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YIELD OPTIMIZATION FOR ARBITRARY STATISTICAL DISTRIBUTIONS

PART I: THEORY

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Abstract This paper generalizes certain analytical formulas for yield and yield sensitivities so that design centering and yield optimization can be effectively carried out employing given statistical parameter distributions. The tolerance region of possible outcomes is discretized into a set of orthotopic cells. A suitable weight is assigned to each cell in conjunction with an assumed uniform distribution on the cell. Explicit formulas for yield and its sensitivities w.r.t. nominal parameter values and component tolerances are presented for linear cuts and sensitivities of these cuts based upon approximations of the boundary of the constraint region. To avoid unnecessary evaluations of circuit responses, e.g., integrations for nonlinear circuits, multidimensional quadratic interpolation is performed. Sparsity is exploited in the determination of these quadratic models leading to reduced computation as well as increased accuracy.

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I. INTRODUCTION

The aim of this paper (see also Part II [1]) is to present some theoretical concepts leading to the most general approach currently available for automatic optimization of production yield which avoids the use of the Monte Carlo method. Thus, the design centering and/or optimal tolerance assignment which is to be performed takes explicitly into account statistical distributions and possible parameter correlations.

The approach is based on the work of Bandler, Liu and Tromp [2] and represents a generalization of the work of Bandler and Abdel-Malek [3,4]. The presentation is directed to a nonlinear programming method of solution, and can be associated with original ideas suggested by a number of other researchers [5-9].

Following a brief review of the centering and tolerancing problem, the multidimensional modeling approach adopted by Bandler and Abdel-Malek [3] is outlined in Section II. Section III organizes the determination of suitable quadratic approximations to the constraints of the problem. Sparsity associated with the selection of the required base points is exploited to reduce computation and increase accuracy. This is particularly opportune for a large number of variables. Furthermore, an approach is suggested aimed at reusing available function values when the interpolation region is relocated according, e.g., to required updating of the approximations as forced by the optimization process and accuracy.

Section IV derives exact formulas for production yield and its sensitivities for arbitrary discretized distributions implied by linear cuts of the tolerance region. It is shown how these cuts may be obtained from the quadratic constraint approximations. It is further shown how they are involved in dynamic updating for yield recalculation as required by optimization.

Simple illustrative examples validate the formulas presented. Part II of this paper applies this material to the optimization of yield for a current switch emitter follower [1].

II. FUNDAMENTAL CONCEPTS AND DEFINITIONS

A design can be described by a nominal parameter vector $\underset{\sim}{\phi}^0$ and a tolerance vector $\underset{\sim}{\varepsilon}$, where

$$\underset{\sim}{\phi}^0 \triangleq \begin{bmatrix} \phi_1^0 \\ \phi_2^0 \\ \cdot \\ \cdot \\ \cdot \\ \phi_k^0 \end{bmatrix}, \qquad \underset{\sim}{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_k \end{bmatrix} \tag{1}$$

and k is the number of designable parameters [2]. The tolerance vector $\underset{\sim}{\varepsilon}$ may be used to define the extremes of the tolerance region or the standard deviation, etc. It is assumed that the parameters can be varied continuously. Some of these vector elements may be set to zero or held constant.

An outcome $\{\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}, \underset{\sim}{\mu}\}$ of a design $\{\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}\}$ implies a point in the parameter space given by

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu}, \tag{2}$$

where

$$\underset{\sim}{E} \triangleq \begin{bmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \ddots & \\ & & & \epsilon_k \end{bmatrix}, \quad \underset{\sim}{\mu} \triangleq \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix} \tag{3}$$

and where $\underset{\sim}{\mu}$ is a <u>random vector</u> distributed according to the <u>joint probability distribution function</u> (PDF). The PDF might extend as far as $(-\infty, \infty)$, however, for all practical cases it is possible to consider a finite <u>tolerance region</u> R_ϵ such that

$$\int_{R_\epsilon} F(\underset{\sim}{\phi})\, d\phi_1\, d\phi_2 \ldots d\phi_k \simeq 1 , \tag{4}$$

where $F(\underset{\sim}{\phi})$ is the PDF.

For the sake of simplicity as well as the implications of independent design parameters, there is no loss of generality to consider R_ϵ to be an <u>orthotope</u> (multidimensional generalization of rectangle) defined by

$$R_\epsilon \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\, \underset{\sim}{\mu} ,\ \underset{\sim}{\mu} \in R_\mu\} , \tag{5}$$

where

$$R_\mu \triangleq \{\underset{\sim}{\mu} \mid -1 \le \mu_i \le 1,\ i = 1, 2, \ldots, k\} . \tag{6}$$

This orthotope is centered at $\underset{\sim}{\phi}^0$ and has edges of length $2\epsilon_i$, $i = 1, 2, \ldots, k$. The extreme points of R_ϵ are called <u>vertices</u> and the set of vertices is defined by [2]

$$R_v \triangleq \{\underset{\sim}{\phi} \mid \phi_i = \phi_i^0 + \varepsilon_i \mu_i,\ \mu_i \in \{-1,1\},\ i = 1, 2, \ldots, k\} . \tag{7}$$

The number of these vertices is 2^k and the following enumeration scheme used by Bandler [10] will be considered. For a vertex

$$\underset{\sim}{\phi}^r = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\ \underset{\sim}{\mu}^r,\ \mu_i^r \in \{-1,1\} \tag{8}$$

we have

$$r = 1 + \sum_{i=1}^{k} \left(\frac{\mu_i^r+1}{2}\right) 2^{i-1}. \tag{9}$$

The <u>constraint region</u> (or <u>feasible region</u>) itself is given by

$$R_c \triangleq \{\underset{\sim}{\phi} \mid g_i(\underset{\sim}{\phi}) \geq 0,\ i = 1, 2, \ldots, m_c\} , \tag{10}$$

where m_c is the number of constraints g_i. The production or manufacturing yield is simply defined by

$$Y \triangleq N/M , \tag{11}$$

where M is the total number of outcomes and N is the number of outcomes $\underset{\sim}{\phi}$ which satisfy the specifications, i.e., for which $\underset{\sim}{\phi} \in R_c$.

<u>Interpolation by Multidimensional Polynomials</u>

An approximate representation of a constraint $g(\underset{\sim}{\phi})$ by using its values at a finite set of points is possible [11,12]. These points are called <u>nodes</u> or <u>base points</u>, and denoted by

$$\underset{\sim}{\phi}^n \ , \ n = 1, 2, \ldots, N_b \ ,$$

where N_b is the number of base points.

Interpolation can be done by means of a linear combination of the set of all possible monomials. Hence,

$$g(\underset{\sim}{\phi}) \simeq \sum_{j=1}^{N} a_j \ \Phi_j(\underset{\sim}{\phi}) \tag{12}$$

where a_j, j = 1, 2, ..., N, are unknown coefficients,

$$\Phi_j \triangleq (\phi_1 - \bar{\phi}_1)^{\alpha_1} (\phi_2 - \bar{\phi}_2)^{\alpha_2} \ldots (\phi_k - \bar{\phi}_k)^{\alpha_k} \ , \ \sum_{i=1}^{k} \alpha_i \leq m \ , \tag{13}$$

or

$$\Phi_j \triangleq (\phi_1)^{\alpha_1} (\phi_2)^{\alpha_2} \ldots (\phi_k)^{\alpha_k} \ , \ \sum_{i=1}^{k} \alpha_i \leq m \ , \tag{14}$$

m is the degree of the interpolating polynomial, k the number of independent variables, i.e., number of components of $\underset{\sim}{\phi}$, α_i, i = 1, 2, ..., k, are nonnegative integers and $\bar{\underset{\sim}{\Phi}}$ may be any reference point. The number of such monomials is given by

$$N = \frac{(m+k)!}{m!k!} \ . \tag{15}$$

If the number of base points N_b is such that

$$N_b = N \ , \tag{16}$$

exact evaluation of the coefficients a_j, j = 1, 2, ..., N, to force the approximation to coincide with the actual function at the base points, i.e.,

$$P(\underset{\sim}{\phi}^n) = g(\underset{\sim}{\phi}^n) \ , \ n = 1, 2, \ldots, N \ , \tag{17}$$

where

$$P(\underset{\sim}{\phi}) = \sum_{j=1}^{n} b_j \, \Phi_j(\underset{\sim}{\phi}) \tag{18}$$

is possible.

