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FLOPT4 - A PROGRAM FOR LEAST PTH OPTIMIZATION WITH EXTRAPOLATION TO MINIMAX SOLUTIONS

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FLOPT4 - A PROGRAM FOR LEAST PTH OPTIMIZATION

WITH EXTRAPOLATION TO MINIMAX SOLUTIONS

J.W. Bandler and D. Sinha

Abstract

FLOPT4 is a package of subroutines primarily for solving least pth optimization problems. Its main features include Fletcher's quasi-Newton subroutine, a least pth objective formulation subroutine, an extrapolation procedure and a scheme for dropping inactive functions. With appropriate utilization of these feature, the program can solve a wide variety of optimization problems. These may range from unconstrained problems, problems subject to inequality or equality constraints to nonlinear minimax approximation problems. In solving constrained problems, the user may, for example, use the Fiacco-McCormick method with extrapolation or the Bandler-Charalambous minimax formulation and least pth approximation, also with extrapolation. The program has been used on a CDC 6400 computer. Several detailed examples of varying complexity are used to illustrate the versatility of the program. A FORTRAN IV listing is included. FLOPT4 replaces a previous package on which it is based, namely, FLOPT2.

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I. INTRODUCTION

FLOPT4 is a package of subroutines primarily for solving least pth optimization problems. Its main features include a modification of the 1972 version of Fletcher's quasi-Newton subroutine [1], a least pth objective formulation subroutine, an extrapolation procedure and a scheme for dropping inactive functions [2]. With appropriate utilization of these features, the program can solve a wide variety of optimization problems. These may range from unconstrained problems, problems subject to inequality/equality constraints to minimax problems in general.

In solving constrained problems, the user may use the Fiacco-McCormick method with extrapolation [3] or use the Bandler-Charalambous minimax formulation [4] and least pth approximation. Using the p-algorithm [2], the program solves minimax problems that can be formulated with a least pth objective.

The program FLOPT4 is an improved version of the program FLOPT2 [5]. Section IV deals with the specific improvements made in this program. The program has been used on a CDC 6400 computer and is written in FORTRAN IV. It requires at least 37,000 octal words of computer memory. Several examples of varying complexity have been included in this report to illustrate the versatility of this program. FLOPT4 updates and supercedes FLOPT2. In comparison with FLOPT2, it provides considerable savings in execution time and storage requirements.

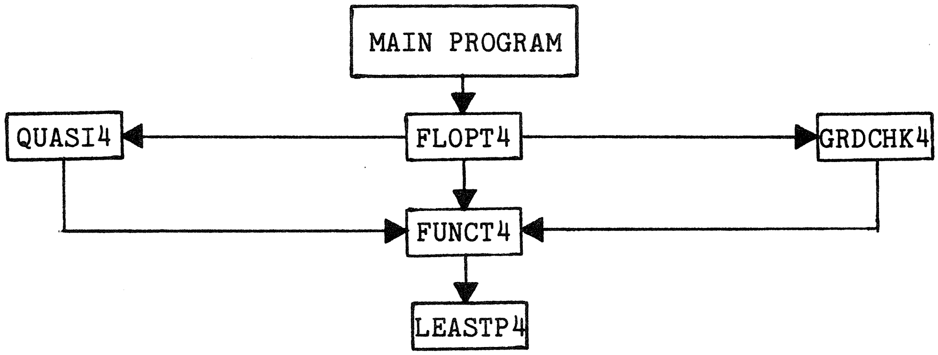

Fig. 1 Overall organization of the subroutines

Figure 1, with arrows emanating from calling subprograms and leading to called subroutines, highlights the overall organization of the program units.

In order to use the FLOPT4 package, the user has to provide the main program and a subroutine called FUNCT4. A discussion of the various subroutines, the variables, common blocks, etc., in the following sections and some completely worked out examples will familiarize the user with the details necessary to use this package successfully.

II. VARIABLES

This section adequately describes all the variables used in this program. The essential information regarding the dimensions, initialization and default values of these variables is also provided in Table I in a condensed form.

Integer variables

ID equals 0 if input data is not to be printed

IEX equals 0 when extrapolation is not to be performed

IEXIT a flag used by QUASI4 to stop the program execution and print a message if the chosen value of EPS is too small (IEXIT=2) or, if MAX has been exceeded (IEXIT=3). IEXIT=1 indicates a normal exit and no message is printed

IFINIS equals N when the projected minimax solution has converged to the true solution within EPS. This may be used as a stopping criterion in the main program

IFN counts the function evaluations

IGK equals 1 when gradient check is required

IH used as the index of a DO loop in the main program that calls FLOPT4 IK times

IK the number of times FLOPT4 is called from the main program. It corresponds to the number of p values used in least pth approximation or the number of r values used in the Fiacco-McCormick method

IPT the results of the unconstrained minimization are printed for the first and the last iterations of QUASI4 as well as after every IPT iterations within QUASI4. It must be noted that IPT=0 suppresses the entire printout. When IPT=0, JPRINT has no influence on printing

IREDU equals 0 when the scheme for choosing active functions is not used

JD an array which identifies the active functions

JORDER the highest order of the estimate of the minimax solution determined by the extrapolation procedure

JPRINT offers the following options for printing:
0 extrapolation estimates will not be printed
1 extrapolation estimates of the minimax solution and the error functions will be printed
2 in addition to the above printout, the multipliers and the normalized errors at the next estimated least pth solution will also be printed

JV used in subroutine LEASTP4. JV=1 results in the calculation of both the gradients and the multipliers. If JV=0 only the gradients are calculated

M nonzero if the initial value of X is to be read by FLOPT4

MAX maximum permissible number of function evaluations. Execution stops if max is exceeded

MODE for MODE=1 an identity matrix is the initial estimate of the Hessian in subroutine QUASI4. For MODE=3 the initial estimate of the Hessian is a matrix which is in the decomposed form LDL (TRANSPOSE) and has been generated by the last call to QUASI4

N the number of variables in the problem. N.GE.2

NA the number of active functions. If the reduction scheme is used, a function whose multiplier V does not equal or exceed ETA at the starting point of an optimization (except the first) is considered inactive and dropped from future consideration. When the reduction scheme is not used, NA is set equal to NR by FLOPT4

NR the number of error functions in the problem. When the least pth objective formulation is not being used and, for example, the Fiacco-McCormick method is used, the default value NR=1 should be used

Real Variables

EM equals the maximum of the error functions

EPS this array of N elements is used for testing the convergence of the solution of the unconstrained optimization as well as the projected minimax solution

ER an array of NR elements containing the values of the error functions. Array EN contains the normalized values and array ES contains the normalized values raised to power p

EST user's guess of the optimal objective function value

ETA used by the reduction scheme to select active functions, i.e., those functions with multiplier values .GE. ETA

FACTOR	multiplies p to update its value for a subsequent iteration in least pth approximation. It divides r in the Fiacco-McCormick method
G	an array of N elements storing the gradient vector at X
GE	an array of N*NR elements storing the partial derivatives of the error functions when least pth approximation is used
H	this array of N*(N+1)/2 elements stores the current estimate of the Hessian matrix at X
P	the parameter of least pth approximation. Also equals r in the Fiacco-McCormick method
U	value of the unconstrained objective function
V	an array storing the multipliers of the active functions if the reduction scheme is used
W	an array of 4*N elements used as working space
X	an array of N elements in which the current estimate of the solution is stored. An initial approximation must be set in X on entry. When the extrapolation procedure is used, an estimate of the next minimum in the sequence will be stored in X at the end of each iteration of FLOPT4
XB	an array of N elements in which the best estimate of the minimax solution currently available is stored
XE	an array of N*(JORDER+1)*IK elements in which estimates of the minimax solution corresponding to different orders are stored for each call of FLOPT4

TABLE I ESSENTIAL INFORMATION ON DIMENSIONS, INITIALIZATION AND DEFAULT VALUES

VARIABLE NAME	INITIALIZED BY USER (1)	DIMENSIONS IN MAIN PROG.	DIMENSIONS IN FUNCT4	DEFAULT VALUE (2)
EN	-	NR	NR	-
EPS	yes	N	-	-
ER	-	-	NR	-
ES	-	-	NR	-
EST	yes	-	-	0.
ETA	yes	-	-	.0005
FACTOR	yes	-	-	2.
G	-	N	N	-
GE	-	-	(N,NR)	-
H	-	N(N+1)/2	-	-
ID	yes	-	-	1.
IEX	yes	-	-	1.
IGK	yes	-	-	1.
IH	yes	-	-	1.
IK	yes	-	-	1.
IPT	yes	-	-	10.
IREDU	yes	-	-	1.
JD	-	NR	NR	-
JORDER	yes	-	-	3.
JPRINT	yes	-	-	2.
M	yes	-	-	-
MAX	yes	-	-	200.
NR	yes	-	-	1.
P	yes	-	-	2.
V	-	NR	NR	-
W	-	4N	-	-
X	yes (3)	N	N	-
XB	-	N	-	-
XE	-	(N,JORDER+1,IK)	-	-

(1) Variables may be initialized by any means other than a data statement in the main program.

(2) The user may take advantage of the default values to avoid initializing some variables.

(3) For M=1, X is read by FLOPT4 in free format.

III. SUBROUTINES

The user has to supply the main program in which the necessary initialization is performed and subroutine FLOPT4 is called. The number of times FLOPT4 is called depends on the number of p values in the case of least pth approximation, or the number of r values in the case of the Fiacco-McCormick method. For a problem of unconstrained minimization involving only one function, it will be enough to call FLOPT4 only once. In addition, the DIMENSION and COMMON cards should be included.

The user also has to supply a subroutine called FUNCT4. The input for this subroutine is variable X and it must return the values of the objective function and the gradient vector. In the case of least pth approximation this subroutine must define the error functions and their partial derivatives in order to utilize subroutine LEASTP4 to generate the objective function, the gradient vector and the multiplier vector.

Enough examples have been included in this report to illustrate these subroutines. A brief description of other subroutines is given as follows:

FLOPT4 (EN, EPS, G, H, JD, V, W, X, XB, XE)

This subroutine reads in the starting value of X, calls another subroutine for the unconstrained minimization, performs extrapolation, selection of active functions and outputs the results.

QUASI4 (N, X, U, G, H, W, EST, EPS, MODE, MAX, IPT, IEXIT, IFN,
EN, JD, V)

This subroutine is a modification of the 1972 version of Fletcher's unconstrained minimization method. The details of the original subroutine may be found in [1]. The initial estimate of the Hessian for the first iteration of FLOPT4 is an identity matrix. For subsequent iterations of FLOPT4 the updated Hessian is used.

LEASTP4 (EN, ER, ES, G, GE, JD, U, V)

This subroutine formulates the objective function U for the least pth approximation method and calculates the gradient vector.

GRDCHK4 (N, X, G, W, EN, JD, V)

This subroutine performs the gradient check by first determining the numerical gradient by perturbation and then calculating the percentage error between the analytical gradient (provided by the user) and the numerical gradient. A percentage error exceeding 10% results in the termination of the program with a message.

IV. FLOPT4 VERSUS FLOPT2

The FLOPT4 package in terms of the basic technique of optimization is essentially the same as FLOPT2, but a reorganization of the program has resulted in a considerably reduced execution time and savings in the storage requirement.

Those features of FLOPT4 which are different from FLOPT2 are summarized as follows:

(1) The 3-dimensional array XE(N, IK, JORDER+1) of FLOPT2 which stores the estimates of the minimax solution has been dimensioned as XE(N, JORDER+1, IK) in FLOPT4 and each element of this array has been referenced by a single subscripted variable in subroutine FLOPT4. In other words, this multi-dimensional array has been treated as a vector. A standard practice in all good programs, it has contributed to saving in execution time here.

The user may still use a three-dimensional variable XE in the main program for his manipulations. If extrapolation is not used, XE(1) is sufficient in the main program and not XE(N, IK, 1) as required by FLOPT2.

(2) In least pth approximation with the reduction scheme in operation, the multipliers for the error functions are calculated at the starting point of an iteration of FLOPT4. The reduction scheme is not used until after the first iteration.

(3) All the undimensioned integer or real variables that the user could possibly require for manipulations in the main program belong to the two numbered common blocks. Hence, to access a FLOPT4 variable from

some subprogram one only needs to include a COMMON statement.

(4) The argument list of subroutine FLOPT4 is more managable with only those variables in it which require to be dimensioned. FLOPT4 does not impose any limit on the number of variables or the number of error functions in the problem.

(5) Subroutines EXTRAP and RESULT of FLOPT2 have been eliminated and their operations are performed by subroutine FLOPT4. This has eliminated some repetitious calculations and tests.

(6) Printing of results has been modified and frequent referencing of FORMAT statements cut down by using WRITE statements sparingly.

(7) Using the reduction scheme in least pth approximation may sometimes, at some stage, lead to only one active function, which may be undesirable. There is a way to handle this kind of situation. It is illustrated in Example 7 of Section V.

(8) Three minor errors were detected in FLOPT4 which have been rectified in this revised edition. These errors were:

a. At the Bandler-Charalambous least pth solution, the normalized errors were sometimes not being printed at the exact solution point but at a point within the epsilon neighbourhood.

b. The normalized errors for functions having a zero multiplier were not being calculated and therefore the printed values were not correct.

c. In subroutine GRDCHK4 the gradient vector was being perturbed by a small quantity under certain circumstances.

V. EXAMPLES

Several examples have been presented in this section to illustrate the flexibility and power of this program. For each example, a complete listing of the main program, subroutine FUNCT4 and the output has been provided.

Example 1: Rosenbrock's function [7]

Minimize

$$U = 100\ (x_1^2 - x_2)^2 + (1 - x_1)^2.$$

The function has a minimum value of zero at $x_1 = x_2 = 1$. The starting point used was $x_1 = -1.2$, $x_2 = 1.0$.

It is not necessary to use least pth approximation for this problem. Extrapolation and the reduction scheme are also not required.

```
      PROGRAM TST ( INPUT,OUTPUT,TAPE5= INPUT,TAPE6=OUTPUT)
C
C     MAIN PROGRAM OF EXAMPLE 1
C
C     THIS IS A PROBLEM OF UNCONSTRAINED MINIMIZATION INVOLVING ONLY ONE
C     FUNCTION. IT IS ENOUGH TO CALL FLOPT4 ONLY ONCE IN ORDER TO OBTAIN
C     THE DESIRED SOLUTION
C
      DIMENSION X(2),G(2),H(3),W(8),EPS(2),XB(1),XE(1),V(1),EN(1),JD(1)
C
      COMMON /1/ ID,IEX,IFINIS,IGK,IH,IK,IPT,IREDU,JORDER,JPRINT,M,MAX
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      IREDU=0
      READ (5,*) N,M,IEX,EPS(1),EPS(2)
      CALL FLOPT4 (EN,EPS,G,H,JD,V,W,X,XB,XE)
      STOP
      END
```

```
      SUBROUTINE FUNCT4 (EN,G,JD,U,V,X)
C
C     ROSENBROCK"S FUNCTION
C
C     THE ROLE OF THIS SUBROUTINE IS TO RETURN THE VALUE OF THE OBJECT-
C     IVE FUNCTION AND THE GRADIENT VECTOR AT A GIVEN POINT X. AS LEAST
C     PTH APPROXIMATION WILL NOT BE USED FOR THIS PROBLEM, SUBROUTINE
C     LEASTP4 NEED NOT BE CALLED AND, THEREFORE, THE OBJECTIVE FUNCTION
C     AND THE GRADIENT VECTOR MUST BE DEFINED HERE
C
      DIMENSION X(2),G(2),EN(1),JD(1),V(1)
C
      A=X(1)*X(1)
      B=A-X(2)
      C=1.0-X(1)
C
C     THE OBJECTIVE FUNCTION IS DEFINED HERE
C
      U=100.*B*B+C*C
C
C     THE GRADIENT VECTOR IS DEFINED HERE
C
      G(1)=400.*X(1)*(A-X(2))-C-C
      G(2)=-200.*B
      RETURN
      END
```

INPUT DATA FOR EXAMPLE 1

```
2,1,0,1.E-8,1.E-8
-1.2,1.0
```

INPUT DATA FOR BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

NUMBER OF ERROR FUNCTIONS .... NR = 1

PREDICTED FUNCTION LOWER BOUND .... EST = 0.

STARTING POINT AND TEST QUANTITIES FOR THE TERMINATION CRITERION

	VARIABLE VECTOR X(I)		TEST VECTOR EPS(I)
1	-.12000000E+01	1	.10000000E-07
2	.10000000E+01	2	.10000000E-07

GRADIENT CHECK AT THE STARTING POINT

	ANALYTICAL GRADIENT VECTOR G(I)		NUMERICAL GRADIENT VECTOR G(I)		PERCENTAGE ERROR VECTOR YP(I)
1	-.21560000E+03	1	-.21560000E+03	1	.25461507E-06
2	-.88000000E+02	2	-.88000000E+02	2	.70497208E-07

GRADIENTS ARE O.K.

ITERATION NO. 1 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .20000000E+01

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	1	.24200000E+02	1	-.12000000E+01	1	-.21560000E+03
			2	.10000000E+01	2	-.88000000E+02
10	14	.12820268E+01	1	-.86272932E-01	1	-.32748854E+01
			2	-.24500359E-01	2	-.63886756E+01
20	27	.14762556E+00	1	.64388633E+00	1	.30030459E+01
			2	.40016442E+00	2	-.28850382E+01
30	40	.61476392E-04	1	.99327613E+00	1	.14679176E+00
			2	.98619416E+00	2	-.80662112E-01
37	47	.25874638E-24	1	.10000000E+01	1	-.76028073E-11
			2	.10000000E+01	2	.42632564E-11

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .080 SECONDS

Example 2: Beale constrained function [8]

Minimize

$$f = 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$$

subject to

$$x_i \geq 0, \quad i = 1,2,3$$

$$3 - x_1 - x_2 - 2x_3 \geq 0.$$

The function has a minimum f = 1/9 at $\underset{\sim}{x} = [4/3 \;\; 7/9 \;\; 4/9]^T$. The SUMT method of Fiacco and McCormick [3] was used to transform the constrained problem into an unconstrained problem by defining

$$U = f(\underset{\sim}{x}) - r \sum_{i=1}^{3} \ln g_i(\underset{\sim}{x}).$$

The objective function U was minimized w.r.t. $\underset{\sim}{x}$ for a strictly decreasing sequence of r values together with extrapolation. The starting point was $\underset{\sim}{x} = [1 \quad 2 \quad 1]^T$. A COMMON block named USER was used in the main program and subroutine FUNCT4 to transfer the value of parameter r, a weighting factor WT and an indicator IGRAD. WT was used in the formulation of the unconstrained objective function only when the process drifted into the nonfeasible region. IGRAD is an indicator to control the printing of the original objective function and constraints at the extrapolated solution, which is available from the argument list of subroutine FLOPT4. The reduction scheme cannot be used in this example.

With the sequence of r values 10^{-2}, $2\text{x}10^{-3}$, $4\text{x}10^{-4}$, $8\text{x}10^{-5}$, $1.6\text{x}10^{-5}$ and 3rd order extrapolation, 33 function evaluations were necessary to obtain an accuracy of eight digits in the objective function and parameter values.

```
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C     MAIN PROGRAM OF EXAMPLE 2
C
C     THIS PROBLEM INVOLVES CONSTRAINED MINIMIZATION USING THE FIACCO-
C     MCCORMICK METHOD. FLOPT4 WILL HAVE TO BE CALLED AS MANY TIMES AS
C     THE USER WISHES TO UPDATE THE VALUE OF PARAMETER R. IT MUST BE
C     NOTED THAT THE UPDATING OF R IS DONE WITHIN FLOPT4 PROVIDED NR
C     EQUALS 1, ITS DEFAULT VALUE. PARAMETER IGRAD CONTROLS THE PRINTING
C     OF THE FUNCTION AND THE CONSTRAINTS
C
      DIMENSION  X(3),G(3),EPS(3),XB(3),H(6),W(12),XE( 3,4,6),
     1           V(1),EN(1),JD(1)
C
      COMMON /USER/ R,WT,IGRAD
      COMMON /1/ ID,IEX,IFINIS,IGK,IH,IK,IPT,IREDU,JORDER,JPRINT,M,MAX
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      N=3
      M=1
      IK=6
      WT=1.E+16
      P=.01
      FACTOR=5.
      IPT=5
      IREDU=0
      EPS(1)=1.E-8
      EPS(2)=1.E-8
      EPS(3)=1.E-8
C
      DO 1 IH=1,IK
      IGRAD=1
      CALL FLOPT4 (EN,EPS,G,H,JD,V,W,X,XB,XE)
      M=0
      IGRAD=0
      CALL FUNCT4 (EN,G,JD,U,V,XB)
      IF (IFINIS.EQ.N) CALL EXIT
1     CONTINUE
C
      STOP
      END
```

```
      SUBROUTINE FUNCT4 (EN,G,JD,U,V,X)
C
C     BEALE FUNCTION
C
C     THE ROLE OF THIS SUBROUTINE IS TO RETURN THE VALUE OF THE OBJECT-
C     IVE FUNCTION AND THE GRADIENT VECTOR AT A GIVEN POINT X. AS LEAST
C     PTH APPROXIMATION WILL NOT BE USED FOR THIS PROBLEM, SUBROUTINE
C     LEASTP4 NEED NOT BE CALLED AND, THEREFORE, THE OBJECTIVE FUNCTION
C     AND THE GRADIENT VECTOR MUST BE DEFINED HERE
C
      DIMENSION X(3),G(3),C(4),GF(3),GC(3,4),EN(1),JD(1),V(1)
C
      COMMON /USER/ R,WT,IGRAD
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      R=P
      B=X(1)+X(1)
      D=X(2)+X(2)
      E=X(3)+X(3)
      F=9.+B*(X(1)+X(2)+X(3)-4.)+D*(X(2)-3.)+X(3)*X(3)-E-E
      C(1)=X(1)
      C(2)=X(2)
      C(3)=X(3)
      C(4)=3.-X(1)-X(2)-E
      IF (IGRAD.EQ.0) GO TO 6
      GF(1)=-8.+B+B+D+E
      GF(2)=-6.+D+D+B
      GF(3)=-4.+B+E
      GC(1,1)=1.
      GC(2,1)=0.
      GC(3,1)=0.
      GC(1,2)=0.
      GC(2,2)=1.
      GC(3,2)=0.
      GC(1,3)=0.
      GC(2,3)=0.
      GC(3,3)=1.
      GC(1,4)=-1.
      GC(2,4)=-1.
      GC(3,4)=-2.
      S1=0.
      S2=0.
C
      DO 2 I=1,4
      IF (C(I).LT.1.E-6) GO TO 1
      S1=S1-ALOG(C(I))
      GO TO 2
1     S2=S2+WT*C(I)*C(I)
2     CONTINUE
C
C     THE OBJECTIVE FUNCTION IS DEFINED HERE
C
      U=F+R*S1+S2
C
C     THE GRADIENT VECTOR IS DEFINED HERE
C
      DO 5 J=1,3
      S3=0.
      S4=0.
      DO 4 I=1,4
      IF (C(I).LT.1.E-6) GO TO 3
      S3=S3-GC(J,I)/C(I)
      GO TO 4
3     S4=S4+(WT+WT)*C(I)*GC(J,I)
4     CONTINUE
5     G(J)=GF(J)+S3*R+S4
C
      RETURN
6     PRINT 7, F,(I,X(I),I,C(I),I=1,3),C(4)
      RETURN
C
7     FORMAT (*-SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLE
     1M*/1X,57(*-*)/*0ORIGINAL OBJECTIVE FUNCTION *,12(*.*),* F(X) =*,E1
```

```
     25.8/1H0,31X,*SOLUTION*,11X,*CONSTRAINT*/31X,*VECTOR X(I)*,8X,*VECT FUN 740
     3OR C(I)*//25X,3(I4,E15.8,I4,E15.8/25X),22X,*4*,E15.8)              FUN 750
C                                                                         FUN 760
      END                                                                 FUN 770
```

INPUT DATA FOR EXAMPLE 2

```
1.,2.,1.
```

INPUT DATA FOR BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

NUMBER OF ERROR FUNCTIONS .... NR = 1

MAXIMUM ORDER OF EXTRAPOLATION .... JORDER = 3

INITIAL VALUE OF THE PARAMETER .... P = .10000000E-01

MULTIPLYING FACTOR FOR P .... FACTOR = .50000000E+01

PREDICTED FUNCTION LOWER BOUND .... EST = 0.

