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A COMPLEX LAGRANGIAN APPROACH WITH APPLICATIONS TO POWER NETWORK SENSITIVITY ANALYSIS

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Abstract

The well-known Lagrangian approach, traditionally described in real form, for calculating first-order changes and gradients of functions of interest subject to equality constraints is generalized and applied in a compact complex form. Hence, general complex functions and constraints can be handled directly while maintaining the original complex mode of formulation. The theoretical foundations of the approach are stated. An application to power network sensitivity analysis and gradient evaluation is presented.

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I. INTRODUCTION

In sensitivity calculations for electrical networks [1-5], the first-order change of a function of both system state and control variables is required to be expressed solely in terms of first-order changes of the control variables. This expression is useful for determining total derivatives of the function w.r.t. control variables. The state and control variables (hence, their first-order changes) are related through a set of equality constraints which may represent network flow equations.

In the real form, the Lagrangian approach has been successfully applied to power system analysis and design problems where Lagrange multipliers obtained by solving a set of adjoint equations are used to relate first-order changes of a real function to those of the control variables.

In some cases, the set of equality constraints is described basically in a compact complex form, e.g., the power flow equations in electrical power systems. Moreover, first-order changes of a complex function may be required. The application of the Lagrangian approach [3] requires separation of real and imaginary parts of the equality constraints as well as the function of interest which may alter the ease and compactness of formulation.

The study presented in this paper exploits a compact complex notation to describe and formulate the Lagrangian approach in the complex form so that complex functions and constraints may be directly handled.

The description and theoretical bases of the notation used, the complex formulation of the Lagrangian approach and an important application to power system sensitivity analysis and design are presented successively in the paper.

II. THE CONJUGATE NOTATION

We denote by C and R, respectively, the field of complex numbers and the field of real numbers. The vector space over C, of n-tuples $(\zeta_1, \ldots, \zeta_n)$, $\zeta_i \in C$ is denoted by C^n. Similarly, R^n stands for the vector space over R, of n-tuples $(\zeta_{1m}, \ldots, \zeta_{nm})$, m=1, 2 and $\zeta_{im} \in R$. Also, we write

$$\underset{\sim}{\zeta} = \underset{\sim}{\zeta}_1 + j\, \underset{\sim}{\zeta}_2, \tag{1}$$

where $\underset{\sim}{\zeta}$ is a column vector of components ζ_i given by

$$\zeta_i = \zeta_{i1} + j\, \zeta_{i2}, \tag{2}$$

$\underset{\sim}{\zeta}_1, \underset{\sim}{\zeta}_2 \in R^n$, $\zeta_{i1}, \zeta_{i2} \in R$, $i = 1, 2, \ldots, n$.

Formal Partial Derivatives

For a continuously differentiable complex valued function f on an open set $\Omega \subset C^n$ (f possesses derivatives [6] w.r.t. all the variables ζ_{i1} and ζ_{i2}), we define the formal [7] or symbolic [8] partial derivatives

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}} \triangleq \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} - j \frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)/2 \tag{3}$$

and

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \triangleq \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} + j \frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)/2, \tag{4}$$

where, $\partial f/\partial \underset{\sim}{\zeta}$, $\partial f/\partial \underset{\sim}{\zeta}^*$, $\partial f/\partial \underset{\sim}{\zeta}_1$ and $\partial f/\partial \underset{\sim}{\zeta}_2$ are column vectors.

Note that in formal derivatives, the Cauchy-Riemann differential equations may be written [7] as

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} = \underset{\sim}{0}. \tag{5}$$

First-Order Change in Terms of Formal Derivatives

We consider the nonsingular transformation

$$\begin{bmatrix} \underset{\sim}{\zeta}_1 \\ \underset{\sim}{\zeta}_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \underset{\sim}{1}^n & \underset{\sim}{1}^n \\ -\underset{\sim}{j}^n & \underset{\sim}{j}^n \end{bmatrix} \begin{bmatrix} \underset{\sim}{\zeta} \\ \underset{\sim}{\zeta}^* \end{bmatrix}, \tag{6}$$

where $\underset{\sim}{1}^n$ is the identity matrix of order n and

$$\underset{\sim}{j}^n \triangleq j\, \underset{\sim}{1}^n. \tag{7}$$

Equation (6) may be written in the perturbed form

$$\begin{bmatrix} \delta\underset{\sim}{\zeta}_1 \\ \delta\underset{\sim}{\zeta}_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \underset{\sim}{1}^n & \underset{\sim}{1}^n \\ -\underset{\sim}{j}^n & \underset{\sim}{j}^n \end{bmatrix} \begin{bmatrix} \delta\underset{\sim}{\zeta} \\ \delta\underset{\sim}{\zeta}^* \end{bmatrix}, \tag{8}$$

