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A COMPLEX LAGRANGIAN APPROACH WITH APPLICATIONS TO POWER NETWORK SENSITIVITY ANALYSIS

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Abstract

The well-known Lagrangian approach, traditionally described in real form, for calculating first-order changes and gradients of functions of interest subject to equality constraints is generalized and applied in a compact complex form. Hence, general complex functions and constraints can be handled directly while maintaining the original complex mode of formulation. Applications in power network sensitivity analysis and gradient evaluation are presented.

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I. INTRODUCTION

In sensitivity calculations for electrical networks [1-4], the first-order change of a function of both system state and control variables is required to be expressed solely in terms of first-order changes of the control variables. This expression is useful for determining total derivatives of the function w.r.t. control variables. The state and control variables (hence, their first-order changes) are related through a set of equality constraints which may represent network flow equations.

In the real form, the Lagrangian approach has been successfully applied [3] to power system analysis and design problems where Lagrange multipliers obtained by solving a set of adjoint equations are used to relate first-order changes of a real function to those of the control variables.

In some cases, the set of equality constraints is described basically in a compact complex form, e.g., the power flow equations in electrical power systems. Moreover, first-order changes of a complex function may be required. The application of the Lagrangian approach [3] requires separation of real and imaginary parts of the equality constraints as well as the function of interest which may alter the ease and compactness of formulation.

The study presented in this paper exploits the conjugate notation [4] to describe and formulate the Lagrangian approach in the complex form so that complex functions and constraints may be directly handled.

The description and theoretical bases of the notation used, the complex formulation of the Lagrangian approach and some important applications to power system sensitivity analysis and design are presented successively in the paper.

II. NOTATION

We denote by $\mathcal{C}$ and $\mathcal{R}$, respectively, the field of complex numbers and the field of real numbers. The vector space over $\mathcal{C}$, of n-tuples $(\zeta_1, \ldots, \zeta_n)$, $\zeta_i \in \mathcal{C}$ is denoted by $\mathcal{C}^n$. Similarly, $\mathcal{R}^n$ stands for the vector space over $\mathcal{R}$, of n-tuples $(\zeta_{1m}, \ldots, \zeta_{nm})$, m=1, 2 and $\zeta_{im} \in \mathcal{R}$.

In the conjugate notation [4], a complex variable ζ_i,

$$\zeta_i = \zeta_{i1} + j\,\zeta_{i2} \tag{1}$$

and its complex conjugate ζ_i^* replace, as independent quantities, the real and imaginary parts of the variable. Hence, for a continuously differentiable complex valued function f on an open set $\Omega \subset \mathcal{C}^n$, we may define the formal [5] or symbolic [6] partial derivatives

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}} \triangleq \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} - j\,\frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)/2 \tag{2}$$

and

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \triangleq \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} + j\,\frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)/2, \tag{3}$$

where

$$\underset{\sim}{\zeta} = \underset{\sim}{\zeta}_1 + j\,\underset{\sim}{\zeta}_2 \tag{4}$$

is a column vector of components ζ_i of (1), i = 1, 2, ..., n and $\partial f/\partial \underset{\sim}{\zeta}$, $\partial f/\partial \underset{\sim}{\zeta}^*$, $\partial f/\partial \underset{\sim}{\zeta}_1$ and $\partial f/\partial \underset{\sim}{\zeta}_2$ are column vectors.

The first-order variation of the function f can be expressed [4] as

$$\delta f = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^T \delta \underset{\sim}{\zeta} + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^T \delta \underset{\sim}{\zeta}^*, \tag{5}$$

where δ denotes first-order change and T denotes transposition.

It can be shown that, for arbitrary $\underset{\sim}{\zeta}$, if

$$\underset{\sim}{\mu}^T \underset{\sim}{\zeta} + \overline{\underset{\sim}{\mu}}^T \underset{\sim}{\zeta}^* = \underset{\sim}{\mu}^{\bullet T} \underset{\sim}{\zeta} + \overline{\underset{\sim}{\mu}}^{\bullet T} \underset{\sim}{\zeta}^*, \tag{6a}$$

where $\underset{\sim}{\mu}$, $\overline{\underset{\sim}{\mu}}$, $\underset{\sim}{\mu}^{\bullet}$ and $\overline{\underset{\sim}{\mu}}^{\bullet}$ are appropriate vectors of complex scalars, then

$$\underset{\sim}{\mu} = \underset{\sim}{\mu}^{\bullet} \text{ and } \overline{\underset{\sim}{\mu}} = \overline{\underset{\sim}{\mu}}^{\bullet}. \tag{6b}$$

