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CENTERING, TOLERANCING, TUNING AND MINIMAX DESIGN EMPLOYING BIQUADRATIC MODELS

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Abstract This paper exploits the biquadratic behaviour w.r.t. a variable exhibited in the frequency domain by certain lumped, linear circuits. Boundary points of the constraint region of acceptable designs are explicitly calculated w.r.t. any such variable at any sample point in the frequency domain. An algorithm to exactly determine the constraint region itself for the general nonconvex case is presented and illustrated. This type of analysis leads to the determination of circuit tunability and to decisions on design center and tolerance assignment. A globally convergent and extremely efficient minimax algorithm is developed and tested to optimize the frequency response w.r.t. any circuit parameter.

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I. INTRODUCTION

A number of researchers have considered properties of response or constraint functions w.r.t. one designable variable at a time in the contexts of sensitivity evaluation of linear circuits [1-6] and the prediction of worst cases in design centering and tolerance assignment [7-11]. The bilinear behaviour of certain linear circuits has been used to derive relationships between, e.g., first-order and large change sensitivities. In the tolerance problem, attempts have been made to find conditions which satisfy the common assumption that worst cases occur at extremes of parameter uncertainty intervals.

In this paper we exploit the resulting biquadratic function obtained from the modulus squared of the bilinear function to produce some new results. In particular, at any point in the frequency domain we can explicitly calculate boundary points of the constraint region of acceptable designs. These boundary points are further utilized to exactly determine the constraint region itself for the general nonconvex case. Our analysis leads to the explicit determination of circuit tunability. Furthermore, design centering and tolerance assignment w.r.t. each parameter at a time is facilitated.

We present some theoretical ideas for predicting worst cases. A globally convergent and extremely efficient minimax algorithm is derived and stated. Examples employing a realistic tunable active filter demonstrate the optimization of the frequency response w.r.t. a circuit parameter.

II. THEORY

For certain lumped, linear circuits, we can express the circuit response as a bilinear functon in a variable parameter ϕ (see, for example, Fidler [1])

$$f(\phi) = \frac{u + a\,\phi}{1 + b\,\phi}\,, \tag{1}$$

where f is the circuit response at a particular frequency s, while u, a and b are complex constants in general. The variable ϕ does not necessarily have the value of the parameter, but it may take the value of the parameter p referred to a reference value p^0. Hence, we take

$$\phi = p - p^0\,. \tag{2}$$

It is to be noted that while u and a may assume zero value, b is never zero for all practical problems.

Three analyses are consequently required to obtain the complex constants in (1). In order to save computational effort, we may proceed as follows.

1. Find f(0) by performing the corresponding circuit analysis, i.e., for $\phi = 0$.
2. Find $d = \partial f(0)/\partial\phi$ by performing an adjoint analysis [12]. In this process an LU factorization is saved by not using a new value of ϕ.
3. Find $f_1 = f(\phi_1)$, where ϕ_1 is a chosen value of ϕ different than zero, by performing the corresponding circuit analysis.

Thus, we can simply derive the following expressions:

$$u = f(0)\,, \tag{3}$$

$$a = d\,f_1/(f_1-u) - u/\phi_1\,, \tag{4}$$

$$b = d/(f_1-u) - 1/\phi_1\,. \tag{5}$$

Alternatively, we might use the analysis at three different values of ϕ in order to find u, a and b. Considering $\phi = 0$, $\phi = \phi_1$ and $\phi = \phi_2$, for example, we get

$$u = f_0 \ , \tag{6}$$

$$a = \frac{f_1 f_2(\phi_1-\phi_2)-f_0(\phi_1 f_1-\phi_2 f_2)}{\phi_1\phi_2(f_1-f_2)} \ , \tag{7}$$

$$b = \frac{(\phi_1 f_2-\phi_2 f_1)-f_0(\phi_1-\phi_2)}{\phi_1\phi_2(f_1-f_2)} \ , \tag{8}$$

where

$$f_0 = f(0),\ f_1 = f(\phi_1) \text{ and } f_2 = f(\phi_2) \ . \tag{9}$$

Since the magnitude of the response $|f|$ or functions of this magnitude are often of interest[†], we may write

$$|f(\phi)|^2 = \frac{|u|^2+2\ R(u^* a)\phi+|a|^2\phi^2}{1+2\ R(b)\phi+|b|^2\phi^2} \ , \tag{10}$$

where u^* is the complex conjugate of u and $R(\cdot)$ denotes the real part of $(\cdot)$.

In order to simplify the following derivations, we write (10) in the form

$$F = \frac{A+2B\phi+C\phi^2}{1+2D\phi+E\phi^2} \ . \tag{11}$$

Hence,

$$\lim_{\phi\to\pm\infty} F = \frac{C}{E} \ , \quad E \neq 0 \ . \tag{12}$$

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† Insertion loss and reflection coefficient specifications can be expressed as specifications on the magnitude of the response.

To find the values of ϕ at which $F = S$, where S is a certain specification, we replace F by S in (11). Then

$$(SE-C)\phi^2 + 2(SD-B)\phi + S - A = 0 . \qquad (13)$$

When $S \neq C/E$, (13) has two finite roots given by

$$r_{1,2} = -\beta \pm \sqrt{\beta^2-(S-A)/(SE-C)} , \qquad (14)$$

where

$$\beta = (SD-B)/(SE-C) . \qquad (15)$$

The following two cases will arise.

(a) Real Roots ($r_1 \leq r_2$)

The value of F will satisfy the inequality

$$F \gtrless S \quad \text{for all } \phi \in [r_1, r_2] \quad \text{if} \quad S \gtrless C/E . \qquad (16)$$

If the specification S is such that

$$S = C/E , \quad E \neq 0 , \qquad (17)$$

a single root is obtained from (13) and is given by

$$r = -(C-AE)/2(CD-BE) . \qquad (18)$$

By perturbing S and looking for the average of the resulting two roots, given by $-\beta$, we investigate whether $-\beta$ tends to $\pm\infty$. Hence, we derive the inequalities

$$F \gtrless S \quad \text{for all } \phi \in [r, \infty] \quad \text{if} \quad BE \gtrless CD , \qquad (19)$$

$$F \gtrless S \quad \text{for all } \phi \in [-\infty, r] \quad \text{if} \quad BE \lessgtr CD . \qquad (20)$$

(b) Imaginary Roots

The value of F will satisfy the inequality

$$F \lessgtr S \text{ for all } \phi \in (-\infty,\infty) \text{ if } S \gtrless C/E . \qquad (21)$$

Fig. 1 illustrates these inequalities.

The inequalities (16), (19), (20) and (21) lead to the identification of certain intervals on the real axis such that if ϕ belongs to any of these intervals the specifications will be satisfied. This is the subject of the next section.

III. VALID PARAMETER INTERVALS

Consider the set of specifications

$$e_i \triangleq w_i(F_i - S_i) \leq 0 \ , \ i = 1, 2, \ldots, m \ , \tag{22}$$

where

$$w_i = \begin{cases} -1 \text{ for lower specification } S_i, \\ \ \ 1 \text{ for upper specification } S_i, \end{cases} \tag{23}$$

and m is, for example, the number of frequency points taken into consideration.