STARTING POINT AND TEST QUANTITIES FOR THE TERMINATION CRITERION

	VARIABLE VECTOR X(I)		TEST VECTOR EPS(I)
1	.10000000E+01	1	.10000000E-07
2	.20000000E+01	2	.10000000E-07
3	.10000000E+01	3	.10000000E-07

GRADIENT CHECK AT THE STARTING POINT

	ANALYTICAL GRADIENT VECTOR G(I)		NUMERICAL GRADIENT VECTOR G(I)		PERCENTAGE ERROR VECTOR YP(I)
1	.40000000E+17	1	.40000000E+17	1	.64000000E-06
2	.40000000E+17	2	.40000000E+17	2	0.
3	.80000000E+17	3	.80000000E+17	3	.64000000E-12

GRADIENTS ARE O.K.

ITERATION NO. 1 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .10000000E-01

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	1	.40000000E+17	1	.10000000E+01	1	.40000000E+17
			2	.20000000E+01	2	.40000000E+17
			3	.10000000E+01	3	.80000000E+17
5	8	.35870261E+00	1	.89685764E+00	1	-.58276986E+04
			2	.10724335E+01	2	-.58263763E+04
			3	.51535441E+00	3	-.11654096E+05
10	13	.11885189E+00	1	.13312048E+01	1	-.23505458E+00
			2	.77884307E+00	2	-.23505773E+00
			3	.44497608E+00	3	-.47011140E+00
11	16	.11885189E+00	1	.13312048E+01	1	-.23505458E+00
			2	.77884307E+00	2	-.23505773E+00
			3	.44497608E+00	3	-.47011140E+00

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .063 SECONDS

SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM

ORIGINAL OBJECTIVE FUNCTION .... F(X) = .11111593E+00

	SOLUTION VECTOR X(I)		CONSTRAINT VECTOR C(I)
1	.13312048E+01	1	.13312048E+01
2	.77884307E+00	2	.77884307E+00
3	.44497608E+00	3	.44497608E+00
		4	.53290705E-13

```
ITERATION NO.   2 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)
-------------------------------------------------------------------
WITH PARAMETER P =   .20000000E-02
---------------------------------

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD
-----------------------------------------------------------------

ITER. FUNC. OBJECTIVE           VARIABLE            GRADIENT
 NO.  EVAL. FUNCTION            VECTOR X(I)         VECTOR G(I)

  0    17   .11266312E+00   1  .13312048E+01   1 -.10660431E+04
                            2  .77884307E+00   2 -.10660389E+04
                            3  .44497608E+00   3 -.21320803E+04

  3    21   .11266004E+00   1  .13329053E+01   1  .70829481E+02
                            2  .77799185E+00   2  .70829481E+02
                            3  .44455142E+00   3  .14165896E+03

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS   .023  SECONDS

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION
----------------------------------------------------------------

            VARIABLE
            VECTOR X(I)

            ORDER 1

         1  .13333304E+01
         2  .77777905E+00
         3  .44444526E+00


SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM
---------------------------------------------------------

ORIGINAL OBJECTIVE FUNCTION ........... F(X) =  .11111111E+00

                               SOLUTION            CONSTRAINT
                               VECTOR X(I)         VECTOR C(I)

                            1  .13333304E+01   1  .13333304E+01
                            2  .77777905E+00   2  .77777905E+00
                            3  .44444526E+00   3  .44444526E+00
                                               4  .49737992E-13
```

ITERATION NO. 3 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .40000000E-03

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	22	.11142093E+00	1	.13332454E+01	1	-.10660368E+04
			2	.77782161E+00	2	-.10660368E+04
			3	.44446649E+00	3	-.21320737E+04
2	25	.11142093E+00	1	.13332475E+01	1	-.71277010E+02
			2	.77782071E+00	2	-.71277010E+02
			3	.44446591E+00	3	-.14255402E+03

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .024 SECONDS

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2
1	.13333330E+01	1	.13333331E+01
2	.77777793E+00	2	.77777788E+00
3	.44444453E+00	3	.44444450E+00

SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM

ORIGINAL OBJECTIVE FUNCTION .... F(X) = .11111111E+00

	SOLUTION VECTOR X(I)		CONSTRAINT VECTOR C(I)
1	.13333331E+01	1	.13333331E+01
2	.77777788E+00	2	.77777788E+00
3	.44444450E+00	3	.44444450E+00
		4	.78159701E-13

ITERATION NO. 4 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .80000000E-04

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	26	.11117308E+00	1	.13333160E+01	1	-.19897420E+04
			2	.77778645E+00	2	-.19897420E+04
			3	.44444878E+00	3	-.39794840E+04
2	29	.11117308E+00	1	.13333161E+01	1	.70831948E+02
			2	.77778639E+00	2	.70831949E+02
			3	.44444874E+00	3	.14166390E+03

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .021 SECONDS

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.13333333E+01	1	.13333333E+01	1	.13333333E+01
2	.77777781E+00	2	.77777781E+00	2	.77777781E+00
3	.44444445E+00	3	.44444445E+00	3	.44444445E+00

SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM

ORIGINAL OBJECTIVE FUNCTION .... F(X) = .11111111E+00

	SOLUTION VECTOR X(I)		CONSTRAINT VECTOR C(I)
1	.13333333E+01	1	.13333333E+01
2	.77777781E+00	2	.77777781E+00
3	.44444445E+00	3	.44444445E+00
		4	.92370556E-13

ITERATION NO. 5 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .16000000E-04

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	30	.11112350E+00	1	.13333299E+01	1	-.27002846E+04
			2	.77777953E+00	2	-.27002846E+04
			3	.44444531E+00	3	-.54005693E+04
2	32	.11112350E+00	1	.13333299E+01	1	-.22224280E+00
			2	.77777949E+00	2	-.22224279E+00
			3	.44444530E+00	3	-.44448559E+00

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .019 SECONDS

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.13333333E+01	1	.13333334E+01	1	.13333334E+01
2	.77777777E+00	2	.77777777E+00	2	.77777777E+00
3	.44444444E+00	3	.44444444E+00	3	.44444444E+00

SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM

ORIGINAL OBJECTIVE FUNCTION .... F(X) = .11111111E+00

	SOLUTION VECTOR X(I)		CONSTRAINT VECTOR C(I)
1	.13333334E+01	1	.13333334E+01
2	.77777777E+00	2	.77777777E+00
3	.44444444E+00	3	.44444444E+00
		4	.56843419E-13

ITERATION NO. 6 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .32000000E-05

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	33	.11111359E+00	1	.13333327E+01	1	-.19186876E+04
			2	.77777811E+00	2	-.19186876E+04
			3	.44444461E+00	3	-.38373752E+04
2	35	.11111359E+00	1	.13333326E+01	1	-.22222634E+00
			2	.77777812E+00	2	-.22222634E+00
			3	.44444462E+00	3	-.44445267E+00

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .020 SECONDS

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.13333333E+01	1	.13333333E+01	1	.13333333E+01
2	.77777778E+00	2	.77777778E+00	2	.77777778E+00
3	.44444444E+00	3	.44444444E+00	3	.44444444E+00

SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM

ORIGINAL OBJECTIVE FUNCTION .... F(X) = .11111111E+00

	SOLUTION VECTOR X(I)		CONSTRAINT VECTOR C(I)
1	.13333333E+01	1	.13333333E+01
2	.77777778E+00	2	.77777778E+00
3	.44444444E+00	3	.44444444E+00
		4	.78159701E-13

Example 3: A minimax example [9]

Minimize the maximum of the following three functions

$$e_1 = x_1^2 + x_2^4$$

$$e_2 = (2-x_1)^2 + (2-x_2)^2$$

$$e_3 = 2\ \exp(-x_1 + x_2).$$

The minimax solution is defined by the functions e_1 and e_2 at the point $x_1 = 1.13904$, $x_2 = 0.89956$ where $e_1 = e_2 = 1.95222$ and $e_3 = 1.57408$. Using least pth approximation with p = 4, 16, 64, 256, 1024, 46 function evaluations yielded $x_1 = 1.1390346$, $x_2 = 0.8995623$. All the three functions were used in the initial objective formulation. The reduction scheme selected two active functions after the first two iterations.

```
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C     MAIN PROGRAM OF EXAMPLE 3
C
C     THIS IS A MINIMAX PROBLEM INVOLVING THREE FUNCTIONS. LEAST PTH
C     APPROXIMATION, IN CONJUNCTION WITH EXTRAPOLATION AND THE REDUCTION
C     SCHEME, WILL BE USED HERE IN ORDER TO OBTAIN THE DESIRED SOLUTION.
C     FLOPT4 WILL HAVE TO BE CALLED AS MANY TIMES AS THE USER WISHES TO
C     UPDATE PARAMETER P
C
      DIMENSION X(2),G(2),H(3),W(8),EPS(2),XB(2),XE(40),V(3),EN(3),JD(3)
C
      COMMON /1/ ID,IEX,IFINIS,IGK,IH,IK,IPT,IREDU,JORDER,JPRINT,M,MAX
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      N=2
      NR=3
      M=1
      IGK=0
      IK=5
      P=4.
      FACTOR=4.
      EPS(1)=1.E-8
      EPS(2)=1.E-8
C
      DO 1 IH=1,IK
      CALL FLOPT4 (EN,EPS,G,H,JD,V,W,X,XB,XE)
      IF(IFINIS.EQ.N) CALL EXIT
1     M=0
C
      STOP
      END
```

```
      SUBROUTINE FUNCT4 (EN,G,JD,U,V,X)
C
C     A MINIMAX EXAMPLE
C
C     THE ROLE OF THIS SUBROUTINE IS TO RETURN THE VALUE OF THE OBJECT-
C     IVE FUNCTION, THE GRADIENT VECTOR AND THE MULTIPLIER VECTOR AT A
C     GIVEN POINT X. AS LEAST PTH APPROXIMATION WILL BE USED FOR THIS
C     PROBLEM, SUBROUTINE LEASTP4 WILL BE CALLED TO GENERATE THESE
C     QUANTITIES. HOWEVER, THE ERROR FUNCTIONS AND THEIR PARTIAL DERI-
C     VATIVES, WHICH ARE REQUIRED BY SUBROUTINE LEASTP4 AS AN INPUT,
C     MUST BE DEFINED HERE
C
      DIMENSION X(2),G(2),ER(3),GE(2,3),ES(3),EN(3),JD(3),V(3)
C
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      Y1=X(1)*X(1)
      Y2=X(2)*X(2)
      Y3=X(1)+X(1)
      Y4=X(2)+X(2)
C
      DO 4 I=1,NA
      K=JD(I)
      GO TO (1,2,3) ,K
1     ER(1)=Y1+Y2*Y2
      GE(1,1)=Y3
      GE(2,1)=(Y2+Y2)*Y4
      GO TO 4
2     ER(2)=8.-4.*(X(1)+X(2))+Y1+Y2
      GE(1,2)=-4.+Y3
      GE(2,2)=-4.+Y4
      GO TO 4
3     ER(3)=2.*EXP(-X(1)+X(2))
      GE(1,3)=-ER(3)
      GE(2,3)=ER(3)
4     CONTINUE
C
      CALL LEASTP4 (EN,ER,ES,G,GE,JD,U,V)
      RETURN
      END
```

INPUT DATA FOR EXAMPLE 3

```
2.0,2.0
```

INPUT DATA FOR BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

NUMBER OF ERROR FUNCTIONS .... NR = 3

MAXIMUM ORDER OF EXTRAPOLATION .... JORDER = 3

INITIAL VALUE OF THE PARAMETER .... P = .40000000E+01

MULTIPLYING FACTOR FOR P .... FACTOR = .40000000E+01

TOLERANCE ETA FOR INACTIVE FUNCTIONS .... ETA = .50000000E-03

PREDICTED FUNCTION LOWER BOUND .... EST = 0.

STARTING POINT AND TEST QUANTITIES FOR THE TERMINATION CRITERION

	VARIABLE VECTOR X(I)		TEST VECTOR EPS(I)
1	.20000000E+01	1	.10000000E-07
2	.20000000E+01	2	.10000000E-07

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .20000000E+02

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.99990001E+00	1	.10000000E+01
2	0.	2	0.
3	.99990001E-04	3	.10000000E+00

ITERATION NO. 1 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .40000000E+01

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	1	.20000500E+02	1	.20000000E+01	1	.39977002E+01
			2	.20000000E+01	2	.31999600E+02
10	13	.24033042E+01	1	.12008090E+01	1	-.24628529E-07
			2	.82623537E+00	2	.54009733E-07

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .041 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .20164297E+01

	NORM. ERROR VECTOR EN(I)
1	.94621380E+00
2	.10000000E+01
3	.68198003E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .20164297E+01

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.29177586E+00	1	.94621380E+00
2	.70667687E+00	2	.10000000E+01
3	.15472745E-02	3	.68198003E+00

ITERATION NO. 2 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .16000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	15	.20606621E+01	1	.12008090E+01	1	-.40069412E+00
			2	.82623537E+00	2	-.98117056E+00
8	24	.20392604E+01	1	.11424333E+01	1	.11910686E-08
			2	.89020827E+00	2	-.43954481E-07

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .033 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19670583E+01

	NORM. ERROR VECTOR EN(I)
1	.98276892E+00
2	.10000000E+01
3	.79008317E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1
1	.11229748E+01
2	.91153256E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19539346E+01

	NORM. ERROR VECTOR EN(I)
1	.99873001E+00
2	.10000000E+01
3	.82849862E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19570593E+01

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.41343885E+00	1	.99454984E+00
2	.58655953E+00	2	.10000000E+01
3	.16262417E-05	3	.81878502E+00

ITERATION NO. 3 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .64000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	27	.19734408E+01	1	.11278394E+01	1	-.86170883E-01
			2	.90620149E+00	2	-.46110024E-01
6	33	.19731820E+01	1	.11377074E+01	1	-.10197023E-10
			2	.89892092E+00	2	-.12117136E-10

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .025 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19559236E+01

	NORM. ERROR VECTOR EN(I)
1	.99561005E+00
2	.10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2
1	.11361321E+01	1	.11370093E+01
2	.90182513E+00	2	.90117797E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19523318E+01

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.99991344E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19531062E+01

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.43375008E+00	1	.99895926E+00
2	.56624992E+00	2	.10000000E+01

ITERATION NO. 4 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .25600000E+03

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	36	.19574500E+01	1	.11371427E+01	1	.10336383E-01
			2	.90064404E+00	2	.23885606E-01
6	43	.19574433E+01	1	.11387066E+01	1	.14525699E-05
			2	.89940088E+00	2	.20629525E-05

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .024 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19531448E+01

	NORM. ERROR VECTOR EN(I)
1	.99890565E+00
2	.10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.11390396E+01	1	.11392335E+01	1	.11392688E+01
2	.89956087E+00	2	.89940992E+00	2	.89938186E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19522326E+01

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.99999279E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19524501E+01

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.43173437E+00	1	.99973170E+00
2	.56826563E+00	2	.10000000E+01

ITERATION NO. 5 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .10240000E+04

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	46	.19535280E+01	1	.11391171E+01	1	.54386023E-02
			2	.89939531E+00	2	.58703554E-02
6	52	.19535279E+01	1	.11389550E+01	1	-.30598460E-07
			2	.89952021E+00	2	-.38927848E-07

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .025 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19524543E+01

	NORM. ERROR VECTOR EN(I)
1	.99972661E+00
2	.10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.11390378E+01	1	.11390377E+01	1	.11390346E+01
2	.89955999E+00	2	.89955993E+00	2	.89956231E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19522246E+01

	NORM. ERROR VECTOR EN(I)
1	.99999986E+00
2	.10000000E+01

Example 4: Rosen-Suzuki function [8]

Minimize

$$f = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4$$

subject to

$$-x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 + 8 \geq 0$$

$$-x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_1 + x_4 + 10 \geq 0$$

$$-2x_1^2 - x_2^2 - x_3^2 - 2x_1 + x_2 + x_4 + 5 \geq 0.$$

The function has a minimum $f = -44$ at $\underset{\sim}{x} = [0\ 1\ 2\ -1]^T$. The Bandler-Charalambous technique [4] was used to transform the nonlinear programming problem into an unconstrained minimax problem. The value of the parameter α was 10. Using least pth approximation with p = 4, 12, 36, 108, 324 and 972, 71 function evaluations were required to obtain the following solution:

$$f = -44.000000$$

$$x_1 = -1.5 \times 10^{-8}$$

$$x_2 = 1.0000001$$

$$x_3 = 2.0000000$$

$$x_4 = -1.0000000\ .$$

This problem has also been solved with p = 10 and FACTOR = 2. All the other parameters were the same. 75 function evaluations and 10 iterations were required to arrive at the solution:

$$f = -44.000000$$

$$x_1 = -1.4 \times 10^{-8}$$

$$x_2 = 1.0000000$$

$$x_3 = 2.0000000$$

$$x_4 = -0.99999996\ .$$

```
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C     MAIN PROGRAM OF EXAMPLE 4
C
C     THIS PROBLEM INVOLVES CONSTRAINED MINIMIZATION. THE CONSTRAINED
C     PROBLEM HAS BEEN CONVERTED TO A MINIMAX PROBLEM BY THE BANDLER-
C     CHARALAMBOUS TECHNIQUE. LEAST PTH APPROXIMATION, IN CONJUNCTION
C     WITH EXTRAPOLATION AND THE REDUCTION SCHEME, WILL BE USED HERE
C     TO GET THE DESIRED SOLUTION. PARAMETER IGRAD CONTROLS THE PRINTING
C     OF THE FUNCTION AND THE CONSTRAINTS
C
      DIMENSION X(4),G(4),H(10),W(16),EPS(4),XB(4),XE(4,4,6),
     1          EN(4),JD(4),V(4)
C
      COMMON /USER/ IGRAD
      COMMON /1/ ID,IEX,IFINIS,IGK,IH,IK,IPT,IREDU,JORDER,JPRINT,M,MAX
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      N=4
      NR=4
      READ (5,*) EST,ETA,P,FACTOR,EPS
      M=1
      IK=6
C
      DO 1 IH=1,IK
      IGRAD=1
      CALL FLOPT4 (EN,EPS,G,H,JD,V,W,X,XB,XE)
      M=0
      IGRAD=0
      CALL FUNCT4 (EN,G,JD,U,V,XB)
      IF(IFINIS.EQ.N) CALL EXIT
1     CONTINUE
C
      STOP
      END
```

```
      SUBROUTINE FUNCT4 (EN,G,JD,U,V,X)
C
C     ROSEN-SUZUKI FUNCTION
C
C     THE ROLE OF THIS SUBROUTINE IS TO RETURN THE VALUE OF THE OBJECT-
C     IVE FUNCTION, THE GRADIENT VECTOR AND THE MULTIPLIER VECTOR AT A
C     GIVEN POINT X. AS LEAST PTH APPROXIMATION WILL BE USED FOR THIS
C     PROBLEM, SUBROUTINE LEASTP4 WILL BE CALLED TO GENERATE THESE
C     QUANTITIES. HOWEVER, THE ERROR FUNCTIONS AND THEIR PARTIAL DERI-
C     VATIVES, WHICH ARE REQUIRED BY SUBROUTINE LEASTP4 AS AN INPUT,
C     MUST BE DEFINED HERE
C
      DIMENSION X(4),G(4),C(3),GF(4),GC(4,3),ER(4),GE(4,4),ES(4),
     1          EN(4),JD(4),V(4)
C
      COMMON /USER/ IGRAD
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      DATA ALFA/10.0/
      B=X(1)*X(1)
      R=X(2)*X(2)
      D=X(3)*X(3)
      E=X(4)*X(4)
      BB=X(1)+X(1)
      RR=X(2)+X(2)
      DD=X(3)+X(3)
      EE=X(4)+X(4)
      F=B+R+D+D+E-5.*(X(1)+X(2))-21.*X(3)+7.*X(4)
      IF (IGRAD.EQ.0) GO TO 10
      GF(1)=BB-5.
      GF(2)=RR-5.
      GF(3)=DD+DD-21.
      GF(4)=EE+7.
C
      DO 9 I=1,NA
      K=JD(I)
      GO TO (1,3,5,7), K
1     C(1)=-B-R-D-E-X(1)+X(2)-X(3)+X(4)+8.
      ER(1)=F-ALFA*C(1)
      GC(1,1)=-BB-1.
      GC(2,1)=-RR+1.
      GC(3,1)=-DD-1.
      GC(4,1)=-EE+1.
C
      DO 2 J=1,4
2     GE(J,1)=GF(J)-ALFA*GC(J,1)
C
      GO TO 9
3     C(2)=-B-R-R-D-E-E+X(1)+X(4)+10.
      ER(2)=F-ALFA*C(2)
      GC(1,2)=GC(1,1)+2.
      GC(2,2)=-RR-RR
      GC(3,2)=GC(3,1)+1.
      GC(4,2)=-EE-EE+1.
C
      DO 4 J=1,4
4     GE(J,2)=GF(J)-ALFA*GC(J,2)
C
      GO TO 9
5     C(3)=-B-B-R-D-BB+X(2)+X(4)+5.
      ER(3)=F-ALFA*C(3)
      GC(1,3)=GC(1,1)+GC(1,1)
      GC(2,3)=GC(2,1)
      GC(3,3)=GC(3,1)+1.
      GC(4,3)=1.
C
      DO 6 J=1,4
6     GE(J,3)=GF(J)-ALFA*GC(J,3)
C
      GO TO 9
7     ER(4)=F
C
      DO 8 J=1,4
```

```
8       GE(J,4)=GF(J)
C
9       CONTINUE
C
        CALL LEASTP4 (EN,ER,ES,G,GE,JD,U,V)
        RETURN
10      C(1)=-B-R-D-E-X(1)+X(2)-X(3)+X(4)+8.
        C(2)=-B-R-R-D-E-E+X(1)+X(4)+10.
        C(3)=-B-B-R-D-BB+X(2)+X(4)+5.
        PRINT 11, F,(I,X(I),I,C(I),I=1,3),X(4)
        RETURN
C
11      FORMAT (*-SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLE
       1M*/1X,57(*-*)/*0ORIGINAL OBJECTIVE FUNCTION *,12(*.*),* F(X) =*,E1
       25.8/1H0,31X,*SOLUTION*,11X,*CONSTRAINT*/31X,*VECTOR X(I)*,8X,*VECT
       3OR C(I)*//25X,3(I4,E15.8,I4,E15.8/25X),*    4*,E15.8)
C
        END
```

INPUT DATA FOR EXAMPLE 4

```
-100.0,0.0001,4.0,3.0,1.0E-8,1.0E-8,1.0E-8,1.0E-8
0.0,0.0,0.0,0.0
```

INPUT DATA FOR BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

NUMBER OF ERROR FUNCTIONS .... NR = 4

MAXIMUM ORDER OF EXTRAPOLATION .... JORDER = 3

INITIAL VALUE OF THE PARAMETER .... P = .40000000E+01

MULTIPLYING FACTOR FOR P .... FACTOR = .30000000E+01

TOLERANCE ETA FOR INACTIVE FUNCTIONS .... ETA = .10000000E-03

PREDICTED FUNCTION LOWER BOUND .... EST = -.10000000E+03

STARTING POINT AND TEST QUANTITIES FOR THE TERMINATION CRITERION

	VARIABLE VECTOR X(I)		TEST VECTOR EPS(I)
1	0.	1	.10000000E-07
2	0.	2	.10000000E-07
3	0.	3	.10000000E-07
4	0.	4	.10000000E-07

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE
NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.10000000E-09

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.24414062E-47	1	.80000000E+12
2	.10000000E-47	2	.10000000E+13
3	.16000000E-46	3	.50000000E+12
4	.10000000E+01	4	.10000000E+01

GRADIENT CHECK AT THE STARTING POINT

	ANALYTICAL GRADIENT VECTOR G(I)		NUMERICAL GRADIENT VECTOR G(I)		PERCENTAGE ERROR VECTOR YP(I)
1	-.50000000E+01	1	-.50000000E+01	1	.17053026E-11
2	-.50000000E+01	2	-.50000000E+01	2	.17053026E-11
3	-.21000000E+02	3	-.21000000E+02	3	.16240977E-11
4	.70000000E+01	4	.70000000E+01	4	.12180733E-11

GRADIENTS ARE O.K.