For a pure real function f, we write

$$\delta f = \delta f^* = (\delta f)^*, \tag{7}$$

or, using (5),

$$\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^T \delta \underset{\sim}{\zeta} + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^T \delta \underset{\sim}{\zeta}^* = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^{*T} \delta \underset{\sim}{\zeta}^* + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^{*T} \delta \underset{\sim}{\zeta}, \tag{8}$$

hence, from (6)

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}} = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^*. \tag{9}$$

Also, for a pure imaginary function f, we write

$$\delta f = - \delta f^* = - (\delta f)^*, \tag{10}$$

or

$$\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^T \delta \underset{\sim}{\zeta} + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^T \delta \underset{\sim}{\zeta}^* = - \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^{*T} \delta \underset{\sim}{\zeta}^* - \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^{*T} \delta \underset{\sim}{\zeta}, \tag{11}$$

hence, from (6)

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}} = - \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^*. \tag{12}$$

We remark [5] that the terminology of formal derivatives arises because of the possibility of obtaining them formally using the ordinary differentiation rules. The use of the conjugate notation facilitates the derivations and subsequent formulation of the equations to be solved.

III. THE COMPLEX LAGRANGIAN CONCEPT

In this section, we formulate the Lagrangian approach in the general complex case.

We consider, as before, a complex function f of a set of complex variables $\underset{\sim}{\zeta}$ and their complex conjugate $\underset{\sim}{\zeta}^*$. We write

$$\underset{\sim}{\zeta} = \begin{bmatrix} \underset{\sim}{\zeta}_x \\ \underset{\sim}{\zeta}_u \end{bmatrix} , \tag{13}$$

where the variables $\underset{\sim}{\zeta}$ have been classified as n_x state variables $\underset{\sim}{\zeta}_x$ and n_u control variables $\underset{\sim}{\zeta}_u$. The state and control variables are related through the set of n_x complex equality constraints

$$\underset{\sim}{h}\ (\underset{\sim}{\zeta},\underset{\sim}{\zeta}^*) = \underset{\sim}{0} \ . \tag{14}$$

The first-order change of f is written, using (5), in the form

$$\delta f = [\ \underset{\sim}{f}_{\zeta x}^T\ \ \underset{\sim}{\bar{f}}_{\zeta x}^T\ ] \begin{bmatrix} \delta\underset{\sim}{\zeta}_x \\ \\ \delta\underset{\sim}{\zeta}_x^* \end{bmatrix} + [\ \underset{\sim}{f}_{\zeta u}^T\ \ \underset{\sim}{\bar{f}}_{\zeta u}^T\ ] \begin{bmatrix} \delta\underset{\sim}{\zeta}_u \\ \\ \delta\underset{\sim}{\zeta}_u^* \end{bmatrix} , \tag{15}$$

where $\underset{\sim}{f}_{\zeta x}$, $\underset{\sim}{\bar{f}}_{\zeta x}$, $\underset{\sim}{f}_{\zeta u}$ and $\underset{\sim}{\bar{f}}_{\zeta u}$ stand for $\partial f/\partial\underset{\sim}{\zeta}_x$, $\partial f/\partial\underset{\sim}{\zeta}_x^*$, $\partial f/\partial\underset{\sim}{\zeta}_u$ and $\partial f/\partial\underset{\sim}{\zeta}_u^*$, respectively.

We write (14) in the perturbed form

$$\delta\underset{\sim}{h}\ (\underset{\sim}{\zeta},\underset{\sim}{\zeta}^*) = \underset{\sim}{0} \tag{16}$$

or

$$[\underset{\sim}{H}_{\zeta x}\ \overline{\underset{\sim}{H}}_{\zeta x}] \begin{bmatrix} \delta\underset{\sim}{\zeta}_x \\ \\ \delta\underset{\sim}{\zeta}_x^* \end{bmatrix} + [\underset{\sim}{H}_{\zeta u}\ \overline{\underset{\sim}{H}}_{\zeta u}] \begin{bmatrix} \delta\underset{\sim}{\zeta}_x \\ \\ \delta\underset{\sim}{\zeta}_u^* \end{bmatrix} = \underset{\sim}{0}, \tag{17}$$