According to the inequalities derived in Section II, it is possible to define a unique <u>continuous</u> interval I_i so that if the specification is satisfied on I_i then it is violated for all $\phi \notin I_i$ and vice versa. The logical variable t_i is defined by

$$t_i = \text{True} \quad \text{if} \quad I_i \equiv \{\phi | e_i \leq 0\} \ , \tag{24}$$

or

$$t_i = \text{False} \quad \text{if} \quad I_i \equiv \{\phi | e_i > 0\} \ . \tag{25}$$

A check can be made to investigate the possibility of meeting the m specifications of (22) simultaneously by adjusting the parameter ϕ only. This investigation can be carried out by finding the feasible region of ϕ given by

$$R_S = \bigcap_{t_i = \text{True}} I_i - \bigcup_{t_i = \text{False}} I_i \ . \tag{26}$$

If R_S, the feasible region of ϕ with the specification S, is empty, it is impossible to meet this specification by adjusting the single variable ϕ only.

It is to be noted that R_S is not necessarily a continuous interval. In general,

$$R_S = \bigcup_{\ell=1}^{k} [\check{\phi}_\ell, \hat{\phi}_\ell] , \qquad (27)$$

where k is the number of closed intervals $[\check{\phi}_\ell, \hat{\phi}_\ell]$. A flow diagram is shown in Fig. 2 which provides k and the intervals $[\check{\phi}_\ell, \hat{\phi}_\ell]$, $\ell = 1, 2, \ldots, k$, as well as the indices of the functions F_i which actually define the extreme points of each interval. These indices are denoted $\check{i}_\ell$ and $\hat{i}_\ell$ for the lower and upper extremes, respectively.

Parameter Centering

Having obtained R_S it is possible now to center the parameter ϕ at

$$\phi^0 = (\hat{\phi}_j + \check{\phi}_j)/2 ,$$

where

$$(\hat{\phi}_j - \check{\phi}_j) \geq (\hat{\phi}_\ell - \check{\phi}_\ell), \ \ell = 1, 2, \ldots, k.$$

The corresponding tolerance will be

$$\varepsilon = (\hat{\phi}_j - \check{\phi}_j)/2 .$$

In words,we choose the largest continuous interval in R_S and center ϕ^0 at its middle.

If several parameters are subject to centering, this process can be successively carried out for each parameter independently (see Butler [13]).

Tuning

Finding the feasible region R_S is of particular importance in the case of single parameter tuning (trimming). It provides an inexpensive check on the tunability of an outcome of the manufacturing process. An outcome will be tunable if

$$[\check{\phi}_t, \hat{\phi}_t] \cap R_S \neq \emptyset , \tag{28}$$

where $[\check{\phi}_t, \hat{\phi}_t]$ is the tuning range of ϕ. This interval can be used to initialize $\check{\phi}_1$ and $\hat{\phi}_1$ in Fig. 2.

IV. EXTREMES OF A BIQUADRATIC FUNCTION

The stationary points of F, see (11), are given by

$$\frac{dF}{d\phi} = 0 , \tag{29}$$

where

$$\frac{dF}{d\phi} = 2 \frac{(B-AD)+(C-AE)\phi+(CD-BE)\phi^2}{(1+2D\phi+E\phi^2)^2} . \tag{30}$$

For finite stationary points, we solve the quadratic equation

$$(CD-BE)\phi^2 + (C-AE)\phi + (B-AD) = 0 . \tag{31}$$

In general, there are two stationary points [8], given by the roots of (31). In the case of

$$CD - BE = 0 , \tag{32}$$

there is only one stationary point given by

$$\phi = - (B-AD)/(C-AE) . \tag{33}$$

To prove that these stationary points are extremes of F (maximum or minimum), we rewrite (11) as

$$(1+ 2D\phi + E\phi^2)F = A + 2B\phi + C\phi^2 , \tag{34}$$

where we have assumed that the denominator has no real roots. Differentiating (34) w.r.t. ϕ we get

$$(1+2D\phi+E\phi^2) \frac{dF}{d\phi} + 2(D+E\phi)F = 2(B+C\phi) , \tag{35}$$

and differentiating (35) w.r.t. ϕ we get

$$(1+2D\phi+E\phi^2)\frac{d^2F}{d\phi^2} + 4(D+E\phi)\frac{dF}{d\phi} + 2EF = 2C \ . \tag{36}$$

Thus, for a stationary point

$$\frac{d^2F}{d\phi^2} = 2\,\frac{C-EF}{1+2D\phi+E\phi^2} \ . \tag{37}$$

If a stationary point is an inflection point, i.e.,

$$\frac{d^2F}{d\phi^2} = 0 \ , \tag{38}$$

then (37) leads to

$$F = \frac{C}{E} \ . \tag{39}$$

The finite point at which F = C/E is obtained by replacing F by C/E in (11) to get

$$\phi = -\frac{C-AE}{2(CD-BE)} \ . \tag{40}$$

Using (35), a stationary point satisfies

$$F = \frac{B+C\phi}{D+E\phi} \ . \tag{41}$$

Hence, for a finite stationary point to be an inflection point (39) and (41) have to be satisfied simultaneously for a finite value of ϕ. This is true if

$$BE = CD \ . \tag{42}$$

Collecting the information we have so far, a finite stationary point ϕ which happens to be an inflection point is given by (40), however, (42) indicates that ϕ is infinite unless

$$C - AE = 0 \ . \tag{43}$$

Substituting for C from (43) into (42) and assuming that $E \neq 0$, we get

$$B = AD \; . \tag{44}$$

But, (42), (43) and (44) make $dF/d\phi$, see (30), equal to zero everywhere. This is the special case of a constant function F=A which is of no interest.

To summarize, the stationary points of a biquadratic function which has no real poles are extreme points. This result gives information to the designer about the behaviour of the function on a certain interval. It indicates that the function is monotonic at any interval as long as no stationary points fall within this interval. If a minimax solution is to be defined by one function (Section VI), finding the extremes of this function is essential.

V. IMPLICATIONS OF A POLE

So far we have avoided the situation in which $f(\phi)$ has a real pole. A pole of $F = |f|^2$ of order two w.r.t. ϕ at

$$\phi = -1/b \tag{45}$$

is possible only if b is real, otherwise the zeros of the denominator of (10) are complex. Similarly, the numerator of (10) indicates that a real zero of order two w.r.t. ϕ at

$$\phi = -\,(u^*a)/|u|^2|a|^2 \tag{46}$$

exists if (u^*a) is real.

Two design requirements are now of interest. If a pole is to be placed at this particular frequency (at which $f(\phi)$ is obtained), we have no choice but to take ϕ as given by (45). A finite specification S at this particular frequency will not cause the denominator to be zero at the values of ϕ considered and hence all derivations in Sections II, III and IV are valid. Regarding the stationary points we add that

$$\frac{dF}{d\phi} = 2 \frac{(b\phi+1)^2(R(u^*a)+|a|^2\phi)-b(b\phi+1)(|u|^2+2 R(u^*a)\phi+|a|^2\phi^2)}{(b\phi+1)^4} . \qquad (47)$$

Thus, one of the zeros of the numerator will be $\phi = -1/b$, which is a point of infinite gradient and the stationary point is

$$\phi = \frac{b|u|^2- R(u^*a)}{|a|^2-b R(u^*a)} = \frac{AD-B}{C-DB} . \qquad (48)$$

If $C-DB \neq 0$, this point will be a minimum of F since $d^2F/d\phi^2$ is positive at this point and given by

$$\frac{d^2F}{d\phi^2} = \frac{2}{(1+b\phi)^4} |ub-a|^2 . \qquad (49)$$

VI. THE ONE-DIMENSIONAL MINIMAX ALGORITHM

A one-dimensional minimax algorithm based upon the foregoing theory is now developed. The algorithm is guaranteed to converge to the global minimax optimum. See the Appendix for theorems on the convergence of the algorithm. Fig. 3 illustrates the algorithm. The following steps set it out in detail.

