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Abstract This paper reviews some recent algorithms for minimax and near-minimax approximation and the application of these algorithms to optimum system modelling and electrical network design. The philosophy of system modelling is discussed in length, including various techniques involved in implementing the models. Automated modelling and design of high-order systems is shown to be feasible, and the present state of the art in minimax circuit design is considered in detail.

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INTRODUCTION

Computer-aided design techniques are increasingly being employed in system designs, and various optimization methods are now available [1]. Minimax (or Chebyshev) designs are gaining importance, and recently both direct search [2] and gradient methods [3] have been proposed. Least pth objectives for large values of p [4] have been formulated to give near-minimax results on a number of problems [5] using efficient gradient minimization methods [6,7]. A new gradient algorithm called the grazor search has been developed and applied to problems of microwave network design [8,9] and system modelling [5]. Experience indicates the suitability of choosing minimax objectives for computer-aided system designs.

In this paper the problem of minimax optimization has been extended to include constraints involving the variable parameters [10]. Lower-order modelling of high-order systems has been performed for minimax objectives on the basis of input-output information of the system using efficient optimization techniques. A generalized objective function is defined, by means of which it is possible to get model responses which are optimal from the point of view of transient and steady-state errors. Suitable constraints can be imposed [11] when required, and the whole modeling process can be automated and performed on-line through a small or medium-sized computer. The importance of testing a proposed or design solution for optimality conditions is also emphasized [12,13].

MINIMAX METHODS

Both direct search and gradient methods exist for solving the unconstrained minimax optimization problem of minimizing

$$U(\underset{\sim}{\phi}) \triangleq \max_{i \in I} y_i(\underset{\sim}{\phi}) \tag{1}$$

where $I \triangleq \{1,2,\ldots,n\}$ is an index set relating to discrete elements corresponding to the i, $\underset{\sim}{\phi}$ is a vector of variable parameters, and the y_i are, in general, nonlinear differentiable functions. It is desired to find a point $\underset{\sim}{\check{\phi}}$ such that

$$U(\underset{\sim}{\check{\phi}}) = \min_{\underset{\sim}{\phi}} \max_{i \in I} y_i(\underset{\sim}{\phi}) \tag{2}$$

In direct search strategies, the minimax problem has been explored using pattern search and razor search [2,14]. The razor search method due to Bandler and Macdonald is based on pattern search [15]. A few random moves are used in an effort to negotiate certain kinds of "razor-sharp" valleys in multi-dimensional space. This method has been used to optimize microwave networks where the objective was to minimize the maximum deviation of some network response from an ideal response specification.

Of the gradient strategies, there are methods involving the penalty function approach [16], linear programming [3,17], quadratic programming [18], and a method proposed by Bandler and Lee-Chan [19]. The minimax algorithm due to Osborne and Watson [3] is very similar to the one proposed by Ishizaki and Watanabe [17], and deals with minimax formulations by following two steps - a linear programming part that provides a given step in the parameter space, followed by a linear search along the direction of the step. The algorithm works very well in many problems, but in cases where the linear approximation is not very good in the vicinity of the optimum, the method may fail to converge toward the optimum for successive iterations.

Whenever efficient methods of finding derivatives are not available, direct search methods are useful. For electrical networks in particular it is now possible to evaluate the derivatives of network responses with respect to network parameters rather easily using the adjoint network approach [20,21], and the gradient methods are thus more suited for such cases. The quadratic programming methods are usually more time-consuming than solution of linear programming problems, while penalty function methods rely on suitable function minimization algorithms.

NEAR-MINIMAX METHODS

As is well-known to network designers, least pth approximation for sufficiently large values of p can result in an optimal solution very close to the optimal minimax solution [22]. When appropriate error functions are raised to a power p, the objective function may be ill-conditioned for values of p greater than or equal to about 10. Bandler and Charalambous have given a unified approach to the least pth approximation problems, as encountered in network and system design, having upper and lower response specifications, e.g., as in filter design [4,23]. The ill-conditioning is removed by proper scaling, and least pth optimization has been carried out for extremely large values of p, typically 10^3 to 10^6. This approach has been extensively used in a variety of computer-aided network design problems [24-26].