ITERATION NO. 1 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .40000000E+01

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION	I	VARIABLE VECTOR X(I)	I	GRADIENT VECTOR G(I)
0	1	-.10000000E-09	1	0.	1	-.50000000E+01
			2	0.	2	-.50000000E+01
			3	0.	3	-.21000000E+02
			4	0.	4	.70000000E+01
10	15	-.34790263E+02	1	.58461974E-02	1	-.33658142E-03
			2	.86667337E+00	2	.44348325E-03
			3	.18395873E+01	3	.38969131E-03
			4	-.68144639E+00	4	-.82145183E-03
14	20	-.34790264E+02	1	.58761870E-02	1	.90958689E-08
			2	.86664727E+00	2	.11091500E-06
			3	.18396046E+01	3	-.61137996E-07
			4	-.68134639E+00	4	-.66182865E-07

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .109 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.39780104E+02

I	NORM. ERROR VECTOR EN(I)
1	.14374876E+01
2	.18822826E+01
3	.12609990E+01
4	.10000000E+01

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.39780104E+02

I	MULTIPLIER VECTOR V(I)	I	NORM. ERROR VECTOR EN(I)
1	.11946931E-01	1	.14374876E+01
2	.47020533E-03	2	.18822826E+01
3	.57534518E-01	3	.12609990E+01
4	.93004834E+00	4	.10000000E+01

SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM

ORIGINAL OBJECTIVE FUNCTION .... F(X) = -.39780104E+02

I	SOLUTION VECTOR X(I)	I	CONSTRAINT VECTOR C(I)
1	.58761870E-02	1	.17403301E+01
2	.86664727E+00	2	.35097294E+01
3	.18396046E+01	3	.10382569E+01
4	-.68134639E+00		

ITERATION NO. 2 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .12000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	22	-.39540428E+02	1	.58761870E-02	1	-.38812182E+01
			2	.86664727E+00	2	-.27940926E+01
			3	.18396046E+01	3	-.11281383E+02
			4	-.68134639E+00	4	.48570648E+01
9	33	-.41016091E+02	1	-.86347186E-02	1	.43203542E-06
			2	.96144478E+00	2	-.87099572E-07
			3	.19362345E+01	3	.99214134E-06
			4	-.89504638E+00	4	-.38155254E-06

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .068 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.42466733E+02

	NORM. ERROR VECTOR EN(I)
1	.11564132E+01
2	.14465312E+01
3	.10965788E+01
4	.10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1
1	-.15890171E-01
2	.10088435E+01
3	.19845495E+01
4	-.10018964E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.43754893E+02

	NORM. ERROR VECTOR EN(I)
1	.10178350E+01
2	.12286375E+01
3	.10187454E+01
4	.10000000E+01

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.43329592E+02

MULTIPLIER NORM. ERROR

	VECTOR V(I)		VECTOR EN(I)
1	.81590705E-01	1	.10639294E+01
2	.57891767E-04	2	.13013247E+01
3	.15889768E+00	3	.10444118E+01
4	.75945373E+00	4	.10000000E+01

SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM

ORIGINAL OBJECTIVE FUNCTION .... F(X) = -.43754893E+02

	SOLUTION VECTOR X(I)		CONSTRAINT VECTOR C(I)
1	-.15890171E-01	1	.78036943E-01
2	.10088435E+01	2	.10004009E+01
3	.19845495E+01	3	.82020483E-01
4	-.10018964E+01		

ITERATION NO. 3 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .36000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION	VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)	
0	36	-.42999677E+02	1	-.13471687E-01	1	-.12520636E+01
			2	.99304395E+00	2	-.71502375E+00
			3	.19684445E+01	3	-.31712968E+01
			4	-.96627972E+00	4	.12263833E+01
8	46	-.43017368E+02	1	-.56458459E-02	1	.14771431E-06
			2	.99344200E+00	2	.13950216E-06
			3	.19752190E+01	3	-.36938138E-06
			4	-.96840728E+00	4	.14834597E-06

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .055 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.43469679E+02

	NORM. ERROR VECTOR EN(I)
1	.10527264E+01
3	.10340112E+01
4	.10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2
1	-.41514096E-02	1	-.26840643E-02
2	.10094406E+01	2	.10095152E+01
3	.19947112E+01	3	.19959814E+01
4	-.10050877E+01	4	-.10054867E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.43956734E+02

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
3	.10021980E+01
4	.10007604E+01

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.43815285E+02

MULTIPLIER VECTOR V(I) NORM. ERROR VECTOR EN(I)

1	.11154165E+00	1	.10171864E+01
3	.18588024E+00	3	.10123877E+01
4	.70257811E+00	4	.10000000E+01

SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM

ORIGINAL OBJECTIVE FUNCTION .... F(X) = -.43990157E+02

	SOLUTION VECTOR X(I)		CONSTRAINT VECTOR C(I)
1	-.26840643E-02	1	-.33422673E-02
2	.10095152E+01	2	.94763136E+00
3	.19959814E+01	3	.63194314E-02
4	-.10054867E+01		

ITERATION NO. 4 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .10800000E+03

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION	VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)	
0	49	-.43672308E+02	1	-.37800171E-02	1	-.25326934E+00
			2	.10041520E+01	2	-.22200099E-01
			3	.19889665E+01	3	-.22677063E+00
			4	-.99309731E+00	4	-.11729395E+00
6	57	-.43672996E+02	1	-.19320503E-02	1	-.52053601E-09
			2	.99776477E+00	2	-.13002567E-09
			3	.19915804E+01	3	-.77918891E-09
			4	-.98925257E+00	4	-.32715692E-10

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .049 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.43820176E+02

	NORM. ERROR VECTOR EN(I)
1	.10178713E+01
3	.10115111E+01
4	.10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	-.75152496E-04	1	.43437964E-03	1	.55431979E-03
2	.99992616E+00	2	.99873685E+00	2	.99832230E+00
3	.19997611E+01	3	.20003924E+01	3	.20005620E+01
4	-.99967521E+00	4	-.99899864E+00	4	-.99874911E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.43994466E+02

	NORM. ERROR VECTOR EN(I)
1	.10005666E+01
3	.10000000E+01
4	.10000982E+01

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.43938995E+02

MULTIPLIER VECTOR V(I)	NORM. ERROR VECTOR EN(I)

1	.92744180E-01	1	.10062661E+01
3	.20539898E+00	3	.10037997E+01
4	.70185684E+00	4	.10000000E+01

SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM

ORIGINAL OBJECTIVE FUNCTION .... F(X) = -.43998785E+02

	SOLUTION VECTOR X(I)		CONSTRAINT VECTOR C(I)
1	.55431979E-03	1	.20609613E-02
2	.99832230E+00	2	.10112621E+01
3	.20005620E+01	3	-.43186879E-03
4	-.99874911E+00		

ITERATION NO. 5 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .32400000E+03

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	60	-.43891010E+02	1	-.32373025E-03	1	.16959610E-01
			2	.99826436E+00	2	-.41776907E-01
			3	.19975051E+01	3	-.21373448E+00
			4	-.99565767E+00	4	.19847077E+00
5	69	-.43891072E+02	1	-.65027450E-03	1	-.28056737E-05
			2	.99924898E+00	2	-.21985562E-05
			3	.19971753E+01	3	-.13218740E-05
			4	-.99639469E+00	4	-.19059202E-05

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .042 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.43939719E+02

	NORM. ERROR VECTOR EN(I)
1	.10059897E+01
3	.10038567E+01
4	.10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	-.93865907E-05	1	-.11658525E-05	1	-.17917602E-04
2	.99999109E+00	2	.99999921E+00	2	.10000478E+01
3	.19999728E+01	3	.19999993E+01	3	.19999842E+01
4	-.99996576E+00	4	-.10000021E+01	4	-.10000407E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.43999325E+02

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
3	.10000190E+01
4	.10000165E+01

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = -.43979898E+02

MULTIPLIER VECTOR V(I) NORM. ERROR VECTOR EN(I)

1	.10114926E+00	1	.10019908E+01
3	.19982581E+00	3	.10012891E+01
4	.69902493E+00	4	.10000000E+01

SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM

ORIGINAL OBJECTIVE FUNCTION .... F(X) = -.44000051E+02

	SOLUTION VECTOR X(I)		CONSTRAINT VECTOR C(I)
1	-.17917602E-04	1	-.72646544E-04
2	.10000478E+01	2	.99965106E+00
3	.19999842E+01	3	.10770593E-04
4	-.10000407E+01		

```
ITERATION NO.   6 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)
--------------------------------------------------------------------
WITH PARAMETER P =   .97200000E+03
---------------------------------

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD
-----------------------------------------------------------------

ITER. FUNC. OBJECTIVE            VARIABLE             GRADIENT
 NO.  EVAL. FUNCTION            VECTOR X(I)          VECTOR G(I)

  0     72 -.43963699E+02   1 -.22770364E-03   1  .41218468E-03
                            2  .99977624E+00   2  .47491311E-02
                            3  .19990474E+01   3  .26569524E-01
                            4 -.99881895E+00   4 -.21672048E-01


  5     79 -.43963699E+02   1 -.21748731E-03   1 -.84652545E-08
                            2  .99974907E+00   2  .10178890E-08
                            3  .19990564E+01   3 -.25015154E-07
                            4 -.99879576E+00   4  .10300076E-07

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS   .046   SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE
----------------------------------------------------------

MAXIMUM OF THE ERROR FUNCTIONS ........... EM = -.43979868E+02

                                                 NORM. ERROR
                                                VECTOR EN(I)

                                               1  .10020002E+01
                                               3  .10012878E+01
                                               4  .10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION
----------------------------------------------------------------

            VARIABLE             VARIABLE             VARIABLE
           VECTOR X(I)          VECTOR X(I)          VECTOR X(I)

            ORDER 1              ORDER 2              ORDER 3

         1 -.10937215E-05   1 -.57112809E-07   1 -.14468974E-07
         2  .99999911E+00   2  .10000001E+01   2  .10000001E+01
         3  .19999969E+01   3  .19999999E+01   3  .20000000E+01
         4 -.99999629E+00   4 -.10000001E+01   4 -.10000000E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS ........... EM = -.44000000E+02

                                                 NORM. ERROR
                                                VECTOR EN(I)

                                               1  .10000000E+01
                                               3  .10000000E+01
                                               4  .10000000E+01


SOLUTION OF THE ORIGINAL CONSTRAINED MINIMIZATION PROBLEM
---------------------------------------------------------

ORIGINAL OBJECTIVE FUNCTION ............ F(X) = -.44000000E+02

                             SOLUTION             CONSTRAINT
                            VECTOR X(I)          VECTOR C(I)

                            1 -.14468974E-07   1 -.26398936E-07
                            2  .10000001E+01   2  .99999940E+00
                            3  .20000000E+01   3  .61238836E-08
                            4 -.10000000E+01
```

Example 5: A microwave circuit example

The design of a three-section 100-percent relative bandwidth 10:1 transmission-line transformer [10] is considered. In this case, we let the error function e_i be the modulus of the reflection coefficient samples at the 11 normalized frequencies (w.r.t. 1 GHz)

$\{0.5, 0.6, 0.7, 0.77, 0.9, 1.0, 1.1, 1.23, 1.3, 1.4, 1.5\}$.

Gradient vectors with respect to section lengths and characteristic impedances are obtained using the adjoint network method. Using 3rd order extrapolation and the reduction scheme with p = 8, 48, 288, 1728, we get a reflection coefficient magnitude of 0.19729 (optimal to 5 figures). The necessary effort required is summarized in Table II. A total of 495 network analyses were required, which was about 38% less than what would be required if the reduction scheme was not used. Note that the sample points are read from the main program and passed to subroutine FUNCT4 via a COMMON block named USER. At the end of each iteration of FLOPT4 the responses of the transformer at the local solution and the extrapolated solution are printed. In subroutine FUNCT4 the error functions and their gradients are obtained from the subroutine NET which defines the reflection coefficient of the transformer.

TABLE II COMPUTATIONAL EFFORT FOR THE TRANSFORMER PROBLEM

Parameter p	Function evaluations x	Number of error functions	=	Number of network analyses
8	29	11		319
48	16	4		64
288	16	4		64
1728	12	4		48
10368	10	4		40
Total	83		Total	535

```
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C     MAIN PROGRAM OF EXAMPLE 5
C
C     THIS IS A MINIMAX PROBLEM INVOLVING ELEVEN FUNCTIONS. LEAST PTH
C     APPROXIMATION, IN CONJUNCTION WITH EXTRAPOLATION AND THE REDUCTION
C     SCHEME, WILL BE USED HERE IN ORDER TO OBTAIN THE DESIRED SOLUTION.
C     FLOPT4 WILL HAVE TO BE CALLED AS MANY TIMES AS THE USER WISHES TO
C     UPDATE PARAMETER P
C
      DIMENSION GRAD(6)
      DIMENSION X(6),G(6),H(21),W(24),EPS(6),XB(6),XE(6,4,5),
     1          EN(11),JD(11),V(11)
C
      COMMON /USER/ WN(11)
      COMMON /1/ ID,IEX,IFINIS,IGK,IH,IK,IPT,IREDU,JORDER,JPRINT,M,MAX
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      IGK=0
      IK=5
      M=1
      N=6
      NR=11
      READ 4, (WN(I),I=1,NR)
      READ (5,*) P,FACTOR,EPS
C
      DO 3 IH=1,IK
      CALL FLOPT4 (EN,EPS,G,H,JD,V,W,X,XB,XE)
      PRINT 6
C
      DO 2 I=1,NA
      K=JD(I)
      S=WN(K)
C
      DO 1 J=1,N
1     W(J)=XE(J,1,IH)
C
      CALL NET (W,S,ARHO,ATNG,GRAD,0)
      CALL NET (XB,S,ARHO,ATNB,GRAD,0)
2     PRINT 5, K,S,K,ATNG,K,ATNB
C
      IF (IFINIS.EQ.N) CALL EXIT
3     M=0
C
      STOP
C
4     FORMAT (8F10.0)
C
5     FORMAT (6X,3(I4,E15.8))
C
6     FORMAT (*-RESPONSES OF THE THREE SECTION TRANSMISSION-LINE TRANSFO
     1RMER*/1X,60(*-*)/1H0,30X,*REFLECTION*,12X,*BEST*/12X,*FREQUENCY*,1
     20X,*COEFFICIENT*,8X,*REF. COEF.*/12X,*VECTOR S(I)*,8X,*VECTOR ATNG
     3(I)*,5X,*VECTOR ATNB(I)*/)
C
      END
```

```
      SUBROUTINE FUNCT4 (EN,G,JD,U,V,X)
C
C     A MICROWAVE CIRCUIT EXAMPLE
C
C     THE ROLE OF THIS SUBROUTINE IS TO RETURN THE VALUE OF THE OBJECT-
C     IVE FUNCTION, THE GRADIENT VECTOR AND THE MULTIPLIER VECTOR AT A
C     GIVEN POINT X. AS LEAST PTH APPROXIMATION WILL BE USED FOR THIS
C     PROBLEM, SUBROUTINE LEASTP4 WILL BE CALLED TO GENERATE THESE
C     QUANTITIES. HOWEVER, THE ERROR FUNCTIONS AND THEIR PARTIAL DERI-
C     VATIVES, WHICH ARE REQUIRED BY SUBROUTINE LEASTP4 AS AN INPUT,
C     MUST BE DEFINED HERE
C
      DIMENSION X(6),G(6),ER(11),GE(6,11),ES(11),EN(11),JD(11),V(11)
      DIMENSION GRAD(6)
C
      COMMON /USER/ WN(11)
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      DO 1 I=1,NA
      K=JD(I)
      CALL NET (X,WN(K),ARHO,ATN,GRAD,1)
      ER(K)=ATN
      DO 1 J=1,N
1     GE(J,K)=GRAD(J)
C
      CALL LEASTP4(EN,ER,ES,G,GE,JD,U,V)
      RETURN
      END
```

```
      SUBROUTINE NET (AX,S,ARHO,ATN,GRAD,IGRAD)
C
C     THREE SECTION 10 TO 1 TRANSFORMER
C
      COMPLEX A,B,C,D,VG,RHO,CJRHO,TVG,XIG
      COMPLEX XI(21),V(21),G(20)
C
      DIMENSION AX(1),GRAD(1),THETA(20),XL(20),Z(20)
C
      DATA XLQ,FACT/7.4948125,0.2095844728/
      BETA=FACT*S
      M=3
      MP1=M+1
C
      DO 1 I=1,M
      J=I+I
      XL(I)=XLQ*AX(J-1)
1     Z(I)=AX(J)
C
      RG=1.0
      RL=10.0
      XI(MP1)=CMPLX(1.0,0.0)
      V(MP1)=RL*XI(MP1)
C
      DO 2 J=1,M
      I=M+1-J
      IP1=I+1
      THETA(I)=BETA*XL(I)
      TH=THETA(I)
      CTH=COS(TH)
      STH=SIN(TH)
      A=CMPLX(CTH,0.)
      B=CMPLX(0.,(Z(I)*STH))
      C=CMPLX(0.,(STH/Z(I)))
      D=A
      V(I)=A*V(IP1)+B*XI(IP1)
2     XI(I)=C*V(IP1)+D*XI(IP1)
C
      XIG=-XI(1)
      VG=V(1)-XIG*RG
      RHO=1.+(RG+RG)*XIG/VG
      CJRHO=CONJG(RHO)
      AR=CJRHO*RHO
      ATN=SQRT(AR)
      IF (IGRAD.EQ.0) RETURN
      TVG=(RG+RG)/VG
C
      DO 3 I=1,M
      TH=THETA(I)
      J=I+I
      IP1=I+1
      G(J)=(V(I)*XI(I)-V(IP1)*XI(IP1))/(VG*Z(I))
      J1=J-1
3     G(J1)=BETA*(V(I)*XI(IP1)-V(IP1)*XI(I))/(VG*SIN(TH))*XLQ
C
      M2=M+M
C
      DO 4 I=1,M2
4     GRAD(I)=REAL(TVG*CJRHO*G(I))/ATN
C
      RETURN
      END
```

INPUT DATA FOR EXAMPLE 5

```
 .5       .6        .7        .77       .9        1.0       1.1       1. 23
 1.3      1.4       1.5
8.0,6.0,1.0E-8,1.0E-8,1.0E-8,1.0E-8,1.0E-8,1.0E-8
0.8,1.5,1.2,3.0,0.8,6.0
```

INPUT DATA FOR BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

NUMBER OF ERROR FUNCTIONS .... NR = 11

MAXIMUM ORDER OF EXTRAPOLATION .... JORDER = 3

INITIAL VALUE OF THE PARAMETER .... P = .80000000E+01

MULTIPLYING FACTOR FOR P .... FACTOR = .60000000E+01

TOLERANCE ETA FOR INACTIVE FUNCTIONS .... ETA = .50000000E-03

PREDICTED FUNCTION LOWER BOUND .... EST = 0.

