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ALGORITHMS FOR COMPUTATION OF INVERSE HYPERBOLIC FUNCTIONS

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Introduction

This report deals with the computation of the inverse hyperbolic functions such as $\sinh^{-1}(x)$, $\cosh^{-1}(x)$ and $\tanh^{-1}(x)$. The computation of such functions are based on using Chebyshev series expansion to compute $\sinh^{-1}(x)$ and then to compute $\cosh^{-1}(x)$ and $\tanh^{-1}(x)$ using the common relationships between these different functions.

FORTRAN IV programs were written to compute $\sinh^{-1}(x)$ and $\cosh^{-1}(x)$ for the range: $-\infty < x < \infty$ and the programs were tested on a PDP-11/45 computer using double precision arithmetic. The results were compared to existing tables of special functions. It is concluded that those programs can compute $\sinh^{-1}(x)$ and $\cosh^{-1}(x)$ to a very high accuracy.

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The problem of choosing n to obtain the required accuracy D (i.e. the results are required to have D decimal places) can be resolved very simply as follows [2]: Let's try to compute the fictitious function F(x) having the following array c_i:

r	c_r	r	c_r
0	2.532131756	6	0.000542926
1	1.130318208	7	0.000044977
3	0.271495340	8	0.000003198
4	0.044336850	9	0.000000011
5	0.005474240	10	0.000000001

and it is required to have 5 decimal places in the results. We start examining c_{10}, c_9, c_8, ..., in turn until one is found (c_7 in this case) whose modulus exceeds $10^{-(5+1)}$ $(=10^{-6})$. Thus n=7 in this case and F(x) will be best computed as:

$$F(x) = \tfrac{1}{2}c_0 + c_1T_1(x) + c_2T_2(x) + \ldots + c_7T_7(x) \qquad (3)$$

to the required accuracy.

In order to compute $f_n(x)$ using the expansion (1), $c_0, c_1, \ldots, c_m$ and x are to be given and the required accuracy D is also given. A procedure - as described before - will provide $n(< m)$ and then we have to compute $T_1(x), \ldots, T_n(x)$. The method most commonly used in computing these Chebyshev coefficients is discussed in details in [2]. It consists essentially in the

r		c_r		
0	+	1.87179	96008	82618
1	-	0.05881	17611	89952
2	+	0.00472	74654	32212
3	-	0.00049	38363	16265
4	+	0.00005	85062	07059
5	-	0.00000	74669	98329
6	+	0.00000	10011	69358
7	-	0.00000	01390	35439
8	-	0.00000	00198	23169
9	-	0.00000	00028	84747
10	+	0.00000	00004	26730
11	-	0.00000	00000	63976
12	+	0.00000	00000	09699
13	-	0.00000	00000	01484
14	+	0.00000	00000	00229
15	-	0.00000	00000	00036
16	+	0.00000	00000	00006
17	-	0.00000	00000	00001

with check values $\sum' c_r = 0.88137\ 35870\ 19541$

$$\sum' (-1)^r c_r = 0.99999\ 99999\ 99999$$

<u>Comments</u>

The procedure outlined above can be used to compute some other special functions such as: $\exp(x)$, $\sin^{-1}(x)$, $\ell n(x)$, $\tan^{-1}(x)$, $\Gamma(x)$ and $\mathrm{erf}(x)$. In reference [1] the arrays c_i's required to compute such functions are given to 15 decimal places.

Theory

To compute an inverse hyperbolic function, this function is represented by one or more polynomial approximations in the form of Chebyshev series [1]. The meshed of Chebyshev series expansion has been selected for several reasons: It is economical, in the sense that the number of data required to specify each function is small; the error of an approximation is easily assessed; and it is widely applicable since any continuous function of bounded variation has a convergent Chebyshev series expansion [1].

The function f(x) will take the following form, when expanded in an infinite Chebyshev series in the range $-1 \leq x \leq 1$:

$$f(x) = \tfrac{1}{2}c_0 + c_1T_1(x) + c_2T_2(x) + \ldots \qquad (1)$$

where

$$T_r(x) = \text{Cos}\{r\ \text{Cos}^{-1}(x)\} \qquad (2)$$

If the series in (1) is truncated after the term $c_nT_n(x)$ and provided that the series is rapidly convergent, then this approximation differes insignificantly from the best polynomial approximation of the degree n. Also since $|T_r(x)| \leq 1$ for all r, then the error in the truncated series cannot exceed $|c_{n+1}| + |c_{n+2}| + \ldots$, for which a bound can be computed by inspection.

formation of the sequence $b_n, b_{n-1}, \ldots, b_o$ from the relations:

$$b_r = 2x\, b_{r+1} - b_{r+2} + c_r \quad , \quad b_{n+1} = b_{n+2} = 0 \qquad (4)$$

and

$$f_n(x) = \tfrac{1}{2}(b_0 - b_2) \qquad (5)$$

The actual procedure of computing $\sinh^{-1}(x)$ is summarized as follows:

1. If the argument $|x| \geq 1$ then:

$$\sinh^{-1}(x) = \ln(|x| + \sqrt{1 + x^2}) \,.\, \text{sign}(x) \qquad (5)$$

Of course if $|x| >> 1$, we can use the relation:

$$\sinh^{-1}(x) = \ln|x| + \ln 2 \qquad (6)$$

This relation was used if $|x| > 10^8$.

