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A UNIFIED APPROACH TO POWER SYSTEM SENSITIVITY ANALYSIS AND PLANNING

PART I: FAMILY OF ADJOINT SYSTEMS

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Abstract

Efficient sensitivity analysis and gradient evaluation, essential in power system studies such as optimal power flow, contingency analysis and planning, is the subject of this paper. We present an approach based upon a generalized adjoint network concept. It exploits all the powerful features of Tellegen's theorem by suitable extensions through which the a.c. load flow model can be used without any approximations. We introduce the conjugate notation used in formulating the Tellegen expressions for general complex functions. We also introduce the concept of group terms which facilitate control of the adjoint system so that a wide variety of particular cases can be handled. We derive and tabulate standard sensitivity expressions common to all relevant power system studies.

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I. INTRODUCTION

Efficient sensitivity analysis and gradient evaluation are essential in power system studies such as optimal power flow, contingency analysis and planning. A number of relevant papers [1-6] have dealt with appropriate computational approaches. Previous work based upon Tellegen's theorem [7-9] approximates the a.c. load flow model.

In this paper (see also Part II), we present a general approach to power system sensitivity analysis and planning based upon a generalized adjoint network concept. Our new theory exploits all the powerful features of Tellegen's theorem by suitable extensions through which the a.c. load flow model can be used without any approximations. Hence, all applications of interest can be handled.

Part I of this paper introduces the conjugate notation employed throughout. It is used in formulating the Tellegen expressions for general complex functions. These expressions are augmented via complex coefficients, in general. We also introduce the concept of group terms which facilitate control of the adjoint system so that a wide variety of particular cases can be handled.

We denote branch voltages and currents and their complex conjugates as basic variables since the theory is expressed in them. Transformations from these basic variables to element variables of practical interest to power system engineers are developed.

We derive and tabulate standard sensitivity expressions common to all relevant power system studies.

II. CONJUGATE NOTATION

In this paper we use a special notation which we shall call the conjugate notation. A complex variable and its complex conjugate replace, as independent quantities, the real and imaginary parts of the variable. The use of conjugate notation facilitates the derivations and subsequent formulation of the equations to be solved.

Consider a complex function f of a set of complex variables assembled as the column vector $\underset{\sim}{\zeta}$. Let f_1 and f_2 be the real and imaginary parts, respectively, of f and $\underset{\sim}{\zeta}_1$ and $\underset{\sim}{\zeta}_2$ the real and imaginary parts, respectively, of $\underset{\sim}{\zeta}$. Then the first-order variation of f is expressed as

$$\delta f(\underset{\sim}{\zeta}_1, \underset{\sim}{\zeta}_2) = \delta f_1(\underset{\sim}{\zeta}_1, \underset{\sim}{\zeta}_2) + j\delta f_2(\underset{\sim}{\zeta}_1, \underset{\sim}{\zeta}_2)$$

$$= \left(\frac{\partial f_1}{\partial \underset{\sim}{\zeta}_1} \right)^T \delta\underset{\sim}{\zeta}_1 + \left(\frac{\partial f_1}{\partial \underset{\sim}{\zeta}_2} \right)^T \delta\underset{\sim}{\zeta}_2 + j\left\{ \left(\frac{\partial f_2}{\partial \underset{\sim}{\zeta}_1} \right)^T \delta\underset{\sim}{\zeta}_1 + \left(\frac{\partial f_2}{\partial \underset{\sim}{\zeta}_2} \right)^T \delta\underset{\sim}{\zeta}_2 \right\}, \qquad (1)$$

where, $\partial f_i/\underset{\sim}{\zeta}_j$; i, j = 1, 2, are column vectors of appropriate partial derivatives assembled corresponding to the components of $\underset{\sim}{\zeta}_1$ and $\underset{\sim}{\zeta}_2$. T denotes transposition. Using conjugate notation we obtain

$$\delta f(\underset{\sim}{\zeta}, \underset{\sim}{\zeta}^*) = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}} \right)^T \delta\underset{\sim}{\zeta} + \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^T \delta\underset{\sim}{\zeta}^*, \qquad (2)$$

where * denotes the complex conjugate and partial derivatives of f w.r.t. $\underset{\sim}{\zeta}$ and $\underset{\sim}{\zeta}^*$ are again assembled in appropriate column vectors. Note that (1) and (2) are equivalent.

Let us consider the arbitrary example

$$f = \zeta^2 + \zeta^* = (\zeta_1^2 - \zeta_2^2 + \zeta_1) + j(2\zeta_1\zeta_2 - \zeta_2).$$

In this case, (1) leads to

$$\delta f = [(2\zeta_1 + 1)\delta\zeta_1 - 2\zeta_2\,\delta\zeta_2] + j[2\zeta_2\,\delta\zeta_1 + (2\zeta_1 - 1)\delta\zeta_2],$$

and (2) leads to

$$\delta f = 2\zeta\ \delta\zeta + \delta\zeta^*.$$

Note also that, for real functions, we can show that

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}} = \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^*, \tag{3}$$

and for imaginary functions

$$\frac{\partial f}{\partial \underset{\sim}{\zeta}} = - \left(\frac{\partial f}{\partial \underset{\sim}{\zeta}^*} \right)^*. \tag{4}$$

III. TELLEGEN'S TERMS AND GROUP TERMS FOR THE A.C. POWER MODEL

In this paper variables are basically expressed in terms of the complex voltage V and complex current I associated with the given network. We use ^ to distinguish the corresponding variables associated with the topologically similar adjoint network.

Tellegen's theorem [7], which depends solely upon Kirchhoff's laws and the topology of the network, states that

$$\sum_b \hat{I}_b V_b = 0 \tag{5a}$$

and

$$\sum_b \hat{V}_b I_b = 0, \tag{5b}$$

where the summation is taken over all branches, subscript b denoting the bth branch.

Since the V_b and $\hat{V}_b$ of (5) satisfy Kirchhoff's voltage law (KVL), the V_b^* and $\hat{V}_b^*$ also satisfy KVL. Similarly, since the I_b and $\hat{I}_b$ of (5) satisfy Kirchhoff's current law (KCL), the I_b^* and $\hat{I}_b^*$ also satisfy KCL. Hence, in addition to (5) the following valid variations of Tellegen's theorem can be considered [9]

$$\sum_b \hat{I}_b^* V_b^* = 0, \quad \text{(5c)}$$

$$\sum_b \hat{V}_b^* I_b^* = 0, \quad \text{(5d)}$$

$$\sum_b \hat{I}_b V_b^* = 0, \quad \text{(5e)}$$

$$\sum_b \hat{V}_b I_b^* = 0, \quad \text{(5f)}$$

$$\sum_b \hat{I}_b^* V_b = 0, \quad \text{(5g)}$$

$$\sum_b \hat{V}_b^* I_b = 0. \quad \text{(5h)}$$

Note that, in the case of identical original and adjoint networks, we set $\hat{V}_b = V_b$ and $\hat{I}_b = I_b$ in (5).

In addition to Tellegen's terms (5) we also consider valid expressions in terms of certain groups of elements in the form

$$\sum_{b \in B_k} C_b^k = 0 \quad \text{(6a)}$$

and

$$\sum_{b \in B_k} C_b^{k*} = 0, \tag{6b}$$

where C_b^k and C_b^{k*} are complex functions of the variables V_b and I_b and their complex conjugates V_b^* and I_b^*, and B_k is the set of branch elements forming the kth group.

An example of the group terms (6) is the KVL for a local loop of the network. The number of the group terms considered in a practical problem is usually small.

The extended Tellegen's sum is now written as

$$\sum_b [\alpha \hat{I}_b V_b + \bar{\alpha} \hat{I}_b^* V_b^* - \beta \hat{V}_b I_b - \bar{\beta} \hat{V}_b^* I_b^* + \xi \hat{I}_b V_b^* + \bar{\xi} \hat{I}_b^* V_b$$

$$- \nu \hat{V}_b I_b^* - \bar{\nu} \hat{V}_b^* I_b + \sum_k \Gamma_k \lambda_{bk} C_b^k + \sum_k \bar{\Gamma}_k \lambda_{bk} C_b^{k*}] = 0, \tag{7}$$

where the terms (5) and (6) have been adjoined in an appropriate sequence via the complex coefficients α, $\bar{\alpha}$, β, $\bar{\beta}$, ξ, $\bar{\xi}$, ν, $\bar{\nu}$, Γ_k and $\bar{\Gamma}_k$,

$$\lambda_{bk} = \begin{cases} 0 & \text{if } b \notin B_k \\ 1 & \text{if } b \in B_k \end{cases} . \tag{8}$$

Note that in cases where

$$\bar{\alpha} = \alpha^*, \tag{9a}$$

$$\bar{\beta} = \beta^*, \tag{9b}$$

$$\bar{\xi} = \xi^*, \tag{9c}$$

$$\bar{\nu} = \nu^*, \tag{9d}$$

and

$$\overline{\Gamma}_k = \Gamma_k^* \text{ for all } k, \tag{9e}$$

the extended Tellegen's sum (7) is a real quantity.

In sensitivity analysis first-order changes are of prime interest. The sum (7) is written in terms of first-order changes in V and I as

$$\sum_b [\alpha \hat{I}_b \delta V_b + \overline{\alpha} \hat{I}_b^* \delta V_b^* - \beta \hat{V}_b \delta I_b - \overline{\beta} \hat{V}_b^* \delta I_b^* + \xi \hat{I}_b \delta V_b^* + \overline{\xi} \hat{I}_b^* \delta V_b$$

$$- \nu \hat{V}_b \delta I_b^* - \overline{\nu} \hat{V}_b^* \delta I_b + \sum_k \Gamma_k \lambda_{bk} (C_{bv}^k \delta V_b + \overline{C}_{bv}^k \delta V_b^* + C_{bi}^k \delta I_b$$

$$+ \overline{C}_{bi}^k \delta I_b^*) + \sum_k \overline{\Gamma}_k \lambda_{bk} (C_{bv}^{k*} \delta V_b + \overline{C}_{bv}^{k*} \delta V_b^* + C_{bi}^{k*} \delta I_b + \overline{C}_{bi}^{k*} \delta I_b^*)] = 0,$$

(10a)

or

$$\sum_b [(\alpha \hat{I}_b + \overline{\xi} \hat{I}_b^* + \sum_k \Gamma_k \lambda_{bk} C_{bv}^k + \sum_k \overline{\Gamma}_k \lambda_{bk} C_{bv}^{k*}) \delta V_b$$

$$+ (\overline{\alpha} \hat{I}_b^* + \xi \hat{I}_b + \sum_k \Gamma_k \lambda_{bk} \overline{C}_{bv}^k + \sum_k \overline{\Gamma}_k \lambda_{bk} \overline{C}_{bv}^{k*}) \delta V_b^*$$

$$+ (-\beta \hat{V}_b - \overline{\nu} \hat{V}_b^* + \sum_k \Gamma_k \lambda_{bk} C_{bi}^k + \sum_k \overline{\Gamma}_k \lambda_{bk} C_{bi}^{k*}) \delta I_b$$

$$+ (-\overline{\beta} \hat{V}_b^* - \nu \hat{V}_b + \sum_k \Gamma_k \lambda_{bk} \overline{C}_{bi}^k + \sum_k \overline{\Gamma}_k \lambda_{bk} \overline{C}_{bi}^{k*}) \delta I_b^*] = 0, \tag{10b}$$

where C_{bu}^k, C_{bu}^{k*}, $\overline{C}_{bu}^k$ and $\overline{C}_{bu}^{k*}$ stand for $\partial C_b^k/\partial U$, $\partial C_b^{k*}/\partial U$, $\partial C_b^k/\partial U^*$, $\partial C_b^{k*}/\partial U^*$, respectively, u denoting v or i and U denoting V or I.