STARTING POINT AND TEST QUANTITIES FOR THE TERMINATION CRITERION

	VARIABLE VECTOR X(I)		TEST VECTOR EPS(I)
1	.80000000E+00	1	.10000000E-07
2	.15000000E+01	2	.10000000E-07
3	.12000000E+01	3	.10000000E-07
4	.30000000E+01	4	.10000000E-07
5	.80000000E+00	5	.10000000E-07
6	.60000000E+01	6	.10000000E-07

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .38813233E+00

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.74908394E-02	1	.59179736E+00
2	.37189956E-06	2	.17145885E+00
3	.22114333E-01	3	.67754936E+00
4	.19015225E+00	4	.88663403E+00
5	.49790480E+00	5	.10000000E+01
6	.23236218E+00	6	.90913294E+00
7	.37282032E-01	7	.72325952E+00
8	.11045459E-02	8	.46585921E+00
9	.23727914E-03	9	.38438299E+00
10	.37926024E-03	10	.40759033E+00
11	.10972098E-01	11	.62071597E+00

ITERATION NO. 1 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .80000000E+01

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	1	.42348353E+00	1	.80000000E+00	1	-.41877277E+00
			2	.15000000E+01	2	.10293966E+00
			3	.12000000E+01	3	.53785205E+00
			4	.30000000E+01	4	.27081936E-01
			5	.80000000E+00	5	-.46087917E+00
			6	.60000000E+01	6	-.52074793E-01
10	13	.23717033E+00	1	.98853603E+00	1	.23454561E-02
			2	.15962692E+01	2	-.35983322E-02
			3	.10005957E+01	3	.16644382E-01
			4	.31016004E+01	4	-.15584970E-02
			5	.98771091E+00	5	.21035375E-02
			6	.60106821E+01	6	-.67063616E-02
20	23	.23663055E+00	1	.98827693E+00	1	-.46778728E-07
			2	.16286836E+01	2	.36413508E-07
			3	.10000448E+01	3	-.34872042E-06
			4	.31622779E+01	4	.10649350E-06
			5	.98827691E+00	5	-.12989557E-06
			6	.61399279E+01	6	-.56382467E-07
24	28	.23663055E+00	1	.98827692E+00	1	-.44795321E-10
			2	.16286836E+01	2	.16635330E-09
			3	.10000448E+01	3	.20152899E-09
			4	.31622777E+01	4	-.11886634E-09
			5	.98827692E+00	5	-.17833563E-09
			6	.61399279E+01	6	.76257461E-10

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS 1.584 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .21016566E+00

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.13524902E+00
3	.79604299E+00
4	.94283789E+00
5	.64637268E+00
6	.88930229E-01
7	.49284972E+00
8	.87151540E+00
9	.78133494E+00
10	.20960966E+00
11	.85713276E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .21016566E+00

MULTIPLIER NORM. ERROR

	VECTOR V(I)		VECTOR EN(I)
1	.94225439E+00	1	.10000000E+01
2	.18561179E-41	2	.13524902E+00
3	.16562323E-04	3	.79604299E+00
4	.55865731E-01	4	.94283789E+00
5	.75396630E-09	5	.64637268E+00
6	.33770685E-50	6	.88930229E-01
7	.16767363E-14	7	.49284972E+00
8	.12804757E-02	8	.87151540E+00
9	.67663745E-05	9	.78133494E+00
10	.25229611E-32	10	.20960966E+00
11	.57607189E-03	11	.85713276E+00

RESPONSES OF THE THREE SECTION TRANSMISSION-LINE TRANSFORMER

	FREQUENCY VECTOR S(I)		REFLECTION COEFFICIENT VECTOR ATNG(I)		BEST REF. COEF. VECTOR ATNB(I)
1	.50000000E+00	1	.21016566E+00	1	.21016566E+00
4	.77000000E+00	4	.19815215E+00	4	.19815215E+00
8	.12300000E+01	8	.18316261E+00	8	.18316261E+00
11	.15000000E+01	11	.18013987E+00	11	.18013987E+00

ITERATION NO. 2 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .48000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION	I	VARIABLE VECTOR X(I)	I	GRADIENT VECTOR G(I)
0	30	.21042625E+00	1	.98827692E+00	1	-.32166058E+00
			2	.16286836E+01	2	-.38213464E+00
			3	.10000448E+01	3	-.61530438E+00
			4	.31622777E+01	4	.47769859E-10
			5	.98827692E+00	5	-.32166058E+00
			6	.61399279E+01	6	.10136543E+00
10	42	.20273572E+00	1	.99833503E+00	1	-.32685447E-06
			2	.16347838E+01	2	.48595062E-06
			3	.99990690E+00	3	.24500227E-05
			4	.31622776E+01	4	.10425922E-06
			5	.99833425E+00	5	-.23459434E-05
			6	.61170151E+01	6	-.22673862E-06
13	46	.20273572E+00	1	.99833467E+00	1	.28301261E-07
			2	.16347840E+01	2	.67392060E-08
			3	.99990687E+00	3	-.21148905E-06
			4	.31622777E+01	4	.56464052E-08
			5	.99833466E+00	5	.85286616E-09
			6	.61170160E+01	6	-.20012528E-08

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .387 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19838094E+00

I	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
4	.99861498E+00
8	.98861852E+00
11	.97671613E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

I	VARIABLE VECTOR X(I) ORDER 1
1	.10003462E+01
2	.16360041E+01
3	.99987927E+00
4	.31622777E+01
5	.10003462E+01
6	.61124337E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19864119E+00

NORM. ERROR
VECTOR EN(I)

1	.98677189E+00
4	.99704574E+00
8	.10000000E+01
11	.98969127E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19822324E+00

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.37019862E-01	1	.99084419E+00
4	.41529741E+00	4	.99919658E+00
8	.52346662E+00	8	.10000000E+01
11	.24216101E-01	11	.98938502E+00

RESPONSES OF THE THREE SECTION TRANSMISSION-LINE TRANSFORMER

	FREQUENCY VECTOR S(I)		REFLECTION COEFFICIENT VECTOR ATNG(I)		BEST REF. COEF. VECTOR ATNB(I)
1	.50000000E+00	1	.19838094E+00	1	.19601354E+00
4	.77000000E+00	4	.19810618E+00	4	.19805435E+00
8	.12300000E+01	8	.19612307E+00	8	.19864119E+00
11	.15000000E+01	11	.19376186E+00	11	.19659345E+00

ITERATION NO. 3 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .28800000E+03

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	49	.19866925E+00	1	.10000110E+01	1	.11410896E+00
			2	.16358008E+01	2	.33121793E+00
			3	.99988387E+00	3	-.17845789E+00
			4	.31622777E+01	4	.24854624E-05
			5	.10000110E+01	5	.11410546E+00
			6	.61131974E+01	6	-.88630838E-01
10	64	.19818901E+00	1	.99972821E+00	1	.10907595E-07
			2	.16347196E+01	2	-.14811164E-07
			3	.99998454E+00	3	-.14886033E-07
			4	.31622776E+01	4	-.28451078E-08
			5	.99972821E+00	5	.93407877E-08
			6	.61172570E+01	6	.46142574E-08

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .358 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19746869E+00

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
4	.99977416E+00
8	.99814134E+00
11	.99617447E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2
1	.10000069E+01	1	.99999722E+00
2	.16347067E+01	2	.16346696E+01
3	.10000001E+01	3	.10000035E+01
4	.31622776E+01	4	.31622776E+01
5	.10000069E+01	5	.99999722E+00
6	.61173052E+01	6	.61174444E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19732510E+00

	NORM. ERROR VECTOR EN(I)
1	.99998803E+00
4	.99969572E+00
8	.99965559E+00
11	.10000000E+01

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19734622E+00

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.42928032E+00	1	.10000000E+01
4	.26721784E+00	4	.99972571E+00
8	.15807680E+00	8	.99942203E+00
11	.14542503E+00	11	.99937378E+00

RESPONSES OF THE THREE SECTION TRANSMISSION-LINE TRANSFORMER

	FREQUENCY VECTOR S(I)		REFLECTION COEFFICIENT VECTOR ATNG(I)		BEST REF. COEF. VECTOR ATNB(I)
1	.50000000E+00	1	.19746869E+00	1	.19732274E+00
4	.77000000E+00	4	.19742409E+00	4	.19726506E+00
8	.12300000E+01	8	.19710166E+00	8	.19725714E+00
11	.15000000E+01	11	.19671326E+00	11	.19732510E+00

ITERATION NO. 4 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .17280000E+04

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	67	.19744282E+00	1	.99995261E+00	1	-.11365723E-02
			2	.16346788E+01	2	-.88272695E-01
			3	.10000003E+01	3	.12076778E-01
			4	.31622776E+01	4	-.25900216E-07
			5	.99995261E+00	5	-.11365723E-02
			6	.61174099E+01	6	.23588025E-01
8	78	.19744011E+00	1	.99995486E+00	1	-.57740310E-08
			2	.16347092E+01	2	.10678849E-07
			3	.99999743E+00	3	-.20623997E-07
			4	.31622777E+01	4	-.45971076E-10
			5	.99995486E+00	5	-.58294124E-08
			6	.61172959E+01	6	-.28309100E-08

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .272 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19732020E+00

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
4	.99996250E+00
8	.99969126E+00
11	.99936393E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.10000002E+01	1	.10000000E+01	1	.10000000E+01
2	.16347071E+01	2	.16347071E+01	2	.16347073E+01
3	.10000000E+01	3	.10000000E+01	3	.99999998E+00
4	.31622777E+01	4	.31622777E+01	4	.31622777E+01
5	.10000002E+01	5	.10000000E+01	5	.10000000E+01
6	.61173037E+01	6	.61173037E+01	6	.61173030E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19729078E+00

	NORM. ERROR VECTOR EN(I)
1	.99999848E+00
4	.99999984E+00
8	.10000000E+01
11	.99999837E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19729544E+00

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.34780374E+00	1	.10000000E+01
4	.32971603E+00	4	.99999485E+00
8	.20666821E+00	8	.99994980E+00
11	.11581203E+00	11	.99989394E+00

RESPONSES OF THE THREE SECTION TRANSMISSION-LINE TRANSFORMER

	FREQUENCY VECTOR S(I)		REFLECTION COEFFICIENT VECTOR ATNG(I)		BEST REF. COEF. VECTOR ATNB(I)
1	.50000000E+00	1	.19732020E+00	1	.19729048E+00
4	.77000000E+00	4	.19731280E+00	4	.19729075E+00
8	.12300000E+01	8	.19725928E+00	8	.19729078E+00
11	.15000000E+01	11	.19719469E+00	11	.19729046E+00

ITERATION NO. 5 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .10368000E+05

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	81	.19731553E+00	1	.99999249E+00	1	.20270835E-05
			2	.16347076E+01	2	.24585231E-02
			3	.99999956E+00	3	-.42860671E-03
			4	.31622777E+01	4	.68666988E-09
			5	.99999249E+00	5	.20271895E-05
			6	.61173019E+01	6	-.65698372E-03
4	89	.19731553E+00	1	.99999248E+00	1	.18332037E-06
			2	.16347075E+01	2	.64976376E-07
			3	.99999957E+00	3	.40349677E-06
			4	.31622776E+01	4	-.53514491E-08
			5	.99999248E+00	5	.18027738E-06
			6	.61173024E+01	6	-.17572634E-07

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .196 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19729555E+00

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
4	.99999375E+00
8	.99994857E+00
11	.99989403E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.10000000E+01	1	.10000000E+01	1	.10000000E+01
2	.16347071E+01	2	.16347071E+01	2	.16347071E+01
3	.10000000E+01	3	.10000000E+01	3	.10000000E+01
4	.31622776E+01	4	.31622776E+01	4	.31622776E+01
5	.10000000E+01	5	.10000000E+01	5	.10000000E+01
6	.61173036E+01	6	.61173036E+01	6	.61173036E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19729063E+00

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
4	.10000000E+01
8	.10000000E+01
11	.10000000E+01

RESPONSES OF THE THREE SECTION TRANSMISSION-LINE TRANSFORMER

	FREQUENCY VECTOR S(I)		REFLECTION COEFFICIENT VECTOR ATNG(I)		BEST REF. COEF. VECTOR ATNB(I)
1	.50000000E+00	1	.19729555E+00	1	.19729063E+00
4	.77000000E+00	4	.19729432E+00	4	.19729063E+00
8	.12300000E+01	8	.19728541E+00	8	.19729063E+00
11	.15000000E+01	11	.19727465E+00	11	.19729063E+00

Example 6: An unconstrained minimization problem

This example has been presented here to illustrate the kind of output produced by FLOPT4 when the gradients are checked at the starting point (IGK=1) and found to be incorrect.

The problem of Example 1 has been repeated here with an erroneous definition of the gradient vector.

```
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C     MAIN PROGRAM OF EXAMPLE 6
C
C     THE PROBLEM OF EXAMPLE 1 HAS BEEN REPEATED HERE WITH THE ONLY EX-
C     CEPTION THAT THE GRADIENT VECTOR HAS BEEN DELIBERATELY DEFINED
C     WRONG
      DIMENSION X(2),G(2),H(3),W(8),EPS(2),XB(1),XE(1),V(1),EN(1),JD(1)
C
      COMMON /1/ ID,IEX,IFINIS,IGK,IH,IK,IPT,IREDU,JORDER,JPRINT,M,MAX
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      IREDU=0
      READ (5,*) N,M,IEX,EPS(1),EPS(2)
      CALL FLOPT4 (EN,EPS,G,H,JD,V,W,X,XB,XE)
      STOP
      END
```

```
      SUBROUTINE FUNCT4 (EN,G,JD,U,V,X)
C
C     ROSENBROCK"S FUNCTION
C
C     THE ROLE OF THIS SUBROUTINE IS TO RETURN THE VALUE OF THE OBJECT-
C     IVE FUNCTION AND THE GRADIENT VECTOR AT A GIVEN POINT X. AS LEAST
C     PTH APPROXIMATION WILL NOT BE USED FOR THIS PROBLEM, SUBROUTINE
C     LEASTP4 NEED NOT BE CALLED AND, THEREFORE, THE OBJECTIVE FUNCTION
C     AND THE GRADIENT VECTOR MUST BE DEFINED HERE. NOTICE THAT THE
C     DEFINITION OF THE GRADIENT VECTOR HERE IS NOT CORRECT
C
      DIMENSION X(2), G(2), EN(1), JD(1), V(1)
C
      A=X(1)*X(1)
      B=A-X(2)
      C=1.0-X(1)
C
C     THE OBJECTIVE FUNCTION IS DEFINED HERE
C
      U=100.*B*B+C*C
C
C     THE GRADIENT VECTOR IS WRONGLY DEFINED HERE
C
      G(1)=40.*X(1)*(A-X(2))-C
      G(2)=20.*B
      RETURN
      END
```

INPUT DATA FOR EXAMPLE 6

```
2,1,0,1.0E-8,1.0E-8
-1.2,1.0
```

INPUT DATA FOR BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

NUMBER OF ERROR FUNCTIONS .... NR = 1

PREDICTED FUNCTION LOWER BOUND .... EST = 0.

STARTING POINT AND TEST QUANTITIES FOR THE TERMINATION CRITERION

	VARIABLE VECTOR X(I)		TEST VECTOR EPS(I)
1	-.12000000E+01	1	.10000000E-07
2	.10000000E+01	2	.10000000E-07

GRADIENT CHECK AT THE STARTING POINT

	ANALYTICAL GRADIENT VECTOR G(I)		NUMERICAL GRADIENT VECTOR G(I)		PERCENTAGE ERROR VECTOR YP(I)
1	-.23320000E+02	1	-.21560000E+03	1	.89183673E+02
2	.88000000E+01	2	-.88000000E+02	2	.11000000E+03

YOUR PROGRAM HAS BEEN TERMINATED BECAUSE GRADIENTS ARE INCORRECT
PLEASE CHECK THEM AGAIN

Example 7: A minimax problem

The problem of Example 3 is repeated here to illustrate how it is possible to use the reduction scheme (with least pth approximation) and yet be safeguarded against the undesirable possibility of being left with only one "active" function.

One way to achieve this is the following:

```
      MAIN PROGRAM
      ......
      M = 1
      DO 3  IH = 1,  IK
      CALL FLOPT4 (.....)
1     IF (NA.GT.1) GO TO 3
      NA = NR
      ETA = ETA/2
      JV = 1
      CALL FUNCT4 (.....)
      DO 2  J = 1,  NA
2     JD(J) = J
      CALL AGAIN (.....)
      GO TO 1
3     M = 0
      ......
      ......
```

A simpler scheme has been used for this example. The chosen initial value of ETA is 1.0 which is, obviously, very unreasonable.

```
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C     MAIN PROGRAM OF EXAMPLE 7
C
C     THE PROBLEM OF EXAMPLE 3 HAS BEEN REPEATED HERE WITH THE EXCEPTION
C     THAT THE CHOICE OF ETA IS UNREASONABLE MAKING IT DIFFICULT TO SE-
C     LECT ACTIVE FUNCTIONS. ONE WAY TO HANDLE THIS PROBLEM IS PRESENT-
C     ED HERE
C
      DIMENSION X(2),G(2),H(3),W(8),EPS(2),XB(2),XE(40),V(3),EN(3),JD(3)
C
      COMMON /1/ ID,IEX,IFINIS,IGK,IH,IK,IPT,IREDU,JORDER,JPRINT,M,MAX
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      N=2
      NR=3
      M=1
      IGK=0
      IK=5
      P=4.
      FACTOR=4.
      EPS(1)=1.E-8
      EPS(2)=1.E-8
      ETA=1.
C
      DO 3 IH=1,IK
      CALL FLOPT4 (EN,EPS,G,H,JD,V,W,X,XB,XE)
      IF(IFINIS.EQ.N) CALL EXIT
    1 IF (NA.GT.1) GO TO 3
      NA=NR
      DO 2 J=1,NA
    2 JD(J)=J
      ETA=ETA/2.
      CALL AGAIN (EN,EPS,G,H,JD,V,W,X,XB,XE)
      GO TO 1
    3 M=0
C
      STOP
      END
```

```
      SUBROUTINE FUNCT4 (EN,G,JD,U,V,X)
C
C     A MINIMAX EXAMPLE
C
C     THE ROLE OF THIS SUBROUTINE IS TO RETURN THE VALUE OF THE OBJECT-
C     IVE FUNCTION, THE GRADIENT VECTOR AND THE MULTIPLIER VECTOR AT A
C     GIVEN POINT X. AS LEAST PTH APPROXIMATION WILL BE USED FOR THIS
C     PROBLEM, SUBROUTINE LEASTP4 WILL BE CALLED TO GENERATE THESE
C     QUANTITIES. HOWEVER, THE ERROR FUNCTIONS AND THEIR PARTIAL DERI-
C     VATIVES, WHICH ARE REQUIRED BY SUBROUTINE LEASTP4 AS AN INPUT,
C     MUST BE DEFINED HERE
C
      DIMENSION X(2),G(2),ER(3),GE(2,3),ES(3),EN(3),JD(3),V(3)
C
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      Y1=X(1)*X(1)
      Y2=X(2)*X(2)
      Y3=X(1)+X(1)
      Y4=X(2)+X(2)
C
      DO 4 I=1,NA
      K=JD(I)
      GO TO (1,2,3) ,K
1     ER(1)=Y1+Y2*Y2
      GE(1,1)=Y3
      GE(2,1)=(Y2+Y2)*Y4
      GO TO 4
2     ER(2)=8.-4.*(X(1)+X(2))+Y1+Y2
      GE(1,2)=-4.+Y3
      GE(2,2)=-4.+Y4
      GO TO 4
3     ER(3)=2.*EXP(-X(1)+X(2))
      GE(1,3)=-ER(3)
      GE(2,3)=ER(3)
4     CONTINUE
C
      CALL LEASTP4 (EN,ER,ES,G,GE,JD,U,V)
      RETURN
      END
```

INPUT DATA FOR EXAMPLE 7

```
2.0,2.0
```

INPUT DATA FOR BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

NUMBER OF ERROR FUNCTIONS .... NR = 3

MAXIMUM ORDER OF EXTRAPOLATION .... JORDER = 3

INITIAL VALUE OF THE PARAMETER .... P = .40000000E+01

MULTIPLYING FACTOR FOR P .... FACTOR = .40000000E+01

TOLERANCE ETA FOR INACTIVE FUNCTIONS .... ETA = .10000000E+01

PREDICTED FUNCTION LOWER BOUND .... EST = 0.

STARTING POINT AND TEST QUANTITIES FOR THE TERMINATION CRITERION

	VARIABLE VECTOR X(I)		TEST VECTOR EPS(I)
1	.20000000E+01	1	.10000000E-07
2	.20000000E+01	2	.10000000E-07

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .20000000E+02

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.99990001E+00	1	.10000000E+01
2	0.	2	0.
3	.99990001E-04	3	.10000000E+00

```
ITERATION NO.    1 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)
---------------------------------------------------------------------
WITH PARAMETER P =   .40000000E+01
---------------------------------

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD
-----------------------------------------------------------------

ITER. FUNC. OBJECTIVE            VARIABLE             GRADIENT
 NO.  EVAL. FUNCTION            VECTOR X(I)          VECTOR G(I)

  0     1   .20000500E+02   1  .20000000E+01    1  .39977002E+01
                            2  .20000000E+01    2  .31999600E+02


 10    13   .24033042E+01   1  .12008090E+01    1 -.24628529E-07
                            2  .82623537E+00    2  .54009733E-07

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS    .043  SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE
----------------------------------------------------------

MAXIMUM OF THE ERROR FUNCTIONS ........... EM =  .20164297E+01

                                                  NORM. ERROR
                                                 VECTOR EN(I)

                                              1  .94621380E+00
                                              2  .10000000E+01
                                              3  .68198003E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE
--------------------------------------------------------------
NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)
-----------------------------------------------------------------

MAXIMUM OF THE ERROR FUNCTIONS ........... EM =  .20164297E+01

                               MULTIPLIER         NORM. ERROR
                              VECTOR V(I)        VECTOR EN(I)

                            1  .29177586E+00   1  .94621380E+00
                            2  .70667687E+00   2  .10000000E+01
                            3  .15472745E-02   3  .68198003E+00
```

ITERATION NO. 2 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .16000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	15	.20606621E+01	1	.12008090E+01	1	-.40069412E+00
			2	.82623537E+00	2	-.98117056E+00
8	25	.20373916E+01	1	.11336182E+01	1	.82721582E-07
			2	.89695971E+00	2	.37807499E-07

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .032 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19673154E+01

	NORM. ERROR VECTOR EN(I)
1	.98223674E+00
2	.10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

VARIABLE
VECTOR X(I)

ORDER 1

1	.11112212E+01
2	.92053449E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19551735E+01

	NORM. ERROR VECTOR EN(I)
1	.99882273E+00
2	.10000000E+01

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19580107E+01

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.41180088E+00	1	.99444478E+00
2	.58819912E+00	2	.10000000E+01

ITERATION NO. 3 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .64000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	28	.19743141E+01	1	.11168204E+01	1	-.11496663E+00
			2	.91464079E+00	2	-.94795243E-02
6	35	.19731820E+01	1	.11377074E+01	1	-.77947626E-07
			2	.89892092E+00	2	-.11059698E-06

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .026 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19559236E+01

	NORM. ERROR VECTOR EN(I)
1	.99561005E+00
2	.10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2
1	.11390705E+01	1	.11409271E+01
2	.89957465E+00	2	.89817733E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19525159E+01

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.99974569E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19529988E+01

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.44163372E+00	1	.99908427E+00
2	.55836628E+00	2	.10000000E+01

ITERATION NO. 4 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .25600000E+03

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	38	.19574495E+01	1	.11400352E+01	1	.47636500E-01
			2	.89842873E+00	2	.52203107E-01
5	44	.19574433E+01	1	.11387066E+01	1	-.48672922E-09
			2	.89940086E+00	2	-.55564193E-09

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .023 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19531448E+01

	NORM. ERROR VECTOR EN(I)
1	.99890565E+00
2	.10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.11390397E+01	1	.11390376E+01	1	.11390076E+01
2	.89956084E+00	2	.89955992E+00	2	.89958186E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19522280E+01

	NORM. ERROR VECTOR EN(I)
1	.99999586E+00
2	.10000000E+01

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19524567E+01

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.42973566E+00	1	.99972374E+00
2	.57026434E+00	2	.10000000E+01

ITERATION NO. 5 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .10240000E+04

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	47	.19535279E+01	1	.11389342E+01	1	-.29200651E-02
			2	.89953539E+00	2	-.35981829E-02
4	52	.19535279E+01	1	.11389550E+01	1	-.22391426E-11
			2	.89952021E+00	2	-.28748486E-11

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .022 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19524543E+01

	NORM. ERROR VECTOR EN(I)
1	.99972661E+00
2	.10000000E+01

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.11390378E+01	1	.11390377E+01	1	.11390377E+01
2	.89955999E+00	2	.89955994E+00	2	.89955994E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .19522245E+01

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.10000000E+01

Example 8: The Beale problem [8]

This problem has been solved in Example 2 using the SUMT method of Fiacco and McCormick. In this example, the conversion of the constrained problem into an unconstrained minimax objective has been carried out using the Bandler-Charalambous technique. 31 necessary function evaluations are required here as compared to 33 in Example 2, to get the same accuracy in the optimal solution.

