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GENERALIZED POWER NETWORK SENSITIVITIES

PART II: ANALYSIS

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Abstract

Generalized sensitivity expressions for calculating first-order changes and gradients of functions of interest in different power system applications are derived. We utilize the special complex notation and the transformations between different modes of formulation described in Part I of the paper to compactly derive the required sensitivity expressions. These generalized sensitivity expressions are common to all modes of formulation, e.g., polar and cartesian, common to both real and complex functions and common to all real and complex variables defined in a particular study. The Jacobian matrix of the load flow solution by the Newton-Raphson method is used to define the adjoint system of linear equations required to be solved.

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I. INTRODUCTION

Sensitivity analysis in power system studies [1] attempts to determine (or to estimate) the effect of variations in control variables defined in a particular study on performance functions of interest. According to the type of study performed, variations in control variables can be either pre-specified or post-determined. In contingency studies, for example, the effect of pre-specified variations in control variables such as line admittances and generator powers on system states and other performance functions is determined. This effect is usually estimated using first-order changes supplied via a suitable sensitivity analysis.

In power system studies seeking optimal solutions [2] subject to system constraints, e.g., the economic dispatch and minimum loss problems, post-determined variations in control variables from the base case or a starting point are required. These variations are obtained using suitable constrained optimization techniques to which the gradients of performance functions as well as constraints are supplied. These gradients must be calculated in an efficient way using a suitable sensitivity analysis.

A first step towards a generalized adjoint network approach to power system sensitivity analysis and gradient evaluation has been accomplished in Part I of the paper. Different modes of formulation have been described and transformations to a standard complex mode have been derived.

The standard complex mode of formulation utilizes the conjugate notation to compactly describe the perturbed power network equations. The complex mode will be used to derive the generalized sensitivity expressions in an efficient, straightforward and compact way. With the aid of the transformation formulas derived in Part I, expressions relating the solution of adjoint systems corresponding to different modes can be derived. Hence, the user may solve the adjoint system exploiting the Jacobian of the load flow already available and then obtain the required first-order changes and derivatives of real or complex functions from the generalized sensitivity expressions derived in this part.

II. BACKGROUND

In part I of the paper, we have shown that the set of linear equations to be solved in the well-known Newton-Raphson iterative method can be written compactly in the complex form

$$[\underset{\sim}{K} \quad \underset{\sim}{\bar{K}}] \begin{bmatrix} \delta \underset{\sim}{V}_M \\ \\ \delta \underset{\sim}{V}_M^* \end{bmatrix} = \underset{\sim}{d} \, , \tag{1}$$

where

$$\underset{\sim}{K} = \underset{\sim}{K}_1 + j \, \underset{\sim}{K}_2 \, , \tag{2}$$

$$\underset{\sim}{\bar{K}} = \underset{\sim}{\bar{K}}_1 + j \, \underset{\sim}{\bar{K}}_2 \tag{3}$$

are n x n complex matrices, n denoting the number of buses of the power network,

$$\underset{\sim}{V}_M = \underset{\sim}{V}_{M1} + j \, \underset{\sim}{V}_{M2} \tag{4}$$

is a column vector of the bus voltages, δ denotes first-order change, $*$ denotes the complex conjugate and

$$\underset{\sim}{d} = \underset{\sim}{d}_1 + j \, \underset{\sim}{d}_2 \tag{5}$$

is a vector containing perturbations of control variables of the power system.

Using cartesian coordinates, (1) has the form

$$\underset{\sim}{K}^{crt} \begin{bmatrix} \delta \underset{\sim}{V}_{M1} \\ \\ \delta \underset{\sim}{V}_{M2} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{d}_1 \\ \\ -\underset{\sim}{d}_2 \end{bmatrix}, \tag{6}$$

where the 2n x 2n matrix of coefficients $\underset{\sim}{K}^{crt}$ which constitutes the Jacobian matrix of the load flow problem in rectangular form is given from (47) of Part I.

Using polar coordinates, (1) has the form

$$\underset{\sim}{K}^{plr} \begin{bmatrix} \delta \underset{\sim}{\delta} \\ \\ \delta |\underset{\sim}{V}| \end{bmatrix} = \begin{bmatrix} \underset{\sim}{d}_1 \\ \\ -\underset{\sim}{d}_2 \end{bmatrix}, \tag{7}$$

where $\underset{\sim}{\delta}$ and $|\underset{\sim}{V}|$ are vectors of phase angles and magnitudes of $\underset{\sim}{V}_M$ of (4) and the 2n x 2n matrix of coefficients $\underset{\sim}{K}^{plr}$ which constitutes the Jacobian matrix of the load flow problem in polar form is given from (69) of Part I.

III. SENSITIVITY CALCULATIONS

In this section, we derive the required sensitivity expressions using the compact complex form (1). We exploit the relationships derived in the previous section to provide flexibility in solving the resulting adjoint system of equations in other modes of formulation.

Standard Complex Form

We write (1) in the form

$$\begin{bmatrix} \underset{\sim}{K} & \overline{\underset{\sim}{K}} \\ \\ \overline{\underset{\sim}{K}}^{*} & \underset{\sim}{K}^{*} \end{bmatrix} \begin{bmatrix} \delta \underset{\sim}{V}_M \\ \\ \delta \underset{\sim}{V}_M^{*} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{d} \\ \\ \underset{\sim}{d}^{*} \end{bmatrix}. \tag{8}$$

It can be shown [3] that the matrix of coefficients of (8), denoted by $\underset{\sim}{K}^{cmp}$, has the same rank as that of (6) and the system of equations (8) is consistent if and only if the system (6) is consistent.

For a real function f, we may write, using (12) of Part I,

$$\delta f = [\hat{\underset{\sim}{\mu}}^T \;\; \hat{\underset{\sim}{\mu}}^{*T}] \begin{bmatrix} \delta \underset{\sim}{V}_M \\ \delta \underset{\sim}{V}_M^* \end{bmatrix} + \delta f_\rho , \qquad (9)$$

where we have defined

$$\hat{\underset{\sim}{\mu}} \triangleq \frac{\partial f}{\partial \underset{\sim}{V}_M} \qquad (10)$$

and used

$$\frac{\partial f}{\partial \underset{\sim}{V}_M} = \left(\frac{\partial f}{\partial \underset{\sim}{V}_M^*} \right)^* . \qquad (11)$$

δf_ρ denotes the change in f due to changes in other variables in terms of which f may be explicitly expressed. Hence, from (8)

$$\delta f = [\hat{\underset{\sim}{\mu}}^T \;\; \hat{\underset{\sim}{\mu}}^{*T}] \begin{bmatrix} \underset{\sim}{K} & \overline{\underset{\sim}{K}} \\ \overline{\underset{\sim}{K}}^* & \underset{\sim}{K}^* \end{bmatrix}^{-1} \begin{bmatrix} \underset{\sim}{d} \\ \underset{\sim}{d}^* \end{bmatrix} + \delta f_\rho \qquad (12)$$

or

$$\delta f = [\hat{\underset{\sim}{V}}^T \;\; \hat{\underset{\sim}{V}}^{*T}] \begin{bmatrix} \underset{\sim}{d} \\ \underset{\sim}{d}^* \end{bmatrix} + \delta f_\rho , \qquad (13)$$

where

$$\begin{bmatrix} \underset{\sim}{K}^T & \overline{\underset{\sim}{K}}^{*T} \\ \overline{\underset{\sim}{K}}^T & \underset{\sim}{K}^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}} \\ \hat{\underset{\sim}{V}}^* \end{bmatrix} = \begin{bmatrix} \hat{\underset{\sim}{\mu}} \\ \hat{\underset{\sim}{\mu}}^* \end{bmatrix} \qquad (14)$$

or, simply

$$[\underset{\sim}{K}^T \quad \underset{\sim}{\bar{K}}^{*T}] \begin{bmatrix} \hat{\underset{\sim}{V}} \\ \hat{\underset{\sim}{V}}^* \end{bmatrix} = \hat{\underset{\sim}{\mu}} \ . \tag{15}$$

Hence, the first-order change of the real function f and corresponding gradients can be evaluated by solving (14) and substituting into (13).