where δ denotes first-order change. Note that

$$\delta\underset{\sim}{\zeta}^* = (\delta\underset{\sim}{\zeta})^*. \tag{9}$$

The first-order change of f is given by

$$\delta f = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} \right)^T \delta\underset{\sim}{\zeta}_1 + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)^T \delta\underset{\sim}{\zeta}_2 \tag{10}$$

or, using (8),

$$\delta f = \frac{1}{2}\left[\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} \right)^T - \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)^T \underset{\sim}{j}^n\right] \delta\underset{\sim}{\zeta} + \frac{1}{2}\left[\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_1} \right)^T + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}_2} \right)^T \underset{\sim}{j}^n\right] \delta\underset{\sim}{\zeta}^*. \tag{11}$$

Hence, from (3) and (4)

$$\delta f = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^T \delta\underset{\sim}{\zeta} + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^T \delta\underset{\sim}{\zeta}^*. \tag{12}$$

Equation (12) expresses δf in terms of the variations in $\underset{\sim}{\zeta}$ and $\underset{\sim}{\zeta}^*$ using the formal derivatives $\partial f/\partial \underset{\sim}{\zeta}$ and $\partial f/\partial \underset{\sim}{\zeta}^*$ of (3) and (4), respectively.

Pure Real and Pure Imaginary Functions

For arbitrary $\underset{\sim}{\zeta}$, if

$$\underset{\sim}{\mu}^T \underset{\sim}{\zeta} + \overline{\underset{\sim}{\mu}}^T \underset{\sim}{\zeta}^* = \underset{\sim}{\mu}^{\bullet T} \underset{\sim}{\zeta} + \overline{\underset{\sim}{\mu}}^{\bullet T} \underset{\sim}{\zeta}^*, \tag{13}$$

where $\underset{\sim}{\mu}$, $\overline{\underset{\sim}{\mu}}$, $\underset{\sim}{\mu}^\bullet$ and $\overline{\underset{\sim}{\mu}}^\bullet$ are appropriate vectors of complex scalars, then, by equating the real and imaginary parts of both sides of (13) and subsequently equating the coefficients of $\underset{\sim}{\zeta}_1$ and $\underset{\sim}{\zeta}_2$ (since $\underset{\sim}{\zeta}$ is arbitrary), we get

$$\underset{\sim}{\mu} = \underset{\sim}{\mu}^\bullet \text{ and } \overline{\underset{\sim}{\mu}} = \overline{\underset{\sim}{\mu}}^\bullet . \tag{14}$$

For a pure real function f, we write

$$\delta f = \delta f^* = (\delta f)^* \tag{15}$$

or, using (12),

$$\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^T \delta \underset{\sim}{\zeta} + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^T \delta \underset{\sim}{\zeta}^* = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^{*T} \delta \underset{\sim}{\zeta}^* + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^{*T} \delta \underset{\sim}{\zeta}, \tag{16}$$

hence, from (13) and (14)

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}} = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^* . \tag{17}$$

Also, for a pure imaginary function f, we write

$$\delta f = - \delta f^* = - (\delta f)^*, \tag{18}$$

or

$$\left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^T \delta \underset{\sim}{\zeta} + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^T \delta \underset{\sim}{\zeta}^* = - \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^{*T} \delta \underset{\sim}{\zeta}^* - \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^{*T} \delta \underset{\sim}{\zeta} , \tag{19}$$

hence, from (13) and (14)

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}} = - \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^* . \qquad (20)$$

Remark

We remark [7] that the terminology of formal derivatives arises because of the possibility of obtaining them formally using the ordinary differentiation rules. The use of the conjugate notation facilitates the derivations and subsequent formulation of the equations to be solved.

III. THE COMPLEX LAGRANGIAN CONCEPT

In this section, we formulate the Lagrangian approach in the general complex case.