<u>Step 1</u> Find u_i, a_i and b_i, $i = 1, 2, \ldots, m$.

<u>Comment</u> These complex constants are obtained through (3), (4) and (5) or by (6), (7) and (8). Hence, a biquadratic function of the form (10) (equivalent to (11)) for each $i = 1, 2, \ldots, m$, is available.

<u>Step 2</u> Initialize ϕ.

<u>Step 3</u> Find

$$\delta = \max_i e_i(\phi) \tag{50}$$

<u>Step 4</u> Find $[\check{\phi}_\ell, \hat{\phi}_\ell]$ and $\check{i}_\ell$, $\hat{i}_\ell$, $\ell = 1, 2, \ldots, k$, using the specifications

$$e_i \leq \delta \;, \; i = 1, 2, \ldots, m. \tag{51}$$

<u>Comment</u> This is carried out using the flow diagram of Fig. 2. If all functions are convex, k will always be one.

<u>Step 5</u> Find $\check{g}_\ell$ and $\hat{g}_\ell$, $\ell = 1, 2, \ldots, k$, given by

$$\check{g}_\ell = w_{\check{i}_\ell} \left. \frac{dF_{\check{i}_\ell}}{d\phi} \right|_{\check{\phi}_\ell} , \tag{52}$$

$$\hat{g}_\ell = w_{\hat{i}_\ell} \left. \frac{dF_{\hat{i}_\ell}}{d\phi} \right|_{\hat{\phi}_\ell} . \tag{53}$$

<u>Comment</u> These are simply the sensitivities at the extreme points of each valid interval.

<u>Step 6</u> If $k = 1$, set $j \leftarrow 1$ and go to Step 8.

<u>Step 7</u> Find j such that

$$\Delta_j \geq \Delta_\ell \;, \quad \ell = 1, 2, \ldots, k, \tag{54}$$

where

$$\Delta_\ell = \hat{g}_\ell \check{g}_\ell (\hat{\phi}_\ell - \check{\phi}_\ell)/(\check{g}_\ell - \hat{g}_\ell) \; . \tag{55}$$

<u>Comment</u> In this step we select the jth interval which appears to be the most promising interval in terms of the expected improvement in the minimax optimum based on linearization. Δ_ℓ will always be positive.

<u>Step 8</u> Set

$$\phi \leftarrow (\check{g}_j\check{\phi}_j-\hat{g}_j\hat{\phi}_j)/(\check{g}_j-\hat{g}_j) \text{ if } \check{i}_j \neq \hat{i}_j \ . \tag{56}$$

<u>Comment</u> If the extremes of the jth interval are defined by two different functions, the new value ϕ is taken as the intersection of the linear approximation to the two functions.

<u>Step 9</u> Set ϕ to the minimizing point of the function $w_{\hat{i}_j} F_{\hat{i}_j}$ if $\check{i}_j$ = $\hat{i}_j$.

<u>Comment</u> The minimum of a function F_i is one of the roots of (31) for which (37) is positive after multiplying by $w_{\hat{i}_j}$.

<u>Step 10</u> Set

$$\phi \leftarrow (\check{\phi}_j+\hat{\phi}_j)/2 \text{ if } \phi \notin (\check{\phi}_j,\hat{\phi}_j) \ . \tag{57}$$

<u>Comment</u> This is a default value to obviate any numerical problem which may arise in Step 7 or Step 8, for example, $\hat{g}_j = 0$.

<u>Step 11</u> Stop if k = 1 and if

$$(\hat{\phi}_1-\check{\phi}_1) \leq \varepsilon \ |\phi| \ , \tag{58}$$

where ε is a prescribed small positive number.

Step 12 Go to Step 3.

Comment It is anticipated (see Appendix) that, eventually, k will be one, i.e., a single global minimum is reached.

VII. EXAMPLE

A tunable active filter [14] has been chosen to implement the theory and algorithms. The filter is shown in Fig. 4 and its equivalent circuit in Fig. 5. The specifications w.r.t. frequency on the modulus squared of the transfer function $F = |V_2/V_g|^2$ are

$$F \leq 0.5 \text{ for } f/f_0 \leq 1-10/f_0,$$

$$F \leq 1.21 \text{ for } 1-10/f_0 \leq f/f_0 \leq 1+10/f_0,$$

$$F \leq 0.5 \text{ for } f/f_0 \geq 1+10/f_0,$$

$$F \geq 0.5 \text{ for } 1-8/f_0 \leq f/f_0 \leq 1+8/f_0,$$

$$F \geq 1 \text{ for } f = f_0 \text{ Hz},$$

where f_0 is the center frequency. Using the one pole roll-off model for the operational amplifiers, given by

$$A(s) = \frac{A_0 \omega_a}{s+\omega_a},$$

where s is the complex frequency, A_0 is the d.c. gain and ω_a the 3 dB radian bandwidth, the nodal equations are

$$\begin{bmatrix} G_1+G_g & 0 & -G_1 & 0 \\ 0 & G_2+G_3+sC_2+A_2G_3 & -sC_2 & -G_2+A_1A_2G_3 \\ -G_1 & -sC_2 & G_1+G_4+sC_1+sC_2 & -sC_1 \\ 0 & -G_2 & -sC_1 & G_2+sC_1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} G_gV_g \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

According to equations (3), (4), (5) and (10) a biquadratic model in R_4 was obtained at each sample frequency. The normalized sample frequencies are taken as 1 and 1 $\pm$ $10/f_0$ for the relevant upper specifications, 1 and 1 $\pm$ $8/f_0$ for the relevant lower specifications. This leads to six error functions e_i, i = 1, 2, ..., 6. The range of R_4 for which the specifications are satisfied is that for which $e_i \leq 0$, i = 1, 2, ..., 6. The maximum of the error functions e_i versus R_4 is shown in Fig. 6. A single run of a program implementing the flow diagram of Fig. 2 indicated that the filter is tunable for the specifications defined at a center frequency of 100 Hz. It meets these specifications if

$$R_4 \in [181.126, \quad 187.166]$$

and with other circuit parameters fixed at values given in Table 1. It is also tunable around a center frequency of 700 Hz (see Fig. 7) and meets the specifications if

$$R_4 \in [3.4881, \quad 3.5012] .$$

To find $\min_{R_4} \max_i e_i$, we are faced with the local minima in Fig. 6. The convergence of other algorithms to the global minimum depends upon the starting point. For the proposed algorithm the results are shown in Table II for different starting points and at different center frequencies. Note how small is the number of iterations required.

When R_1 was altered to the value 14 kΩ the filter is not tunable as is determined by one run of the program. The optimum value of R_4, however, was obtained in only two iterations (see Table II). In fact, the algorithm converged in the first iteration since the optimum is defined by one function, however, the second iteration was performed to satisfy the stopping criterion.

VIII. CONCLUSIONS

Implications of the bilinear behaviour of certain linear circuits in the frequency domain have been investigated. The explicit determination of the points defining the boundary of the feasible region w.r.t. one parameter led to results on centering and tolerance assignment as well as a considerably simple check on the tunability of an outcome of the manufacturing process by adjusting a single parameter at a time. Detection of worst cases within an interval for any circuit parameter, of course, is also facilitated.

The proposed minimax algorithm is not only extremely efficient but is also globally convergent. It has been shown how few iterations are required for convergence to the global minimax optimum from different starting points even when local minima exist. In fact, difficulties arising out of multiple local minima have been observed by the present authors in implementing a one-dimensional version of the minimax algorithm of Madsen et al. [15].