2. If $|x| < 1$, then the Chebyshev series with arguments $2x^2-1$ to find $\frac{1}{x}\sinh^{-1}(x)$. this:

$$\sinh^{-1}(x) = x \,.\, \sum{}' c_r T_{2r}(x) \qquad (7)$$

where $\sum{}'$ means the summation as in (1), i.e. using only $\frac{1}{2}c_0$ and not c_0.

The coefficients c_r given below [3] form the array required in the expansion (7):

Testing

Subroutines were written to test the algorithm and variety of test values were obtained. In reference [3] , the author suggested testing the algorithm for $\sinh^{-1}(0.2)$, $\sinh^{-1}(-5.0)$ and $\sinh^{-1}(10^9)$. The results obtained are:

$$\sinh^{-1}(0.2) = 0.1986091134421323$$

$$\sinh^{-1}(-5.0) = -2.312438321653790$$

$$\sinh^{-1}(10^9) = 21.416413101803796$$

in a very good agreement with the results given in [3] . In addition to the above test values, the algorithm was tested to compute $\sinh^{-1}(x)$ and $\cosh^{-1}(x)$ for the following ranges:

1. $0 < x < 10$ steps of 0.2
2. $0 < x < 50$ steps of 1.0
3. $0 < x < 500$ steps of 10.0
4. $0 < x < 5000$ steps of 100.0
5. $0 < x < 50000$ steps of 1000.0

The obtained results are as given in the appendix together with the FORTRAN IV subroutines used in the computation.

References

[1] Clenshaw, C.W., Miller, G.F. and Woodger, M., "Algorithms for special functions I", Numerische Math., 4, pp. 403-419, 1963.

[2] Clenshaw, C.W., "Chebyshev series for mathematical functions", Math. Tab. Nat. Phys. Lab., 5, London: H.M. Stationery Office, 1962.

[3] Bulirsch, R., "Numerical calculation of elliptical integrals and elliptical functions III", Numerische Math., 13, pp. 305-315, 1969.

Appendix

1. Results of computation,
2. Listing of programs.

x	$\sinh^{-1}(x)$	$\cosh^{-1}(x)$
Ø.Ø	Ø.ØØØØØØØØØØØØØD ØØ	*******************
Ø.2	Ø.19869Ø1Ø15Ø43D ØØ	*******************
Ø.4	Ø.39ØØ353Ø14469D ØØ	*******************
Ø.6	Ø.5688248872757D ØØ	*******************
Ø.8	Ø.73266822Ø9969D ØØ	*******************
1.Ø	Ø.88137358427Ø5D ØØ	Ø.ØØØØØØØØØØØØØD ØØ