IV. ELEMENT VARIABLES VIA BASIC VARIABLES

The perturbed Tellegen's sum (10) has been written in terms of first-order changes of V_b, V_b^*, I_b and I_b^*. We shall call these variables the basic variables and denote them by the vector

$$\underset{\sim}{w}_b = \begin{bmatrix} \underset{\sim}{w}_{bv} \\ \hline \underset{\sim}{w}_{bi} \end{bmatrix} \triangleq \begin{bmatrix} V_b \\ V_b^* \\ \hline I_b \\ I_b^* \end{bmatrix} . \qquad (11)$$

Now, for each element, and according to its type, another set of variables called the element variables is of practical interest. The element variables will be denoted by the vector $\underset{\sim}{z}_b$ of four components describing the practical state $\underset{\sim}{x}_b$ and control $\underset{\sim}{u}_b$ variables associated with element b as

$$\underset{\sim}{z}_b = \begin{bmatrix} \underset{\sim}{x}_b \\ \underset{\sim}{u}_b \end{bmatrix} , \qquad (12)$$

where $\underset{\sim}{x}_b$ and $\underset{\sim}{u}_b$ are two component vectors. $\delta\underset{\sim}{z}_b$ can be expressed in terms of $\delta\underset{\sim}{w}_b$ in the form

$$\delta\underset{\sim}{z}_b = \begin{bmatrix} \delta\underset{\sim}{x}_b \\ \delta\underset{\sim}{u}_b \end{bmatrix} = \underset{\sim}{J}_b \delta\underset{\sim}{w}_b , \qquad (13)$$

where

$$\underset{\sim}{J}_b \triangleq \left(\frac{\partial \underset{\sim}{z}_b^T}{\partial \underset{\sim}{w}_b} \right)^T \qquad (14)$$

is the Jacobian matrix.

From (13)

$$\delta \underset{\sim}{w}_b = \underset{\sim}{J}_b^{-1} \; \delta \underset{\sim}{z}_b . \tag{15}$$

A term of (10) associated with the bth branch is written in the more convenient form

$$\hat{\underset{\sim}{f}}_b^T \; \delta \underset{\sim}{w}_b , \tag{16}$$

where

$$\hat{\underset{\sim}{f}}_b = \begin{pmatrix} \hat{\underset{\sim}{f}}_{bi} \\ \hline \hat{\underset{\sim}{f}}_{bv} \end{pmatrix} \triangleq \begin{pmatrix} \alpha \; \hat{I}_b + \overline{\xi} \; \hat{I}_b^* + \sum_k \Gamma_k \; \lambda_{bk} \; C_{bv}^k + \sum_k \overline{\Gamma}_k \; \lambda_{bk} \; C_{bv}^{k*} \\ \overline{\alpha} \; \hat{I}_b^* + \xi \; \hat{I}_b + \sum_k \Gamma_k \; \lambda_{bk} \; \overline{C}_{bv}^k + \sum_k \overline{\Gamma}_k \; \lambda_{bk} \; \overline{C}_{bv}^{k*} \\ \hline -\beta \; \hat{V}_b - \overline{\nu} \; \hat{V}_b^* + \sum_k \Gamma_k \; \lambda_{bk} \; C_{bi}^k + \sum_k \overline{\Gamma}_k \; \lambda_{bk} \; C_{bi}^{k*} \\ -\overline{\beta} \; \hat{V}_b^* - \nu \; \hat{V}_b + \sum_k \Gamma_k \; \lambda_{bk} \; \overline{C}_{bi}^k + \sum_k \overline{\Gamma}_k \; \lambda_{bk} \; \overline{C}_{bi}^{k*} \end{pmatrix} , \tag{17}$$

hence (10b) is written as

$$\sum_b \hat{\underset{\sim}{f}}_b^T \; \delta \underset{\sim}{w}_b = 0, \tag{18}$$

or using (15) as

$$\sum_b \hat{\underset{\sim}{f}}_b^T \; \underset{\sim}{J}_b^{-1} \; \delta \underset{\sim}{z}_b = 0 \tag{19a}$$

or

$$\sum_b ((\underset{\sim}{J}_b^{-1})^T \; \hat{\underset{\sim}{f}}_b)^T \; \delta \underset{\sim}{z}_b = 0. \tag{19b}$$

V. TRANSFORMED ADJOINT VARIABLES AND NETWORK SENSITIVITIES

Let

$$\hat{\eta}_b = \begin{bmatrix} \hat{\eta}_{bx} \\ \hat{\eta}_{bu} \end{bmatrix}$$

$$\triangleq (J_b^{-1})^T \hat{f}_b \tag{20}$$

be transformed adjoint variables associated with the bth branch, where $\hat{\eta}_{bx}$ and $\hat{\eta}_{bu}$ are two component vectors, then from (19b)

$$\sum_b \hat{\eta}_b^T \delta z_b = 0 \tag{21}$$

or from (12)

$$\sum_b (\hat{\eta}_{bx}^T \delta x_b + \hat{\eta}_{bu}^T \delta u_b) = 0. \tag{22}$$

Now, for a general complex function f of all state vectors x_b and all control vectors u_b we set

$$\hat{\eta}_{bx} = \frac{\partial f}{\partial x_b}, \tag{23}$$

hence

$$\delta f = \sum_b [(\frac{\partial f}{\partial x_b})^T \delta x_b + (\frac{\partial f}{\partial u_b})^T \delta u_b]$$

$$= \sum_b [\hat{\eta}_{bx}^T \delta x_b + (\frac{\partial f}{\partial u_b})^T \delta u_b]. \tag{24}$$

Then, from (22),

$$\delta f = \sum_b [(\frac{\partial f}{\partial \underset{\sim}{u}_b})^T - \hat{\underset{\sim}{\eta}}_{bu}^T] \delta \underset{\sim}{u}_b , \tag{25}$$

so that

$$\frac{df}{d\underset{\sim}{u}_b} = \frac{\partial f}{\partial \underset{\sim}{u}_b} - \hat{\underset{\sim}{\eta}}_{bu} . \tag{26}$$

In the case when $\underset{\sim}{u}_b$ is a function of some real design variables we write

$$\delta \underset{\sim}{u}_b = \sum_i \frac{\partial \underset{\sim}{u}_b}{\partial \zeta_{bi}} \Delta\zeta_{bi} , \tag{27}$$

where ζ_{bi} is the ith design variable associated with $\underset{\sim}{u}_b$ and $\Delta\zeta_{bi}$ denotes the change in ζ_{bi}. In practice, ζ_{bi} represent, for example, the parameters of shunt control elements and phase shifting transformers. From (25)

$$\frac{df}{d\zeta_{bi}} = [(\frac{\partial f}{\partial \underset{\sim}{u}_b})^T - \hat{\underset{\sim}{\eta}}_{bu}^T] \frac{\partial \underset{\sim}{u}_b}{\partial \zeta_{bi}} . \tag{28}$$

Note that (23) defines the adjoint elements while (26) or (28) provides the required gradients.

VI. GENERAL FORMULATION

We define an adjoint vector analogous to $\underset{\sim}{w}_b$ of (11) as

$$\hat{\underset{\sim}{w}}_b = \begin{pmatrix} \hat{\underset{\sim}{w}}_{bv} \\ \hline \hat{\underset{\sim}{w}}_{bi} \end{pmatrix} \triangleq \begin{pmatrix} \hat{V}_b \\ \hat{V}_b^* \\ \hline \hat{I}_b \\ \hat{I}_b^* \end{pmatrix} , \tag{29}$$

and write the matrix $(\underset{\sim}{J}_b^{-1})^T$ of (20) in a partitioned form

$$(\underset{\sim}{J}_b^{-1})^T = \begin{pmatrix} \underset{\sim}{M}_{11}^b & \underset{\sim}{M}_{12}^b \\ \underset{\sim}{M}_{21}^b & \underset{\sim}{M}_{22}^b \end{pmatrix} , \tag{30}$$

where $\underset{\sim}{M}_{11}^b$, $\underset{\sim}{M}_{12}^b$, $\underset{\sim}{M}_{21}^b$ and $\underset{\sim}{M}_{22}^b$ are 2x2 matrices.

Using (17) and (30) the vectors $\hat{\underset{\sim}{\eta}}_{bx}$ and $\hat{\underset{\sim}{\eta}}_{bu}$ of (20) are given by

$$\hat{\underset{\sim}{\eta}}_{bx} = \underset{\sim}{M}_{11}^b \hat{\underset{\sim}{f}}_{bi} + \underset{\sim}{M}_{12}^b \hat{\underset{\sim}{f}}_{bv} \tag{31}$$

and

$$\hat{\underset{\sim}{\eta}}_{bu} = \underset{\sim}{M}_{21}^b \hat{\underset{\sim}{f}}_{bi} + \underset{\sim}{M}_{22}^b \hat{\underset{\sim}{f}}_{bv} \tag{32}$$

The vectors $\hat{\underset{\sim}{f}}_{bi}$ and $\hat{\underset{\sim}{f}}_{bv}$ are written in terms of $\underset{\sim}{w}_b$ and $\hat{\underset{\sim}{w}}_b$ as

$$\hat{\underset{\sim}{f}}_{bi} = \underset{\sim}{\Lambda}_i^b \underset{\sim}{w}_{bi} + \bar{\underset{\sim}{\Lambda}}_i \hat{\underset{\sim}{w}}_{bi} \tag{33}$$

and

$$\hat{\underset{\sim}{f}}_{bv} = \underset{\sim}{\Lambda}_v^b \, \underset{\sim}{w}_{bv} + \overline{\underset{\sim}{\Lambda}}_v \, \hat{\underset{\sim}{w}}_{bv}, \tag{34}$$

where $\underset{\sim}{\Lambda}_i^b$, $\overline{\underset{\sim}{\Lambda}}_i$, $\underset{\sim}{\Lambda}_v^b$ and $\overline{\underset{\sim}{\Lambda}}_v$ are 2x2 matrices. The elements of $\overline{\underset{\sim}{\Lambda}}_v$ and $\overline{\underset{\sim}{\Lambda}}_i$ consist of the adjoint coefficients α, $\overline{\alpha}$, ξ, $\overline{\xi}$, β, $\overline{\beta}$, ν and $\overline{\nu}$.