The least pth approximation problem can effectively be tackled by efficient gradient minimization techniques such as the Fletcher-Powell method [6], Jacobson-Oksman algorithm [27], and a more recent method due to Fletcher [7]. These methods have been compared critically for near-minimax approximation problems in the area of lower-order modelling of high-order systems [5,28].

The problem of minimizing (1) can be reformulated as a least pth approximation problem as follows [29]. Suppose at least one of the functions $y_i(\underset{\sim}{\phi})$ is positive. Then, since $U(\underset{\sim}{\phi}) > 0$,

$$U(\underset{\sim}{\phi}) = \lim_{p\to\infty} U(\underset{\sim}{\phi}) \left(\sum_{i\in I} \left(\frac{w_i y_i(\underset{\sim}{\phi})}{U(\underset{\sim}{\phi})} \right)^p \right)^{\frac{1}{p}} \tag{3}$$

where

$$w_i = \begin{cases} 0 & \text{for } y_i < 0 \\ 1 & \text{for } y_i \geq 0 \end{cases} \tag{4}$$

Suppose all the functions y_i are negative. Then, since $U(\underset{\sim}{\phi}) < 0$,

$$U(\underset{\sim}{\phi}) = \lim_{p\to-\infty} U(\underset{\sim}{\phi}) \left(\sum_{i\in I} \left(\frac{w_i y_i(\underset{\sim}{\phi})}{U(\underset{\sim}{\phi})} \right)^p \right)^{\frac{1}{p}} \tag{5}$$

where

$$w_i = 1 \qquad \text{for all } y_i < 0 \tag{6}$$

Therefore, the minimization function is chosen as

$$f(\underset{\sim}{\phi}) = U(\underset{\sim}{\phi})\Big(\sum_{i\in I}\Big(\frac{w_i y_i(\underset{\sim}{\phi})}{U(\underset{\sim}{\phi})}\Big)^q\Big)^{\frac{1}{q}} \tag{7}$$

where the w_i are given by (4) or (6), and

$$q \triangleq \frac{U(\underset{\sim}{\phi})}{|U(\underset{\sim}{\phi})|}\, p \begin{cases} 1 < p < \infty & \text{for } U > 0 \\ 1 \leq p < \infty & \text{for } U < 0 \end{cases} \tag{8}$$

GRAZOR SEARCH METHOD

A new algorithm called the grazor search method has been developed [8,30] in which gradient information of one or more of the highest ripples among the functions $y_i(\underset{\sim}{\phi})$, $i\varepsilon I$ is used to produce a downhill direction by solving a suitable linear programming problem. A linear search follows to find the minimum in that direction, and the procedure is repeated. This type of descent process is repeated with as many ripples as necessary until a minimax solution is reached to some desired accuracy.

Let $\hat{y}_\ell(\underset{\sim}{\phi})$, $\ell\varepsilon L$ be the largest local discrete maxima (ripples) of $y_i(\underset{\sim}{\phi})$, $i\varepsilon I$, in descending magnitude, where $L \triangleq \{1,2,\ldots,n_r\}$. The grazor search method consists of solving a linear program at the point $\underset{\sim}{\phi}^j$

$$\text{maximize } \alpha_{k_r+1}(\underset{\sim}{\phi}^j) \geq 0 \tag{9}$$

subject to

$$- \underset{\sim}{\nabla}^T \hat{y}_i(\underset{\sim}{\phi}^j) \sum_{\ell \varepsilon J} \alpha_\ell^j \underset{\sim}{\nabla} \hat{y}_\ell(\underset{\sim}{\phi}^j) \leq - \alpha_{k_r+1}^j \quad i\varepsilon J \tag{10}$$

$$\alpha_i^j \geq 0 \qquad i\varepsilon J \tag{11}$$

$$\sum_{\ell \varepsilon J} \alpha_\ell^j = 1 \tag{12}$$

where $\hat{y}_\ell(\underset{\sim}{\phi}^j)$, $\ell\varepsilon J$ are the highest ripples under consideration ($k_r \leq n_r$), and $J \triangleq \{1,2,\ldots,k_r\}$. Next define

$$\Delta\underset{\sim}{\phi}^j = - \sum_{\ell \varepsilon J} \alpha_\ell^j \underset{\sim}{\nabla} \hat{y}_\ell(\underset{\sim}{\phi}^j) \tag{13}$$

which is normalized to

$$\Delta\underset{\sim}{\phi}_n^j = \Delta\underset{\sim}{\phi}^j / \|\Delta\underset{\sim}{\phi}^j\| \tag{14}$$

