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ALGORITHMS FOR TOLERANCE AND SECOND-ORDER SENSITIVITIES OF CASCADED STRUCTURES

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Abstract An exact and efficient approach to network analysis for cascaded structures has been suggested by Bandler et al. They demonstrated that it is useful for sensitivity and tolerance analyses, in particular, for a multiple of simultaneous large changes in design parameter values. This paper extends their work to second-order sensitivities, as well as to the evaluation of the response and its first-order sensitivity at the vertices of a tolerance region located in the space of toleranced design parameters. This information is needed in a worst-case search algorithm for design centering and tolerance assignment. A substantial saving in computational effort is achieved by using the new approach over the basic approach of reanalyzing the circuit at every vertex.

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I. INTRODUCTION

A newly developed approach for the analysis of cascaded networks (using the chain matrix) has been used efficiently to perform response evaluation as well as simultaneous and arbitrary large-change sensitivity [1]. This paper shows how first- and second-order sensitivities of the response w.r.t. the variable parameters can be obtained using this new approach.

In tolerance assignment, where tolerances on elements are optimized, the response and its first-order sensitivity at the vertices of the tolerance region [2] are needed by the optimization algorithms. This information is very useful if a worst-case search algorithm has to identify the worst vertex. Using the new approach this information can be obtained very easily and with minimum effort.

Two specific algorithms are presented. One is designed for evaluating first- and second-order sensitivities of the response and the other for evaluating the response and its sensitivities at the vertices of a tolerance region. An example is given along with a comparison between the new approach and the conventional way (the reanalysis) for evaluating the response and its sensitivites at the vertices.

II. THEORETICAL FOUNDATION

Consider the two-port element depicted in Fig. 1. The basic iteration, also summarized by Table I, is $\bar{\mathbf{y}} = \mathbf{A}\,\mathbf{y}$, where $\mathbf{A}$ is the transmission or chain matrix, $\mathbf{y}$ contains the output voltage and current and $\bar{\mathbf{y}}$ the corresponding input quantities. Table I presents some of the principal concepts involved in the following analyses. Fig. 2 depicts a cascaded network with appropriate terminations.

Forward analysis consists of initializing a $\bar{\underset{\sim}{u}}^T$ row vector as either [1 0], [0 1] or a suitable linear combination and successively premultiplying each constant chain matrix by the resulting row vector until an element of interest or a termination is reached.

Reverse analysis, which is similar to conventional analysis of cascaded networks, proceeds by initializing a $\underset{\sim}{v}$ column vector as either $[1\ 0]^T$ or $[0\ 1]^T$ or a suitable linear combination and successively postmultiplying each constant matrix by the resulting column vector, again until either an element of interest, or a termination is reached.

In summary, assuming a cascade of n two-ports we have

$$\bar{\underset{\sim}{y}}^1 = \underset{\sim}{y}^0 = \underset{\sim}{A}^1\underset{\sim}{A}^2 \ldots \underset{\sim}{A}^i \ldots \underset{\sim}{A}^n\underset{\sim}{y}^n \tag{1}$$

and, applying forward and reverse analyses up to $\underset{\sim}{A}^i$, this reduces to an expression of the form

$$d = \bar{\underset{\sim}{u}}^{1^T} \bar{\underset{\sim}{y}}^1 = c\, \bar{\underset{\sim}{u}}^{i^T} \underset{\sim}{A}^i \underset{\sim}{v}^i , \tag{2}$$

where

$$\underset{\sim}{y}^n = c\, \underset{\sim}{v}^n \tag{3}$$

and c and d relate selected output and input variables of interest explicitly with $\underset{\sim}{A}^i$.

The typical formula will, therefore, contain factors of the form

$$\text{function evaluation:} \qquad \bar{\underset{\sim}{u}}^T\underset{\sim}{A}\,\underset{\sim}{v} \Longrightarrow Q , \tag{4}$$

$$\text{first-order sensitivity:} \qquad \bar{\underset{\sim}{u}}^T\delta\underset{\sim}{A}\,\underset{\sim}{v} \Longrightarrow \delta Q , \tag{5}$$

partial derivative: $$\bar{\underset{\sim}{u}}^T \frac{\partial \underset{\sim}{A}}{\partial \phi} \underset{\sim}{v} \Longrightarrow Q' , \qquad (6)$$

large-change sensitivity: $$\bar{\underset{\sim}{u}}^T \Delta \underset{\sim}{A} \underset{\sim}{v} \Longrightarrow \Delta Q , \qquad (7)$$

where the parameter ϕ is contained in $\underset{\sim}{A}$. A full reverse analysis taking

$$[\underset{\sim}{v}_1^n \quad \underset{\sim}{v}_2^n] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yields

$$[\underset{\sim}{v}_1^i \quad \underset{\sim}{v}_2^i] = \underset{\sim}{A}^{i+1} \underset{\sim}{A}^{i+2} \ldots \underset{\sim}{A}^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and a corresponding full forward analysis taking

$$[\bar{\underset{\sim}{u}}_1^1 \quad \bar{\underset{\sim}{u}}_2^1]^T = [\underset{\sim}{u}_1^0 \quad \underset{\sim}{u}_2^0]^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yields

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underset{\sim}{A}^1 \underset{\sim}{A}^2 \ldots \underset{\sim}{A}^{i-1} = [\bar{\underset{\sim}{u}}_1^i \quad \bar{\underset{\sim}{u}}_2^i]^T .$$

Reference Planes

In considering more than one element in the cascade we divide the network into subnetworks by reference planes. These in turn are chosen so that no more than one element is to be explicitly considered between any pair of reference planes. In Fig. 3 the elements $\underset{\sim}{A}^k$, $\underset{\sim}{A}^i$ and $\underset{\sim}{A}^j$ are considered in the kth, the ith and the jth subnetworks, respectively. Note that the superscripts of $\underset{\sim}{A}$ here, and from now on, denote the subnetwork and not the element. Forward and reverse analyses are initiated at the reference planes. A forward iteration of the structure of Fig. 3 is illustrated in Fig. 4, where equivalent (Thevenin) sources are iteratively determined. Reverse iteration is shown in Fig. 5, where equivalent (Norton) sources are iteratively determined.

