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THE COMPLEX ADJOINT APPROACHES TO NETWORK SENSITIVITIES

J.W. Bandler and M.A. El-Kady

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FACULTY OF ENGINEERING
McMASTER UNIVERSITY
HAMILTON, ONTARIO, CANADA

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J.W. Bandler, Fellow, IEEE, and M.A. El-Kady, Member, IEEE

Abstract

We present a comprehensive comparison between the widely used Lagrange multiplier and Tellegen's theorem approaches to sensitivity calculations in electrical networks. The two approaches are described on a unified basis using the conjugate notation. Different aspects of comparison can thereby be investigated. The linear electronic circuit analysis case is seen to be a special case.

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The authors are with the Group on Simulation, Optimization and Control and the Department of Electrical and Computer Engineering, McMaster University, Hamilton, Canada L8S 4L7.

M.A. El-Kady is also with Ontario Hydro, Toronto, Canada.

I. INTRODUCTION

Sensitivity calculations are performed routinely in electrical network analysis and design to supply first-order changes and gradients of functions of interest w.r.t. practically defined control or design variables.

Two approaches, namely the Lagrange multiplier approach [1,2] and Tellegen's theorem approach [3,4], are intensively used for sensitivity calculations in both electronic and power networks. Methods based on the two approaches have been described and applied [1-4] on an individual basis. A combination between the two approaches has been proposed in [5].

The material presented in this paper aims at investigating relationships between the two approaches. This investigation is accomplished by employing common bases of description and analysis through which the required aspects of comparison can be clearly stated.

We state the notation used and the basic formulation in Section II. In Sections III and IV, we describe, on a unified basis, the application of the Lagrange multiplier and the Tellegen's theorem approaches to sensitivity analysis of electrical networks. A comprehensive discussion of some aspects of comparison is then presented in Section V.

II. BASIC FORMULATION

We denote by f a single valued continuous complex function of $2n_x$ system complex state variables $(\underset{\sim}{x}, \underset{\sim}{x}^*)$ and $2n_u$ complex control variables $(\underset{\sim}{u}, \underset{\sim}{u}^*)$ arranged as column vectors. We also denote by $\underset{\sim}{h}$ a set of n_x complex equality constraints relating $(\underset{\sim}{x}, \underset{\sim}{x}^*)$ to $(\underset{\sim}{u}, \underset{\sim}{u}^*)$.

Using the conjugate notation [4-6], the first-order change of f is written as

$$\delta f = \underset{\sim}{f}_x^T \, \delta \underset{\sim}{x} + \overline{\underset{\sim}{f}}_x^T \, \delta \underset{\sim}{x}^* + \underset{\sim}{f}_u^T \, \delta \underset{\sim}{u} + \overline{\underset{\sim}{f}}^T \, \delta \underset{\sim}{u}^*, \tag{1}$$

where δ denotes first-order change, T denotes transposition and $\underset{\sim}{f}_x$ $\overline{\underset{\sim}{f}}_x$, $\underset{\sim}{f}_u$ and $\overline{\underset{\sim}{f}}_u$ denote the formal derivatives [5] $\partial f/\partial \underset{\sim}{x}$, $\partial f/\partial \underset{\sim}{x}^*$, $\partial f/\partial \underset{\sim}{u}$ and $\partial f/\partial \underset{\sim}{u}^*$, respectively. Also, the first-order change of $\underset{\sim}{h}$ is written as

$$\delta \underset{\sim}{h} = \underset{\sim}{H}_x \, \delta \underset{\sim}{x} + \overline{\underset{\sim}{H}}_x \, \delta \underset{\sim}{x}^* + \underset{\sim}{H}_u \, \delta \underset{\sim}{u} + \overline{\underset{\sim}{H}}_u \, \delta \underset{\sim}{u}^* = \underset{\sim}{0}, \tag{2}$$

where $\underset{\sim}{H}_x$, $\overline{\underset{\sim}{H}}_x$, $\underset{\sim}{H}_u$ and $\overline{\underset{\sim}{H}}_u$ stand for $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{x})^T$, $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{x}^*)^T$, $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{u})^T$ and $(\partial \underset{\sim}{h}^T/\partial \underset{\sim}{u}^*)^T$, respectively.