Cartesian Coordinates

Similarly to (9), we may write, using the rectangular formulation

$$\delta f = [\hat{\underset{\sim}{\mu}}_r^T \quad \hat{\underset{\sim}{\mu}}_s^T] \begin{bmatrix} \delta \underset{\sim}{V}_{M1} \\ \delta \underset{\sim}{V}_{M2} \end{bmatrix} + \delta f_\rho \ , \tag{16}$$

where we have defined

$$\hat{\underset{\sim}{\mu}}_r \triangleq \frac{\partial f}{\partial \underset{\sim}{V}_{M1}} \tag{17}$$

and

$$\hat{\underset{\sim}{\mu}}_s \triangleq \frac{\partial f}{\partial \underset{\sim}{V}_{M2}} \ . \tag{18}$$

Hence, from (47) of Part I

$$\delta f = [\hat{\underset{\sim}{V}}_r^T \quad \hat{\underset{\sim}{V}}_s^T] \begin{bmatrix} \underset{\sim}{d}_1 \\ -\underset{\sim}{d}_2 \end{bmatrix} + \delta f_\rho \ , \tag{19}$$

where

$$\begin{bmatrix} (\underset{\sim}{K}_1+\underset{\sim}{\bar{K}}_1)^T & -(\underset{\sim}{K}_2+\underset{\sim}{\bar{K}}_2)^T \\ (-\underset{\sim}{K}_2+\underset{\sim}{\bar{K}}_2)^T & (-\underset{\sim}{K}_1+\underset{\sim}{\bar{K}}_1)^T \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_r \\ \hat{\underset{\sim}{V}}_s \end{bmatrix} = \begin{bmatrix} \hat{\underset{\sim}{\mu}}_r \\ \hat{\underset{\sim}{\mu}}_s \end{bmatrix} . \tag{20}$$

Observe that the matrix of coefficients of (20) is the transpose of the Jacobian matrix of the load flow problem in rectangular form (6).

Theorem 1

(a) The solution vectors $\hat{\underset{\sim}{V}}_r$ and $\hat{\underset{\sim}{V}}_s$ of the adjoint system of equations (20) are given by

$$\hat{\underset{\sim}{V}}_r = 2\ \mathrm{Re}\{\hat{\underset{\sim}{V}}\}$$

and

$$\hat{\underset{\sim}{V}}_s = 2\ \mathrm{Im}\{\hat{\underset{\sim}{V}}\}.$$

where $\hat{\underset{\sim}{V}}$ is given from (14).

(b) The RHS vectors $\hat{\underset{\sim}{\mu}}_r$ and $\hat{\underset{\sim}{\mu}}_s$ of the adjoint system of equations (20) are given by

$$\hat{\underset{\sim}{\mu}} = \underset{\sim}{L}_1^T\ \hat{\underset{\sim}{\mu}}_r + \underset{\sim}{L}_2^T\ \hat{\underset{\sim}{\mu}}_s,$$

where $\hat{\underset{\sim}{\mu}}$ is given by (10) and $\underset{\sim}{L}_1$ and $\underset{\sim}{L}_2$ are given by (43) of Part I.

Proof

Comparing (13) and (19), and using (5), we get

$$\hat{\underset{\sim}{V}} = (\hat{\underset{\sim}{V}}_r + j\ \hat{\underset{\sim}{V}}_s)/2\ . \tag{21}$$

From (21), the first part of the theorem is proved. Now, multiplying (20) from the left by the transpose of $\underset{\sim}{L}^q$ of (43) of Part I and using the relation

$$2\begin{bmatrix} (\underset{\sim}{K}_1 + \overline{\underset{\sim}{K}}_1)^T & -(\underset{\sim}{K}_2 + \overline{\underset{\sim}{K}}_2)^T \\ (-\underset{\sim}{K}_2 + \overline{\underset{\sim}{K}}_2)^T & (-\underset{\sim}{K}_1 + \overline{\underset{\sim}{K}}_1)^T \end{bmatrix} = \begin{bmatrix} \underset{\sim}{K}^{qT} & \underset{\sim}{K}^{q*T} \\ \overline{\underset{\sim}{K}}^{qT} & \overline{\underset{\sim}{K}}^{q*T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{j} \\ \underset{\sim}{1} & -\underset{\sim}{j} \end{bmatrix} , \qquad (22)$$

it follows from (52) of Part I and (21) that

$$\begin{bmatrix} \underset{\sim}{K}^T & \overline{\underset{\sim}{K}}^{*T} \\ \overline{\underset{\sim}{K}}^T & \underset{\sim}{K}^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}} \\ \hat{\underset{\sim}{V}}^{*} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{L}_1^T & \underset{\sim}{L}_2^T \\ \underset{\sim}{L}_1^{*T} & \underset{\sim}{L}_2^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{\mu}}_r \\ \hat{\underset{\sim}{\mu}}_s \end{bmatrix} , \qquad (23)$$

hence, from (14)

$$\begin{bmatrix} \hat{\underset{\sim}{\mu}} \\ \hat{\underset{\sim}{\mu}}^{*} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{L}_1^T & \underset{\sim}{L}_2^T \\ \underset{\sim}{L}_1^{*T} & \underset{\sim}{L}_2^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{\mu}}_r \\ \hat{\underset{\sim}{\mu}}_s \end{bmatrix} \qquad (24)$$

or, simply

$$\hat{\underset{\sim}{\mu}} = [\underset{\sim}{L}_1^T \quad \underset{\sim}{L}_2^T] \begin{bmatrix} \hat{\underset{\sim}{\mu}}_r \\ \hat{\underset{\sim}{\mu}}_s \end{bmatrix} \qquad \blacksquare \qquad (25)$$

The relationship (25) could also be derived by applying, formally, the chain rule of differentiation using the definitions (10), (17) and (18).

Observe that equation (21) relates the solution of the adjoint system (20) to that of (15), and equation (25) relates the RHS of (20) to that of (15).

Polar Coordinates

Using the polar formulation, we may write

$$\delta f = [\hat{\underset{\sim}{\mu}}_s^T \quad \hat{\underset{\sim}{\mu}}_v^T] \begin{bmatrix} \delta \underset{\sim}{\delta} \\ \\ \delta |\underset{\sim}{V}| \end{bmatrix} + \delta f_\rho \ , \tag{26}$$

where we have defined

$$\hat{\underset{\sim}{\mu}}_\sigma \triangleq \frac{\partial f}{\partial \underset{\sim}{\delta}} \tag{27}$$

and

$$\hat{\underset{\sim}{\mu}}_v \triangleq \frac{\partial f}{\partial |\underset{\sim}{V}|} \ . \tag{28}$$

Hence, from (7)

$$\delta f = [\hat{\underset{\sim}{V}}_\delta^T \quad \hat{\underset{\sim}{V}}_v^T] \begin{bmatrix} \underset{\sim}{d}_1 \\ \\ -\underset{\sim}{d}_2 \end{bmatrix} + \delta f_\rho \ , \tag{29}$$

where

$$\begin{bmatrix} \underset{\sim}{K}_1^{PT} & -\underset{\sim}{K}_2^{PT} \\ \\ \overline{\underset{\sim}{K}}_1^{PT} & -\overline{\underset{\sim}{K}}_2^{PT} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_\delta \\ \\ \hat{\underset{\sim}{V}}_v \end{bmatrix} = \begin{bmatrix} \hat{\underset{\sim}{\mu}}_\delta \\ \\ \hat{\underset{\sim}{\mu}}_v \end{bmatrix} \ . \tag{30}$$

The matrix of coefficients of (30) is the transpose of the Jacobian matrix of the load flow problem in the polar form [4].

Theorem 2

(a) The solution vectors $\hat{\underset{\sim}{V}}_\delta$ and $\hat{\underset{\sim}{V}}_v$ of the adjoint system of equations (30) are given by

$$\hat{\underset{\sim}{V}}_\delta = 2 \ \mathrm{Re}\{\hat{\underset{\sim}{V}}\}$$

and

$$\hat{\underset{\sim}{V}}_v = 2\ \mathrm{Im}\{\hat{\underset{\sim}{V}}\},$$

where $\hat{\underset{\sim}{V}}$ is given from (14).