III. NETWORK FUNCTIONS IN TERMS OF ELEMENTS UNDER CONSIDERATION

Performing forward analysis from the source of the ith subnetwork to the input of $\underset{\sim}{A}^i$ and reverse analysis from the load to the output of $\underset{\sim}{A}^i$ we have

$$V_S^i = (\bar{\underset{\sim}{u}}_1 + Z_S^i \bar{\underset{\sim}{u}}_2)^T \underset{\sim}{A}^i (V_L^i \underset{\sim}{v}_1 + (Y_L^i V_L^i - I_L^i)\underset{\sim}{v}_2) = V_L^k + Z_S^i I_S^i \tag{8}$$

and the current through the voltage source of the ith subnetwork

$$I_S^i = \bar{\underset{\sim}{u}}_2^T \underset{\sim}{A}^i (V_L^i \underset{\sim}{v}_1 + (Y_L^i V_L^i - I_L^i)\underset{\sim}{v}_2) = V_L^k Y_L^k - I_L^k . \tag{9}$$

From (8), letting $I_L^i = 0$ and $Y_L^i = 0$, we have $I_S^j = 0$ and the Thevenin voltage

$$V_S^j = V_L^i = \frac{V_S^i}{(\bar{\underset{\sim}{u}}_1 + Z_S^i \bar{\underset{\sim}{u}}_2)^T \underset{\sim}{A}^i \underset{\sim}{v}_1} = \frac{V_S^i}{Q_{11}^i + Z_S^i Q_{21}^i} , \tag{10}$$

where the Q terms have been defined in (4). See also Table II. Letting $V_S^i = 0$ and $Y_L^i = 0$, we have $I_S^j = -I_L^i$ and the output impedance

$$Z_S^j = \frac{V_L^i}{I_L^i} = \frac{(\bar{\underset{\sim}{u}}_1 + Z_S^i \bar{\underset{\sim}{u}}_2)^T \underset{\sim}{A}^i \underset{\sim}{v}_2}{(\bar{\underset{\sim}{u}}_1 + Z_S^i \bar{\underset{\sim}{u}}_2)^T \underset{\sim}{A}^i \underset{\sim}{v}_1} = \frac{Q_{12}^i + Z_S^i Q_{22}^i}{Q_{11}^i + Z_S^i Q_{21}^i} , \tag{11}$$

where, again, the Q terms of (4) are used to obtain a compact expression (see Table II). These expressions for V_S^j and Z_S^j permit equivalent Thevenin sources to be moved in a forward iteration.

From (8) and (9), letting $I_L^i = 0$ and $Z_S^i = 0$ we have $I_L^k = 0$ and the input admittance

$$Y_L^k = \frac{I_S^i}{V_S^i} = \frac{\bar{\underset{\sim}{u}}_2^T \underset{\sim}{A}^i(\underset{\sim}{v}_1 + Y_L^i \underset{\sim}{v}_2)}{\bar{\underset{\sim}{u}}_1^T \underset{\sim}{A}^i(\underset{\sim}{v}_1 + Y_L^i \underset{\sim}{v}_2)} = \frac{Q_{21}^i + Y_L^i Q_{22}^i}{Q_{11}^i + Y_L^i Q_{12}^i} \, . \tag{12}$$

Letting $V_S^i = 0$ and $Z_S^i = 0$, we have $V_L^k = 0$ and the Norton current

$$I_L^k = -I_S^i = -I_L^i (Y_L^k \bar{\underset{\sim}{u}}_1 - \bar{\underset{\sim}{u}}_2)^T \underset{\sim}{A}^i \underset{\sim}{v}_2 = -I_L^i (Y_L^k Q_{12}^i - Q_{22}^i) \, . \tag{13}$$

These expressions for I_L^k and Y_L^k permit equivalent Norton sources to be moved (if desired) in a reverse iteration.

A special case of (10) applicable to Fig. 2 is

$$V_L = \frac{V_S}{\bar{\underset{\sim}{u}}_1^T \underset{\sim}{A} \underset{\sim}{v}_1} = \frac{V_S}{Q_{11}} \, . \tag{14}$$

Table III gives some useful formulas which can be obtained for variations in a particular element $\underset{\sim}{A}$. We note, for example, that, since $\underset{\sim}{A}$ is arbitrary and at most only one full analysis yields all Q_{11}, δQ_{11}, Q_{11}' and ΔQ_{11}, the corresponding V_L, δV_L, $\partial V_L/\partial\phi$ and ΔV_L w.r.t. all possible parameters anywhere in the cascade can be evaluated exactly for one network analysis.

IV. SECOND-ORDER SENSITIVITIES

The first-order sensitivity of V_L w.r.t. a variable parameter ϕ_1 is given using (14) by

$$\frac{\partial V_L}{\partial \phi_1} = \frac{-V_S \dfrac{\partial Q_{11}}{\partial \phi_1}}{Q_{11}^2} \, . \tag{15}$$

Differentiating (15) w.r.t. ϕ_2 we get

$$\frac{\partial^2 V_L}{\partial\phi_2\partial\phi_1} = - V_S \frac{\partial}{\partial\phi_2}\left[\frac{\partial Q_{11}}{\partial\phi_1} / Q_{11}^2\right]$$

$$= - V_L \left[\frac{Q_{11}\dfrac{\partial^2 Q_{11}}{\partial\phi_2\partial\phi_1} - 2\dfrac{\partial Q_{11}}{\partial\phi_1}\dfrac{\partial Q_{11}}{\partial\phi_2}}{Q_{11}^2}\right] . \qquad (16)$$

The evaluation of $\partial Q_{11}/\partial\phi_1$ and $\partial Q_{11}/\partial\phi_2$ is straightforward (see Table III). For the evaluation of the term $\partial^2 Q_{11}/\partial\phi_2\partial\phi_1$, we assume that the variables are numbered consecutively from the source end to the load end so that this term is expressed, for example, by

$$\frac{\partial^2 Q_{11}}{\partial\phi_2\partial\phi_1} = \frac{\partial}{\partial\phi_1}(\bar{\underset{\sim}{u}}_1^T)\frac{\partial \underset{\sim}{A}}{\partial\phi_2}\underset{\sim}{v}_1 . \qquad (17)$$

Note that $\bar{\underset{\sim}{u}}_1^T$ is a function of a certain chain matrix which contains the variable ϕ_1, $\underset{\sim}{A}$ is the chain matrix containing ϕ_2 and $\underset{\sim}{v}_1$ is evaluated at the reference plane following $\underset{\sim}{A}$.

The following algorithm, which is similar to Algorithm 2 in [1] can be used to obtain the first- and second-order sensitivities of V_L w.r.t. the design variables. Fig. 6 illustrates the main stages of the algorithm.

<u>Algorithm 1 First- and Second-Order Sensitivities</u>

<u>Step 1</u> Initialize $\underset{\sim}{u}^0$ and $\underset{\sim}{v}$.

Set $i \leftarrow 1$, $m \leftarrow 1$, $q \leftarrow 0$, $r \leftarrow 1$, $j \leftarrow n$.

<u>Comment</u> n is the total number of elements in the cascade.

Step 2 If $i = \ell_m$ go to Step 6.

Comment ℓ_m is an element of L, an index set containing superscripts of the k matrices containing the k variable parameters and ordered consecutively.