We consider, as before, a complex function f of a set of complex variables $\underset{\sim}{\zeta}$ and their complex conjugate $\underset{\sim}{\zeta}^*$. We write

$$\underset{\sim}{\zeta} = \begin{bmatrix} \underset{\sim}{\zeta}_x \\ \underset{\sim}{\zeta}_u \end{bmatrix} , \qquad (21)$$

where the variables $\underset{\sim}{\zeta}$ have been classified as n_x state variables $\underset{\sim}{\zeta}_x$ and n_u control variables $\underset{\sim}{\zeta}_u$. The state and control variables are related through the set of n_x complex equality constraints

$$\underset{\sim}{h} (\underset{\sim}{\zeta}, \underset{\sim}{\zeta}^*) = \underset{\sim}{0} . \qquad (22)$$

Complex Perturbed Form of Function and Equality Constraints

The first-order change of f is written, using (12), in the form

$$\delta f = [\; \underset{\sim}{f}_{\zeta x}^T \;\; \overline{\underset{\sim}{f}}_{\zeta x}^T \;] \begin{bmatrix} \delta \underset{\sim}{\zeta}_x \\ \delta \underset{\sim}{\zeta}_x^* \end{bmatrix} + [\; \underset{\sim}{f}_{\zeta u}^T \;\; \overline{\underset{\sim}{f}}_{\zeta u}^T \;] \begin{bmatrix} \delta \underset{\sim}{\zeta}_u \\ \delta \underset{\sim}{\zeta}_u^* \end{bmatrix} , \qquad (23)$$

where $\underset{\sim}{f}_{\zeta x}$, $\overline{\underset{\sim}{f}}_{\zeta x}$, $\underset{\sim}{f}_{\zeta u}$ and $\overline{\underset{\sim}{f}}_{\zeta u}$ stand for $\partial f/\partial \underset{\sim}{\zeta}_x$, $\partial f/\partial \underset{\sim}{\zeta}_x^*$, $\partial f/\partial \underset{\sim}{\zeta}_u$ and $\partial f/\partial \underset{\sim}{\zeta}_u^*$, respectively.

We write (22) in the perturbed form

$$\delta \underset{\sim}{h} (\underset{\sim}{\zeta}, \underset{\sim}{\zeta}^{*}) = \underset{\sim}{0} \tag{24}$$

or

$$[\underset{\sim}{H}_{\zeta x}\ \underset{\sim}{\overline{H}}_{\zeta x}] \begin{bmatrix} \delta \underset{\sim}{\zeta}_x \\ \delta \underset{\sim}{\zeta}_x^{*} \end{bmatrix} + [\underset{\sim}{H}_{\zeta u}\ \underset{\sim}{\overline{H}}_{\zeta u}] \begin{bmatrix} \delta \underset{\sim}{\zeta}_u \\ \delta \underset{\sim}{\zeta}_u^{*} \end{bmatrix} = \underset{\sim}{0}, \tag{25}$$

where $\underset{\sim}{H}_{\zeta x}$, $\underset{\sim}{\overline{H}}_{\zeta x}$, $\underset{\sim}{H}_{\zeta u}$ and $\underset{\sim}{\overline{H}}_{\zeta u}$ stand for $(\partial \underset{\sim}{h}^T / \partial \underset{\sim}{\zeta}_x)^T$, $(\partial \underset{\sim}{h}^T / \partial \underset{\sim}{\zeta}_x^{*})^T$, $(\partial \underset{\sim}{h}^T / \partial \underset{\sim}{\zeta}_u)^T$ and $(\partial \underset{\sim}{h}^T / \partial \underset{\sim}{\zeta}_u^{*})^T$, respectively. Using the complex conjugate of (25), we may write

$$\begin{bmatrix} \underset{\sim}{H}_{\zeta x} & \underset{\sim}{\overline{H}}_{\zeta x} \\ \underset{\sim}{\overline{H}}_{\zeta x}^{*} & \underset{\sim}{H}_{\zeta x}^{*} \end{bmatrix} \begin{bmatrix} \delta \underset{\sim}{\zeta}_x \\ \delta \underset{\sim}{\zeta}_x^{*} \end{bmatrix} = - \begin{bmatrix} \underset{\sim}{d}_u \\ \underset{\sim}{d}_u^{*} \end{bmatrix}, \tag{26}$$

where

$$\underset{\sim}{d}_u \triangleq \underset{\sim}{H}_{\zeta u}\ \delta \underset{\sim}{\zeta}_u + \underset{\sim}{\overline{H}}_{\zeta u}\ \delta \underset{\sim}{\zeta}_u^{*}. \tag{27}$$

Matrix Rank and Consistency Conditions

The following two theorems justify the analytical aspects of the sets of complex equations of the form (26) expressed in terms of complex variables and their complex conjugate.