The following system of simultaneous linear equations results.

$$\begin{bmatrix} \Phi_1(\underset{\sim}{\phi}^1) & \Phi_2(\underset{\sim}{\phi}^1) & \cdots & \Phi_N(\underset{\sim}{\phi}^1) \\ \Phi_1(\underset{\sim}{\phi}^2) & \Phi_2(\underset{\sim}{\phi}^2) & \cdots & \Phi_N(\underset{\sim}{\phi}^2) \\ \vdots & & & \\ \Phi_1(\underset{\sim}{\phi}^N) & \Phi_2(\underset{\sim}{\phi}^N) & \cdots & \Phi_N(\underset{\sim}{\phi}^N) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} g(\underset{\sim}{\phi}^1) \\ g(\underset{\sim}{\phi}^2) \\ \vdots \\ g(\underset{\sim}{\phi}^N) \end{bmatrix} . \tag{19}$$

The solution of (19) exists if the system of equations is linearly independent. This is satisfied if the set of base points is degree-m independent [13].

III. EXPLOITING SPARSITY IN QUADRATIC INTERPOLATION

Interpolation by Quadratic Polynomials

In order to minimize the computational effort to obtain a quadratic polynomial approximation, the number of base points required will be chosen to be equal to the number of unknown coefficients, i.e., interpolation will be adopted. Replacing m by 2 in (15) the number of base points is

$$N = (k+1)(k+2)/2 \, . \tag{20}$$

Let R_i be the interpolation region defined by

$$R_i \triangleq \{\underset{\sim}{\phi} \mid \delta_i \geq |\phi_i - \bar{\phi}_i| , i = 1, 2, \ldots, k\} , \tag{21}$$

where $\bar{\underset{\sim}{\phi}}$ is the center of the interpolation region and δ_i, i = 1, 2, ..., k, are parameters defining the size of the interpolation region. The quadratic polynomial approximation can be expressed in terms of the monomials (13) or (14) as

$$P(\underset{\sim}{\phi}) = a_0 + \underset{\sim}{a}^T(\underset{\sim}{\phi}-\bar{\underset{\sim}{\phi}}) + \frac{1}{2}(\underset{\sim}{\phi}-\bar{\underset{\sim}{\phi}})^T \underset{\sim}{H}(\underset{\sim}{\phi}-\bar{\underset{\sim}{\phi}}) \tag{22}$$

or

$$P(\underset{\sim}{\phi}) = b_1 (\phi_1)^2 + b_2 (\phi_2)^2 + \ldots + b_k (\phi_k)^2 + b_{k+1} \phi_1\phi_2$$

$$+ b_{k+2} \phi_1 \phi_3 + \ldots + b_{N-k-1} \phi_{k-1} \phi_k$$

$$+ b_{N-k} \phi_1 + b_{N-k+1} \phi_2 + \ldots + b_{N-1} \phi_k + b_N, \tag{23}$$

where $\underset{\sim}{H}$ is the Hessian matrix of the quadratic approximation and is given by

$$\underset{\sim}{H} = \underset{\sim}{\nabla} \underset{\sim}{\nabla}^T P(\underset{\sim}{\phi}) , \tag{24}$$

$$\underset{\sim}{\nabla} = \begin{bmatrix} \frac{\partial}{\partial \phi_1} \\ \frac{\partial}{\partial \phi_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial}{\partial \phi_k} \end{bmatrix} . \tag{25}$$

The relations between the coefficients in (22) and (23) are given by

$$b_i = h_{ii}/2 \ , \ i = 1, 2, \ldots, k \ , \tag{26}$$

$$b_\ell = h_{ij}, \ \ell = j - i + \sum_{p=1}^{i} (k-p+1), \ i < j \ , \tag{27}$$

$$b_{N-k-1+i} = a_i - \sum_{j=1}^{k} h_{ij} \bar{\phi}_j \ , \ i = 1, 2, \ldots, k \ , \tag{28}$$

$$b_N = a_0 - \sum_{i=1}^{k} a_i \bar{\phi}_i + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} h_{ij} \bar{\phi}_i \bar{\phi}_j \ , \tag{29}$$

where N is given by (20).

Sparsity and Choice of Base Points

If we have freedom in choosing the base points, we can save computational effort, particularly if the number of variables k is large. In general, the matrix of monomials in (19) is full, however it is possible to make it sparse by using the following choice of base points. Let

$$[\underset{\sim}{\phi}^1 \ \underset{\sim}{\phi}^2 \ \ldots \ \underset{\sim}{\phi}^N] = \underset{\sim}{D} \ [\underset{\sim}{1}_k \ -\underset{\sim}{1}_k \ \underset{\sim}{B} \ \underset{\sim}{0}_k] + [\bar{\underset{\sim}{\phi}} \ \bar{\underset{\sim}{\phi}} \ \ldots \ \bar{\underset{\sim}{\phi}}] \ , \tag{30}$$

where

$\underset{\sim}{D}$ is a k x k diagonal matrix with diagonal elements δ_i,

$\underset{\sim}{1}_k$ is a k-dimensional identity matrix,

$\underset{\sim}{0}_k$ is a zero vector of dimension k,

$\underset{\sim}{B}$ is a k x L matrix having the structure

$$
\mathbf{B} = \left[\begin{array}{c:c:c:c:c}
\mathbf{u}_{k-1}^T & \mathbf{0}_{k-2}^T & \mathbf{0}_{k-3}^T & & \\
\hdashline
 & \mathbf{u}_{k-2}^T & \mathbf{0}_{k-3}^T & & \\
\cline{2-3}
 & & \mathbf{u}_{k-3}^T & & \mathbf{0}_{k-2} \\
\cline{3-3}
 & & & \cdots & \\
\mathbf{T}_{k-1} & & & & \\
 & \mathbf{T}_{k-2} & & & \\
 & & \mathbf{T}_{k-3} & & \mathbf{u}_1 \\
 & & & & \mathbf{T}_1
\end{array}\right] , \tag{31}
$$

where $\mathbf{u}_j$ is a column vector of dimension j and having components u_{ij} such that

$$
0 < | u_{ij} | \leq 1, \; i = 1, 2, \ldots, j \; , \tag{32}
$$

$\mathbf{T}_j$ is a diagonal matrix of dimension j with diagonal elements T_{ij} satisfying

$$
0 < | T_{ij} | \leq 1, \; i = 1, 2, \ldots, j \; , \tag{33}
$$

and

$$
L = k(k-1)/2 \; . \tag{34}
$$

According to this choice of base points

$$
a_0 = g(\boldsymbol{\phi}^N) \; , \tag{35}
$$

and the system of simultaneous linear equations given in (36) is solved for the vector $\mathbf{a}$ and the symmetric matrix $\mathbf{H}$.