Step 3 $\underset{\sim}{u}^{0T} \leftarrow \underset{\sim}{u}^{0T} \underset{\sim}{A}^{i}$.

If m = 1 go to Step 4.

$\underset{\sim}{u}^{1T} \leftarrow \underset{\sim}{u}^{1T} \underset{\sim}{A}^{i}$.

.
.
.

$\underset{\sim}{u}^{qT} \leftarrow \underset{\sim}{u}^{qT} \underset{\sim}{A}^{i}$.

Comment $\underset{\sim}{u}^1, \underset{\sim}{u}^2, \ldots, \underset{\sim}{u}^q$ are working arrays used to proceed with the evaluation of the gradients of $\underset{\sim}{u}^0$ w.r.t. the q variables already passed by the forward analysis.

Step 4 Set $i \leftarrow i+1$.

Step 5 If $i = \ell_m$ go to Step 6.

Go to Step 3.

Step 6 $\underset{\sim}{x}^m \leftarrow \underset{\sim}{u}^0$.

If m = 1 go to Step 10.

Comment Once a variable element is reached the $\underset{\sim}{u}^0$ is stored in $\underset{\sim}{x}^m$ to be used in the calculation of the first-order sensitivity.

Step 7 Set $p \leftarrow 1$.

Step 8 $\underset{\sim}{w}^r \leftarrow \underset{\sim}{u}^p$.

$r \leftarrow r+1$.

If p = q go to Step 10.

Comment The w arrays are used to store the appropriate gradients of $\underset{\sim}{u}^0$, namely, $\underset{\sim}{u}^1, \underset{\sim}{u}^2, \ldots,$ for the calculation of second-order sensitivities.

<u>Step 9</u> Set $p \leftarrow p+1$.

Go to Step 8.

<u>Step 10</u> If m=k go to Step 12.

$\utilde{u}^{mT} \leftarrow \utilde{u}^{0T} \frac{\partial \utilde{A}^i}{\partial \phi_m}$.

$\utilde{u}^{0T} \leftarrow \utilde{u}^{0T} \utilde{A}^i$.

$\utilde{u}^{1T} \leftarrow \utilde{u}^{1T} \utilde{A}^i$.

.
.
.

$\utilde{u}^{(m-1)T} \leftarrow \utilde{u}^{(m-1)T} \utilde{A}^i$.

<u>Comment</u> At this step a new $\utilde{u}$ is introduced which is equal to $\utilde{u}^0$ multiplied by the derivative of $\utilde{A}^i$ w.r.t. ϕ_m, where $\utilde{A}^i$ is a function of ϕ_m only.

<u>Step 11</u> Set $i \leftarrow i+1$.

$m \leftarrow m+1$.

$q \leftarrow q+1$.

Go to Step 5.

<u>Step 12</u> Set $r \leftarrow r-1$.

If $n = \ell_k$ go to Step 15.

<u>Step 13</u> $\utilde{v} \leftarrow \utilde{A}^j \utilde{v}$.

$j \leftarrow j-1$.

<u>Comment</u> This step is concerned with the reverse analysis.

<u>Step 14</u> If $j = \ell_m$ go to Step 15.

Go to Step 13.

<u>Step 15</u> Calculate $\partial Q/\partial \phi_m$ and $\partial^2 Q/\partial \phi_m^2$.

<u>Comment</u> At this point the first-order derivative of Q w.r.t. ϕ_m can be evaluated, since $\utilde{u}^0$ and $\utilde{v}$ at the reference planes before and after the element are known. $\partial^2 Q/\partial \phi_m^2$ is evaluated using $\utilde{u}^0$, $\utilde{v}$

and $\partial^2 \underset{\sim}{A}^j / \partial \phi_m^2$.

Step 16 If m = 1 stop.

Set $s \leftarrow m - 1$. $p \leftarrow 1$.

Step 17 Calculate $\partial^2 Q / \partial \phi_s \partial \phi_m$.

If p=q go to Step 19.

Comment $\partial^2 Q / \partial \phi_s \partial \phi_m$ is evaluated using the appropriate $\underset{\sim}{w}^r$, $\partial \underset{\sim}{A}^j / \partial \phi_m$ and $\underset{\sim}{v}$, where s = 1, ..., m-1.

Step 18 Set $p \leftarrow p + 1$.

$r \leftarrow r - 1$.

$s \leftarrow s - 1$.

Go to Step 17.

Step 19 Set $q \leftarrow q - 1$.

$m \leftarrow m - 1$.

$r \leftarrow r - 1$.

Go to Step 14.

V. THE EVALUATION OF V_L AND ITS SENSITIVITIES W.R.T. DESIGN PARAMETERS AT ALL VERTICES OF THE TOLERANCE REGION

Algorithms concerned with finding worst vertices of the tolerance region need the value of the response at the vertices [3] as well as the sensitivity of this reponse w.r.t. the design parameters [4,5]. Each parameter will have a tolerance associated with it so that it can have one of two values $\phi + \varepsilon$ or $\phi - \varepsilon$, where ε is the tolerance [2]. The number of vertices of the tolerance region is 2^k, where k is the number of variable parameters, which includes all different combinations of parameter values.

Assume that we have partitioned the network by reference planes into subnetworks such that each subnetwork contains one chain matrix containing a variable parameter. Each reference plane is chosen to fall immediately after a variable element.

The Thevenin voltage/impedance of the ith subnetwork is considered as the source voltage/impedance of the (i+1)th subnetwork, given by (10) and (11), respectively, where j = i+1. We have to note here that the terms Q^i_{11}, Q^i_{21}, Q^i_{12} and Q^i_{22} are as defined in (4) with $\tilde{v}_1$ and $\tilde{v}_2$ set to $\tilde{e}_1$ and $\tilde{e}_2$, respectively, since the appropriate reference plane immediately follows the element $\tilde{A}^i$. The number of pairs of terms V_S^{i+1} and Z_S^{i+1} to be evaluated is 2^i, since each subnetwork contains one variable element with two extreme values (assuming that each $\tilde{A}^i$ contains only one variable parameter).