(b) The RHS vectors $\hat{\underset{\sim}{\mu}}_\delta$ and $\hat{\underset{\sim}{\mu}}_v$ of the adjoint system of equations (30) are given by

$$\hat{\underset{\sim}{\mu}} = \underset{\sim}{L}_\delta^T\, \hat{\underset{\sim}{\mu}}_\delta + \underset{\sim}{L}_v^T\, \hat{\underset{\sim}{\mu}}_v\ ,$$

where $\hat{\underset{\sim}{\mu}}$ is given by (10) and $\underset{\sim}{L}_\delta$ and $\underset{\sim}{L}_v$ are given by (59) and (60) of Part I.

Proof

Comparing (13) and (29), and using (5), we get

$$\hat{\underset{\sim}{V}} = (\hat{\underset{\sim}{V}}_\delta + j\ \hat{\underset{\sim}{V}}_v)/2\ . \tag{31}$$

From (31), the first part of the theorem is proved. Now, multiplying (30) from left by the transpose of $\underset{\sim}{L}^p$ of (58) of Part I and using the relation

$$2\begin{bmatrix} \underset{\sim}{K}_1^{PT} & -\underset{\sim}{K}_2^{PT} \\ \bar{\underset{\sim}{K}}_1^{PT} & -\bar{\underset{\sim}{K}}_2^{PT} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{K}^{PT} & \underset{\sim}{K}^{P*T} \\ \bar{\underset{\sim}{K}}^{PT} & \bar{\underset{\sim}{K}}^{P*T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{j} \\ \underset{\sim}{1} & -\underset{\sim}{j} \end{bmatrix}, \tag{32}$$

it follows from (70) of Part I and (96) that

$$\begin{bmatrix} \underset{\sim}{K}^T & \bar{\underset{\sim}{K}}^{*T} \\ \bar{\underset{\sim}{K}}^T & \underset{\sim}{K}^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}} \\ \hat{\underset{\sim}{V}}^* \end{bmatrix} = \begin{bmatrix} \underset{\sim}{L}_\delta^T & \underset{\sim}{L}_v^T \\ \underset{\sim}{L}_\delta^{*T} & \underset{\sim}{L}_v^{*T} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{\mu}}_\delta \\ \hat{\underset{\sim}{\mu}}_v \end{bmatrix}, \tag{33}$$

hence, from (24)

$$\begin{bmatrix} \underset{\sim}{\hat{\mu}} \\ \underset{\sim}{\hat{\mu}}^{*} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{L}_{\delta}^{T} & \underset{\sim}{L}_{v}^{T} \\ \underset{\sim}{L}_{\delta}^{*T} & \underset{\sim}{L}_{v}^{*T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\hat{\mu}}_{\delta} \\ \underset{\sim}{\hat{\mu}}_{v} \end{bmatrix} \tag{34}$$

or, simply

$$\underset{\sim}{\hat{\mu}} = [\underset{\sim}{L}_{\delta}^{T} \quad \underset{\sim}{L}_{v}^{T}] \begin{bmatrix} \underset{\sim}{\hat{\mu}}_{\delta} \\ \underset{\sim}{\hat{\mu}}_{v} \end{bmatrix} \tag{35}$$

[redacted]

Again, the relationship (35) could also be derived by applying, formally, the chain rule of differentiation using the definitions (10), (27) and (28).

Equation (31) relates the solution of the adjoint system (30) to that of (15), and equation (35) relates the RHS of (30) to that of (15).

Remarks

We remark that using (21) or (31), the adjoint system can be formulated and solved in a convenient mode, preferably the same formulation as the original load flow problem, while the first-order change of f and corresponding gradients may be derived compactly using the adjoint variables $\underset{\sim}{\hat{V}}$. On the other hand, the relations (25) and (35) allow the use of more elegant formal derivatives which, in many cases, facilitate the formulation. For example, consider the function

$$f = \sigma |V_i - V_j|^2 = \sigma (V_i - V_j)(V_i^* - V_j^*), \tag{36}$$

where V_i and V_j are the ith and jth components of $\underset{\sim}{V}_M$, respectively, and σ is a real scalar or variable. Note that f of (36) may represent, for example, the power loss in line ij. For the polar formulation, $\underset{\sim}{\hat{\mu}}_v$ and $\underset{\sim}{\hat{\mu}}_\delta$ of (30) are calculated as follows. The ith and jth components of $\underset{\sim}{\hat{\mu}}_\delta$

and $\hat{\underset{\sim}{\mu}}_V$ are given by

$$\hat{\mu}_{\delta i} = \sigma[-2\ (|V_i|\cos\delta_i - |V_j|\cos\delta_j)\,|V_i|\sin\delta_i$$
$$+2\ (|V_i|\sin\delta_i - |V_j|\sin\delta_j)\,|V_i|\cos\delta_i],$$

$$\hat{\mu}_{\delta j} = \sigma[\ 2\ (|V_i|\cos\delta_i - |V_j|\cos\delta_j)\,|V_j|\sin\delta_j$$
$$-2\ (|V_i|\sin\delta_i - |V_j|\sin\delta_j)\,|V_j|\cos\delta_j],$$

$$\hat{\mu}_{vi} = \sigma[\ 2\ (|V_i|\cos\delta_i - |V_j|\cos\delta_j)\cos\delta_i$$
$$+2\ (|V_i|\sin\delta_i - |V_j|\sin\delta_j)\sin\delta_i]$$

and

$$\hat{\mu}_{vj} = \sigma[\ 2\ (|V_i|\cos\delta_i - |V_j|\cos\delta_j)\cos\delta_j$$
$$-2\ (|V_i|\sin\delta_i - |V_j|\sin\delta_j)\sin\delta_j].$$

All other components are zero. On the other hand, one may calculate

$$\hat{\underset{\sim}{\mu}} = \sigma \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ (V_i^* - V_j^*) \\ \cdot \\ \cdot \\ \cdot \\ -(V_i^* - V_j^*) \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

and use (34) to calculate $\hat{\underset{\sim}{\mu}}_V$ and $\hat{\underset{\sim}{\mu}}_\delta$, where $(\underset{\sim}{L}^{pT})^{-1}$ is the transpose of $(\underset{\sim}{L}^p)^{-1}$ of (63) of Part I. In this example, the derivation of the formal derivatives is clearly easier.

We also remark that other forms of power flow equations can be handled in a similar way. The previous theorems can be easily generalized for other formulations provided that transformations similar to (43) and (58) of Part I are defined.

We illustrate the foregoing concepts by the two simple examples considered in Part I.

Example 1

For the first system as shown in Fig. 1 of Part I, the load flow solution is given by

$$V_1 = 0.7352 - j\,0.2041$$

and

$$S_2 = 5.6705 + j\,1.0706.$$

S_2 is the injected power at bus 2.

The Jacobian matrix of the complex form (1) is given by

$$[\underset{\sim}{K}\ \overline{\underset{\sim}{K}}] = \begin{bmatrix} (8.0852-j12.0097) & (-8.4934+13.4802) & (-5.2623+j5.5411) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

of the cartesian form (6) by

$$\underset{\sim}{K}^{crt} = \begin{bmatrix} 2.8229 & -8.4934 & 17.5508 & -13.4802 \\ 0 & 1 & 0 & 0 \\ 6.4686 & -13.4802 & -13.3475 & 8.4934 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and of the polar form (7) by

$$\underset{\sim}{K}^{plr} = \begin{bmatrix} 13.4802 & -13.4802 & -1.9745 & -8.4934 \\ 0 & 0 & 0 & 1 \\ -8.4934 & 8.4934 & 9.8031 & -13.4802 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now, consider the function

$$f = |V_1|^2 = V_1 V_1^* .$$

From (10),

$$\hat{\underset{\sim}{\mu}} = \begin{bmatrix} V_1^* \\ 0 \end{bmatrix} = \begin{bmatrix} 0.7352 + j0.2041 \\ 0 \end{bmatrix},$$

and (15) has the solution

$$\hat{\underset{\sim}{V}} = \begin{bmatrix} 0.0562 + j0.0892 \\ 1.6788 + j0.0 \end{bmatrix}.$$

Also, for the polar formulation, we have from (27) and (28)

$$\hat{\underset{\sim}{\mu}}_\delta = \underset{\sim}{0}$$

and

$$\hat{\underset{\sim}{\mu}}_V = \begin{bmatrix} 2|V_1| \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5261 \\ 0 \end{bmatrix},$$

and (30) has the solution

$$\hat{\underset{\sim}{V}}_\delta = \begin{bmatrix} 0.1123 \\ 3.3577 \end{bmatrix}$$

and

$$\hat{\underset{\sim}{V}}_V = \begin{bmatrix} 0.1783 \\ 0 \end{bmatrix}.$$

Note that the $\hat{\underset{\sim}{V}}_\delta$ and $\hat{\underset{\sim}{V}}_V$ obtained for the polar formulation and $\hat{\underset{\sim}{V}}$ satisfy (31).

