THE LOGICAL MODALITIES AND THEIR SEMANTICS

by

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ABSTRACT

This thesis is an interpretation of the logical modalities with the formulation of a suitable semantics. The system devised, QS4.3*, is based on the modal system S4.3, with the defining axiom, $\Box[\Box p \supset \Box q] \vee \Box[\Box q \supset \Box p]$, and the addition of quantifiers. The "accessibility relation" for the system is reflexive, transitive and anti-symmetric which defines a partial-ordering of possible partial worlds, rather than of the usual possible worlds. The need for a partial-ordering allows for a fuller interpretation of truth in all/some possible partial worlds with respect to states-of-affairs and sub-states-of-affairs. This is possible with a restricted meaning of the possible world as a possible partial world.

PREFACE

The original impetus behind the thesis was a general dissatisfaction with what appeared to be going on in modal logic. The "hint" I found in Wittgenstein's <u>Philosophical Investigations</u>: "a smiling mouth smiles only in a human face"; it made me wonder if such <u>intensional</u> considerations ought not be brought into an analysis of modal logics, especially when it comes to considerations of possibility, so that what is construed as possible has some ring of truth and relates to the way things tend to go in the actual world; namely, from past to future, and from a consideration of larger states of affairs to smaller and smaller states of affairs. There would be many occurrences which could logically be possible, but could also not <u>be</u>. On the other hand, there is a sense in which anything is possible. Exactly what such claims could mean puzzled me, since current modal logics did not appear to be too well grounded philosophically, even though their algebraic analysis seemed to please everyone.

Originally it was thought that the main way to approach the topic was to concentrate exclusively on the genesis of possible worlds. This approach would easily have made use of a theory of actuality/potentiality. This voyage into metaphysics, though, would not have left much room for the logic.

The problem of essential and accidental attribution seems to me not to be a logical question, but one which requires a clear analysis of the nature of possibility (this might have brought us back to the actual/potential) so that essential attribution and necessary statements become connected. If an individual, I, has an attribute, P, essentially, then in all possible worlds it has P, so that the statement "I has P" is necessarily true; i.e. "I has P" must be true in all possible worlds. Similarly, in order to understand what an accidental attribute is we must be able to make sense of situations in which I lacks P (i.e. has some other attribute, without loss of identity). What happens here is that if there is no theory of essential and accidental attribution, we admit of an individual having any accidental attribute we please (and even some other individual's essential ones), without any concern for what is being said. There is a connexion between accidental/possible and essential/necessary which can only be understood through an analysis of the nature of possibiltiy.

This thesis is an analysis of possibility and presents a semantics based not on possible worlds, but possible partial worlds, involving the "nesting" of such partial worlds, thus requiring that the accessibility be at least anti-symmetric.

In formulating this thesis in terms of content, I thank Dr. N.L. Wilson, who patiently helped improve it, especially with some difficulties concerning the anti-symmetry of the accessibility relation. I would also like to thank Dr. M. Radner for some interesting discussions on certain aspects of this thesis concerning epistemology. A former teacher, Dr. Yukio Kachi, University of Utah, and his colleague, Dr. Zeno Vendlar, Rice University, provided me with some interesting and useful insights. I have also benefitted from discussions with my good friend Malcolm Lake on issues related to the philosophical analysis of the nature of necessity and possibility.

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INTRODUCTION

There is a branch of logic which deals with the nature of necessity and possibility called alethic logic, or more commonly, modal logic. A treatment of modal logic in a semantic (model-theoretic) sense has been available with the work mainly of S. Kripke and others. Although there are numerous treatments and discussions — by Lewis, Quine, Carnap, Marcus, and others — such a semantic treatment has been dubbed "possible worlds semantics" after the central intuitive ideas of Leibniz' possible worlds. Leibniz recognized that, in a theological context, God had a choice about which world he would actualise; from this, reasoned Leibniz, there must be some plethora of possible worlds (possibly infinite in number) in addition to the one world actually chosen [Leibniz, Selections]. Kripke took this notion and related it to a semantic analysis of necessary and contingent truth resulting in the view of a necessary proposition as being true in all possible worlds. The idea here is that possible worlds, being sufficiently uniform, allow certain propositions to be determined as either true or false; that is, the truth value of a given proposition as determinable [as true or as false] in all, or in any arbitrary subset of these, possible worlds.