Differentiating (10) w.r.t. ϕ_h, where ϕ_h does not belong to $\tilde{A}^i$, but V^i_S and Z^i_S are functions of ϕ_h (i.e., ϕ_h is in a subnetwork h before the ith subnetwork) we get

$$\frac{\partial V_S^{i+1}}{\partial \phi_h} = \frac{(Q^i_{11}+Z^i_S\, Q^i_{21})\,\dfrac{\partial V^i_S}{\partial \phi_h} - V^i_S\,\dfrac{\partial Z^i_S}{\partial \phi_h}\,Q^i_{21}}{(Q^i_{11} + Z^i_S\, Q^i_{21})^2} \,, \qquad (18)$$

and differentiating (11) w.r.t. ϕ_h, we get

$$\frac{\partial Z_S^{i+1}}{\partial \phi_h} = \frac{(Q^i_{11}+Z^i_S\, Q^i_{21})\,\dfrac{\partial Z^i_S}{\partial \phi_h}\,Q^i_{22} - (Q^i_{21}+Z^i_S\, Q^i_{22})\,\dfrac{\partial Z^i_S}{\partial \phi_h}\,Q^i_{21}}{(Q^i_{11} + Z^i_S\, Q^i_{21})^2}$$

$$= \frac{\partial Z_S^i}{\partial \phi_h} \frac{(Q_{11}^i Q_{22}^i - Q_{12}^i Q_{21}^i)}{(Q_{11}^i + Z_S^i Q_{21}^i)^2} . \tag{19}$$

On the other hand, the derivatives w.r.t. ϕ_i which is contained in $\underset{\sim}{A}^i$ (Z_S^i and V_S^i are not functions of ϕ_i), are

$$\frac{\partial V_S^{i+1}}{\partial \phi_i} = \frac{-V_S^i \left(\frac{\partial Q_{11}^i}{\partial \phi_i} + Z_S^i \frac{\partial Q_{21}^i}{\partial \phi_i}\right)}{(Q_{11}^i + Z_S^i Q_{21}^i)^2} \tag{20}$$

and

$$\frac{\partial Z_S^{i+1}}{\partial \phi_i} = \frac{(Q_{11}^i + Z_S^i Q_{21}^i)\left(\frac{\partial Q_{21}^i}{\partial \phi_i} + Z_S^i \frac{\partial Q_{22}^i}{\partial \phi_i}\right) - (Q_{12}^i + Z_S^i Q_{22}^i)\left(\frac{\partial Q_{11}^i}{\partial \phi_i} + Z_S^i \frac{\partial Q_{21}^i}{\partial \phi_i}\right)}{(Q_{11}^i + Z_S^i Q_{21}^i)^2} , \tag{21}$$

where $\frac{\partial Q_{11}^i}{\partial \phi_i}$, $\frac{\partial Q_{21}^i}{\partial \phi_i}$, $\frac{\partial Q_{12}^i}{\partial \phi_i}$ and $\frac{\partial Q_{22}^i}{\partial \phi_i}$ correspond to (6) and Table II. This sensitivity information is carried and through the analysis for each subnetwork. The number of variables for which sensitivities of V_S^{i+1} and Z_S^{i+1} exist at the (i+1)th subnetwork is i so that $2^i.i$ sensitivity calculations are performed. Having Y_L and I_L as zeros, the expression relating V_L and the last sets of V_S and Z_S, is given by (10), so that 2^k values for V_L and its sensitivities can be obtained from appropriate values of V_S, Z_S and $\underset{\sim}{A}$.

Fig. 7 shows an example of the stages involved in the algorithm to obtain the response and its sensitivities at the vertices (3 variables ==> 8 vertices) of the tolerance region.

Algorithm 2 Response Value and its Derivatives w.r.t. All Variable Parameters, at All Vertices of the Tolerance Region

Step 1 Initialize $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$ and $\underset{\sim}{v}$.

Set $i \leftarrow 1$, $m \leftarrow 1$, $j \leftarrow n$.

Step 2 If $i = \ell_m$ go to Step 6.

Step 3 $\overline{\underset{\sim}{u}}{}_1^T \leftarrow \overline{\underset{\sim}{u}}{}_1^T \underset{\sim}{A}^i$.

$\overline{\underset{\sim}{u}}{}_2^T \leftarrow \overline{\underset{\sim}{u}}{}_2^T \underset{\sim}{A}^i$.

Set $i \leftarrow i + 1$.

Step 4 If $i = \ell_m$ go to Step 5.

Go to Step 3.

Step 5 If m=k go to Step 7.

Step 6 Calculate V_S, Z_S ,

$$\frac{\partial V_S}{\partial \phi_1}, \ldots, \frac{\partial V_S}{\partial \phi_m},$$

$$\frac{\partial Z_S}{\partial \phi_1}, \ldots, \frac{\partial Z_S}{\partial \phi_m},$$

2^m sets all together.

Set $m \leftarrow m + 1$.

$i \leftarrow i + 1$.

Initialize $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ and go to Step 4.

Step 7 If $n = \ell_k$ go to Step 10.

Step 8 $\underset{\sim}{v} = \underset{\sim}{A}^j \underset{\sim}{v}$.

Set $j \leftarrow j-1$.

Step 9 If $j = \ell_k$ go to Step 10.

Go to Step 8.

<u>Step 10</u> Calculate Q, $\partial Q/\partial \phi_1$, ..., $\partial Q/\partial \phi_k$ 2^k times.

Stop.

VI. EXAMPLE

The cascaded seven-section bandpass filter shown in Fig. 8 [6] was considered. All sections are quarter-wave at 2.175 GHz. The optimal minimax characteristic impedances [7] are taken as nominal values. They are

$$Z_1 = Z_7 = 0.606595$$
$$Z_2 = Z_6 = 0.303547$$
$$Z_3 = Z_5 = 0.722287$$
$$Z_4 = 0.235183$$

The sensitivity of the output voltage V_L w.r.t. length ℓ_4 of the fourth section and the sensitivity w.r.t. Z_4 are evaluated at a normalized frequency of 0.5 as

$$\frac{\partial V_L}{\partial \ell_4} = -0.2804064 + j0.5161026$$

$$\frac{\partial V_L}{\partial Z_4} = -2.617364 + j4.817395$$

Without any further effort (since the two parameters belong to the same element) we obtain

$$\frac{\partial^2 V_L}{\partial Z_4 \partial \ell_4} = 11.71675 + j5.415667$$

Table IV compares the results obtained by this method and the one obtained by the adjoint network method [8]. Taking two parameters in different elements, for example Z_4 and Z_5, we obtain the second-order term

$$\frac{\partial^2 V_L}{\partial Z_4 \partial Z_5} = -30.12383 - j7.516802$$

A tolerance of ±0.03 on Z_1, Z_4 and Z_5 was chosen. Algorithm 2 was used to evaluate V_L, $\partial V_L/\partial Z_1$, $\partial V_L/\partial Z_4$ and $\partial V_L/\partial Z_5$ at the eight vertices of the tolerance region (2^3 vertices where 3 is the number of toleranced variables). The results are tabulated in Table V. They were checked individually by reanalyzing the circuit at each vertex.

VII. DISCUSSION AND CONCLUSIONS

The calculation of the first- and second-order sensitivities of a circuit reponse involves one additional analysis of the adjoint network (assuming the analysis of the original network has already been performed) and k(k+1)/2 analyses to find second-order sensitivities calculated by finite differences. A more efficient approach is to calculate these second-order sensitivities using the adjoint network concept by performing only k analyses. Using the new approach for the analysis of cascaded structures, however, less than k analyses are performed and no additional memory is required.