When dealing with electrical networks, $\underset{\sim}{x}$, $\underset{\sim}{x}^*$, $\underset{\sim}{u}$ and $\underset{\sim}{u}^*$ may be classified [4] into 2-component subvectors $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$, respectively, associated with different element (branch) types, b denoting the bth branch. The elements of $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ may constitute complex conjugate pairs of network variables. In general, $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ constitute node branch variables $\underset{\sim}{x}_m$ and $\underset{\sim}{u}_m$ and line branch variables $\underset{\sim}{x}_t$ and $\underset{\sim}{u}_t$. For example, $\underset{\sim}{x}_m$ may represent node voltages in a typical linear electronic network. In this case the components of $\underset{\sim}{x}_m$ are V_m and V_m^*.

In power networks $\underset{\sim}{x}_m$ and $\underset{\sim}{u}_m$ are further classified [4,6] into vectors associated with load ($\underset{\sim}{x}_\ell$, $\underset{\sim}{u}_\ell$), generator ($\underset{\sim}{x}_g$, $\underset{\sim}{u}_g$) and slack generator ($\underset{\sim}{x}_n$, $\underset{\sim}{u}_n$) branches. An element of $\underset{\sim}{x}_\ell$, $\underset{\sim}{x}_g$, $\underset{\sim}{x}_n$ and $\underset{\sim}{x}_t$ may be defined as V_ℓ, $Q_g + j\delta_g$, $P_n + jQ_n$ and I_t, respectively, where P_m and Q_m are the real and reactive powers associated with bus m and $V_m = |V_m| \angle \delta_m$. A corresponding element of $\underset{\sim}{u}_\ell$, $\underset{\sim}{u}_g$, $\underset{\sim}{u}_n$ and $\underset{\sim}{u}_t$ is defined, respectively, as $P_\ell + jQ_\ell$, $P_g + j|V_g|$, V_n and Y_t.

In general, we write

$$\{\underset{\sim}{x}, \underset{\sim}{x}^*\} = \{\underset{\sim}{x}_b\} = \{\underset{\sim}{x}_m, \underset{\sim}{x}_t\} \tag{3}$$

and

$$\{\underset{\sim}{u}, \underset{\sim}{u}^*\} = \{\underset{\sim}{u}_b\} = \{\underset{\sim}{u}_m, \underset{\sim}{u}_t\}. \tag{4}$$

In this formulation, we have assumed that the number of state or control variables defined is $2n_B$, n_B denoting the number of branches in the network. This assumption is made to simplify the comparison between the Lagrange multiplier and Tellegen's theorem approaches performed in the following sections. Both of these approaches can, however, be applied [2,5] for a general number of state variables.

III. LAGRANGE MULTIPLIER APPROACH

In this approach, we use (2) and its complex conjugate to write the first-order change δf of (1) in the form [5]

$$\delta f = (\underset{\sim}{f}_u - \underset{\sim}{H}_u^T \underset{\sim}{\lambda} - \overline{\underset{\sim}{H}}_u^{*} \overline{\underset{\sim}{\lambda}})^T \delta \underset{\sim}{u} + (\overline{\underset{\sim}{f}}_u - \overline{\underset{\sim}{H}}_u^T \underset{\sim}{\lambda} - \underset{\sim}{H}_u^{*T} \overline{\underset{\sim}{\lambda}})^T \delta \underset{\sim}{u}^*, \tag{5}$$

where $\underset{\sim}{\lambda}$ and $\overline{\underset{\sim}{\lambda}}$ are vectors of Lagrange multipliers obtained by solving the adjoint equations

$$\underset{\sim}{H}_x^T \underset{\sim}{\lambda} + \overline{\underset{\sim}{H}}_x^{*T} \overline{\underset{\sim}{\lambda}} = \underset{\sim}{f}_x, \tag{6a}$$

$$\overline{\underset{\sim}{H}}_x^T \underset{\sim}{\lambda} + \underset{\sim}{H}_x^{*T} \overline{\underset{\sim}{\lambda}} = \overline{\underset{\sim}{f}}_x. \tag{6b}$$