This problem has also been solved by the program DISOPT3 [12], which utilizes the Charalambous algorithm [13].

```
      PROGRAM TST(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C     MAIN PROGRAM FOR EXAMPLE 8
C
      DIMENSION X(3), G(3), H(6), W(12), EPS(3), XB(3), XE(3,4,5)
      DIMENSION EN(5), JD(5), V(5)
C
      COMMON /1/ ID,IEX,IFINIS,ICK,IH,IK,IPT,IREDU,JORDER,JPRINT,M,MAX
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      DATA EPS/3*.1E-6/,P/10./
C
      ETA=.1E-3
      IK=5
      M=1
      N=3
      NR=5
C
      DO 1 IH=1,IK
      CALL FLOPT4 (EN,EPS,G,H,JD,V,W,X,XB,XE)
      M=0
      IF (IFINIS.EQ.N) CALL EXIT
1     CONTINUE
C
      STOP
      END
```

```
      SUBROUTINE FUNCT4 (EN,G,JD,U,V,X)
C
C     THE BEALE PROBLEM
C
      DIMENSION CONS(5), GCONS(3,5), X(3)
      DIMENSION G(3), ER(5), GE(3,5), ES(5), EN(5), JD(5), V(5)
C
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      DATA AL/1./
C
1     ER(1)=9.-8.*X(1)-6.*X(2)-4.*X(3)+2.*(X(1)**2+X(2)**2)+X(3)**2+2.*X
     1(1)*(X(2)+X(3))
      GE(1,1)=-8.+4.*X(1)+2.*(X(2)+X(3))
      GE(2,1)=-6.+4.*X(2)+2.*X(1)
      GE(3,1)=-4.+2.*X(3)+2.*X(1)
C
      DO 10 I=1,NA
      J=JD(I)
      GO TO (10,2,4,6,8,10), J
C
2     CONS(2)=X(1)
      GCONS(1,2)=1.
      GCONS(2,2)=0.
      GCONS(3,2)=0.
      ER(2)=ER(1)-AL*CONS(2)
      DO 3 IJ=1,3
      GE(IJ,2)=GE(IJ,1)-AL*GCONS(IJ,2)
3     CONTINUE
      GO TO 10
C
4     CONS(3)=X(2)
      GCONS(1,3)=0.
      GCONS(2,3)=1.
      GCONS(3,3)=0.
      ER(3)=ER(1)-AL*CONS(3)
      DO 5 IJ=1,3
      GE(IJ,3)=GE(IJ,1)-AL*GCONS(IJ,3)
5     CONTINUE
      GO TO 10
C
6     CONS(4)=X(3)
      GCONS(1,4)=0.
      GCONS(2,4)=0.
      GCONS(3,4)=1.
      ER(4)=ER(1)-AL*CONS(4)
      DO 7 IJ=1,3
      GE(IJ,4)=GE(IJ,1)-AL*GCONS(IJ,4)
7     CONTINUE
      GO TO 10
C
8     CONS(5)=3.-X(1)-X(2)-2.*X(3)
      GCONS(1,5)=-1.
      GCONS(2,5)=-1.
      GCONS(3,5)=-2.
      ER(5)=ER(1)-AL*CONS(5)
      DO 9 IJ=1,3
      GE(IJ,5)=GE(IJ,1)-AL*GCONS(IJ,5)
9     CONTINUE
C
10    CONTINUE
C
      CALL LEASTP4 (EN,ER,ES,G,GE,JD,U,V)
      RETURN
      END
```

INPUT DATA FOR BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

NUMBER OF ERROR FUNCTIONS .... NR = 5

MAXIMUM ORDER OF EXTRAPOLATION .... JORDER = 3

INITIAL VALUE OF THE PARAMETER .... P = .10000000E+02

MULTIPLYING FACTOR FOR P .... FACTOR = .20000000E+01

TOLERANCE ETA FOR INACTIVE FUNCTIONS .... ETA = .10000000E-03

PREDICTED FUNCTION LOWER BOUND .... EST = 0.

STARTING POINT AND TEST QUANTITIES FOR THE TERMINATION CRITERION

	VARIABLE VECTOR X(I)		TEST VECTOR EPS(I)
1	.50000000E+00	1	.10000000E-06
2	.50000000E+00	2	.10000000E-06
3	.50000000E+00	3	.10000000E-06

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .22500000E+01

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.80267124E+00	1	.10000000E+01
2	.65026893E-01	2	.77777778E+00
3	.65026893E-01	3	.77777778E+00
4	.65026893E-01	4	.77777778E+00
5	.22480846E-02	5	.55555556E+00

GRADIENT CHECK AT THE STARTING POINT

	ANALYTICAL GRADIENT VECTOR G(I)		NUMERICAL GRADIENT VECTOR G(I)		PERCENTAGE ERROR VECTOR YP(I)
1	-.44054831E+01	1	-.44054831E+01	1	.29421663E-06
2	-.33244442E+01	2	-.33244443E+01	2	.78415396E-06
3	-.22392689E+01	3	-.22392689E+01	3	.76857088E-06

GRADIENTS ARE O.K.

ITERATION NO. 1 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .10000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	1	.23000048E+01	1	.50000000E+00	1	-.44054831E+01
			2	.50000000E+00	2	-.33244442E+01
			3	.50000000E+00	3	-.22392689E+01
10	13	.11700904E+00	1	.13382190E+01	1	.75654061E-06
			2	.77452060E+00	2	.35536738E-06
			3	.43630180E+00	3	.15634088E-05
11	14	.11700904E+00	1	.13382190E+01	1	.76119008E-08
			2	.77452070E+00	2	-.10787137E-07
			3	.43630175E+00	3	.17278964E-07

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .064 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .11439206E+00

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	-.10698530E+02
3	-.57707558E+01
4	-.28140913E+01
5	.87187180E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .11439206E+00

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.93947461E+00	1	.10000000E+01
2	0.	2	-.10698530E+02
3	0.	3	-.57707558E+01
4	0.	4	-.28140913E+01
5	.60525393E-01	5	.87187180E+00

ITERATION NO. 2 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .20000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	16	.11474972E+00	1	.13382190E+01	1	-.15855900E+00
			2	.77452070E+00	2	-.15855902E+00
			3	.43630175E+00	3	-.31711801E+00
6	22	.11405888E+00	1	.13357135E+01	1	-.77302187E-09
			2	.77619099E+00	2	.25279292E-08
			3	.44047748E+00	3	-.18778336E-08

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .033 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .11270356E+00

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
5	.93664314E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I)
	ORDER 1
1	.13332081E+01
2	.77786129E+00
3	.44465320E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .11140339E+00

	NORM. ERROR VECTOR EN(I)
1	.99662702E+00
5	.10000000E+01

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .11186402E+00

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.77349069E+00	1	.10000000E+01
5	.22650931E+00	5	.96976337E+00

ITERATION NO. 3 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .40000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	25	.11258462E+00	1	.13344608E+01	1	.90812539E-02
			2	.77702614E+00	2	.90812677E-02
			3	.44256534E+00	3	.18162506E-01
3	29	.11258398E+00	1	.13345082E+01	1	.92374498E-08
			2	.77699451E+00	2	-.46070481E-07
			3	.44248632E+00	3	.23991204E-07

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .026 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .11189574E+00

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
5	.96850070E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2
1	.13333029E+01	1	.13333345E+01
2	.77779803E+00	2	.77777694E+00
3	.44449516E+00	3	.44444248E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .11111190E+00

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
5	.99996794E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .11150084E+00

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.78026152E+00	1	.10000000E+01
5	.21973848E+00	5	.98428490E+00

```
ITERATION NO.    4 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)
---------------------------------------------------------------------
WITH PARAMETER P =   .80000000E+02
---------------------------------

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD
------------------------------------------------------------------

ITER. FUNC. OBJECTIVE           VARIABLE            GRADIENT
 NO.  EVAL. FUNCTION           VECTOR X(I)         VECTOR G(I)

  0    32  .11184721E+00   1  .13339174E+01    1 -.14622176E-03
                           2  .77738836E+00    2 -.14632295E-03
                           3  .44347098E+00    3 -.29243340E-03


  2    35  .11184721E+00   1  .13339170E+01    1 -.21506698E-08
                           2  .77738864E+00    2 -.51633387E-07
                           3  .44347163E+00    3  .64697580E-09

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS    .021  SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE
----------------------------------------------------------

MAXIMUM OF THE ERROR FUNCTIONS ........... EM =  .11150058E+00

                                                  NORM. ERROR
                                                 VECTOR EN(I)

                                               1  .10000000E+01
                                               5  .98429535E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION
-----------------------------------------------------------------

              VARIABLE            VARIABLE            VARIABLE
             VECTOR X(I)         VECTOR X(I)         VECTOR X(I)

              ORDER 1             ORDER 2             ORDER 3

          1  .13333258E+01   1  .13333335E+01   1  .13333333E+01
          2  .77778276E+00   2  .77777767E+00   2  .77777778E+00
          3  .44445694E+00   3  .44444420E+00   3  .44444444E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS ........... EM =  .11111111E+00

                                                  NORM. ERROR
                                                 VECTOR EN(I)

                                               1  .10000000E+01
                                               5  .99999998E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE
---------------------------------------------------------------
NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)
------------------------------------------------------------------

MAXIMUM OF THE ERROR FUNCTIONS ........... EM =  .11130514E+00

                                MULTIPLIER          NORM. ERROR
                               VECTOR V(I)         VECTOR EN(I)

                           1  .77894247E+00    1  .10000000E+01
                           5  .22105753E+00    5  .99215894E+00
```

ITERATION NO. 5 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .16000000E+03

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	38	.11147906E+00	1	.13336243E+01	1	-.17491354E-06
			2	.77758382E+00	2	-.20645937E-06
			3	.44395959E+00	3	-.34666013E-06
1	39	.11147906E+00	1	.13336243E+01	1	-.17491354E-06
			2	.77758382E+00	2	-.20645937E-06
			3	.44395959E+00	3	-.34666013E-06

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .018 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .11130514E+00

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
5	.99215894E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.13333315E+01	1	.13333334E+01	1	.13333333E+01
2	.77777901E+00	2	.77777776E+00	2	.77777778E+00
3	.44444754E+00	3	.44444441E+00	3	.44444444E+00

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .11111111E+00

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
5	.99999998E+00

Example 9: The Wong problem 1 [12, 13]

Minimize

$$f = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10\,x_5^6 + 7\,x_6^2 + x_7^4 - 4\,x_6\,x_7 - 10\,x_6 - 8\,x_7$$

subject to

$$-2\,x_1^2 - 3\,x_2^4 - x_3 - 4\,x_4^2 - 5\,x_5 + 127 \geq 0$$

$$-7\,x_1 - 3\,x_2 - 10\,x_3^2 - x_4 + x_5 + 282 \geq 0$$

$$-23\,x_1 - x_2^2 - 6\,x_6^2 + 8\,x_7 + 196 \geq 0$$

$$-4\,x_1^2 - x_2^2 + 3\,x_1\,x_2 - 2\,x_3^2 - 5\,x_6 + 11\,x_7 \geq 0\,.$$

The optimal solution found in 94 necessary function evaluations is

$$f = 680.63006$$

$x_1 = 2.3304993$ $\quad$ $x_2 = 1.9513724$

$x_3 = -0.47754129$ $\quad$ $x_4 = 4.3657262$

$x_5 = -0.62448697$ $\quad$ $x_6 = 1.0381310$

$x_7 = 1.5942267$.

This problem has also been solved by the program DISOPT3 [12] using the Charalambous algorithm [13].

```
      PROGRAM TST(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C     MAIN PROGRAM FOR EXAMPLE 9
C
      DIMENSION X(7), G(7), H(28), W(28), EPS(7), XB(7), XE(7,4,15)
      DIMENSION EN(5), JD(5), V(5)
C
      COMMON /1/ ID,IEX,IFINIS,IGK,IH,IK,IPT,IREDU,JORDER,JPRINT,M,MAX
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      DATA EPS/7*.1E-6/
C
      ETA=.1E-3
      IK=15
      M=1
      N=7
      NR=5
      P=10.
C
      DO 1 IH=1,IK
      CALL FLOPT4 (EN,EPS,G,H,JD,V,W,X,XB,XE)
      M=0
      IF (IFINIS.EQ.N) CALL EXIT
1     CONTINUE
C
      STOP
      END
```

```
      SUBROUTINE FUNCT4 (EN,G,JD,U,V,X)
C
C     THE FIRST WONG PROBLEM
C
      DIMENSION CONS(5), GCONS(7,5), X(7)
      DIMENSION G(7), ER(5), GE(7,5), ES(5), EN(5), JD(5), V(5)
C
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      DATA AL/10./
C
1     ER(1)=(X(1)-10.)**2+5.*(X(2)-12.)**2+X(3)**4+3.*(X(4)-11.)**2+10.*
     1X(5)**6+7.*X(6)**2+X(7)**4-4.*X(6)*X(7)-10.*X(6)-8.*X(7)
      GE(1,1)=2.*(X(1)-10.)
      GE(2,1)=10.*(X(2)-12.)
      GE(3,1)=4.*X(3)**3
      GE(4,1)=6.*(X(4)-11.)
      GE(5,1)=60.*X(5)**5
      GE(6,1)=14.*X(6)-4.*X(7)-10.
      GE(7,1)=4.*X(7)**3-4.*X(6)-8.
C
      DO 10 I=1,NA
      J=JD(I)
      GO TO (10,2,4,6,8,10), J
C
2     CONS(2)=-2.*X(1)**2-3.*X(2)**4-X(3)-4.*X(4)**2-5.*X(5)+127.
      GCONS(1,2)=-4.*X(1)
      GCONS(2,2)=-12.*X(2)**3
      GCONS(3,2)=-1.
      GCONS(4,2)=-8.*X(4)
      GCONS(5,2)=-5.
      GCONS(6,2)=0.
      GCONS(7,2)=0.
      ER(2)=ER(1)-AL*CONS(2)
      DO 3 IJ=1,7
      GE(IJ,2)=GE(IJ,1)-AL*GCONS(IJ,2)
3     CONTINUE
      GO TO 10
C
4     CONS(3)=-7.*X(1)-3.*X(2)-10.*X(3)**2-X(4)+X(5)+282.
      GCONS(1,3)=-7.
      GCONS(2,3)=-3.
      GCONS(3,3)=-20.*X(3)
      GCONS(4,3)=-1.
      GCONS(5,3)=1,
      GCONS(6,3)=0.
      GCONS(7,3)=0.
      ER(3)=ER(1)-AL*CONS(3)
      DO 5 IJ=1,7
      GE(IJ,3)=GE(IJ,1)-AL*GCONS(IJ,3)
5     CONTINUE
      GO TO 10
C
6     CONS(4)=-23.*X(1)-X(2)**2-6.*X(6)**2+8.*X(7)+196.
      GCONS(1,4)=-23.
      GCONS(2,4)=-2.*X(2)
      GCONS(3,4)=0.
      GCONS(4,4)=0.
      GCONS(5,4)=0.
      GCONS(6,4)=-12.*X(6)
      GCONS(7,4)=8.
      ER(4)=ER(1)-AL*CONS(4)
      DO 7 IJ=1,7
      GE(IJ,4)=GE(IJ,1)-AL*GCONS(IJ,4)
7     CONTINUE
      GO TO 10
C
8     CONS(5)=-4.*X(1)**2-X(2)**2+3.*X(1)*X(2)-2.*X(3)**2-5.*X(6)+11.*X(
     17)
      GCONS(1,5)=-8.*X(1)+3.*X(2)
      GCONS(2,5)=-2.*X(2)+3.*X(1)
      GCONS(3,5)=-4.*X(3)
      GCONS(4,5)=0.
```

```
      GCONS(5,5)=0.
      GCONS(6,5)=-5.
      GCONS(7,5)=11.
      ER(5)=ER(1)-AL*CONS(5)
      DO 9  IJ=1,7
      GE(IJ,5)=GE(IJ,1)-AL*GCONS(IJ,5)
9     CONTINUE
C
10    CONTINUE
C
      CALL LEASTP4 (EN,ER,ES,G,GE,JD,U,V)
      RETURN
      END
```

INPUT DATA FOR BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

NUMBER OF ERROR FUNCTIONS .... NR = 5

MAXIMUM ORDER OF EXTRAPOLATION .... JORDER = 3

INITIAL VALUE OF THE PARAMETER .... P = .10000000E+02

MULTIPLYING FACTOR FOR P .... FACTOR = .20000000E+01

TOLERANCE ETA FOR INACTIVE FUNCTIONS .... ETA = .10000000E-03

PREDICTED FUNCTION LOWER BOUND .... EST = 0.

STARTING POINT AND TEST QUANTITIES FOR THE TERMINATION CRITERION

	VARIABLE VECTOR X(I)		TEST VECTOR EPS(I)
1	.10000000E+01	1	.10000000E-06
2	.20000000E+01	2	.10000000E-06
3	0.	3	.10000000E-06
4	.40000000E+01	4	.10000000E-06
5	0.	5	.10000000E-06
6	.10000000E+01	6	.10000000E-06
7	.10000000E+01	7	.10000000E-06

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .71400000E+03

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.58967193E+00	1	.10000000E+01
2	.79023599E-01	2	.81792717E+00
3	0.	3	-.27114846E+01
4	0.	4	-.13949580E+01
5	.33130447E+00	5	.94397759E+00

GRADIENT CHECK AT THE STARTING POINT

	ANALYTICAL GRADIENT VECTOR G(I)		NUMERICAL GRADIENT VECTOR G(I)		PERCENTAGE ERROR VECTOR YP(I)
1	-.82089809E+01	1	-.82089809E+01	1	.84484030E-06
2	-.78706625E+01	2	-.78706635E+01	2	.12717307E-04
3	.10185471E+01	3	.10186341E+01	3	.85354873E-02
4	-.13334000E+02	4	-.13334000E+02	4	.36818028E-06
5	.50927356E+01	5	.50749804E+01	5	.34985686E+00
6	.18500121E+02	6	.18500119E+02	6	.11884043E-04
7	-.49448364E+02	7	-.49448368E+02	7	.97549721E-05

GRADIENTS ARE O.K.

ITERATION NO. 1 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .10000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION	I	VARIABLE VECTOR X(I)	I	GRADIENT VECTOR G(I)
0	1	.75272643E+03	1	.10000000E+01	1	-.82089809E+01
			2	.20000000E+01	2	-.78706625E+01
			3	0.	3	.10185471E+01
			4	.40000000E+01	4	-.13334000E+02
			5	0.	5	.50927356E+01
			6	.10000000E+01	6	.18500121E+02
			7	.10000000E+01	7	-.49448364E+02
10	18	.72279326E+03	1	.15734901E+01	1	.68053690E-01
			2	.19242857E+01	2	.20852880E+01
			3	-.22006145E+00	3	-.26795681E-01
			4	.42261497E+01	4	.46566243E+00
			5	-.62977844E+00	5	.14031064E+00
			6	.75710378E+00	6	-.66032387E-01
			7	.18654161E+01	7	-.26998046E-01
17	25	.72279067E+03	1	.15736285E+01	1	.34121678E-07
			2	.19207655E+01	2	.48348462E-06
			3	-.21294121E+00	3	-.20519360E-08
			4	.42321180E+01	4	.17061075E-06
			5	-.63089198E+00	5	-.48806691E-07
			6	.76103566E+00	6	-.20185967E-07
			7	.18670823E+01	7	-.16438693E-07

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .217 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .70497960E+03

I	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.81647971E+00
3	-.26867146E+01
4	-.13770732E+01
5	.82814841E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .70497960E+03

I	MULTIPLIER VECTOR V(I)	I	NORM. ERROR VECTOR EN(I)
1	.96120810E+00	1	.10000000E+01
2	.16661930E-01	2	.81647971E+00
3	0.	3	-.26867146E+01
4	0.	4	-.13770732E+01
5	.22129973E-01	5	.82814841E+00

ITERATION NO. 2 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .20000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	27	.70637558E+03	1	.15736285E+01	1	-.13911965E+02
			2	.19207655E+01	2	-.84681639E+02
			3	-.21294121E+00	3	-.62608345E-01
			4	.42321180E+01	4	-.34104050E+02
			5	-.63089198E+00	5	-.50364928E+01
			6	.76103566E+00	6	-.55454872E+01
			7	.18670823E+01	7	.12200072E+02
10	37	.69988856E+03	1	.18524144E+01	1	.12718312E-06
			2	.19405490E+01	2	-.27348487E-06
			3	-.30706863E+00	3	.10913675E-06
			4	.43115828E+01	4	.83639490E-07
			5	-.62706610E+00	5	.40416712E-06
			6	.90630109E+00	6	.12440817E-06
			7	.17382191E+01	7	.18544247E-06

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .104 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .69277596E+03

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.90360000E+00
5	.88895490E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

VARIABLE VECTOR X(I)

ORDER 1

1	.21312004E+01
2	.19603325E+01
3	-.40119605E+00
4	.43910476E+01
5	-.62324022E+00
6	.10515665E+01
7	.16093558E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68182823E+03

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.99993471E+00
5	.96119571E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68715023E+03

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.85208903E+00	1	.10000000E+01
2	.11238990E+00	2	.95061805E+00
5	.35521071E-01	5	.92363425E+00

ITERATION NO. 3 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .40000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	40	.68990545E+03	1	.19918074E+01	1	-.28711160E+01
			2	.19504407E+01	2	.31061814E+01
			3	-.35413234E+00	3	.46014664E+00
			4	.43513152E+01	4	.91736109E+00
			5	-.62515316E+00	5	.13260758E+00
			6	.97893380E+00	6	-.10978023E+01
			7	.16737875E+01	7	.26815267E+01
8	49	.68977962E+03	1	.20589753E+01	1	.11169729E-04
			2	.19477146E+01	2	.62048062E-04
			3	-.38290811E+00	3	-.20627274E-05
			4	.43437981E+01	4	.23667146E-04
			5	-.62552911E+00	5	-.66757103E-05
			6	.97505214E+00	6	-.64973137E-06
			7	.16673484E+01	7	-.16033522E-04

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .094 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68662918E+03

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.95072390E+00
5	.93515985E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2
1	.22655361E+01	1	.23103147E+01
2	.19548802E+01	2	.19530628E+01
3	-.45874759E+00	3	-.47793144E+00
4	.43760135E+01	4	.43710021E+01
5	-.62399212E+00	5	-.62424275E+00
6	.10438032E+01	6	.10412154E+01
7	.15964776E+01	7	.15921849E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68203314E+03

	NORM. ERROR VECTOR EN(I)
1	.99781848E+00
2	.10000000E+01
5	.99454856E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68357402E+03

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.83823232E+00	1	.10000000E+01
2	.11807730E+00	2	.97579823E+00
5	.43690378E-01	5	.96374640E+00

ITERATION NO. 4 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .80000000E+02

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	52	.68508348E+03	1	.21790476E+01	1	.83771998E-01
			2	.19506159E+01	2	.56353999E+01
			3	-.42802180E+00	3	.11907215E+00
			4	.43580265E+01	4	.21584491E+01
			5	-.62485460E+00	5	.30945052E+00
			6	.10084572E+01	6	-.14742736E+00
			7	.16303032E+01	7	.33706958E+00
7	60	.68508086E+03	1	.21850080E+01	1	.26064779E-06
			2	.19501147E+01	2	.60300573E-04
			3	-.42818606E+00	3	.15490972E-05
			4	.43564925E+01	4	.23750063E-04
			5	-.62492518E+00	5	.83854566E-05
			6	.10073024E+01	6	.15363276E-05
			7	.16310953E+01	7	-.35007218E-05

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .087 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68359979E+03

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.97511755E+00
5	.96457122E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.23110408E+01	1	.23262091E+01	1	.23284797E+01
2	.19525147E+01	2	.19517262E+01	2	.19515352E+01
3	-.47346401E+00	3	-.47836949E+00	3	-.47843207E+00
4	.43691868E+01	4	.43669112E+01	4	.43663268E+01
5	-.62432124E+00	5	-.62443095E+00	5	-.62445784E+00
6	.10395527E+01	6	.10381359E+01	6	.10376960E+01
7	.15948423E+01	7	.15942972E+01	7	.15945989E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68078268E+03