For the set of terms considered in Tellegen's sum (7) the matrices $\underset{\sim}{\Lambda}_i^b$, $\overline{\underset{\sim}{\Lambda}}_i$, $\underset{\sim}{\Lambda}_v^b$ and $\overline{\underset{\sim}{\Lambda}}_v$ are given from (17) by

$$\underset{\sim}{\Lambda}_i^b = \begin{pmatrix} \sum_k \Gamma_k \lambda_{bk} C_{bv}^k / I_b & \sum_k \overline{\Gamma}_k \lambda_{bk} C_{bv}^{k*} / I_b^* \\ \sum_k \Gamma_k \lambda_{bk} \overline{C}_{bv}^k / I_b & \sum_k \overline{\Gamma}_k \lambda_{bk} \overline{C}_{bv}^{k*} / I_b^* \end{pmatrix}, \tag{35a}$$

$$\overline{\underset{\sim}{\Lambda}}_i = \begin{pmatrix} \alpha & \overline{\xi} \\ \xi & \overline{\alpha} \end{pmatrix}, \tag{35b}$$

$$\underset{\sim}{\Lambda}_v^b = \begin{pmatrix} \sum_k \Gamma_k \lambda_{bk} C_{bi}^k / V_b & \sum_k \overline{\Gamma}_k \lambda_{bk} C_{bi}^{k*} / V_b^* \\ \sum_k \Gamma_k \lambda_{bk} \overline{C}_{bi}^k / V_b & \sum_k \overline{\Gamma}_k \lambda_{bk} \overline{C}_{bi}^{k*} / V_b^* \end{pmatrix}, \tag{35c}$$

and

$$\overline{\underset{\sim}{\Lambda}}_v = - \begin{pmatrix} \beta & \overline{\nu} \\ \nu & \overline{\beta} \end{pmatrix}. \tag{35d}$$

Note that if C_b^k of (6a) has the form

$$C_b^k = \tilde{V}_b \, \tilde{I}_b, \tag{36}$$

where

$$\tilde{V}_b = \pm V_b \tag{37a}$$

and

$$\tilde{I}_b = \pm I_b, \tag{37b}$$

the elements of $\underset{\sim}{\Lambda}_i^b$ and $\underset{\sim}{\Lambda}_v^b$ consist solely of the adjoining coefficients Γ_k and $\overline{\Gamma}_k$. Note also that $\tilde{V}_b = \pm 1$ in (35a) and $\tilde{I}_b = \pm 1$ in (35c) lead to corresponding zero matrices.

For use later we now define

$$N_{ib}^k = \Gamma_k \; C_{bv}^k + \overline{\Gamma}_k \; C_{bv}^{k*}, \tag{38a}$$

$$\overline{N}_{ib}^k = \Gamma_k \; \overline{C}_{bv}^k + \overline{\Gamma}_k \; \overline{C}_{bv}^{k*}, \tag{38b}$$

$$N_{vb}^k = \Gamma_k \; C_{bi}^k + \overline{\Gamma}_k \; C_{bi}^{k*}, \tag{38c}$$

and

$$\overline{N}_{vb}^k = \Gamma_k \; \overline{C}_{bi}^k + \overline{\Gamma}_k \; \overline{C}_{bi}^{k*}. \tag{38d}$$

Using (33) and (34), equation (31) is written as

$$\overline{\underset{\sim}{\theta}}_{bi} \; \hat{\underset{\sim}{w}}_{bi} = \overline{\underset{\sim}{\theta}}_{bv} \; \hat{\underset{\sim}{w}}_{bv} + \underset{\sim}{\theta}_b, \tag{39}$$

where the 2x2 matrices $\bar{\underset{\sim}{\theta}}_{bi}$ and $\bar{\underset{\sim}{\theta}}_{bv}$ are given by

$$\bar{\underset{\sim}{\theta}}_{bi} = \underset{\sim}{M}^{b}_{11} \bar{\underset{\sim}{\Lambda}}_{i} \tag{40a}$$

and

$$\bar{\underset{\sim}{\theta}}_{bv} = - \underset{\sim}{M}^{b}_{12} \bar{\underset{\sim}{\Lambda}}_{v}, \tag{40b}$$

and the vector $\underset{\sim}{\theta}_b$ is given by

$$\underset{\sim}{\theta}_b = \hat{\underset{\sim}{\eta}}_{bx} - \underset{\sim}{M}^{b}_{11} \underset{\sim}{\Lambda}^{b}_{i} \underset{\sim}{w}_{bi} - \underset{\sim}{M}^{b}_{12} \underset{\sim}{\Lambda}^{b}_{v} \underset{\sim}{w}_{bv}. \tag{41}$$

Note that the choice of the coefficients α, $\bar{\alpha}$, ... etc. is subject to the consistency of (39).

VII. POWER SYSTEM ELEMENT VARIABLES

We consider the total number of branches to be n_B consisting of n_L loads, n_G generators, one slack generator and $n_T = n_B - n_L - n_G -1$ other branch elements.

The buses are ordered such that subscripts $\ell = 1, 2, \ldots, n_L$ identify load branches, $g = n_L+1, \ldots, n_L+n_G$ identify generator branches and $n = n_L + n_G + 1$ identifies the slack generator branch. Subscripts $t = n + 1, \ldots, n_B$ are used to identify other branches.

The element variables for a load are usually defined as

$$\bar{\underset{\sim}{z}}_\ell = \begin{pmatrix} \underset{\sim}{x}_\ell \\ -- \\ \underset{\sim}{u}_\ell \end{pmatrix} \triangleq \begin{pmatrix} |V_\ell| \\ \delta_\ell \\ ---- \\ P_\ell \\ Q_\ell \end{pmatrix} = \begin{pmatrix} (V_\ell V_\ell^*)^{1/2} \\ \tan^{-1}[j(V_\ell^* - V_\ell)/(V_\ell + V_\ell^*)] \\ ---------------------- \\ (V_\ell I_\ell^* + V_\ell^* I_\ell)/2 \\ j(V_\ell^* I_\ell - V_\ell I_\ell^*)/2 \end{pmatrix}, \qquad (42a)$$

or, for example, as

$$\tilde{\underset{\sim}{z}}_\ell \triangleq \begin{pmatrix} V_\ell \\ V_\ell^* \\ S_\ell \\ S_\ell^* \end{pmatrix} = \begin{pmatrix} V_\ell \\ V_\ell^* \\ V_\ell I_\ell^* \\ V_\ell^* I_\ell \end{pmatrix}. \qquad (42b)$$

The element variables for a generator are usually defined as

$$\bar{\underset{\sim}{z}}_g = \begin{pmatrix} \underset{\sim}{x}_g \\ -- \\ \underset{\sim}{u}_g \end{pmatrix} \triangleq \begin{pmatrix} \delta_g \\ Q_g \\ ---- \\ |V_g| \\ P_g \end{pmatrix} = \begin{pmatrix} \tan^{-1}\ [j(V_g^* - V_g)/(V_g + V_g^*)] \\ j(V_g^* I_g - V_g I_g^*)/2 \\ ---------------------- \\ (V_g V_g *)^{1/2} \\ (V_g I_g^* + V_g^* I_g)/2 \end{pmatrix}, \qquad (43a)$$

or, for example, as

$$\tilde{\underset{\sim}{z}}_g \triangleq \begin{pmatrix} V_g \\ I_g \\ |V_g|^2 \\ S_g + S_g^* \end{pmatrix} = \begin{pmatrix} V_g \\ I_g \\ V_g V_g^* \\ V_g I_g^* + V_g^* I_g \end{pmatrix}. \qquad (43b)$$

The element variables for the slack generator are usually defined as

$$\bar{\underset{\sim}{z}}_n = \begin{pmatrix} \underset{\sim}{x}_n \\ \hline \underset{\sim}{u}_n \end{pmatrix} \triangleq \begin{pmatrix} P_n \\ Q_n \\ \hline |V_n| \\ \delta_n \end{pmatrix} = \begin{pmatrix} (V_n I_n^* + V_n^* I_n)/2 \\ j(V_n^* I_n - V_n I_n^*)/2 \\ \hline (V_n V_n^*)^{1/2} \\ \tan^{-1}[j(V_n^* - V_n)/(V_n + V_n^*)] \end{pmatrix}, \qquad (44a)$$

or, for example, as

$$\tilde{\underset{\sim}{z}}_n \triangleq \begin{pmatrix} I_n \\ I_n^* \\ V_n \\ V_n^* \end{pmatrix}. \qquad (44b)$$

For other branches the element variables are defined according to the element type. The element variables for a transmission element, for example, may be defined as

$$\bar{\underset{\sim}{z}}_t = \begin{pmatrix} \underset{\sim}{x}_t \\ \hline \underset{\sim}{u}_t \end{pmatrix} \triangleq \begin{pmatrix} \mathrm{Re}\{I_t\} \\ \mathrm{Im}\{I_t\} \\ \hline G_t \\ B_t \end{pmatrix} = \begin{pmatrix} (I_t + I_t^*)/2 \\ j(I_t^* - I_t)/2 \\ \hline (I_t/V_t + I_t^*/V_t^*)/2 \\ j(I_t^*/V_t^* - I_t/V_t)/2 \end{pmatrix}, \qquad (45a)$$

or as

$$\tilde{z}_t \triangleq \begin{bmatrix} I_t \\ I_t^* \\ Y_t \\ Y_t^* \end{bmatrix} = \begin{bmatrix} I_t \\ I_t^* \\ I_t/V_t \\ I_t^*/V_t^* \end{bmatrix} . \tag{45b}$$

Let $\bar{z}_b$ be a general vector containing $\bar{z}_\ell$, $\bar{z}_g$, $\bar{z}_n$ and $\bar{z}_t$ of (42a), (43a), (44a) and (45a), respectively. Also, let $\tilde{z}_b$ be a general vector containing $\tilde{z}_\ell$, $\tilde{z}_g$, $\tilde{z}_n$ and $\tilde{z}_t$ of (42b), (43b), (44b) and (45b), respectively.

Using the results of the Appendix the corresponding matrices $\bar{\theta}_{bi}$ and $\bar{\theta}_{bv}$ and vector θ_b for different power system elements are shown in Table I for the set of element variables $\bar{z}_b$ and in Table II for the set of element variables $\tilde{z}_b$.

It is important to notice that $\bar{\theta}_{bi}$, $\bar{\theta}_{bv}$ and θ_b of Tables I and II are common to all relevant power system studies as long as the element variables considered are $\bar{z}_b$ and $\tilde{z}_b$, respectively.