where $\underset{\sim}{H}_{\zeta x}$, $\overline{\underset{\sim}{H}}_{\zeta x}$, $\underset{\sim}{H}_{\zeta u}$ and $\overline{\underset{\sim}{H}}_{\zeta u}$ stand for $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}_x)^T$, $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}_x^*)^T$, $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}_u)^T$ and $(\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}_u^*)^T$, respectively. Using the complex conjugate of (17), we may write

$$\begin{bmatrix} \delta\underset{\sim}{\zeta}_x \\ \\ \delta\underset{\sim}{\zeta}_x^* \end{bmatrix} = - \begin{bmatrix} \underset{\sim}{H}_{\zeta x} & \overline{\underset{\sim}{H}}_{\zeta x} \\ \\ \overline{\underset{\sim}{H}}_{\zeta x}^* & \underset{\sim}{H}_{\zeta x}^* \end{bmatrix}^{-1} \begin{bmatrix} \underset{\sim}{H}_{\zeta u} & \overline{\underset{\sim}{H}}_{\zeta u} \\ \\ \overline{\underset{\sim}{H}}_{\zeta u}^* & \underset{\sim}{H}_{\zeta u}^* \end{bmatrix} \begin{bmatrix} \delta\underset{\sim}{\zeta}_u \\ \\ \delta\underset{\sim}{\zeta}_u^* \end{bmatrix} . \tag{18}$$

It can be shown that the inverted matrix in (18) has full rank if and only if the system of equations (16) represent $2n_x$ independent conditions.

Using (18), δf of (15) is written in the form

$$\delta f = \{ [\ \underset{\sim}{f}_{\zeta u}^T\ \overline{\underset{\sim}{f}}_{\zeta u}^T\ ] - [\ \underset{\sim}{\lambda}^T\ \overline{\underset{\sim}{\lambda}}^T\ ] \begin{bmatrix} \underset{\sim}{H}_{\zeta u} & \overline{\underset{\sim}{H}}_{\zeta u} \\ \\ \overline{\underset{\sim}{H}}_{\zeta u}^* & \underset{\sim}{H}_{\zeta u}^* \end{bmatrix} \} \begin{bmatrix} \delta\underset{\sim}{\zeta}_u \\ \\ \delta\underset{\sim}{\zeta}_u^* \end{bmatrix} , \tag{19}$$

where

$$\begin{bmatrix} \underset{\sim}{H}_{\zeta x}^T & \overline{\underset{\sim}{H}}_{\zeta x}^{*T} \\ \\ \overline{\underset{\sim}{H}}_{\zeta x}^T & \underset{\sim}{H}_{\zeta x}^{*T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\lambda} \\ \\ \overline{\underset{\sim}{\lambda}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}_{\zeta x} \\ \\ \overline{\underset{\sim}{f}}_{\zeta x} \end{bmatrix} . \tag{20}$$

Hence, the total formal derivatives of f are given, from (19), by

$$\frac{df}{d\underset{\sim}{\zeta}_u} = \underset{\sim}{f}_{\zeta u} - \underset{\sim}{H}_{\zeta u}^T \underset{\sim}{\lambda} - \overline{\underset{\sim}{H}}_{\zeta u}^{*T} \overline{\underset{\sim}{\lambda}} \tag{21}$$

and

$$\frac{df}{d\underset{\sim}{\zeta}_u^*} = \bar{\underset{\sim}{f}}_{\zeta u} - \underset{\sim}{\bar{H}}^T_{\zeta u}\, \underset{\sim}{\lambda} - \underset{\sim}{H}^{*T}_{\zeta u}\, \bar{\underset{\sim}{\lambda}} \; . \tag{22}$$

The complex Lagrange multipliers $\underset{\sim}{\lambda}$ and $\bar{\underset{\sim}{\lambda}}$ of (21) and (22) are obtained by solving the set of complex adjoint equations (20).