$$
\left(\begin{array}{ccccc|cccc|ccccc}
 & & \underset{\sim}{D}\,\underset{\sim}{D} & & & & \underset{\sim}{0} & & & & & \underset{\sim}{D} & & \\
\hline
 & & \underset{\sim}{D}\,\underset{\sim}{D} & & & & \underset{\sim}{0} & & & & & -\underset{\sim}{D} & & \\
\hline
(\zeta_1^2)^2 & (\zeta_2^1)^2 & 0 & \cdots & 0 & \zeta_1^2\zeta_2^1 & & & & \zeta_1^2 & \zeta_2^1 & 0 & \cdots & 0 \\
(\zeta_1^3)^2 & 0 & (\zeta_3^1)^2 & \cdots & 0 & & \zeta_1^3\zeta_3^1 & & & \zeta_1^3 & 0 & \zeta_3^1 & \cdots & 0 \\
\vdots & & & & & & & \ddots & & \vdots & & & & \\
(\zeta_1^k)^2 & 0 & 0 & \cdots & (\zeta_k^1)^2 & & & \zeta_1^k\zeta_k^1 & & \zeta_1^k & 0 & 0 & \cdots & \zeta_k^1 \\
0 & (\zeta_2^3)^2 & (\zeta_3^2)^2 & \cdots & 0 & & & & \zeta_2^3\zeta_3^2 & 0 & \zeta_2^3 & \zeta_3^2 & \cdots & 0 \\
\vdots & & & & & & & & \ddots & & & & & \\
0 & 0 & \cdots & (\zeta_{k-1}^k)^2 & (\zeta_k^{k-1})^2 & & & & \zeta_{k-1}^k\,\zeta_k^{k-1} & 0 & 0 & \cdots & \zeta_{k-1}^k & \zeta_k^{k-1}
\end{array}\right)
\left(\begin{array}{c}
h_{11}/2 \\ h_{22}/2 \\ \vdots \\ h_{kk}/2 \\ \hline h_{12} \\ h_{13} \\ \vdots \\ h_{k-1,k} \\ \hline a_1 \\ a_2 \\ \vdots \\ a_k
\end{array}\right)
=
\left(\begin{array}{c}
g(\underset{\sim}{\phi}^1)-g(\underset{\sim}{\phi}^N) \\ g(\underset{\sim}{\phi}^2)-g(\underset{\sim}{\phi}^N) \\ \vdots \\ \\ \\ \\ \\ \\ \\ \\ g(\underset{\sim}{\phi}^{N-1})-g(\underset{\sim}{\phi}^N)
\end{array}\right)
\qquad (36)
$$

We have taken

$$\zeta_i^j = u_{j-i,k-i}\,\delta_i\ ,\ \zeta_j^i = T_{j-i,k-i}\,\delta_j\ ,\ i < j\ . \tag{37}$$

The structure of the coefficient matrix of (36) emphasizing its sparsity is shown in Fig. 1. Hence, solving (36) reduces to the following

$$h_{ii} = [g(\underset{\sim}{\phi}^i) + g(\underset{\sim}{\phi}^{k+i}) - 2g(\underset{\sim}{\phi}^N)]/\delta_i^2\ , \tag{38}$$

$$a_i = [g(\underset{\sim}{\phi}^i) - g(\underset{\sim}{\phi}^{k+i})]/2\delta_i\ ,\ i = 1, 2, \ldots, k\ , \tag{39}$$

$$h_{ij} = h_{ji} = [g(\underset{\sim}{\phi}^\ell) - g(\underset{\sim}{\phi}^N) - (\zeta_i^j)^2\frac{h_{ii}}{2} - (\zeta_j^i)^2\frac{h_{jj}}{2}$$

$$- \zeta_i^j a_i - \zeta_j^i a_j]/\zeta_i^j\,\zeta_j^i\ , \tag{40}$$

where

$$\ell = k + j - i + \sum_{p=1}^{i} (k - p + 1)\ ,\ j > i\ . \tag{41}$$

Subsequently, the number of multiplications or divisions required to obtain the approximation is reduced to $5k^2 - 2k$ instead of $(N^3 + 3N^2 - N)/3$ for Gauss elimination, where N is defined in (20).

Fig. 2 shows the choice of base points in two dimensions and three dimensions [3].

If we are not completely free in choosing the base points, for example, if the function evaluation is expensive and some evaluations for parameter values inside the interpolation region are known, the matrix of monomials can appropriately be arranged. Assuming that the resulting matrix of monomials will not be singular, we replace the bottom rows of the matrix of monomials by the monomials of these known, n say, base points. No singularity will result, for example, if the rows introduced are independent and full. This arrangement in the matrix of monomials is shown in Fig. 3. In solving the resulting system of simultaneous equations, we proceed with finding the polynomial coefficients using (38), (39) and (40) until we come to the full part of the matrix, i.e., the last n equations. The unknown coefficients beyond this point should be found by solving n simultaneous linear equations, for example, by Gauss elimination.

Example

Consider the approximation of the function

$$g(\underset{\sim}{\phi}) = (\phi_3)^2 + 5\phi_2\phi_3 + \phi_1 + 2\phi_2 + \phi_3 + 3 ,$$

where

$$\underset{\sim}{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} .$$

The execution time using a CDC 6400 computer to evaluate the approximation using equations (38), (39) and (40) is 0.005 s compared with 0.066 s using Gauss elimination. Using equal step size δ for the interpolation region, the Euclidean norm of the errors in the coefficients of the approximating polynomial is plotted against δ in Fig. 4.

IV. EVALUATION OF YIELD AND ITS SENSITIVITIES

The Linear Cut [3]

In order to obtain the linear cuts required for yield evaluation [3], consider linearizing the quadratic constraints at a point $\underset{\sim}{\phi}^a$ which may, for example, be the nominal point $\underset{\sim}{\phi}^0$ or a vertex $\underset{\sim}{\phi}^r$. Hence, the linear cut based upon the ℓth constraint is given by

$$g_\ell(\underset{\sim}{\phi}^a) + (\underset{\sim}{\phi} - \underset{\sim}{\phi}^a)^T \underset{\sim}{\nabla} g_\ell(\underset{\sim}{\phi}^a) \geq 0 \ . \tag{42}$$

Define a reference vertex $\underset{\sim}{\phi}^r$ by

$$\underset{\sim}{\phi}^r = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\ \underset{\sim}{\mu}^r, \tag{43}$$

where

$$\mu_j^r = -\text{sign}\left(\frac{\partial g_\ell(\underset{\sim}{\phi}^a)}{\partial \phi_j}\right) \quad , \ j = 1, 2, \ldots, k \ . \tag{44}$$

The distance from the reference vertex to the point of intersection with the ℓth cut along the orthotope edge in the jth direction is

$$\alpha_j^\ell = \mu_j^r \left[g_\ell(\underset{\sim}{\phi}^a) + (\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^a)^T \underset{\sim}{\nabla} g_\ell(\underset{\sim}{\phi}^a)\right] \Big/ \left\{\frac{\partial g_\ell(\underset{\sim}{\phi}^a)}{\partial \phi_j}\right\}. \tag{45}$$

