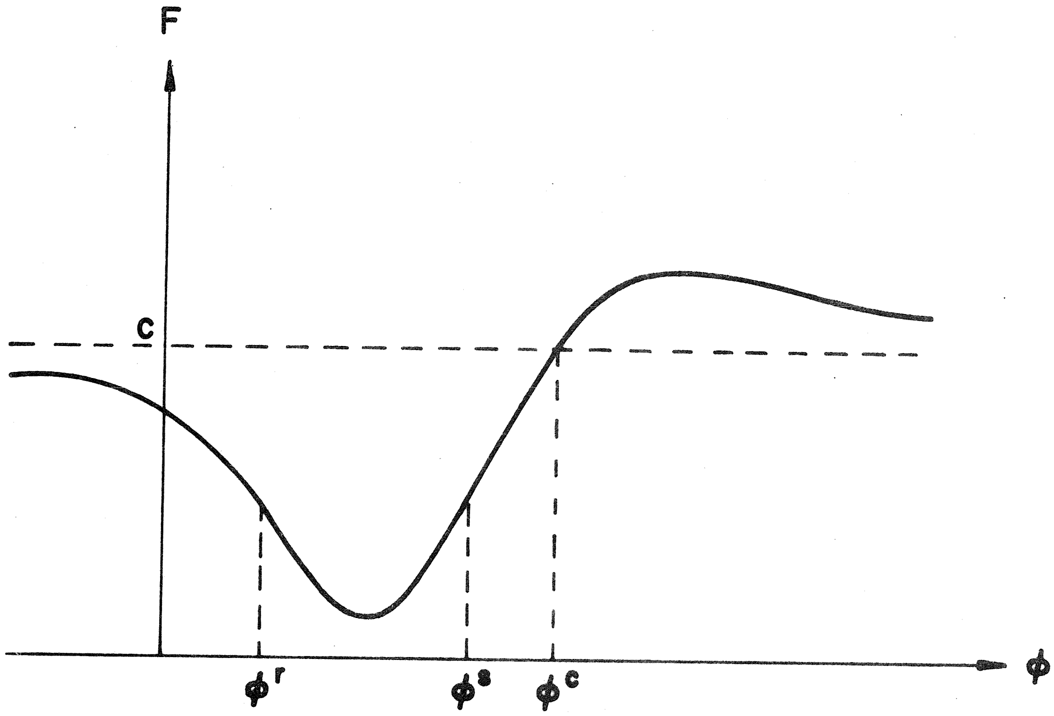

Fig. 1 A general biquadratic function.