Hence, from (5)

$$\frac{df}{d\underset{\sim}{u}} = \underset{\sim}{f}_u - \underset{\sim}{H}_u^T \underset{\sim}{\lambda} - \overline{\underset{\sim}{H}}_u^{*T} \overline{\underset{\sim}{\lambda}}, \tag{7a}$$

$$\frac{df}{d\underset{\sim}{u}^*} = \overline{\underset{\sim}{f}}_u - \overline{\underset{\sim}{H}}_u^T \underset{\sim}{\lambda} - \underset{\sim}{H}_u^{*T} \overline{\underset{\sim}{\lambda}}. \tag{7b}$$

In practice, we solve the $2n_x$ complex adjoint equations (6) for the Lagrange multipliers $\underset{\sim}{\lambda}$ and $\overline{\underset{\sim}{\lambda}}$ which are then substituted into (7) to obtain the required total formal derivatives of f w.r.t. control variables.

For use later, we now describe the approach in a slightly different way. We employ the classifications of (3) and (4) to define the change of an element-local Lagrangian term as

$$\delta L_b \triangleq (\underset{\sim}{\lambda}^T [\underset{\sim}{h}_{bx} \quad \overline{\underset{\sim}{h}}_{bx}] + \overline{\underset{\sim}{\lambda}}^T [\overline{\underset{\sim}{h}}^*_{bx} \quad \underset{\sim}{h}^*_{bx}]) \, \delta \underset{\sim}{x}_b + (\underset{\sim}{\lambda}^T [\underset{\sim}{h}_{bu} \quad \overline{\underset{\sim}{h}}_{bu}]$$
$$+ \overline{\underset{\sim}{\lambda}}^T [\overline{\underset{\sim}{h}}^*_{bu} \quad \underset{\sim}{h}^*_{bu}]) \, \delta \underset{\sim}{u}_b, \tag{8}$$

where

$$\underset{\sim}{H}_x = [\underset{\sim}{h}_{1x} \ \cdots \ \underset{\sim}{h}_{n_B x}], \tag{9a}$$

$$\overline{\underset{\sim}{H}}_x = [\overline{\underset{\sim}{h}}_{1x} \ \cdots \ \overline{\underset{\sim}{h}}_{n_B x}] \tag{9b}$$

and

$$\underset{\sim}{H}_u \triangleq [\underset{\sim}{h}_{1u} \ \cdots \ \underset{\sim}{h}_{n_B u}], \tag{10a}$$

$$\overline{\underset{\sim}{H}}_u \triangleq [\overline{\underset{\sim}{h}}_{1u} \ \cdots \ \overline{\underset{\sim}{h}}_{n_B u}], \tag{10a}$$

$\underset{\sim}{h}_{bx}$ and $\underset{\sim}{h}_{bu}$ being n_B vectors.

We also define

$$\delta L \triangleq \sum_b \delta L_b, \tag{11}$$

hence, from (2) and (8)

$$\delta L = 0. \tag{12}$$

Using (8), (12) and

$$\delta f = \sum_b ([f_{bx} \quad \overline{f}_{bx}] \, \delta \underset{\sim}{x}_b + [f_{bu} \quad \overline{f}_{bu}] \, \delta \underset{\sim}{u}_b) \tag{13}$$

we may write, from (11)

$$\delta L = \delta f - \sum_b \{([f_{bx} \quad \bar{f}_{bx}] - \underset{\sim}{\lambda}^T [\underset{\sim}{h}_{bx} \quad \underset{\sim}{\bar{h}}_{bx}] - \underset{\sim}{\bar{\lambda}}^T [\underset{\sim}{\bar{h}}^*_{bx} \quad \underset{\sim}{h}^*_{bx}])\, \delta \underset{\sim}{x}_b +$$

$$([f_{bu} \quad \bar{f}_{bu}] - \underset{\sim}{\lambda}^T [\underset{\sim}{h}_{bu} \quad \underset{\sim}{\bar{h}}_{bu}] - \underset{\sim}{\bar{\lambda}}^T [\underset{\sim}{\bar{h}}^*_{bu} \quad \underset{\sim}{h}^*_{bu}])\, \delta \underset{\sim}{u}_b\} . \tag{14}$$