Starting at $\underset{\sim}{\phi}^j$, one or more steps are taken in the direction of $\Delta\underset{\sim}{\phi}_n^j$ until an improved point is obtained for a step equal to $\Delta\underset{\sim}{\phi}^o$. Next, a method based on golden section search to find the γ^{j*} corresponding to the constrained minimum value of $U(\underset{\sim}{\phi}^j + \gamma^j \Delta\underset{\sim}{\phi}^o)$ is used. The jth iteration ends by setting

$$\underset{\sim}{\phi}^{j+1} = \underset{\sim}{\phi}^j + \gamma^{j*} \Delta\underset{\sim}{\phi}^o \tag{15}$$

This method is guaranteed to converge under certain conditions [8]. The algorithm has been successfully applied to problems of cascaded lumped LC filter designs, antenna modelling circuit optimization, cascaded noncommensurate transmission-line network designs, and modelling high-order control systems [5,28]. The method has been observed to be more reliable than the Osborne and Watson algorithm, and more efficient than the razor search method in many circuit design problems [8].

CONSTRAINED MINIMAX OPTIMIZATION

It is now possible to extend the spectrum of minimax optimization problems to include constraints involving the variable parameters [10]. Consider the problem of minimizing (1) subject to

$$g_j(\underset{\sim}{\phi}) \geq 0 \qquad j \in M \tag{16}$$

where $M \triangleq \{1,2,\ldots,m\}$ and g_j are nonlinear function of the parameters in general. This problem reduces to minimizing ϕ_{k+1} subject to (16) and

$$\phi_{k+1} - y_i(\underset{\sim}{\phi}) \geq 0 \qquad i \in I \tag{17}$$

The above problem can be reformulated as an unconstrained minimax problem by two methods as follows.

<u>Formulation 1</u>

The problem can be reformulated [31,32] as minimizing with respect to $\underset{\sim}{\phi}$, ϕ_{k+1} the function

$$V(\underset{\sim}{\phi},\phi_{k+1},\underset{\sim}{\alpha}) = \max_{\substack{i \in I \\ j \in M}} [\phi_{k+1},\ \phi_{k+1} - \alpha_1(\phi_{k+1} - y_i(\underset{\sim}{\phi})),\ \phi_{k+1} - \alpha_{j+1} g_j(\underset{\sim}{\phi})] \tag{18}$$

where

$$\underset{\sim}{\alpha} \triangleq [\alpha_1 \ \alpha_2 \ \dots \ \alpha_{m+1}]^T \tag{19}$$

$$\alpha_j > 0 \qquad j = 1,2,\dots,m+1 \tag{20}$$

For a large enough value of $\underset{\sim}{\alpha}$ one can obtain, in principle, the exact optimal solution for the original problem by minimizing this reformulated objective function.

When implementing this scheme one can, for the problem defined earlier, slightly modify the formulation in order to save on computational effort, so that the minimization function chosen is

$$V'(\underset{\sim}{\phi},\phi_{k+1},\underset{\sim}{\alpha}) = \max_{\substack{\ell\epsilon L \\ j\epsilon M}} [\phi_{k+1},\ \phi_{k+1} - \alpha_1(\phi_{k+1} - \hat{y}_\ell(\underset{\sim}{\phi})),\ \phi_{k+1} - \alpha_{j+1}g_j(\underset{\sim}{\phi})] \tag{21}$$