Example 2

For the second system as shown in Fig. 2 of Part I, the load flow solution is given by

$$\delta_1 = -0.1995 \text{ rad},$$

$$Q_1 = 1.9929$$

and

$$S_2 = 4.2742 - j1.7131.$$

The Jacobian matrix of the complex form (1) is given by

$$[\underset{\sim}{K}\ \bar{\underset{\sim}{K}}] = \begin{bmatrix} (2.3920-j9.4199) & (-4.4300+j8.2864) & (2.1938+j8.4398) & (-4.4300-j8.2864) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

of the cartesian form (6) by

$$\underset{\sim}{K}^{crt} = \begin{bmatrix} 4.5858 & -8.8600 & 17.8597 & -16.5729 \\ 0 & 1 & 0 & 0 \\ 0.9802 & 0 & -0.1982 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and of the polar form (7) by

$$\underset{\sim}{K}^{plr} = \begin{bmatrix} 16.5729 & -16.5729 & 0.9556 & -8.8600 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now, consider the function

$$f = \delta_1 = \tan^{-1} \left[\frac{V_1 + V_1^*}{j(V_1^* - V_1)}\right] .$$

From (10)

$$\hat{\underset{\sim}{\mu}} = \begin{bmatrix} -j/(2V_1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1101 - j0.5445 \\ 0 \end{bmatrix}$$

and (15) has the solution

$$\hat{\underset{\sim}{V}} = \begin{bmatrix} 0.0302 - j0.0288 \\ 0.2673 + j0.5 \end{bmatrix}$$

Also, for the polar formulation, we have from (27) and (28)

$$\hat{\underset{\sim}{\mu}}_\delta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\hat{\underset{\sim}{\mu}}_V = \underset{\sim}{0}$$

and (30) has the solution

$$\hat{\underset{\sim}{V}}_{\delta} = \begin{bmatrix} 0.0603 \\ 0.5346 \end{bmatrix}$$

and

$$\hat{\underset{\sim}{V}}_{v} = \begin{bmatrix} -0.0577 \\ 1.0 \end{bmatrix} .$$

Observe that the obtained $\hat{\underset{\sim}{V}}_{\delta}$ and $\hat{\underset{\sim}{V}}_{v}$ for the polar formulation and $\hat{\underset{\sim}{V}}$ satisfy (31).

IV. GRADIENT CALCULATIONS

In the previous section, we have derived the adjoint systems in different modes of formulation and investigated the relationships between the corresponding excitation and solution vectors. In power system studies such as contingency analysis, the first-order change of f is of prime interest. The first-order change δf can be calculated from (13), (19) and (29). On the other hand, the derivatives of f w.r.t. control variables are required to be calculated, for example, in planning studies.

In the following, we consider the buses to be ordered such that subscripts $\ell=1, 2, \ldots, n_L$ identify load buses, $g = n_L+1, \ldots, n_L + n_G$ identify generator buses and $n = n_L + n_G+1$ identifies the slack bus.

The vector $\underset{\sim}{d}$ of (1) is now partitioned into subvectors associated with the sets of load, generator and slack buses of appropriate dimension in the form

$$\underset{\sim}{d} = \begin{bmatrix} \underset{\sim}{d}_L \\ \underset{\sim}{d}_G \\ d_n \end{bmatrix} , \qquad (37)$$

where $\underset{\sim}{d}_L$ has elements d_ℓ given from (26) of Part I by

$$d_\ell = \delta S_\ell^* - V_\ell^* \underset{\sim}{V}_M^T \delta \underset{\sim}{y}_\ell , \tag{38}$$

$\underset{\sim}{y}_\ell^T$ representing the corresponding row of the bus admittance matrix $\underset{\sim}{Y}_T$, $\underset{\sim}{d}_G$ has elements d_g given by (41) of Part I and d_n is δV_n^* from (27) of Part I. Also, the vector $\hat{\underset{\sim}{V}}$ of (13) is partitioned correspondingly in the form

$$\hat{\underset{\sim}{V}} = \begin{bmatrix} \hat{\underset{\sim}{V}}_L \\ \underset{\sim}{V}_G \\ \hat{V}_n \end{bmatrix} . \tag{39}$$

Note that the above formulation leads to expressing the vector $\underset{\sim}{d}$ solely in terms of variations in control variables, the gradients in terms of which can be obtained by writing (13) in the form

$$\delta f = \hat{\underset{\sim}{V}}_L^T \underset{\sim}{d}_L + \hat{\underset{\sim}{V}}_G^T \underset{\sim}{d}_G + \hat{V}_n d_n + \left(\frac{\partial f}{\partial \underset{\sim}{\rho}}\right)^T \delta \underset{\sim}{\rho}$$

$$+ \hat{\underset{\sim}{V}}_L^{*T} \underset{\sim}{d}_L^* + \hat{\underset{\sim}{V}}_G^{*T} \underset{\sim}{d}_G^* + \hat{V}_n^* d_n^* + \left(\frac{\partial f}{\partial \underset{\sim}{\rho}}\right)^{*T} \delta \underset{\sim}{\rho}^* . \tag{40}$$

The first term of (40) is given, using (38), by

$$\hat{\underset{\sim}{V}}_L^T \underset{\sim}{d}_L = \sum_{\ell=1}^{n_L} \hat{V}_\ell d_\ell$$

$$= \sum_{\ell=1}^{n_L} (\hat{V}_\ell \delta S_\ell^*) - \sum_{\ell=1}^{n_L} \sum_{m=1}^{n} (\hat{V}_\ell V_\ell^* V_m \delta Y_{\ell m}) , \tag{41}$$

where $Y_{\ell m}$ is an element of $\underset{\sim}{Y}_T$, which is assumed, for simplicity, to be a symmetric admittance matrix, or

$$\hat{\underset{\sim}{V}}_L^T \underset{\sim}{d}_L = \sum_{\ell=1}^{n_L} (\hat{V}_\ell \delta S_\ell^*) + \sum_{\ell=1}^{n_L} \sum_{\substack{m=1 \\ m\neq\ell}}^{n} \hat{V}_\ell V_\ell^* (V_m - V_\ell) \delta y_{\ell m}$$

$$- \sum_{\ell=1}^{n} (\hat{V}_{\ell} \; V_{\ell}^{*} \; V_{\ell} \; \delta y_{\ell 0}) \; , \tag{42}$$

where $y_{\ell m}$ denotes the admittance of line ℓm connecting load bus ℓ with bus m (=ℓ, g or n), and $y_{\ell 0}$ is the shunt admittance at bus ℓ. The second term of (40) is given, using (41) of Part I by

$$\hat{\underset{\sim}{V}}_{G}^{T} \; \underset{\sim}{d}_{G} = \sum_{g=n_L+1}^{n-1} \hat{V}_g \; d_g$$

$$= \sum_{g=n_L+1}^{n-1} \hat{V}_g \; (\delta P_g - j \; \delta|V_g|)$$

$$- \sum_{g=n_L+1}^{n-1} \sum_{m=1}^{n} \hat{V}_g \; \mathrm{Re} \; \{ V_g^{*} \; V_m \; \delta Y_{gm} \} \tag{43}$$

or

$$\hat{\underset{\sim}{V}}_{G}^{T} \; \underset{\sim}{d}_{G} = \sum_{g=n_L+1}^{n-1} \hat{V}_g (\delta P_g - j\delta|V_g|) + \sum_{g=n_L+1}^{n-1} \sum_{\substack{m=1 \\ m \neq g}}^{n} \hat{V}_g \; \mathrm{Re} \; \{V_g^{*} (V_m - V_g) \; \delta y_{gm}\}$$

$$- \sum_{g=n_L+1}^{n-1} \hat{V}_g \; \mathrm{Re} \; \{V_g^{*} \; V_g \; \delta y_{g0}\} \; , \tag{44}$$

where y_{gm} denotes the admittance of line gm connecting generator bus g with bus m (=ℓ, g or n), and y_{g0} is the shunt admittance at bus g. The third term of (40) is given, using (27) of Part I by

$$\hat{V}_n \; d_n = \hat{V}_n \; \delta V_n^{*} \; . \tag{45}$$

The fourth term of (40) is simply the first-order change of f due to changes in other variables $\underset{\sim}{\rho}$ in terms of which the function f may be explicitly expressed.