The algorithm for evaluating the response and its sensitivities at the vertices of the tolerance region proved to be very efficient. The seven-section filter example was run with tolerances on the characteristic impedances of the stubs and transmission lines (all seven). It took 0.269 s CPU time to evaluate the response (only) at the $128(2^7)$ vertices. Using the conventional method of reanalyzing the circuit for different component values would take 0.074 x 128 = 9.472 s CPU, where one analysis is performed in approximately 0.074 s. For the case of evaluating the response and its sensitivities at vertices discussed in Section VI, it took 0.118 s CPU time compared with 8 x 0.074 = 0.592 s for 8 analyses. The savings in computational effort is substantial.

REFERENCES

[1] J.W. Bandler, M.R.M. Rizk and H.L. Abdel-Malek, "New results in network simulation, sensitivity and tolerance analysis for cascaded structures", IEEE Trans. Microwave Theory Tech., vol. MTT-26, 1978.

[2] J.W. Bandler and P.C. Liu, "Automated network design with optimal tolerances", IEEE Trans. Circuits and Systems, vol. CAS-21, 1974, pp. 219-222.

[3] K.H. Leung and R. Spence, "Multiparameter large-change sensitivity analysis and systematic exploration", IEEE Trans. Circuits and Systems, vol. CAS-22, 1975, pp. 796-804.

[4] J.W. Bandler, P.C. Liu and J.H.K. Chen, "Worst case network tolerance optimization", IEEE Trans. Microwave Theory Tech., vol. MTT-23, 1975, pp. 630-641.

[5] H. Tromp, "Generalized worst case design, with applications to microwave networks", Doctoral Thesis (in Dutch), Faculty of Engineering, University of Ghent, Ghent, Belgium, 1978.

[6] M.C. Horton and R.J. Wenzel, "General theory and design of optimum quarter-wave TEM filters", IEEE Trans. Microwave Theory Tech., vol. MTT-13, 1965, pp. 316-327.

[7] J.W. Bandler, C. Charalambous, J.H.K. Chen and W.Y. Chu, "New results in the least pth approach to minimax design", IEEE Trans. Microwave Theory Tech., vol. MTT-24, 1976, pp. 116-119.

[8] M.R.M. Rizk, "Efficient simulation of circuits for amplitude and delay optimization in the frequency domain", M.Eng. Thesis, McMaster University, Hamilton, Canada, 1975.

TABLE I

PRINCIPAL CONCEPTS INVOLVED IN THE ANALYSES

Concept	Definition	Implication
Basic iteration	$\bar{\underset{\sim}{y}} = \underset{\sim}{A}\,\underset{\sim}{y}$	$\underset{\sim}{y} \Longrightarrow \bar{\underset{\sim}{y}}$
Forward operation	$\bar{\underset{\sim}{u}}^T \underset{\sim}{A} = \underset{\sim}{u}^T$	$\bar{\underset{\sim}{u}}^T \underset{\sim}{y} = \bar{\underset{\sim}{u}}^T \underset{\sim}{A}\underset{\sim}{y} = \underset{\sim}{u}^T \underset{\sim}{y}$
Reverse operation	$\bar{\underset{\sim}{v}} = \underset{\sim}{A}\underset{\sim}{v}$	$\underset{\sim}{y} = c\underset{\sim}{v} \Longrightarrow \bar{\underset{\sim}{y}} = c\bar{\underset{\sim}{v}}$
Voltage selector	$\underset{\sim}{e}_1 \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\underset{\sim}{e}_1 \Longrightarrow \underset{\sim}{u}_1$ or $\underset{\sim}{v}_1$
Current selector	$\underset{\sim}{e}_2 \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\underset{\sim}{e}_2 \Longrightarrow \underset{\sim}{u}_2$ or $\underset{\sim}{v}_2$
Equivalent source	$\underset{\sim}{y} = \begin{bmatrix} V_S - Z_S I_S \\ I_S \end{bmatrix}$	$\underset{\sim}{e}_1^T \underset{\sim}{y} = V_S - Z_S I_S$, $\underset{\sim}{e}_2^T \underset{\sim}{y} = I_S$
Equivalent load	$\underset{\sim}{y} = \begin{bmatrix} V_L \\ Y_L V_L - I_L \end{bmatrix}$	$\underset{\sim}{y} = V_L \underset{\sim}{e}_1 + (Y_L V_L - I_L)\underset{\sim}{e}_2$

TABLE II

NOTATION AND IMPLIED INITIAL CONDITIONS

Factor	Identification	Initial Conditions Forward	Initial Conditions Reverse
$\bar{u}_1^T$ (*) v_1	$(\dagger)_{11}$	voltage	voltage
$\bar{u}_1^T$ (*) v_2	$(\dagger)_{12}$	voltage	current
$\bar{u}_2^T$ (*) v_1	$(\dagger)_{21}$	current	voltage
$\bar{u}_2^T$ (*) v_2	$(\dagger)_{22}$	current	current

(*) denotes either A, δA, $\partial A/\partial\phi$ or ΔA

(†) denotes Q, δQ, Q' or ΔQ, as taken from (4), (5), (6) or (7), respectively

TABLE III

FUNCTIONS OF OUTPUT VOLTAGE V_L FOR CHANGES IN $\underset{\sim}{A}$ ONLY

Variable	Output
$\underset{\sim}{A}$	$V_L = \dfrac{V_S}{Q_{11}}$
$\delta\underset{\sim}{A}$	$\delta V_L = -\dfrac{V_L^2}{V_S}\ \delta Q_{11}$
$\dfrac{\partial \underset{\sim}{A}}{\partial\phi}$	$\dfrac{\partial V_L}{\partial\phi} = -\dfrac{V_L^2}{V_S}\ Q'_{11}$
$\Delta\underset{\sim}{A}$	$\Delta V_L = -\dfrac{V_L^2}{V_L+V_S/\Delta Q_{11}}$

TABLE IV

COMPARISON OF SECOND-ORDER SENSITIVITIES WITH DIFFERENT APPROACHES

Term	Adjoint Network	1st Order Sensitivity by Adjoint Network 2nd Order Sensitivity by Perturbation	The New Approach
$\frac{\partial^2 V_L}{\partial Z_4 \partial \ell_4}$	11.71675+j5.415667	11.713232+j5.431066	11.71675+j5.415667