VIII. THE ADJOINT EQUATIONS

In this section we derive the adjoint equations. We write the matrices $\bar{\theta}_{bi}$ and $\bar{\theta}_{bv}$ and vector θ_b of (39) in the form

$$\bar{\theta}_{bi} = \begin{bmatrix} \tilde{\phi}_b^1 & \bar{\phi}_b^1 \\ \tilde{\phi}_b^2 & \bar{\phi}_b^2 \end{bmatrix} , \tag{46a}$$

$$\bar{\underset{\sim}{\theta}}_{bv} = \begin{pmatrix} \tilde{\psi}_b^1 & \bar{\psi}_b^1 \\ \tilde{\psi}_b^2 & \bar{\psi}_b^2 \end{pmatrix} , \tag{46b}$$

and

$$\underset{\sim}{\theta}_b = \begin{pmatrix} \hat{W}_b^{S1} \\ \hat{W}_b^{S2} \end{pmatrix} , \tag{46c}$$

hence, the adjoint current-voltage relationship for element b has, from (39), the form

$$\tilde{\phi}_b^k \hat{I}_b + \bar{\phi}_b^k \hat{I}_b^* = \tilde{\psi}_b^k \hat{V}_b + \bar{\psi}_b^k \hat{V}_b^* + \hat{W}_b^{Sk}, \tag{47}$$

where k = 1, 2 denotes the first and second complex equations of (39), respectively, or, when separated into real and imaginary parts,

$$(\tilde{\phi}_{b1}^i+\bar{\phi}_{b1}^i)\hat{I}_{b1} + (\bar{\phi}_{b2}^i-\tilde{\phi}_{b2}^i)\hat{I}_{b2} = (\tilde{\psi}_{b1}^i+\bar{\psi}_{b1}^i)\hat{V}_{b1} + (\bar{\psi}_{b2}^i-\tilde{\psi}_{b2}^i)\hat{V}_{b2} + \hat{W}_{b1}^{Si}; \quad i = 1,2 \tag{48a}$$

and

$$(\tilde{\phi}_{b2}^j+\bar{\phi}_{b2}^j)\hat{I}_{b1} + (\tilde{\phi}_{b1}^j-\bar{\phi}_{b1}^j)\hat{I}_{b2} = (\tilde{\psi}_{b2}^j+\bar{\psi}_{b2}^j)\hat{V}_{b1} + (\tilde{\psi}_{b1}^j-\bar{\psi}_{b1}^j)\hat{V}_{b2} + \hat{W}_{b2}^{Sj}; \quad j = 1,2 \tag{48b}$$

where

$$\tilde{\phi}_b^k = \tilde{\phi}_{b1}^k + j\tilde{\phi}_{b2}^k, \tag{49a}$$

$$\bar{\phi}_b^k = \bar{\phi}_{b1}^k + j\bar{\phi}_{b2}^k, \tag{49b}$$

$$\tilde{\psi}_b^k = \tilde{\psi}_{b1}^k + j\tilde{\psi}_{b2}^k, \tag{49c}$$

$$\bar{\psi}_b^k = \bar{\psi}_{b1}^k + j\bar{\psi}_{b2}^k, \tag{49d}$$

$$\hat{V}_b = \hat{V}_{b1} + j\hat{V}_{b2}, \tag{49e}$$

$$\hat{I}_b = \hat{I}_{b1} + j\hat{I}_{b2}, \tag{49f}$$

and

$$\hat{W}_b^{Sk} = \hat{W}_{b1}^{Sk} + j\hat{W}_{b2}^{Sk}; \quad k = 1,\ 2. \tag{49g}$$

In order to uniquely define the adjoint currents $\hat{I}_b$ in terms of the adjoint voltages $\hat{V}_b$ the system of four linear equations (48) has rank 2. Two of the four equations are used to describe the adjoint element. We write these two equations in the form

$$\phi_{11}^b \hat{I}_{b1} + \phi_{12}^b \hat{I}_{b2} = \psi_{11}^b \hat{V}_{b1} + \psi_{12}^b \hat{V}_{b2} + \hat{W}_{b1}^S \tag{50a}$$

and

$$\phi_{21}^b \hat{I}_{b1} + \phi_{22}^b \hat{I}_{b2} = \psi_{21}^b \hat{V}_{b1} + \psi_{22}^b \hat{V}_{b2} + \hat{W}_{b2}^S, \tag{50b}$$

where

$$\phi_{11}^b = \tilde{\phi}_{b1}^i + \bar{\phi}_{b1}^i \tag{51a}$$

$$\phi_{12}^b = \bar{\phi}_{b2}^i - \tilde{\phi}_{b2}^i, \tag{51b}$$

$$\psi_{11}^b = \tilde{\psi}_{b1}^i + \bar{\psi}_{b1}^i, \tag{51c}$$

$$\psi_{12}^b = \bar{\psi}_{b2}^i - \tilde{\psi}_{b2}^i, \tag{51d}$$

$$\hat{W}_{b1}^S = \hat{W}_{b1}^{Si}; \qquad i = 1 \text{ or } 2, \tag{51e}$$

and

$$\phi_{21}^b = \tilde{\phi}_{b2}^j + \bar{\phi}_{b2}^j, \tag{52a}$$

$$\phi_{22}^b = \tilde{\phi}_{b1}^j - \bar{\phi}_{b1}^j, \tag{52b}$$

$$\psi_{21}^{b} = \tilde{\psi}_{b2}^{j} + \overline{\psi}_{b2}^{j}, \tag{52c}$$

$$\psi_{22}^{b} = \tilde{\psi}_{b1}^{j} - \overline{\psi}_{b1}^{j}, \tag{52d}$$

$$\hat{W}_{b2}^{S} = \hat{W}_{b2}^{Sj}; \qquad j = 1 \text{ or } 2. \tag{52e}$$

Equation (50) is written for transmission elements in the form

$$\hat{I}_{t1} = \overline{Y}_{t1}\,\hat{V}_{t1} - \overline{Y}_{t2}\,\hat{V}_{t2} + \hat{I}_{t1}^{S} \tag{53a}$$

and

$$\hat{I}_{t2} = \tilde{Y}_{t2}\,\hat{V}_{t1} + \tilde{Y}_{t1}\,\hat{V}_{t2} + \hat{I}_{t2}^{S}, \tag{53b}$$

where

$$\begin{pmatrix} \overline{Y}_{t1} & -\overline{Y}_{t2} \\ \tilde{Y}_{t2} & \tilde{Y}_{t1} \end{pmatrix} = \frac{1}{\Delta_t} \begin{pmatrix} \phi_{22}^{t} & -\phi_{12}^{t} \\ -\phi_{21}^{t} & \phi_{11}^{t} \end{pmatrix} \begin{pmatrix} \psi_{11}^{t} & \psi_{12}^{t} \\ \psi_{21}^{t} & \psi_{22}^{t} \end{pmatrix} \tag{54a}$$

and

$$\begin{pmatrix} \hat{I}_{t1}^{S} \\ \hat{I}_{t2}^{S} \end{pmatrix} = \frac{1}{\Delta_t} \begin{pmatrix} \phi_{22}^{t} & -\phi_{12}^{t} \\ -\phi_{21}^{t} & \phi_{11}^{t} \end{pmatrix} \begin{pmatrix} \hat{W}_{t1}^{S} \\ \hat{W}_{t2}^{S} \end{pmatrix} \tag{54b}$$

and where

$$\Delta_t \triangleq \phi_{11}^{t}\,\phi_{22}^{t} - \phi_{12}^{t}\,\phi_{21}^{t} \neq 0. \tag{54c}$$

We define the complex quantities

$$\overline{Y}_t \triangleq \overline{Y}_{t1} + j\overline{Y}_{t2} \tag{55a}$$

and

$$\tilde{Y}_t \triangleq \tilde{Y}_{t1} + j\tilde{Y}_{t2}. \quad (55b)$$

Equation (53) is written in the matrix form

$$\begin{pmatrix} \bar{\mathbf{Y}}^p_{T1} & -\bar{\mathbf{Y}}^p_{T2} \\ \tilde{\mathbf{Y}}^p_{T2} & \tilde{\mathbf{Y}}^p_{T1} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{V}}_{T1} \\ \hat{\mathbf{V}}_{T2} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{I}}_{T1} - \hat{\mathbf{I}}^S_{T1} \\ \hat{\mathbf{I}}_{T2} - \hat{\mathbf{I}}^S_{T2} \end{pmatrix}, \quad (56)$$

where $\bar{\mathbf{Y}}^p_{T1}$, $\bar{\mathbf{Y}}^p_{T2}$, $\tilde{\mathbf{Y}}^p_{T1}$ and $\tilde{\mathbf{Y}}^p_{T2}$ are diagonal matrices consisting of the $\bar{Y}_{t1}$, $\bar{Y}_{t2}$, $\tilde{Y}_{t1}$ and $\tilde{Y}_{t2}$, respectively, and $\hat{\mathbf{V}}_{T1}$, $\hat{\mathbf{V}}_{T2}$, $\hat{\mathbf{I}}_{T1}$, $\hat{\mathbf{I}}_{T2}$, $\hat{\mathbf{I}}^S_{T1}$ and $\hat{\mathbf{I}}^S_{T2}$ are vectors of components $\hat{V}_{t1}$, $\hat{V}_{t2}$, $\hat{I}_{t1}$, $\hat{I}_{t2}$, $\hat{I}^S_{t1}$ and $\hat{I}^S_{t2}$, respectively. For later use, let

$$\bar{\mathbf{Y}}^p_T \triangleq \bar{\mathbf{Y}}^p_{T1} + j\bar{\mathbf{Y}}^p_{T2}, \quad (57a)$$

$$\tilde{\mathbf{Y}}^p_T \triangleq \tilde{\mathbf{Y}}^p_{T1} + j\tilde{\mathbf{Y}}^p_{T2}, \quad (57b)$$

$$\hat{\mathbf{V}}_T \triangleq \hat{\mathbf{V}}_{T1} + j\hat{\mathbf{V}}_{T2}, \quad (58a)$$

$$\hat{\mathbf{I}}_T \triangleq \hat{\mathbf{I}}_{T1} + j\hat{\mathbf{I}}_{T2}, \quad (58b)$$

and

$$\hat{\mathbf{I}}^S_T \triangleq \hat{\mathbf{I}}^S_{T1} + j\hat{\mathbf{I}}^S_{T2}. \quad (59)$$

Now we define the 2nx2n matrices

$$\underset{\sim}{\Phi} = \begin{bmatrix} \underset{\sim}{\Phi}_{11} & \underset{\sim}{\Phi}_{12} \\ \underset{\sim}{\Phi}_{21} & \underset{\sim}{\Phi}_{22} \end{bmatrix} \tag{60}$$

and

$$\underset{\sim}{\Psi} = \begin{bmatrix} \underset{\sim}{\Psi}_{11} & \underset{\sim}{\Psi}_{12} \\ \underset{\sim}{\Psi}_{21} & \underset{\sim}{\Psi}_{22} \end{bmatrix} , \tag{61}$$

where

$$\underset{\sim}{\Phi}_{11} \triangleq \text{diag}\ \{\phi_{11}^{m}\}, \tag{62a}$$

$$\underset{\sim}{\Phi}_{12} \triangleq \text{diag}\ \{\phi_{12}^{m}\}, \tag{62b}$$

$$\underset{\sim}{\Phi}_{21} \triangleq \text{diag}\ \{\phi_{21}^{m}\}, \tag{62c}$$

$$\underset{\sim}{\Phi}_{22} \triangleq \text{diag}\ \{\phi_{22}^{m}\}, \tag{62d}$$

$$\underset{\sim}{\Psi}_{11} \triangleq \text{diag}\ \{\psi_{11}^{m}\}, \tag{63a}$$

$$\underset{\sim}{\Psi}_{12} \triangleq \text{diag}\ \{\psi_{12}^{m}\}, \tag{63b}$$

$$\underset{\sim}{\Psi}_{21} \triangleq \text{diag}\ \{\psi_{21}^{m}\}, \tag{63c}$$

and

$$\underset{\sim}{\Psi}_{22} \triangleq \text{diag}\ \{\psi_{22}^{m}\} \tag{63d}$$

are nxn diagonal matrices, m can be ℓ, g or n.