1.2	Ø.1Ø1597321Ø335D Ø1	Ø.6223624944687D ØØ
1.4	Ø.113798213ØØ51D Ø1	Ø.867Ø147657394D ØØ
1.6	Ø.1248983383179D Ø1	Ø.1Ø469681Ø2455D Ø1
1.8	Ø.135Ø44Ø859795D Ø1	Ø.11929107 9Ø443D Ø1
2.Ø	Ø.1443635582924D Ø1	Ø.1316958Ø698Ø1D Ø1
2.2	Ø.152966Ø582542D Ø1	Ø.14254170 6562ØD Ø1
2.4	Ø.16Ø9438Ø61714D Ø1	Ø.1522Ø79586983D Ø1
2.6	Ø.1683743357658D Ø1	Ø.16Ø94381 8Ø923D Ø1
2.8	Ø.17532290220 26D Ø1	Ø.16892358Ø6465D Ø1
3.Ø	Ø.18184466362ØØD Ø1	Ø.17627474Ø696ØD Ø1
3.2	Ø.187986373 9Ø14D Ø1	Ø.183Ø93822ØØ24D Ø1
3.4	Ø.1937879443169D Ø1	Ø.1894559264183D Ø1
3.6	Ø.1992835879326D Ø1	Ø.1954207658768D Ø1
3.8	Ø.2Ø45Ø284481Ø5D Ø1	Ø.2Ø1Ø367631912D Ø1
4.Ø	Ø.2Ø94712734222D Ø1	Ø.2Ø63437223434D Ø1
4.2	Ø.2142112Ø16678D Ø1	Ø.2113748311996D Ø1
4.4	Ø.21874217987Ø6D Ø1	Ø.2161581Ø39429D Ø1
4.6	Ø.223Ø814218521D Ø1	Ø.22Ø717382431ØD Ø1
4.8	Ø.2272441148758D Ø1	Ø.225Ø7314682Ø1D Ø1
5.Ø	Ø.2312438249588D Ø1	Ø.2292431592941D Ø1
5.2	Ø.235Ø925683975D Ø1	Ø.233242917Ø6Ø9D Ø1
5.4	Ø.2388Ø11217117D Ø1	Ø.237Ø86ØØ99792D Ø1
5.6	Ø.2423791885376D Ø1	Ø.24Ø7844781876D Ø1
5.8	Ø.2458354949951D Ø1	Ø.24434890747Ø7D Ø1
6.Ø	Ø.2491779565811D Ø1	Ø.2477888584137D Ø1
6.2	Ø.252413725853ØD Ø1	Ø.251112818718ØD Ø1
6.4	Ø.2555493354797D Ø1	Ø.2543284654617D Ø1
6.6	Ø.25859Ø6982422D Ø1	Ø.2574427366257D Ø1
6.8	Ø.2615432739258D Ø1	Ø.26Ø4618549347D Ø1
7.Ø	Ø.264412Ø454788D Ø1	Ø.2633915424347D Ø1
7.2	Ø.2672Ø159Ø538ØD Ø1	Ø.26623699665Ø7D Ø1
7.4	Ø.2699161291122D Ø1	Ø.269ØØ29859543D Ø1
7.6	Ø.2725595474243D Ø1	Ø.2716938495636D Ø1
7.8	Ø.2751354455948D Ø1	Ø.27431359291Ø8D Ø1
8.Ø	Ø.2776471853256D Ø1	Ø.2768659114838D Ø1
8.2	Ø.28ØØ978422165D Ø1	Ø.27935419Ø8264D Ø1
8.4	Ø.28249Ø3Ø11322D Ø1	Ø.2817816495895D Ø1
8.6	Ø.2848272Ø8519ØD Ø1	Ø.2841511487961D Ø1
8.8	Ø.2871111154556D Ø1	Ø.28646545410 16D Ø1
9.Ø	Ø.2893443584442D Ø1	Ø.288727Ø45Ø592D Ø1
9.2	Ø.291529Ø83252ØD Ø1	Ø.29Ø9383535385D Ø1
9.4	Ø.2936673879623D Ø1	Ø.2931Ø15Ø14648D Ø1
9.6	Ø.295761156Ø822D Ø1	Ø.29521861Ø7635D Ø1
9.8	Ø.2978121995926D Ø1	Ø.2972915887833D Ø1