Note that in the real case when the function f and constraints $\underset{\sim}{h}$ are all pure real, the application of (9) results in the complex conjugate relationships $\bar{\underset{\sim}{f}}_{\zeta x} = \underset{\sim}{f}^*_{\zeta x}$ and $\underset{\sim}{\bar{H}}_{\zeta x} = \underset{\sim}{H}^*_{\zeta x}$ and (20) is reduced to a system of n_x complex equations in the real variables $(\underset{\sim}{\lambda} + \bar{\underset{\sim}{\lambda}})$. The solution of this system of equations is then substituted into (21) and (22) which form a complex conjugate pair since, from (9), $\bar{\underset{\sim}{f}}_{\zeta u} = \underset{\sim}{f}^*_{\zeta u}$ and $\underset{\sim}{\bar{H}}_{\zeta u} = \underset{\sim}{H}^*_{\zeta u}$.

We have stated the Lagrangian approach in the complex form and derived the corresponding adjoint equations to be solved for the Lagrange multipliers so that the required formal derivatives (21) and (22) may be obtained. In the following two sections, we consider some applications of the complex Lagrangian approach in power system analysis and design.

IV. APPLICATION TO POWER NETWORK ANALYSIS

The complex Lagrangian approach described in the previous section can be applied, for example, to power network sensitivity calculations. The set of complex equality constraints (14) may represent the power flow equations of the form

$$\underset{\sim}{h} = \underset{\sim}{S}^*_M - \underset{\sim}{E}^*_M\, \underset{\sim}{Y}_T\, \underset{\sim}{V}_M = \underset{\sim}{0} \; , \tag{23}$$

where $\underset{\sim}{S}_M$ is a vector of the bus powers, $\underset{\sim}{V}_M$ is a vector of bus voltages, $\underset{\sim}{Y}_T$ is the bus admittance matrix of dimension nxn, n denoting number of buses in the power network and $\underset{\sim}{E}_M$ is a diagonal matrix of components of $\underset{\sim}{V}_M$ in a corresponding order.

The vectors $\underset{\sim}{\zeta}_x$ and $\underset{\sim}{\zeta}_u$ of (13) are defined as

$$\underset{\sim}{\zeta}_x \triangleq \begin{bmatrix} \underset{\sim}{V}_L \\ S_n \end{bmatrix} \tag{24}$$

and

$$\underset{\sim}{\zeta}_u \triangleq \begin{bmatrix} \underset{\sim}{S}_L \\ V_n \end{bmatrix} , \tag{25}$$

where we have classified, for simplicity, the buses as load-type buses of voltages $\underset{\sim}{V}_L$ and powers $\underset{\sim}{S}_L$ and a slack bus of voltage V_n and power S_n. We write (23) in the corresponding partitioned form

$$\begin{bmatrix} \underset{\sim}{h}_L \\ h_n \end{bmatrix} = \begin{bmatrix} \underset{\sim}{S}_L^* \\ S_n^* \end{bmatrix} - \begin{bmatrix} \underset{\sim}{E}_L & \underset{\sim}{0} \\ \underset{\sim}{0} & V_n^* \end{bmatrix} \begin{bmatrix} \underset{\sim}{Y}_{LL} & \underset{\sim}{Y}_{LN} \\ \underset{\sim}{Y}_{LN}^T & Y_{nn} \end{bmatrix} \begin{bmatrix} \underset{\sim}{V}_L \\ V_n \end{bmatrix} , \tag{26}$$

where the symmetric bus admittance matrix has been partitioned into $\underset{\sim}{Y}_{LL}$, $\underset{\sim}{Y}_{LN}$, $\underset{\sim}{Y}_{LN}^T$ and Y_{nn} of appropriate dimensions.

The matrices $\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}_x$, $\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}_x^*$, $\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}_u$ and $\partial\underset{\sim}{h}^T/\partial\underset{\sim}{\zeta}_u^*$ are given, respectively, by

$$\frac{\partial\underset{\sim}{h}^T}{\partial\underset{\sim}{\zeta}_x} = \begin{bmatrix} \dfrac{\partial\underset{\sim}{h}_L^T}{\partial\underset{\sim}{V}_L} & \dfrac{\partial h_n}{\partial\underset{\sim}{V}_L} \\ \dfrac{\partial\underset{\sim}{h}_L^T}{\partial S_n} & \dfrac{\partial h_n}{\partial S_n} \end{bmatrix} , \tag{27}$$

$$\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{\zeta}_x^*} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{h}_L^T}{\partial \underset{\sim}{V}_L^*} & \dfrac{\partial h_n}{\partial \underset{\sim}{V}_L^*} \\ \dfrac{\partial \underset{\sim}{h}_L^T}{\partial S_n^*} & \dfrac{\partial h_n}{\partial S_n^*} \end{bmatrix}, \tag{28}$$

