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OPTIMAL CENTERING, TOLERANCING AND YIELD DETERMINATION USING MULTIDIMENSIONAL APPROXIMATIONS

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Abstract

A method is described for efficient optimal design centering and tolerance assignment. In order to overcome the obstacle of scarcity of simulation programs incorporating both the efficient analysis of performance and its sensitivities, a suitable modelling of the functions involved using low-order multidimensional approximations is used. As a result, rapid and accurate determination of design solutions are facilitated, even with relatively inefficiently written analysis programs or with experimentally obtained data. An efficient technique for evaluating the multidimensional approximations and their derivatives is also given. Formulas for yield and yield sensitivities in the case of independent designable parameters, assuming uniform distribution of outcomes between tolerance extremes, are also presented. In addition, this procedure facilitates an inexpensive yield estimate using Monte Carlo analysis in conjunction with the multidimensional approximations. Simple circuit examples illustrate worst-case design and design with yields of less than 100%. The examples also provide verification of the formulas and algorithms.

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I. INTRODUCTION

The optimal tolerance problem, which is also known as the design centering and tolerance assignment problem, is now an integral part of the design process. Design centering is the process of defining a set of nominal parameter values either to maximize the allowable parameter tolerances, in the worst-case design, or to maximize the yield for known but unavoidable statistical fluctuations. Several approaches have been applied to solve this problem. The nonlinear programming approach was used by Bandler et. al. [1, 2] and by Pinel and Roberts [3]. The branch and bound approach was discussed by Karafin [4]. A method which makes use of Monte Carlo analysis was used by Elias [5]. An approach involving approximations of the feasible region is that used by Director and Hachtel [6].

The approach described in this work can make use of any simulation program, whether efficiently written or not, or not containing sensitivity information, for the purpose of design centering and yield determination or optimization. Nonlinear programming is used to inscribe an orthotope inside the feasible region by minimizing a suitable scalar objective function. This orthotope will actually be the optimum tolerance region for a worst-case design problem with independent variables. In the process of inscribing this orthotope an updated sequence of second-order multidimensional polynomial approximations describing the different constraints in certain critical regions are obtained. These second-order approximations can further be used, for example, for inexpensive statistical circuit analysis in the parameter space without any need for the usual multitude of circuit simulations. They can be used for yield determination or optimization directly in the case of independent designable parameters assuming uniform distribution of outcomes between tolerance extremes. The readily differentiable polynomial approximations are also used for solving the nonlinear programming problem using an efficient gradient technique.

This paper describes a method for choosing interpolation base points in order to guarantee a one-dimensionally convex feasible region if the interpolated region is so. It contains an efficient technique for evaluating the approximations and their derivatives at different vertices in different well-chosen interpolation regions. In the case of independent designable parameters and assuming uniform distribution of outcomes between tolerance extremes, formulas for yield and yield sensitivities are given for the linear constraint case as well as their extension in the quadratic constraint case.

Some illustrative examples are also included. A two-section quarter-wave transmission-line transformer is used to explain how a worst-case design is obtained and, further, is used for yield determination and optimization. A worst-case design and a well-centered design for yield less than 100% for a three-section lowpass LC filter as well as a check using Monte Carlo analysis are included. A practical example of a non-ideal two-section waveguide transformer is described. The worst-case design as well as yield determination for the enlarged tolerance region and a comparison between execution times for the Monte Carlo analysis applied to the actual constraints and the approximated constraints are given.

II. OPTIMAL CENTERING AND TOLERANCING

The tolerance assignment problem can be stated as: minimize some cost function

$$C(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon})$$

subject, for example, to the constraint on yield

$$Y(\underset{\sim}{\phi}^0, \underset{\sim}{\varepsilon}) \geq Y_L \quad , \tag{1}$$

where

$$\underset{\sim}{\phi}^0 \triangleq \begin{bmatrix} \phi_1^0 \\ \phi_2^0 \\ \vdots \\ \phi_k^0 \end{bmatrix} \geq \underset{\sim}{0} \quad , \qquad \underset{\sim}{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_k \end{bmatrix} \geq \underset{\sim}{0} \ . \tag{2}$$

k is the number of designable parameters, $\underset{\sim}{\phi}^0$ is the nominal point, $\underset{\sim}{\varepsilon}$ is the tolerance vector and Y_L is a yield specification.

R_c is the constraint region defined by m_c functions $g_i(\underset{\sim}{\phi})$ and given by

$$R_c \triangleq \{\underset{\sim}{\phi} \mid g_i(\underset{\sim}{\phi}) \geq 0, \ i = 1, 2, \ldots, m_c\} \ . \tag{3}$$

Thus, for the worst-case design [1, 7], sometimes called the 100% yield, it is required that

$$R_\varepsilon \subset R_c \quad , \tag{4}$$

where R_ε is the tolerance region given by

$$R_\varepsilon \triangleq \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu}, \ -1 \leq \mu_i \leq 1, \ i = 1, 2, \ldots, k\}, \tag{5}$$

where $\underset{\sim}{E}$ is a k x k matrix with diagonal elements set to ε_i.

For a one-dimensionally convex region [7] it is sufficient that the set of all vertices R_v satisfy the following condition

$$R_v \subset R_c \ , \tag{6}$$

where R_v is defined by

$$R_v \stackrel{\Delta}{=} \{\underset{\sim}{\phi} \mid \underset{\sim}{\phi} = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu},\ \mu_i \in \{-1, 1\},\ i = 1, 2, \ldots, k\} . \quad (7)$$

III. INTERPOLATION BY QUADRATIC POLYNOMIAL

An approximate representation of a function $f(\underset{\sim}{\phi})$ by using its values at a finite set of points is possible [8, 9]. These points are called nodes or base points, and denoted by

$$\underset{\sim}{\phi}^n ,\ n = 1, 2, \ldots, N .$$

Interpolation can be done by means of a linear combination of the set of all possible monomials. Hence,

$$f(\underset{\sim}{\phi}) \simeq \sum_{\nu=1}^{N} a_\nu \Phi_\nu , \quad (8)$$

where

$$\Phi_\nu \stackrel{\Delta}{=} \phi_1^{\alpha_1} \phi_2^{\alpha_2} \ldots \phi_k^{\alpha_k} , \quad \sum_{i=1}^{k} \alpha_i \leq m \quad (9)$$

and m is the degree of the interpolating polynomial, in our case 2. The number of such monomials is given by

$$N = \frac{(m+k)!}{m!k!} . \quad (10)$$

Let

$$\underset{\sim}{\psi}_\nu = [\Phi_\nu(\underset{\sim}{\phi}^1)\ \Phi_\nu(\underset{\sim}{\phi}^2)\ \ldots\ \Phi_\nu(\underset{\sim}{\phi}^N)]^T \quad (11)$$

be an N-dimensional column vector, and

$$\underset{\sim}{A} = [\underset{\sim}{\psi}_1\ \underset{\sim}{\psi}_2\ \ldots\ \underset{\sim}{\psi}_N] \quad (12)$$

be an N x N matrix. In the case of m = 2, $\underset{\sim}{A}$ has the form

$$\underset{\sim}{A} = \left[\begin{array}{cccc|cccc|ccccc} (\phi_1^1)^2 & (\phi_2^1)^2 & \ldots & (\phi_k^1)^2 & \phi_1^1\phi_2^1 & \phi_1^1\phi_3^1 & \ldots & \phi_{k-1}^1\phi_k^1 & \phi_1^1 & \phi_2^1 & \ldots & \phi_k^1 & 1 \\ (\phi_1^2)^2 & (\phi_2^2)^2 & \ldots & (\phi_k^2)^2 & \phi_1^2\phi_2^2 & \phi_1^2\phi_3^2 & \ldots & \phi_{k-1}^2\phi_k^2 & \phi_1^2 & \phi_2^2 & \ldots & \phi_k^2 & 1 \\ \vdots & & & & & & & & & & & & \\ (\phi_1^N)^2 & (\phi_2^N)^2 & \ldots & (\phi_k^N)^2 & \phi_1^N\phi_2^N & \phi_1^N\phi_3^N & \ldots & \phi_{k-1}^N\phi_k^N & \phi_1^N & \phi_2^N & \ldots & \phi_k^N & 1 \end{array}\right] \quad \ldots (13)$$

The values of the polynomial at the base points $\underset{\sim}{\phi}^n$ are given by

$$\underset{\sim}{P}(\underset{\sim}{\phi}^n) = \underset{\sim}{A}\,\underset{\sim}{a} = \underset{\sim}{f}(\underset{\sim}{\phi}^n) \quad , \tag{14}$$

where $\underset{\sim}{a}$ is the unknown coefficient column vector.

The solution of (14) exists if $\underset{\sim}{A}$ is nonsingular. This is satisfied when the set of base points is a degree-2 independent [10]. For a particular choice of base points the quadratic interpolating polynomial will be one-dimensionally convex/concave if the approximated function is so (see Appendix).

Now, let $\underset{\sim}{\overline{\phi}}$ be the centre of the interpolation region and $\underset{\sim}{\delta}$ be a step vector defining the size of the interpolation region in the following manner. For any base point $\underset{\sim}{\phi}^n$, $n = 1, 2, \ldots, N$, we have

$$|\phi_i^n - \overline{\phi}_i| \leq \delta_i \; , \qquad i = 1, 2, \ldots, k \; . \tag{15}$$

The set of base points is given by

$$[\underset{\sim}{\phi}^1 \quad \underset{\sim}{\phi}^2 \ldots \underset{\sim}{\phi}^N] = \underset{\sim}{D} \left[\begin{array}{c|c|c|c} \underset{\sim}{0} & \underset{\sim}{I}_k & -\underset{\sim}{I}_k & \underset{\sim}{B} \end{array} \right] + [\, \underset{\sim}{\overline{\phi}} \; \underset{\sim}{\overline{\phi}} \ldots \underset{\sim}{\overline{\phi}}], \tag{16}$$

where $\underset{\sim}{0}$ is the zero vector of dimension k, $\underset{\sim}{D}$ is a k x k matrix with diagonal elements δ_i, $\underset{\sim}{I}_k$ is a k x k unit matrix, $\underset{\sim}{B}$ is a k x $(\frac{k(k-1)}{2})$ matrix defined by

$$\underset{\sim}{B} = [\underset{\sim}{\mu}^1 \quad \underset{\sim}{\mu}^2 \ldots \underset{\sim}{\mu}^L] \; , \tag{17}$$

in which

$$L = \frac{k(k-1)}{2} \tag{18}$$

and

$$-1 \leq \mu_i^j \leq 1, \qquad j = 1, 2, \ldots, L \; . \tag{19}$$

This choice of base points allows a check for one-dimensional convexity/concavity of the approximated function, since there are three base points along each axis.