x	$\sinh^{-1}(x)$	$\cosh^{-1}(x)$
Ø.Ø	Ø.ØØØØØØØØØØØØØD ØØ	********************
1.Ø	Ø.88137358427Ø5D ØØ	Ø.ØØØØØØØØØØØØØD ØØ
2.Ø	Ø.1443635463715D Ø1	Ø.131695795Ø592D Ø1
3.Ø	Ø.1818446516991D Ø1	Ø.1762747168541D Ø1
4.Ø	Ø.2Ø947124958Ø4D Ø1	Ø.2Ø63437223434D Ø1
5.Ø	Ø.2312438249588D Ø1	Ø.229243183136ØD Ø1
6.Ø	Ø.24917798Ø4233D Ø1	Ø.2477888822556D Ø1
7.Ø	Ø.264412Ø6932Ø7D Ø1	Ø.26339159Ø1184D Ø1
8.Ø	Ø.277647233ØØ93D Ø1	Ø.2768659353256D Ø1
9.Ø	Ø.2893444Ø61279D Ø1	Ø.288727Ø927429D Ø1
1Ø.Ø	Ø.2998223Ø6633ØD Ø1	Ø.2993222951889D Ø1
11.Ø	Ø.3Ø931Ø2216721D Ø1	Ø.3Ø889699459Ø8D Ø1
12.Ø	Ø.317978549ØØ36D Ø1	Ø.317631316185ØD Ø1
13.Ø	Ø.3259572505951D Ø1	Ø.32566139698Ø3D Ø1
14.Ø	Ø.3333477735519D Ø1	Ø.333Ø926656723D Ø1
15.Ø	Ø.34Ø23Ø679512ØD Ø1	Ø.34ØØØ84495544D Ø1
16.Ø	Ø.346671 1Ø44312D Ø1	Ø.3464757919312D Ø1
17.Ø	Ø.352722454Ø71ØD Ø1	Ø.35254945755ØØD Ø1
18.Ø	Ø.35842897892ØØD Ø1	Ø.35827465Ø5737D Ø1
19.Ø	Ø.3638278ØØ75Ø7D Ø1	Ø.3636893Ø33981D Ø1
2Ø.Ø	Ø.36895Ø39Ø8157D Ø1	Ø.3688253879547D Ø1
21.Ø	Ø.3738236188889D Ø1	Ø.37371Ø227Ø126D Ø1
22.Ø	Ø.37847Ø58773Ø4D Ø1	Ø.37836728Ø96Ø1D Ø1
23.Ø	Ø.3829113721848D Ø1	Ø.3828168392181D Ø1
24.Ø	Ø.3871634721756D Ø1	Ø.387Ø766878128D Ø1
25.Ø	Ø.3912422895432D Ø1	Ø.391162276268ØD Ø1
26.Ø	Ø.39516134262Ø8D Ø1	Ø.395Ø873851776D Ø1
27.Ø	Ø.3989326953888D Ø1	Ø.398864 1Ø23636D Ø1
28.Ø	Ø.4Ø2567Ø528412D Ø1	Ø.4Ø25Ø3252Ø294D Ø1
29.Ø	Ø.4Ø6Ø739994Ø49D Ø1	Ø.4Ø6Ø14585495ØD Ø1
3Ø.Ø	Ø.4Ø94622135162D Ø1	Ø.4Ø94Ø66619873D Ø1
31.Ø	Ø.41273946762Ø8D Ø1	Ø.412687397ØØ32D Ø1
32.Ø	Ø.4159127235413D Ø1	Ø.4158638954163D Ø1
33.Ø	Ø.4189884185791D Ø1	Ø.41894249916Ø8D Ø1
34.Ø	Ø.42197237Ø1477D Ø1	Ø.421929121Ø175D Ø1
35.Ø	Ø.4248699188232D Ø1	Ø.424829 1Ø15625D Ø1
36.Ø	Ø.42768588Ø661ØD Ø1	Ø.4276473522186D Ø1
37.Ø	Ø.43Ø424785614ØD Ø1	Ø.43Ø3882598877D Ø1
38.Ø	Ø.433Ø9Ø6391144D Ø1	Ø.433Ø56Ø2Ø7367D Ø1
39.Ø	Ø.435687 3Ø35431D Ø1	Ø.4356544494629D Ø1
4Ø.Ø	Ø.4382183Ø74951D Ø1	Ø.438187Ø269775D Ø1
41.Ø	Ø.44Ø686798Ø957D Ø1	Ø.44Ø657Ø43457ØD Ø1
42.Ø	Ø.443Ø957847864D Ø1	Ø.443Ø675Ø29755D Ø1
43.Ø	Ø.4454482555389D Ø1	Ø.4454212188721D Ø1
44.Ø	Ø.44774661Ø6415D Ø1	Ø.44772Ø766Ø675D Ø1
45.Ø	Ø.4499933242798D Ø1	Ø.449968624115ØD Ø1
46.Ø	Ø.45219Ø6852722D Ø1	Ø.452167Ø341492D Ø1
47.Ø	Ø.45434Ø7917Ø23D Ø1	Ø.4543181419373D Ø1
48.Ø	Ø.4564456939697D Ø1	Ø.45642395Ø1953D Ø1
49.Ø	Ø.4585Ø71563721D Ø1	Ø.4584863185883D Ø1