APPENDIX

The global convergence of the one-dimensional minimax algorithm is proved through the following two theorems.

<u>Theorem 1</u> If ϕ^n is an optimal value of ϕ after n iterations then

$$\delta^{n+1} < \delta^n .$$

<u>Proof</u> According to Step 3 of the algorithm

$$\delta^n \triangleq \max_i e_i(\phi^n) .$$

Knowing that inequalities (16), (19), (20) and (21) may assume equality only at an extreme point of the closed intervals $[\check{\phi}_\ell, \hat{\phi}_\ell]$, $\ell = 1, 2, \ldots, k$, and since ϕ^{n+1} is secured to be an interior point as shown by Step 10, we have

$$e_i(\phi^{n+1}) < \delta^n , \quad i = 1, 2, \ldots, m .$$

Thus,

$$\delta^{n+1} < \delta^n .$$

Q.E.D.

<u>Theorem 2</u> If ϕ^* is a unique global minimax optimum, then for sufficiently small ε, used in the stopping criterion of Step 11, we have

$$|\phi^a - \phi^*| \leq \varepsilon |\phi^a| , \qquad \text{(A1)}$$

where ϕ^a is the value of ϕ to which the algorithm has converged.

<u>Proof</u> The proof is divided into two parts. First consider the case where only one interval (k=1) is found by the algorithm. Thus, inequalities (16), (19), (20) and (21) guarantee that ϕ^* will always belong to this interval which is defined to be the feasible region of (51), where δ is given by (50), i.e.,

$$e_i(\phi^*) \leq e_i(\phi^n) \leq \delta^n ,$$

where ϕ^n is the value of ϕ after n iterations.

According to (58), we have

$$(\hat{\phi}_1 - \check{\phi}_1) \leq \varepsilon |\phi^a| ,$$

But,

$$\phi^a, \phi^* \in [\check{\phi}_1, \hat{\phi}_1] .$$

Thus,

$$|\phi^a-\phi^*| \le |\hat{\phi}_1-\check{\phi}_1| \le \varepsilon|\phi^a| \ .$$

Secondly, assume $k > 1$. We show that for sufficiently small ε the number of intervals k will be reduced. There is no loss of generality if we assume k=2 and that $\phi^* \in [\check{\phi}_2, \hat{\phi}_2]$. Let $\phi \in [\check{\phi}_1, \hat{\phi}_1]$ and $\Delta_1 > \Delta_2$ so that the algorithm will select interval 1 according to (54). Thus,

$$\hat{g}_1\check{g}_1(\hat{\phi}_1-\check{\phi}_1)/(\check{g}_1-\hat{g}_1) > \hat{g}_2\check{g}_2(\hat{\phi}_2-\check{\phi}_2)/(\check{g}_2-\hat{g}_2)$$

or

$$(\hat{\phi}_2-\check{\phi}_2) < \frac{\hat{g}_1\check{g}_1(\check{g}_2-\hat{g}_2)}{\hat{g}_2\check{g}_2(\check{g}_1-\hat{g}_1)} (\hat{\phi}_1-\check{\phi}_1) \ . \tag{A2}$$

But, Theorem 1 implies a strictly montonic decrease in δ and hence a strictly montonic decrease in $(\hat{\phi}_1-\check{\phi}_1)$. Thus using (58) and choosing ε to be sufficiently small we contradict (A2) since it is impossible that $(\hat{\phi}_2-\check{\phi}_2)$ tends to zero while it contains the global minimax optimum. This proves that at a certain stage, interval 1 will be dropped from the algorithm so that the algorithm will converge as in the first part of the proof. Q.E.D.

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FIGURE CAPTIONS

Fig. 1 Illustration of a biquadratic function with regions in which $F \geq S$ or $F \leq S$.

Fig. 2 Flow diagram for checking one-dimensional tunability or for determining the extremes of each continuous interval defining R_S.

Fig. 3 Illustration of the behaviour of the one-dimensional minimax algorithm. Note that the algorithm switches from interval 1 to interval 2, based on predictions of the decrease in the maximum.

Fig. 4 Tunable active filter.

Fig. 5 Equivalent circuit for nodal analysis.

Fig. 6 $\max_{1 \leq i \leq 6} e_i$ versus the tuning resistor R_4 for specifications defined around $f_0 = 100$ Hz indicating the active functions (and hence active frequency points).

Fig. 7 $\max_{1 \leq i \leq 6} e_i$ versus R_4 for specifications defined around $f_0 = 700$ Hz for two cases (a) $R_1 = 12.446$ kΩ, (b) $R_1 = 14$ kΩ.