IV. EFFICIENT CALCULATION OF POLYNOMIAL AND GRADIENTS AT VERTICES

The method used for computing the polynomial and its gradients at the vertices exploits the simple properties of a quadratic approximation. The following two equations are used to obtain the polynomial value and its gradients at any vertex $\underset{\sim}{\phi}^r$ using values at another vertex $\underset{\sim}{\phi}^s$.

$$P(\underset{\sim}{\phi}^r) = P(\underset{\sim}{\phi}^s) + (\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^s)^T \underset{\sim}{\nabla} P(\underset{\sim}{\phi}^s) + \frac{1}{2}(\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^s)^T \underset{\sim}{H}(\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^s) \tag{20}$$

$$\underset{\sim}{\nabla} P(\underset{\sim}{\phi}^r) = \underset{\sim}{\nabla} P(\underset{\sim}{\phi}^s) + \underset{\sim}{H}(\underset{\sim}{\phi}^r - \underset{\sim}{\phi}^s) \tag{21}$$

where $\underset{\sim}{H}$ is the Hessian matrix for the quadratic approximation.

Let $\underset{\sim}{\phi}^r$ and $\underset{\sim}{\phi}^s$ be related as follows

$$\underset{\sim}{\phi}^r = \underset{\sim}{\phi}^s + 2\varepsilon_i \underset{\sim}{e}_i \tag{22}$$

where $\underset{\sim}{e}_i$ is the unit vector in the ith direction.

Hence, we have

$$r = s + 2^{i-1} \tag{23}$$

according to the following vertex enumeration scheme:

$$r = 1 + \sum_{i=1}^{k} \frac{(\mu_i^r + 1)}{2} 2^{i-1} \quad , \qquad \mu_i^r \in \{-1, 1\} \tag{24}$$

where

$$\underset{\sim}{\phi}^r = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\, \underset{\sim}{\mu}^r \quad . \tag{25}$$

Then (20) and (21) reduce to

$$P(\underset{\sim}{\phi}^r) = P(\underset{\sim}{\phi}^s) + 2\varepsilon_i \nabla_i P(\underset{\sim}{\phi}^s) + 2\varepsilon_i^2 H_{ii} \tag{26}$$

$$\underset{\sim}{\nabla} P(\underset{\sim}{\phi}^r) = \underset{\sim}{\nabla} P(\underset{\sim}{\phi}^s) + 2\varepsilon_i \underset{\sim}{H}_i \quad , \tag{27}$$

where H_{ii} is the ith diagonal element of $\underset{\sim}{H}$ and $\underset{\sim}{H}_i$ is the ith column of $\underset{\sim}{H}$.

If $\underset{\sim}{\phi}^r$ and $\underset{\sim}{\phi}^s$ fall into two different interpolation regions, which is the case if $\varepsilon_i > \delta_i$ (see Fig. 1), (26) and (27) can not be used because of the different polynomials.

Now, let $\underset{\sim}{H}^{\ell}$, $\ell = 1, 2, \ldots, N_{in}$ denote the Hessian matrix at the different interpolation regions, where N_{in} is the number of interpolation regions.

Define the set I as

$$I \triangleq \{i \mid \varepsilon_i \leq \delta_i\} \quad . \tag{28}$$

It is clear that if n_i is the number of elements of I, then

$$N_{in} = 2^{k-n_i} \quad . \tag{29}$$

The efficient algorithm is described by the following steps.

Step 1. Compute $P^{\ell}(\underset{\sim}{\phi}^s)$ and $\underset{\sim}{\nabla} P^{\ell}(\underset{\sim}{\phi}^s)$ for all $s \in S$, where

$$S = \{s \mid s = 1 + \sum_{i=1}^{k} \frac{(\mu_i^s + 1)}{2} 2^{i-1},\ \mu_i^s = -1 \text{ if } i \in I,\ \mu_i^s \in \{-1,1\} \text{ if } i \notin I\} \tag{30}$$

$$\ell = 1 + \sum_{i=1}^{k} \frac{(\mu_i^s + 1)}{2} \, 2^{\sum_{j=1}^{i} p_j - 1} \tag{31}$$

$$p_j = \begin{cases} 0 & \text{if } j \in I \\ 1 & \text{if } j \notin I \end{cases} \tag{32}$$

Step 2. If I is empty stop.

Step 3. Set $i \leftarrow i_1$ where $i_1 \in I$ and $i_1 \leq i$ for all $i \in I$.

Step 4. Find $T = \varepsilon_i + \varepsilon_i$.

Step 5. Find the vectors $\underset{\sim}{G}_i^{\ell} = T\, \underset{\sim}{H}_i^{\ell}$ for all ℓ defined by (31).

Step 6. For all $s \in S$ and for all ℓ, calculate

$$P^{\ell}(\underset{\sim}{\phi}^r) = P^{\ell}(\underset{\sim}{\phi}^s) + T\, \nabla_i\, P^{\ell}(\underset{\sim}{\phi}^s) + \varepsilon_i\, G_{ii}^{\ell} \tag{33}$$

$$\underset{\sim}{\nabla} P^{\ell}(\underset{\sim}{\phi}^r) = \underset{\sim}{\nabla} P^{\ell}(\underset{\sim}{\phi}^s) + \underset{\sim}{G}_i^{\ell} \tag{34}$$

where r is defined by (23) and G^{ℓ}_{ii} is the ith element of $\underset{\sim}{G}^{\ell}_{i}$.

Step 7. Set $$S \leftarrow S \cup \{r \mid r = s + 2^{i-1},\ s \in S\} \quad , \tag{35}$$

$$I \leftarrow I - \{1, 2, \ldots, i\} \tag{36}$$

and return back to step 2.

This scheme is illustrated in Fig. 2 for different cases. The computational effort required for considering all vertices compared to that required for one vertex only is shown in Table I.

V. ALGORITHM FOR WORST-CASE DESIGN

Approximation is only done for complicated functions (objective, responses or constraints) or functions for which gradient information is not available. Choose initial values for $\underset{\sim}{\phi}^0$, $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\delta}$.

Step 1. Set $\bar{\underset{\sim}{\phi}}$, the centre of the interpolation region, to $\underset{\sim}{\phi}^0$.

Until $\delta_i \geq \varepsilon_i$, $i = 1, 2, \ldots, k$, set $\delta_i \leftarrow 4\delta_i$.

Step 2. A set of base points $\underset{\sim}{\phi}^n$, $n = 1, 2, \ldots, N$, are chosen to satisfy (15) and (16).

Step 3. Interpolation is carried out in this region by solving the system of linear equations (14).

Step 4. A worst case design is to be obtained with respect to these approximations.

Step 5. Set $\underset{\sim}{\phi}^0$ and $\underset{\sim}{\varepsilon}$ to the optimum values obtained in step 4.

Step 6. If $|\phi^0_i - \bar{\phi}_i| > 1.5\ \delta_i$ for any $i = 1, 2, \ldots, k$, go to Step 1, otherwise set $\underset{\sim}{\delta} \leftarrow \underset{\sim}{\delta}/4$ and one of the following two cases results:

i) If $\delta_i \geq \varepsilon_i$, $i = 1, 2, \ldots, k$, set $\bar{\underset{\sim}{\phi}} = \underset{\sim}{\phi}^0$ and go to Step 2.

ii) If $\delta_i < \varepsilon_i$ for any $i = 1, 2, \ldots, k$, then interpolation is done around the centre points $\bar{\underset{\sim}{\phi}}^{\ell}$, where

$$\bar{\phi}^{\ell} \in R_a \triangleq \{\phi \mid \phi = \phi^0 + P\, E\, \mu^{\ell},\ \mu_i^{\ell} \in \{-1,1\},\ i = 1, 2, \ldots, k\}, \quad (37)$$

where P is a k x k diagonal matrix with elements p_i along the diagonal. Thus,

$$N_{in} = 2^{\left(\sum_{i=1}^{k} p_i\right)} , \quad (38)$$

where p_i is defined by (32). (This will reduce computation in formulation and solving (14).)

Step 7. The step size δ is reduced to $\delta/4$ only if all active vertices satisfy the following condition:

$$|\phi_i^r - \bar{\phi}_i^{\ell}| \leq 2\,\delta_i \quad , \quad i = 1, 2, \ldots, k, \quad (39)$$

where

$$\mu_i^r = \mu_i^{\ell} \qquad \text{for } i \notin I \quad . \quad (40)$$

Step 8. This procedure is performed several times until components of δ become smaller than certain prescribed values.