Accordingly, we have

$$\frac{\partial \alpha_j^\ell}{\partial \phi_i^0} = \mu_j^r \left(\frac{\partial g_\ell(\underset{\sim}{\phi}^a)}{\partial \phi_i} + (\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^a)^T \underset{\sim}{H}_i\right) \Big/ \left(\frac{\partial g_\ell(\underset{\sim}{\phi}^a)}{\partial \phi_j}\right)$$

$$- \mu_j^r \left[g_\ell(\underset{\sim}{\phi}^a) + (\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^a)^T \underset{\sim}{\nabla} g_\ell(\underset{\sim}{\phi}^a) \right] H_{ji} \Big/ \left(\frac{\partial g_\ell(\underset{\sim}{\phi}^a)}{\partial \phi_j} \right)^2 , \quad (46)$$

where

$$\underset{\sim}{H} = \begin{bmatrix} \frac{\partial^2 g_\ell(\underset{\sim}{\phi})}{\partial \phi_1^2} & \frac{\partial^2 g_\ell(\underset{\sim}{\phi})}{\partial \phi_1 \partial \phi_2} & \cdots & \frac{\partial^2 g_\ell(\underset{\sim}{\phi})}{\partial \phi_1 \partial \phi_k} \\ \frac{\partial^2 g_\ell(\underset{\sim}{\phi})}{\partial \phi_2 \partial \phi_1} & \frac{\partial^2 g_\ell(\underset{\sim}{\phi})}{\partial \phi_2^2} & \cdots & \frac{\partial^2 g_\ell(\underset{\sim}{\phi})}{\partial \phi_2 \partial \phi_k} \\ \vdots & & & \\ \frac{\partial^2 g_\ell(\underset{\sim}{\phi})}{\partial \phi_k \partial \phi_1} & \frac{\partial^2 g_\ell(\underset{\sim}{\phi})}{\partial \phi_k \partial \phi_2} & \cdots & \frac{\partial^2 g_\ell(\underset{\sim}{\phi})}{\partial \phi_k^2} \end{bmatrix} , \quad (47)$$

is the <u>Hessian matrix</u> which is a constant matrix for a quadratic function $g_\ell(\underset{\sim}{\phi})$, $\underset{\sim}{H}_i$ is the ith column of $\underset{\sim}{H}$ and H_{ji} is an element of $\underset{\sim}{H}$. In deriving (46) it is assumed that $(\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^a)$ is independent of ϕ_i^0, i = 1, 2, ..., k.

<u>The General Distribution</u>

As described in Section II, we can assume that all outcomes will lie within the tolerance orthotope R_ε. This orthotope is now partitioned into a set of orthocells $R(i_1, i_2, \ldots, i_k)$ as shown in Fig. 5, where $i_j = 1, 2, \ldots, n_j$, n_j is the number of intervals in the jth direction and j = 1, 2, ..., k. A <u>weighting factor</u> $W(i_1, i_2, \ldots, i_k)$ is assigned to each orthocell and is given by

$$W(\underset{\sim}{i}) = w(\underset{\sim}{i})/V(R(\underset{\sim}{i})) , \quad (48)$$

where

$$\underset{\sim}{i} = (i_1, i_2, \ldots, i_k) , \tag{49}$$

$$w(\underset{\sim}{i}) = \int_{R(\underset{\sim}{i})} F(\underset{\sim}{\phi}) \, dv , \tag{50}$$

$$V(R(\underset{\sim}{i})) = \int_{R(\underset{\sim}{i})} dv = \prod_{j=1}^{k} \epsilon_{j,i_j} , \tag{51}$$

$$dv = d\phi_1 \, d\phi_2 \ldots d\phi_k , \tag{52}$$

$\epsilon_{1,i_1}, \epsilon_{2,i_2}, \ldots, \epsilon_{k,i_k}$ are the dimensions of the orthocell and $F(\underset{\sim}{\phi})$ is the joint probability distribution function (PDF). $W(\underset{\sim}{i})$ is seen to be the probability per unit volume that an outcome falls within the $\underset{\sim}{i}$th cell, whereas $w(\underset{\sim}{i})$ is simply the corresponding probability. The weighting factors $W(\underset{\sim}{i})$ can be obtained by sampling the parameters or from a histogram if the PDF is not available.

In principle, the problem of finding the yield is now reduced to finding the contribution to the yield given by all of these orthocells. A formula for the <u>weighted nonfeasible hypervolume</u> with respect to the ℓth constraint is constructed and is given by [4]

$$V^{\ell} = \left(\frac{1}{k!} \prod_{j=1}^{k} \alpha_j^{\ell} \right) \left(\sum_{i_1=1}^{n_1+1} \sum_{i_2=1}^{n_2+1} \ldots \sum_{i_k=1}^{n_k+1} \Delta W(\underset{\sim}{i}) \, (\delta^{\ell}(\underset{\sim}{i}))^k \right) , \tag{53}$$

where, for indexing with respect to $\underset{\sim}{\phi}^r$, i.e., numbering starts at this vertex (see Fig. 5), α_j^{ℓ} is the distance from the reference vertex to the point of intersection of the ℓth linear cut with the orthotope edge in the jth direction,

$$\delta^{\ell}(\underset{\sim}{i}) = \max \left\{ 0,\ 1- \sum_{j=1}^{k} \frac{1}{\alpha_j^{\ell}} \sum_{p=1}^{i_j} \varepsilon_{j,p-1} \right\}, \tag{54}$$

$$\varepsilon_{j,0} = 0\ ,\ j = 1,\ 2,\ \ldots,\ k\ , \tag{55}$$

$$\Delta W(\underset{\sim}{i}) = W(\underset{\sim}{i}) - \sum_{j=1}^{k} W(\underset{\sim}{i}-\underset{\sim}{e}_j) + \sum_{j=1}^{k-1} \sum_{p=j+1}^{k} W(\underset{\sim}{i}- \underset{\sim}{e}_j- \underset{\sim}{e}_p) - \ldots$$

$$+ (-1)^k\, W(\underset{\sim}{i} - \underset{\sim}{e}_1 - \underset{\sim}{e}_2 - \ldots - \underset{\sim}{e}_k)\ , \tag{56}$$

$$\underset{\sim}{e}_j = (0,\ 0,\ \ldots,\ 0,\ \underset{\substack{\vdots \\ j}}{1},\ 0,\ \ldots,\ 0) \tag{57}$$

and where

$$W(\underset{\sim}{i}) = 0 \text{ if } i_j = 0 \text{ or } i_j = n_j+1 \text{ for any } j. \tag{58}$$

Assuming no overlapping of nonfeasible regions defined by different cuts inside the orthotope R_ε , i.e,

$$R_i \underset{i \neq j}{\cap} R_j = \emptyset \tag{59}$$

where

$$R_\ell = \{\underset{\sim}{p} \mid g_\ell(\underset{\sim}{p}) < 0\} \cap R_\varepsilon\ , \tag{60}$$

the yield can be expressed as

$$Y = 1 - \sum_{\ell=1}^{m} V^{\ell} , \tag{61}$$

where m is the number of linear cuts.

Independent Parameters

In the case of independent parameters, (53) can be written as [4]

$$V^{\ell} = \left(\frac{1}{k!} \prod_{j=1}^{k} \alpha_j^{\ell} \right) \left(\sum_{i_1=1}^{n_1+1} \Delta W_1(i_1) \sum_{i_2=1}^{n_2+1} \Delta W_2(i_2) \cdots \right.$$

$$\left. \sum_{i_k=1}^{n_k+1} \Delta W_k(i_k) \; (\delta^{\ell}(\underset{\sim}{i}))^k \right) , \tag{62}$$

where $\underset{\sim}{i}$ and $\delta^{\ell}(\underset{\sim}{i})$ are as defined in (49) and (54), respectively, and where

$$\Delta W_j(i_j) = W_j(i_j) - W_j(i_j-1) \; , \; j = 1, 2, \ldots, k, \tag{63}$$

$$W_j(0) = W_j(n_j+1) = 0 \; , \qquad j = 1, 2, \ldots, k, \tag{64}$$

$$W_j(i_j) = w_j(i_j)/\epsilon_{j,i_j} \; , \qquad i_j = 1, 2, \ldots, n_j, \tag{65}$$

$$w_j(i_j) = \int_{R_j(i_j)} f_j(\phi_j) \, d\phi_j \; , \; i_j = 1, 2, \ldots, n_j, \tag{66}$$

$f_j(\phi_j)$ is the PDF of the jth parameter and $R_j(i_j)$ is the ith interval for that parameter. Similarly the yield will be given by (61).