Equations (42), (44) and (45) provide useful information for gradient evaluation since they provide direct expressions w.r.t. the control variables of interest. The derivatives of the function f w.r.t. the control variables are obtained as follows, where we temporarily assume that $\underset{\sim}{\rho}$ does not contain such control variables.

Load Bus Control Variables

From (42) and its complex conjugate, the derivatives of f w.r.t. the demand S_ℓ and S_ℓ^* at load bus ℓ is given by

$$\frac{df}{dS_\ell} = \hat{V}_\ell^* \tag{46}$$

and

$$\frac{df}{dS_\ell^*} = \hat{V}_\ell \ . \tag{47}$$

Generator Bus Control Variables

From (44) and its complex conjugate, the derivatives of f w.r.t. the real generated power P_g and the voltage magnitude $|V_g|$ at generator bus g are given by

$$\frac{df}{d\tilde{S}_g} = \hat{V}_g^* \tag{48}$$

and

$$\frac{df}{d\tilde{S}_g^*} = \hat{V}_g \ , \tag{49}$$

$\tilde{S}_g$ is given by (30) of Part I, namely

$$\tilde{S}_g = P_g + j\,|V_g| \ . \tag{50}$$

Slack Bus Control Variables

From (45) and its complex conjugate, the derivatives of f w.r.t. the slack bus voltage V_n and V_n^* are given by

$$\frac{df}{dV_n} = \hat{V}_n^* \tag{51}$$

and

$$\frac{df}{dV_n^*} = \hat{V}_n \ . \tag{52}$$

In practice, the phase angle of the slack bus voltage is set to zero as a reference angle. Hence, the slack bus has only one effective real control variable.

<u>Line Control Variables</u>

The derivatives of f w.r.t. line control variables $y_{ij`}$ can be obtained from (42) and (44) and their complex conjugate as follows. For $y_{\ell\ell`}$, between load buses ℓ and $\ell`$, we have from (42) and its complex conjugate

$$\frac{df}{dy_{\ell\ell`}} = (\hat{V}_\ell V_\ell^* - \hat{V}_{\ell`} V_{\ell`}^*)\ (V_{\ell`} - V_\ell) \tag{53}$$

and

$$\frac{df}{dy_{\ell\ell`}^*} = (\hat{V}_\ell^* \hat{V}_\ell - \hat{V}_{\ell`}^* V_{\ell`})\ (V_{\ell`}^* - V_\ell^*) \ . \tag{54}$$

For $y_{\ell 0}$ between load bus ℓ and ground, we have from (42) and its complex conjugate

$$\frac{df}{dy_{\ell 0}} = - \hat{V}_\ell V_\ell^* V_\ell \tag{55}$$

and

$$\frac{df}{dy_{\ell 0}^*} = - \hat{V}_\ell^* V_\ell V_\ell^* \ . \tag{56}$$

For $y_{gg`}$ between generator buses g and g`, we have from (44) and its complex conjugate

$$\frac{df}{dy_{gg`}} = (\hat{V}_{g1} V_g^* - \hat{V}_{g`1} V_{g`}^*)\ (V_{g`} - V_g) \tag{57}$$

and

$$\frac{df}{dy^*_{gg'}} = (\hat{V}_{g1} V_g - \hat{V}_{g'1} V_{g'}) (V^*_{g'} - V^*_g) , \tag{58}$$

where

$$\hat{V}_m = \hat{V}_{m1} + j \hat{V}_{m2} \tag{59}$$

and m is a bus index. For y_{g0} between generator bus g and ground, we have from (44)

$$\frac{df}{dy_{g0}} = \frac{df}{dy^*_{g0}} = - \hat{V}_{g1} V^*_g V_g . \tag{60}$$

For $y_{\ell g}$ between load bus ℓ and generator bus g, we have from (42) and (44) and their complex conjugate

$$\frac{df}{dy_{\ell g}} = (\hat{V}_{g1} V^*_g - \hat{V}_\ell V^*_\ell) (V_\ell - V_g) \tag{61}$$

and

$$\frac{df}{dy^*_{\ell g}} = (\hat{V}_{g1} V_g - \hat{V}^*_\ell V_\ell) (V^*_\ell - V^*_g) . \tag{62}$$

For $y_{\ell n}$ between load bus ℓ and the slack bus n, we have from (42) and its complex conjugate

$$\frac{df}{dy_{\ell n}} = \hat{V}_\ell V^*_\ell (V_n - V_\ell) \tag{63}$$

and

$$\frac{df}{dy^*_{\ell n}} = \hat{V}^*_\ell V_\ell (V^*_n - V^*_\ell) . \tag{64}$$

Finally, for y_{gn} between generator bus g and the slack bus n, we have from (44) and its complex conjugate

$$\frac{df}{dy_{gn}} = \hat{V}_{g1} V^*_g (V_n - V_g) \tag{65}$$

and

$$\frac{df}{dy_{gn}^*} = \hat{V}_{g1} \; V_g \; (V_n^* - V_g^*) \; . \tag{66}$$

Special Considerations

If $\underset{\sim}{\rho}$ of (40) contains some of the above control variables, the partial derivatives of f w.r.t. appropriate control variables must be added to the expressions obtained.

When any of the control variables u_k is a function of some real design variables we write

$$\delta u_k = \sum_i \frac{\partial u_k}{\partial \zeta_{ki}} \; \Delta\zeta_{ki} \; , \tag{67}$$

where ζ_{ki} is the ith design variable associated with u_k and $\Delta\zeta_{ki}$ denotes the change in ζ_{ki}. Hence,

$$\frac{df}{d\zeta_{ki}} = \frac{df}{du_k} \frac{\partial u_k}{\partial \zeta_{ki}} \; . \tag{68}$$

The control variables associated with other power system components, e.g., transformers, which are represented in the bus admittance matrix $\underset{\sim}{Y}_T$ can be easily considered. The corresponding sensitivity expressions may be derived in a similar straightforward manner.

Equations (46)-(49), (51)-(56) and (60)-(66) compactly define the required formal derivatives of the real function f w.r.t. complex control variables. In practice, gradients w.r.t. real and imaginary parts of the defined control variables are of direct interest. These gradients are simply obtained from

$$\frac{df}{du_{k1}} = 2 \text{ Re } \{\frac{df}{du_k}\} \tag{69}$$

and

$$\frac{df}{du_{k2}} = -2 \text{ Im } \{\frac{df}{du_k}\} , \tag{70}$$

where the complex control variable u_k is given by

$$u_k = u_{k1} + j\ u_{k2} . \tag{71}$$

Table I summarizes the derived expressions of function gradients w.r.t. real control variables of practical interest.