VI. YIELD ESTIMATION AND YIELD SENSITIVITIES

The Linear Constraints Case

An estimate of the yield in the case of uncorrelated uniformly distributed parameters is given by

$$Y = 1 - \frac{\sum_{\ell} V^{\ell}}{2^k \prod_{i=1}^{k} \varepsilon_i} , \quad (41)$$

where

$$V^{\ell} = \left\{ \frac{1}{k!} \prod_{j=1}^{k} \alpha_j^{\ell} \right\} \left\{ 1 + \sum_{\nu=1}^{k} (-1)^{\nu} \sum_{\beta=1}^{n_{\nu}^{\ell}} \left(1 - \sum_{1}^{\nu} \frac{2\varepsilon_{i_\beta}}{\alpha_{i_\beta}^{\ell}} \right)^k \right\} \quad (42)$$

is the nonfeasible hypervolume in the tolerance region according to the ℓth constraint given by

$$g_\ell(\underset{\sim}{\phi}) = \underset{\sim}{\phi}^T \underset{\sim}{q}^\ell - c \geq 0 \ . \tag{43}$$

n_ν^ℓ is the number of vertices differing in ν parameters i_β from the nonfeasible reference vertex $\underset{\sim}{\phi}^r = \underset{\sim}{\phi}^0 + \underset{\sim}{E}\,\underset{\sim}{\mu}^r$ and do not satisfy the ℓth constraint. We take

$$\mu_i^r = - \text{sign}(q_i^\ell), \ i = 1, 2, \ldots, k \ . \tag{44}$$

$\sum_{\nu=1}^{k} n_\nu^\ell$ is the total number of vertices which do not satisfy the ℓth constraint. α_j^ℓ is the distance from the reference vertex to the point of intersection along the jth direction (α_j^ℓ may be greater than $2\,\varepsilon_j$). Fig. 3 illustrates some cases for volume calculation when k = 3. The assumption of no overlapping of nonfeasible regions defined by different constraints inside the orthotope is required in order to use (41), i.e.,

$$R_\varepsilon \cap \underset{i \neq j}{R_i \cap R_j} = \emptyset, \quad i, j = 1, 2, \ldots, m_c, \tag{45}$$

where

$$R_i \triangleq \{\underset{\sim}{\phi} \mid g_i(\underset{\sim}{\phi}) < 0\} \tag{46}$$

and $\emptyset$ is the empty set.

In order to find V^ℓ the intersections between the hyperplane $g_\ell(\underset{\sim}{\phi}) = 0$ and the orthotope edges are required. Any of these intersections is obtained by solving the linear equation

$$\underset{\sim}{\phi}^T \underset{\sim}{q}^\ell - c = 0 \tag{47}$$

knowing that the ϕ_j are fixed along a certain edge. They are given by

$$\phi_j = \phi_j^0 + \varepsilon_j\,\mu_j^r \ , \quad \mu_j^r \in \{-1, 1\}, \ j = 1, 2, \ldots, i-1, i+1, \ldots, k. \tag{48}$$

Hence,

$$\phi_i = \frac{c - \sum_{j \neq i} q_j^\ell\,\phi_j}{q_i^\ell} \ , \quad i = 1, 2, \ldots, k \ . \tag{49}$$

Then

$$\alpha_i^\ell = | \phi_i^r - \phi_i | = \mu_i^r (\phi_i^r - \phi_i) \quad , \quad i = 1, 2, \ldots, k \quad . \tag{50}$$

The yield sensitivities are calculated according to the gradients of these k intersections.

$$\frac{\partial Y}{\partial \phi_i^0} = - \frac{1}{2^k \prod_{j=1}^{k} \varepsilon_j} \sum_\ell \frac{\partial V^\ell}{\partial \phi_i^0} \tag{51}$$

$$\frac{\partial Y}{\partial \varepsilon_i} = \left(\frac{1}{\varepsilon_i} \sum_\ell V^\ell - \sum_\ell \frac{\partial V^\ell}{\partial \varepsilon_i} \right) / \left(2^k \prod_{j=1}^{k} \varepsilon_j \right) \tag{52}$$

where

$$\frac{\partial V^\ell}{\partial \phi_i^0} = \left\{ \frac{1}{k!} \left[\mu_i^r \prod_{\substack{j=1 \\ j \neq i}}^{k} \alpha_j^\ell + \sum_{\substack{p=1 \\ p \neq i}}^{k} \left(\mu_p^r \frac{q_i^\ell}{q_p^\ell} \prod_{\substack{j=1 \\ j \neq p}}^{k} \alpha_j^\ell \right) \right] \right\} X$$

$$+ Z \left\{ k \sum_{\nu=1}^{k} (-1)^\nu \sum_{\beta=1}^{n_\nu^\ell} \left(1 - \sum_1^\nu \frac{2\varepsilon_{i_\beta}}{\alpha_{i_\beta}^\ell} \right)^{k-1} \right.$$

$$\left. \cdot \left(\sum_1^\nu \frac{q_i^\ell}{q_{i_\beta}^\ell} \frac{2\varepsilon_{i_\beta} \mu_{i_\beta}^r}{(\alpha_{i_\beta}^\ell)^2} \right) \right\} \tag{53}$$

and where

$$X = 1 + \sum_{\nu=1}^{k} (-1)^\nu \sum_{\beta=1}^{n_\nu^\ell} \left(1 - \sum_1^\nu \frac{2\varepsilon_{i_\beta}}{\alpha_{i_\beta}^\ell} \right)^k \tag{54}$$

$$Z = \frac{1}{k!} \prod_{j=1}^{k} \alpha_j^\ell \tag{55}$$

$$\frac{\partial V^{\ell}}{\partial \varepsilon_i} = \left\{ \frac{1}{k!} \left[\prod_{\substack{j=1 \\ j\neq i}}^{k} \alpha_j^{\ell} + \mu_i^r \sum_{\substack{p=1 \\ p\neq i}}^{k} \mu_p^r \left(\frac{q_i^{\ell}}{q_p^{\ell}} \prod_{\substack{j=1 \\ j\neq p}}^{k} \alpha_j^{\ell} \right) \right] \right\} X$$

$$+ Z \left\{ k \sum_{\nu=1}^{k} (-1)^{\nu} \sum_{\beta=1}^{n_{\nu}^{\ell}} \left(1 - \sum_{1}^{\nu} \frac{2\varepsilon_{i_\beta}}{\alpha_{i_\beta}^{\ell}} \right)^{k-1} \right.$$

$$\left. \cdot \left[-\frac{2\sigma}{\alpha_i^{\ell}} + \mu_i^r \sum_{1}^{\nu} \mu_{i_\beta}^r \frac{q_i^{\ell}}{q_{i_\beta}^{\ell}} \frac{2\varepsilon_{i_\beta}}{(\alpha_{i_\beta}^{\ell})^2} \right] \right\} , \qquad (56)$$

where $\sigma = 1$ if $i = i_\beta$,otherwise $\sigma = 0$.

It is to be noted that the gradients are discontinuous when a vertex $\underset{\sim}{\phi}^s$ say satisfies the equation $g_\ell(\underset{\sim}{\phi}^s) = 0$ for any constraint.

The Quadratic Constraints Case

The procedure described before, in which a quadratic approximation is obtained for each constraint, can also be used for yield estimation. Since the region in which there is an active vertex for the worst-case design is the most probable location for violating the constraints, the approximation performed there will be a reasonable one.

It is possible to obtain a linear approximation using least squares and the base points used for the quadratic interpolation. In such a case, we can follow the same procedure described before for the linear constraint case, however, a better procedure for the case of k distinct points of intersection between the orthotope edges and the hypersurface $g_\ell(\underset{\sim}{\phi})$ is given below.

Consider the intersections between the hypersurface $g_\ell(\underset{\sim}{\phi}) = 0$ and the orthotope edges. Any of these intersections is obtained by solving a quadratic equation. The quadratic polynomial approximation is expressed along the orthotope edge in the form

$$\phi_i^2 + 2\,\phi_i\,\xi(\phi_1, \phi_2, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_k)$$

$$+ \eta(\phi_1, \phi_2, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_k) = 0, \tag{57}$$

where ξ and η are constant functions, ϕ_i being the only variable along that edge of the orthotope and $\phi_j = \phi_j^0 + \varepsilon_j\,\mu_j^r$, $\mu_i^r \in \{-1,1\}$, $j \neq i$.

Thus,

$$\lambda_i = -\xi \pm \sqrt{\xi^2 - \eta} \quad , \quad \phi_i^0 - \varepsilon_i \leq \lambda_i \leq \phi_i^0 + \varepsilon_i \quad . \tag{58}$$

A hyperplane is constructed containing k distinct points of intersection between the approximated constraint and the orthotope edges. The equation of this hyperplane is given by

$$\det \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_k & 1 \\ \phi_1^1 & \phi_2^1 & \cdots & \phi_k^1 & 1 \\ \phi_1^2 & \phi_2^2 & \cdots & \phi_k^2 & 1 \\ \vdots & & & & \\ \phi_1^k & \phi_2^k & \cdots & \phi_k^k & 1 \end{bmatrix} = 0 \quad , \tag{59}$$

where $\tilde{\phi}^j$, $j = 1, 2, \ldots, k$ are the vectors representing the points of intersection.

The yield sensitivities are calculated according to the gradients of the k intersections.

$$\frac{\partial \lambda_i}{\partial \phi_j} = -\frac{\partial \xi}{\partial \phi_j} \pm \frac{1}{2\sqrt{\xi^2 - \eta}} \left(2\xi \frac{\partial \xi}{\partial \phi_j} - \frac{\partial \eta}{\partial \phi_j} \right) \quad , \quad j \neq i \tag{60}$$

$$\frac{\partial \lambda_i}{\partial \phi_i} = 0 \quad . \tag{61}$$

Thus, if α_i is the distance from the vertex $\tilde{\phi}^r$ to the point of intersection along the orthotope edge in the ith direction, then

$$\frac{\partial \alpha_i}{\partial \phi_j^0} = - \mu_i^r \frac{\partial \lambda_i}{\partial \phi_j} \quad , \quad j \neq i \quad , \tag{62}$$

$$\frac{\partial \alpha_i}{\partial \varepsilon_j} = \mu_j^r \frac{\partial \alpha_i}{\partial \phi_j^0} \quad , \quad j \neq i \quad , \tag{63}$$

$$\frac{\partial \alpha_i}{\partial \phi_i^0} = \mu_i^r \quad , \tag{64}$$

$$\frac{\partial \alpha_i}{\partial \varepsilon_i} = 1 \quad . \tag{65}$$

VII. EXAMPLES

Example 1

Consider a 2-section 10:1 quarter-wave lossless transmission-line transformer [1]. The worst-case tolerance optimization problem denoted by PO of impedences Z_1 and Z_2 over 100% bandwidth is shown in Table II, for two different objective cost functions. The constraint region and the resulting optimum solutions in the two cases are shown in Fig. 4 and Fig. 5. An equal value of δ_1 and δ_2 was used.