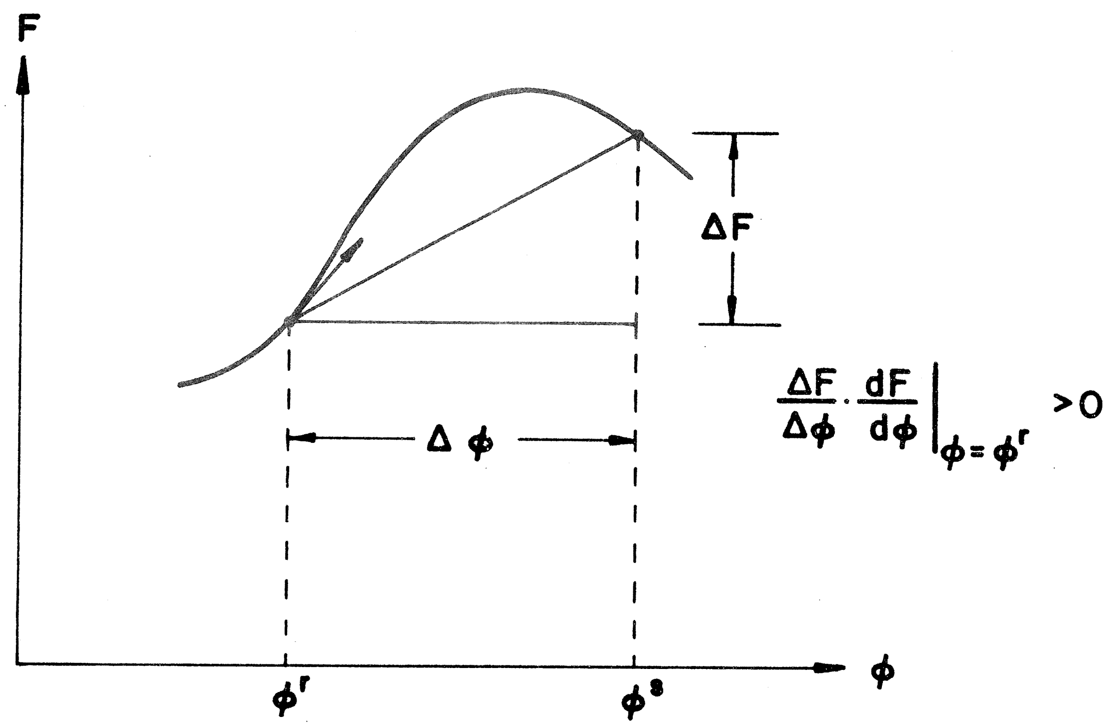

Fig. 2 Illustration of pseudoconcavity on an interval.

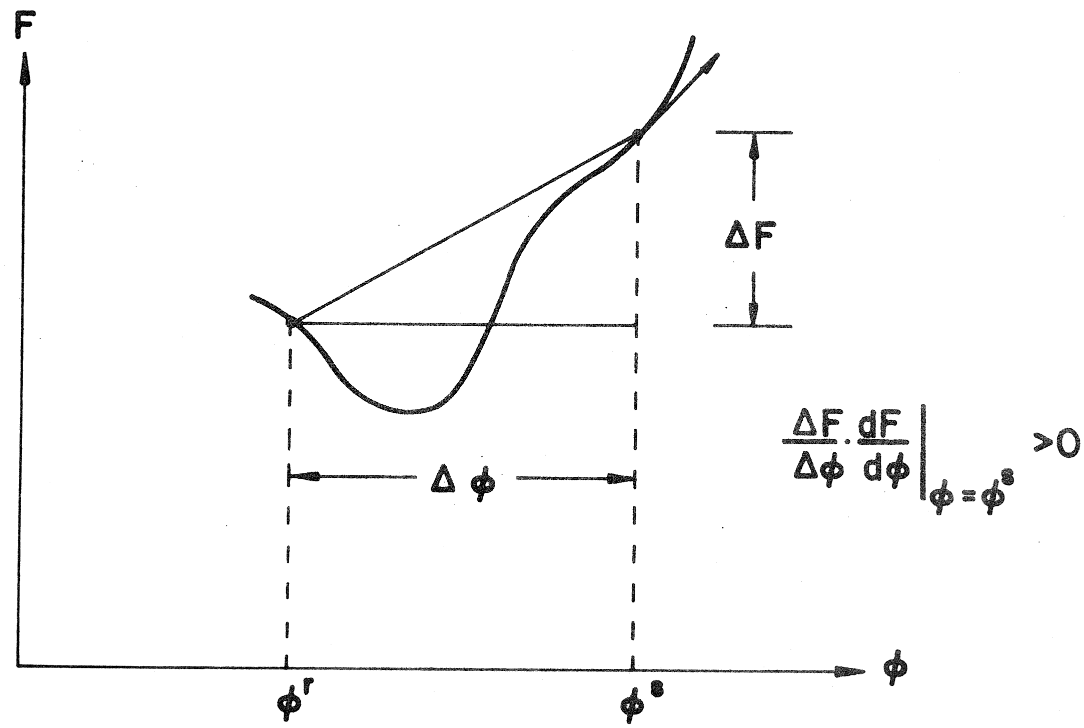

Fig. 3 Illustration of pseudoconvexity on an interval.