This notion, though intuitively adequate, is not sufficiently well understood. It assumes that any proposition is determinable under any and all circumstances; it could be more appropriate to consider such propositions as being determinable in some subset of these possible worlds and indeterminable in the rest. With that type of proviso we introduce the notion of "relevance". That is, a proposition is necessary if and only if it is true in all <u>relevant</u> possible worlds. Relevancy, in such a case, can mean a number of things such as "all possible worlds epistemically accessible to the inhabitants of some base world", or "all possible worlds within the range of the human powers of conception", and so on.

The claim made by the proponents of this possible worlds semantics is that it supplies a forceful explanation of truth in general and modalized truth in particular, through the construction of suitable models. These models are of a possible <u>state of</u> the world, not of some metaphorically fanciful cosmic entity (though from the talk about the issue, one would think they could be found in the <u>New Galactic Catalogue</u>). Possible worlds can be no more than counterfactual situations; Kripke has noted this in "Identity and Necessity" [Munitz, <u>Identity</u>, esp 147/8]. One problem claimed to be associated with modal logic is of identifying a certain entity in a counterfactual situation. For example, if one says: "If Mr. G. were such-and-such then. . .", how does one ascertain that Mr. G. in such a statement is <u>the</u> Mr. G. that lives next door and about whom one is speaking. This, though, is a gross misunderstanding of the problem. In such a statement, Mr. G. is <u>given</u> in the discourse.

If possible worlds are no more than counterfactual situations, then, they are governed by the same physical laws which govern this world, the actual one [q.v. Sudbury, <u>Laws</u>]. Given that, then, we would want to impose on our semantics considerations which respect what it means to say that a proposition is true, or could be true. The human faculty of imagination is often invoked as specifying what amounts to an explanation of possibilia, almost as though human imagination had some privileged acess to knowledge of what exists or can exist. What is impossible is often identified as being "inconceivable", or "unimaginable", since logical contradictions are hard to imagine — for example, something being red and non-red all over at the same time. However, there are varieties of impossibility which do not embrace the prohibition against being inconceivable. What counts as logical possibility can turn out to be rather vague, if we do not pay close enough attention to what the modalities are all about. This discussion, then, hinges on a clear understanding of what it means to conceive or to imagine something as being possible. Such an understanding produces a modified concept of what a semantic explanation of modal logic must say.

The semantic construction which arises from these initial considerations captures what is felt to be a philosophically more acceptable explanation of the relation of accessibility, based on the notion of Hintikka's model-sets, or partial state-descriptions.

Chapter I sets the stage for such analysis, wherein we investigate, in an historical-dialectical manner, initial attempts to investigate modal idioms and establish a rigorous modal logic. Kripke's writings form the core of chapter II, since his constructions more so than previous ones deal with a generalized notion of logical truth, rather than the syntactic notion of provability and "being deducible from". This chapter evaluates the various mechanisms of a model-theoretic approach and presents what is felt to be a more philosophically satisfying account, which relies quite strongly on meta-linguistic considerations. The final chapter expounds the philosophical argumention which underlies the semantic system of chapter II.

CHAPTER I

THE FOUNDING OF MODAL LOGIC

§1 Introduction

In this investigation, we are going to be torn between two philosophical perspectives. One asks the very basic question about whether modal logic actually makes any sense at all; this is the Bergmann position [Phil. Sig.] which questions its rôle in philosophical analysis and philosophy of logic, as well the other is the position which raises basic questions about the aspect of modality concerning a "logic of entailment", which attempts to make sense of "necessity" in the syntactic sense of deducibility, or of "B following from A". The second major perspective takes it *cum grano sale* that modal logic has utility and proceeds to investigate it through suitable model-theoretic techniques — the so-called "possible worlds semantics". These models represent a way of interpreting modal logic (also with quantifiers) and the general model-theoretic approach is to study the logic under all possible interpretations. One interpretation will be the interpretation incorporating the elements of the real or the actual world. As one makes changes in the actual world one constructs alternatives to it. For example, counterfactual situations represent alternate states of the world and hence, by incorporating them into a model, a possible world. Possible worlds form the intuition upon which semantic truth is based: as we have noted, necessarily true propositions are true in all possible worlds or true in some and (in some way) non-false in the rest, while possibly true ones are true in some and false (or indeterminable) in the rest (or possibly not-flase in the rest -- this based on $\Box p \supset \Diamond p$).