Subsequently, the approximation obtained at the two active vertices shown in Fig. 4 was used for yield optimization. A rough estimate of δ was obtained in the following way. For a yield constraint

$$Y \geq 90\%$$

the nonfeasible hypervolume (it is area in this example) is given approximately by

$$A \simeq (1 - 0.9)\ (2\varepsilon_1)\ (2\varepsilon_2).$$

The area cut off by each constraint is

$$A' \simeq \frac{1}{2} A \quad .$$

But, assuming equal intersections $\alpha = \alpha_1 = \alpha_2$,

$$A' = \frac{1}{2}\alpha^2 .$$

Hence,

$$\alpha \simeq \sqrt{0.1(2\varepsilon_1)\ (2\varepsilon_2)} = 0.27 \quad ,$$

where ε_1 and ε_2 are the worst-case absolute tolerances. The approximation with $\delta = 0.1$ was used for solving the following two problems:

P1

$$\text{Minimize } C_1 = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}$$

subject to

$$Y \geq 90\%$$

P2

$$\text{Minimize } C_2 = \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) / Y$$

The optimum solutions for P1 and P2 are shown in Table III and Fig. 6. The program used for solving the nonlinear optimization problem is FLNLP2 [11]. Because of the convex feasible region the values of yield obtained are lower bounds for the true yield.

Example 2

A normalized 3-components LC lowpass ladder network, terminated with equal load and source resistances of 1Ω, is considered [1]. Although this filter is symmetric, a 3-dimensional approximation was required in order to perform the yield optimization technique described before.

Using equal step size δ for all components, a worst-case solution was first obtained with final $\delta = 0.01$. The base points used are given by (16) with

$$\underset{\sim}{B} = \begin{bmatrix} 0.5 & -0.5 & 1.0 \\ 0.8 & 0.8 & 1.0 \\ -0.5 & 0.5 & 1.0 \end{bmatrix} .$$

The final solution is given in Table IV. The active frequency point constraints at the solution were 0.55, 1.0 and 2.5 rad./sec. Now, consider the optimization problem given by

$$\text{Minimize} \quad \frac{L_1^0}{\varepsilon_1} + \frac{L_2^0}{\varepsilon_2} + \frac{C^0}{\varepsilon_C}$$

subject to

$$Y \geq 96\% \ .$$

In a similar way to the previous example an estimate of $\delta = 0.04$ was obtained. The quadratic approximation obtained with $\delta = 0.04$ after and before averaging symmetric coefficients is shown in Table VI. Symmetry between L_1 and L_2 was used for reducing computation in finding the value and the gradients of the intersections between the orthotope edges and the quadratic constraints. The results are shown in Table IV and in Fig. 7.

To check our results a uniformly distributed set of 10,000 points was generated inside the tolerance region. The results are shown in Table V. Also shown is the computation time saving when the approximation is used for statistical analysis instead of the exact constraint.

Example 3

Consider a practical example of a nonideal two-section waveguide transformer [12, 13]. The general situation is illustrated by Fig. 8. The two-section transformer was optimized with a design specification of a reflection coefficient of 0.05 over 500 MHz centered at 6.175 GHz. Table VII shows the dimensions of the input and output waveguides and the width of the two sections. The program given in [13] was used to obtain the reflection coefficient. It should be noted that the program calculates only the reflection coefficient. No sensitivities are provided. An equal absolute tolerance ε was assumed for the heights and lengths of the two sections. The assumption is reasonable if they are machined in the same way. The objective is to maximize ε. All vertices of the tolerance region were considered and the efficient method to obtain the values of the relevant constraints and their gradients was applied. The optimum nominal point and tolerances for the worst-case design is given in Table VIII. The active vertices at the worst-case solution indicate that the reflection coefficient is more sensitive to the error in b_1.

To gain an impression of the utility of our approach we show in Table IX the effect of assuming $\varepsilon = 0.01$, keeping other parameters at the appropriate values in Tables VII and VIII. Based on a uniform distribution, 500 Monte Carlo analyses were conducted with both the quadratic model and with the actual response program. The model yields excellent results 11 times faster.

VIII. CONCLUSIONS

It is felt that a significant step has been taken in bridging the gap between available analysis programs, which may or may not be efficiently written and probably do not supply derivative information, and the advancing art of optimal centering, tolerancing, and tuning. Thus, efficient gradient methods, which are essential in such general design problems, can be usefully employed.

The yield optimization technique described for quadratic constraints can be extended for general nonlinear constraints. The efficient technique for calculation of the function and gradients at the different vertices may also be implemented with a large-change sensitivity algorithm.

Yield estimation for other statistical distributions, different from the uniform distribution, can be done efficiently using the Monte

Carlo method and the quadratic approximations for the constraints. Avoiding the use of the Monte Carlo method entirely is still a topic for further research.

APPENDIX

Theorem If there exist three distinct base points $\underset{\sim}{\phi}^1$, $\underset{\sim}{\phi}^2$ and $\underset{\sim}{\phi}^3$ in the ith direction, i.e.,

$$\underset{\sim}{\phi}^j = \underset{\sim}{\phi}^1 + c_j \underset{\sim}{e}_i \qquad \text{(A1)}$$

where c_j, j = 2, 3 are scalars, and $\underset{\sim}{e}_i$ is the unit vector in the ith direction, then the interpolating polynomial is one-dimensionally convex/concave in the ith variable if the interpolated function is so.

Proof

Assume that

$$P(\lambda \underset{\sim}{\phi}^a + (1-\lambda) \underset{\sim}{\phi}^b) \gtrless \lambda P(\underset{\sim}{\phi}^a) + (1-\lambda) P(\underset{\sim}{\phi}^b), \; 0<\lambda<1 \qquad \text{(A2)}$$

where $\underset{\sim}{\phi}^b = \underset{\sim}{\phi}^a + c \underset{\sim}{e}_i$ and c is a scalar, i.e., $P(\underset{\sim}{\phi})$ is not one-dimensionally convex/concave in the ith variable.

$$P(\underset{\sim}{\phi}^a + (1-\lambda) c \underset{\sim}{e}_i) \gtrless \lambda P(\underset{\sim}{\phi}^a) + (1-\lambda) P(\underset{\sim}{\phi}^a + c \underset{\sim}{e}_i)$$

$$\gtrless P(\underset{\sim}{\phi}^a) + (1-\lambda) c \underset{\sim}{e}_i^T \underset{\sim}{\nabla} P(\underset{\sim}{\phi}^a) + \tfrac{1}{2}(1-\lambda) c^2 \underset{\sim}{e}_i^T \underset{\sim}{H} \underset{\sim}{e}_i$$

$$P(\underset{\sim}{\phi}^a) + (1-\lambda) c \underset{\sim}{e}_i^T \underset{\sim}{\nabla} P(\underset{\sim}{\phi}^a) + \tfrac{1}{2}(1-\lambda)^2 c^2 \underset{\sim}{e}_i^T \underset{\sim}{H} \underset{\sim}{e}_i$$

$$\gtrless P(\underset{\sim}{\phi}^a) + (1-\lambda) c \underset{\sim}{e}_i^T \underset{\sim}{\nabla} P(\underset{\sim}{\phi}^a) + \tfrac{1}{2}(1-\lambda) c^2 \underset{\sim}{e}_i^T \underset{\sim}{H} \underset{\sim}{e}_i$$

Thus,

$$(1-\lambda)^2 \underset{\sim}{e}_i^T \underset{\sim}{H} \underset{\sim}{e}_i \gtrless (1-\lambda) \underset{\sim}{e}_i^T \underset{\sim}{H} \underset{\sim}{e}_i$$

but since $0 < (1-\lambda) < 1$, hence,

$$\underset{\sim}{e}_i^T \underset{\sim}{H} \underset{\sim}{e}_i \lessgtr 0 \quad . \qquad \text{(A3)}$$

Without any loss of generality we can assume the base points to be such that,

$$\underset{\sim}{\phi}^3 = \gamma \underset{\sim}{\phi}^1 + (1-\gamma)\underset{\sim}{\phi}^2, \quad 0 < \gamma < 1 \qquad \text{(A4)}$$

Then,

$$P(\underset{\sim}{\phi}^3) = P(\gamma\, \underset{\sim}{\phi}^1 + (1-\gamma)\underset{\sim}{\phi}^2)$$

$$= P(\underset{\sim}{\phi}^1 + (1-\gamma)\beta\, \underset{\sim}{e}_i)$$

where $\underset{\sim}{\phi}^2 = \underset{\sim}{\phi}^1 + \beta\, \underset{\sim}{e}_i$ and β is a scalar.

$$P(\underset{\sim}{\phi}^3) = P(\underset{\sim}{\phi}^1) + (1-\gamma)\beta\, \underset{\sim}{e}_i^T\, \underset{\sim}{\nabla}\, P(\underset{\sim}{\phi}^1) + \frac{1}{2}(1-\gamma)^2\, \beta^2\, \underset{\sim}{e}_i^T\, \underset{\sim}{H}\, \underset{\sim}{e}_i$$

$$= \gamma\, P(\underset{\sim}{\phi}^1) + (1-\gamma)\, P(\underset{\sim}{\phi}^2) - \frac{1}{2}\,\gamma(1-\gamma)\beta^2\, \underset{\sim}{e}_i^T\, \underset{\sim}{H}\, \underset{\sim}{e}_i$$

But, using (A3),

$$P(\underset{\sim}{\phi}^3) \gtrless \gamma\, P(\underset{\sim}{\phi}^1) + (1-\gamma)\, P(\underset{\sim}{\phi}^2) \qquad \text{(A5)}$$

i.e.,

$$f(\underset{\sim}{\phi}^3) \gtrless \gamma\, f(\underset{\sim}{\phi}^1) + (1-\gamma)\, f(\underset{\sim}{\phi}^2) \qquad \text{(A6)}$$

which contradicts that $f(\underset{\sim}{\phi})$ is one-dimensionally convex/concave in the ith variable. Hence, the assumption (A2) is never true.