Example 3

Using the values of $\hat{\underset{\sim}{V}}$ obtained, we have for the first system

$$\frac{df}{dP_1} = 2\ \hat{V}_{11} = 0.1123 ,$$

$$\frac{df}{dQ_1} = 2\ \hat{V}_{12} = 0.1783 ,$$

$$\frac{df}{dV_{21}} = 2\ \hat{V}_{21} = 3.3577 ,$$

$$\frac{df}{B_{10}} = 2|V_1|^2\ \hat{V}_{12} = 0.1038 ,$$

$$\frac{df}{dG_{12}} = 2 \text{ Re } \{\hat{V}_1 V_1\ (V_2 - V_1)\} = -0.0192 ,$$

and

$$\frac{df}{dB_{12}} = -2 \text{ Im } \{\hat{V}_1 V_1^*\ (V_2 - V_1)\} = -0.0502 ,$$

where $G_{mm`}$ and $B_{mm`}$ denote, respectively, the conductance and susceptance of line mm` connecting buses m and m`, m`=0 denotes the ground.

Example 4

For the second system, we have

$$\frac{df}{dP_1} = 2\ \hat{V}_{11} = 0.0603\ ,$$

$$\frac{df}{d|V_1|} = 2\ \hat{V}_{12} = -0.0577\ ,$$

$$\frac{df}{dV_{21}} = 2\ \hat{V}_{21} = 0.5346\ ,$$

$$\frac{df}{dB_{10}} = 0.0\ ,$$

$$\frac{df}{dG_{12}} = 2\ \hat{V}_{11}\ \mathrm{Re}\ \{V_1^*\ (V_2 - V_1)\} = 0.0044$$

and

$$\frac{df}{dB_{12}} = -2\ \hat{V}_{11}\ \mathrm{Im}\ \{V_1^* V_2\} = -0.0108.$$

The gradients obtained can be easily checked by small perturbations about the base case values.

V. SENSITIVITY OF COMPLEX FUNCTIONS

In the previous sections, we have derived the required sensitivity expressions and gradients for a general real function. The relationships between different modes of formulation have been investigated and expressions relating the RHS and solution vector of corresponding adjoint systems have been derived.

The sensitivities of a general complex function can be obtained using the previous formulas derived simply by considering the real and imaginary parts separately. In this case, only the RHS of the adjoint system of equations has to be changed. In other words, only one forward and one backward substitutions are required for each real function, provided that the LU factors of the formed matrix of coefficients are stored and that the base case point remains unchanged.

In this section, we show how the compact complex formulation can be exploited to formulate the adjoint system corresponding to a general complex function and to derive the required sensitivities. The relationships between different modes of formulation are again investigated for the complex function case.

For a complex function f, we may write, using (12) of Part I

$$\delta f = [\hat{\mu}^T \quad \hat{\bar{\mu}}^T] \begin{bmatrix} \delta V_M \\ \\ \delta V_M^* \end{bmatrix} + \delta f_\rho \quad , \tag{72}$$

where we have defined

$$\hat{\mu} \triangleq \frac{\partial f}{\partial V_M} \tag{73}$$

and

$$\hat{\bar{\mu}} \triangleq \frac{\partial f}{\partial V_M^*}, \tag{74}$$

δf_ρ being the change in f due to changes in other variables in terms of which f may be explicitly expressed. Hence, from (8)

$$\delta f = [\hat{\mu}^T \quad \hat{\bar{\mu}}^T] \begin{bmatrix} K & \bar{K} \\ \\ \bar{K}^* & K^* \end{bmatrix}^{-1} \begin{bmatrix} d \\ \\ d^* \end{bmatrix} + \delta f_\rho \tag{75}$$

or

$$\delta f = [\hat{V}^T \quad \hat{\bar{V}}^T] \begin{bmatrix} d \\ \\ d^* \end{bmatrix} + \delta f_\rho \quad , \tag{76}$$

where

$$\begin{bmatrix} \underset{\sim}{K}^T & \underset{\sim}{\overline{K}}^{*T} \\ \underset{\sim}{\overline{K}}^T & \underset{\sim}{K}^{*T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\hat{V}} \\ \underset{\sim}{\hat{\overline{V}}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\hat{\mu}} \\ \underset{\sim}{\hat{\overline{\mu}}} \end{bmatrix} \tag{77}$$

which represents the adjoint system of equations to be solved. The first-order change of the complex function f can be evaluated by solving (77) and substituting into (76).

The relationships between the adjoint solution of different modes of formulation are derived as follows. Let

$$f = f_1 + j\, f_2, \tag{78}$$

hence

$$\delta f = \delta f_1 + j\, \delta f_2\ , \tag{79}$$

and let $\underset{\sim}{\hat{V}}_r^1$ and $\underset{\sim}{\hat{V}}_s^1$ be the solution vector of the adjoint system (20) using cartesian coordinates for the real function f_1. Similarly, let $\underset{\sim}{\hat{V}}_r^2$ and $\underset{\sim}{\hat{V}}_s^2$ be the solution vector of (20) for the real function f_2. Hence, using (19) and (76), one may write

$$\underset{\sim}{\hat{V}}^T \underset{\sim}{d} + \underset{\sim}{\hat{\overline{V}}}^T \underset{\sim}{d}^* = (\underset{\sim}{\hat{V}}_r^{1T}\, \underset{\sim}{d}_1 - \underset{\sim}{\hat{V}}_s^{1T}\, \underset{\sim}{d}_2) + j(\underset{\sim}{\hat{V}}_r^{2T}\, \underset{\sim}{d}_1 - \underset{\sim}{\hat{V}}_s^{2T}\, \underset{\sim}{d}_2), \tag{80}$$

hence, from (5),

$$\underset{\sim}{\hat{V}} = (\underset{\sim}{\hat{V}}_r^1 - \underset{\sim}{\hat{V}}_s^2)/2 + j(\underset{\sim}{\hat{V}}_s^1 + \underset{\sim}{\hat{V}}_r^2)/2 \tag{81}$$

and

$$\underset{\sim}{\hat{\overline{V}}} = (\underset{\sim}{\hat{V}}_r^1 + \underset{\sim}{\hat{V}}_s^2)/2 + j(-\underset{\sim}{\hat{V}}_s^1 + \underset{\sim}{\hat{V}}_r^2)/2\ . \tag{82}$$

Equations (81) and (82) relate the solutions of the adjoint system (20) for both f_1 and f_2 to the solution of (77) for the complex function f.

Similarly, let $\underset{\sim}{\hat{V}}_\delta^1$ and $\underset{\sim}{\hat{V}}_v^1$ be the solution vector of the adjoint system (30) using polar coordinates for the real function f_1. Also, let $\underset{\sim}{\hat{V}}_\delta^2$ and $\underset{\sim}{\hat{V}}_v^2$ be the solution vector of (30) for the real function f_2. Hence, using (29) and (76), one may write

$$\hat{\underset{\sim}{V}}^T \underset{\sim}{d} + \hat{\bar{\underset{\sim}{V}}}^T \underset{\sim}{d}^* = (\hat{\underset{\sim}{V}}_\delta^{1T} \underset{\sim}{d}_1 - \hat{\underset{\sim}{V}}_v^{1T} \underset{\sim}{d}_2) + j(\hat{\underset{\sim}{V}}_\delta^{2T} \underset{\sim}{d}_1 - \hat{\underset{\sim}{V}}_v^{2T} \underset{\sim}{d}_2), \tag{83}$$

hence, from (5)

$$\hat{\underset{\sim}{V}} = (\hat{\underset{\sim}{V}}_\delta^1 - \hat{\underset{\sim}{V}}_v^2)/2 + j(\hat{\underset{\sim}{V}}_v^1 + \hat{\underset{\sim}{V}}_\delta^2)/2 \tag{84}$$

and

$$\hat{\bar{\underset{\sim}{V}}} = (\hat{\underset{\sim}{V}}_\delta^1 + \hat{\underset{\sim}{V}}_v^2)/2 + j(-\hat{\underset{\sim}{V}}_v^1 + \hat{\underset{\sim}{V}}_\delta^2)/2 . \tag{85}$$

Equations (84) and (85) relate the solutions of the adjoint system (30) for both f_1 and f_2 to the solution of (77) for the complex function f.