TABLE V

THE RESPONSE V_L AND ITS SENSITIVITIES AT THE VERTICES OF THE TOLERANCE REGION AT NORMALIZED FREQUENCY 0.7

Vertex	V_L	$\partial V_L/\partial Z_1$	$\partial V_L/\partial Z_4$	$\partial V_L/\partial Z_5$	Sign of Tolerance Extreme		
1	0.49135+j0.02351	-0.02450+j0.05953	0.26004-j1.15934	0.02549+j0.32944	-	-	-
2	0.48819+j0.02571	-0.07761+j0.01588	0.28346-j1.05326	0.00954+j0.34878	+	-	-
3	0.49679-j0.04862	0.03751+j0.15916	-0.06631-j0.94430	0.04534+j0.29165	-	+	-
4	0.49677-j0.04046	-0.03384+j0.11417	-0.00426-j0.87724	0.03578+j0.31848	+	+	-
5	0.49209+j0.04341	-0.04367+j0.08072	0.29407-j1.19530	-0.00103+j0.33324	-	-	+
6	0.48786+j0.04670	-0.09378+j0.03123	0.32067-j1.07952	-0.02042+j0.35007	+	-	+
7	0.49889-j0.03101	0.02608+j0.18868	-0.05742-j0.97346	0.02462+j0.29494	-	+	+
8	0.49818-j0.02127	-0.04526+j0.13735	0.01132-j0.90191	0.01113+j0.32057	+	+	+

Figure Captions

Fig. 1 Notation for an element in the chain, indicating reference directions and voltage and current variables.

Fig. 2 Cascaded network with appropriate terminations.

Fig. 3 Subnetwork i cascaded with subnetworks k (at source end) and j (at load end).

Fig. 4 Forward iteration for Fig. 3, transferring an equivalent source accounting for design variables from subnetwork k from one reference plane to the other.

Fig. 5 Reverse iteration for Fig. 3, transferring an equivalent source accounting for design variables from subnetwork j from one reference plane to the other.

Fig. 6 Illustration for a cascade of 6 two-ports of the principal stages in the calculations of first- and second-order sensitivities w.r.t. three variable elements.

$$g_i \triangleq \partial Q_{11}/\partial \phi_i, \; i = 1, 2, 3,$$

$$S_{ij} \triangleq \partial^2 Q_{11}/\partial \phi_i \partial \phi_j, \; i,j = 1,2,3.$$

Fig. 7 Illustration of the principal stages of Algorithm 2.

Fig. 8 Seven-section filter containing unit elements and stubs [6]. All sections are quarter-wave at 2.175 GHz.