Equation (50) is written for the bus elements using (60)-(63) in the matrix form

$$\begin{bmatrix} \underset{\sim}{\Phi}_{11} & \underset{\sim}{\Phi}_{12} \\ \underset{\sim}{\Phi}_{21} & \underset{\sim}{\Phi}_{22} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{I}}^B_{M1} \\ \hat{\underset{\sim}{I}}^B_{M2} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\Psi}_{11} & \underset{\sim}{\Psi}_{12} \\ \underset{\sim}{\Psi}_{21} & \underset{\sim}{\Psi}_{22} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}^B_{M1} \\ \hat{\underset{\sim}{V}}^B_{M2} \end{bmatrix} + \begin{bmatrix} \hat{\underset{\sim}{W}}^{SB}_{M1} \\ \hat{\underset{\sim}{W}}^{SB}_{M2} \end{bmatrix}. \quad (64)$$

where $\hat{\underset{\sim}{I}}^B_{M1}$, $\hat{\underset{\sim}{I}}^B_{M2}$, $\hat{\underset{\sim}{V}}^B_{M1}$, $\hat{\underset{\sim}{V}}^B_{M2}$, $\hat{\underset{\sim}{W}}^{SB}_{M1}$ and $\hat{\underset{\sim}{W}}^{SB}_{M2}$ are vectors of components $\hat{I}_{m1}$, $\hat{I}_{m2}$, $\hat{V}_{m1}$, $\hat{V}_{m2}$, $\hat{W}^S_{m1}$ and $\hat{W}^S_{m2}$, respectively. We let

$$\hat{\underset{\sim}{I}}^B_M \triangleq \hat{\underset{\sim}{I}}^B_{M1} + j\hat{\underset{\sim}{I}}^B_{M2}, \quad (65a)$$

$$\hat{V}^B_M \triangleq \hat{\underset{\sim}{V}}^B_{M1} + j\hat{\underset{\sim}{V}}^B_{M2} \quad (65b)$$

and

$$\hat{\underset{\sim}{W}}^{SB}_M \triangleq \hat{\underset{\sim}{W}}^{SB}_{M1} + j\hat{\underset{\sim}{W}}^{SB}_{M2} \quad (66)$$

KCL is written as

$$\left[\underset{\sim}{A}_M \;\middle|\; \underset{\sim}{A}_T\right] \begin{bmatrix} \hat{\underset{\sim}{I}}^B_M \\ \hline \hat{\underset{\sim}{I}}_T \end{bmatrix} = \underset{\sim}{0}, \quad (67)$$

where

$$\underset{\sim}{A} \triangleq \left[\underset{\sim}{A}_M \;\middle|\; \underset{\sim}{A}_T\right] \quad (68)$$

is the reduced incidence matrix of dimension nxn_B (n buses, n_B branches) whose elements a_{ij} are given by

$a_{ij} = 1$ if branch j is incident at bus i and oriented away from it,

$a_{ij} = -1$ if branch j is incident at bus i and oriented toward it, and

$a_{ij} = 0$ if branch j is not incident at bus i.

Now we define

$$\bar{\underset{\sim}{Y}}_T = \bar{\underset{\sim}{Y}}_{T1} + j\bar{\underset{\sim}{Y}}_{T2} \triangleq \underset{\sim}{A}_T \, \bar{\underset{\sim}{Y}}_T^p \, \underset{\sim}{A}_T^T, \tag{69a}$$

$$\tilde{\underset{\sim}{Y}}_T = \tilde{\underset{\sim}{Y}}_{T1} + j\tilde{\underset{\sim}{Y}}_{T2} \triangleq \underset{\sim}{A}_T \, \tilde{\underset{\sim}{Y}}_T^p \, \underset{\sim}{A}_T^T, \tag{69b}$$

$$\hat{\underset{\sim}{J}}_M = \hat{\underset{\sim}{J}}_{M1} + j\hat{\underset{\sim}{J}}_{M2} \triangleq \underset{\sim}{A}_T \, \hat{\underset{\sim}{I}}_T^S \tag{70}$$

and

$$\hat{\underset{\sim}{W}}_M^S = \hat{\underset{\sim}{W}}_{M1}^S + j\hat{\underset{\sim}{W}}_{M2}^S \triangleq \underset{\sim}{A}_M \, \hat{\underset{\sim}{W}}_M^{SB}. \tag{71}$$

Also the bus voltages

$$\hat{\underset{\sim}{V}}_M = \hat{\underset{\sim}{V}}_{M1} + j\hat{\underset{\sim}{V}}_{M2} \triangleq \underset{\sim}{A}_M \, \hat{\underset{\sim}{V}}_M^B, \tag{72}$$

are related to $\hat{\underset{\sim}{V}}_T$ through the relationship

$$\hat{\underset{\sim}{V}}_T = \underset{\sim}{A}_T^T \, \hat{\underset{\sim}{V}}_M. \tag{73}$$

Eliminating $\hat{\underset{\sim}{I}}_T$ and $\hat{\underset{\sim}{I}}_M^B$ from (56), (64) and (67) and using (72) and (73) we arrive at the final set of adjoint equations to be solved in the form

$$\begin{bmatrix} (\underset{\sim}{\Phi}_{11}\bar{\underset{\sim}{Y}}_{T1}+\underset{\sim}{\Phi}_{12}\tilde{\underset{\sim}{Y}}_{T2}+\underset{\sim}{\Psi}_{11}) & (-\underset{\sim}{\Phi}_{11}\bar{\underset{\sim}{Y}}_{T2}+\underset{\sim}{\Phi}_{12}\tilde{\underset{\sim}{Y}}_{T1}+\underset{\sim}{\Psi}_{12}) \\ (\underset{\sim}{\Phi}_{21}\bar{\underset{\sim}{Y}}_{T1}+\underset{\sim}{\Phi}_{22}\tilde{\underset{\sim}{Y}}_{T2}+\underset{\sim}{\Psi}_{21}) & (-\underset{\sim}{\Phi}_{21}\bar{\underset{\sim}{Y}}_{T2}+\underset{\sim}{\Phi}_{22}\tilde{\underset{\sim}{Y}}_{T1}+\underset{\sim}{\Psi}_{22}) \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_{M1} \\ \hat{\underset{\sim}{V}}_{M2} \end{bmatrix} = - \begin{bmatrix} \underset{\sim}{\Phi}_{11}\hat{\underset{\sim}{J}}_{M1}+\underset{\sim}{\Phi}_{12}\hat{\underset{\sim}{J}}_{M2}+\hat{\underset{\sim}{W}}^{S}_{M1} \\ \underset{\sim}{\Phi}_{21}\hat{\underset{\sim}{J}}_{M1}+\underset{\sim}{\Phi}_{22}\hat{\underset{\sim}{J}}_{M2}+\hat{\underset{\sim}{W}}^{S}_{M2} \end{bmatrix}. \tag{74}$$

Note that multiplying (56) from left by the matrix

$$\bar{\underset{\sim}{A}}_T = \begin{bmatrix} \underset{\sim}{A}_T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A}_T \end{bmatrix}, \tag{75}$$

substituting $\hat{\underset{\sim}{V}}_T$ from (73) and using (67), (69) and (70) we get

$$\begin{bmatrix} \bar{\underset{\sim}{Y}}_{T1} & -\bar{\underset{\sim}{Y}}_{T2} \\ \tilde{\underset{\sim}{Y}}_{T2} & \tilde{\underset{\sim}{Y}}_{T1} \end{bmatrix} \begin{bmatrix} \hat{\underset{\sim}{V}}_{M1} \\ \hat{\underset{\sim}{V}}_{M2} \end{bmatrix} = - \begin{bmatrix} \hat{\underset{\sim}{I}}_{M1} + \hat{\underset{\sim}{J}}_{M1} \\ \hat{\underset{\sim}{I}}_{M2} + \hat{\underset{\sim}{J}}_{M2} \end{bmatrix}, \tag{76}$$

where

$$\hat{\underset{\sim}{I}}_M = \hat{\underset{\sim}{I}}_{M1} + j\hat{\underset{\sim}{I}}_{M2} = \underset{\sim}{A}_M \hat{\underset{\sim}{I}}^B_M. \tag{77}$$

The form of (76) is that of the conventional nodal equations. It can be used for solution purposes if the RHS is voltage independent, e.g., as for typical linear electronic circuit cases.

IX. GRADIENT CALCULATIONS

The solution of the adjoint system (74) provides the adjoint variables $\hat{\underset{\sim}{w}}_b$ of (29). The required gradients are then calculated using (26) or (28). The vector $\hat{\underset{\sim}{\eta}}_{bu}$ is obtained from (32) where $\hat{\underset{\sim}{f}}_{bi}$ and $\hat{\underset{\sim}{f}}_{bv}$ are calculated from (33), (34) and (35). Using the results of the Appendix matrices $\underset{\sim}{M}^b_{21}$ and $\underset{\sim}{M}^b_{22}$ of (32) for different power system elements are shown in Table III for the set of element variables $\bar{\underset{\sim}{z}}_b$ and in Table IV for the set of element variables $\tilde{\underset{\sim}{z}}_b$.

X. CONCLUSIONS

This paper has laid the foundation of an exact adjoint network approach to general power system sensitivity analysis and planning problems. A family of adjoint systems of equations has been derived so that a wide variety of special problems can be handled. We have overcome the difficulties which have prevented previous workers from applying Tellegen's theorem to the a.c. power flow model in general and without any approximations.