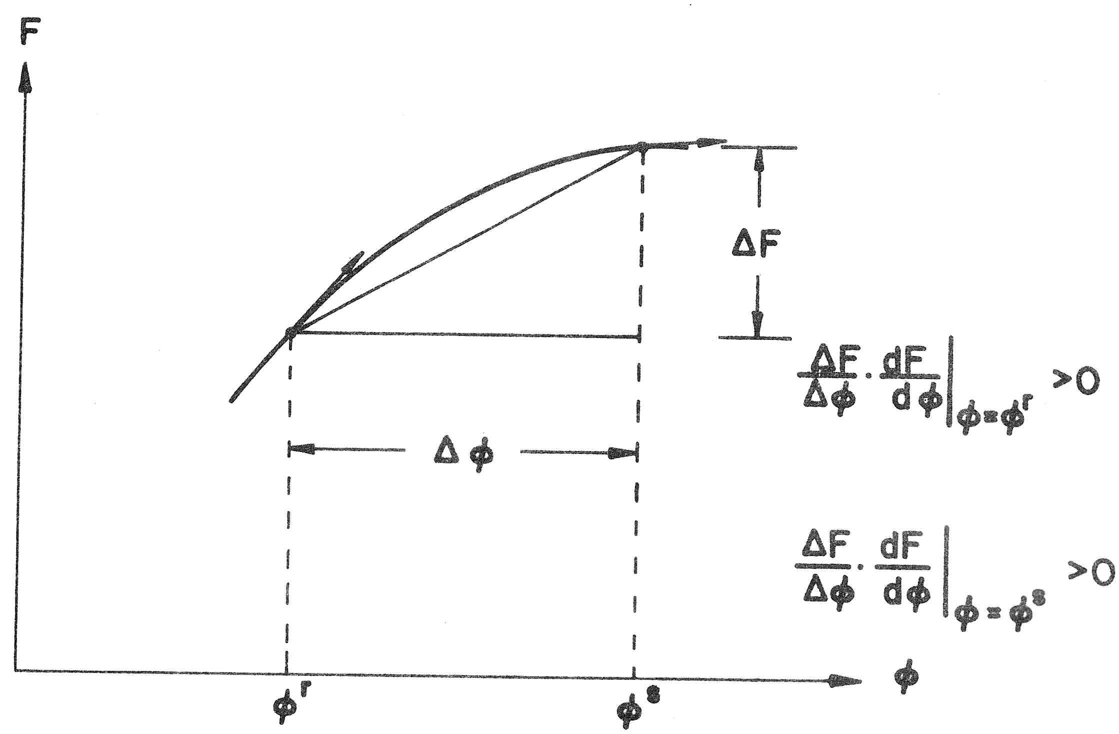

Fig. 4 Illustration of monotonicity on an interval.

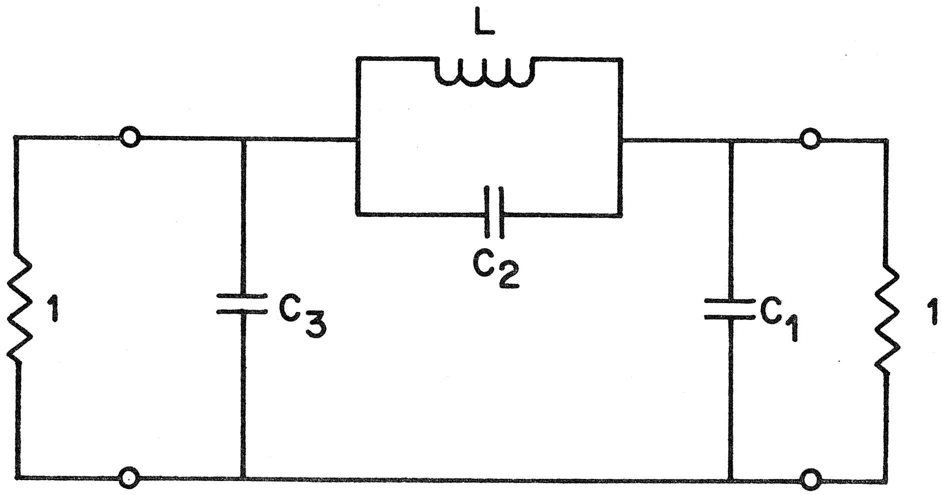

Fig. 5 An LC lowpass example.