Modern modal logic has its roots in the work of C. I. Lewis. His analysis construes necessity in the context of deducibility. There still remains the ordinary language usage of the modal operators and exactly what they mean. If modal idioms embody linguistic redundancy, then ordinary quantified logic can handle it; if, on the other hand, modal idioms reflect legitimate distinctions in language, then there is much to be gained by an analysis of them.

In order to appreciate the need for modal logic we can examine a number of statements and see how the various types of logic represent their logical structure.

Consider these two sentences:

1. Some known sentences are possibly false.
2. It is possible that some known sentences are false.

In the first sentence, we can have a predicate G: 'possibly false':

3. $(\exists x)$ [Kx & Gx]

but, 3 does not completely reveal the logical structure, if by appending 'possibly' to 'false' we were trying to say something more about the sentence. If we want this then we should want 'Gx' to be an abbreviation of the conjunction of two more logically primitive predicates: That of being a false sentence and that of being a possible one, i.e., Px & Fx:

3'. $(\exists x)$ [Kx & Px & Fx].

this procedure of treating possibility as a type of special predicate is inadequate when one considers 2, which says,

4. it is possible that $(\exists x)$ [Kx & Fx]

To be consistent with 3', 4 should become:

4'. P(($\exists$x)[Kx & Fx]).

'P' being an operator operating, in this case, on sentences, while in 3' it took sentence names, unless, in 4', 'P' takes names of sentences. However, if that is the case, the whole point of talking about modal distinctions seems rather pointless. Consequently, we want to write 4' in such a way as to reveal the <u>logical</u> force of appending 'it is possible that' to a sentence. It must be defined in such a way that it can handle logically intact sentences, and predicates, and in such a way that it reveals something about the statement's logical structure. This is the reason why we feel compelled to seek a logical modal operator, which operates not as a predicate does in taking names as values, but as a sentence operator taking predicates or logically complete statements as values, or more precisely, the truth values of such entities.

We end up with an operator which "alters", according to the interpretation we give to it, the truth value of the operand in such a way as to reflect on "normal", ordinary language use of it. In this way, we claim to be formalising completely a modalized statement. The inadequacy of the various types of logic to reveal complete logical structure can be revealed with some examples. Consider,

(a) it is necessary that if something is going to happen tomorrow, then something is going to happen tomorrow;

this can be formalised in a number of ways. In non-modal, propositional logic we have:

(a') p

since the modal operator applies to the whole statement, we cannot extract the logical structure of the contained statement. In modal, non-quantified logic, we have

(a'') $\Box(p \supset p)$

but since we cannot utilise quantificational logic the structure of the contained sentence cannot be made explicit. However, in quantified modal logic, we have

(a''') $\Box[(\exists x) Hx \supset (\exists x)Hx]$, where H: ① is going to happen. As another example consider these two:

(b) Some things are going to happen tomorrow.

(c) Some things are going to happen tomorrow of necessity.

The two cannot be distinguished except in quantified modal logic, since in ordinary propositional logic they are

(b') p

(c') q

in quantificational logic they are

(b'') $(\exists x)Hx$, where 'H' as in (a''')

(c'') $(\exists x)Gx$, where G: ③ is going to happen tomorrow of necessity,

and in non-quantificational modal logic, the need to make distinctions becomes clear:

(b''') r

(c''') s

the two above not sufficiently different even though "$s \neq \Box r$", because we must recognize the distinction between (i) of the things which are going to happen tomorrow, some are necessary, and (ii) it is necessary that something will happen tomorrow. It is only in a quantified modal logic that the logical structure of (b) and (c) is made clear:

(b'''') $(\exists x)Hx$

(c'''') $\Box(\exists x)Hx$

These examples seem to indicate that modal logic in the general sense is needed in order to express the complete logical structure of ordinary language.