We have derived and tabulated standard sensitivity expressions common to all relevant power system studies. Part II of the paper [10] addresses, in detail, an important special class of adjoint systems applicable to the evaluation of sensitivities w.r.t. all design and control parameters of most functions of practical interest. Numerical examples are provided in that part.

The concepts stated in the paper are general. While they have been applied with power systems in mind, they are applicable to other systems as well.

APPENDIX

Loads

For a load the Jacobian $\underset{\sim}{J}_\ell$ using the set of element variables $\underset{\sim}{\bar{z}}_\ell$ of (42a) is given by

$$\underset{\sim}{J}_\ell = \left(\begin{array}{cc|cc} V_\ell^*/[2(V_\ell V_\ell^*)^{1/2}] & V_\ell/[2(V_\ell V_\ell^*)^{1/2}] & 0 & 0 \\ -j/(2V_\ell) & j/(2V_\ell^*) & 0 & 0 \\ \hline I_\ell^*/2 & I_\ell/2 & V_\ell^*/2 & V_\ell/2 \\ -jI_\ell^*/2 & jI_\ell/2 & jV_\ell^*/2 & -jV_\ell/2 \end{array}\right), \qquad \text{(A1)}$$

hence

$$\underset{\sim}{J}_\ell^{-1} = \left(\begin{array}{cc|cc} (V_\ell/V_\ell^*)^{1/2} & jV_\ell & 0 & 0 \\ (V_\ell^*/V_\ell)^{1/2} & -jV_\ell^* & 0 & 0 \\ \hline -I_\ell/(V_\ell V_\ell^*)^{1/2} & jI_\ell & 1/V_\ell^* & -j/V_\ell^* \\ -I_\ell^*/(V_\ell V_\ell^*)^{1/2} & -jI_\ell^* & 1/V_\ell & j/V_\ell \end{array}\right), \qquad \text{(A2)}$$

so that

$$\underset{\sim}{\bar{\theta}}_{\ell i} = \begin{pmatrix} (\alpha V_\ell+\xi V_\ell^*)/(V_\ell V_\ell^*)^{1/2} & (\bar{\xi} V_\ell+\bar{\alpha} V_\ell^*)/(V_\ell V_\ell^*)^{1/2} \\ j(\alpha V_\ell-\xi V_\ell^*) & j(\bar{\xi} V_\ell-\bar{\alpha} V_\ell^*) \end{pmatrix}, \qquad \text{(A3)}$$

$$\bar{\underset{\sim}{\Theta}}_{\ell v} = \begin{bmatrix} -(\beta I_\ell + \nu I_\ell^*)/(V_\ell V_\ell^*)^{1/2} & -(\bar{\nu} I_\ell + \bar{\beta} I_\ell^*)/(V_\ell V_\ell^*)^{1/2} \\ j(\beta I_\ell - \nu I_\ell^*) & j(\bar{\nu} I_\ell - \bar{\beta} I_\ell^*) \end{bmatrix}, \quad \text{(A4)}$$

$$\underset{\sim}{M}_{11}^{\ell} \, \underset{\sim}{\Lambda}_{i}^{\ell} \, \underset{\sim}{w}_{\ell i} = \begin{bmatrix} \sum_k \lambda_{\ell k} [N_{i\ell}^k (V_\ell / V_\ell^*)^{1/2} + \bar{N}_{i\ell}^k (V_\ell^* / V_\ell)^{1/2}] \\ \sum_k j\lambda_{\ell k} [N_{i\ell}^k V_\ell - \bar{N}_{i\ell}^k V_\ell^*] \end{bmatrix}, \quad \text{(A5)}$$

$$\underset{\sim}{M}_{12}^{\ell} \, \underset{\sim}{\Lambda}_{v}^{\ell} \, \underset{\sim}{w}_{\ell v} = \begin{bmatrix} \sum_k \lambda_{\ell k} [-N_{v\ell}^k I_\ell / (V_\ell V_\ell^*)^{1/2} - \bar{N}_{v\ell}^k I_\ell^* / (V_\ell V_\ell^*)^{1/2}] \\ \sum_k j\lambda_{\ell k} [N_{v\ell}^k I_\ell - \bar{N}_{v\ell}^k \; I_\ell^*] \end{bmatrix}. \quad \text{(A6)}$$

Using the set of element variables $\tilde{\underset{\sim}{z}}_\ell$ of (42b) the Jacobian is given by

$$\underset{\sim}{J}_\ell = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline I_\ell^* & 0 & 0 & V_\ell \\ 0 & I_\ell & V_\ell^* & 0 \end{array}\right], \quad \text{(A7)}$$

hence

$$\underset{\sim}{J}_\ell^{-1} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & -I_\ell/V_\ell^* & 0 & 1/V_\ell^* \\ -I_\ell^*/V_\ell & 0 & 1/V_\ell & 0 \end{array}\right], \quad \text{(A8)}$$

so that

$$\bar{\underset{\sim}{\theta}}_{\ell i} = \begin{bmatrix} \alpha & \bar{\xi} \\ \xi & \bar{\alpha} \end{bmatrix} , \qquad (A9)$$

$$\bar{\underset{\sim}{\theta}}_{\ell v} = \begin{bmatrix} -\nu I_\ell^*/V_\ell & -\bar{\beta} I_\ell^*/V_\ell \\ -\beta I_\ell/V_\ell^* & -\bar{\nu} I_\ell/V_\ell^* \end{bmatrix} , \qquad (A10)$$

$$\underset{\sim}{M}_{11}^{\ell} \; \underset{\sim}{\Lambda}_{i}^{\ell} \; \underset{\sim}{w}_{\ell i} = \begin{bmatrix} \sum_k \lambda_{\ell k} N_{i\ell}^{k} \\ \sum_k \lambda_{\ell k} \bar{N}_{i\ell}^{k} \end{bmatrix} , \qquad (A11)$$

$$\underset{\sim}{M}_{12}^{\ell} \; \underset{\sim}{\Lambda}_{v}^{\ell} \; \underset{\sim}{w}_{\ell v} = \begin{bmatrix} \sum_k -\lambda_{\ell k} \bar{N}_{v\ell}^{k} \; I_\ell^*/V_\ell \\ \sum_k -\lambda_{\ell k} N_{v\ell}^{k} \; I_\ell/V_\ell^* \end{bmatrix} . \qquad (A12)$$

<u>Generators</u>

For a generator the Jacobian $\underset{\sim}{J}_g$ using the set of element variables $\bar{\underset{\sim}{z}}_g$ of (43a) is given by

$$\underset{\sim}{J}_g = \left(\begin{array}{cc|cc} -j/(2V_g) & j/(2V_g^*) & 0 & 0 \\ -jI_g^*/2 & jI_g/2 & jV_g^*/2 & -jV_g/2 \\ \hline V_g^*/[2(V_gV_g^*)^{1/2}] & V_g/[2(V_gV_g^*)^{1/2}] & 0 & 0 \\ I_g^*/2 & I_g/2 & V_g^*/2 & V_g/2 \end{array}\right), \quad \text{(A13)}$$

hence

$$\underset{\sim}{J}_g^{-1} = \left(\begin{array}{cc|cc} jV_g & 0 & V_g/(V_gV_g^*)^{1/2} & 0 \\ -jV_g^* & 0 & V_g^*/(V_gV_g^*)^{1/2} & 0 \\ \hline jI_g & -j/V_g^* & -I_g/(V_gV_g^*)^{1/2} & 1/V_g^* \\ -jI_g^* & j/V_g & -I_g^*/(V_gV_g^*)^{1/2} & 1/V_g \end{array}\right), \quad \text{(A14)}$$

so that

$$\underset{\sim}{\bar{\Theta}}_{gi} = \begin{pmatrix} j(\alpha V_g - \xi V_g^*) & j(\bar{\xi} V_g - \bar{\alpha} V_g^*) \\ 0 & 0 \end{pmatrix}, \quad \text{(A15)}$$

$$\underset{\sim}{\bar{\Theta}}_{gv} = \begin{pmatrix} j(\beta I_g - \nu I_g^*) & j(\bar{\nu} I_g - \bar{\beta} I_g^*) \\ j(-\beta/V_g^* + \nu/V_g) & j(-\bar{\nu}/V_g^* + \bar{\beta}/V_g) \end{pmatrix}, \quad \text{(A16)}$$

$$\underset{\sim}{M}_{11}^{g}\ \underset{\sim}{\Lambda}_{i}^{g}\ \underset{\sim}{w}_{gi} = \begin{pmatrix} \sum_k j\lambda_{gk}[N_{ig}^{k} V_g - \bar{N}_{ig}^{k} V_g^{*}] \\ \\ 0 \end{pmatrix}, \tag{A17}$$

$$\underset{\sim}{M}_{12}^{g}\ \underset{\sim}{\Lambda}_{v}^{g}\ \underset{\sim}{w}_{gv} = \begin{pmatrix} \sum_k j\lambda_{gk}[N_{vg}^{k} I_g - \bar{N}_{vg}^{k} I_g^{*}] \\ \\ \sum_k j\lambda_{gk}[-N_{vg}^{k}/V_g^{*} + \bar{N}_{vg}^{k}/V_g] \end{pmatrix}. \tag{A18}$$

Using the set of element variables $\tilde{\underset{\sim}{z}}_g$ of (43b) the Jacobian is given by

$$\underset{\sim}{J}_g = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline V_g^{*} & V_g & 0 & 0 \\ I_g^{*} & I_g & V_g^{*} & V_g \end{array}\right), \tag{A19}$$

hence

$$\underset{\sim}{J}_g^{-1} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -V_g^{*}/V_g & 0 & 1/V_g & 0 \\ \hline 0 & 1 & 0 & 0 \\ (I_g V_g^{*}/V_g^2 - I_g^{*}/V_g) & -V_g^{*}/V_g & -I_g/V_g^2 & 1/V_g \end{array}\right), \tag{A20}$$

so that

$$\bar{\underset{\sim}{\theta}}_{gi} = \begin{pmatrix} (\alpha-\xi V_g^*/V_g) & (\bar{\xi}-\bar{\alpha}V_g^*/V_g) \\ 0 & 0 \end{pmatrix}, \quad \text{(A21)}$$

$$\bar{\underset{\sim}{\theta}}_{gv} = \begin{pmatrix} \nu(I_g V_g^*/V_g^2 - I_g^*/V_g) & \bar{\beta}(I_g V_g^*/V_g^2 - I_g^*/V_g) \\ \beta-\nu V_g^*/V_g & \bar{\nu}-\bar{\beta}V_g^*/V_g \end{pmatrix}, \quad \text{(A22)}$$

$$\underset{\sim}{M}_{11}^{g} \, \underset{\sim}{\Lambda}_{i}^{g} \, \underset{\sim}{w}_{gi} = \begin{pmatrix} \sum_k \lambda_{gk}[N_{ig}^k - \bar{N}_{ig}^k \; V_g^*/V_g] \\ 0 \end{pmatrix}, \quad \text{(A23)}$$

$$\underset{\sim}{M}_{12}^{g} \, \underset{\sim}{\Lambda}_{v}^{g} \, \underset{\sim}{w}_{gv} = \begin{pmatrix} \sum_k \lambda_{gk} \bar{N}_{vg}^k (I_g V_g^*/V_g^2 - I_g^*/V_g) \\ \sum_k \lambda_{gk}[N_{vg}^k - \bar{N}_{vg}^k \; V_g^*/V_g] \end{pmatrix}. \quad \text{(A24)}$$