Corollary

A quadratic polynomial is one-dimensionally convex/concave if and only if all of the diagonal elements of the Hessian matrix are nonnegative/nonpositive. The proof follows from inequality (A3).

It is to be noted that the number of base points required to keep the one-dimensional convexity/concavity is 2k+1 which is less than the required number of base points (k+1)(k+2)/2.

This corollary indicates whether the approximate constraint region is one-dimensionally convex or not.

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TABLE I

COMPUTATIONAL EFFORT FOR EVALUATION OF THE QUADRATIC POLYNOMIAL AND ITS DERIVATIVES

Description	Number of additions	Number of multiplications
At one vertex only	$\frac{1}{2}k(3k+5)$	$\frac{3}{2}k(k+1)$
At all vertices using original formula	$2^{k-1}k(3k+5)$	$3 \times 2^{k-1}k(k+1)$
At all the vertices using the efficient scheme	$2^{k-n_i}[\frac{1}{2}k(3k+5)+(k+2)(2^{n_i}-1)]+n_i$	$2^{k-n_i}[\frac{3}{2}k(k+1)+n_i(k+1)+2^{n_i}-1]$
At all the vertices using the efficient scheme when $n_i = k$	$\frac{1}{2}k(3k+7)+(k+2)(2^k-1)$	$\frac{5}{2}k(k+1)+2^k-1$

TABLE II

WORST-CASE DESIGN OF THE TWO-SECTION 10:1 QUARTER-WAVE TRANSFORMER

Cost Function	Z_1^0	Z_2^0	ε_1/Z_1^0 (%)	ε_2/Z_2^0 (%)	δ	N.O.F.E.*	CDC Time (sec)
C_1	2.5637	5.5048	14.678	9.007	0.4	18	7.213
C_1	2.5234	5.4379	14.988	9.081	0.1	24	9.533
C_2	2.1515	4.7350	12.715	12.697	0.4	12	2.468
C_2	2.1494	4.7305	12.687	12.700	0.1	18	2.959

Starting values $Z_1^0 = 2.2361$, $Z_2^0 = 4.4721$, $\varepsilon_1 = 0.2$ and $\varepsilon_2 = 0.4$

Frequency points used 0.5, 0.6, ..., 1.5 GHz

Objective cost functions $C_1 = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}$, $C_2 = \frac{Z_1^0}{\varepsilon_1} + \frac{Z_2^0}{\varepsilon_2}$

Reflection coefficient specification $|\rho| \leq 0.55$

*N.O.F.E. denotes the number of function evaluations

TABLE III

YIELD DETERMINATION AND OPTIMIZATION OF THE TWO-SECTION 10:1 QUARTER-WAVE TRANSFORMER

Problem	Z_1^0	Z_2^0	ε_1/Z_1^0 (%)	ε_2/Z_2^0 (%)	Objective	Yield (%)
P1	2.5273	5.3998	21.09	13.51	3.2465	90.0
P2	2.5290	5.1513	31.44	22.13	3.2597	65.5

TABLE IV

WORST-CASE AND YIELD CONSTRAINED RESULTS OF THE LC LOWPASS FILTER

Yield (%)	L_1^0	L_2^0	C^0	ε_1/L_1^0 (%)	ε_2/L_2^0 (%)	ε_C/C^0 (%)
100	1.999	1.998	0.9058	9.88	9.89	7.60
96	1.997	1.997	0.9033	11.23	11.23	12.46

Frequency points used 0.45, 0.5, 0.55, 1.0 in the passband and 2.5 in the stopband

Objective cost function is $\frac{L_1^0}{\varepsilon_1} + \frac{L_2^0}{\varepsilon_2} + \frac{C^0}{\varepsilon_C}$

Insertion loss specification $|\rho| \leq 1.5$ dB in the passband and $|\rho| \geq 25$ dB in the stopband

TABLE V

COMPARISON OF METHODS OF YIELD ESTIMATION FOR THE LC LOWPASS FILTER

Description	Yield (%)	CDC Time (sec)
Exact Constraints	96.59	20.98
Approximate constraints	96.58	10.43

Yield estimation using a set of 10,000 uniformly distributed points inside the tolerance region for the case of 96% yield according to the hyperplane approximation. All of the five frequency points were used.

TABLE VI

COEFFICIENTS OF THE QUADRATIC APPROXIMATION AROUND ACTIVE VERTICES

Freq. point	State	L_1^2	L_2^2	C^2	L_1L_2	L_1C	L_2C	L_1	L_2	C	-
0.55	before	-.06847	-.06847	-.57056	.3301	.92247	.93855	-1.67845	-1.69182	-.46249	3.8375
	after	-.06847	-.06847	-.57056	.3301	.93051	.93051	-1.68513	-1.68513	-.46249	3.8375
1.0	before	-1.12188	-1.16702	-9.98122	.21439	-8.16357	-8.30295	10.2144	10.51832	44.18607	-33.86206
	after	-1.14445	-1.14445	-9.98122	.21439	-8.23326	-8.23326	10.36637	10.36637	44.18607	-33.86206
2.5	before	-1.38601	-1.42228	-9.90167	.39487	-.92910	-.947315	10.19142	10.32736	32.94001	-46.93184
	after	-1.40414	-1.40414	-9.90167	.39487	-.93821	-.93821	10.25939	10.25939	32.94001	-46.93184

Coefficients of the quadratic approximation obtained at active vertices with a step $\delta = 0.04$. The table shows the coefficients obtained by the algorithm and the coefficients used for yield optimization after averaging symmetric coefficients.

TABLE VII

FIXED PARAMETERS AND SPECIFICATIONS FOR THE TWO-SECTION WAVEGUIDE TRANSFORMER

Description	Width (cm)	Height (cm)	Length (cm)
Input guide	3.48488	0.508	∞
First section	3.6	variable	variable
Second section	3.8	variable	variable
Output guide	4.0386	2.0193	∞

Frequency points used 5.925, 6.175, 6.425 GHz
Reflection coefficient specification $|\rho| < 0.05$
Minimax solution (no tolerances) $|\rho| = 0.00443$

TABLE VIII

RESULTS CONTRASTING THE TOLERANCED SOLUTION AND THE MINIMAX SOLUTION WITH NO TOLERANCES FOR THE TWO-SECTION WAVEGUIDE TRANSFORMER

Description	b_1 (cm)	b_2 (cm)	ℓ_1 (cm)	ℓ_2 (cm)	ε (cm)	number of complete response evaluations	CDC Time (sec)
Toleranced optimum	0.72917	1.41782	1.51317	1.39463	0.00687	45	10
Minimax optimum	0.71315	1.39661	1.56044	1.51621	0	-	-

TABLE IX

COMPARISON OF METHODS OF YIELD ESTIMATION FOR THE TWO-SECTION WAVEGUIDE TRANSFORMER

Number of points	Tolerance ε	Yield(%)		CDC Time (sec)	
		Approx.	Actual	Approx.	Actual
500	0.01	99.4	100	< 0.5	5.7

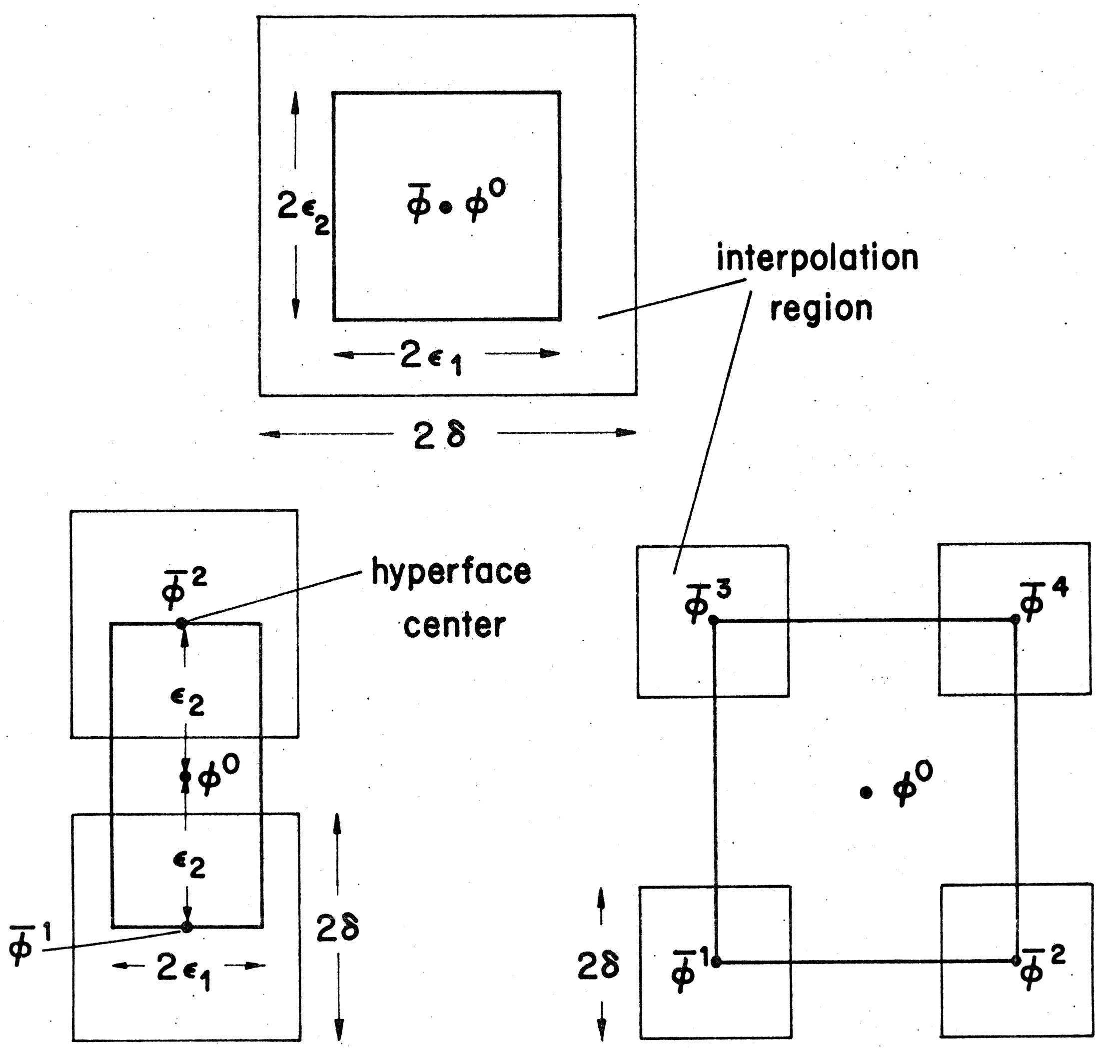

Fig. 1 Three situations created by certain step sizes $\delta = \delta_1 = \delta_2$ and tolerances. The different interpolation regions and their centers are indicated.