Yield Sensitivities

Formulas for yield sensitivities can be derived assuming that the probabilities $w(\underset{\sim}{i})$ are independent of $\underset{\sim}{\phi}^0$ as long as the ratios between ε_{j,i_j} , $i_j = 1, 2, \ldots, n_j$, are fixed for each parameter $j = 1, 2, \ldots, k$. This is true, for example, if the sizes of the orthocells are fixed.

Let

$$\kappa_{j,i_j} = \varepsilon_{j,i_j}/\varepsilon_j \;, \tag{67}$$

hence,

$$\sum_{i_j=1}^{n_j} \kappa_{j,i_j} = 2 \;, \; j = 1, 2, \ldots, k \;. \tag{68}$$

The yield sensitivities are now given by

$$\frac{\partial Y}{\partial \phi_i^0} = - \sum_{\ell=1}^{m} \frac{\partial V^\ell}{\partial \phi_i^0} \;, \tag{69}$$

$$\frac{\partial Y}{\partial \varepsilon_i} = - \sum_{\ell=1}^{m} \frac{\partial V^\ell}{\partial \varepsilon_i} \;, \tag{70}$$

where

$$\frac{\partial V^\ell}{\partial \phi_i^0} = \left(\frac{1}{k!} \sum_{j=1}^{k} \frac{\partial \alpha_j^\ell}{\partial \phi_i^0} \prod_{\substack{p=1 \\ p \neq j}}^{k} \alpha_p^\ell \right) B + A \left(k \sum_{i_1=1}^{n_1+1} \sum_{i_2=1}^{n_2+1} \cdots \sum_{i_k=1}^{n_k+1} \Delta W(\underset{\sim}{i}) \right.$$

$$(\delta^{\ell}(\underset{\sim}{i}))^{k-1}\frac{\partial\delta^{\ell}(\underset{\sim}{i})}{\partial\phi_i^0}\Bigg)\Bigg), \qquad (71)$$

$$\frac{\partial V^{\ell}}{\partial\varepsilon_i} = \left(\frac{\mu_i^r}{k!}\sum_{j=1}^{k}\frac{\partial\alpha_j^{\ell}}{\partial\phi_i^0}\prod_{\substack{p=1\\p\neq j}}^{k}\alpha_p^{\ell}\right)B + A\left(\sum_{i_1=1}^{n_1+1}\sum_{i_2=1}^{n_2+1}\cdots\sum_{i_k=1}^{n_k+1}\left(k\,\Delta W(\underset{\sim}{i})\right.\right.$$

$$\left.\left.(\delta^{\ell}(\underset{\sim}{i}))^{k-1}\frac{\partial\delta^{\ell}(\underset{\sim}{i})}{\partial\varepsilon_i} + (\delta^{\ell}(\underset{\sim}{i}))^{k}\frac{\partial\Delta W(\underset{\sim}{i})}{\partial\varepsilon_i}\right)\right), \qquad (72)$$

where $\frac{\partial\Delta W(\underset{\sim}{i})}{\partial\varepsilon_i}$ is obtained by replacing $W(\cdot)$ by $\frac{\partial W(\cdot)}{\partial\varepsilon_i}$ in (56) and where

$$\frac{\partial W(\underset{\sim}{i})}{\partial\varepsilon_i} = -\,W(\underset{\sim}{i})\frac{\kappa_{i,i_i}}{\varepsilon_{i,i_i}}, \qquad (73)$$

$$A = \frac{1}{k!}\prod_{j=1}^{k}\alpha_j^{\ell}, \qquad (74)$$

$$B = \sum_{i_1=1}^{n_1+1}\sum_{i_2=1}^{n_2+1}\cdots\sum_{i_k=1}^{n_k+1}\Delta W(\underset{\sim}{i})\,(\delta^{\ell}(\underset{\sim}{i}))^{k} \qquad (75)$$

and where

$$\frac{\partial\delta^{\ell}(\underset{\sim}{i})}{\partial\phi_i^0} = \begin{cases} 0 & \text{if } \delta^{\ell}(\underset{\sim}{i}) = 0, \\ \\ \displaystyle\sum_{j=1}^{k}\frac{1}{(\alpha_j^{\ell})^2}\frac{\partial\alpha_j^{\ell}}{\partial\phi_i^0}\sum_{p=1}^{i_j}\varepsilon_{j,p-1} & \text{if } \delta^{\ell}(\underset{\sim}{i}) > 0, \end{cases} \qquad (76)$$

$$\frac{\partial\delta^{\ell}(\underset{\sim}{i})}{\partial\varepsilon_i} = \mu_i^r\frac{\partial\delta^{\ell}(\underset{\sim}{i})}{\partial\phi_i^0} - \sum_{j=1}^{k}\frac{1}{\alpha_j^{\ell}}\sum_{p=1}^{i_j}\kappa_{j,p-1}. \qquad (77)$$

The formulas for $\partial\alpha_j^\ell/\partial\phi_i^0$ and for μ_i^r are given by (46) and (44), respectively.

The case of independent parameters is obtained by substituting

$$\Delta W(\underset{\sim}{i}) = \prod_{j=1}^{k} \Delta W_j(i_j) \tag{78}$$

in (71), (72) and (75).

Example for Yield

In order to illustrate the calculation of the weighted hyper-volume, consider the two-dimensional example shown in Table I. The weighted volume is given by

$$V = \left(\frac{1}{2} \times 12 \times 3\right) \left(\sum_{i_1=1}^{4} \sum_{i_2=1}^{3} \Delta W(i_1, i_2)\ (\delta(i_1, i_2))^2\right)$$

$$= 1813/3600 \ .$$

The same example can be considered as if the parameters are independent as shown in Table II and Table III. Here, the weighted volume is given by

$$V = \left(\frac{1}{2} \times 12 \times 3\right) \left(\sum_{i_1=1}^{4} \Delta W(i_1) \sum_{i_2=1}^{3} \Delta W(i_2)\ (\delta(i_1, i_2))^2\right) ,$$

where the δ are as given in Table I. Hence,

$$V = 1813/3600 \ .$$

Example for Yield Sensitivities

Assuming that the sizes of the orthocells are fixed, the sensitivities of the weighted hypervolume with respect to the nominal parameter vector $\underset{\sim}{\phi}^0$ can be evaluated. The location of $\underset{\sim}{\phi}^0$ itself is not important. It is the relative location of the constraint with respect to the orthotope that matters. The constraint can be considered as

$$\phi_1/12 - \phi_2/3 \geq 0 \ .$$

According to (46) we have

$$\frac{\partial \alpha_1}{\partial \phi_1^0} = -1 \ ,$$

$$\frac{\partial \alpha_1}{\partial \phi_2^0} = (-1)\ (-1/3)/(1/12) = 4 \ ,$$

$$\frac{\partial \alpha_2}{\partial \phi_1^0} = (1)\ (1/12)/(-1/3) = -1/4 \ ,$$

and

$$\frac{\partial \alpha_2}{\partial \phi_2^0} = 1 \ .$$

Using (76), the values of $\partial \delta^{\ell}(\underset{\sim}{i})/\partial \phi_i^0$ are given in Table IV and Table V. Substituting in (71) we get

$$\frac{\partial V}{\partial \phi_1^0} = -43/720 ,$$

$$\frac{\partial V}{\partial \phi_2^0} = 43/180 .$$

These sensitivities were verified using the central difference approach with $\Delta\phi_i^0 = 10^{-3}$, $i = 1, 2$. An agreement of 6 digits was obtained.

V. CONCLUSIONS

The exploitation of sparsity in choosing the base points reduces the computational effort required for interpolation significantly.

The yield estimation technique presented provides an inexpensive yield determination without the need for the multitude of circuit simulations required in the Monte Carlo method. The method approximates the integration of the PDF over the feasible region. In addition, the availability of yield sensitivities permit the use of efficient gradient optimization techniques (see Part II [1]).