Consider, now, this example:

It is necessary that Socrates be human.

That is (in propositional logic):

$\Box p$

Here we meet the problem of essential and accidental attribution. In order to make full sense of this kind of statement, we must make an appela to some meaningful concept of this type of attribution by which one can determine what are the essential aspects of Socrates and what are the accidental ones. Aristotelian essentialism is the more comprehensive analysis of essentialism; to understand the necessity of Socrates' humanity as a necessary property means that if he were to lose that property, he would cease to <u>be</u>. That is, an individual has a property essentially, if it ceases to exist as <u>that</u> individual when it loses that essential property; it may also go out of existence completely. The doctrine of essentialism wrapped up here is that of "coming to be and passing away" — individuals come to be because of <u>their</u> existing properties which are essential, and pass away as that individual when such properties are lost.

A clear understanding of essentialism can provide one way out of the morass of modality, but only with a complete philosophical analysis of the nature of the modalities. The essential property quesiton is one thing; the other involves an interpretation of $\Box[p\supset q]$ which is used by C. I. Lewis to characterise deduction.

The following provides some perspective on problems to be encountered in an analysis of this. The general orientation is to show that $\Box[p\supset q]$ means that q is deducible from p. The modal logicians' commitment, or non-commitment, to essentialism arises only in the non-logical philosophical investigations of modal logic. In this regard, it is perhaps to a pure logician a red-herring of a problem; however, in the context of this thesis, such questions are seen as having priority -- it is not a question of the logic being committed to essentialism (though it certainly is committed to its meaningfulness) but of whether the essentialists' arguments are accounted for in the logic (modal logic appearing to be the main candidate in this matter).

We begin with Lewis as the source of modern work on modal logic, and show how various subsequent logicians have attempted to ground a philosophy of modal logic.

§2 C. I. Lewis

C. I. Lewis was the first to attempt to come to grips with the problem of implication and deduction along modal lines in [Symbolic]. B. Russell has made an attempt to analyze the two modalities in an unpublished manuscript "Necessity and Possibility".[1] There is much to be said against modal logic as G. Bergmann has noted in "The Philosophical Significance of Modal Logic" [vide Phil. Sig.]. Bergmann's overriding concern was to show that modal logic had no use in philosophical analysis [Phil. Sig., 466]. As J. Hintikka has noted, Bergmann's protestations are founded on his premise

1. In the Bertrand Russell Archives, McMaster University, Hamilton.

that "all the earlier deductive systems of modal logic [failed] to be based on a satisfactory semantical . . . characterization of validity (logical truth)" [Models, 84]. However, with the development of a suitable semantics for modal logic, Bergmann's objections no longer hold, although there is still debate over analytic/synthetic distinctions.

As we mentioned, Lewis was the first to attempt to analyze necessity for use in a logistic by coming to grips with the nature of deduction. His motivation was a dislike of Russell's material implication as an alleged account of deducibility. Lewis called his type of implication, based on deduction, "strict implication". The Kneale's observe in The Development of Logic that the merit of Lewis' modal system is not so much that it was a better way to reason, but that it attempted to grasp the meaning of "necessity", and hence, to make sense of alethic modality. Clearly, then, Lewis believed modal logic to be of utility and have scope beyond the questions of algebraic concern.

In the following, we analyze Lewis' notion of necessity as he presents it in terms of "entailment". In essence, the argument to follow will be concerned with entailment as having any meaning and necessity as being a useful operator in this context. The main source of difficulties though, centre on Lewis' insensitivity to the "use/mention" distinction which caused him to confuse the object language and the metalanguage, especially with respect to the object language's

'-3' and the metalanguage's (for Lewis) "strict implication".