	NORM. ERROR VECTOR EN(I)
1	.99976590E+00
2	.10000000E+01
5	.99931225E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68210371E+03

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.84466903E+00	1	.10000000E+01
2	.11420775E+00	2	.98757209E+00
5	.41123225E-01	5	.98128747E+00

ITERATION NO. 5 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .16000000E+03

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	63	.68282375E+03	1	.22544576E+01	1	.86728645E-02
			2	.19509563E+01	2	.11389989E+01
			3	-.45268512E+00	3	.25754122E-01
			4	.43617945E+01	4	.43631843E+00
			5	-.62467317E+00	5	.62536075E-01
			6	.10227519E+01	6	-.42338393E-01
			7	.16128634E+01	7	.92043526E-01
8	71	.68282369E+03	1	.22549916E+01	1	.90046795E-08
			2	.19509112E+01	2	.12564396E-06
			3	-.45238659E+00	3	-.14177230E-07
			4	.43616268E+01	4	.33573140E-07
			5	-.62468149E+00	5	.43428841E-08
			6	.10228742E+01	6	.11284072E-07
			7	.16127662E+01	7	.23079816E-08

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .091 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68210586E+03

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.98750181E+00
5	.98140153E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.23249751E+01	1	.23296198E+01	1	.23301071E+01
2	.19517078E+01	2	.19514389E+01	2	.19513978E+01
3	-.47658713E+00	3	-.47762816E+00	3	-.47752226E+00
4	.43667612E+01	4	.43659527E+01	4	.43658158E+01
5	-.62443779E+00	5	-.62447664E+00	5	-.62448317E+00
6	.10384459E+01	6	.10380770E+01	6	.10380685E+01
7	.15944371E+01	7	.15943020E+01	7	.15943027E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68064813E+03

	NORM. ERROR VECTOR EN(I)
1	.99997387E+00
2	.10000000E+01
5	.99988161E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

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NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)
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MAXIMUM OF THE ERROR FUNCTIONS ........... EM =  .68136557E+03

                                 MULTIPLIER          NORM. ERROR
                                VECTOR V(I)          VECTOR EN(I)

                             1  .84717271E+00     1  .10000000E+01
                             2  .11375837E+00     2  .99374518E+00
                             5  .39068923E-01     5  .99043177E+00
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ITERATION NO. 6 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .32000000E+03

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	74	.68171880E+03	1	.22918850E+01	1	-.15115159E-01
			2	.19511952E+01	2	.27436963E+00
			3	-.46484250E+00	3	.90855705E-02
			4	.43638459E+01	4	.10347760E+00
			5	-.62457635E+00	5	.14812327E-01
			6	.10305189E+01	6	-.16803494E-01
			7	.16035512E+01	7	.36959135E-01
7	81	.68171880E+03	1	.22919974E+01	1	.42775736E-06
			2	.19511878E+01	2	-.47719380E-04
			3	-.46484565E+00	3	-.13415771E-05
			4	.43638200E+01	4	-.18204105E-04
			5	-.62457741E+00	5	-.26147294E-05
			6	.10305374E+01	6	.20694402E-05
			7	.16035291E+01	7	-.45011840E-05

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .085 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68136548E+03

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.99373726E+00
5	.99045768E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.23290032E+01	1	.23303459E+01	1	.23304497E+01
2	.19514645E+01	2	.19513833E+01	2	.19513754E+01
3	-.47730470E+00	3	-.47754389E+00	3	-.47753185E+00
4	.43660131E+01	4	.43657637E+01	4	.43657367E+01
5	-.62447333E+00	5	-.62448517E+00	5	-.62448639E+00
6	.10382006E+01	6	.10381188E+01	6	.10381248E+01
7	.15942921E+01	7	.15942438E+01	7	.15942355E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68063199E+03

	NORM. ERROR VECTOR EN(I)
1	.99999728E+00
2	.10000000E+01
5	.99998567E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68099715E+03

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.84817302E+00	1	.10000000E+01
2	.11377895E+00	2	.99686613E+00
5	.38048030E-01	5	.99516138E+00

ITERATION NO. 7 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .64000000E+03

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	84	.68117239E+03	1	.23110379E+01	1	-.47842904E-02
			2	.19512931E+01	2	.59000056E-01
			3	-.47116092E+00	3	.22066851E-02
			4	.43648141E+01	4	.22104726E-01
			5	-.62453021E+00	5	.31664747E-02
			6	.10343403E+01	6	-.41839851E-02
			7	.15988898E+01	7	.92121355E-02
4	91	.68117239E+03	1	.23110526E+01	1	.28903864E-05
			2	.19512922E+01	2	.20649834E-04
			3	-.47116388E+00	3	-.62454380E-06
			4	.43648110E+01	4	.80345787E-05
			5	-.62453039E+00	5	.85171207E-06
			6	.10343421E+01	6	.63448842E-07
			7	.15988872E+01	7	.10503603E-05

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .067 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68099712E+03

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.99686530E+00
5	.99516479E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.23301079E+01	1	.23304761E+01	1	.23304947E+01
2	.19513966E+01	2	.19513740E+01	2	.19513727E+01
3	-.47748210E+00	3	-.47754124E+00	3	-.47754086E+00
4	.43658021E+01	4	.43657318E+01	4	.43657272E+01
5	-.62448338E+00	5	-.62448673E+00	5	-.62448695E+00
6	.10381469E+01	6	.10381290E+01	6	.10381305E+01
7	.15942453E+01	7	.15942297E+01	7	.15942277E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68063023E+03

	NORM. ERROR VECTOR EN(I)
1	.99999975E+00
2	.10000000E+01
5	.99999865E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68081343E+03

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.84866416E+00	1	.10000000E+01
2	.11385504E+00	2	.99843190E+00
5	.37480799E-01	5	.99756560E+00

ITERATION NO. 8 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .12800000E+04

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	94	.68090071E+03	1	.23207244E+01	1	-.94941346E-03
			2	.19513355E+01	2	.10899357E-01
			3	-.47434504E+00	3	.41783343E-03
			4	.43652787E+01	4	.40766839E-02
			5	-.62450821E+00	5	.58305119E-03
			6	.10362383E+01	6	-.79762656E-03
			7	.15965598E+01	7	.17578272E-02
2	98	.68090071E+03	1	.23207259E+01	1	-.55340590E-05
			2	.19513354E+01	2	-.70891442E-04
			3	-.47434541E+00	3	-.26336005E-05
			4	.43652784E+01	4	-.27538901E-04
			5	-.62450822E+00	5	-.43117762E-05
			6	.10362384E+01	6	.16509011E-06
			7	.15965595E+01	7	.12995829E-05

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .049 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68081343E+03

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.99843183E+00
5	.99756593E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.23303991E+01	1	.23304962E+01	1	.23304990E+01
2	.19513786E+01	2	.19513726E+01	2	.19513724E+01
3	-.47752695E+00	3	-.47754190E+00	3	-.47754199E+00
4	.43657458E+01	4	.43657270E+01	4	.43657263E+01
5	-.62448605E+00	5	-.62448694E+00	5	-.62448697E+00
6	.10381348E+01	6	.10381307E+01	6	.10381309E+01
7	.15942318E+01	7	.15942273E+01	7	.15942269E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68063007E+03

	NORM. ERROR VECTOR EN(I)
1	.99999999E+00
2	.10000000E+01
5	.99999990E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68072170E+03

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.84891438E+00	1	.10000000E+01
2	.11390881E+00	2	.99921571E+00
5	.37176813E-01	5	.99877876E+00

ITERATION NO. 9 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .25600000E+04

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	101	.68076526E+03	1	.23255998E+01	1	-.18814290E-03
			2	.19513547E+01	2	.12023054E-02
			3	-.47594182E+00	3	.56105951E-04
			4	.43655048E+01	4	.44224587E-03
			5	-.62449748E+00	5	.62948693E-04
			6	.10371852E+01	6	-.12051266E-03
			7	.15953938E+01	7	.26703178E-03
2	105	.68076526E+03	1	.23255999E+01	1	.36806531E-04
			2	.19513547E+01	2	.43187506E-03
			3	-.47594161E+00	3	.48525802E-05
			4	.43655048E+01	4	.16851135E-03
			5	-.62449748E+00	5	.23885024E-04
			6	.10371852E+01	6	-.36458407E-05
			7	.15953938E+01	7	.90798455E-05

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .048 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68072170E+03

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.99921571E+00
5	.99877879E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.23304740E+01	1	.23304989E+01	1	.23304993E+01
2	.19513740E+01	2	.19513724E+01	2	.19513724E+01
3	-.47753780E+00	3	-.47754142E+00	3	-.47754135E+00
4	.43657312E+01	4	.43657263E+01	4	.43657263E+01
5	-.62448674E+00	5	-.62448697E+00	5	-.62448697E+00
6	.10381319E+01	6	.10381310E+01	6	.10381310E+01
7	.15942280E+01	7	.15942268E+01	7	.15942267E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68063006E+03

	NORM. ERROR VECTOR EN(I)
1	.99999999E+00
2	.10000000E+01
5	.99999998E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68067587E+03

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.84903810E+00	1	.10000000E+01
2	.11394219E+00	2	.99960781E+00
5	.37019713E-01	5	.99938834E+00

ITERATION NO. 10 OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

WITH PARAMETER P = .51200000E+04

UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METHOD

ITER. NO.	FUNC. EVAL.	OBJECTIVE FUNCTION		VARIABLE VECTOR X(I)		GRADIENT VECTOR G(I)
0	108	.68069762E+03	1	.23280464E+01	1	.14529036E-03
			2	.19513637E+01	2	.22098540E-02
			3	-.47674104E+00	3	.36358955E-04
			4	.43656162E+01	4	.85780832E-03
			5	-.62449219E+00	5	.12269026E-03
			6	.10376582E+01	6	-.33015442E-04
			7	.15948104E+01	7	.73086469E-04
1	110	.68069762E+03	1	.23280464E+01	1	-.79820027E-04
			2	.19513637E+01	2	-.50375356E-05
			3	-.47674102E+00	3	.10716002E-04
			4	.43656161E+01	4	-.91708518E-05
			5	-.62449219E+00	5	-.14070191E-05
			6	.10376582E+01	6	-.30508284E-04
			7	.15948104E+01	7	.67486503E-04

EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS .038 SECONDS

AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOVE

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68067587E+03

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.99960780E+00
5	.99938834E+00

ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAPOLATION

	VARIABLE VECTOR X(I) ORDER 1		VARIABLE VECTOR X(I) ORDER 2		VARIABLE VECTOR X(I) ORDER 3
1	.23304929E+01	1	.23304993E+01	1	.23304993E+01
2	.19513728E+01	2	.19513724E+01	2	.19513724E+01
3	-.47754043E+00	3	-.47754130E+00	3	-.47754129E+00
4	.43657275E+01	4	.43657262E+01	4	.43657262E+01
5	-.62448691E+00	5	-.62448697E+00	5	-.62448697E+00
6	.10381312E+01	6	.10381310E+01	6	.10381310E+01
7	.15942270E+01	7	.15942267E+01	7	.15942267E+01

NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOLUTION

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .68063006E+03

	NORM. ERROR VECTOR EN(I)
1	.10000000E+01
2	.99999999E+00
5	.99999999E+00

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

```
-------------------------------------------------------------
NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)
-----------------------------------------------------------------

MAXIMUM OF THE ERROR FUNCTIONS ........... EM =   .68065296E+03

                               MULTIPLIER           NORM. ERROR
                              VECTOR V(I)          VECTOR EN(I)

                           1  .84910757E+00     1  .10000000E+01
                           2  .11395352E+00     2  .99980389E+00
                           5  .36938907E-01     5  .99969390E+00
```

Example 10: The Wong problem 2 [12, 13]

Minimize

$$f = x_1^2 + x_2^2 + x_1 x_2 - 14 x_1 - 16 x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 + (x_5 - 3)^2 + 2(x_6 - 1)^2 + 5 x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45 .$$

subject to

$$-3(x_1 - 2)^2 - 4(x_2 - 3)^2 - 2 x_3^2 + 7x_4 + 120 \geq 0$$

$$-5x_1^2 - 8x_2 - (x_3 - 6)^2 + 2x_4 + 40 \geq 0$$

$$-0.5(x_1 - 8)^2 - 2(x_2 - 4)^2 - 3x_5^2 + x_6 + 30 \geq 0$$

$$-x_1^2 - 2(x_2 - 2)^2 + 2x_1 x_2 - 14x_5 + 6x_6 \geq 0$$

$$-4x_1 - 5x_2 + 3x_7 - 9x_8 + 105 \geq 0$$

$$-10x_1 + 8x_2 + 17x_7 - 2x_8 \geq 0$$

$$3x_1 - 6x_2 - 12(x_9 - 8)^2 + 7x_{10} \geq 0$$

$$8x_1 - 2x_2 - 5x_9 + 2x_{10} + 12 \geq 0 .$$

The optimal solution found in 101 necessary function evaluations is

$$f = 24.306211$$

$x_1 = 2.1719964$	$x_2 = 2.3636829$
$x_3 = 8.7739257$	$x_4 = 5.0959844$
$x_5 = 0.99065477$	$x_6 = 1.4305740$
$x_7 = 1.3216442$	$x_8 = 9.8287258$
$x_9 = 8.2800917$	$x_{10} = 8.3759266$.

This problem has also been solved by the program DISOPT3 [12] using the Charalambous algorithm [13].

```
      PROGRAM TST(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C     MAIN PROGRAM FOR EXAMPLE 10
C
      DIMENSION X(10), G(10), H(55), W(40), EPS(10), XB(10), XE(10,4,15)
      DIMENSION EN(9), JD(9), V(9)
C
      COMMON /1/ ID,IEX,IFINIS,IGK,IH,IK,IPT,IREDU,JORDER,JPRINT,M,MAX
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      DATA EPS/10*.1E-6/, IREDU/0 /
C
      ETA=.1E-3
      IK=15
      M=1
      N=10
      NR=9
      P=10.
C
      DO 1 IH=1,IK
      CALL FLOPT4 (EN,EPS,G,H,JD,V,W,X,XB,XE)
      M=0
      IF (IFINIS.EQ.N) CALL EXIT
1     CONTINUE
C
      STOP
      END
```

```
      SUBROUTINE FUNCT4 (EN,G,JD,U,V,X)
C
C     THE SECOND WONG PROBLEM
C
      DIMENSION CONS(9), GCONS(10,9), X(10)
      DIMENSION G(10), ER(9), GE(10,9), ES(9), EN(9), JD(9), V(9)
C
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
      DATA AL/10./
C
1     ER(1)=X(1)**2+X(2)**2+X(1)*X(2)-14.*X(1)-16.*X(2)+(X(3)-10.)**2+4.
     1*(X(4)-5.)**2+(X(5)-3.)**2+2.*(X(6)-1.)**2+5.*X(7)**2+7.*(X(8)-11.
     2)**2+2.*(X(9)-10.)**2+(X(10)-7.)**2+45.
      GE(1,1)=2.*X(1)+X(2)-14.
      GE(2,1)=2.*X(2)+X(1)-16.
      GE(3,1)=2.*(X(3)-10.)
      GE(4,1)=8.*(X(4)-5.)
      GE(5,1)=2.*(X(5)-3.)
      GE(6,1)=4.*(X(6)-1.)
      GE(7,1)=10.*X(7)
      GE(8,1)=14.*(X(8)-11.)
      GE(9,1)=4.*(X(9)-10.)
      GE(10,1)=2.*(X(10)-7.)
C
      DO 18 I=1,NA
      J=JD(I)
      GO TO (18,2,4,6,8,10,12,14,16,18), J
C
2     CONS(2)=-3.*(X(1)-2.)**2-4.*(X(2)-3.)**2-2.*X(3)**2+7.*X(4)+120.
      GCONS(1,2)=-6.*(X(1)-2.)
      GCONS(2,2)=-8.*(X(2)-3.)
      GCONS(3,2)=-4.*X(3)
      GCONS(4,2)=7.
      GCONS(5,2)=0.
      GCONS(6,2)=0.
      GCONS(7,2)=0.
      GCONS(8,2)=0.
      GCONS(9,2)=0.
      GCONS(10,2)=0.
      ER(2)=ER(1)-AL*CONS(2)
      DO 3 IJ=1,10
      GE(IJ,2)=GE(IJ,1)-AL*GCONS(IJ,2)
3     CONTINUE
      GO TO 18
C
4     CONS(3)=-5.*X(1)**2-8.*X(2)-(X(3)-6.)**2+2.*X(4)+40.
      GCONS(1,3)=-10.*X(1)
      GCONS(2,3)=-8.
      GCONS(3,3)=-2.*(X(3)-6.)
      GCONS(4,3)=2.
      GCONS(5,3)=0.
      GCONS(6,3)=0.
      GCONS(7,3)=0.
      GCONS(8,3)=0.
      GCONS(9,3)=0.
      GCONS(10,3)=0.
      ER(3)=ER(1)-AL*CONS(3)
      DO 5 IJ=1,10
      GE(IJ,3)=GE(IJ,1)-AL*GCONS(IJ,3)
5     CONTINUE
      GO TO 18
C
6     CONS(4)=-.5*(X(1)-8.)**2-2.*(X(2)-4.)**2-3.*X(5)**2+X(6)+30.
      GCONS(1,4)=8.-X(1)
      GCONS(2,4)=-4.*(X(2)-4.)
      GCONS(3,4)=0.
      GCONS(4,4)=0.
      GCONS(5,4)=-6.*X(5)
      GCONS(6,4)=1.
      GCONS(7,4)=0.
      GCONS(8,4)=0.
      GCONS(9,4)=0.
```

```
      GCONS(10,4)=0.
      ER(4)=ER(1)-AL*CONS(4)
      DO 7 IJ=1,10
      GE(IJ,4)=GE(IJ,1)-AL*GCONS(IJ,4)
7     CONTINUE
      GO TO 18
C
8     CONS(5)=-X(1)**2-2.*(X(2)-2.)**2+2.*X(1)*X(2)-14.*X(5)+6.*X(6)
      GCONS(1,5)=-2.*X(1)+2.*X(2)
      GCONS(2,5)=-4.*(X(2)-2.)+2.*X(1)
      GCONS(3,5)=0.
      GCONS(4,5)=0.
      GCONS(5,5)=-14.
      GCONS(6,5)=6.
      GCONS(7,5)=0.
      GCONS(8,5)=0.
      GCONS(9,5)=0.
      GCONS(10,5)=0.
      ER(5)=ER(1)-AL*CONS(5)
      DO 9 IJ=1,10
      GE(IJ,5)=GE(IJ,1)-AL*GCONS(IJ,5)
9     CONTINUE
      GO TO 18
C
10    CONS(6)=-4.*X(1)-5.*X(2)+3.*X(7)-9.*X(8)+105.
      GCONS(1,6)=-4.
      GCONS(2,6)=-5.
      GCONS(3,6)=0.
      GCONS(4,6)=0.
      GCONS(5,6)=0.
      GCONS(6,6)=0.
      GCONS(7,6)=3.
      GCONS(8,6)=-9.
      GCONS(9,6)=0.
      GCONS(10,6)=0.
      ER(6)=ER(1)-AL*CONS(6)
      DO 11 IJ=1,10
      GE(IJ,6)=GE(IJ,1)-AL*GCONS(IJ,6)
11    CONTINUE
      GO TO 18
C
12    CONS(7)=-10.*X(1)+8.*X(2)+17.*X(7)-2.*X(8)
      GCONS(1,7)=-10.
      GCONS(2,7)=8.
      GCONS(3,7)=0.
      GCONS(4,7)=0.
      GCONS(5,7)=0.
      GCONS(6,7)=0.
      GCONS(7,7)=17.
      GCONS(8,7)=-2.
      GCONS(9,7)=0.
      GCONS(10,7)=0.
      ER(7)=ER(1)-AL*CONS(7)
      DO 13 IJ=1,10
      GE(IJ,7)=GE(IJ,1)-AL*GCONS(IJ,7)
13    CONTINUE
      GO TO 18
C
14    CONS(8)=3.*X(1)-6.*X(2)-12.*(X(9)-8.)**2+7.*X(10)
      GCONS(1,8)=3.
      GCONS(2,8)=-6.
      GCONS(3,8)=0.
      GCONS(4,8)=0.
      GCONS(5,8)=0.
      GCONS(6,8)=0.
      GCONS(7,8)=0.
      GCONS(8,8)=0.
      GCONS(9,8)=-24.*(X(9)-8.)
      GCONS(10,8)=7.
      ER(8)=ER(1)-AL*CONS(8)
      DO 15 IJ=1,10
      GE(IJ,8)=GE(IJ,1)-AL*GCONS(IJ,8)
15    CONTINUE
```

```
      GO TO 18
C
16    CONS(9)=8.*X(1)-2.*X(2)-5.*X(9)+2.*X(10)+12.
      GCONS(1,9)=8.
      GCONS(2,9)=-2.
      GCONS(3,9)=0.
      GCONS(4,9)=0.
      GCONS(5,9)=0.
      GCONS(6,9)=0.
      GCONS(7,9)=0.
      GCONS(8,9)=0.
      GCONS(9,9)=-5.
      GCONS(10,9)=2.
      ER(9)=ER(1)-AL*CONS(9)
      DO 17 IJ=1,10
      GE(IJ,9)=GE(IJ,1)-AL*GCONS(IJ,9)
17    CONTINUE
C
18    CONTINUE
C
      CALL LEASTP4 (EN,ER,ES,G,GE,JD,U,V)
      RETURN
      END
```

INPUT DATA FOR BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

NUMBER OF ERROR FUNCTIONS .... NR = 9

MAXIMUM ORDER OF EXTRAPOLATION .... JORDER = 3

INITIAL VALUE OF THE PARAMETER .... P = .10000000E+02

MULTIPLYING FACTOR FOR P .... FACTOR = .20000000E+01

TOLERANCE ETA FOR INACTIVE FUNCTIONS .... ETA = .10000000E-03

PREDICTED FUNCTION LOWER BOUND .... EST = 0.

STARTING POINT AND TEST QUANTITIES FOR THE TERMINATION CRITERION

	VARIABLE VECTOR X(I)		TEST VECTOR EPS(I)
1	.20000000E+01	1	.10000000E-06
2	.30000000E+01	2	.10000000E-06
3	.50000000E+01	3	.10000000E-06
4	.50000000E+01	4	.10000000E-06
5	.10000000E+01	5	.10000000E-06
6	.20000000E+01	6	.10000000E-06
7	.70000000E+01	7	.10000000E-06
8	.30000000E+01	8	.10000000E-06
9	.60000000E+01	9	.10000000E-06
10	.10000000E+02	10	.10000000E-06

MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT OF THE

NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTATION)

MAXIMUM OF THE ERROR FUNCTIONS .... EM = .75300000E+03

	MULTIPLIER VECTOR V(I)		NORM. ERROR VECTOR EN(I)
1	.35981771E+00	1	.10000000E+01
2	0.	2	-.39442231E+00
3	.18100311E+00	3	.93359894E+00
4	.10075534E+00	4	.88047809E+00
5	.20846257E+00	5	.94687915E+00
6	0.	6	-.92961487E-02
7	0.	7	-.55378486E+00
8	.86549506E-01	8	.86719788E+00
9	.63411770E-01	9	.84063745E+00

GRADIENT CHECK AT THE STARTING POINT

	ANALYTICAL GRADIENT VECTOR G(I)		NUMERICAL GRADIENT VECTOR G(I)		PERCENTAGE ERROR VECTOR YP(I)
1	.12220232E+02	1	.12220231E+02	1	.59130649E-05
2	.10989327E+02	2	.10989326E+02	2	.44140188E-05
3	-.16074643E+02	3	-.16074643E+02	3	.43660215E-05
4	-.42948443E+01	4	-.42948439E+01	4	.89290122E-05
5	.37032160E+02	5	.37032160E+02	5	.24412952E-05
6	-.11186646E+02	6	-.11186646E+02	6	.36602335E-06
7	.82458588E+02	7	.82458588E+02	7	.17770470E-06
8	-.13193374E+03	8	-.13193374E+03	8	.67666749E-06
9	-.67731597E+02	9	-.67731596E+02	9	.56040835E-06
10	-.23412785E+01	10	-.23412787E+01	10	.11514134E-04

GRADIENTS ARE O.K.