The better the description of the boundary of the constraint region by linear cuts the more accurate is the yield estimate. It is possible to describe a constraint defining the boundary of the feasible region by a different cut in each orthocell, however, the computational effort will increase.

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TABLE I

EXAMPLE TO ILLUSTRATE CALCULATION OF WEIGHTED HYPERVOLUME BY THE GENERAL FORMULA

Orthocell dimensions			0	1	2	3	4
		i_1	0	1	2	3	4
		ε_{1,i_1}	0	3.0	3.0	2.0	-
i_2	ε_{2,i_2}						
0	0	w,W	0	0	0	0	0
1	2.0	w	0	18/100	12/100	3/10	0
		W	0	3/100	1/50	3/40	0
		ΔW	-	3/100	-1/100	11/200	-3/40
		δ	-	1	3/4	1/2	1/3
2	3.0	w	0	12/100	8/100	2/10	0
		W	0	1/75	2/225	1/30	0
		ΔW	-	-1/60	1/180	-11/360	1/24
		δ	-	1/3	1/12	0	0
3	-	w,W	0	0	0	0	0
		ΔW	-	-1/75	1/225	-11/450	1/30
		δ	-	0	0	0	0

Reference vertex $\underset{\sim}{\phi}^r$ given by $\mu_1^r = -1$, $\mu_2^r = 1$

Intersections of the linear constraint are $\alpha_1 = 12$, $\alpha_2 = 3$

Weighted volume V = 1813/3600

TABLE II

LENGTHS AND WEIGHTS OF FIRST PARAMETER INTERVALS

i_1	ε_{1,i_1}	$w(i_1)$	$W(i_1)$	$\Delta W(i_1)$
0	0.0	0	0	-
1	3.0	3/10	1/10	1/10
2	3.0	2/10	1/15	-1/30
3	2.0	5/10	1/4	11/60
4	-	0	0	-1/4

TABLE III

LENGTHS AND WEIGHTS OF SECOND PARAMETER INTERVALS

i_2	ε_{2,i_2}	$w(i_2)$	$W(i_2)$	$\Delta W(i_2)$
0	0.0	0	0	-
1	2.0	6/10	3/10	3/10
2	3.0	4/10	2/15	-1/6
3	-	0	0	-2/15

TABLE IV

VALUES OF $\partial\delta^{\ell}(i_1,i_2)/\partial\phi_1^0$

i_2 \ i_1	1	2	3	4
1	0	-1/48	-1/24	-1/18
2	-1/18	-11/144	0	0
3	0	0	0	0

TABLE V

VALUES OF $\partial\delta^{\ell}(i_1,i_2)/\partial\phi_2^0$

i_2 \ i_1	1	2	3	4
1	0	1/12	1/6	2/9
2	2/9	11/36	0	0
3	0	0	0	0

FIGURE CAPTIONS

Fig. 1 The structure of the coefficient matrix of (36).

Fig. 2 Arrangement of the base points w.r.t. the centers of interpolation regions in (a) two dimensions ($\underset{\sim}{\phi}^5$ is a random base point) and (b) three dimensions ($\underset{\sim}{\phi}^7$, $\underset{\sim}{\phi}^8$ and $\underset{\sim}{\phi}^9$ are random base points). To exploit sparsity $\underset{\sim}{\phi}^7$, $\underset{\sim}{\phi}^8$ and $\underset{\sim}{\phi}^9$ should be, respectively, placed in the planes containing $\{\bar{\underset{\sim}{\phi}}, \underset{\sim}{\phi}^1, \underset{\sim}{\phi}^2\}$, $\{\bar{\underset{\sim}{\phi}}, \underset{\sim}{\phi}^1, \underset{\sim}{\phi}^3\}$ and $\{\bar{\underset{\sim}{\phi}}, \underset{\sim}{\phi}^2, \underset{\sim}{\phi}^3\}$.

Fig. 3 The arrangement of the matrix of monomials for a restricted selection of base points.

Fig. 4 Errors in computing the coefficients of the quadratic approximation using dense and sparse matrix approaches.

Fig. 5 Two-dimensional illustration of the partitioning of the tolerance region into cells indicating the dimensions and weighting of those cells relevant to the calculation of the weighted nonfeasible hypervolume.

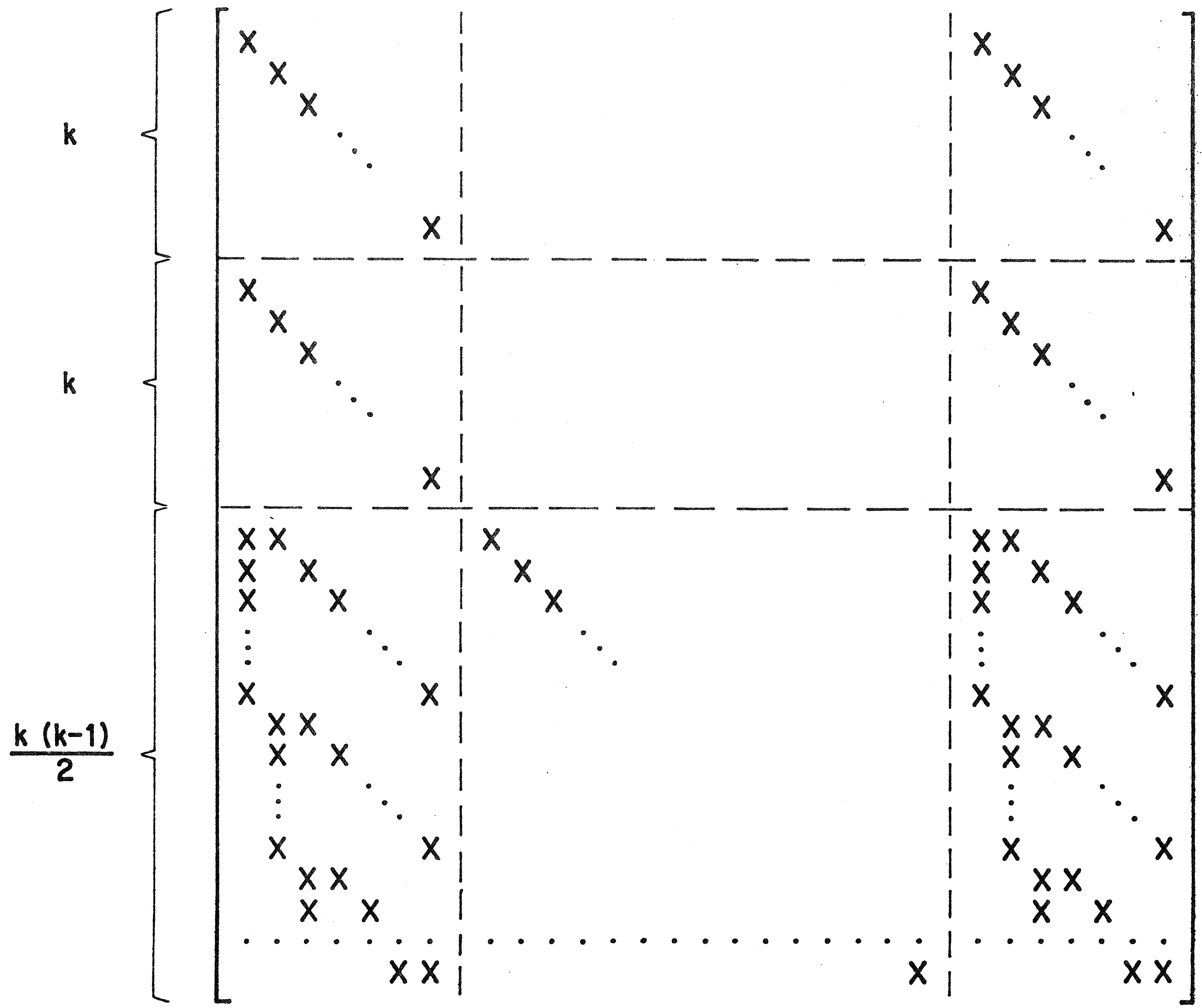
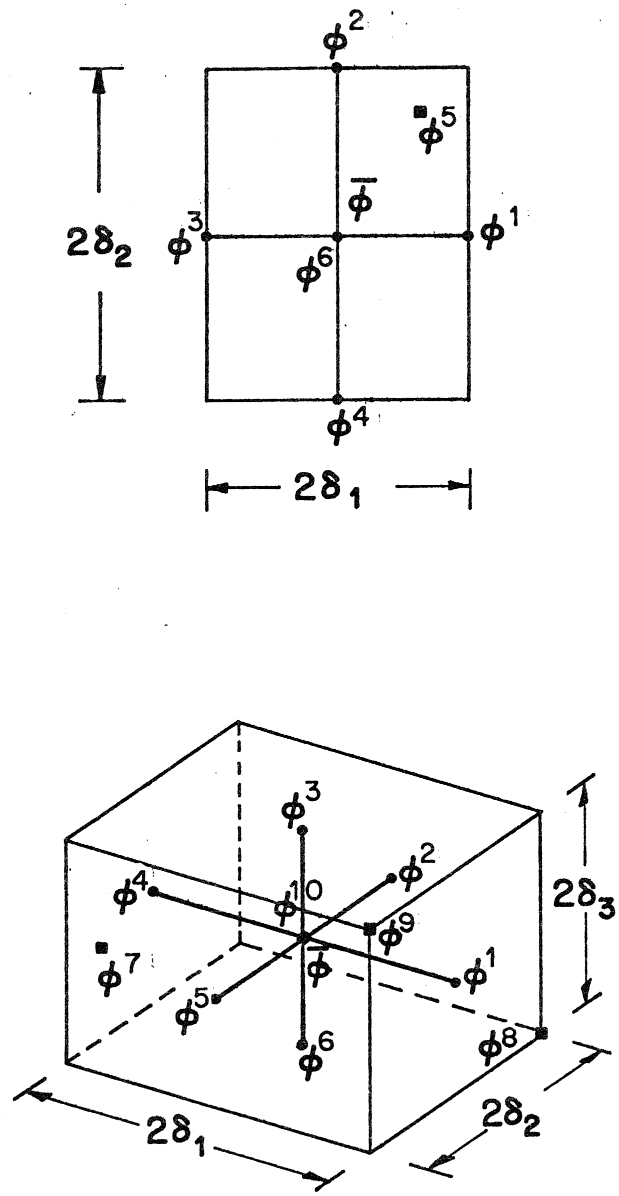