We begin here by considering 'if-then'; the truth table is:

p	q	$p \supset q$
t	t	t
t	f	f
f	t	t
f	f	t

Notice that $p \supset q$ is true for q true and p either true or false; and that it is true for p false and q either true or false. Putting this another way, we have:

(a) a true proposition is materially implied by any proposition, whether true or false;

i.e. to say that p materially implies q is just to say that the conditional '$p \supset q$' is true;

(b) a false proposition materially implies any proposition whether true or false,

(a) and (b) are represented in the sentential calsulus as:

(a') $p \supset [q \supset p]$

(b') $-p \supset [p \supset q]$

These two are traditionally referred to as the paradoxes of material implication. It follows readily from (a') and (b') that for any pair of propositions:

(c) $[p \supset q] \vee [q \supset p]$

The above usage is consistent with the concept of "imply" as used by A. N. Whitehead and B. Russell in

[Principia, Part I, Sect. A, *1.01, 1.1], though Lewis did not question this type of implication. Hughes and Cresswell have said of Lewis that he maintained [Introduction, 215]:

> There is another, stronger sense of "imply", a sense in which when we say that p implies q, we mean that q follows from p; and that in this s nse of "imply" it is not the case that every true proposition whatsoever, or that every false proposition implied any proposition whatsoever.

Under such an interpretation of '$\supset$', (c) is not a tautology, the grounding of this non-tautologousness being the interpretation one assigns to implication. Lewis referred to this as "strict implication" -- p strictly implies q if and only if it is impossible that p be true and q be false, (there are other definitions of entailment which are even stronger).

The main concern of strict implication is in the context of the deduction of the consequent from the antecedent; i.e., of q following from p. Strict implication intends to answer the question: can there be a relation unambiguously determined, which holds of p and q if and only if q is deducible from p [Symbolic, 236]. The answer is in the affirmative; it is the converse of deducibility. As Lewis and Langford say [Symbolic, 235]:

> The chief business of a canon of deduction is to delineate correctly the properties of that relation which holds between any premise, or set of premises, and a conclusion which can validly be inferred. Commonly this relation is called 'implication'.

In the overall sense of modal logic, we must come to grips with "necessarily, p then q" as well as such statements as "necessarily, that p" and "necessarily, for some x, that Fx".

Consider the rule of detachment (modus ponens) where '⊃' has been replaced by 'I', but which has, as yet, no (truth-tabular) interpretation, but which must satisfy certain criteria to be adequate; e.g.:

[(p & [pIq]) Iq] must be a theorem

'I' is characterized generally as follows:

1. When the antecedent p in any relation pIq is assertable as true, the consequent must also be assertable as true.

The requirement here is that pIq not be true when p is true and q is false.

2. pIq will hold when q is deducible from p and fail to hold when q is not deducilbe from p.

pIq holds for any pair of true propositions p and q; accordingly, p and q would be interdeducible. Non-deducibility would obtain only when one was true and the other false. The undesirable conslusion is that every true proposition is deducible from every other true proposition. On this basis, pIq could never mean "q is deducible from p", since it could conceivably hold when p and q were true, but q not deducilbe from p.

> Since all . . . true implications hold whenever p and q are bothetrue, every such relation is too inclusive in its meaning to be equivalent to 'q is deducible from p'. [Symbolic, 289]

What is required, then, is that pIq hold when and only when p and q are true and q is deducible from p, our only concern being for valid deductions.

To clarify this, one must distinguish "pIq is true" from "pIq is a true conditional". Lewis and Langford write [Symbolic, 244]:

> When pIq is true but not tautological, q can be deduced from the two premises p and pIq, but only because '(p and pIq)Iq' is a tautology. When pIq is a tautology, q can be deduced from p.

From this, the nature of 'I' follows:

> 'I' must be some connective '$\otimes$' such that '$p \otimes q$' means "p true and q false is a logically impossible combination";

that is,

$$-\Diamond[p \mathbin{\&} -q]$$

or,

$$\Box[p \supset q]$$

which is "it is necessary that if p then q", and becomes

$$p \prec q$$

in Lewis' notation.

This is the meaning that is to be attached to strict implication, and it holds when q is deducible from p. With the definition:

$$p \prec q =\mathrm{df} \quad \Box[p \supset q]$$

one can specify its fundamental set of truth tables for each of the $(2^{2^2}-1)$ logically possible assignments of logically possible assignments of logically possible truth values to the formula within its scope.