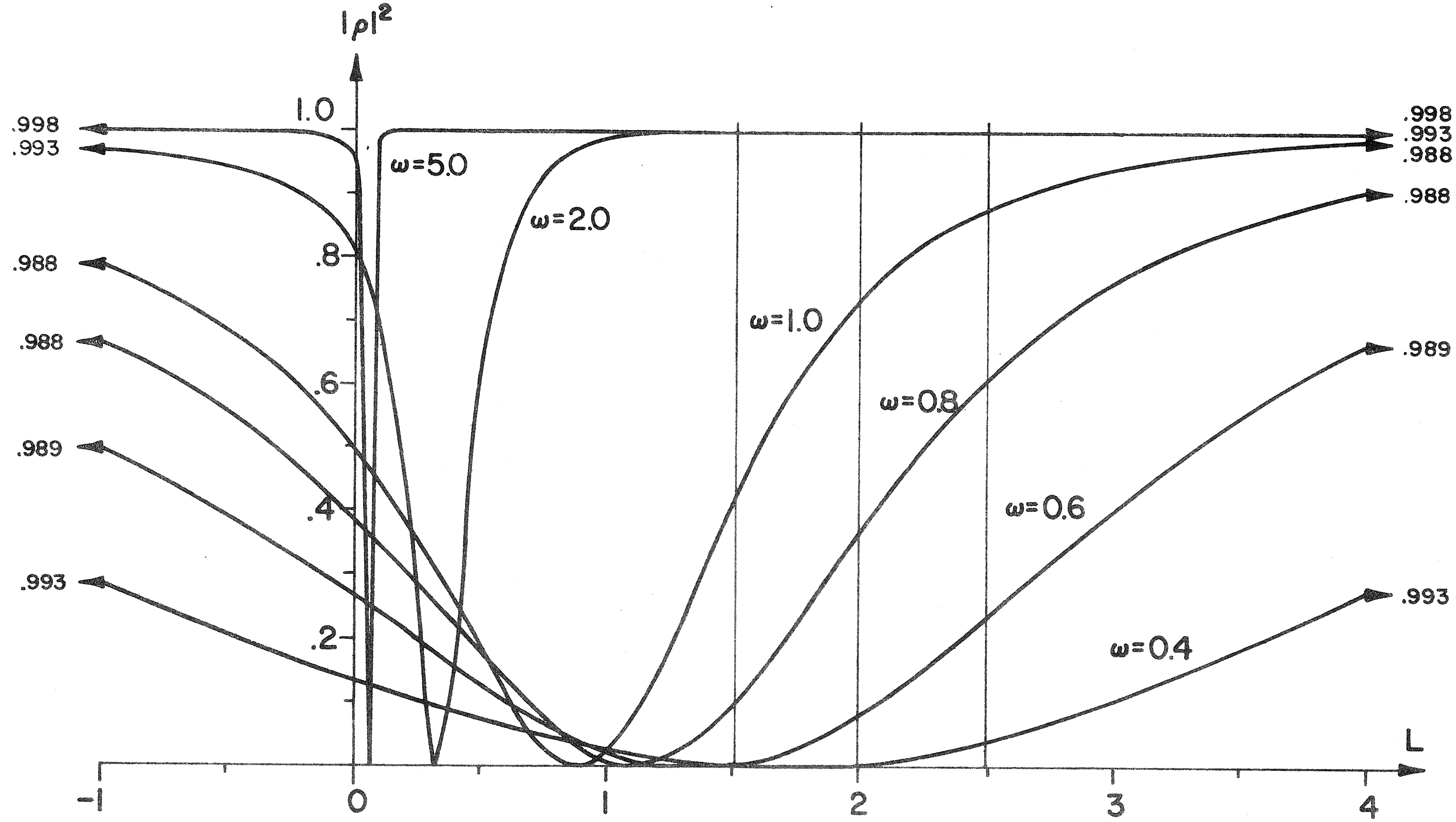

Fig. 6(a) $|\rho|^2$ vs L for the example.