Theorem 1

Let $\underset{\sim}{\theta}, \underset{\sim}{\overline{\theta}} \in C^{m \times n}$, where

$$\underset{\sim}{\theta} = \underset{\sim}{\theta}_1 + j\ \underset{\sim}{\theta}_2 \tag{28}$$

and

$$\underset{\sim}{\overline{\theta}} = \underset{\sim}{\overline{\theta}}_1 + j\ \underset{\sim}{\overline{\theta}}_2, \tag{29}$$

$\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \underset{\sim}{\bar{\theta}}_1, \underset{\sim}{\bar{\theta}}_2 \in R^{m\times n}$. Then the two matrices $\underset{\sim}{\theta}^c \in C^{2m\times 2n}$ and $\underset{\sim}{\theta}^r \in R^{2m\times 2n}$ defined as

$$\underset{\sim}{\theta}^c \triangleq \begin{bmatrix} \underset{\sim}{\theta} & \underset{\sim}{\bar{\theta}} \\ \underset{\sim}{\bar{\theta}}^* & \underset{\sim}{\theta}^* \end{bmatrix} \tag{30}$$

and

$$\underset{\sim}{\theta}^r \triangleq \begin{bmatrix} (\underset{\sim}{\theta}_1 + \underset{\sim}{\bar{\theta}}_1) & (\underset{\sim}{\bar{\theta}}_2 - \underset{\sim}{\theta}_2) \\ (\underset{\sim}{\theta}_2 + \underset{\sim}{\bar{\theta}}_2) & (\underset{\sim}{\theta}_1 - \underset{\sim}{\bar{\theta}}_1) \end{bmatrix} \tag{31}$$

have the same rank.

Proof

Let $\underset{\sim}{1}^\ell$ be the identity matrix of order ℓ and

$$\underset{\sim}{j}^\ell \triangleq j\, \underset{\sim}{1}^\ell, \tag{32}$$

and define the two unitary matrices

$$\underset{\sim}{U}_L \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} \underset{\sim}{1}^m & \underset{\sim}{j}^m \\ \underset{\sim}{1}^m & -\underset{\sim}{j}^m \end{bmatrix} \tag{33}$$

and

$$\underset{\sim}{U}_R \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} \underset{\sim}{1}^n & \underset{\sim}{1}^n \\ -\underset{\sim}{j}^n & \underset{\sim}{j}^n \end{bmatrix} \tag{34}$$

Since $\underset{\sim}{U}_L$ and $\underset{\sim}{U}_R$ are nonsingular, hence [9]

$$\text{rank}[\underset{\sim}{U}_L\, \underset{\sim}{\theta}^r\, \underset{\sim}{U}_R] = \text{rank}[\underset{\sim}{\theta}^r].$$

But

$$\underset{\sim}{U}_L\, \underset{\sim}{\theta}^r\, \underset{\sim}{U}_R = \underset{\sim}{\theta}^c,$$

hence

$$\text{rank}[\underset{\sim}{\theta}^{r}] = \text{rank}[\underset{\sim}{\theta}^{c}] \qquad \square$$

<u>Theorem 2</u>

Let $\underset{\sim}{\theta}$, $\overline{\underset{\sim}{\theta}} \in C^{mxn}$ given by (28) and (29), and let $\underset{\sim}{\theta} \in C^{m}$ and $\underset{\sim}{w} \in C^{n}$, where

$$\underset{\sim}{\theta} = \underset{\sim}{\theta}_1 + j\ \underset{\sim}{\theta}_2 \tag{35}$$

and

$$\underset{\sim}{w} = \underset{\sim}{w}_1 + j\ \underset{\sim}{w}_2, \tag{36}$$

$\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2 \in R^{m}$ and $\underset{\sim}{w}_1$, $\underset{\sim}{w}_2 \in R^{n}$. then the system of complex linear equations

$$\underset{\sim}{\theta}\ \underset{\sim}{w} + \overline{\underset{\sim}{\theta}}\ \underset{\sim}{w}^{*} = \underset{\sim}{\theta} \tag{37}$$

has a solution $\underset{\sim}{w}$ if and only if

$$\text{rank}[(\underset{\sim}{\theta}^{c},\ \underset{\sim}{\theta}^{c})] = \text{rank}[\underset{\sim}{\theta}^{c}],$$

where $\underset{\sim}{\theta}^{c} \in C^{2m}$ is defined as

$$\underset{\sim}{\theta}^{c} \triangleq \begin{bmatrix} \underset{\sim}{\theta} \\ \underset{\sim}{\theta}^{*} \end{bmatrix} \tag{38}$$

and $\underset{\sim}{\theta}^{c}$ is given by (30).