For gradient calculations, we proceed as before and use the partitioned forms (37), (39) and

$$\hat{\bar{\underset{\sim}{V}}} = \begin{bmatrix} \hat{\bar{\underset{\sim}{V}}}_L \\ \hat{\bar{\underset{\sim}{V}}}_G \\ \hat{\bar{V}}_n \end{bmatrix}, \tag{86}$$

and we write (13) in the form

$$\delta f = \hat{\underset{\sim}{V}}_L^T \underset{\sim}{d}_L + \hat{\underset{\sim}{V}}_G^T \underset{\sim}{d}_G + \hat{V}_n d_n + (\frac{\partial f}{\partial \underset{\sim}{\rho}})^T \delta \underset{\sim}{\rho}$$

$$+ \hat{\bar{\underset{\sim}{V}}}_L^T \underset{\sim}{d}_L^* + \hat{\bar{\underset{\sim}{V}}}_G^T \underset{\sim}{d}_G^* + \hat{\bar{V}}_n d_n^* + (\frac{\partial f}{\partial \underset{\sim}{\rho}^*})^T \delta \underset{\sim}{\rho}. \tag{87}$$

The first, second and third terms of (87) are given by (42), (44) and (45) respectively. The fifth term of (87) is given, using (38), by

$$\hat{\bar{\underset{\sim}{V}}}_L^T \underset{\sim}{d}_L^* = \sum_{\ell=1}^{n_L} (\hat{\bar{V}}_\ell \, \delta S_\ell) + \sum_{\ell=1}^{n_L} \sum_{\substack{m=1 \\ m\neq\ell}}^{n} \hat{\bar{V}}_\ell \, V_\ell (V_m^* - V_\ell^*) \, \delta y_{\ell m}^*$$

$$- \sum_{\ell=1}^{n} \hat{\bar{V}}_\ell \, V_\ell \, V_\ell^* \, \delta y_{\ell 0}^*. \tag{88}$$

Also, the sixth term of (87) is given, using (41) of Part I by

$$\hat{\underset{\sim}{V}}_G^T \underset{\sim}{d}_G^* = \sum_{g=n_L+1}^{n-1} \hat{\bar{V}}_g(\delta P_g + j\delta|V_g|) + \sum_{g=n_L+1}^{n-1} \sum_{\substack{m=1 \\ m\neq g}}^{n} \hat{\bar{V}}_g \, \mathrm{Re}\{V_g^*(V_m - V_g)\ \delta y_{gm}\}$$

$$- \sum_{g=n_L+1}^{n-1} \hat{\bar{V}}_g \, \mathrm{Re}\{V_g^* V_g \ \delta y_{g0}\} \tag{89}$$

and the seventh term of (87) is given, using (27) of Part I by

$$\hat{\bar{V}}_n d_n^* = \hat{\bar{V}}_n \delta V_n \ . \tag{90}$$

Equations (42), (44), (45), (88), (89) and (90) provide useful information for gradient evaluation of the complex function f w.r.t. the control variables of interest. Under the assumption that $\underset{\sim}{\rho}$ does not contain such control variables, the derivatives of the complex function f are obtained as follows.

<u>Load Bus Control Variables</u>

From (42) and (88), the derivatives of f w.r.t. the demand S_ℓ and S_ℓ^* at load bus ℓ is given by

$$\frac{df}{dS_\ell} = \hat{\bar{V}}_\ell \tag{91}$$

and

$$\frac{df}{dS_\ell^*} = \hat{V}_\ell \ . \tag{92}$$

<u>Generator Bus Control Variables</u>

From (44) and (89), the derivatives of f w.r.t. the generator control variables are given by

$$\frac{df}{d\tilde{S}_g} = \hat{\bar{V}}_g \tag{93}$$

and

$$\frac{df}{d\tilde{S}_g^*} = \hat{V}_g \ , \tag{94}$$

where $\tilde{S}_g$ is given by (50).

Slack Bus Control Variables

From (45) and (90), the derivatives of f w.r.t. the slack bus voltage V_n and V_n^* are given by

$$\frac{df}{dV_n} = \hat{\bar{V}}_n \tag{95}$$

and

$$\frac{df}{dV_n^*} = \hat{V}_n \ . \tag{96}$$

Line Control Variables

The derivatives of f w.r.t. line control variables y_{ij} can be obtained from (42), (44), (88) and (89) as follows. For $y_{\ell\ell'}$ between load buses ℓ and ℓ', we have from (42) and (88)

$$\frac{df}{dy_{\ell\ell'}} = (\hat{V}_\ell \ V_\ell^* - \hat{V}_{\ell'} \ V_{\ell'}^*) \ (V_{\ell'} - V_\ell) \tag{97}$$

and

$$\frac{df}{dy_{\ell\ell'}^*} = (\hat{\bar{V}}_\ell \ V_\ell - \hat{\bar{V}}_{\ell'} \ V_{\ell'}) \ (V_{\ell'}^* - V_\ell^*) \ . \tag{98}$$

For $y_{\ell 0}$ between load bus ℓ and ground, we have from (42) and (88)

$$\frac{df}{dy_{\ell 0}} = - \hat{V}_{\ell} \; V^*_{\ell} \; V_{\ell} \tag{99}$$

and

$$\frac{df}{dy^*_{\ell 0}} = - \hat{\bar{V}}_{\ell} \; V_{\ell} \; V^*_{\ell} \; . \tag{100}$$

For $y_{gg'}$ between generator buses g and g', we have from (44) and (89)

$$\frac{df}{dy_{gg'}} = \frac{1}{2} \left[(\hat{V}_g + \hat{\bar{V}}_g) V^*_g - (\hat{V}_{g'} + \hat{\bar{V}}_{g'}) V^*_{g'} \right] (V_{g'} - V_g) \tag{101}$$

and

$$\frac{df}{dy_{gg'}} = \frac{1}{2} \left[(\hat{V}_g + \hat{\bar{V}}_g) V_g - (\hat{V}_{g'} + \hat{\bar{V}}_{g'}) V_{g'} \right] (V^*_{g'} - V^*_g) \; . \tag{102}$$

For y_{g0} between generator bus g and ground, we have from (44) and (89)

$$\frac{df}{dy_{g0}} = \frac{df}{dy^*_{g0}} = - \frac{1}{2} (\hat{V}_g + \hat{\bar{V}}_g) \; V^*_g V_g \; . \tag{103}$$

For $y_{\ell g}$ between load bus ℓ and generator bus g, we have from (42), (44), (88) and (89)

$$\frac{df}{dy_{\ell g}} = \left[\frac{1}{2} (\hat{V}_g + \hat{\bar{V}}_g) \; V^*_g - \hat{V}_{\ell} \; V^*_{\ell}\right] (V_{\ell} - V_g) \tag{104}$$

and

$$\frac{df}{dy^*_{\ell g}} = \left[\frac{1}{2} (\hat{V}_g + \hat{\bar{V}}_g) \; V_g - \hat{\bar{V}}_{\ell} \; V_{\ell}\right] (V^*_{\ell} - V^*_g) \; . \tag{105}$$

For $y_{\ell n}$ between load bus ℓ and the slack bus n, we have from (42) and (88)

$$\frac{df}{dy_{\ell n}} = \hat{V}_{\ell} \; V^*_{\ell} \; (V_n - V_{\ell}) \tag{106}$$

and

$$\frac{df}{dy^*_{\ell n}} = \hat{\hat{V}}_\ell \; V_\ell \; (V^*_n - V^*_\ell) \; . \qquad (107)$$

Finally, for y_{gn} between generator bus g and the slack bus n, we have from (44) and (89)

$$\frac{df}{dy_{gn}} = \frac{1}{2} (\hat{V}_g + \hat{\hat{V}}_g) \; V^*_g \; (V_n - V_g) \qquad (108)$$

and

$$\frac{df}{dy^*_{gn}} = \frac{1}{2} (\hat{V}_g + \hat{\hat{V}}_g) \; V_g \; (V^*_n - V^*_g) \; . \qquad (109)$$

Remarks

If $\underset{\sim}{\rho}$ of (87) contains any of the above control variables, the partial derivatives of f w.r.t. appropriate control variables must be added to the expressions (91)-(109).