$$\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{\zeta}_u} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{h}_L^T}{\partial \underset{\sim}{S}_L} & \dfrac{\partial h_n}{\partial \underset{\sim}{S}_L} \\ \dfrac{\partial \underset{\sim}{h}_L^T}{\partial V_n} & \dfrac{\partial h_n}{\partial V_n} \end{bmatrix}, \tag{29}$$

and

$$\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{\zeta}_u^*} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{h}_L^T}{\partial \underset{\sim}{S}_L^*} & \dfrac{\partial h_n}{\partial \underset{\sim}{S}_L^*} \\ \dfrac{\partial \underset{\sim}{h}_L^T}{\partial V_n^*} & \dfrac{\partial h_n}{\partial V_n^*} \end{bmatrix}, \tag{30}$$

Using (26)-(30), the matrices $\underset{\sim}{H}_{\zeta x}$, $\overline{\underset{\sim}{H}}_{\zeta x}$, $\underset{\sim}{H}_{\zeta u}$ and $\overline{\underset{\sim}{H}}_{\zeta u}$ of (17) are given, respectively, by

$$\underset{\sim}{H}_{\zeta x} = \begin{bmatrix} -(\underset{\sim}{E}_L^* \, \underset{\sim}{Y}_{LL}) & \underset{\sim}{0} \\ -(V_n^* \, \underset{\sim}{Y}_{LN}^T) & 0 \end{bmatrix}, \tag{31}$$

$$\overline{\underset{\sim}{H}}_{\zeta x} = \begin{bmatrix} -\,\mathrm{diag}\,\{\underset{\sim}{I}_L\} & \underset{\sim}{0} \\ \underset{\sim}{0} & 1 \end{bmatrix}, \tag{32}$$

$$\underset{\sim}{H}_{\zeta u} = \begin{bmatrix} \underset{\sim}{0} & -(\underset{\sim}{E}_L^* \, \underset{\sim}{Y}_{LN}) \\ \underset{\sim}{0} & -V_n^* \, Y_{nn} \end{bmatrix} \tag{33}$$

and

$$\bar{\mathbf{H}}_{\zeta u} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_n \end{bmatrix} , \tag{34}$$

where the bus currents

$$\mathbf{I}_M = \begin{bmatrix} \mathbf{I}_L \\ I_n \end{bmatrix} \tag{35}$$

are given by

$$\mathbf{I}_M = \mathbf{Y}_T \, \mathbf{V}_M. \tag{36}$$

Hence, for a given function f with the formal derivatives $\mathbf{f}_{\zeta x}$ and $\bar{\mathbf{f}}_{\zeta x}$, the adjoint system of equations (20) is formed using (31) and (32) and solved for the Lagrange multipliers $\boldsymbol{\lambda}$ and $\bar{\boldsymbol{\lambda}}$. The total formal derivatives of f w.r.t. the control variables are then calculated from (21) and (22) using (33) and (34).

We remark that the choice of $\mathbf{V}_L$ and S_n as the only control variables $\boldsymbol{\zeta}_u$ has been made for simplicity. We could equally well define other control variables e.g., line admittances.

Note also that the voltage-controlled buses or generator-type buses [3] can be included by defining complex conjugate pairs of state variables, e.g.,

$$\zeta_x^g \triangleq Q_g + j \, \delta_g \tag{37}$$

and of control variables, e.g.,

$$\zeta_u^g \triangleq P_g + j \, |V_g| , \tag{38}$$

where the generator bus power S_g is given by

$$S_g = P_g + j\, Q_g \tag{39}$$

and the generator bus voltage V_g is given by

$$V_g = |V_g| \angle \delta_g . \tag{40}$$

The modification required to include other control and state variables can be performed in a straightforward manner.

V. THE ELEMENT-LOCAL LAGRANGIAN APPROACH

In this section, we consider an important application of the complex Lagrangian concept stated in section III. This application is concerned with the approach presented in [4] for sensitivity evaluation of electrical networks. This approach utilizes Tellegen's theorem with suitable extensions to obtain sensitivity expressions for the network elements which allow first-order changes and gradients of functions of interest w.r.t. control variables to be effectively calculated.