Fig. 1

(a)

(b)



ϕ^2
ϕ^5
$\bar{\phi}$
ϕ^3
ϕ^1
$2\delta_2$
ϕ^6
ϕ^4
$2\delta_1$
ϕ^3
ϕ^2
ϕ^4
ϕ^{10}
$2\delta_3$
ϕ^9
ϕ^7
$\bar{\phi}$
ϕ^1
ϕ^5
ϕ^6
ϕ^8
$2\delta_2$
$2\delta_1$

Fig. 2

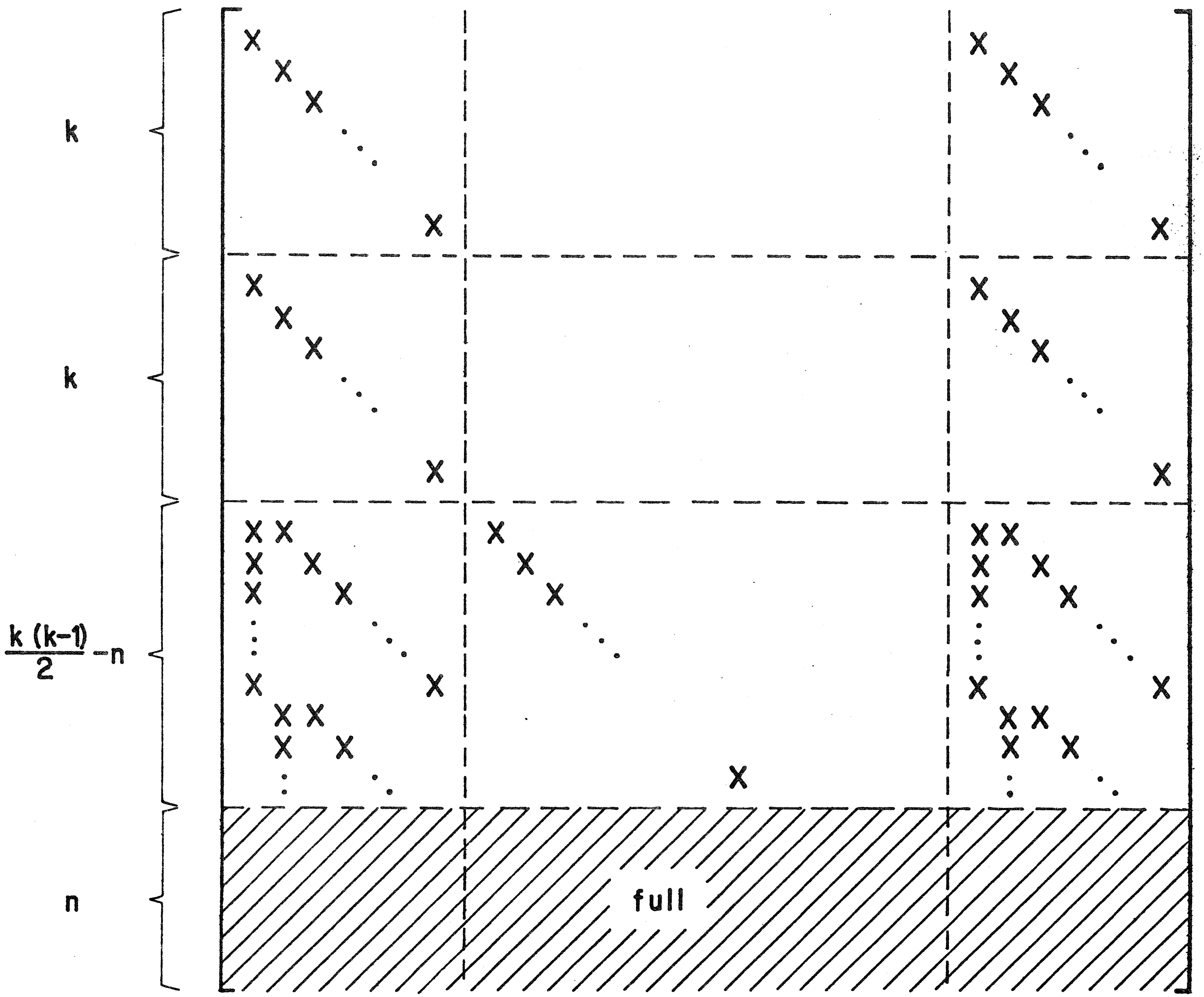
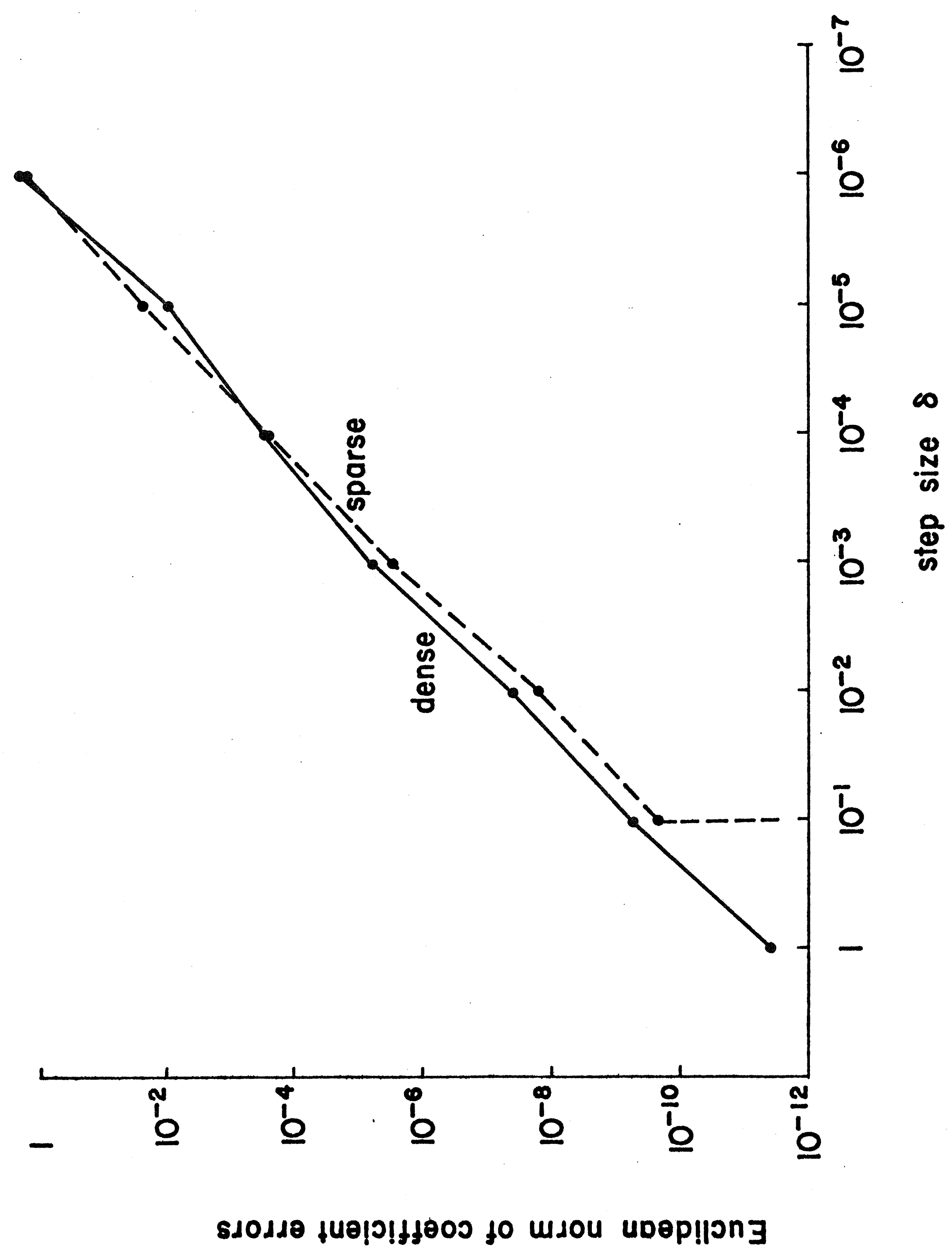