<u>Formulation 2</u>

Minimize with respect to $\underset{\sim}{\phi}$ the function

$$W(\underset{\sim}{\phi},\underset{\sim}{w}) = \max_{\substack{i\epsilon I \\ j\epsilon M}}[y_i(\underset{\sim}{\phi}),\ - w_j g_j(\underset{\sim}{\phi})] \tag{22}$$

where

$$\underset{\sim}{w} \triangleq [w_1 \ w_2 \ \dots \ w_m]^T \tag{23}$$

$$w_j > 0 \qquad j\epsilon M \tag{24}$$

For purposes of practical implementation, as long as $U(\underset{\sim}{\phi}) > 0$ and one wishes to apply nonzero weights only to violated constraints of

(16), the minimization function may be chosen as

$$W'(\underset{\sim}{\phi},\underset{\sim}{w}') = \max_{\substack{\ell\varepsilon L \\ j\varepsilon M}} [\hat{y}_\ell(\underset{\sim}{\phi}), \; - w_j' \, g_j(\underset{\sim}{\phi})] \tag{25}$$

where

$$\underset{\sim}{w}' \triangleq [w_1' \; w_2' \; \ldots \; w_m']^T \tag{26}$$

$$\left.\begin{aligned} w_j' &> 0 \quad \text{for} \quad g_j(\underset{\sim}{\phi}) < 0 \\ w_j' &= 0 \quad \text{for} \quad g_j(\underset{\sim}{\phi}) \geq 0 \end{aligned}\right\} \quad j\varepsilon M \tag{27}$$

The advantage of this formulation is apparent when $U > 0$ implies that certain specifications are violated and $U < 0$ implies that they are satisfied. In this case, comparison with violated and satisfied constraints seems appropriate.

By proper choices of the elements of $\underset{\sim}{\alpha}$, $\underset{\sim}{w}$ or $\underset{\sim}{w}'$, the reformulated functions V, V', W or W' can be minimized by choosing a suitable minimax algorithm, say, the grazor search method.

INVESTIGATION OF OPTIMALITY CONDITIONS

Once a solution for minimax objectives is obtained, it may be required to investigate the solution for conditions for minimax optimality [33] so as to verify whether the solution is optimal or not. Consider the problem of minimization of (1). If the $\hat{y}_\ell(\underset{\sim}{\phi})$, $\ell\varepsilon J$ are taken as equal, then for $\underset{\sim}{\phi}$ to satisfy the necessary conditions for a minimax optimum, there exist $u_\ell \geq 0$ for $\ell\varepsilon J$ such that

$$\sum_{\ell \in J} u_\ell \, \underset{\sim}{\nabla} \hat{y}_\ell(\underset{\sim}{\phi}) = \underset{\sim}{0} \tag{28}$$

$$\sum_{\ell \in J} u_\ell = 1 \tag{29}$$

When testing the optimality conditions at a point $\underset{\sim}{\phi}$, an attempt is made to solve (28) and (29) for $k_r = 1,2,\ldots$ until for a value of $k_r^*(\leq n_r)$ (28) and (29) are satisfied. If this is not possible the necessary conditions are not satisfied.

Though the necessary optimality conditions may seem to be straightforward to verify, it is both tedious and difficult to implement in practice. An investigation has been performed [12] and a computer program has been developed [13] to test a solution for the necessary conditions for a minimax optimum by two different formulations. One uses a linear programming approach, and the other the solution of a set of linear independent equations.

Method 1

Equations (28) and (29) are solved here by minimizing $u_{k_r+1} \geq 0$ such that (29) is satisfied and

$$\left| \sum_{\ell \in J} u_\ell \frac{\partial \hat{y}_\ell}{\partial \phi_i} \right| \leq u_{k_r+1} \qquad i \in K \tag{30}$$

where $K \triangleq \{1, 2, \ldots, k\}$. Linear programming ensures that $u_\ell \geq 0$ for $\ell = 1,2,\ldots,k_r+1$.

Method 2

Here, a set of independent equations

$$\sum_{\ell \varepsilon J} u_\ell \frac{\partial \hat{y}_\ell}{\partial \phi_i} = 0 \qquad i \varepsilon K' \tag{31}$$

and (29) are solved, where K' is a suitable subset of K.

There is no guarantee, however, that $u_\ell \geq 0$ for $\ell \varepsilon J$. When k_r-1 is greater than the number of elements of K', the system of equations (29) and (31) have more unknowns than equations, and Method 1 is used to get the u_ℓ, $\ell \varepsilon J$.