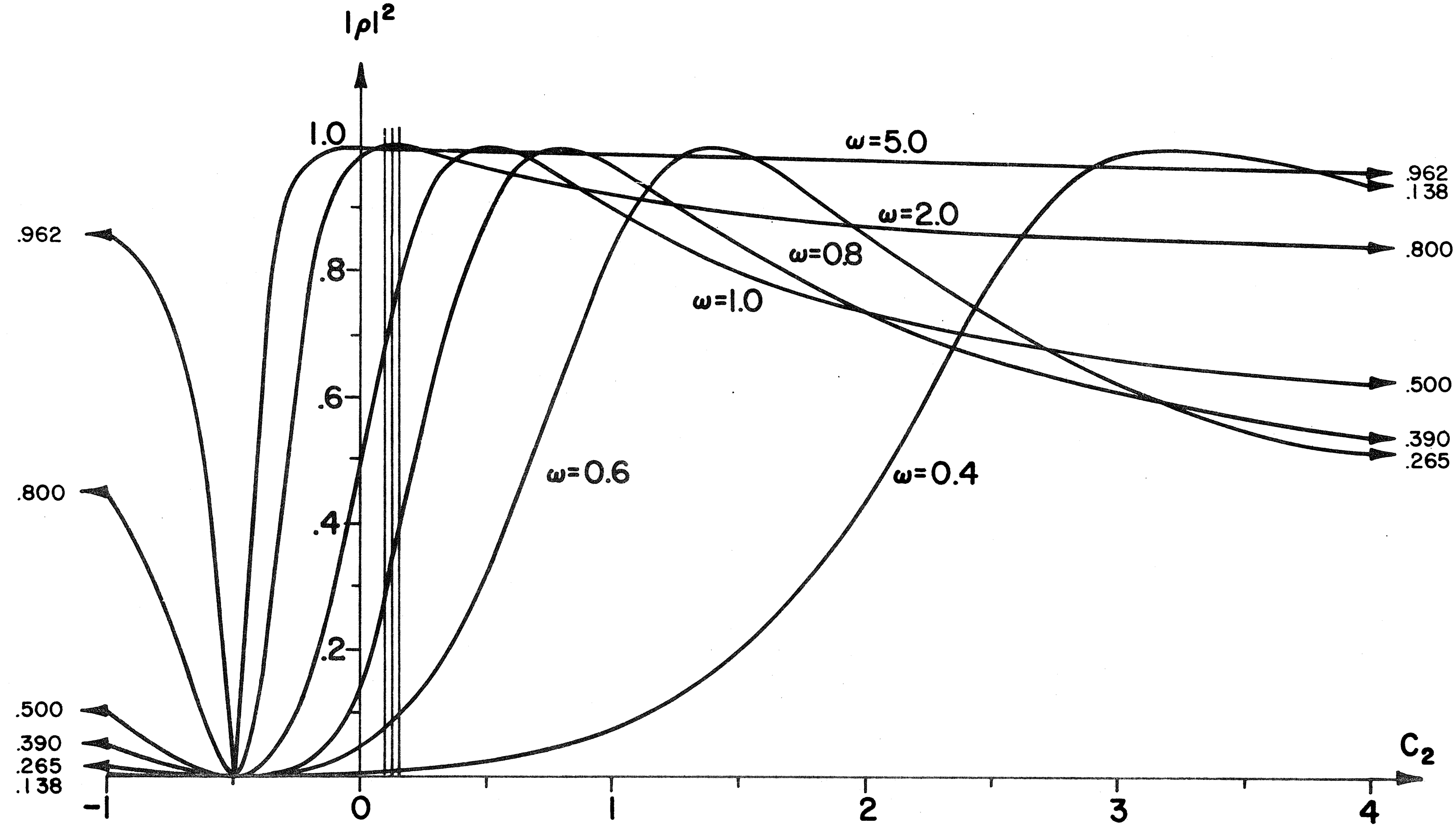

Fig. 6(b) $|\rho|^2$ vs C_2 for the example.