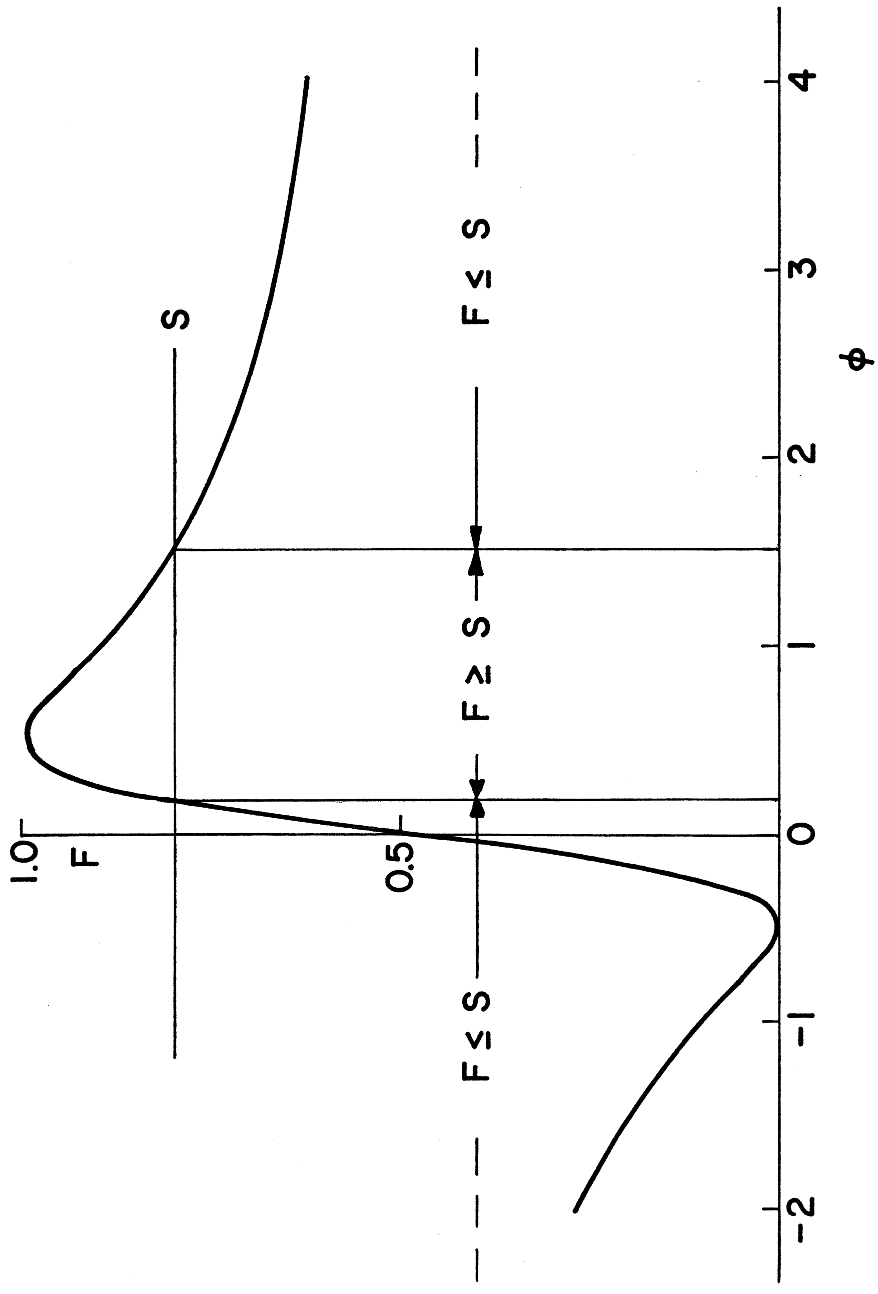

Fig. 1

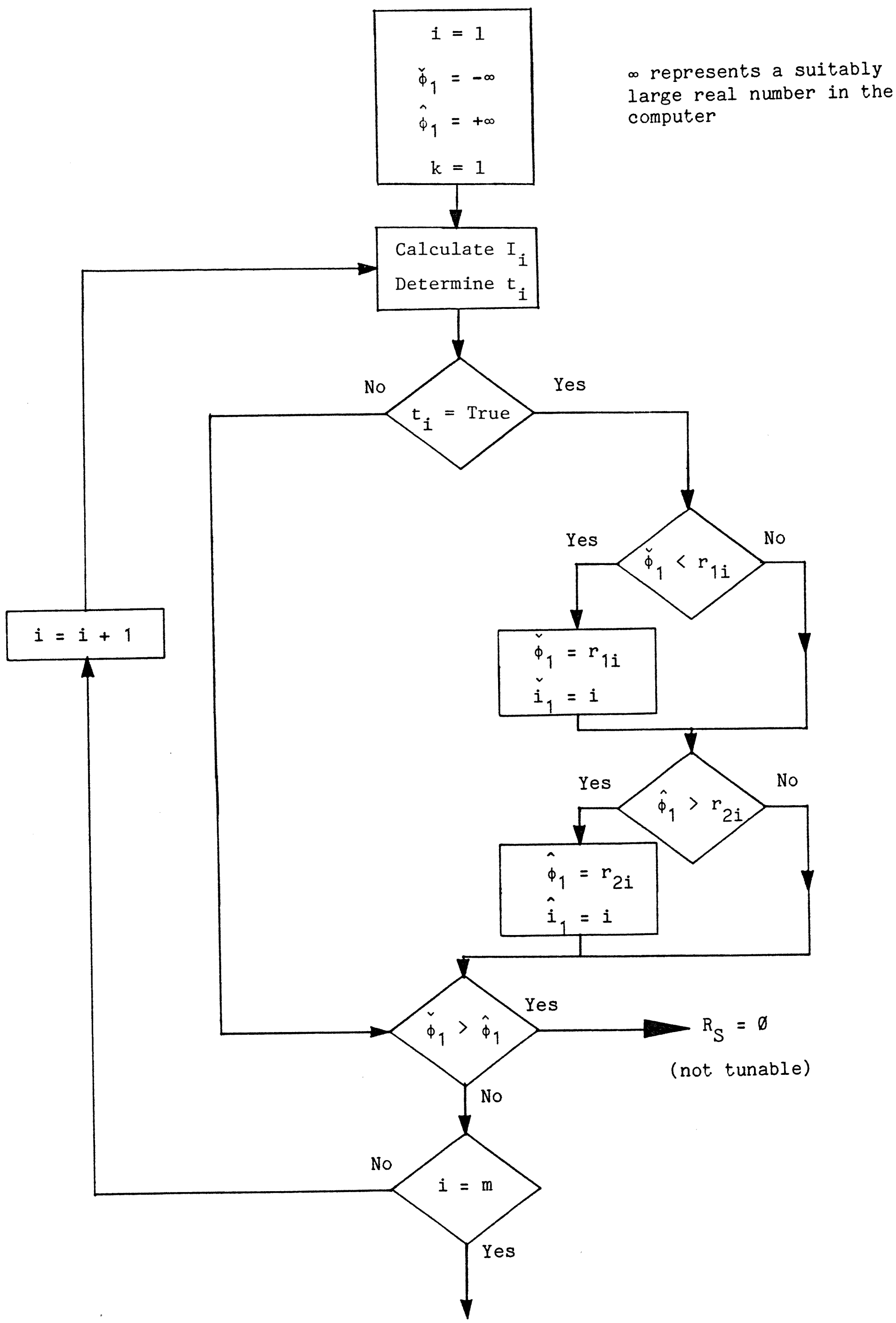
i = 1
φ̌1 = -∞
φ̂1 = +∞
k = 1
∞ represents a suitably large real number in the computer
Calculate Ii
Determine ti
No
Yes
ti = True
Yes
No
φ̌1 < r1i
φ̌1 = r1i
ǐ1 = i
i = i + 1
Yes
No
φ̂1 > r2i
φ̂1 = r2i
î1 = i
Yes
φ̌1 > φ̂1
RS = Ø
(not tunable)
No
No
i = m
Yes