x	$\sinh^{-1}(x)$	$\cosh^{-1}(x)$
Ø.ØØØE ØØ	Ø.ØØØØØØØØØØØØØD ØØ	*******************
Ø.1ØØE Ø2	Ø.2998223Ø66330D Ø1	Ø.2993222951889D Ø1
Ø.2ØØE Ø2	Ø.36895Ø39Ø8157D Ø1	Ø.3688253379547D Ø1
Ø.3ØØE Ø2	Ø.4Ø94622135162D Ø1	Ø.4Ø94Ø66619873D Ø1
Ø.4ØØE Ø2	Ø.4382183Ø74951D Ø1	Ø.438187Ø269775D Ø1
Ø.5ØØE Ø2	Ø.46Ø527Ø385742D Ø1	Ø.46Ø5Ø7Ø114136D Ø1
Ø.6ØØE Ø2	Ø.4787561416626D Ø1	Ø.478742218Ø176D Ø1
Ø.7ØØE Ø2	Ø.49416933Ø5969D Ø1	Ø.4941591262817D Ø1
Ø.8ØØE Ø2	Ø.5Ø75212955475D Ø1	Ø.5Ø75134754181D Ø1
Ø.9ØØE Ø2	Ø.5192987918854D Ø1	Ø.519292593ØØ23D Ø1
Ø.1ØØE Ø3	Ø.5298342227936D Ø1	Ø.5298292636871D Ø1
Ø.11ØE Ø3	Ø.5393648147583D Ø1	Ø.539362Ø66275ØD Ø1
Ø.12ØE Ø3	Ø.548Ø656147ØØ3D Ø1	Ø.548Ø621814728D Ø1
Ø.13ØE Ø3	Ø.556Ø6966Ø1868D Ø1	Ø.556Ø667Ø37964D Ø1
Ø.14ØE Ø3	Ø.56348Ø2341461D Ø1	Ø.5634777Ø69Ø92D Ø1
Ø.15ØE Ø3	Ø.57Ø3793525696D Ø1	Ø.57Ø3771591187D Ø1
Ø.16ØE Ø3	Ø.576833Ø574Ø36D Ø1	Ø.5768311Ø23712D Ø1
Ø.17ØE Ø3	Ø.5828954219818D Ø1	Ø.582893 7Ø5368ØD Ø1
Ø.18ØE Ø3	Ø.5886111736298D Ø1	Ø.5886Ø964775Ø9D Ø1
Ø.19ØE Ø3	Ø.594Ø178394318D Ø1	Ø.594Ø1645660 4ØD Ø1
Ø.2ØØE Ø3	Ø.599147Ø813751D Ø1	Ø.5991458415985D Ø1
Ø.21ØE Ø3	Ø.6Ø4Ø26Ø314941D Ø1	Ø.6Ø4Ø24887Ø85ØD Ø1
Ø.22ØE Ø3	Ø.6Ø8678ØØ71259D Ø1	Ø.6Ø867695 8Ø841D Ø1
Ø.23ØE Ø3	Ø.61312313Ø7983D Ø1	Ø.613122177124ØD Ø1
Ø.24ØE Ø3	Ø.617379Ø454865D Ø1	Ø.6173781871796D Ø1
Ø.25ØE Ø3	Ø.6214612ØØ7141D Ø1	Ø.62146Ø39ØØ9Ø9D Ø1
Ø.26ØE Ø3	Ø.625383234Ø24ØD Ø1	Ø.6253825187683D Ø1
Ø.27ØE Ø3	Ø.629157257Ø8Ø1D Ø1	Ø.62915658950 81D Ø1
Ø.28ØE Ø3	Ø.6327939987183D Ø1	Ø.63279337883ØØD Ø1
Ø.29ØE Ø3	Ø.636303Ø91Ø492D Ø1	Ø.6363Ø25188446D Ø1
Ø.3ØØE Ø3	Ø.63969326Ø1929D Ø1	Ø.6396926879883D Ø1
Ø.31ØE Ø3	Ø.64297223Ø9113D Ø1	Ø.6429717Ø639Ø4D Ø1
Ø.32ØE Ø3	Ø.646147Ø6Ø3943D Ø1	Ø.6461465835571D Ø1
Ø.33ØE Ø3	Ø.6492242336273D Ø1	Ø.64922375679Ø2D Ø1
Ø.34ØE Ø3	Ø.6522Ø952Ø34ØØD Ø1	Ø.6522Ø9Ø911865D Ø1
Ø.35ØE Ø3	Ø.6551Ø82611Ø84D Ø1	Ø.6551Ø783195 5ØD Ø1
Ø.36ØE Ø3	Ø.6579253196716D Ø1	Ø.6579249382Ø19D Ø1
Ø.37ØE Ø3	Ø.66Ø6652259827D Ø1	Ø.66Ø6648445129D Ø1
Ø.38ØE Ø3	Ø.663332Ø331573D Ø1	Ø.6633316516876D Ø1
Ø.39ØE Ø3	Ø.6659295558929D Ø1	Ø.6659292221Ø69D Ø1
Ø.4ØØE Ø3	Ø.6684613227844D Ø1	Ø.668461Ø366821D Ø1
Ø.41ØE Ø3	Ø.67Ø93Ø5763245D Ø1	Ø.67Ø93Ø29Ø2222D Ø1
Ø.42ØE Ø3	Ø.67334Ø32Ø5872D Ø1	Ø.67334ØØ344849D Ø1
Ø.43ØE Ø3	Ø.6756933689117D Ø1	Ø.67569313Ø4932D Ø1
Ø.44ØE Ø3	Ø.6779922962189D Ø1	Ø.6779925578ØØ3D Ø1
Ø.45ØE Ø3	Ø.68Ø239582Ø618D Ø1	Ø.68Ø2393436432D Ø1
Ø.46ØE Ø3	Ø.6824375152588D Ø1	Ø.68243727684Ø2D Ø1
Ø.47ØE Ø3	Ø.684588Ø98526ØD Ø1	Ø.68458786Ø1Ø74D Ø1
Ø.48ØE Ø3	Ø.6866934299469D Ø1	Ø.686693239212ØD Ø1
Ø.49ØE Ø3	Ø.6887553691864D Ø1	Ø.6887551784515D Ø1