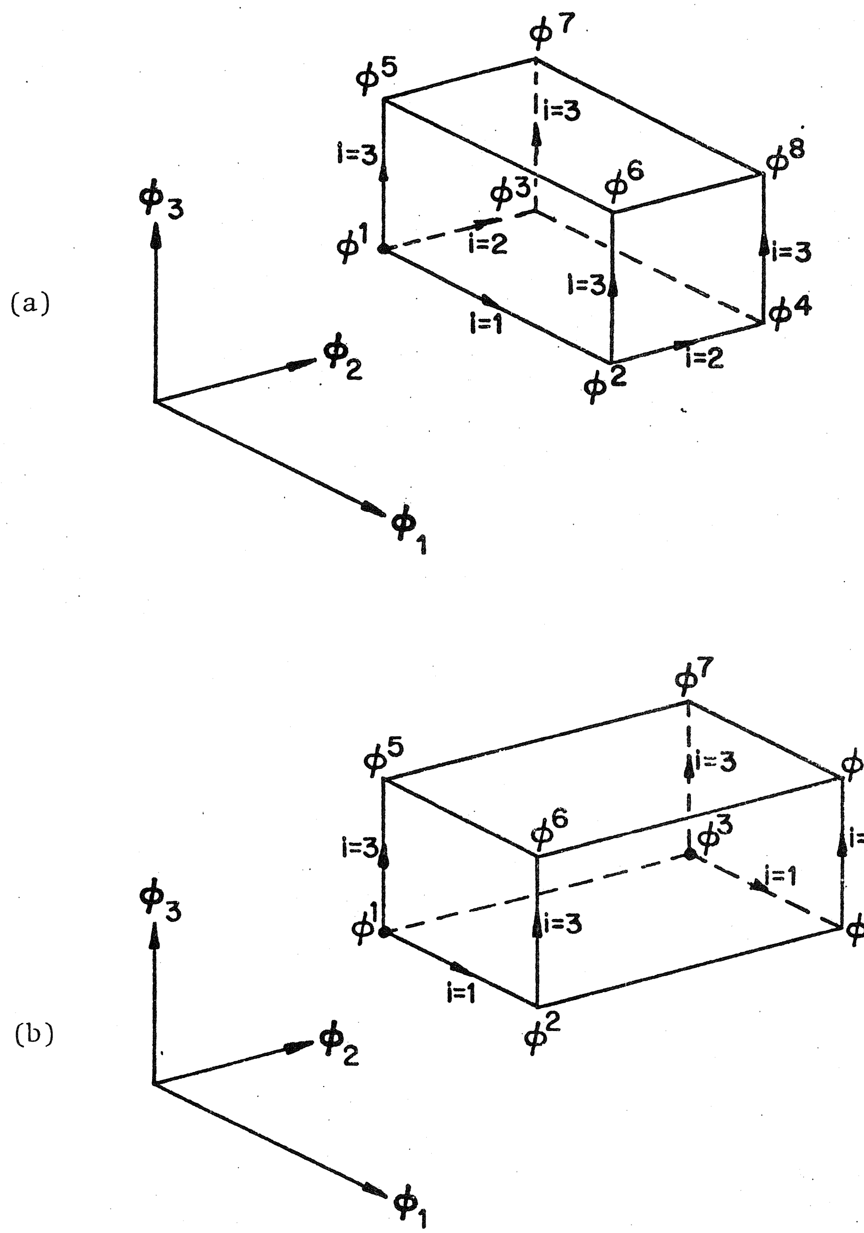

Fig. 2 Illustration of the efficient technique for evaluation of the approximations and their derivatives.
(a) $n_i = 3$, $N_{in} = 1$ and initially $S = \{\underset{\sim}{\phi}^1\}$.
(b) $n_i = 2$, $N_{in} = 2$ and initially $S = \{\underset{\sim}{\phi}^1, \underset{\sim}{\phi}^3\}$.

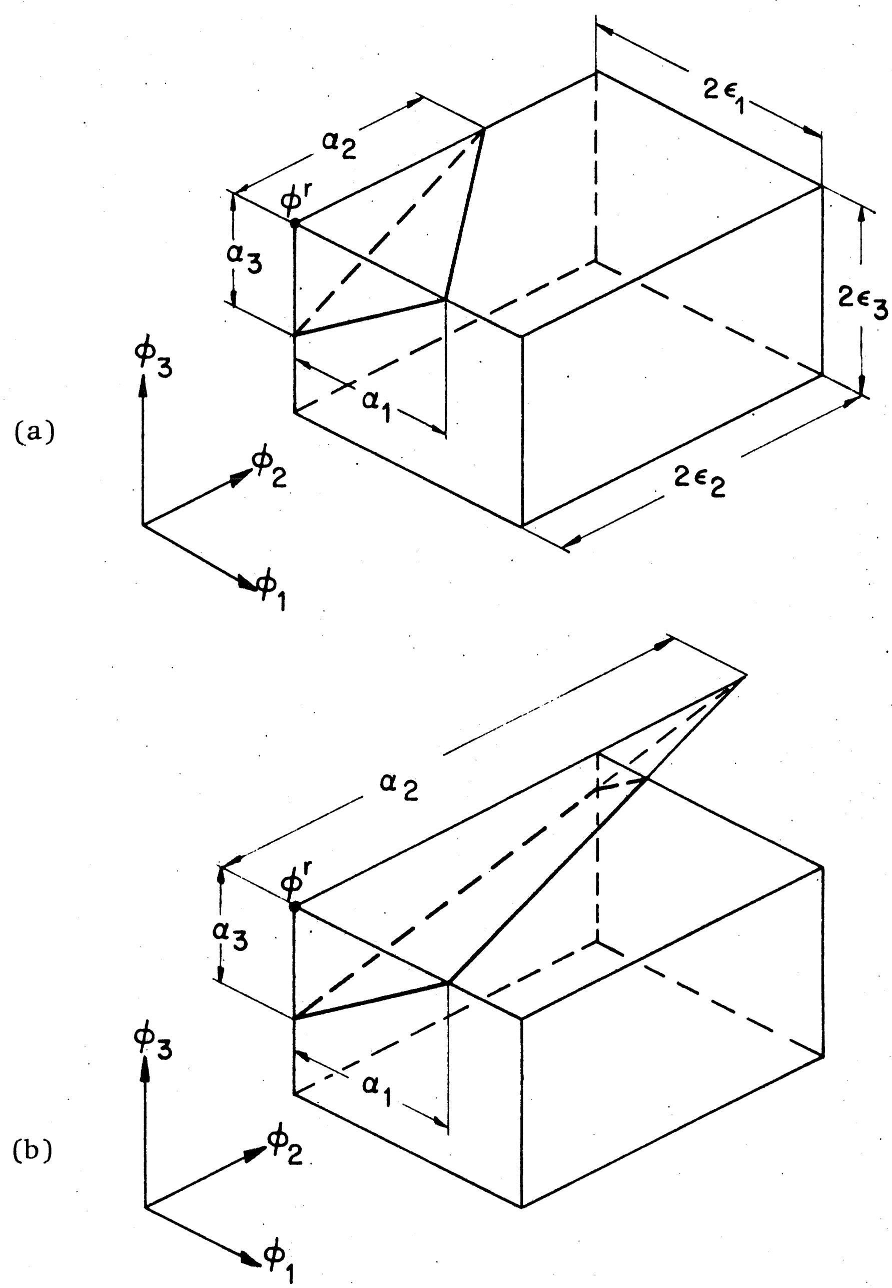

Fig. 3 The nonfeasible volume obtained by a linear constraint.

(a) $V = \frac{1}{3!}\alpha_1 \alpha_2 \alpha_3$.

(b) $V = (\frac{1}{3!}\alpha_1 \alpha_2 \alpha_3)\ [1-(1-\frac{2\varepsilon_2}{\alpha_2})^3]$.