Fig. 2

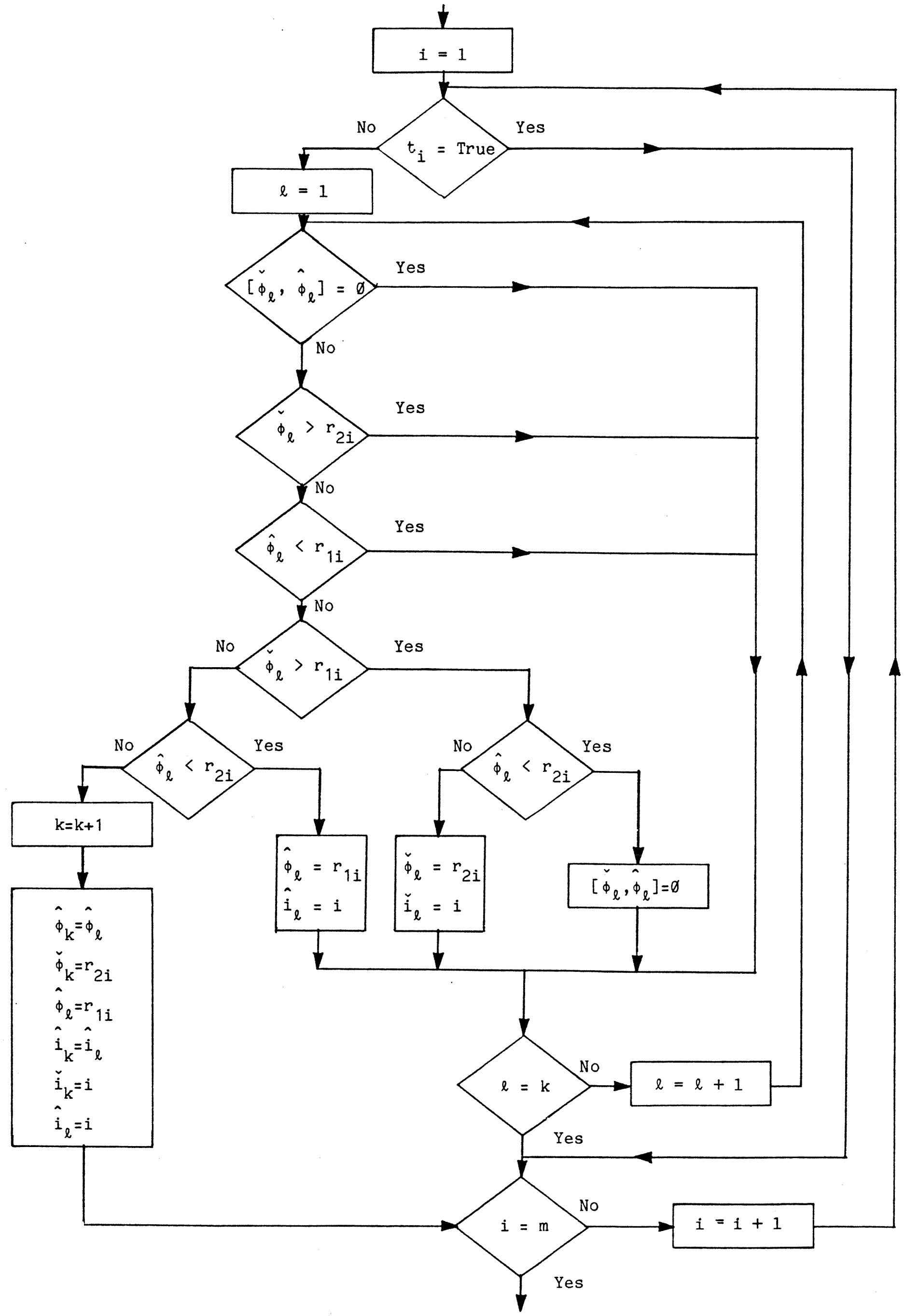

Fig. 2 [continued]

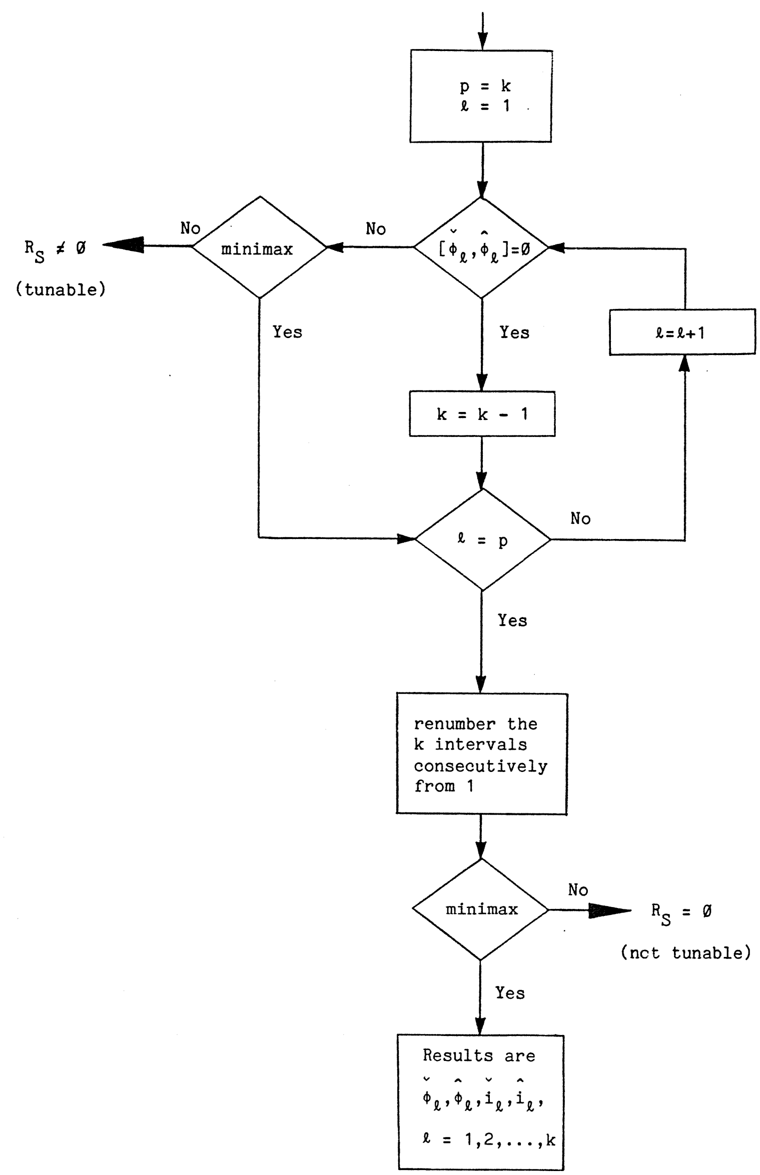

Fig. 2 [continued]

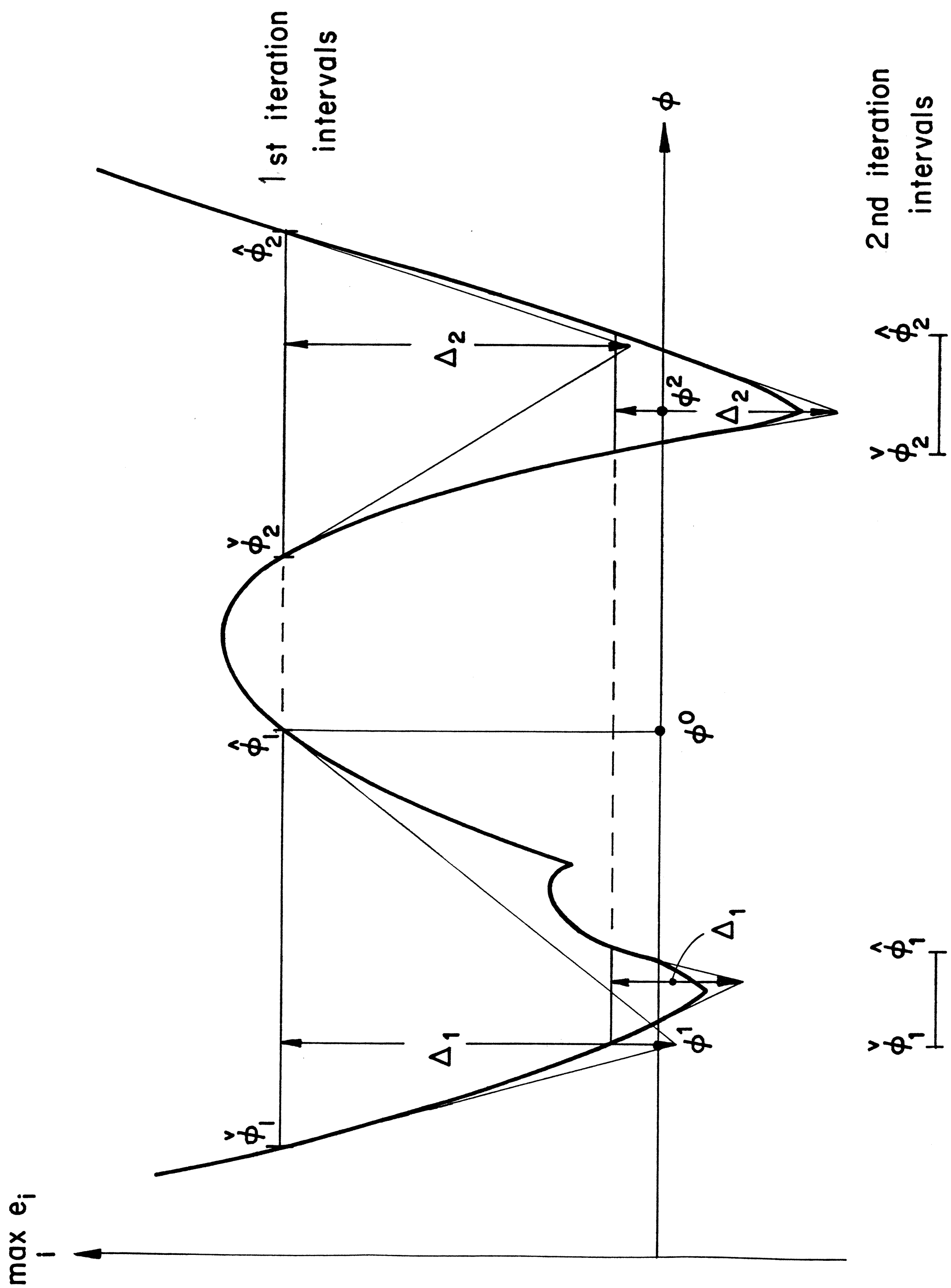
max e_i
1st iteration intervals
2nd iteration intervals
φ
Δ1
Δ2
φ⁰
φ¹
φ²