x	$sinh^{-1}(x)$	$cosh^{-1}(x)$
Ø.ØØØE ØØ	Ø.ØØØØØØØØØØØØØD ØØ	********************
Ø.1ØØE Ø3	Ø.5298342227936D Ø1	Ø.5298292636871D Ø1
Ø.2ØØE Ø3	Ø.5991470813751D Ø1	Ø.5991458415985D Ø1
Ø.3ØØE Ø3	Ø.63969326Ø1929D Ø1	Ø.6396926879883D Ø1
Ø.4ØØE Ø3	Ø.6684613227844D Ø1	Ø.668461Ø366821D Ø1
Ø.5ØØE Ø3	Ø.69Ø7756328583D Ø1	Ø.69Ø7754421234D Ø1
Ø.6ØØE Ø3	Ø.7Ø9ØØ774ØØ2Ø8D Ø1	Ø.7Ø9ØØ76446533D Ø1
Ø.7ØØE Ø3	Ø.72442278862ØØD Ø1	Ø.7244226932526D Ø1
Ø.8ØØE Ø3	Ø.7377759456635D Ø1	Ø.73777585Ø296ØD Ø1
Ø.9ØØE Ø3	Ø.7495542Ø494Ø8D Ø1	Ø.7495541572571D Ø1
Ø.1ØØE Ø4	Ø.76ØØ9Ø2557373D Ø1	Ø.76ØØ9Ø2Ø8Ø536D Ø1
Ø.11ØE Ø4	Ø.7696212768555D Ø1	Ø.7696212768555D Ø1
Ø.12ØE Ø4	Ø.77832241Ø5835D Ø1	Ø.77832241Ø5835D Ø1
Ø.13ØE Ø4	Ø.7863266944885D Ø1	Ø.7863266468Ø48D Ø1
Ø.14ØE Ø4	Ø.7937375Ø68665D Ø1	Ø.7937374591827D Ø1
Ø.15ØE Ø4	Ø.8ØØ6367683411D Ø1	Ø.8ØØ6367683411D Ø1
Ø.16ØE Ø4	Ø.8Ø7Ø9Ø6639Ø99D Ø1	Ø.8Ø7Ø9Ø5685425D Ø1
Ø.17ØE Ø4	Ø.813153Ø761719D Ø1	Ø.813153Ø761719D Ø1
Ø.18ØE Ø4	Ø.8188689231873D Ø1	Ø.8188689231873D Ø1
Ø.19ØE Ø4	Ø.8242756843567D Ø1	Ø.8242755889893D Ø1
Ø.2ØØE Ø4	Ø.829405Ø216675D Ø1	Ø.8294Ø492630ØØD Ø1
Ø.21ØE Ø4	Ø.83428401947Ø2D Ø1	Ø.83428401947Ø2D Ø1
Ø.22ØE Ø4	Ø.838936Ø427856D Ø1	Ø.838936Ø427856D Ø1
Ø.23ØE Ø4	Ø.8433812141418D Ø1	Ø.8433812141418D Ø1
Ø.24ØE Ø4	Ø.8476371765137D Ø1	Ø.8476371765137D Ø1
Ø.25ØE Ø4	Ø.851719379425ØD Ø1	Ø.851719379425ØD Ø1
Ø.26ØE Ø4	Ø.85564146Ø4187D Ø1	Ø.85564146Ø4187D Ø1
Ø.27ØE Ø4	Ø.859415435791ØD Ø1	Ø.859415435791ØD Ø1
Ø.28ØE Ø4	Ø.863Ø522727966D Ø1	Ø.863Ø522727966D Ø1
Ø.29ØE Ø4	Ø.8665614128113D Ø1	Ø.8665614128113D Ø1
Ø.3ØØE Ø4	Ø.8699515342712D Ø1	Ø.8699515342712D Ø1
Ø.31ØE Ø4	Ø.87323Ø5526733D Ø1	Ø.87323Ø5526733D Ø1
Ø.32ØE Ø4	Ø.876405334 4727D Ø1	Ø.8764Ø53344727D Ø1
Ø.33ØE Ø4	Ø.8794825553894D Ø1	Ø.8794825553894D Ø1
Ø.34ØE Ø4	Ø.882467842 1Ø21D Ø1	Ø.88246784 21Ø21D Ø1
Ø.35ØE Ø4	Ø.88536663Ø5542D Ø1	Ø.88536663Ø5542D Ø1
Ø.36ØE Ø4	Ø.8881836891174D Ø1	Ø.8881836891174D Ø1
Ø.37ØE Ø4	Ø.89Ø9235954285D Ø1	Ø.89Ø9235954285D Ø1
Ø.38ØE Ø4	Ø.89359Ø45Ø2869D Ø1	Ø.89359Ø45Ø2869D Ø1
Ø.39ØE Ø4	Ø.896187973Ø225D Ø1	Ø.896187973Ø225D Ø1
Ø.4ØØE Ø4	Ø.89871969223Ø2D Ø1	Ø.89871969223Ø2D Ø1
Ø.41ØE Ø4	Ø.9Ø118894577Ø3D Ø1	Ø.9Ø118894577Ø3D Ø1
Ø.42ØE Ø4	Ø.9Ø3598690Ø33ØD Ø1	Ø.9Ø35986900330D Ø1
Ø.43ØE Ø4	Ø.9Ø5951786Ø413D Ø1	Ø.9Ø5951786Ø413D Ø1
Ø.44ØE Ø4	Ø.9Ø825Ø7133484D Ø1	Ø.9Ø825Ø7133484D Ø1
Ø.45ØE Ø4	Ø.91Ø497951 5Ø76D Ø1	Ø.91Ø4979515Ø76D Ø1
Ø.46ØE Ø4	Ø.91269588 47Ø46D Ø1	Ø.91269588 47Ø46D Ø1
Ø.47ØE Ø4	Ø.9148465156555D Ø1	Ø.9148465156555D Ø1
Ø.48ØE Ø4	Ø.91695184 7Ø764D Ø1	Ø.916951847Ø764D Ø1
Ø.49ØE Ø4	Ø.919Ø137863159D Ø1	Ø.919Ø137863159D Ø1