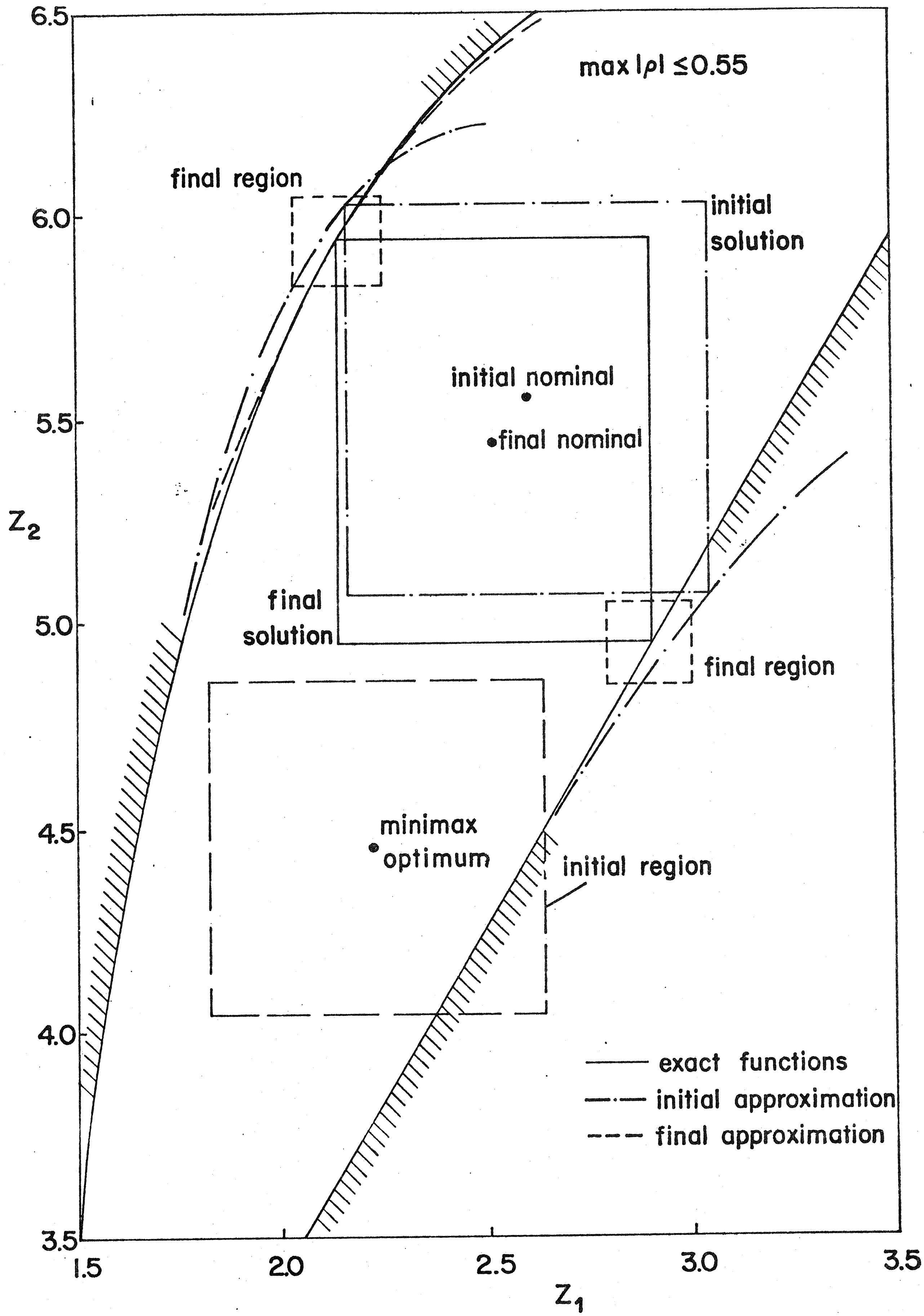

Fig. 4 Minimization of $1/\varepsilon_1 + 1/\varepsilon_2$ for the two-section transformer.

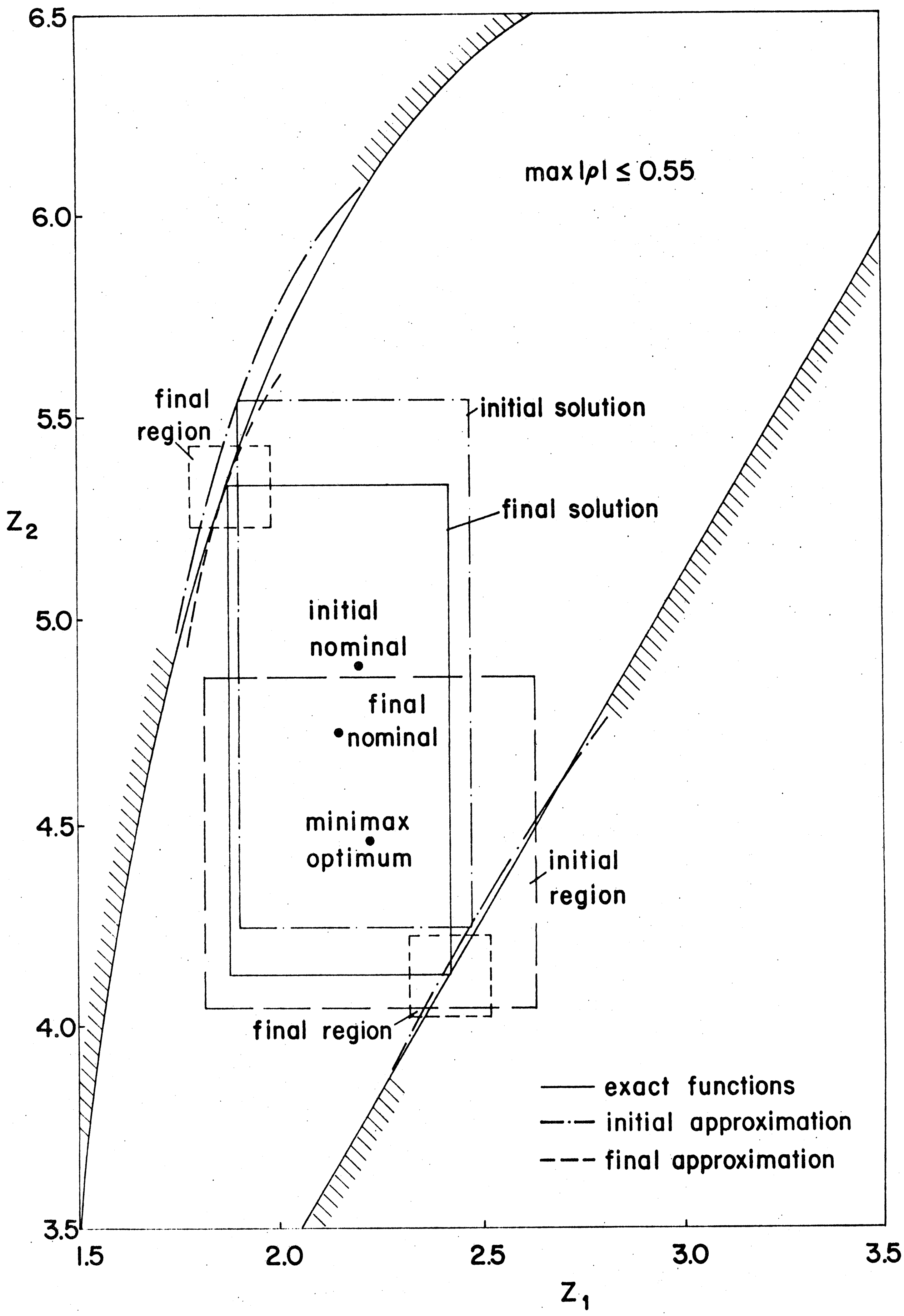

Fig. 5 Minimization of $Z_1^0/\varepsilon_1 + Z_2^0/\varepsilon_2$ for the two-section transformer.

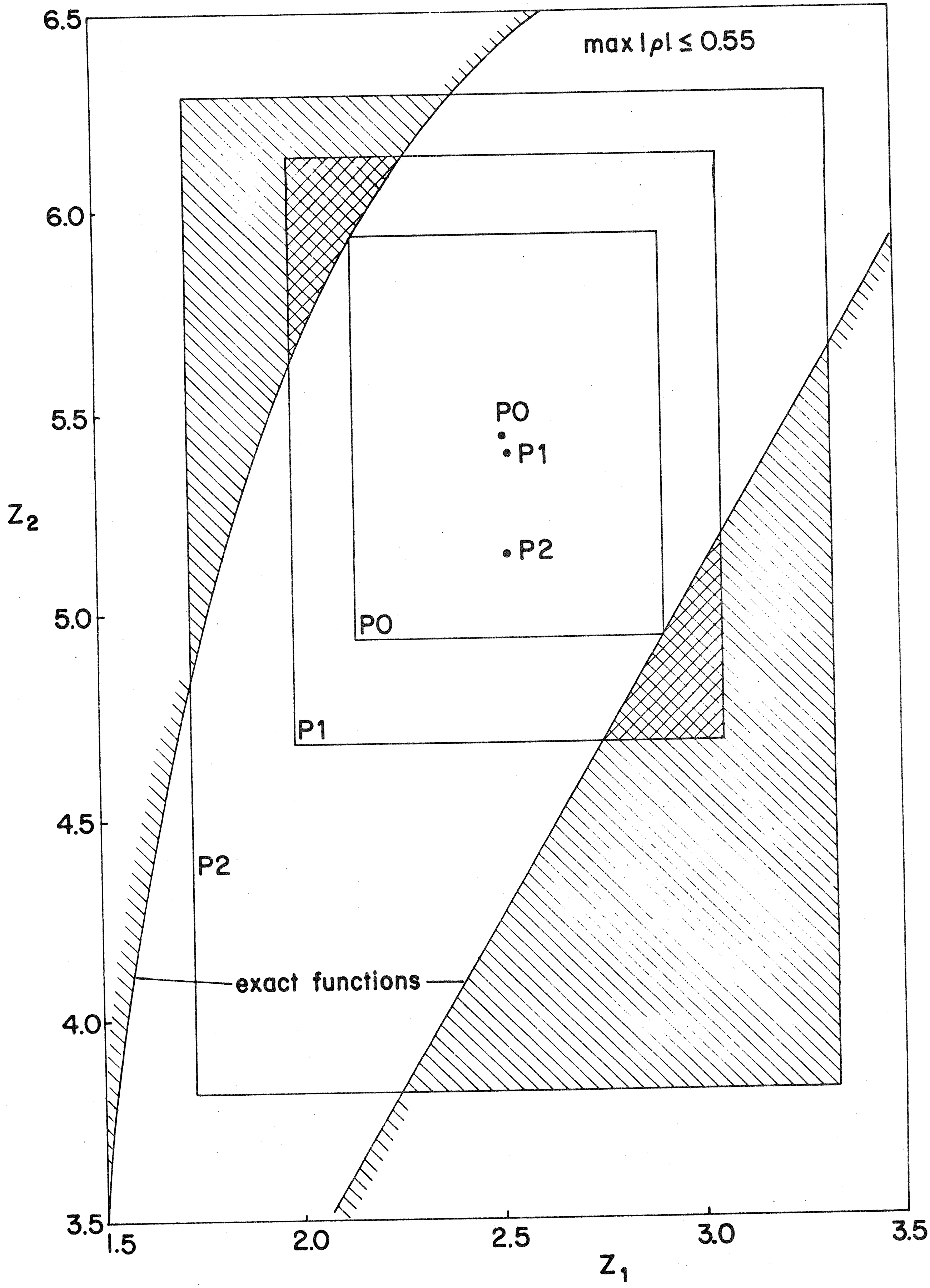

Fig. 6 The optimum tolerance regions and nominal values for the worst-case, 90% yield and optimum yield designs.