The idea is that $\Box p$ is true on every line (of a truth table), if and only if p is true on every line (of the truth table), and false if there is at least one false interpretation. (Each line can represent an interpretation of the relevant formula in some possible world, and hence $\Box p$ is true if and only if p is true on every line (in all possible worlds)).

The set of truth-tables depends on not only the truth-table for '$\Box$', but also '$\supset$', whose fundamental set of truth-tables is given as follows:

p	$\Box$p
t	t

p	$\Box$p
f	f

p	$\Box$p
t	f
f	f

It follows, then, that in any theorem where '$\supset$' is the major connective, '⥽' can be substituted. For example:

1. $q \supset [p \supset q]$

becomes

1'. $q ⥽ [p \supset q]$;

2. $-p \supset [p \supset q]$

becomes

2'. $-p ⥽ [p \supset q]$;

3. $[p \,\&.\, p \supset q] \supset q$

becomes

3'. $[p \,\&.\, p \supset q] ⥽ q$.

The last one, for instance, says that p is deducible from the two premises p and $p \supset q$.

Associated with strict implication are two paradoxes, the so-called paradoxes of strict implication:

4. $\Box -p \prec [p \prec q]$

i.e., if p is necessarily false, then it strictly implies any proposition q.

5. $\Box p \prec [q \prec p]$

i.e., if p is necessarily true, then any proposition q strictly implies it.

Some have objected to these two statements, [Anderson & Belnap, Entail] on the grounds that necessary statements cannot be strictly implied by, or strictly imply, any arbitrary statement, unless there is some meaning relationship between antecedent and consequent similar to what one finds in Ockham's theory of consequences.

We can analyse this in the following way. If there is no meaning relation, then 4 and 5 do appear to generate puzzling circumstances. If one claims that an antecedent's truth follows from its meaning, and the consequent's truth follows from the meaning of the antecedent, then one has a meaning relation, and hence, strict implications obtaining between statements such that if the statement should, due to its meaning, be true, then the consequent will likewise be true. Accordingly, one must rule out as necessary statements:

George is an unmarried man strictly implies if

1 + 1 = 2, then 1 + 1 = 2

in favour of:

George is an unmarried man strictly implies George is a bachelor.

supposedly on some (intensional) notion of meaning synonymy and in order to keep our language levels from becoming confused. In addition to statements such as this, one must also recognise statements involving analytic deduction, as for example, when one says that 'George is a man' is analytically deducible from 'George is a bachelor' on the basis that "bachelor" is the meaning-conjunction of "man" and "unmarred", either conjuct being deducible.

Synonymy is something which seems to be based firmly in intuition yet its explication is not all that clear. Carnap, for example, needs synonymy and defines it in terms of "designation": two expressions are symonyms when the two expressions have the same designations (an extensioanl definition). Carnap's particular formulation allows for cross-language determination of synonymy: such as Hase in German and hare in English which have the same designation. We should note here what Carnap means by designation. In semiotic, one must distinguish among the speaker, the expression, and to what is being referred. The last is called the designatum, being the expression which is said to designate. What is designatum is various; for example, it may be a property, relation, function, or thing (a concrete individual), given by a proper name, such as 'Toronto'.

We can give a sketch of synonymy in the following way; let synonymy be represented by '$=_s$', and M be the function which picks out the meaning of sentences, then:

M[George is an unmarried man] $\neq_s$ M[if 1 + 1 = 2 then 1 + 1 = 2]

while

M[George is an unmarried man] $=_s$ M[George is a bachelor].

and that obviously depending on:

M[unmarried man] $=_s$ M[bachelor]

meaning, of course, that they are interchangeable in sentences including referentially opaque contexts, salva veritate.