<u>Proof</u>

Separating (37) into real and imaginary parts using (28), (29), (35) and (36), we get

$$(\underset{\sim}{\theta}_1 + \overline{\underset{\sim}{\theta}}_1)\ \underset{\sim}{w}_1 + (\overline{\underset{\sim}{\theta}}_2 - \underset{\sim}{\theta}_2)\ \underset{\sim}{w}_2 = \underset{\sim}{\theta}_1 \tag{39}$$

and

$$(\underset{\sim}{\theta}_2 + \overline{\underset{\sim}{\theta}}_2)\ \underset{\sim}{w}_1 + (\underset{\sim}{\theta}_1 - \overline{\underset{\sim}{\theta}}_1)\ \underset{\sim}{w}_2 = \underset{\sim}{\theta}_2 \tag{40}$$

or, using (31),

$$\underset{\sim}{\Theta}^r\ \underset{\sim}{w}^r = \underset{\sim}{\theta}^r, \tag{41}$$

where

$$\underset{\sim}{w}^r \triangleq \begin{bmatrix} \underset{\sim}{w}_1 \\ \\ \underset{\sim}{w}_2 \end{bmatrix} \tag{42}$$

and

$$\underset{\sim}{\theta}^r \triangleq \begin{bmatrix} \underset{\sim}{\theta}_1 \\ \\ \underset{\sim}{\theta}_2 \end{bmatrix}. \tag{43}$$

We define the nonsingular matrix

$$\overline{\underset{\sim}{U}}_R \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} \underset{\sim}{1}^n & \underset{\sim}{1}^n & \underset{\sim}{0} \\ \\ -\underset{\sim}{j}^n & \underset{\sim}{j}^n & \underset{\sim}{0} \\ \\ \underset{\sim}{0} & \underset{\sim}{0} & 2 \end{bmatrix}, \tag{44}$$

hence

$$\text{rank}[\underset{\sim}{U}_L\ (\underset{\sim}{\theta}^r,\ \underset{\sim}{\theta}^r)\ \overline{\underset{\sim}{U}}_R] = \text{rank}[\underset{\sim}{\theta}^r,\ \underset{\sim}{\theta}^r)],$$

where $\underset{\sim}{U}_L$ is given by (33). But

$$\underset{\sim}{U}_L\ (\underset{\sim}{\theta}^r,\ \underset{\sim}{\theta}^r)\ \overline{\underset{\sim}{U}}_R = (\underset{\sim}{\theta}^c,\ \underset{\sim}{\theta}^c),$$

hence

$$\text{rank}[(\underset{\sim}{\theta}^r,\ \underset{\sim}{\theta}^r)] = \text{rank}[(\underset{\sim}{\theta}^c,\ \underset{\sim}{\theta}^c)]. \tag{45}$$

Now, the system of equations (41) has a solution if and only if

$$\text{rank}[(\underset{\sim}{\theta}^r, \underset{\sim}{\theta}^r)] = \text{rank}[\underset{\sim}{\theta}^r],$$

hence the theorem is proved using (45) and Theorem 1 □

Sensitivity Calculations

Now, we write (26) in the form

$$\begin{bmatrix} \delta\underset{\sim}{\zeta}_x \\ \delta\underset{\sim}{\zeta}_x^* \end{bmatrix} = - \begin{bmatrix} \underset{\sim}{H}_{\zeta x} & \overline{\underset{\sim}{H}}_{\zeta x} \\ \overline{\underset{\sim}{H}}{}^*_{\zeta x} & \underset{\sim}{H}^*_{\zeta x} \end{bmatrix}^{-1} \begin{bmatrix} \underset{\sim}{H}_{\zeta u} & \overline{\underset{\sim}{H}}_{\zeta u} \\ \overline{\underset{\sim}{H}}{}^*_{\zeta u} & \underset{\sim}{H}^*_{\zeta u} \end{bmatrix} \begin{bmatrix} \delta\underset{\sim}{\zeta}_u \\ \delta\underset{\sim}{\zeta}_u^* \end{bmatrix} . \tag{46}$$

From Theorem 1, the inverted matrix in (46) has full rank if and only if the system of equations (24) represent $2n_x$ independent conditions.