For each element (branch) b, and according to its type, a set of four complex element variables $\underset{\sim}{z}_b$ is defined describing the practical state $\underset{\sim}{x}_b$ and control $\underset{\sim}{u}_b$ variables associated with it, $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ denoting two component column vectors, and

$$\underset{\sim}{z}_b = \begin{bmatrix} \underset{\sim}{x}_b \\ \underset{\sim}{u}_b \end{bmatrix} . \tag{41}$$

The elements of $\underset{\sim}{z}_b$ may constitute complex conjugate pairs and a slightly different formulation from that of section III is considered here where $\underset{\sim}{z}_b$ may contain elements of both $\underset{\sim}{\zeta}$ and $\underset{\sim}{\zeta}^*$.

The first-order change of a general complex function f of all state vectors $\underset{\sim}{x}_b$ and all control vectors $\underset{\sim}{u}_b$ is given by

$$\delta f = \sum_b (\underset{\sim}{f}_{xb}^T \, \delta \underset{\sim}{x}_b + \underset{\sim}{f}_{ub}^T \, \delta \underset{\sim}{u}_b), \tag{42}$$

where $\underset{\sim}{f}_{xb}$ and $\underset{\sim}{f}_{ub}$ denote the formal derivatives $\partial f/\partial \underset{\sim}{x}_b$ and $\partial f/\partial \underset{\sim}{u}_b$, respectively. The application of Tellegen's theorem results in the identity

$$\sum_b (\hat{\underset{\sim}{\eta}}_{bx}^T \, \delta \underset{\sim}{x}_b + \hat{\underset{\sim}{\eta}}_{bu}^T \, \delta \underset{\sim}{u}_b) = 0, \tag{43}$$

where the 2-component complex vectors $\hat{\underset{\sim}{\eta}}_{bx}$ and $\hat{\underset{\sim}{\eta}}_{bu}$ are linear functions of the adjoint network current variables $\hat{I}_b$ and voltage variables $\hat{V}_b$ and their complex conjugate.

The adjoint network is defined by setting

$$\hat{\underset{\sim}{\eta}}_{bx} = \underset{\sim}{f}_{xb}, \tag{44}$$

hence, from (42) and (43)

$$\delta f = \sum_b (\underset{\sim}{f}_{ub} - \hat{\underset{\sim}{\eta}}_{bu})^T \, \delta \underset{\sim}{u}_b, \tag{45}$$

from which

$$\frac{df}{d\underset{\sim}{u}_b} = \underset{\sim}{f}_{ub} - \hat{\underset{\sim}{\eta}}_{bu}. \tag{46}$$

Now, in the approach [4] described above, only two state variables $\underset{\sim}{x}_b$ and two control variables $\underset{\sim}{u}_b$ are defined for each element. The function f must be expressed solely in terms of the $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$. In some cases, however, the function f may be expressed basically in terms of the $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ as well as other dependent variables $\underset{\sim}{\rho}_b$ which, by themselves, are functions of $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$. The variables $\underset{\sim}{\rho}_b$ may be related to $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ through a set of complicated equality constraints so that the direct expression of $\underset{\sim}{\rho}_b$ in terms of $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ may be difficult or impossible.

In the following, we show how the complex Lagrangian concept stated before can be applied to handle any number of the complex dependent variables $\underset{\sim}{\rho}_b$ in terms of which the function f may be expressed.

We assume that the $n_{\rho b}$ variables $\underset{\sim}{\rho}_b$ associated with element b are related to the element variables $\underset{\sim}{z}_b$ by the set of $n_{\rho b}$ equality constraints

$$\underset{\sim}{h}_b\ (\underset{\sim}{x}_b,\ \underset{\sim}{u}_b,\ \underset{\sim}{\rho}_b) = \underset{\sim}{0} \tag{47}$$

and we denote by δf_b the change in f due to changes in $\underset{\sim}{x}_b$, $\underset{\sim}{u}_b$ and $\underset{\sim}{\rho}_b$, hence

$$\delta f = \sum_b \delta f_b. \tag{48}$$

Now, we apply the element-local Lagrangian concept as follows. We write δf_b as

$$\delta f_b = \underset{\sim}{f}_{xb}^T\ \delta \underset{\sim}{x}_b + \underset{\sim}{f}_{ub}^T\ \delta \underset{\sim}{u}_b + \underset{\sim}{f}_{\rho b}^T\ \delta \underset{\sim}{\rho}_b, \tag{49}$$