MINIMAX SYSTEM MODELLING

System modelling is an area which demands attention primarily because of the complexity and computational effort involved when considering the original system, and the introduction of judiciously chosen models can not only reduce the complexity but also improve the computation time. A number of papers are now available in the area of system modelling [34]. Some methods neglect the modes of the original system which contribute little to the overall response of the system [35] to obtain the model. Other methods [36-38] search in some way for the coefficients of a set of differential or difference equations of specified order, the response of which is approximated as closely as possible to that of the system, when both are driven by the same inputs.

The basic problem considered may be stated as that of finding a transfer function of a given order such that its response is an approximation to the response of the high-order system in some sense. The error criterion may be of the least-squares or the minimax type. For the purpose of this paper, a minimax objective is considered.

In general the transfer function of a given order n may be written as

$$H_{m,n}(s) = \frac{\sum_{i=0}^{m} b_i s^i}{s^n + \sum_{j=1}^{n} a_{n-j} s^{n-j}} \tag{32}$$

where $m \leq n$ for physical systems. The input is chosen as a unit step and the error criterion is to minimize the maximum error between the system and model responses over a specified time-interval [0,T] where the vector of variable model parameters is given by

$$\underset{\sim}{\phi} = [a_0\ a_1\ \dots\ a_{n-1}\ b_0\ b_1\ \dots\ b_m]^T \tag{33}$$

In this paper

- t_i is an ith time instant in [0,T]
- c_i^s is the response of the system at t_i
- $c_i^m(\underset{\sim}{\phi})$ is the response of the approximating model at t_i
- $e_i(\underset{\sim}{\phi}) = c_i^m(\underset{\sim}{\phi}) - c_i^s$ is the error between the system and the model responses at t_i
- c_∞^s is the steady-state value of the system
- c_∞^m is the steady-state value of the model

The usual approximation problem that has been considered in the past [5,34] assumes that c_∞^m is fixed at a convenient value (usually c_∞^s or c_i^s at t_i=T), so that the objective is to minimize

$$U(\underset{\sim}{\phi}) = \max_{t_i \varepsilon [0,T]} |e_i(\underset{\sim}{\phi})| \tag{34}$$

It may, however, be unacceptable to fix c_∞^m at a certain value, in which case a realistic tradeoff between transient and steady-state errors can be achieved.

NEW APPROACHES TO SYSTEM MODELLING

In this section, some new approaches [11] to minimax system modelling are introduced. The methods suggested make it possible to implement the automated modelling of a high-order system on-line.

A Generalized Objective Function

It is possible to apply the ideas of constrained minimax optimization (discussed in an earlier section) to system modelling so that a generalized objective function can be defined to take into account both the transient and steady-state response errors. The following additional notation is introduced.

$S_{u\infty}$	is the upper bound of the system specifications at steady-state
$S_{\ell\infty}$	is the lower bound of the system specifications at steady-state

$e_{u\infty} = c_\infty^m - S_{u\infty}$ is the error between upper steady-state specifications and model steady-state value

$e_{\ell\infty} = c_\infty^m - S_{\ell\infty}$ is the error between lower system specifications and model steady-state value

The problem may now be formulated into two forms as follows:

The first one minimizes with respect to $\underset{\sim}{\phi}$ and ϕ_{k+1}

$$V(\underset{\sim}{\phi},\phi_{k+1},\alpha,\alpha_{\ell\infty},\alpha_{u\infty}) = \max_{t_i \varepsilon [0,T]} [\phi_{k+1},\phi_{k+1} - \alpha(\phi_{k+1} - |e_i(\underset{\sim}{\phi})|), \phi_{k+1} - \alpha_{\ell\infty}e_{\ell\infty},\phi_{k+1} + \alpha_{u\infty}e_{u\infty}] \tag{35}$$

where α, $\alpha_{\ell\infty}$, $\alpha_{u\infty}$ are positive. If c_∞^m is fixed such that $e_{\ell\infty}$ and $-\ e_{u\infty}$ are positive, the objective function (35) reduces essentially to $U(\underset{\sim}{\phi})$ in (34).