Observe that when $\underset{\sim}{\lambda}$ and $\underset{\sim}{\bar{\lambda}}$ of (14) satisfy (6), namely

$$\underset{\sim}{h}^T_{bx} \underset{\sim}{\lambda} + \underset{\sim}{\bar{h}}^{*T}_{bx} \underset{\sim}{\bar{\lambda}} = f_{bx}, \tag{15a}$$

$$\underset{\sim}{\bar{h}}^T_{bx} \underset{\sim}{\lambda} + \underset{\sim}{h}^{*T}_{bx} \underset{\sim}{\bar{\lambda}} = \bar{f}_{bx}, \tag{15b}$$

then (14) reduces to

$$\delta L = \delta f - \sum_b ([f_{bu} \quad \bar{f}_{bu}]^T - \begin{bmatrix} \underset{\sim}{h}^T_{bu} \\ \underset{\sim}{\bar{h}}^T_{bu} \end{bmatrix} \underset{\sim}{\lambda} - \begin{bmatrix} \underset{\sim}{\bar{h}}^{*T}_{bu} \\ \underset{\sim}{h}^{*T}_{bu} \end{bmatrix} \underset{\sim}{\bar{\lambda}})^T \delta \underset{\sim}{u}_b, \tag{16}$$

hence, from (12)

$$\delta f = \sum_b ([f_{bu} \quad \bar{f}_{bu}]^T - \begin{bmatrix} \underset{\sim}{h}^T_{bu} \\ \underset{\sim}{\bar{h}}^T_{bu} \end{bmatrix} \underset{\sim}{\lambda} - \begin{bmatrix} \underset{\sim}{\bar{h}}^{*T}_{bu} \\ \underset{\sim}{h}^{*T}_{bu} \end{bmatrix} \underset{\sim}{\bar{\lambda}})^T \delta \underset{\sim}{u}_b, \tag{17}$$

so that

$$\frac{df}{d\underset{\sim}{u}_b} = \begin{bmatrix} f_{bu} \\ \bar{f}_{bu} \end{bmatrix} - \begin{bmatrix} \underset{\sim}{h}^T_{bu} \\ \underset{\sim}{\bar{h}}^T_{bu} \end{bmatrix} \underset{\sim}{\lambda} - \begin{bmatrix} \underset{\sim}{\bar{h}}^{*T}_{bu} \\ \underset{\sim}{h}^{*T}_{bu} \end{bmatrix} \underset{\sim}{\bar{\lambda}}, \tag{18}$$

which is a form of (7).

IV. TELLEGEN'S THEOREM APPROACH

In this approach, the application of Tellegen's theorem [4] results in the identity

$$\delta T = 0, \tag{19}$$

where

$$\delta T \triangleq \sum_b \delta T_b, \tag{20}$$

and the element-local Tellegen term δT_b is defined as

$$\delta T_b \triangleq \hat{\eta}_{bx}^T \, \delta x_b + \hat{\eta}_{bu}^T \, \delta u_b \tag{21}$$

and the 2-component vectors $\hat{\eta}_{bx}$ and $\hat{\eta}_{bu}$ are linear functions of the formulated adjoint network current variables $\hat{I}_b$ and voltage variables $\hat{V}_b$ and their complex conjugate. Hence, the $\hat{\eta}_{bx}$ and $\hat{\eta}_{bu}$ are related through Kirchhoff's current and voltage laws formulating a set of network equations. Using (13) and (21), we may write, from (20)

$$\delta T = \delta f - \sum_b \{([f_{bx} \;\; \bar{f}_{bx}] - \hat{\eta}_{bx}^T) \, \delta x_b + ([f_{bu} \;\; \bar{f}_{bu}] - \hat{\eta}_{bu}^T) \, \delta u_b\}. \tag{22}$$