where $\underset{\sim}{f}_{\rho b}$ denotes $\partial f / \partial \rho_b$. Also, we write $\delta \underset{\sim}{h}_b$ as

$$\delta \underset{\sim}{h}_b = \underset{\sim}{H}_{xb}\ \delta \underset{\sim}{x}_b + \underset{\sim}{H}_{ub}\ \delta \underset{\sim}{u}_b + \underset{\sim}{H}_{\rho b}\ \delta \underset{\sim}{\rho}_b = \underset{\sim}{0}, \tag{50}$$

where $\underset{\sim}{H}_{xb}$, $\underset{\sim}{H}_{ub}$ and $\underset{\sim}{H}_{\rho b}$ stand for $(\partial \underset{\sim}{h}_b^T / \partial \underset{\sim}{x}_b)^T$, $(\partial \underset{\sim}{h}_b^T / \partial \underset{\sim}{u}_b)^T$ and $(\partial \underset{\sim}{h}_b^T / \partial \underset{\sim}{\rho}_b)^T$, respectively. Hence

$$\delta \underset{\sim}{\rho}_b = - \underset{\sim}{H}_{\rho b}^{-1}\ (\underset{\sim}{H}_{xb}\ \delta \underset{\sim}{x}_b + \underset{\sim}{H}_{ub}\ \delta \underset{\sim}{u}_b), \tag{51}$$

where $\underset{\sim}{H}_{\rho b}$ is a full rank matrix.

Substituting (51) into (49), we get

$$\delta f_b = (\underset{\sim}{f}_{xb}^T - \underset{\sim}{\lambda}_{\rho b}^T\ \underset{\sim}{H}_{xb})\ \delta \underset{\sim}{x}_b + (\underset{\sim}{f}_{ub}^T - \underset{\sim}{\lambda}_{\rho b}^T\ \underset{\sim}{H}_{ub})\ \delta \underset{\sim}{u}_b, \tag{52}$$

where the element-local Lagrange multipliers $\underset{\sim}{\lambda}_{\rho b}$ are obtained by solving

$$\underset{\sim}{H}^T_{\rho b} \underset{\sim}{\lambda}_{\rho b} = \underset{\sim}{f}_{\rho b}. \tag{53}$$

Equations (48) and (52) express, instead of (42), δf. We therefore define the adjoint network by setting

$$\hat{\underset{\sim}{\eta}}_{bx} = \underset{\sim}{f}_{xb} - \underset{\sim}{H}^T_{xb} \underset{\sim}{\lambda}_{\rho b}, \tag{54}$$

hence, from (43), (48) and (52)

$$\delta f = \sum_b (\underset{\sim}{f}_{ub} - \hat{\underset{\sim}{\eta}}_{bu} - \underset{\sim}{H}^T_{ub} \underset{\sim}{\lambda}_{\rho b})^T \delta \underset{\sim}{u}_b, \tag{55}$$

from which

$$\frac{df}{d\underset{\sim}{u}_b} = \underset{\sim}{f}_{ub} - \hat{\underset{\sim}{\eta}}_{bu} - \underset{\sim}{H}^T_{ub} \underset{\sim}{\lambda}_{\rho b} \tag{56}$$

which is the required formal derivatives of f w.r.t. the complex control variables $\underset{\sim}{u}_b$.

VI. CONCLUSIONS

The far reaching consequences gained by using the compact conjugate notation have been exploited in formulating the Lagrangian approach in the complex form. First-order changes and formal derivatives of complex functions of interest subject to general complex equality constraints can be evaluated, directly, while keeping the original compact complex mode of formulation. Some important applications to power network sensitivity analysis have been studied. The possibility of defining a general number of states associated with a branch in the approach of [4] has been afforded by describing an element-local Lagrangian technique.

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SOC-247

A COMPLEX LAGRANGIAN APPROACH WITH APPLICATIONS TO POWER NETWORK SENSITIVITY ANALYSIS

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Key Words: Complex analysis, conjugate notation, sensitivity analysis, Lagrange multipliers, power flow equations, Tellegen's theorem

Abstract: The well-known Lagrangian approach, traditionally described in real form, for calculating first-order changes and gradients of functions of interest subject to equality constraints is generalized and applied in a compact complex form. Hence, general complex functions and constraints can be handled directly while maintaining the original complex mode of formulation. Applications in power network sensitivity analysis and gradient evaluation are presented.

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