VI. CONCLUSIONS

A package of subroutines, called FLOPT4, for solving least pth optimization problems has been presented. Its features, which include Fletcher's quasi-Newton subroutine, a least pth objective formulation subroutine, an extrapolation procedure and a scheme for dropping inactive functions, make it capable of solving unconstrained problems, constrained problems or nonlinear minimax approximation problems. Several examples have been presented to illustrate the versatility of the program. The mathematical background for the extrapolation procedure to minimax solutions (or the p-algorithm) has been omitted, but is readily available [2], [6], [11].

REFERENCES

[1] R. Fletcher, "FORTRAN subroutines for minimization by quasi-Newton methods", Atomic Energy Research Establishment, Harwell, Berkshire, England, Report AERE-R7125, 1972.

[2] J.W. Bandler, C. Charalambous, J.H.K. Chen and W.Y. Chu, "New results in the least pth approach to minimax design", IEEE Trans. Microwave Theory Tech., vol. MTT-24, Feb. 1976, pp. 116-119. Also Faculty of Engineering, McMaster University, Hamilton, Canada, Mar. 1975, Report SOC-80.

[3] A.V. Fiacco and G.P. McCormick, Nonlinear Programming: Sequential Unconstrained Minimization Techniques. New York: Wiley, 1968.

[4] J.W. Bandler and C. Charalambous, "Nonlinear programming using minimax techniques", J. Optimization Theory and Applications, vol. 13, June 1974, pp. 607-619.

[5] J.W. Bandler and W.Y. Chu, "FLOPT2 - A program for least pth optimization with extrapolation to minimax solutions", Faculty of Engineering, McMaster University, Hamilton, Canada, Apr. 1975, Report SOC-84.

[6] W.Y. Chu, "Extrapolation in least pth approximation and nonlinear programming", Faculty of Engineering, McMaster University, Hamilton, Canada, Dec. 1974, Report SOC-71.

[7] H.H. Rosenbrock, "An automatic method for finding the greatest or least value of a function", Comp. J., vol. 3, Oct. 1960, pp.175-184.

[8] J. Kowalik and M.R. Osborne, Methods for Unconstrained Optimization Problems. New York: Elsevier, 1968.

[9] C. Charalambous and J.W. Bandler, "Nonlinear minimax optimization as a sequence of least pth optimization with finite values of p", Int. J. Systems Science, vol. 7, 1976, pp. 377-391.

[10] J.W. Bandler, T.V. Srinivasan and C. Charalambous, "Minimax optimization of networks by grazor search", IEEE Trans. Microwave Theory Tech., vol. MTT-20, Sept. 1972, pp.596-604.

[11] J.W. Bandler and W.Y. Chu, "Nonlinear programming using least pth optimization with extrapolation", Faculty of Engineering, McMaster University, Hamilton, Canada, Mar. 1975, Report SOC-78.

[12] J.W. Bandler and D. Sinha, "DISOPT3 - a user-oriented package for nonlinear continuous and discrete optimization problems", Faculty of Engineering, McMaster University, Hamilton, Canada, July 1977, Report SOC-174.

[13] C. Charalambous, "Nonlinear least pth optimization and nonlinear programming", Mathematical Programming, vol. 12, Apr. 1977, pp. 195-225.

FORTRAN Listing for Subroutines FLOPT4, LEASTP4, GRDCHK4 and QUASI4

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      SUBROUTINE FLOPT4 (EN,EPS,G,H,JD,V,W,X,XB,XE)
C
C     THIS SUBROUTINE READS IN THE STARTING VALUE OF X, CALLS ANOTHER
C     SUBROUTINE FOR THE UNCONSTRAINED MINIMIZATION, PERFORMS EXTRA-
C     POLATION, SELECTS ACTIVE FUNCTIONS AND OUTPUTS THE RESULT
C
      DIMENSION X(1),XB(1),XE(1),G(1),H(1),W(1),EPS(1),V(1),EN(1),JD(1)
C
C     ALL THE UNDIMENSIONED INTEGER OR REAL VARIABLES THAT THE USER
C     COULD POSSIBILY REQUIRE FOR MANIPULATION WITHIN THE MAIN PROGRAM
C     BELONG TO TWO NUMBERED COMMON BLOCKS
C
      COMMON /1/ ID,IEX,IFINIS,IGK,IH,IK,IPT,IREDU,JORDER,JPRINT,M,MAX
      COMMON /2/ N,NA,JV,P,EM,EST,ETA,FACTOR,NR
C
C     THE DEFAULT VALUES OF THE VARIABLES LISTED IN TABLE 1 OF REPORT
C     SOC-151 ARE ASSIGNED BY THE FOLLOWING DATA STATEMENTS
C
      DATA MAX,JORDER,JPRINT,IPT,ID,IREDU,IEX,IGK/200,3,2,10,4*1/
      DATA EST,ETA,FACTOR,P,IH,IK,NR/0.,.0005,2*2.,3*1/
C
C     A DESCRIPTION OF ALL THE VARIABLES USED BY THIS AS WELL AS OTHER
C     SUBROUTINES NOW FOLLOWS
C
C     ************************ INTEGER VARIABLES ***********************
C
C     ID     EQUALS 0 IF INPUT DATA IS NOT TO BE PRINTED
C
C     IEX    EQUALS 0 WHEN EXTRAPOLATION IS NOT TO BE PERFORMED
C
C     IEXIT  A FLAG USED BY QUASI4 TO STOP THE PROGRAM EXECUTION AND
C            PRINT A MESSAGE IF THE CHOSEN VALUE OF EPS IS TOO SMALL
C            (IEXIT=2) OR, IF MAX HAS BEEN EXCEEDED (IEXIT=3). IEXIT=1
C            INDICATES A NORMAL EXIT AND NO MESSAGE IS PRINTED
C
C     IFINIS EQUALS N WHEN THE PROJECTED MINIMAX SOLUTION HAS CONVERGED
C            TO THE TRUE SOLUTION WITHIN EPS. THIS MAY BE USED AS A
C            STOPPING CRITERION IN THE MAIN PROGRAM
C
C     IFN    COUNTS THE FUNCTION EVALUATIONS
C
C     IGK    EQUALS 1 WHEN GRADIENT CHECK IS REQUIRED
C
C     IH     USED AS THE INDEX OF A DO LOOP IN THE MAIN PROGRAM THAT
C            CALLS FLOPT4 IK TIMES
C
C     IK     THE NUMBER OF TIMES FLOPT4 IS CALLED FROM THE MAIN PROGRAM.
C            IT CORRESPONDS TO THE NUMBER OF P VALUES USED IN LEAST PTH
C            APPROXIMATION OR THE NUMBER OF R VALUES USED IN THE
C            FIACCO-MCCORMICK METHOD
C
C     IPT    THE RESULTS OF THE UNCONSTRAINED MINIMIZATION ARE PRINTED
C            FOR THE FIRST AND THE LAST ITERATIONS OF QUASI4 AS WELL AS
C            AFTER EVERY IPT ITERATIONS WITHIN QUASI4. IT MUST BE NOTED
C            THAT IPT=0 SUPPRESSES THE ENTIRE PRINTOUT. WHEN IPT=0,
C            JPRINT HAS NO INFLUENCE ON PRINTING
C
C     IREDU  EQUALS 0 WHEN THE SCHEME FOR CHOOSING ACTIVE FUNCTIONS
C            IS NOT USED
C
C     JD     AN ARRAY WHICH IDENTIFIES THE ACTIVE FUNCTIONS
C
C     JORDER THE HIGHEST ORDER OF THE ESTIMATE OF THE MINIMAX SOLUTION
C            DETERMINED BY THE EXTRAPOLATION PROCEDURE
C
C     JPRINT OFFERS THE FOLLOWING OPTIONS FOR PRINTING...
C            0 EXTRAPOLATION ESTIMATES WILL NOT BE PRINTED
C            1 EXTRAPOLATION ESTIMATES OF THE MINIMAX SOLUTION AND THE
C              ERROR FUNCTIONS WILL BE PRINTED
C            2 IN ADDITION TO THE ABOVE PRINTOUT, THE MULTIPLIERS AND
C              THE NORMALIZED ERRORS AT THE NEXT ESTIMATED LEAST PTH
C              SOLUTION WILL ALSO BE PRINTED
C
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C       JV      USED IN SUBROUTINE LEASTP4. JV=1 RESULTS IN THE CALCULA-
C               TION OF BOTH THE GRADIENTS AND THE MULTIPLIERS. IF JV=0
C               ONLY THE GRADIENTS ARE CALCULATED
C
C       M       NONZERO IF THE INITIAL VALUE OF X IS TO BE READ BY FLOPT4
C
C       MAX     MAXIMUM PERMISSIBLE NUMBER OF FUNCTION EVALUATIONS.
C               EXECUTION STOPS IF MAX IS EXCEEDED
C
C       MODE    FOR MODE=1 AN IDENTITY MATRIX IS THE INITIAL ESTIMATE OF
C               THE HESSIAN IN SUBROUTINE QUASI4. FOR MODE=3 THE INITIAL
C               ESTIMATE OF THE HESSIAN IS A MATRIX WHICH IS IN THE
C               DECOMPOSED FORM LDL(TRANSPOSE) AND HAS BEEN GENERATED
C               BY THE LAST CALL TO QUASI4
C
C       N       THE NUMBER OF VARIABLES IN THE PROBLEM. N.GE.2
C
C       NA      THE NUMBER OF ACTIVE FUNCTIONS. IF THE REDUCTION SCHEME IS
C               USED, A FUNCTION WHOSE MULTIPLIER V DOES NOT EQUAL OR
C               EXCEED ETA AT THE STARTING POINT OF AN OPTIMIZATION(EXCEPT
C               THE FIRST) IS CONSIDERED INACTIVE AND DROPPED FROM FUTURE
C               CONSIDERATION. WHEN THE REDUCTION SCHEME IS NOT USED, NA
C               IS SET EQUAL TO NR BY FLOPT4
C
C       NR      THE NUMBER OF ERROR FUNCTIONS IN THE PROBLEM. WHEN THE
C               LEAST PTH OBJECTIVE FORMULATION IS NOT BEING USED AND, FOR
C               EXAMPLE, THE FIACCO-MCCORMICK METHOD IS USED, THE DEFAULT
C               VALUE NR=1 SHOULD BE USED
C
C       ***********************   REAL VARIABLES   ***********************
C
C       EM      EQUALS THE MAXIMUM OF THE ERROR FUNCTIONS
C
C       EPS     THIS ARRAY OF N ELEMENTS IS USED FOR TESTING THE CONVER-
C               GENCE OF THE SOLUTION OF THE UNCONSTRAINED OPTIMIZATION AS
C               WELL AS THE PROJECTED MINIMAX SOLUTION
C
C       ER      AN ARRAY OF NR ELEMENTS CONTAINING THE VALUES OF THE ERROR
C               FUNCTIONS. ARRAY EN CONTAINS THE NORMALIZED VALUES AND
C               ARRAY ES CONTAINS THE NORMALIZED VALUES RAISED TO POWER P
C
C       EST     USERS GUESS OF THE OPTIMAL OBJECTIVE FUNCTION VALUE
C
C       ETA     USED BY THE REDUCTION SCHEME TO SELECT ACTIVE FUNCTIONS,
C               I.E., THOSE FUNCTIONS WITH MULTIPLIER VALUES .GE. ETA
C
C       FACTOR  MULTIPLIES P TO UPDATE ITS VALUE FOR A SUBSEQUENT ITERATION
C               IN LEAST PTH APPROXIMATION. IT DIVIDES R IN THE FIACCO-
C               MCCORMICK METHOD
C
C       G       AN ARRAY OF N ELEMENTS STORING THE GRADIENT VECTOR AT X
C
C       GE      AN ARRAY OF N*NR ELEMENTS STORING THE PARTIAL DERIVATIVES
C               OF THE ERROR FUNCTIONS WHEN LEAST PTH APPROXIMATION IS USED
C
C       H       THIS ARRAY OF N*(N+1)/2 ELEMENTS STORES THE CURRENT
C               ESTIMATE OF THE HESSIAN MATRIX AT X
C
C       P       THE PARAMETER OF LEAST PTH APPROXIMATION. ALSO EQUALS R IN
C               THE FIACCO-MCCORMICK METHOD
C
C       U       VALUE OF THE UNCONSTRAINED OBJECTIVE FUNCTION
C
C       V       AN ARRAY STORING THE MULTIPLIERS OF THE ACTIVE FUNCTIONS
C               IF THE REDUCTION SCHEME IS USED
C
C       W       AN ARRAY OF 4*N ELEMENTS USED AS WORKING SPACE
C
C       X       AN ARRAY OF N ELEMENTS IN WHICH THE CURRENT ESTIMATE OF THE
C               SOLUTION IS STORED. AN INITIAL APPROXIMATION MUST BE SET IN
C               X ON ENTRY. WHEN THE EXTRAPOLATION PROCEDURE IS USED, AN
C               ESTIMATE OF THE NEXT MINIMUM IN THE SEQUENCE WILL BE STORED
C               IN X AT THE END OF EACH ITERATION OF FLOPT4
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C
C     XB        AN ARRAY OF N ELEMENTS IN WHICH THE BEST ESTIMATE OF THE
C               MINIMAX SOLUTION CURRENTLY AVAILABLE IS STORED
C
C     XE        AN ARRAY OF N*(JORDER+1)*IK ELEMENTS IN WHICH ESTIMATES OF
C               THE MINIMAX SOLUTION CORRESPONDING TO DIFFERENT ORDERS ARE
C               STORED FOR EACH CALL OF FLOPT4
C
C     THE FOLLOWING SECTION OF THE PROGRAM READS X IN FREE FORMAT, DOES
C     THE NECESSARY INITIALIZATION AND CALLS A SUBROUTINE TO CHECK FOR
C     ANY DISCREPANCY BETWEEN THE GRADIENTS AS DEFINED BY THE USER AND
C     AS CALCULATED NUMERICALLY
C
      IF (M.NE.0) READ (5,*) (X(I),I=1,N)
      IF (IH.NE.1) GO TO 3
      J1=JORDER+1
      NJ1=N*J1
      MODE=1
      IF (IPT.EQ.0) JPRINT=0
      IF (ID.EQ.0) GO TO 1
      WRITE (6,27) NR
      IF (IEX.EQ.1) WRITE (6,29) JORDER
      IF (IK.GT.1) WRITE (6,31) P,FACTOR
      IF (IREDU.EQ.1) WRITE (6,32) ETA
      WRITE (6,33) EST,(I,X(I),I,EPS(I),I=1,N)
1     NA=NR
C
      DO 2 I=1,NA
      JD(I)=I
2     CONTINUE
C
      JV=1
      CALL FUNCT4 (EN,G,JD,U,V,X)
      IFN=1
      IF (IPT.NE.0.AND.NR.GT.1) WRITE (6,22) EM,(I,V(I),I,EN(I),I=1,NA)
      IF (IGK.EQ.1) CALL GRDCHK4 (N,X,G,W,EN,JD,V)
C
C     THE UNCONSTRAINED OPTIMIZATION IS NOW PERFORMED BY CALLING QUASI4
C
3     IF (IPT.EQ.0) GO TO 4
      WRITE (6,23) IH,P
      CALL SECOND (T1)
4     CALL QUASI4 (N,X,U,G,H,W,EST,EPS,MODE,MAX,IPT,IEXIT,IFN,EN,JD,V)
      IF (IPT.EQ.0) GO TO 5
      CALL SECOND (T2)
      WRITE (6,28) T2-T1
      IF (NA.NE.1) CALL FUNCT4 (EN,G,JD,U,V,X)
      IF (NA.NE.1) IFN=IFN+1
      IF (NA.NE.1) WRITE (6,30) EM,(JD(I),EN(JD(I)),I=1,NA)
5     MODE=3
      IF (IEX.EQ.0) GO TO 19
C
C     THE FOLLOWING SECTION OF THE PROGRAM PERFORMS EXTRAPOLATION. XE,
C     THOUGH DECLARED AS A 3-DIMENSIONAL ARRAY IN THE MAIN PROGRAM, IS
C     TREATED AS A VECTOR HERE. THIS IS CONSISTENT WITH THE MANNER IN
C     WHICH IT IS ACTUALLY STORED IN THE COMPUTER MEMORY.THE TOTAL
C     LENGTH OF THE ARRAY XE IS N*J1*IK. NJ1 OR N*J1 ELEMENTS ARE AVAIL-
C     ABLE FOR EACH CALL OF FLOPT4 FROM THE MAIN PROGRAM. THE MINIMAX
C     ESTIMATES OF ORDERS 0, 1, 2, 3,  ................  , JORDER ARE
C     SEQUENTIALLY STORED IN THESE NJ1 ELEMENTS. WITH THIS CLEAR, THE
C     INDEXING USED IN THIS PROGRAM SEGMENT CAN BE UNDERSTOOD EASILY.
C     REFER TO REPORT SOC-71 FOR THE THEORY OF EXTRAPOLATION
C
      INJ1=NJ1*(IH-1)
C
C     0TH ORDER ESTIMATE IS THE RESULT OF THE UNCONSTRAINED
C     OPTIMIZATION ITSELF
C
      DO 6 J=1,N
      XE(INJ1+J)=X(J)
6     CONTINUE
C
      IF (IH.GE.2) GO TO 8
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C
      DO 7 J=1,N
      XB(J)=XE(INJ1+J)
7     CONTINUE
C
      GO TO 19
C
C     HIGHER ORDERS OF THE ESTIMATE ARE BEING CALCULATED NOW
C
8     S=1.
      INJ1=INJ1+N
      IJ=J1
      IF (IH.LT.J1) IJ=IH
      DO 9 L=2,IJ
      LL=L-1
      S=S*FACTOR
C
      DO 9 J=1,N
      INJ1=INJ1+1
      INJ2=INJ1-N
9     XE(INJ1)=(S*XE(INJ2)-XE(INJ2-NJ1))/(S-1.)
C
      IFINIS=0
C
      DO 10 J=1,N
      INJ1=INJ2+J
      IF (ABS(XE(INJ1)-XB(J)).LE.EPS(J)) IFINIS=IFINIS+1
      XB(J)=XE(INJ1)
10    CONTINUE
C
      IF (JPRINT.EQ.0) GO TO 14
      IF (IJ-3) 11,12,13
11    WRITE (6,24) (J,XB(J),J=1,N)
      GO TO 14
12    WRITE (6,25) (J,XE(INJ2-N+J),J,XE(INJ2+J),J=1,N)
      GO TO 14
13    INJ1=INJ1-3*N
      IMIN=JORDER-2
      WRITE (6,26) (I,I=IMIN,JORDER),((J,XE(I+J),I=INJ1,INJ2,N),J=1,N)
14    IF (IH.EQ.IK) GO TO 18
C
C     THE SOLUTION OF THE UNCONSTRAINED OPTIMIZATION CORRESPONDING TO
C     THE NEXT VALUE OF THE PARAMETER IS NOW BEING ESTIMATED BY THE
C     EXTRAPOLATION TECHNIQUE
C
      INJ3=INJ2+NJ1
C
      DO 15 J=1,N
      XE(INJ3+J)=XE(INJ2+J)
15    CONTINUE
C
      SS=S*FACTOR
C
      DO 16 K=2,IJ
      L=IJ+1-K
      INJ4=INJ3
      INJ3=INJ3-N
      SS=SS/FACTOR
      DO 16 J=1,N
      INJ5=INJ3+J
16    XE(INJ5)=((SS-1.)*XE(INJ4+J)+XE(INJ5-NJ1))/SS
C
      DO 17 J=1,N
      X(J)=XE(INJ3+J)
17    CONTINUE
C
18    IF (JPRINT.EQ.0.OR.NR.EQ.1) GO TO 19
      CALL FUNCT4 (EN,G,JD,U,V,XB)
      IFN=IFN+1
      WRITE (6,21) EM,((JD(I),EN(JD(I))),I=1,NA)
19    IF (IH.EQ.IK) RETURN
C
C     THE PARAMETER IS BEING UPDATED HERE. NR=1 IMPLIES THAT THE FIACCO-
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C      MCCORMICK METHOD IS BEING USED                                   FL02930
C                                                                       FL02940
       P=P*FACTOR                                                       FL02950
       IF (NR.EQ.1) P=P/(FACTOR**2)                                     FL02960
C                                                                       FL02970
       JV=1                                                             FL02980
       CALL FUNCT4 (EN,G,JD,U,V,X)                                      FL02990
       IFN=IFN+1                                                        FL03000
       IF (NA.LE.1) RETURN                                              FL03010
       IF (JPRINT.EQ.2) WRITE (6,22) EM,(JD(I),V(JD(I)),JD(I),EN(JD(I)),IFL03020
      1=1,NA)                                                           FL03030
C                                                                       FL03040
C      THE FOLLOWING PROGRAM SEGMENT SELECTS THE ACTIVE FUNCTIONS IF    FL03050
C      IREDU .NE. 0. THIS SEGMENT CAN BE RE-EXECUTED BY THE USER AS MANY FL03060
C      TIMES AS NECESSARY BY USING A CALL AGAIN(EN, ... ,XE) STATEMENT  FL03070
C      IN THE MAIN PROGRAM                                              FL03080
C                                                                       FL03090
       ENTRY AGAIN                                                      FL03100
       IF (IREDU.EQ.0) RETURN                                           FL03110
       K=0                                                              FL03120
C                                                                       FL03130
       DO 20 I=1,NA                                                     FL03140
       J=JD(I)                                                          FL03150
       IF (V(J).LT.ETA) GO TO 20                                        FL03160
       K=K+1                                                            FL03170
       JD(K)=J                                                          FL03180
20     CONTINUE                                                         FL03190
C                                                                       FL03200
       NA=K                                                             FL03210
       RETURN                                                           FL03220
C                                                                       FL03230
21     FORMAT (*0NORMALIZED ERRORS AT THE HIGHEST ORDER OF THE MINIMAX SOFL03240
      1LUTION*/*0MAXIMUM OF THE ERROR FUNCTIONS *,11(*.*),* EM =*,E15.8/1FL03250
      2H0,50X,*NORM. ERROR*/50X,*VECTOR EN(I)*/1H0,43X,99(I4,E15.8/44X)) FL03260
C                                                                       FL03270
22     FORMAT (*0MULTIPLIERS AND NORMALIZED ERRORS AT THE STARTING POINT FL03280
      1OF THE*/1H ,62(*-*)/* NEXT ITERATION OF BANDLER-CHU P-ALGORITHM (FFL03290
      2LOPT4 IMPLEMENTATION)*/1H ,65(*-*)/*0MAXIMUM OF THE ERROR FUNCTIONFL03300
      3S *,11(*.*),* EM =*,E15.8/1H0,31X,*MULTIPLIER*,9X,*NORM. ERROR*/31FL03310
      4X,*VECTOR V(I)*,8X,*VECTOR EN(I)*/1H0,24X,99(I4,E15.8,I4,E15.8/25XFL03320
      5))                                                               FL03330
C                                                                       FL03340
23     FORMAT (*1ITERATION NO. *,I3,* OF BANDLER-CHU P-ALGORITHM (FLOPT4 FL03350
      1IMPLEMENTATION)*/1H ,68(*-*)/* WITH PARAMETER P =*,E15.8/1X,33(*-*FL03360
      2)/*0UNCONSTRAINED OPTIMIZATION USING 1972 VERSION OF FLETCHERS METFL03370
      3HOD*/1H ,65(*-*)/*0ITER. FUNC. OBJECTIVE*,10X,*VARIABLE*,11X,*GRADFL03380
      4IENT*/2X,*NO.  EVAL. FUNCTION*,10X,*VECTOR X(I)*,8X,*VECTOR G(I)*)FL03390
C                                                                       FL03400
24     FORMAT (*0ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAFL03410
      1POLATION*/1H ,64(*-*)/1H0,12X,*VARIABLE*/12X*VECTOR X(I)*/1H0,12X,FL03420
      2*ORDER 1*/1H0,5X,99(I4,E15.8/6X))                                FL03430
C                                                                       FL03440
25     FORMAT (*0ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAFL03450
      1POLATION*/1H ,64(*-*)/2H0 ,2(11X,*VARIABLE*)/4X,2(8X,*VECTOR X(I)*FL03460
      2)/1H0,12X,*ORDER 1*,12X,*ORDER 2*/1H0,5X,99(I4,E15.8,I4,E15.8/6X))FL03470
C                                                                       FL03480
26     FORMAT (*0ESTIMATED MINIMAX SOLUTION BY THE FIACCO-MCCORMICK EXTRAFL03490
      1POLATION*/1H ,64(*-*)/2H0 ,3(11X,*VARIABLE*)/4X,3(8X,*VECTOR X(I)*FL03500
      2)/1H0,3(12X,*ORDER*,I2)/1H0,5X,99(I4,E15.8,I4,E15.8,I4,E15.8/6X)) FL03510
C                                                                       FL03520
27     FORMAT (*1INPUT DATA FOR BANDLER-CHU P-ALGORITHM (FLOPT4 IMPLEMENTFL03530
      1ATION)*/1H ,62(*-*)/*-NUMBER OF ERROR FUNCTIONS *,16(*.*),* NR =*,FL03540
      2I5)                                                              FL03550
C                                                                       FL03560
28     FORMAT (*0EXECUTION TIME FOR THE ABOVE OPTIMIZATION IS *,F6.3,*  SFL03570
      1ECONDS*)                                                         FL03580
C                                                                       FL03590
29     FORMAT (*0MAXIMUM ORDER OF EXTRAPOLATION *,7(*.*),* JORDER =*,I5) FL03600
C                                                                       FL03610
30     FORMAT (*0AT THE BANDLER-CHARALAMBOUS LEAST PTH SOLUTION FOUND ABOFL03620
      1VE*/1H ,58(*-*)/*0MAXIMUM OF THE ERROR FUNCTIONS *,11(*.*),* EM =*FL03630
      2,E15.8/1H0,50X,*NORM. ERROR*/50X,*VECTOR EN(I)*/1H0,43X,99(I4,E15.FL03640
      38/44X))                                                          FL03650
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C
31    FORMAT (*0INITIAL VALUE OF THE PARAMETER *,12(*.*),* P =*,E15.8/*0
     1MULTIPLYING FACTOR FOR P *,13(*.*),* FACTOR =*,E15.8)
C
32    FORMAT (*0TOLERANCE ETA FOR INACTIVE FUNCTIONS .... ETA =*,E15.8)
C
33    FORMAT (*0PREDICTED FUNCTION LOWER BOUND *,10(*.*),* EST =*,E15.8/
     1*0STARTING POINT AND TEST QUANTITIES FOR THE TERMINATION CRITERION
     2*/1H0,31X,*VARIABLE*,14X,*TEST*/31X,*VECTOR X(I)*,8X,*VECTOR EPS(I
     3)*/1H0,24X,99(I4,E15.8,I4,E15.8/25X))
C
      END
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```
      SUBROUTINE LEASTP4 (EN,ER,ES,G,GE,JD,U,V)
C
C     THIS SUBROUTINE FORMULATES THE OBJECTIVE FUNCTION U FOR THE LEAST
C     PTH APPROXIMATION METHOD, CALCULATES THE GRADIENT VECTOR IF JV=0,
C     AND ALSO THE MULTIPLIERS IF JV=1. IT RESETS JV TO 0 AFTER BEING
C     EXECUTED. THIS ROUTINE IS CALLED ONLY FROM SUBROUTINE FUNCT4
C
      DIMENSION G(1),ER(1),EN(1),ES(1),GE(1),V(1),JD(1)
C
      COMMON /2/ N,NA,JV,P,EM
C
C     THE MAXIMUM OF THE ERROR FUNCTIONS IS BEING DETERMINED, AND IF
C     FOUND TO BE ZERO IT WILL BE SET EQUAL TO A SMALL NEGATIVE NUMBER
C
      EM=ER(JD(1))
C
      DO 1 I=2,NA
      EM=AMAX1(EM,ER(JD(I)))
1     CONTINUE
C
      IF (EM.NE.0.) GO TO 3
C
      DO 2 I=1,NA
      J=JD(I)
      ER(J)=ER(J)-1.E-10
2     CONTINUE
C
      EM=EM-1.E-10
C
C     LEAST PTH OBJECTIVE FUNCTION U IS BEING FORMULATED HERE
C
3     Q=SIGN(P,EM)
      S1=0.
C
      DO 5 I=1,NA
      J=JD(I)
      EN(J)=ER(J)/EM
      IF (EM.LT.0.) GO TO 4
      IF (ER(J).LE.0.) GO TO 5
4     ES(J)=EN(J)**Q
      S1=S1+ES(J)
5     CONTINUE
C
      ST=S1**(1./Q)
      U=EM*ST
      IF (JV.EQ.0) GO TO 8
C
C     MULTIPLIERS ARE BEING CALCULATED NOW
C
      DO 7 J=1,NA
      K=JD(J)
      V(K)=0.
      IF (EM.LT.0.) GO TO 6
      IF (ER(K).LE.0.) GO TO 7
6     V(K)=ES(K)/S1
7     CONTINUE
C
C     GRADIENTS ARE CALCULATED BY THE FOLLOWING PROGRAM SEGMENT
C
8     ST=ST/S1
C
      DO 11 I=1,N
      S2=0.
      DO 10 J=1,NA
      K=JD(J)
      IF (EM.LT.0.) GO TO 9
      IF (ER(K).LE.0.) GO TO 10
9     S2=S2+ES(K)*GE((K-1)*N+I)/EN(K)
10    CONTINUE
      G(I)=ST*S2
11    CONTINUE
C
      JV=0
      RETURN
      END
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      SUBROUTINE GRDCHK4 (N,X,G,W,EN,JD,V)
C
C     THIS SUBROUTINE IS CALLED ONLY ONCE BY FLOPT4 AT THE BEGINNING
C     TO VERIFY THAT THE GRADIENT VECTOR AS FORMULATED BY THE USER IS
C     CORRECT. GRADIENT VECTOR IS CALCULATED AT THE STARTING POINT ONCE
C     BY THE USERS DEFINITION AND AGAIN BY NUMERICALLY PERTURBING POINT
C     X. IF THE DIFFERENCE BETWEEN THE TWO VALUES EXCEEDS 10 P.C., THE
C     PROGRAM IS TERMINATED WITH A MESSAGE
C
      DIMENSION X(1),G(1),W(1),EN(1),JD(1),V(1)
C
C     TO CALCULATE G(I), AN ELEMENT OF THE GRADIENT VECTOR, X(I) IS
C     PERTURBED ONCE BY +DX AND ONCE BY -DX, AND THE FUNCTION EVALUATED
C     AT THESE POINTS. A SIMPLE DIVISION YIELDS THE VALUE OF G(I). IT
C     MAY BE NOTED THAT THE FUNCTION IS EVALUATED 2*N TIMES TO YIELD THE
C     ENTIRE GRADIENT VECTOR. IFN IS NOT UPDATED IN THIS ROUTINE
C
      ICON=0
      WRITE (6,3)
C
      DO 1 I=1,N
      Z=X(I)
      DX=1.E-6*X(I)
      IF (ABS(DX).LT.1.E-10) DX=1.E-10
      X(I)=Z+DX
      CALL FUNCT4 (EN,W,JD,U2,V,X)
      X(I)=Z-DX
      CALL FUNCT4 (EN,W,JD,U1,V,X)
      Y=0.5*(U2-U1)/DX
      X(I)=Z
      IF (ABS(Y).LT.1.E-20) Y=1.E-20
      GDUM=G(I)
      IF (ABS(GDUM).LT.1.E-20) GDUM=1.E-20
      YP=ABS((Y-GDUM)/Y)*100.0
      WRITE (6,4) I,G(I),I,Y,I,YP
      IF (YP.GT.10.) ICON=1
1     CONTINUE
C
      IF (ICON.EQ.1) GO TO 2
      WRITE (6,5)
      RETURN
2     WRITE (6,6)
      CALL EXIT
C
3     FORMAT (*-GRADIENT CHECK AT THE STARTING POINT*/1X,36(*-*)/1H0,11X
     1,*ANALYTICAL*,9X,*NUMERICAL*,10X,*PERCENTAGE*/13X,*GRADIENT*,11X,*
     2GRADIENT*,13X,*ERROR*/12X,2(*VECTOR G(I)*,8X),*VECTOR YP(I)*/)
C
4     FORMAT (6X,3(I4,E15.8))
C
5     FORMAT (1H0,///1H ,*GRADIENTS ARE O.K.*)
C
6     FORMAT (1H0,///1H ,*YOUR PROGRAM HAS BEEN TERMINATED BECAUSE GRADI
     1ENTS ARE INCORRECT*/1H0,*PLEASE CHECK THEM AGAIN*)
C
      END
```