The adjoint network is defined by setting

$$\hat{\eta}_{bx} = \begin{bmatrix} f_{bx} \\ \bar{f}_{bx} \end{bmatrix}, \tag{23}$$

hence (22) reduces to

$$\delta T = \delta f - \sum_b \left(\begin{bmatrix} f_{bu} \\ \bar{f}_{bu} \end{bmatrix} - \hat{\eta}_{bu}\right)^T \delta u_b, \tag{24}$$

from which

$$\frac{df}{du_b} = \begin{bmatrix} f_{bu} \\ \bar{f}_{bu} \end{bmatrix} - \hat{\eta}_{bu} \, . \tag{25}$$

In practice, we formulate the adjoint network using (23) and solve the 2n adjoint network equations to get $\hat{\eta}_{bu}$, which are then substituted into (25) to obtain the required total formal derivatives of f w.r.t. complex control variables.

V. ANALOGY AND COMPARISON

In the last two sections, we have described both the complex Lagrange multiplier and Tellegen's theorem approaches to sensitivity calculations in electrical networks. In this section, we investigate the analogous features of the two approaches and state a general comparison between them.

Both approaches have been applied to both real and complex functions [6]. The application of the Lagrangian approach to complex functions in power networks implies the direct solution of $2n_B \times 2n_B$ complex equations of (6) or (15). For real functions $\bar{\underset{\sim}{f}}_x = \underset{\sim}{f}_x^*$ and it is sufficient to solve either (6a) or (6b) where $\bar{\underset{\sim}{\lambda}} = \underset{\sim}{\lambda}^*$. On the other hand, the application of the Tellegen theorem approach to real function sensitivities implies a consistency analysis [6] which depends on the particular network being analyzed. In order to attain a reasonable comparison between the two approaches, only the case of real functions will be considered. The equations derived are general and the case of complex function sensitivities can be analyzed in a similar, straightforward manner.

We remark on the resemblance between the element-local Lagrangian term δL_b of (8) and the element-local Tellegen term δT_b of (21). We also remark on the resemblance between equation (12) formed to satisfy (2) and equation (19) formed by applying Tellegen's theorem. The δf of (14) and (22) is expressed solely in terms of the control variables via defining, respectively, the adjoint systems (15) and (23). The solution of the adjoint network is then used to obtain the total derivatives $df/d\underset{\sim}{u}_b$ from (18) and (25), respectively.

In the complex Lagrange multiplier approach, the adjoint system of equations to be solved for the adjoint variables (Lagrange multipliers) $\underset{\sim}{\lambda}$ and $\overline{\underset{\sim}{\lambda}}$ constitutes a n_B x $2n_B$ complex matrix of coefficients. In general, when other state variables are defined [2], the order of the matrix of coefficients is determined by the total number of state variables defined. On the other hand, the adjoint system of equations in the Tellegen's theorem approach represents a set of network equations and constitutes only a 2n x 2n real matrix of coefficients, n denoting the number of nodes (or buses) in the original network.

The compactness of the adjoint system formulation in the Tellegen's theorem approach is afforded in essence by realizing, when formulating the adjoint equations, Kirchhoff's relations between the different adjoint variables which constitute a fictitious electrical network.

Assuming that the effort required is divided into formulation and solution parts of the adjoint system, we immediately see that the Tellegen's theorem approach sweeps the major effort into the formulation part and results in only 2n real adjoint equations to be solved. In contrast, the Lagrange multiplier approach requires almost nothing to formulate the adjoint system which then constitutes n_x adjoint equations to be solved.

VI. CONCLUSIONS

The two widely used approaches to sensitivity calculations in electrical networks, namely the Lagrange multiplier and Tellegen's theorem approaches have been described and compared. The description has been performed on a unified basis, where we have defined and employed element-local terms in formulating the two approaches so that different aspects of comparison are clearly investigated. The resemblance in formulating the adjoint systems of the two approaches has been discussed.

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