Fig. 3

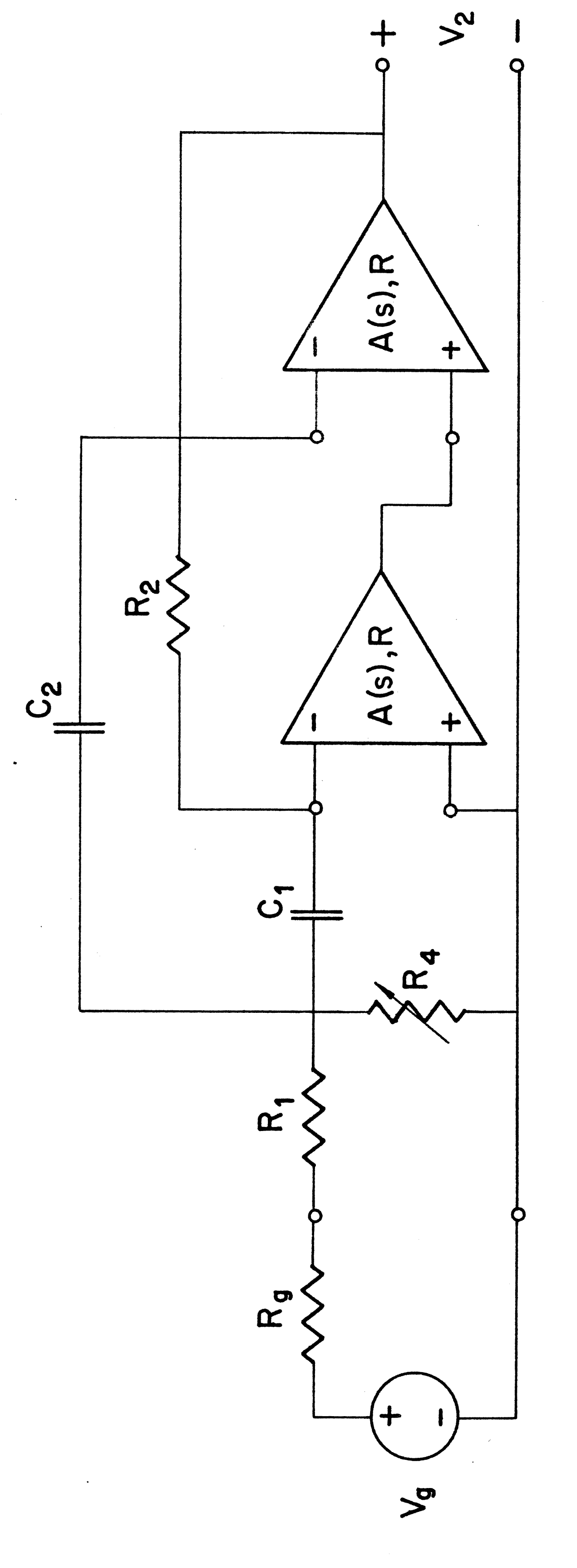

Fig. 4

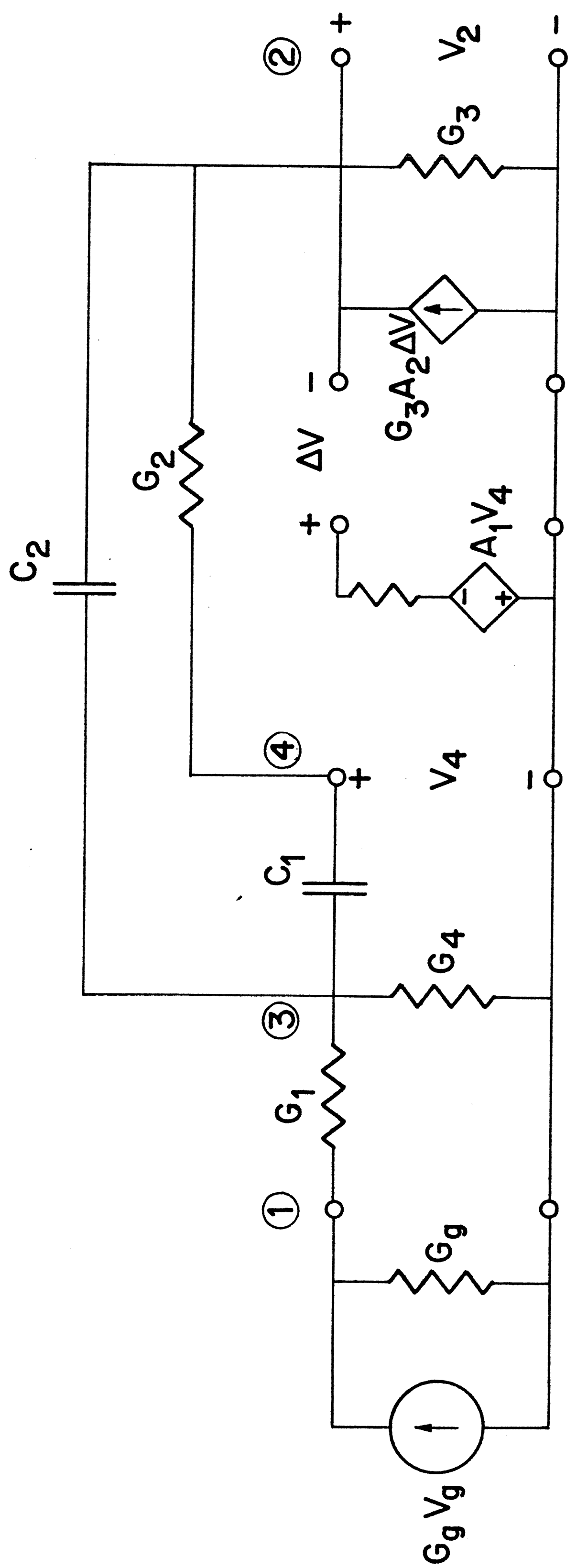

Fig. 5

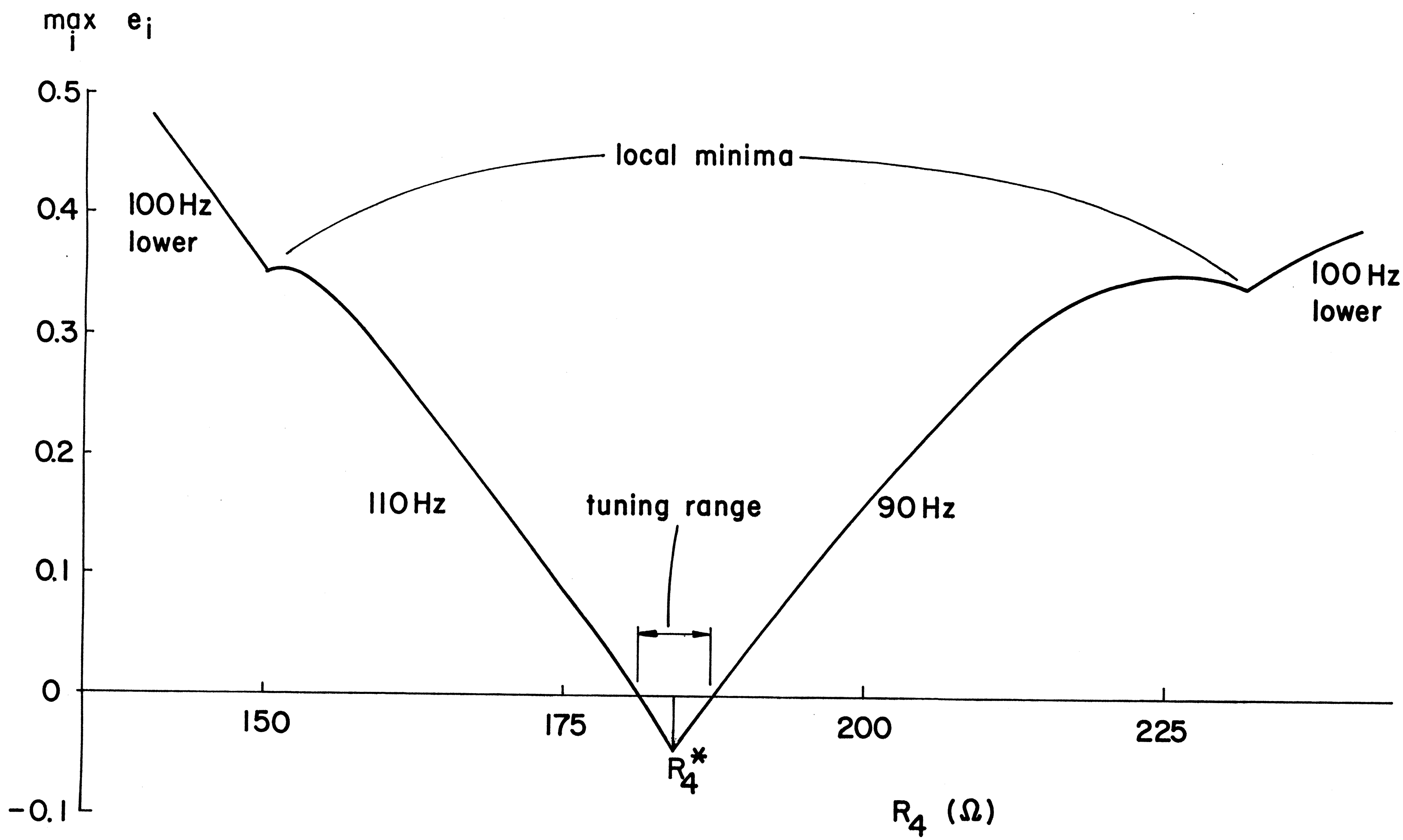
max e_i
i
0.5
0.4
0.3
0.2
0.1
0
-0.1
100Hz lower
local minima
110Hz
tuning range
90Hz
100Hz lower
150
175
200