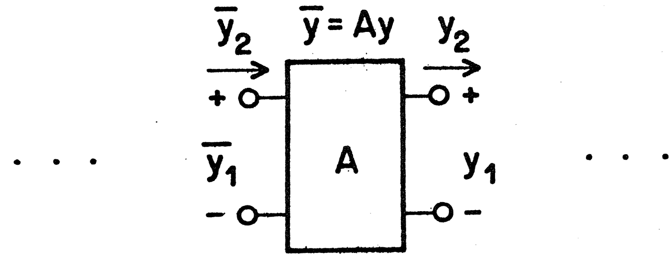

Fig. 1

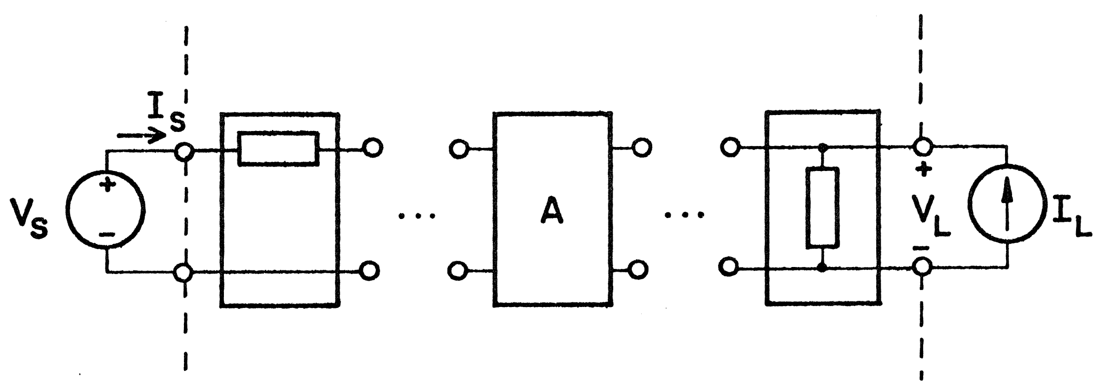

Fig. 2

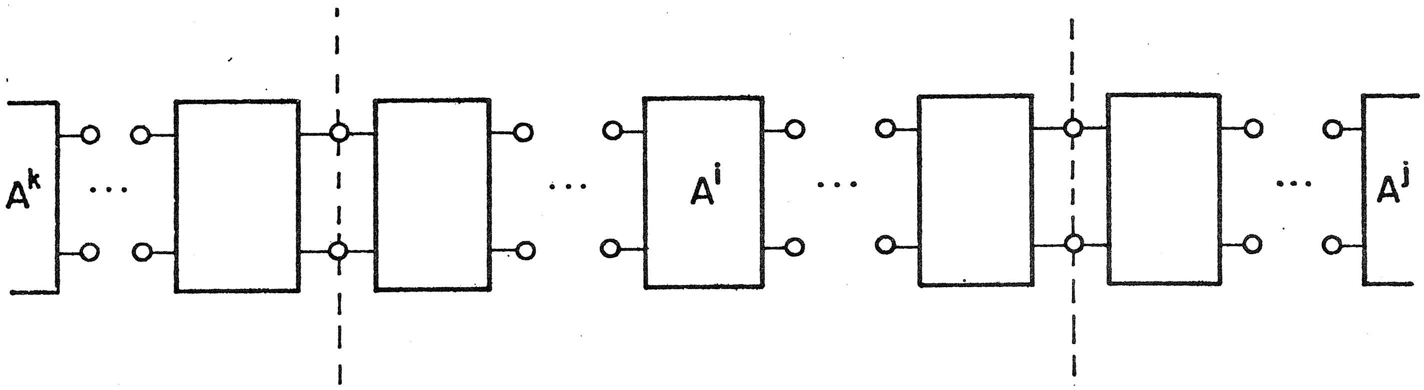

Fig. 3

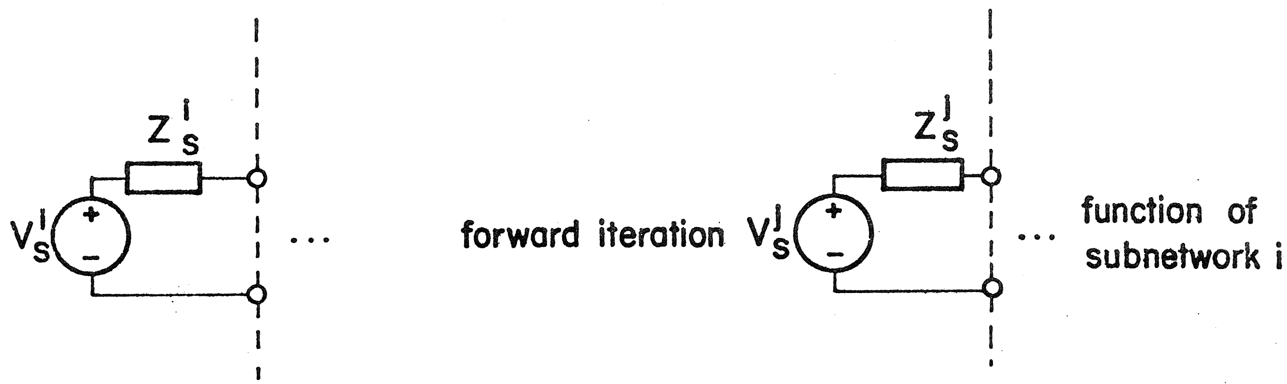

Fig. 4

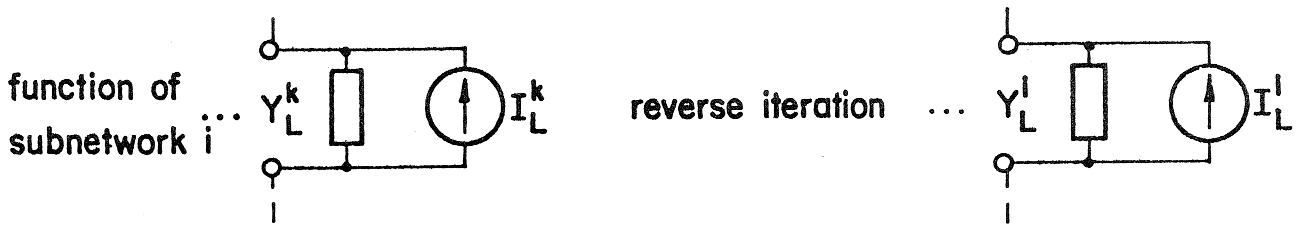

Fig. 5

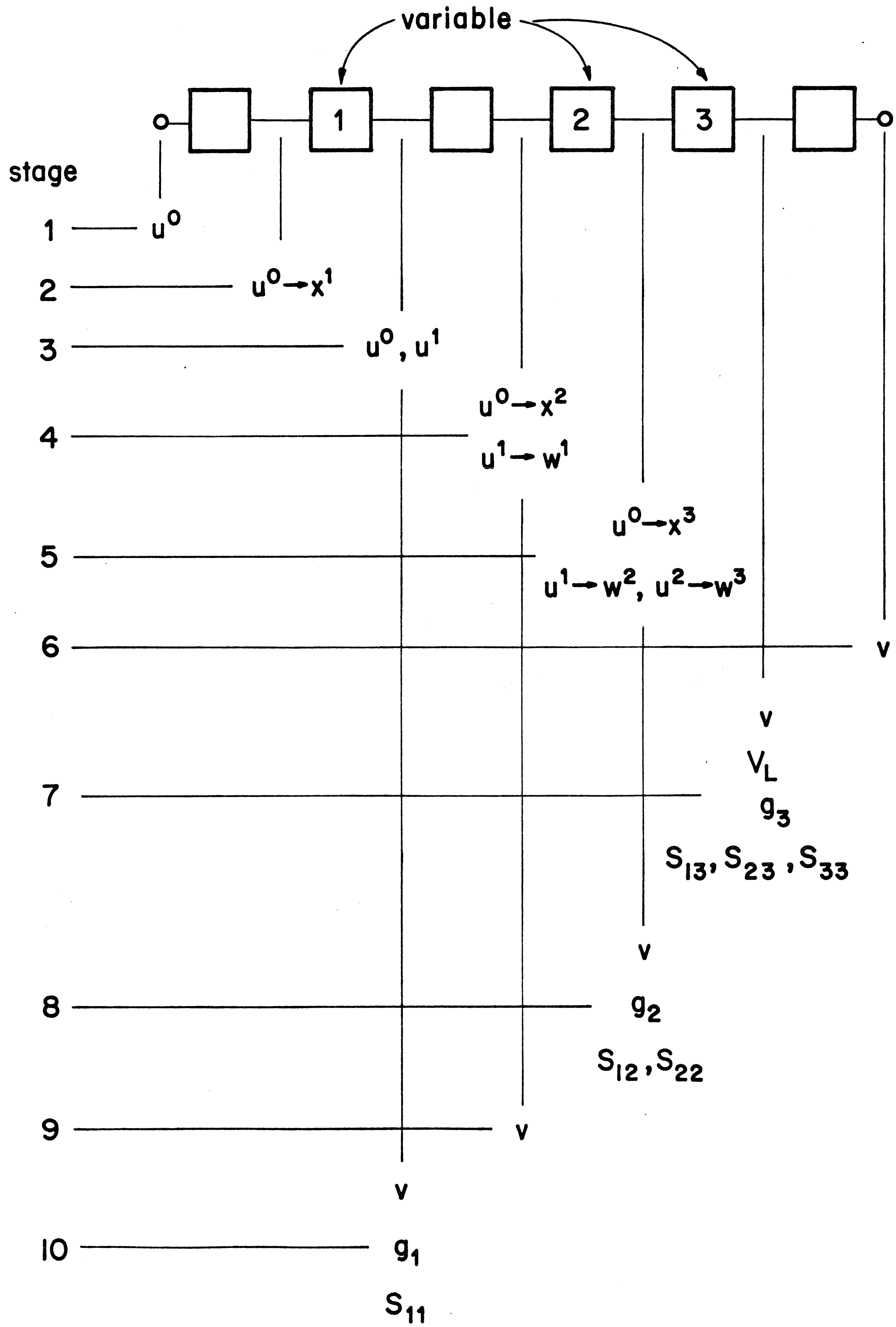
variable
1
2
3
stage
1
2
3
4
5
6
7
8
9
10
u⁰
u⁰→x¹
u⁰, u¹
u⁰→x²
u¹→w¹
u⁰→x³
u¹→w², u²→w³
v
v
V_L
g₃
S₁₃, S₂₃, S₃₃
v
g₂
S₁₂, S₂₂
v
v
g₁
S₁₁