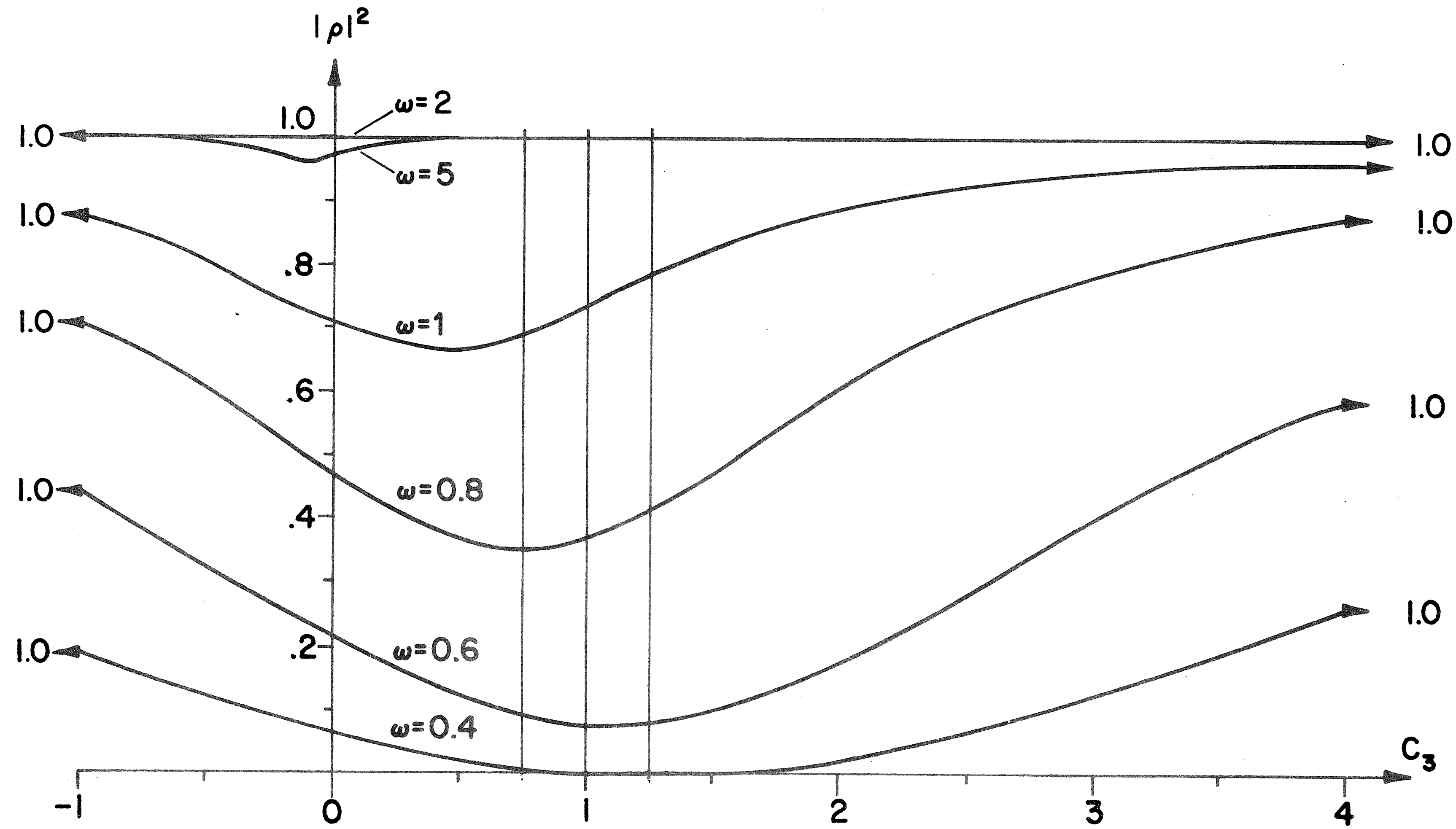

Fig. 6(c) $|\rho|^2$ vs C_3 for the example.