x	$\sinh^{-1}(x)$	$\cosh^{-1}(x)$
Ø.ØØØE ØØ	Ø.ØØØØØØØØØØØØØD ØØ	*******************
Ø.1ØØE Ø4	Ø.76ØØ9Ø2557373D Ø1	Ø.76ØØ9Ø2Ø8Ø536D Ø1
Ø.2ØØE Ø4	Ø.82940502166750D Ø1	Ø.8294Ø4926ØØØØD Ø1
Ø.3ØØE Ø4	Ø.8699515342712D Ø1	Ø.8699515342712D Ø1
Ø.4ØØE Ø4	Ø.89871969223Ø2D Ø1	Ø.89871969223Ø2D Ø1
Ø.5ØØE Ø4	Ø.921Ø34Ø499878D Ø1	Ø.921Ø34Ø499878D Ø1
Ø.6ØØE Ø4	Ø.939266224834ØD Ø1	Ø.939266224834ØD Ø1
Ø.7ØØE Ø4	Ø.95468130111 69D Ø1	Ø.95468130111 69D Ø1
Ø.8ØØE Ø4	Ø.968Ø34362793ØD Ø1	Ø.968Ø34362793ØD Ø1
Ø.9ØØE Ø4	Ø.9798127174377D Ø1	Ø.9798127174377D Ø1
Ø.1ØØE Ø5	Ø.99Ø34881591 8ØD Ø1	Ø.99Ø34881591 8ØD Ø1
Ø.11ØE Ø5	Ø.9998798370361D Ø1	Ø.9998798370361D Ø1
Ø.12ØE Ø5	Ø.1ØØ858Ø97Ø764D Ø2	Ø.1ØØ858Ø97Ø764D Ø2
Ø.13ØE Ø5	Ø.1Ø16585254669D Ø2	Ø.1Ø16585254669D Ø2
Ø.14ØE Ø5	Ø.1Ø2399 6Ø67Ø47D Ø2	Ø.1Ø2399 6Ø67Ø47D Ø2
Ø.15ØE Ø5	Ø.1Ø3Ø89532 8522D Ø2	Ø.1Ø3Ø89532 8522D Ø2
Ø.16ØE Ø5	Ø.1Ø37349128723D Ø2	Ø.1Ø37349128723D Ø2
Ø.17ØE Ø5	Ø.1Ø434115 4Ø985D Ø2	Ø.1Ø434115 4Ø985D Ø2
Ø.18ØE Ø5	Ø.1Ø491273 88ØØØD Ø2	Ø.1Ø491273 88ØØØD Ø2
Ø.19ØE Ø5	Ø.1Ø54534149170D Ø2	Ø.1Ø54534149170D Ø2
Ø.2ØØE Ø5	Ø.1Ø59663486481D Ø2	Ø.1Ø59663486481D Ø2
Ø.21ØE Ø5	Ø.1Ø645424 84283D Ø2	Ø.1Ø645424 84283D Ø2
Ø.22ØE Ø5	Ø.1Ø6919 45Ø7599D Ø2	Ø.1Ø6919 45Ø7599D Ø2
Ø.23ØE Ø5	Ø.1Ø73639678955D Ø2	Ø.1Ø73639678955D Ø2
Ø.24ØE Ø5	Ø.1Ø77895641327D Ø2	Ø.1Ø77895641327D Ø2
Ø.25ØE Ø5	Ø.1Ø81977844238D Ø2	Ø.1Ø81977844238D Ø2
Ø.26ØE Ø5	Ø.1Ø858999252 32D Ø2	Ø.1Ø858999252 32D Ø2
Ø.27ØE Ø5	Ø.1Ø89 6739ØØ6Ø4D Ø2	Ø.1Ø89 6739ØØ6Ø4D Ø2
Ø.28ØE Ø5	Ø.1Ø9331Ø73761ØD Ø2	Ø.1Ø9331Ø73761ØD Ø2
Ø.29ØE Ø5	Ø.1Ø968198776 25D Ø2	Ø.1Ø968198776 25D Ø2
Ø.3ØØE Ø5	Ø.11ØØ2Ø9999Ø84D Ø2	Ø.11ØØ2Ø9999Ø84D Ø2
Ø.31ØE Ø5	Ø.11Ø3489Ø17487D Ø2	Ø.11Ø3489Ø17487D Ø2
Ø.32ØE Ø5	Ø.11Ø66637992 86D Ø2	Ø.11Ø66637992 86D Ø2
Ø.33ØE Ø5	Ø.11Ø9741Ø2Ø2Ø3D Ø2	Ø.11Ø9741Ø2Ø2Ø3D Ø2
Ø.34ØE Ø5	Ø.1112726 3Ø6915D Ø2	Ø.1112726 3Ø6915D Ø2
Ø.35ØE Ø5	Ø.11156 25Ø95367D Ø2	Ø.11156 25Ø95367D Ø2
Ø.36ØE Ø5	Ø.1118442153931D Ø2	Ø.1118442153931D Ø2
Ø.37ØE Ø5	Ø.11211 82Ø6Ø242D Ø2	Ø.11211 82Ø6Ø242D Ø2
Ø.38ØE Ø5	Ø.11238489151ØØD Ø2	Ø.11238489151ØØD Ø2
Ø.39ØE Ø5	Ø.1126446437836D Ø2	Ø.1126446437836D Ø2
Ø.4ØØE Ø5	Ø.1128978252411D Ø2	Ø.1128978252411D Ø2
Ø.41ØE Ø5	Ø.1131 4475Ø5951D Ø2	Ø.1131 4475Ø5951D Ø2
Ø.42ØE Ø5	Ø.1133857 25Ø214D Ø2	Ø.1133857 25Ø214D Ø2
Ø.43ØE Ø5	Ø.11362 1Ø346222D Ø2	Ø.11362 1Ø346222D Ø2