Using (46), δf of (23) is written in the form

$$\delta f = \left\{ [\, \underset{\sim}{f}^T_{\zeta u} \; \overline{\underset{\sim}{f}}{}^T_{\zeta u} \,] - [\, \underset{\sim}{\lambda}^T \; \overline{\underset{\sim}{\lambda}}{}^T \,] \begin{bmatrix} \underset{\sim}{H}_{\zeta u} & \overline{\underset{\sim}{H}}_{\zeta u} \\ \overline{\underset{\sim}{H}}{}^*_{\zeta u} & \underset{\sim}{H}^*_{\zeta u} \end{bmatrix} \right\} \begin{bmatrix} \delta\underset{\sim}{\zeta}_u \\ \delta\underset{\sim}{\zeta}_u^* \end{bmatrix} , \tag{47}$$

where

$$\begin{bmatrix} \underset{\sim}{H}^T_{\zeta x} & \overline{\underset{\sim}{H}}{}^{*T}_{\zeta x} \\ \overline{\underset{\sim}{H}}{}^T_{\zeta x} & \underset{\sim}{H}^{*T}_{\zeta x} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\lambda} \\ \overline{\underset{\sim}{\lambda}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}_{\zeta x} \\ \overline{\underset{\sim}{f}}_{\zeta x} \end{bmatrix} . \tag{48}$$

Hence, the total formal derivatives of f are given, from (47), by

$$\frac{df}{d\underset{\sim}{\zeta}_u} = \underset{\sim}{f}_{\zeta u} - \underset{\sim}{H}^T_{\zeta u}\,\underset{\sim}{\lambda} - \overline{\underset{\sim}{H}}{}^{*T}_{\zeta u}\,\overline{\underset{\sim}{\lambda}} \tag{49}$$

and

$$\frac{df}{d\underset{\sim}{\zeta}_u^*} = \overline{\underset{\sim}{f}}_{\zeta u} - \overline{\underset{\sim}{H}}{}^T_{\zeta u}\,\underset{\sim}{\lambda} - \underset{\sim}{H}^{*T}_{\zeta u}\,\overline{\underset{\sim}{\lambda}} . \tag{50}$$

The complex Lagrange multipliers $\underset{\sim}{\lambda}$ and $\overline{\underset{\sim}{\lambda}}$ of (49) and (50) are obtained by solving the set of complex adjoint equations (48). Theorem 2 provides the consistency conditions of the adjoint equations (48).

Note that in the real case when the function f and constraints $\underset{\sim}{h}$ are all pure real, the application of (17) results in the complex conjugate relationships $\overline{f}_{\underset{\sim}{\zeta}x} = f^*_{\underset{\sim}{\zeta}x}$ and $\overline{H}_{\underset{\sim}{\zeta}x} = H^*_{\underset{\sim}{\zeta}x}$ and (48) reduces to a system of n_x complex equations in the real variables $(\underset{\sim}{\lambda} + \overline{\underset{\sim}{\lambda}})$. The solution of this system of equations is then substituted into (49) and (50) which form a complex conjugate pair since, from (17), $\overline{f}_{\underset{\sim}{\zeta}u} = f^*_{\underset{\sim}{\zeta}u}$ and $\overline{H}_{\underset{\sim}{\zeta}u} = H^*_{\underset{\sim}{\zeta}u}$. Observe that this conclusion agrees with the relationship (17) when applied to the total formal derivatives of (49) and (50).

Note also that the conventional reduced gradients (w.r.t. real control variables) can be obtained from the formal total derivatives using the inverse relationships of (3) and (4), namely

$$\frac{df}{d\underset{\sim}{\zeta}_{u1}} = \frac{df}{d\underset{\sim}{\zeta}_u} + \frac{df}{d\underset{\sim}{\zeta}_u^*} \tag{51}$$

and

$$\frac{df}{d\underset{\sim}{\zeta}_{u2}} = j\left(\frac{df}{d\underset{\sim}{\zeta}_u} - \frac{df}{d\underset{\sim}{\zeta}_u^*}\right) , \tag{52}$$

where $\underset{\sim}{\zeta}_u = \underset{\sim}{\zeta}_{u1} + j\,\underset{\sim}{\zeta}_{u2}$. In the case of a pure real function f, the relations (51) and (52) reduce to

$$\frac{df}{d\underset{\sim}{\zeta}_{u1}} = 2\,\mathrm{Re}\,\left\{\frac{df}{d\underset{\sim}{\zeta}_u}\right\} \tag{53}$$

and

$$\frac{df}{d\underset{\sim}{\zeta}_{u2}} = -\,2\,\mathrm{Im}\,\left\{\frac{df}{d\underset{\sim}{\zeta}_u}\right\} . \tag{54}$$

We have stated the Lagrangian approach in the complex form and derived the corresponding adjoint equations to be solved for the

Lagrange multipliers so that the required formal derivatives (49) and (50) may be obtained. In the following two sections, we consider some applications of the complex Lagrangian approach in power system analysis and design.