225
R*4
R4 (Ω)

Fig. 6

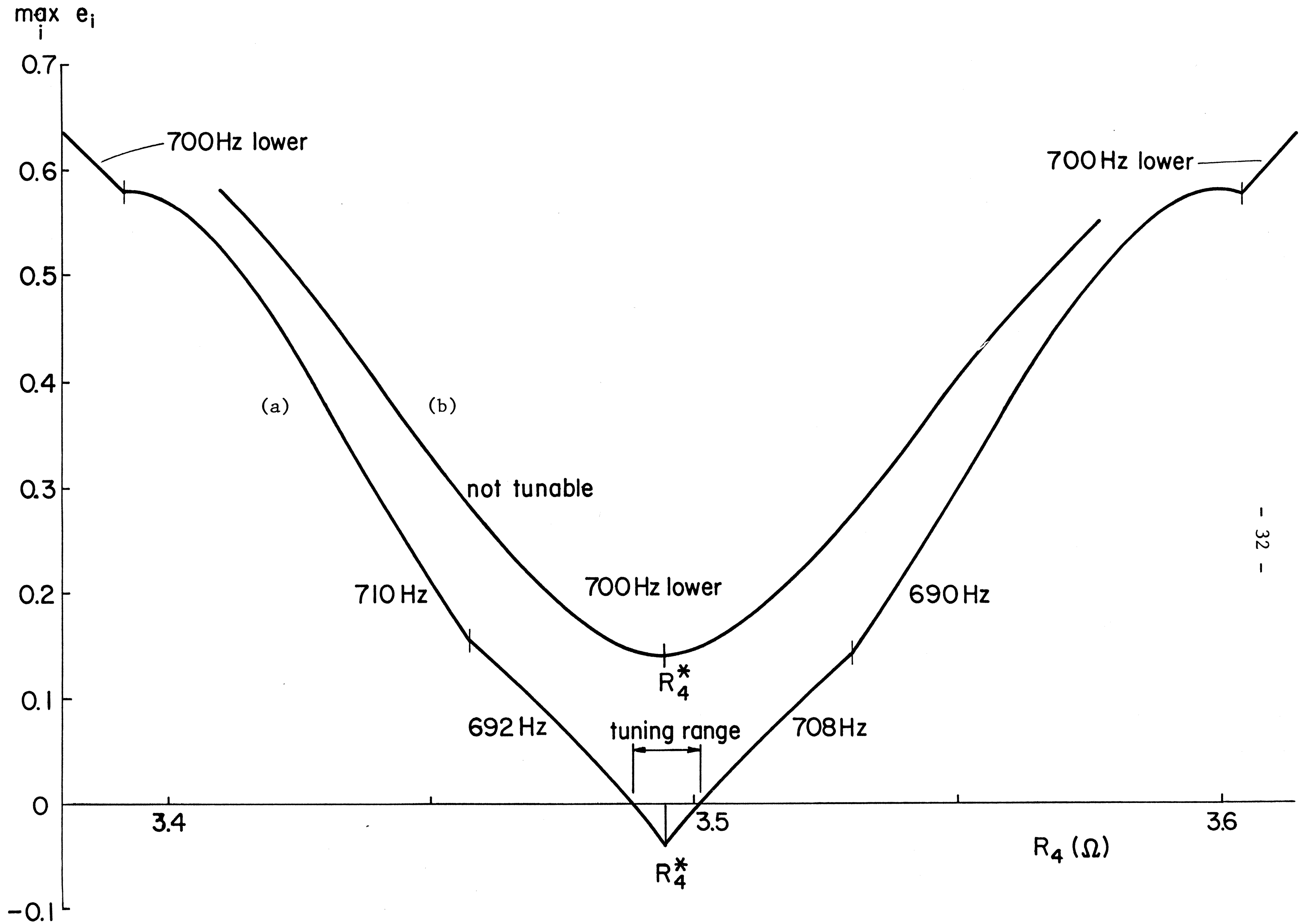
max e_i
i
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
700Hz lower
700Hz lower
(a)
(b)
not tunable
710Hz
700Hz lower
690Hz
R*4
692Hz
tuning range
708Hz
3.4
3.5
3.6
R*4
R4 (Ω)

Fig. 7