The second one minimizes with respect to $\underset{\sim}{\phi}$

$$W(\underset{\sim}{\phi},w_{\ell\infty},w_{u\infty}) = \max_{t_i \varepsilon [0,T]} [|e_i(\underset{\sim}{\phi})|, -\ w_{\ell\infty}e_{\ell\infty},w_{u\infty}e_{u\infty}] \tag{36}$$

where

$$w_{\ell\infty} \begin{cases} = 0 & \text{for } -\ e_{\ell\infty} < 0 \\ > 0 & \text{for } -\ e_{\ell\infty} \geq 0 \end{cases} \tag{37}$$

$$w_{u\infty} \begin{cases} = 0 & \text{for } e_{u\infty} < 0 \\ > 0 & \text{for } e_{u\infty} \geq 0 \end{cases} \tag{38}$$

If c_∞^m is fixed within satisfied specifications the above objective function reduces to $U(\underset{\sim}{\phi})$ in (34).

In cases where suitable constraints - including parameter constraints - are imposed, the above procedure may be used to incorporate this in the objective function. In many cases, it is convenient to choose $S_{\ell\infty} = S_{u\infty} = c_\infty^s$.

Automated Lower-order Models

One of the major problems that is encountered in modelling is to decide whether a certain lower-order model is acceptable or not. If the model is too simple so that computing time for optimizing model parameters is small, the approximation to the original system may be very bad, while if the model is complex, then the very need for system modelling is lost. If one were to strike a reasonable compromise between the speed with which the model is optimized, and the accuracy of the approximation, it would not be unreasonable to devise a scheme whereby one could increase the complexity of the model in an automated fashion after a certain number of iterations or computer time. It is, however, important to keep in mind the desirability of making this increase in complexity as smooth as possible, so that the objective function value is not degraded. Thus, either the number of parameters could be increased for a model with a certain order, or the order of the model itself can be increased.

Let $H^*_{m,n}$ denote an optimized model of the form (32). Three possibilities occur as follows.

(i) Increase in parameters only

$$H^*_{m,n}(s) \rightarrow H_{m+p,n}(s)$$

Here b_{m+p}, b_{m+p-1}, ..., b_{m+1} are initially assumed to be zero so that $H_{m+p,n} = H^*_{m,n}$ in the first iteration.

(ii) Increase in order

$$H^*_{m,n}(s) \rightarrow H_{m+q,n+q}(s)$$

Here q poles of $H_{m+q,n+q}(s)$ are assumed to cancel with q zeroes initially, so that $H_{m+q,n+q} = H^*_{m,n}$ in the first iteration. In this case, initial guesses for q poles (or zeroes) are necessary.

(iii) Increase in order and parameters

$$H^*_{m,n}(s) \rightarrow H_{m+p+q,n+q}(s)$$

Here $b_{m+q+p}, \ldots, b_{m+q+1}$ are assumed to be zero initially and that there is a cancellation of q zeroes and q poles at start, so that $H_{m+p+q,n+q} = H^*_{m,n}$ in the first iteration.

A careful choice of initial parameters can make the increase in model complexity smooth so that the whole modelling procedure can be automated on a small digital computer on-line.

Optimality Conditions

When a certain low-order model is being optimized, it may be useful to investigate intermediate or final solutions after a certain number of iterations of the modelling algorithm, or after a certain convergence criterion is reached, so that one may decide whether to carry on with further optimization, to increase the order of the model, or to terminate altogether. For minimax objectives, it is possible to test the optimality by the procedure outlined in an earlier section [12,13].

CONCLUSIONS

The new ideas presented in this paper have been verified and used in computer-aided design of a variety of electrical networks subject to different objectives and varied constraint specifications. Filters can now easily be designed to meet upper and lower response specifications at predetermined frequencies, within reasonable computing time and desired accuracy. The choice of a circuit model and objective function are as important as the choice of a reliable and efficient optimization technique to give optimal model parameters. If suitable optimization techniques or modelling procedures do not exist for a particular system, the designer is confronted with the task of improving the modelling technique and developing an efficient algorithm to evolve a realistic design. This involves a great deal of experience about the system and an expertise in the state of art methods of computer-aided design.

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