Fig. 3



Euclidean norm of coefficient errors
1
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
step size δ
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
sparse
dense

Fig. 4

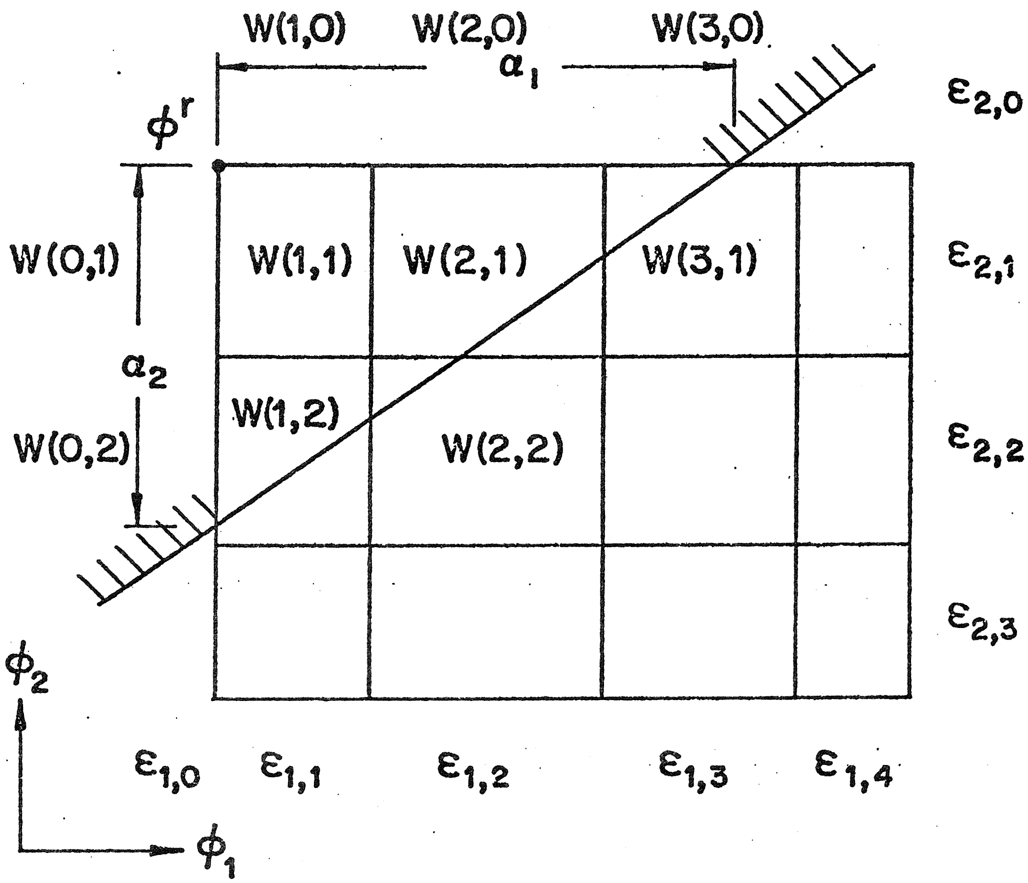

W(1,0)
W(2,0)
W(3,0)
α_1
ϕ^r
$\varepsilon_{2,0}$
W(0,1)
W(1,1)
W(2,1)
W(3,1)
$\varepsilon_{2,1}$
α_2
W(1,2)
W(0,2)
W(2,2)
$\varepsilon_{2,2}$
$\varepsilon_{2,3}$
ϕ_2
$\varepsilon_{1,0}$
$\varepsilon_{1,1}$
$\varepsilon_{1,2}$
$\varepsilon_{1,3}$
$\varepsilon_{1,4}$
ϕ_1

Fig. 5

SOC-184

YIELD OPTIMIZATION FOR ARBITRARY STATISTICAL DISTRIBUTIONS
PART I: THEORY

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Key Words: Yield analysis, design centering, tolerance assignment, multidimensional approximation, sparse matrix methods

Abstract: This paper generalizes certain analytical formulas for yield and yield sensitivities so that design centering and yield optimization can be effectively carried out employing given statistical parameter distributions. The tolerance region of possible outcomes is discretized into a set of orthotopic cells. A suitable weight is assigned to each cell in conjunction with an assumed uniform distribution on the cell. Explicit formulas for yield and its sensitivities w.r.t. nominal parameter values and component tolerances are presented for linear cuts and sensitivities of these cuts based upon approximations of the boundary of the constraint region. To avoid unnecessary evaluations of circuit responses, e.g., integrations for nonlinear circuits, multidimensional quadratic interpolation is performed. Sparsity is exploited in the determination of these quadratic models leading to reduced computation as well as increased accuracy.

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