IV. APPLICATION TO POWER NETWORK ANALYSIS

The complex Lagrangian approach described in the previous section can be applied, for example, to power network sensitivity calculations. The set of complex equality constraints (22) may represent the power flow equations of the form

$$\underset{\sim}{h} = \underset{\sim}{S}_M^* - \underset{\sim}{E}_M^* \underset{\sim}{Y}_T \underset{\sim}{V}_M = \underset{\sim}{0} \ , \tag{55}$$

where $\underset{\sim}{S}_M$ is a vector of the bus powers, $\underset{\sim}{V}_M$ is a vector of bus voltages, $\underset{\sim}{Y}_T$ is the bus admittance matrix of dimension nxn, n denoting number of buses in the power network and $\underset{\sim}{E}_M$ is a diagonal matrix of components of $\underset{\sim}{V}_M$ in a corresponding order.

The vectors $\underset{\sim}{\zeta}_x$ and $\underset{\sim}{\zeta}_u$ of (21) are defined as

$$\underset{\sim}{\zeta}_x \triangleq \begin{bmatrix} \underset{\sim}{V}_L \\ S_n \end{bmatrix} \tag{56}$$

and

$$\underset{\sim}{\zeta}_u \triangleq \begin{bmatrix} \underset{\sim}{S}_L \\ V_n \end{bmatrix} \ , \tag{57}$$

where we have classified, for simplicity, the buses as load-type buses of voltages $\underset{\sim}{V}_L$ and powers $\underset{\sim}{S}_L$ and a slack bus of voltage V_n and power S_n. We write (55) in the corresponding partitioned form

$$\begin{bmatrix} \underset{\sim}{h}_L \\ h_n \end{bmatrix} = \begin{bmatrix} \underset{\sim}{S}_L^* \\ S_n^* \end{bmatrix} - \begin{bmatrix} \underset{\sim}{E}_L & \underset{\sim}{0} \\ \underset{\sim}{0} & V_n^* \end{bmatrix} \begin{bmatrix} \underset{\sim}{Y}_{LL} & \underset{\sim}{Y}_{LN} \\ \underset{\sim}{Y}_{LN}^T & Y_{nn} \end{bmatrix} \begin{bmatrix} \underset{\sim}{V}_L \\ V_n \end{bmatrix}, \tag{58}$$

where the symmetric bus admittance matrix has been partitioned into $\underset{\sim}{Y}_{LL}$, $\underset{\sim}{Y}_{LN}$, $\underset{\sim}{Y}_{LN}^T$ and Y_{nn} of appropriate dimensions.

The matrices $\partial \underset{\sim}{h}^T/\partial \underset{\sim}{\zeta}_x$, $\partial \underset{\sim}{h}^T/\partial \underset{\sim}{\zeta}_x^*$, $\partial \underset{\sim}{h}^T/\partial \underset{\sim}{\zeta}_u$ and $\partial \underset{\sim}{h}^T/\partial \underset{\sim}{\zeta}_u^*$ are given, respectively, by

$$\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{\zeta}_x} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{h}_L^T}{\partial \underset{\sim}{V}_L} & \dfrac{\partial h_n}{\partial \underset{\sim}{V}_L} \\ \dfrac{\partial \underset{\sim}{h}_L^T}{\partial S_n} & \dfrac{\partial h_n}{\partial S_n} \end{bmatrix}, \tag{59}$$

$$\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{\zeta}_x^*} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{h}_L^T}{\partial \underset{\sim}{V}_L^*} & \dfrac{\partial h_n}{\partial \underset{\sim}{V}_L^*} \\ \dfrac{\partial \underset{\sim}{h}_L^T}{\partial S_n^*} & \dfrac{\partial h_n}{\partial S_n^*} \end{bmatrix}, \tag{60}$$

$$\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{\zeta}_u} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{h}_L^T}{\partial \underset{\sim}{S}_L} & \dfrac{\partial h_n}{\partial \underset{\sim}{S}_L} \\ \dfrac{\partial \underset{\sim}{h}_L^T}{\partial V_n} & \dfrac{\partial h_n}{\partial V_n} \end{bmatrix} \tag{61}$$

and

$$\frac{\partial \underset{\sim}{h}^T}{\partial \underset{\sim}{\zeta}_u^*} = \begin{bmatrix} \dfrac{\partial \underset{\sim}{h}_L^T}{\partial \underset{\sim}{S}_L^*} & \dfrac{\partial h_n}{\partial \underset{\sim}{S}_L^*} \\ \dfrac{\partial \underset{\sim}{h}_L^T}{\partial V_n^*} & \dfrac{\partial h_n}{\partial V_n^*} \end{bmatrix}, \tag{62}$$