```
      SUBROUTINE QUASI4 (N,X,U,G,H,W,EST,EPS,MODE,MAX,IPT,IEXIT,IFN,EN,J
     1D,V)
C
C     THIS SUBROUTINE IS BASED ON THE 1972 VERSION OF FLETCHERS METHOD
C     OF UNCONSTRAINED OPTIMIZATION. REFER TO REPORT AERE-R7125 FOR THE
C     THEORETICAL BACKGROUND AND THE ORIGINAL PROGRAM. ALTHOUGH ESSEN-
C     TIALLY THE SAME AS THE ORIGINAL FLETCHERS PROGRAM, SOME MINOR
C     CHANGES HAVE BEEN MADE, FOR EXAMPLE, 1) THE PORTION OF THE ORIGIN-
C     AL PROGRAM WHICH DECOMPOSES H INTO LDL(TRANSPOSE) HAS BEEN RE-
C     MOVED, 2) THE DO LOOP THAT INITIALLY DETERMINES THE SMALLEST ELE-
C     MENT ALONG THE DIAGONAL OF D HAS BEEN REMOVED, AND 3) THE FIRST
C     FUNCTION CALL HAS BEEN ELIMINATED
C
      DIMENSION X(1),G(1),H(1),W(1),EPS(1),EN(1),JD(1),V(1)
C
      COMMON /3/ DMIN
C
C     INITIALIZATION
C
      NP=N+1
      N1=N-1
      NN=N*NP/2
      IS=N
      IU=N
      IV=N+N
      IB=IV+N
      IEXIT=0
      IF (MODE.EQ.3) GO TO 3
C
C     THE INITIAL ESTIMATE OF H, AN IDENTITY MATRIX, IS GENERATED HERE
C
      IJ=NN+1
C
      DO 2 I=1,N
      DO 1 J=1,I
      IJ=IJ-1
      H(IJ)=0.
1     CONTINUE
      H(IJ)=1.
2     CONTINUE
C
      DMIN=1.
C
C     INITIAL PRINTING AND INITIALIZATION
C
3     Z=EST
      ITN=0
      DF=U-EST
      IF (DF.LE.0.0) DF=1.0
4     IF (IPT.EQ.0.OR.MOD(ITN,IPT).NE.0) GO TO 5
      PRINT 37, ITN,IFN,U,(I,X(I),I,G(I),I=1,N)
C
C     AN ITERATION OF QUASI4 BEGINS. IT INVOLVES SELECTION OF ALPHA,
C     THE LINE SEARCH PARAMETER, AND UPDATING OF H FOR THE NEXT ITERA-
C     TION OF QUASI4
C
5     ITN=ITN+1
C
C     THE DIRECTION OF SEARCH, WHICH IS THE PRODUCT OF THE INVERSE OF
C     THE HESSIAN H WITH THE GRADIENT VECTOR G, IS FOUND HERE. THE ELE-
C     MENTS OF THIS VECTOR ARE W(N+1), W(N+2),  ......  , W(2N)
C
      W(1)=-G(1)
C
      DO 7 I=2,N
      IJ=I
      I1=I-1
      Z=-G(I)
      DO 6 J=1,I1
      Z=Z-H(IJ)*W(J)
      IJ=IJ+N-J
6     CONTINUE
      W(I)=Z
```

```
7     CONTINUE
C
      W(IS+N)=W(N)/H(NN)
      IJ=NN
C
      DO 9 I=1,N1
      IJ=IJ-1
      Z=0.
      DO 8 J=1,I
      Z=Z+H(IJ)*W(IS+NP-J)
      IJ=IJ-1
8     CONTINUE
      W(IS+N-I)=W(N-I)/H(IJ)-Z
9     CONTINUE
C
C     THE SCALAR PRODUCT OF G WITH THE DIRECTION OF SEARCH IS NOW FOUND.
C     IT MUST BE NEGATIVE OR ELSE THE FUNCTION CAN NOT BE MINIMIZED ANY
C     FURTHER. GS IS TESTED TO ENSURE THIS
C
      GS=0.
C
      DO 10 I=1,N
      GS=GS+W(IS+I)*G(I)
10    CONTINUE
C
      IEXIT=2
      IF (GS.GE.0.) GO TO 32
C
C     ALPHA, THE LINE SEARCH PARAMETER, WILL NOW BE CALCULATED USING
C     EITHER THE QUADRATIC FIT, THE CUBIC INTERPOLATION, OR THE LINEAR
C     EXTRAPOLATION. AN INEXACT LINE SEARCH IS MADE HERE
C
      GS0=GS
      ALPHA=-2.*DF/GS
      IF (ALPHA.GT.1.) ALPHA=1.
      DF=U
      TOT=0.
11    IEXIT=3
      IF (IFN.EQ.MAX) GO TO 32
      ICON=0
      IEXIT=1
C
      DO 12 I=1,N
      Z=ALPHA*W(IS+I)
      IF (ABS(Z).GE.EPS(I)) ICON=1
      X(I)=X(I)+Z
12    CONTINUE
C
      CALL FUNCT4 (EN,W,JD,FY,V,X)
      IFN=IFN+1
C
C     ELEMENTS W(1),W(2), .... ,W(N) NOW CONTAIN THE GRADIENT VECTOR.
C     GYS, IN THE FOLLOWING SECTION, IS THE SCALAR PRODUCT OF THE GRAD-
C     IENT AT THE NEXT POINT WITH THE PRESENT DIRECTION OF SEARCH
C
      GYS=0.
C
      DO 13 I=1,N
      GYS=GYS+W(I)*W(IS+I)
13    CONTINUE
C
      IF (FY.GE.U) GO TO 14
      IF (ABS(GYS/GS0).LE..9) GO TO 16
      IF (GYS.GT.0.) GO TO 14
C
C     LINEAR EXTRAPOLATION FOR ALPHA IS PERFORMED HERE
C
      TOT=TOT+ALPHA
      Z=10.
      IF (GS.LT.GYS) Z=GYS/(GS-GYS)
      IF (Z.GT.10.) Z=10.
      ALPHA=ALPHA*Z
      U=FY
```

```
      GS=GYS                                                      QUA1470
      GO TO 11                                                    QUA1480
C                                                                 QUA1490
C     CUBIC INTERPOLATION TO FIND ALPHA IS PERFORMED HERE         QUA1500
C                                                                 QUA1510
14    DO 15 I=1,N                                                 QUA1520
      X(I)=X(I)-ALPHA*W(IS+I)                                     QUA1530
15    CONTINUE                                                    QUA1540
C                                                                 QUA1550
      IF (ICON.EQ.0) GO TO 32                                     QUA1560
      Z=3.*(U-FY)/ALPHA+GYS+GS                                    QUA1570
      ZZ=SQRT(Z*Z-GS*GYS)                                         QUA1580
      GZ=GYS+ZZ                                                   QUA1590
      Z=1.-(GZ-Z)/(ZZ+GZ-GS)                                      QUA1600
      ALPHA=ALPHA*Z                                               QUA1610
      GO TO 11                                                    QUA1620
C                                                                 QUA1630
C     THE LINE SEARCH HAS BEEN COMPLETED AND A NEW POINT HAS BEEN OB-  QUA1640
C     TAINED. H MUST BE UPDATED NOW                               QUA1650
C                                                                 QUA1660
16    ALPHA=TOT+ALPHA                                             QUA1670
      U=FY                                                        QUA1680
      IF (ICON.EQ.0) GO TO 30                                     QUA1690
      DF=DF-U                                                     QUA1700
      DGS=GYS-GS0                                                 QUA1710
      LINK=1                                                      QUA1720
C                                                                 QUA1730
C     IF THE FOLLOWING TEST IS TRUE, THE DFP FORMULA WILL BE USED FOR  QUA1740
C     UPDATING H, OTHERWISE, THE COMPLEMENTARY DFP FORMULA WILL BE USED QUA1750
C                                                                 QUA1760
      IF (DGS+ALPHA*GS0.GT.0.) GO TO 18                           QUA1770
C                                                                 QUA1780
      DO 17 I=1,N                                                 QUA1790
      W(IU+I)=W(I)-G(I)                                           QUA1800
17    CONTINUE                                                    QUA1810
C                                                                 QUA1820
      SIG=1./(ALPHA*DGS)                                          QUA1830
      GO TO 25                                                    QUA1840
18    ZZ=ALPHA/(DGS-ALPHA*GS0)                                    QUA1850
      Z=DGS*ZZ-1.                                                 QUA1860
C                                                                 QUA1870
      DO 19 I=1,N                                                 QUA1880
      W(IU+I)=Z*G(I)+W(I)                                         QUA1890
19    CONTINUE                                                    QUA1900
C                                                                 QUA1910
      SIG=1./(ZZ*DGS*DGS)                                         QUA1920
      GO TO 25                                                    QUA1930
20    LINK=2                                                      QUA1940
C                                                                 QUA1950
      DO 21 I=1,N                                                 QUA1960
      W(IU+I)=G(I)                                                QUA1970
21    CONTINUE                                                    QUA1980
C                                                                 QUA1990
      IF (DGS+ALPHA*GS0.GT.0.) GO TO 22                           QUA2000
      SIG=1./GS0                                                  QUA2010
      GO TO 25                                                    QUA2020
22    SIG=-ZZ                                                     QUA2030
      GO TO 25                                                    QUA2040
C                                                                 QUA2050
23    DO 24 I=1,N                                                 QUA2060
      G(I)=W(I)                                                   QUA2070
24    CONTINUE                                                    QUA2080
C                                                                 QUA2090
      GO TO 4                                                     QUA2100
25    W(IV+1)=W(IU+1)                                             QUA2110
C                                                                 QUA2120
      DO 27 I=2,N                                                 QUA2130
      IJ=I                                                        QUA2140
      I1=I-1                                                      QUA2150
      Z=W(IU+I)                                                   QUA2160
      DO 26 J=1,I1                                                QUA2170
      Z=Z-H(IJ)*W(IV+J)                                           QUA2180
      IJ=IJ+N-J                                                   QUA2190
```

```
26    CONTINUE
      W(IV+I)=Z
27    CONTINUE
C
      IJ=1
C
      DO 28 I=1,N
      IVI=IV+I
      IBI=IB+I
      Z=H(IJ)+SIG*W(IVI)*W(IVI)
      IF (Z.LE.0.) Z=DMIN
      IF (Z.LT.DMIN) DMIN=Z
      H(IJ)=Z
      W(IBI)=W(IVI)*SIG/Z
      SIG=SIG-W(IBI)*W(IBI)*Z
      IJ=IJ+NP-I
28    CONTINUE
C
      IJ=1
C
      DO 29 I=1,N1
      IJ=IJ+1
      I1=I+1
      DO 29 J=I1,N
      W(IU+J)=W(IU+J)-H(IJ)*W(IV+I)
      H(IJ)=H(IJ)+W(IB+I)*W(IU+J)
29    IJ=IJ+1
C
      IF (LINK-2) 20,23,23
C
C     THE UPDATING OF H IS NOW COMPLETE AND THE NEXT ITERATION BEGINS
C
30    DO 31 I=1,N
      G(I)=W(I)
31    CONTINUE
C
32    IF (IPT.EQ.0) GO TO 33
      PRINT 37, ITN,IFN,U,(I,X(I),I,G(I),I=1,N)
33    IF (IEXIT-2) 36,34,35
34    PRINT 38, IEXIT
      CALL EXIT
35    PRINT 39, IEXIT
      CALL EXIT
36    RETURN
C
37    FORMAT (1H0,I3,2X,I4,E15.8,99(I4,E15.8,I4,E15.8/25X))
C
38    FORMAT (1H1,*IEXIT =*,I2/1H0,*EPS CHOSEN IS TOO SMALL*)
C
39    FORMAT (1H1,*IEXIT =*,I2/1H0,*MAXIMUM NUMBER OF ALLOWABLE
     1FUNCTION EVALUATIONS HAVE BEEN PERFORMED*)
C
      END
```

SOC-151

FLOPT4 - A PROGRAM FOR LEAST PTH OPTIMIZATION WITH EXTRAPOLATION TO MINIMAX SOLUTIONS

J.W. Bandler and D. Sinha

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Key Words: Unconstrained optimization, gradient minimization methods, penalty function methods, least pth optimization, extrapolation

Abstract: FLOPT4 is a package of subroutines primarily for solving least pth optimization problems. Its main features include Fletcher's quasi-Newton subroutine, a least pth objective formulation subroutine, an extrapolation procedure and a scheme for dropping inactive functions. With appropriate utilization of these features, the program can solve a wide variety of optimization problems. These may range from unconstrained problems, problems subject to inequality or equality constraints to nonlinear minimax approximation problems. In solving constrained problems, the user may, for example, use the Fiacco-McCormick method with extrapolation or the Bandler-Charalambous minimax formulation and least pth approximation, also with extrapolation. The program has been used on a CDC 6400 computer. Several detailed examples of varying complexity are used to illustrate the versatility of the program. A FORTRAN IV listing is included. FLOPT4 replaces a previous package on which it is based, namely, FLOPT2.

Description: Contains Fortran listing, user's manual. Source deck available for $100.00. The listing contains 780 statements of which 366 are comment cards.

Related Work: As for SOC-84. Represents further development of the work presented in SOC-84.

Price: $60.00.