Equations (91)-(109) compactly define the required formal derivatives of the complex function f w.r.t. complex control variables. The gradients of f w.r.t. real and imaginary parts of the control variables are obtained using

$$\frac{df}{du_{k1}} = \frac{df}{du_k} + \frac{df}{du^*_k} \qquad (110)$$

and

$$\frac{df}{du_{k2}} = j \left(\frac{df}{du_k} - \frac{df}{du^*_k} \right) , \qquad (111)$$

where u_k is given by (71).

Expressions of forms (110) and (111) can be directly obtained from (91)-(109).

Example 5

Now, we consider the first 2-bus system and the complex function

$$f = V_1 = V_{11} + j\, V_{12}.$$

Using cartesian coordinates, the adjoint system solutions for V_{11} and V_{12} are given, respectively, by

$$\hat{\underset{\sim}{V}}{}^1_r = \begin{bmatrix} 0.0883 \\ 2.3144 \end{bmatrix},$$

$$\hat{\underset{\sim}{V}}{}^1_s = \begin{bmatrix} 0.1161 \\ 0.2041 \end{bmatrix},$$

$$\hat{\underset{\sim}{V}}{}^2_r = \begin{bmatrix} 0.0428 \\ 0.1117 \end{bmatrix}$$

and

$$\hat{\underset{\sim}{V}}{}^2_s = \begin{bmatrix} -0.0187 \\ 0.7352 \end{bmatrix},$$

hence, from (145) and (146)

$$\hat{\underset{\sim}{V}} = \begin{bmatrix} 0.0535 + j\, 0.0794 \\ 0.7896 + j\, 0.1579 \end{bmatrix}$$

and

$$\hat{\bar{\underset{\sim}{V}}} = \begin{bmatrix} 0.0348 - j\, 0.0366 \\ 1.5248 - j\, 0.0462 \end{bmatrix}.$$

The derivatives of f w.r.t. control variables are calculated, using the derived expressions, as follows. For S_1,

$$\frac{df}{dS_1} = \hat{\bar{V}}_1 = 0.0348 - j\, 0.0366$$

and

$$\frac{df}{dS_1^*} = \hat{V}_1 = 0.0535 + j\,0.0794\ ,$$

hence, from (174) and (175)

$$\frac{df}{dP_1} = 0.0883 - j\,0.0428$$

and

$$\frac{df}{dQ_1} = 0.1161 - j\,0.0187.$$

For V_2,

$$\frac{df}{dV_2} = \hat{\bar{V}}_2 = 1.5248 - j\,0.0462$$

and

$$\frac{df}{dV_2^*} = \hat{V}_2 = 0.7896 + j\,0.1579,$$

hence, from (174)

$$\frac{df}{dV_{21}} = 2.3144 + j\,0.1117.$$

For y_{10},

$$\frac{df}{dy_{10}} = -|V_1|^2\ \hat{V}_1 = -\ 0.0311 - j\,0.0462$$

and

$$\frac{df}{dy_{10}^*} = -|V_1|^2\ \hat{\bar{V}}_1 = -\ 0.0203 + j\,0.0213,$$

hence, from (174) and (175)

$$\frac{df}{dG_{10}} = -\ 0.0514 - j\,0.0249$$

and

$$\frac{df}{dB_{10}} = 0.0676 - j\ 0.0109.$$

For y_{12},

$$\frac{df}{dy_{12}} = \hat{V}_1\ V_1^*\ (V_2 - V_1) = -\ 0.0080 + j\ 0.0231$$

and

$$\frac{df}{dy_{12}^*} = \hat{\bar{V}}_1\ V_1\ (V_2^* - V_1^*) = -\ 0.0022 - j\ 0.0127,$$

hence, from (174) and (175)

$$\frac{df}{dG_{12}} = -\ 0.0102 + j\ 0.0104$$

and

$$\frac{df}{dB_{12}} = -\ 0.0358 - j\ 0.0059\ .$$

VI. CONCLUSIONS

We have derived and tabulated generalized power network sensitivity expressions useful for calculating first-order changes and gradients of functions of interest. The use of these generalized sensitivity expressions requires only the solution of an adjoint system of linear equations, the matrix of coefficients of which is simply the transpose of the Jacobian matrix of the load flow solution by the Newton-Raphson method in any mode of formulation. These generalized sensitivity expressions are applicable to both real and complex modes of performance functions as well as the control variables defined in a particular study.

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TABLE I

DERIVATIVES OF A REAL FUNCTION f W.R.T. CONTROL VARIABLES

Control Variable	Description	Derivative
P_ℓ	demand real power	$2\ \hat{V}_{\ell 1}$
Q_ℓ	demand reactive power	$2\ \hat{V}_{\ell 2}$
P_g	generator real power	$2\ \hat{V}_{g1}$
$\|V_g\|$	generator bus voltage magnitude	$2\ \hat{V}_{g2}$
V_{n1}	real component of slack bus voltage	$2\ \hat{V}_{n1}$
$G_{\ell\ell'}$	conductance between two load buses	$2\ \mathrm{Re}\{(\hat{V}_\ell V_\ell^* - \hat{V}_{\ell'} V_{\ell'}^*)(V_{\ell'} - V_\ell)\}$
$B_{\ell\ell'}$	susceptance between two load buses	$-2\ \mathrm{Im}\{(\hat{V}_\ell V_\ell^* - \hat{V}_{\ell'} V_{\ell'}^*)(V_{\ell'} - V_\ell)\}$
$G_{\ell 0}$	shunt conductance of a load bus	$-2\ \|V_\ell\|^2 \hat{V}_{\ell 1}$
$B_{\ell 0}$	shunt susceptance of a load bus	$2\ \|V_\ell\|^2\ \hat{V}_{\ell 2}$
$G_{gg'}$	conductance between two generator buses	$2\ \mathrm{Re}\{(\hat{V}_{g1} V_g^* - \hat{V}_{g'1} V_{g'}^*)(V_{g'} - V_g)\}$
$B_{gg'}$	susceptance between two generator buses	$-2\ \mathrm{Im}\{(\hat{V}_{g1} V_g^* - \hat{V}_{g'1} V_{g'}^*)(V_{g'} - V_g)\}$
G_{g0}	shunt conductance of a generator bus	$-2\ \|V_g\|^2\ \hat{V}_{g1}$
B_{g0}	shunt susceptance of a generator bus	0
$G_{\ell g}$	conductance between a load and a generator buses	$2\ \mathrm{Re}\{(\hat{V}_{g1} V_g^* - \hat{V}_\ell V_\ell^*)(V_\ell - V_g)\}$
$B_{\ell g}$	susceptance between a load and a generator buses	$-2\ \mathrm{Im}\{(\hat{V}_{g1} V_g^* - \hat{V}_\ell V_\ell^*)(V_\ell - V_g)\}$

Control Variable	Description	Derivative
$G_{\ell n}$	conductance between a load and slack buses	$2\ \mathrm{Re}\{\hat{V}_\ell V_\ell^*(V_n - V_\ell)\}$
$B_{\ell n}$	susceptance between a load and slack buses	$-2\ \mathrm{Im}\{\hat{V}_\ell V_\ell^*(V_n - V_\ell)\}$
G_{gn}	conductance between a generator and slack buses	$2\hat{V}_{g1}\ \mathrm{Re}\{V_g^*(V_n - V_g)\}$
B_{gn}	susceptance between a generator and slack buses	$-2\ \hat{V}_{g1}\ \mathrm{Im}\{V_g^* V_n\}$

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GENERALIZED POWER NETWORK SENSITIVITIES
PART II: ANALYSIS

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Abstract: Generalized sensitivity expressions for calculating first-order changes and gradients of functions of interest in different power system applications are derived. We utilize the special complex notation and the transformations between different modes of formulation described in Part I of the paper to compactly derive the required sensitivity expressions. These generalized sensitivity expressions are common to all modes of formulation, e.g., polar and cartesian, common to both real and complex functions and common to all real and complex variables defined in a particular study. The Jacobian matrix of the load flow solution by the Newton-Raphson method is used to define the adjoint system of linear equations required to be solved.

Description:

Related Work: SOC-237, SOC-242, SOC-247, SOC-248.

Price: $6.00.