Using (58)-(62), the matrices $\underset{\sim}{H}_{\zeta x}$, $\overline{\underset{\sim}{H}}_{\zeta x}$, $\underset{\sim}{H}_{\zeta u}$ and $\overline{\underset{\sim}{H}}_{\zeta u}$ of (25) are given, respectively, by

$$\underset{\sim}{H}_{\zeta x} = \begin{bmatrix} -(\underset{\sim}{E}_L^* \underset{\sim}{Y}_{LL}) & \underset{\sim}{0} \\ -(V_n^* \underset{\sim}{Y}_{LN}^T) & 0 \end{bmatrix}, \tag{63}$$

$$\underset{\sim}{\overline{H}}_{\zeta x} = \begin{bmatrix} -\operatorname{diag}\{\underset{\sim}{I}_L\} & \underset{\sim}{0} \\ \underset{\sim}{0} & 1 \end{bmatrix}, \tag{64}$$

$$\underset{\sim}{H}_{\zeta u} = \begin{bmatrix} \underset{\sim}{0} & -(\underset{\sim}{E}_L^* \underset{\sim}{Y}_{LN}) \\ \underset{\sim}{0} & -V_n^* Y_{nn} \end{bmatrix} \tag{65}$$

and

$$\underset{\sim}{\overline{H}}_{\zeta u} = \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{0} \\ \underset{\sim}{0} & -I_n \end{bmatrix}, \tag{66}$$

where the bus currents

$$\underset{\sim}{I}_M = \begin{bmatrix} \underset{\sim}{I}_L \\ I_n \end{bmatrix} \tag{67}$$

are given by

$$\underset{\sim}{I}_M = \underset{\sim}{Y}_T \underset{\sim}{V}_M. \tag{68}$$

For a given function f with the formal derivatives $\underset{\sim}{f}_{\zeta x}$ and $\underset{\sim}{\overline{f}}_{\zeta x}$, the adjoint system of equations (48) is formed using (63) and (64) and solved for the Lagrange multipliers $\underset{\sim}{\lambda}$ and $\underset{\sim}{\overline{\lambda}}$. The total formal derivatives of f w.r.t. the control variables are then calculated from (49) and (50) using (65) and (66).

We remark that the choice of $\underset{\sim}{V}_L$ and S_n as the only control variables $\underset{\sim}{\zeta}_u$ has been made for simplicity. We could equally well define other control variables, e.g., line admittances represented in the bus admittance matrix. Note also that the voltage-controlled buses or generator-type buses [3] can be included by defining complex conjugate pairs of state variables, e.g.,

$$\zeta_x^g \triangleq Q_g + j\ \delta_g \tag{69}$$

and of control variables, e.g.,

$$\zeta_u^g \triangleq P_g + j\ |V_g|, \tag{70}$$

where the generator bus power S_g is given by

$$S_g = P_g + j\ Q_g \tag{71}$$

and the generator bus voltage V_g is given by

$$V_g = |V_g| \angle \delta_g. \tag{72}$$

The modification required to include other control and state variables can be performed in a straightforward manner.

VI. CONCLUSIONS

We have presented the theoretical foundations of a useful algebraic notation for sensitivity evaluation in the complex mode. The far reaching consequences gained by using the compact conjugate notation have been exploited in formulating the Lagrangian approach in the complex form. First-order changes and formal derivatives of complex functions of interest subject to general complex equality constraints can be evaluated, directly, while keeping the original compact complex mode of formulation. An important application to power network sensitivity analysis has been studied. In this application, the first-order change and the reduced gradients of a general function are evaluated subject to complex equality constraints representing the power flow equations.

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A COMPLEX LAGRANGIAN APPROACH WITH APPLICATIONS TO POWER NETWORK SENSITIVITY ANALYSIS

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Abstract: The well-known Lagrangian approach, traditionally described in real form, for calculating first-order changes and gradients of functions of interest subject to equality constraints is generalized and applied in a compact complex form. Hence, general complex functions and constraints can be handled directly while maintaining the original complex mode of formulation. The theoretical foundations of the approach are stated. An application to power network sensitivity analysis and gradient evaluation is presented.

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