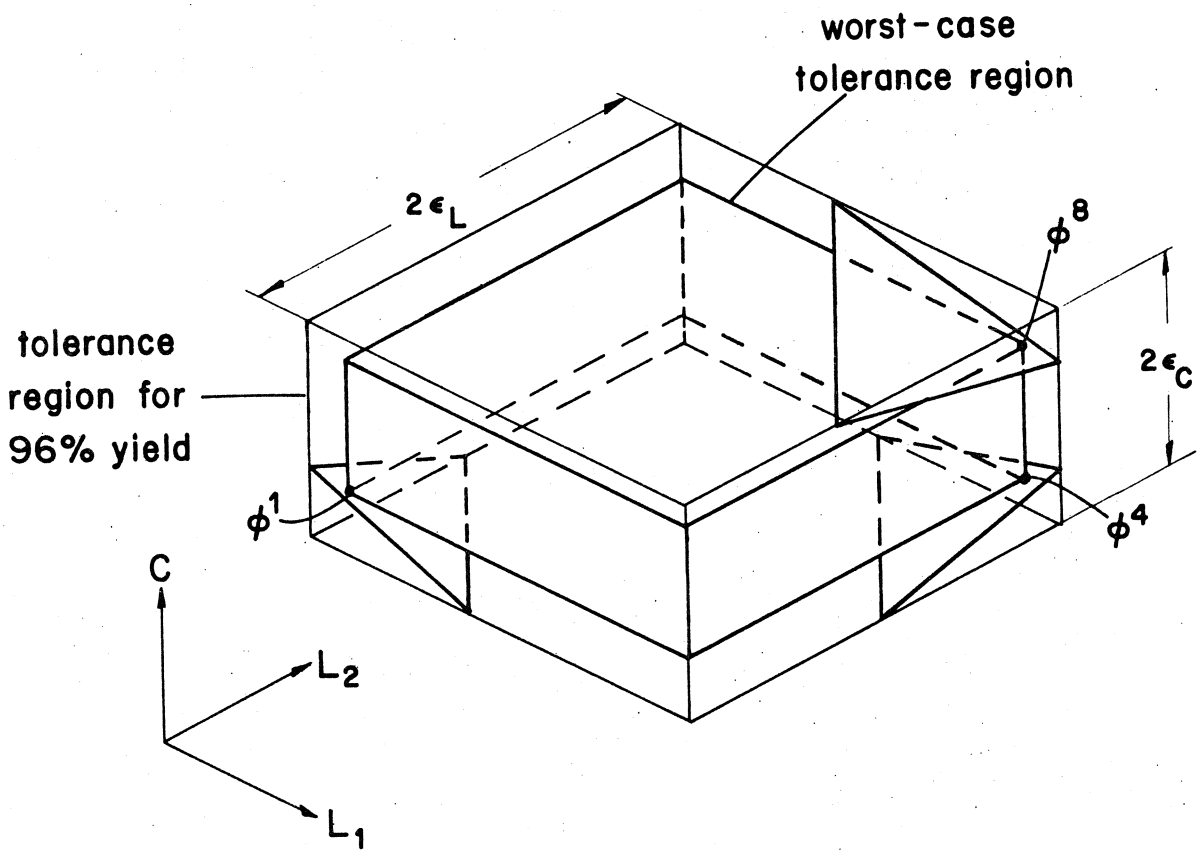

Fig. 7 The tolerance regions for the worst-case design and 96% yield for the LC filter. The linearized active constraints are also shown.

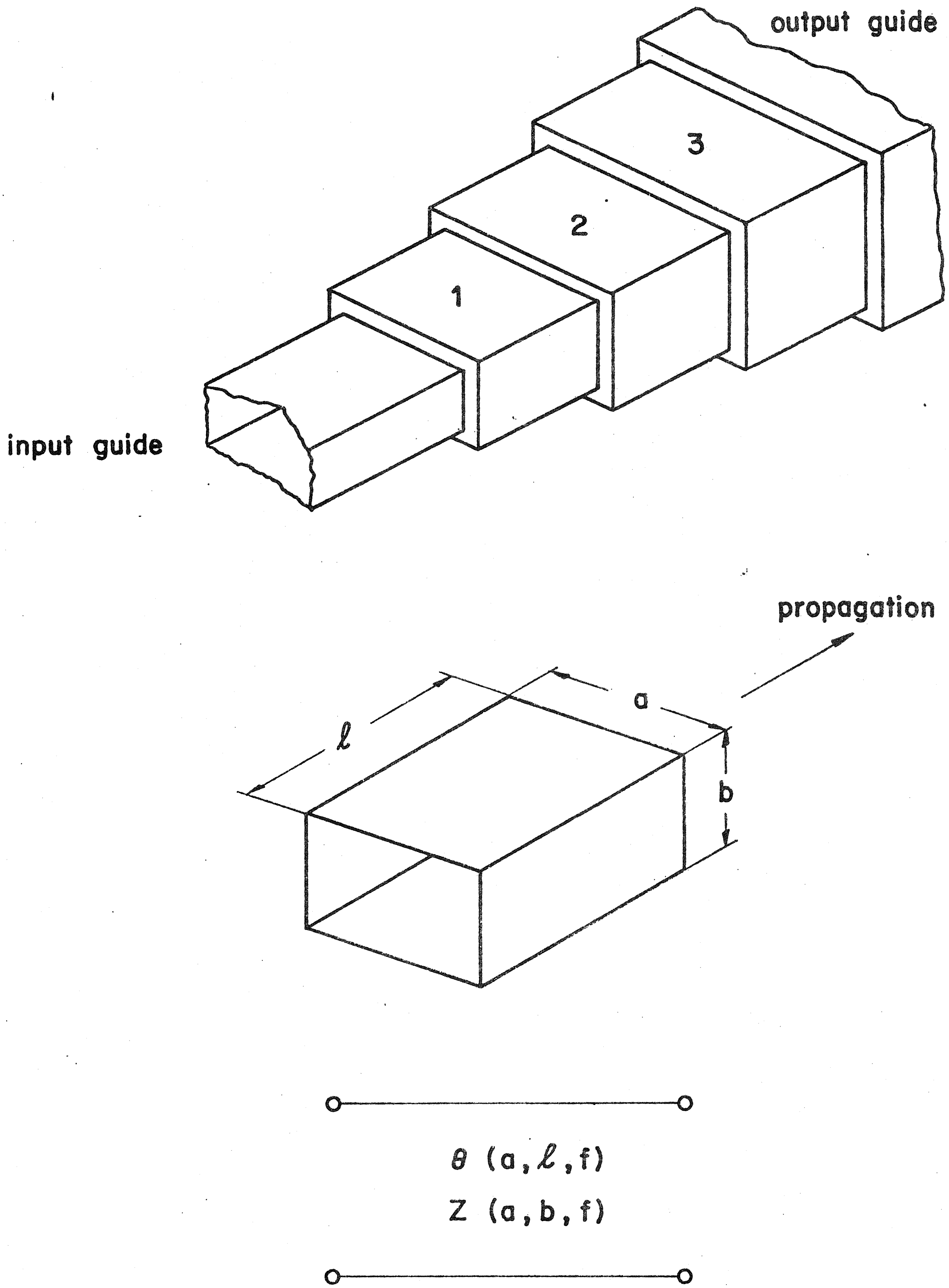

Fig. 8 Illustrations of an inhomogeneous waveguide transformer.

SOC-132

OPTIMAL CENTERING, TOLERANCING AND YIELD DETERMINATION USING MULTI-DIMENSIONAL APPROXIMATIONS

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Key Words: Tolerance assignment, design centering, yield estimation, worst-case design, modeling

Abstract: A method is described for efficient optimal design centering and tolerance assignment. In order to overcome the obstacle of scarcity of simulation programs incorporating both the efficient analysis of performance and its sensitivities, a suitable modelling of the functions involved using low-order multidimensional approximations is used. As a result, rapid and accurate determination of design solutions are facilitated, even with relatively inefficiently written analysis programs or with experimentally obtained data. An efficient technique for evaluating the multidimensional approximations and their derivatives is also given. Formulas for yield and yield sensitivities in the case of independent designable parameters, assuming uniform distribution of outcomes between tolerance extremes, are also presented. In addition, this procedure facilitates an inexpensive yield estimate using Monte Carlo analysis in conjunction with the multidimensional approximations. Simple circuit examples illustrate worst-case design and design with yields of less than 100%. The examples also provide verification of the formulas and algorithms.

Description: Superceded by SOC-173. A paper based on SOC-132 was presented at IEEE International Symposium on Circuits and Systems (Phoenix, Apr. 1977). See also the symposium proceedings, pp. 219-222.

Related Work: As for SOC-1.

Price: $ 5.00.