Slack Generator

For the slack generator the Jacobian $\underset{\sim}{J}_n$ using the set of element variables $\bar{\underset{\sim}{z}}_n$ of (44a) is given by

$$\mathbf{J}_n = \left(\begin{array}{cc|cc} I_n^*/2 & I_n/2 & V_n^*/2 & V_n/2 \\ -jI_n^*/2 & jI_n/2 & jV_n^*/2 & -jV_n/2 \\ \hline V_n^*/[2(V_nV_n^*)^{1/2}] & V_n/[2(V_nV_n^*)^{1/2}] & 0 & 0 \\ -j/(2V_n) & j/(2V_n^*) & 0 & 0 \end{array}\right), \quad \text{(A25)}$$

hence

$$\mathbf{J}_n^{-1} = \left(\begin{array}{cc|cc} 0 & 0 & V_n/(V_nV_n^*)^{1/2} & jV_n \\ 0 & 0 & V_n^*/(V_nV_n^*)^{1/2} & -jV_n^* \\ \hline 1/V_n^* & -j/V_n^* & -I_n/(V_nV_n^*)^{1/2} & jI_n \\ 1/V_n & j/V_n & -I_n^*/(V_nV_n^*)^{1/2} & -jI_n^* \end{array}\right), \quad \text{(A26)}$$

so that

$$\bar{\boldsymbol{\theta}}_{ni} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(A27)}$$

$$\bar{\boldsymbol{\theta}}_{nv} = \begin{pmatrix} (\beta/V_n^*+\nu/V_n) & (\bar{\nu}/V_n^*+\bar{\beta}/V_n) \\ j(-\beta/V_n^*+\nu/V_n) & j(-\bar{\nu}/V_n^*+\bar{\beta}/V_n) \end{pmatrix}, \quad \text{(A28)}$$

$$\mathbf{M}_{11}^n\ \boldsymbol{\Lambda}_i^n\ \mathbf{w}_{ni} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{(A29)}$$

$$\underset{\sim}{M}_{12}^{n} \underset{\sim}{\Lambda}_{n}^{n} \underset{\sim}{w}_{nv} = \begin{pmatrix} \sum_{k} \lambda_{nk}[N_{vn}^{k}/V_{n}^{*}+\overline{N}_{vn}^{k}/V_{n}] \\ \\ \sum_{k} j\lambda_{nk}[-N_{vn}^{k}/V_{n}^{*}+\overline{N}_{vn}^{k}/V_{n}] \end{pmatrix}, \tag{A30}$$

Using the set of element variables $\tilde{\underset{\sim}{z}}_n$ of (44b) the Jacobian is given by

$$\underset{\sim}{J}_{n} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), \tag{A31}$$

hence

$$\underset{\sim}{J}_{n}^{-1} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), \tag{A32}$$

so that

$$\overline{\underset{\sim}{\Theta}}_{ni} = \begin{pmatrix} 0 & 0 \\ \\ 0 & 0 \end{pmatrix}, \tag{A33}$$

$$\bar{\underset{\sim}{\theta}}_{nv} = - \bar{\underset{\sim}{\Lambda}}_{v} = \begin{pmatrix} \beta & \bar{\nu} \\ & \\ \nu & \bar{\beta} \end{pmatrix}, \tag{A34}$$

$$\underset{\sim}{M}^{n}_{11} \underset{\sim}{\Lambda}^{n}_{i} \underset{\sim}{w}_{ni} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}, \tag{A35}$$

$$\underset{\sim}{M}^{n}_{12} \underset{\sim}{\Lambda}^{n}_{v} \underset{\sim}{w}_{nv} = \begin{pmatrix} \sum_k \lambda_{nk} N^{k}_{vn} \\ \\ \sum_k \lambda_{nk} \bar{N}^{k}_{vn} \end{pmatrix}. \tag{A36}$$

<u>Transmission Elements</u>

For a transmission element the Jacobian $\underset{\sim}{J}_t$ using the set of element variables $\bar{\underset{\sim}{z}}_t$ of (45a) is given by

$$\underset{\sim}{J}_t = \left(\begin{array}{cc|cc} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & -j/2 & j/2 \\ \hline -I_t/(2V_t^2) & -I_t^*/(2V_t^*)^2 & 1/(2V_t) & 1/(2V_t^*) \\ jI_t/(2V_t^2) & -jI_t^*/(2V_t^*)^2 & -j/(2V_t) & j/(2V_t^*) \end{array}\right), \tag{A37}$$

hence

$$\underset{\sim}{J}_t^{-1} = \left[\begin{array}{cc|cc} V_t/I_t & jV_t/I_t & -V_t^2/I_t & -jV_t^2/I_t \\ V_t^*/I_t^* & -jV_t^*/I_t^* & -V_t^{*2}/I_t^* & jV_t^{*2}/I_t^* \\ \hline 1 & j & 0 & 0 \\ 1 & -j & 0 & 0 \end{array}\right] , \qquad (A38)$$

so that

$$\bar{\underset{\sim}{\theta}}_{ti} = \begin{bmatrix} (\alpha V_t/I_t+\xi V_t^*/I_t^*) & (\bar{\xi} V_t/I_t+\bar{\alpha} V_t^*/I_t^*) \\ j(\alpha V_t/I_t-\xi V_t^*/I_t^*) & j(\bar{\xi} V_t/I_t-\bar{\alpha} V_t^*/I_t^*) \end{bmatrix} , \qquad (A39)$$

$$\bar{\underset{\sim}{\theta}}_{tv} = \begin{bmatrix} (\beta+\nu) & (\bar{\nu}+\bar{\beta}) \\ j(\beta-\nu) & j(\bar{\nu}-\bar{\beta}) \end{bmatrix} , \qquad (A40)$$

$$\underset{\sim}{M}_{11}^t \, \underset{\sim}{\Lambda}_i^t \, \underset{\sim}{w}_{ti} = \begin{bmatrix} \sum_k \lambda_{tk} \, [N_{it}^k \, V_t/I_t + \bar{N}_{it}^k \, V_t^*/I_t^*] \\ \sum_k j\lambda_{tk} \, [N_{it}^k \, V_t/I_t - \bar{N}_{it}^k \, V_t^*/I_t^*] \end{bmatrix} , \qquad (A41)$$

$$\underset{\sim}{M}_{12}^t \, \underset{\sim}{\Lambda}_v^t \, \underset{\sim}{w}_{tv} = \begin{bmatrix} \sum_k \lambda_{tk} [N_{vt}^k + \bar{N}_{vt}^k] \\ \sum_k j\lambda_{tk} [N_{vt}^k - \bar{N}_{vt}^k] \end{bmatrix} , \qquad (A42)$$

Using the set of element variables $\tilde{z}_t$ of (45b) the Jacobian is given by

$$J_t = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -I_t/V_t^2 & 0 & 1/V_t & 0 \\ 0 & -I_t^*/V_t^{*2} & 0 & 1/V_t^* \end{array}\right), \quad \text{(A43)}$$

hence

$$J_t^{-1} = \left(\begin{array}{cc|cc} V_t/I_t & 0 & -V_t^2/I_t & 0 \\ 0 & V_t^*/I_t^* & 0 & -V_t^{*2}/I_t^* \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), \quad \text{(A44)}$$

so that

$$\bar{\theta}_{ti} = \begin{pmatrix} \alpha\ V_t/I_t & \bar{\xi}\ V_t/I_t \\ \xi\ V_t^*/I_t^* & \bar{\alpha}\ V_t^*/I_t^* \end{pmatrix}, \quad \text{(A45)}$$

$$\bar{\theta}_{tv} = \begin{pmatrix} \beta & \bar{\nu} \\ \nu & \bar{\beta} \end{pmatrix}, \quad \text{(A46)}$$

$$\underset{\sim}{M}^{t}_{11}\ \underset{\sim}{\Lambda}^{t}_{i}\ \underset{\sim}{w}_{ti} = \begin{pmatrix} \sum_{k} \lambda_{tk}\ N^{k}_{it}\ V_t/I_t \\ \\ \sum_{k} \lambda_{tk}\ \overline{N}^{k}_{it}\ V^{*}_{t}/I^{*}_{t} \end{pmatrix} , \qquad \text{(A47)}$$

$$\underset{\sim}{M}^{t}_{12}\ \underset{\sim}{\Lambda}^{t}_{v}\ \underset{\sim}{w}_{tv} = \begin{pmatrix} \sum_{k} \lambda_{tk}\ N^{k}_{vt} \\ \\ \sum_{k} \lambda_{tk}\ \overline{N}^{k}_{vt} \end{pmatrix} . \qquad \text{(A48)}$$

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TABLE Ia

ELEMENTS OF $\underset{\sim}{\bar{\theta}}_{bi}$, $\underset{\sim}{\bar{\theta}}_{bv}$ AND $\underset{\sim}{\theta}_b$ USING ELEMENT VARIABLES $\underset{\sim}{\bar{z}}_b$