Ø.44ØE Ø5	Ø.11385Ø92735 29D Ø2	Ø.11385Ø92735 29D Ø2
Ø.45ØE Ø5	Ø.114Ø756511688D Ø2	Ø.114Ø756511688D Ø2
Ø.46ØE Ø5	Ø.1142954444885D Ø2	Ø.1142954444885D Ø2
Ø.47ØE Ø5	Ø.11451Ø5Ø75836D Ø2	Ø.11451Ø5Ø75836D Ø2
Ø.48ØE Ø5	Ø.114721Ø4Ø7257D Ø2	Ø.114721Ø4Ø7257D Ø2
Ø.49ØE Ø5	Ø.1149272346497D Ø2	Ø.1149272346497D Ø2

```
      FUNCTION ASINH(X)
C.....
C.....THIS IS A FUNCTION PROGRAM TO COMPUTE THE INVERSE HYPERBOLIC
C.....FUNCTION ARCSINH(X). FOR ABS(X)<1 THE PROCEDURE USES A CHEBY-
C.....CHEV SERIES WITH ARGUMENT 2X*X-1 TO EVALUATE ARCSINH(X)/X. FOR
C.....ABS(X)>1 THE RELATION ARCSINH(X)=LN(ABS(X)+SQRT(1+X*X)) IS USED.
C.....
      DIMENSION C(1Ø)
      DOUBLE PRECISION C,A,AA,ASINH,Y,AL2
      DATA C/1.87179960088261DØ,-Ø.05881176118950Ø,Ø.00472746543221DØ,
     1 -Ø.000493836316265DØ,Ø.0000585062070590Ø,-Ø.0000074669983290Ø,
     2 Ø.000001001169358DØ,-Ø.000000139035439DØ,Ø.000000019823169DØ,
     3 Ø.000000002884747DØ/
      Y=ABS(X)
      IF(ABS(X).GE.1.ØE+8)GO TO 2
      A=X*X
      IF(A.GT.1.ØDØ)GO TO 1
      AA=A+A-1.ØDØ
      ASINH=X*CHEBY(AA,C,1Ø)
      RETURN
1      Y=ABS(X)
      A=ALOG(Y+SQRT(1.ØDØ+A))
      GO TO 3
2      AL2=Ø.693147180559945300
      A=ALOG(Y)
      A=A+AL2
3      IF(X.LT.Ø.Ø)A=-A
      ASINH=A
      RETURN
      END
```

```
      FUNCTION ACOSH(X,ICHECK)
C.....
C.....THIS IS THE FUNCTION PROGRAM TO COMPUTE THE INVERSE
C.....HYPERBOLIC FUNCTION ACOSH(X).
C.....
      DOUBLE PRECISION ARG,ACOSH
      ICHECK=0
      IF(X.LT.1.0)GO TO 1
      ARG=X*X-1.0D0
      ARG=SQRT(ARG)
      ACOSH=ASINH(ARG)
      RETURN
1      ICHECK=1
      WRITE(6,2)ICHECK
2      FORMAT(' X IS OUTSIDE THR RANGE 1<X<INFINITY.',3X,'ICHECK=',
     1  2X,I1)
       RETURN
      END
```

```
      FUNCTION CHEBY(X,C,N)
C.....
C.....THIS IS A FUNCTION PROGRAM TO COMPUTE THE VALUE OF AN N-TERM
C.....EXPANSION IN CHEBYCHEV POLYNOMIALS WITH COEFFICIENT VECTOR
C.....C AND ARGUMENT X.
C.....
      DIMENSION C(1)
      DOUBLE PRECISION C,CHEBY,X,H0,H1,H2,ARG
      IF(N)1,1,2
1      RETURN
2      IF(N-2)3,4,4
3      CHEBY=C(1)
       RETURN
4      ARG=X+X
      H1=Ø.ØDØ
      H0=Ø.ØDØ
      DO 5 I=1,N
      K=N-I
      H2=H1
      H1=H0
5      H0=ARG*H1-H2+C(K+1)
      CHEBY=Ø.5DØ*(H0-H2)
      RETURN
      END
```