However, if that is the case, to come to grips with logical deducibility would require a criterion for meaning synonymy and hence, for analyticity. I.e., why are 'unmarried man' and 'bachelor' interchangeable and meaning synonymous? At this level, analyticity and synonymy come out simply as aspects of the language game and mean nothing over and above their use. This operationalist stance is quite tenable in this case since it does answer the question of what are analyticity and synonymy: in a language expressed by a minimumvocabulary, there is no room for operational definitions, but neither is there room for meaning analyticity and synonymy; they arise as functions of an augmented vocabulary (i.e., the minimum vocabulary plus a lexicon of defined words).

The paradoxes of material implication mentioned earlier cease to be "counter-intuitive" if it can be shown that:

$$M[p] =_s M[q]$$

for then analyticity would rest on meaning synonymy and hence, strict implication would go through trivially. The problem, though, that still remains is whether necessarily true propositions incorporating meaning synonymy are analytic, and whether there _are_ analytic statements — is the analytic/synthetic distinction a viable one. It would seem, on the above notion of meaning synonymy, that analytic statements turn out to be based on linguistic convention, which is arrived at through reciprocal and efficacious language use. That is, given some state of affairs that p and q purport to describe — say p: George is an unmarried man; q: George is a bachelor — the use of p and the use of q [by someone] conveys the same meaning [to a listener] and hence, p and q would necessarily have the same truth value: i.e., they would be strictly equivalent. A complete cybernetic modelling may provide the answer, but this is another matter altogether.

Lewis claims that these two statements are not paradoxes, but say something about deduction [_Symbolic_, 252]:

> The tautological character of tautologies is something which ordinary logical procedures assume, and in that sense, all tautologies are already given. This merely reflects the . . . fact that, when deductions are made, logical principles themselves are implicit.

He concludes that they are inevitable and "unavoidable consequences of indispensable rules of inference". [Symbolic, 252]

Hughes and Cresswell note that on Lewis' interpretation of strict implication, these paradoxes do not constitute any violation and, in fact, a correct logic of entailment must include them [Introduction, 336]

Furthermore, strict implication permits the distinguishing of tautologies from mere truths. The corresponding conditional for modus ponens is a tautology:

$$-\Diamond[[p \mathbin{\&}. \ p \supset q] \mathbin{\&} -q].$$

Modus ponens can be taken as a rule of inference, since it does not admit of the drawing of a false conclusion from true premises. Deduction proceeds algorithmically in accordance with such rules. In a logical system with this as a rule of inference, q becomes a necessarily true consequence of the premises, which must also be true. What we want is:

$$[\Box p \mathbin{\&}. \ p \prec q] \prec q.$$

However, this is only valid if the rule of detachment permits $\Box q$ to be concluded given $\Box p$ and $p \prec q$; i.e., if

$$p \prec q . \supset . \Box p \supset \Box q.$$

In such a way, given that the premise set consists in tautologies, only tautologies are to be deduced. Under the first interpretation,

$$p \prec q . \equiv . \Box p \supset \Box q$$

is false, because

$$\Box p \supset \Box q . \equiv . \Diamond -p \vee \Box q$$

which is true for any non-tautologous p or tautologous q, regardless of whether there is any meaning relation. The second interpretation

$$p \prec q . \equiv . - \Diamond [p \,\&\, -q]$$

captures the notion of deducibility, only in so far as it involves a meaning relation between p and q.

The Kneale's have observed [Development, 559]:

> If strict implication justifies deduction there can be no difference, except in vocabulary, between the assertion that inference from the first to the second is valid.

Deduction is invalid when it proceeds from true premises to a false conlusion, and is sometimes invalid when it proceeds from true premises to a true conclusion. Certain rules, such as modus ponens guarantee the validity of the deduction given the truth of the premises.

What we have up to here been doing is showing how Lewis attempted to characterize deduction. His attempt was important and not without merit, and not without error. What Lewis calls deduction and characterized by strict implication is what we want to mean by deduction; however, it does not help in the explication of the logical modalities as he does not provide a (semantical) interpretation of them.

In the following, we consider further the nature of entailment; the arguments given turn on the difficulties of quantified modal logic.

The Kneale's observe that Lewis considers '$\prec$' as though it were a sign of the same grammatical status as '$\supset$'. However, Lewis insists that strict implication is a non-truth-functional relation which holds between a conclusion and a premise set taken as propositions. It is intended to show, now, that because Lewis neglects to consider the difference between a binary predicate and a binary connective, some confusion is generated, with Lewis treating '$\prec$' as relational, but on a par with a statement connective.