	Load Elements	Generator Elements
$\underset{\sim}{\bar{\theta}}_{bi}$	$\begin{pmatrix} (\alpha V_\ell+\xi V_\ell^*)/\lvert V_\ell\rvert & (\bar{\xi} V_\ell+\bar{\alpha} V_\ell^*)/\lvert V_\ell\rvert \\ j(\alpha V_\ell-\xi V_\ell^*) & j(\bar{\xi} V_\ell-\bar{\alpha} V_\ell^*) \end{pmatrix}$	$\begin{pmatrix} j(\alpha V_g-\xi V_g^*) & j(\bar{\xi} V_g-\bar{\alpha} V_g^*) \\ 0 & 0 \end{pmatrix}$
$\underset{\sim}{\bar{\theta}}_{bv}$	$\begin{pmatrix} -(\beta S_\ell^*/V_\ell^*+\nu S_\ell/V_\ell)/\lvert V_\ell\rvert & -(\bar{\nu} S_\ell^*/V_\ell^*+\bar{\beta} S_\ell/V_\ell)/\lvert V_\ell\rvert \\ j(\beta S_\ell^*/V_\ell^*-\nu S_\ell/V_\ell) & j(\bar{\nu} S_\ell^*/V_\ell^*-\bar{\beta} S_\ell/V_\ell) \end{pmatrix}$	$\begin{pmatrix} j(\beta S_g^*/V_g^*-\nu S_g/V_g) & j(\bar{\nu} S_g^*/V_g^*-\bar{\beta} S_g/V_g) \\ j(-\beta/V_g^*+\nu/V_g) & j(-\bar{\nu}/V_g^*+\bar{\beta}/V_g) \end{pmatrix}$
$\underset{\sim}{\theta}_b$	$\begin{pmatrix} \frac{\partial f}{\partial \lvert V_\ell\rvert} - \sum_k \lambda_{\ell k}[V_\ell N_{i\ell}^k+V_\ell^*\bar{N}_{i\ell}^k-S_\ell^* N_{v\ell}^k/V_\ell^*-S_\ell\bar{N}_{v\ell}^k/V_\ell]/\lvert V_\ell\rvert \\ \frac{\partial f}{\partial \delta_\ell} - \sum_k j\lambda_{\ell k}[V_\ell N_{i\ell}^k+V_\ell^*\bar{N}_{i\ell}^k+S_\ell^* N_{v\ell}^k/V_\ell^*-S_\ell\bar{N}_{v\ell}^k/V_\ell] \end{pmatrix}$	$\begin{pmatrix} \frac{\partial f}{\partial \delta_g} - \sum_k j\lambda_{gk}[V_g N_{ig}^k-V_g^*\bar{N}_{ig}^k+S_g^* N_{vg}^k/V_g^*-S_g\bar{N}_{vg}^k/V_g] \\ \frac{\partial f}{\partial Q_g} - \sum_k j\lambda_{gk}[N_{vg}^k/V_g^*+\bar{N}_{vg}^k/V_g] \end{pmatrix}$

TABLE Ib

ELEMENTS OF $\bar{\underset{\sim}{\theta}}_{bi}$, $\bar{\underset{\sim}{\theta}}_{bv}$ AND $\underset{\sim}{\theta}_b$ USING ELEMENT VARIABLES $\bar{\underset{\sim}{z}}_b$

	Slack Generator	Transmission Elements
$\bar{\underset{\sim}{\theta}}_{bi}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} (\alpha/Y_t+\xi/Y_t^*) & (\bar{\xi}/Y_t+\bar{\alpha}/Y_t^*) \\ j(\alpha/Y_t-\xi/Y_t^*) & j(\bar{\xi}/Y_t-\bar{\alpha}/Y_t^*) \end{pmatrix}$
$\bar{\underset{\sim}{\theta}}_{bv}$	$\begin{pmatrix} (\beta/V_n^*+\nu/V_n) & (\bar{\nu}/V_m^*+\bar{\beta}/V_n) \\ j(-\beta/V_n^*+\nu/V_n) & j(-\bar{\nu}/V_n^*+\bar{\beta}/V_n) \end{pmatrix}$	$\begin{pmatrix} (\beta+\nu) & (\bar{\nu}+\bar{\beta}) \\ j(\beta-\nu) & j(\bar{\nu}-\bar{\beta}) \end{pmatrix}$
$\bar{\underset{\sim}{\theta}}_{b}$	$\begin{pmatrix} \frac{\partial f}{\partial P_n} - \sum_k \lambda_{nk}[N_{vn}^k/V_n^*+\bar{N}_{vn}^k/V_n] \\ \frac{\partial f}{\partial Q_n} - \sum_k j\lambda_{nk}[-N_{vn}^k/V_n^*+\bar{N}_{vn}^k/V_n] \end{pmatrix}$	$\begin{pmatrix} \frac{\partial f}{\partial Re\{I_t\}} - \sum_k \lambda_{tk}[N_{it}^k/Y_t+\bar{N}_{it}^k/Y_t^*+N_{vt}^k+\bar{N}_{vt}^k] \\ \frac{\partial f}{\partial Im\{I_t\}} - \sum_k j\lambda_{tk}[N_{it}^k/Y_t-\bar{N}_{it}^k/Y_t^*+N_{vt}^k-\bar{N}_{vt}^k] \end{pmatrix}$

TABLE IIa

ELEMENTS OF $\bar{\underset{\sim}{\theta}}_{bi}$, $\bar{\underset{\sim}{\theta}}_{bv}$ AND $\underset{\sim}{\theta}_b$ USING ELEMENT VARIABLES $\tilde{\underset{\sim}{z}}_b$

	Load Elements	Generator Elements
$\bar{\underset{\sim}{\theta}}_{bi}$	$\begin{bmatrix} \alpha & \bar{\xi} \\ \xi & \bar{\alpha} \end{bmatrix}$	$\begin{bmatrix} (\alpha-\xi V_g^*/V_g) & (\bar{\xi}-\bar{\alpha}V_g^*/V_g) \\ 0 & 0 \end{bmatrix}$
$\bar{\underset{\sim}{\theta}}_{bv}$	$\begin{bmatrix} -\nu S_\ell/V_\ell^2 & -\bar{\beta}S_\ell/V_\ell^2 \\ -\beta S_\ell^*/V_\ell^{*2} & -\bar{\nu}S_\ell^*/V_\ell^{*2} \end{bmatrix}$	$\begin{bmatrix} -j2\nu Q_g/V_g^2 & -j2\bar{\beta}Q_g/V_g^2 \\ \beta-\nu V_g^*/V_g & \bar{\nu}-\bar{\beta}V_g^*/V_g \end{bmatrix}$
$\underset{\sim}{\theta}_b$	$\begin{bmatrix} \frac{\partial f}{\partial V_\ell} - \sum_k \lambda_{\ell k}[N_{i\ell}^k - S_\ell \bar{N}_{v\ell}^k/V_\ell^2] \\ \frac{\partial f}{\partial V_\ell^*} - \sum_k \lambda_{\ell k}[\bar{N}_{i\ell}^k - S_\ell^* N_{v\ell}^k/V_\ell^{*2}] \end{bmatrix}$	$\begin{bmatrix} \frac{\partial f}{\partial V_g} - \sum_k \lambda_{gk}[N_{ig}^k - V_g^*\bar{N}_{ig}^k/V_g - j2Q_g\bar{N}_{vg}^k/V_g^2] \\ \frac{\partial f}{\partial I_g} - \sum_k \lambda_{gk}[N_{vg}^k - \bar{N}_{vg}^k V_g^*/V_g] \end{bmatrix}$

TABLE IIb

ELEMENTS OF $\bar{\theta}_{bi}$, $\bar{\theta}_{bv}$ AND θ_b USING ELEMENT VARIABLES $\tilde{z}_b$

	Slack Generator	Transmission Elements
$\bar{\theta}_{bi}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha/Y_t & \bar{\xi}/Y_t \\ \xi/Y_t^* & \bar{\alpha}/Y_t^* \end{pmatrix}$
$\bar{\theta}_{bv}$	$\begin{pmatrix} \beta & \bar{\nu} \\ \nu & \bar{\beta} \end{pmatrix}$	$\begin{pmatrix} \beta & \bar{\nu} \\ \nu & \bar{\beta} \end{pmatrix}$
θ_b	$\begin{pmatrix} \frac{\partial f}{\partial I_n} - \sum_k \lambda_{nk} N_{vn}^k \\ \frac{\partial f}{\partial I_n^*} - \sum_k \lambda_{nk} \bar{N}_{vn}^k \end{pmatrix}$	$\begin{pmatrix} \frac{\partial f}{\partial I_t} - \sum_k \lambda_{tk}[N_{it}^k/Y_t + N_{vt}^k] \\ \frac{\partial f}{\partial I_t^*} - \sum_k \lambda_{tk}[\bar{N}_{it}^k/Y_t^* + \bar{N}_{vt}^k] \end{pmatrix}$

TABLE IIIa

MATRICES $\underset{\sim}{M}_{21}^{b}$ AND $\underset{\sim}{M}_{22}^{b}$ USING ELEMENT VARIABLES $\bar{\underset{\sim}{z}}_{b}$

	Load Elements	Generator Elements
$\underset{\sim}{M}_{21}^{b}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} V_g/\lvert V_g\rvert & V_g^*/\lvert V_g\rvert \\ 0 & 0 \end{pmatrix}$
$\underset{\sim}{M}_{22}^{b}$	$\begin{pmatrix} 1/V_\ell^* & 1/V_\ell \\ -j/V_\ell^* & j/V_\ell \end{pmatrix}$	$\begin{pmatrix} -S_g^*/(V_g^*\lvert V_g\rvert) & -S_g/(V_g\lvert V_g\rvert) \\ 1/V_g^* & 1/V_g \end{pmatrix}$

TABLE IIIb

MATRICES $\underset{\sim}{M}^b_{21}$ AND $\underset{\sim}{M}^b_{22}$ USING ELEMENT VARIABLES $\underset{\sim}{\bar{z}}_b$

	Slack Generator	Transmission Elements
$\underset{\sim}{M}^b_{21}$	$\begin{pmatrix} V_n/\lvert V_n\rvert & V_n^*/\lvert V_n\rvert \\ jV_n & -jV_n^* \end{pmatrix}$	$\begin{pmatrix} -V_t/Y_t & -V_t^*/Y_t^* \\ -jV_t/Y_t & jV_t^*/Y_t^* \end{pmatrix}$
$\underset{\sim}{M}^b_{22}$	$\begin{pmatrix} -S_n^*/(V_n^*\lvert V_n\rvert) & -S_n/(V_n\lvert V_n\rvert) \\ jS_n^*/V_n^* & -jS_n/V_n \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

TABLE IV

MATRICES $\underset{\sim}{M}^b_{21}$ AND $\underset{\sim}{M}^b_{22}$ USING ELEMENT VARIABLES $\underset{\sim}{\tilde{z}}_b$

	Load Elements	Generator Elements	Slack Generator	Transmission Elements
$\underset{\sim}{M}^b_{21}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1/V_g \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} V_t/Y_t & 0 \\ 0 & V_t^*/Y_t^* \end{pmatrix}$
$\underset{\sim}{M}^b_{22}$	$\begin{pmatrix} 0 & 1/V_\ell \\ 1/V_\ell^* & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -S_g^*/(V_g^* V_g^2) \\ 0 & 1/V_g \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

SOC-237

A UNIFIED APPROACH TO POWER SYSTEM SENSITIVITY ANALYSIS AND PLANNING
PART I: FAMILY OF ADJOINT SYSTEMS

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Abstract: Efficient sensitivity analysis and gradient evaluation, essential in power system studies such as optimal power flow, contingency analysis and planning, is the subject of this paper. We present an approach based upon a generalized adjoint network concept. It exploits all the powerful features of Tellegen's theorem by suitable extensions through which the a.c. load flow model can be used without any approximations. We introduce the conjugate notation used in formulating the Tellegen expressions for general complex functions. We also introduce the concept of group terms which facilitate control of the adjoint system so that a wide variety of particular cases can be handled. We derive and tabulate standard sensitivity expressions common to all relevant power system studies.

Description:

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