Fig. 6

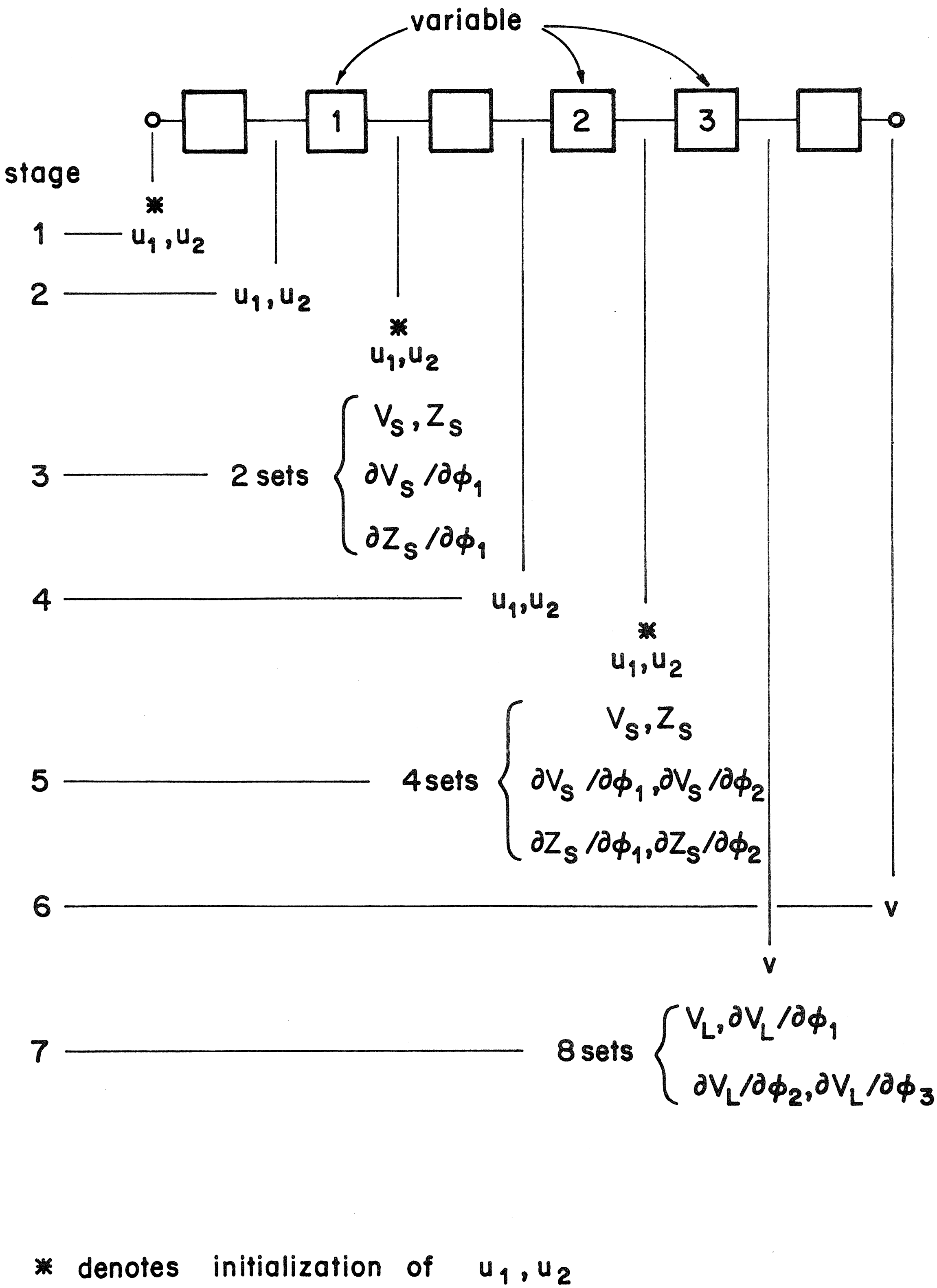
variable
1
2
3
stage
1
2
3
4
5
6
7
*
u1 , u2
u1 , u2
2 sets
Vs , Zs
∂Vs /∂φ1
∂Zs /∂φ1
4 sets
∂Vs /∂φ1 ,∂Vs /∂φ2
∂Zs /∂φ1 ,∂Zs /∂φ2
v
8 sets
VL ,∂VL /∂φ1
∂VL /∂φ2 ,∂VL /∂φ3
* denotes initialization of u1 , u2

Fig. 7

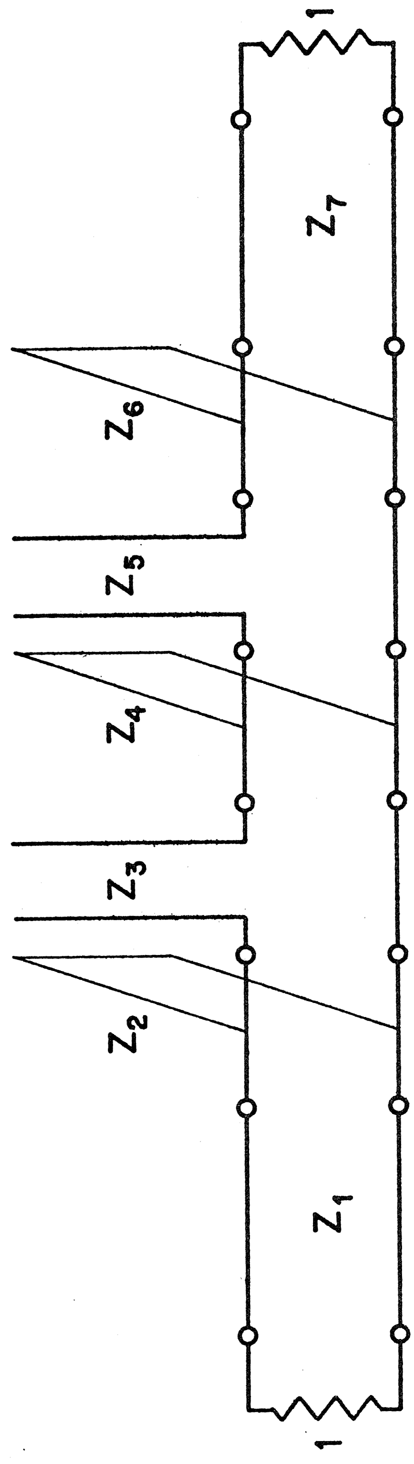

Fig. 8

SOC-217

ALGORITHMS FOR TOLERANCE AND SECOND-ORDER SENSITIVITIES OF CASCADED STRUCTURES

J.W. Bandler and M.R.M. Rizk

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Revised:

Key Words: Sensitivity analysis, tolerance analysis, cascaded networks

Abstract: An exact and efficient approach to network analysis for cascaded structures has been suggested by Bandler et al. They demonstrated that it is useful for sensitivity and tolerance analyses, in particular, for a multiple of simultaneous large changes in design parameter values. This paper extends their work to second-order sensitivities, as well as to the evaluation of the response and its first-order sensitivity at the vertices of a tolerance region located in the space of toleranced design parameters. This information is needed in a worst-case search algorithm for design centering and tolerance assignment. A substantial saving in computational effort is achieved by using the new approach over the basic approach of reanalyzing the circuit at every vertex.

Description:

Related Work: SOC-190.

Price: $ 6.00.