The English language rendering of '$\prec$' is "that . . . entails/strictly implies that . . . ". The blanks are filled by sentences, not names of sentences, the difference here lying in the distinction between a binary predicate flanked by statement names and a binary statement connective flanked by statements. A binary predicate occurs between names of objects and a binary connective between statements, not names of statements. That is:

For 'N_1' and 'N_2', the names of the statements S_1 and S_2, respectively, 'that S_1 entails$_1$ that

S_2' is true if and only if the expression 'N_1 entails$_2$ N_2' is true.

For example, consider:

1. that it is raining entails that the ground is wet

in contrast to:

2. 'it is raining' entails 'the ground is wet'.

The first requires that the entailment be a relation between propositions while the second requires that entailment be a relation between statements.

This is not by any means a trivial point; it is fundamental to logic. Quine writes in [<u>ML</u>, 28]:

> The verb 'implies' belongs between names of statements precisely because, unlike '$\supset$' or 'if-then', it expresses a relation between statements, it is a binary predicate by means of which we talk <u>about</u> semantics.

Accordingly, with the study of the logic of entailment, we invite confusion by using a statement connective, which is a non-relational operator, in the same way as the relation of entailment.

Binary statement connectives, such as '$\supset$', are placed between statements to form new statements. On the other hand, benary, or dyadic, predicates, such as '=', are placed between names (of numbers, for example) to form statements. (Predicates must take names sin ce, for example, '① is equal to ②' lacks a subject and a prepositional object.) Statement connectives can be iterated and still make good sense, while predicates cannot. For example:

$A \supset [B \supset C]$

i.e., if A then if B then C;

but not,

$A > [B > C]$

i.e., A is greater than B is greater than C.

'$\prec$' is a statement connective of the same grammatical type as '&', 'v', or '$\supset$', and is not a binary predicate. It is placed between the same symbols as are truth-functional statement connectives, and can be iterated. Since it is misleading to read '$\supset$' as "implies", as Quine has noted, and due to the definition of '$\prec$', the preferred reading of it is "necessarily, if A then B". For example:

$A \prec [B \prec C]$

i.e., necessarily, if A then necessarily, if B then C.

Hughes and Cresswell fail to be firm on this issue in [<u>Introduction</u>, 23]:

> Another important modal notion is that of <u>entailment</u>. By this, we understand the converse of the relation of following logically from.

i.e., entailment is a relation, yet [<u>Introduction</u>, 24]

> 'Entails' and follows 'logically from' [sic] are dyadic proposition-forming operators which are not truth-functional . . .

which construes 'entails' as performing the same role as any statement connective.

Anderson and Belnap make a similar equivocation. On the one hand, they regard '⥽' as a statement connective and hence flanked by statements [Entail, 79]:

> [We take] the '⥽'; as the formal analogue of the connective 'that . . . entails that . . .'.

yet they insist [Entail, 80]:

> We wish to interpret 'A⥽B' as 'A entails B' or 'B is deducible from A'.

Lewis claims that '⥽' represents deducibility. If that is the case, then the most that one can hope for is that a logic of entailment, or a logic of implication which recognizes the role of modalities, and hence, a making sense of 'B follows logically from A', involves just another dyaidc predicate.

If that is the case, then the most that one can hope for is that of a logic of entailment, or a logic of implication which takes into account distinctions one must make between the object-language and the meta-language. Certain distinctions can be made in the meta-language and counterpart of B follows logically form A as well as the more broad considerations surrounding true statements, and theorems.

Lewis was the first systematically to try to meet such a need; however, he clearly neglected the use/mention distinction which led him to see a solution where he had only provided for confusion. The strength of Lewis' work is not to be denied. In subsequent discussion herein, we shall see how others have attempted to equip suitably the object language.

§3 Identity and Modal Logic

Subsequent to the appearance of Symbolic Logic by C. I. Lewis and C. H. Langford which presented a study of modal propositional logic, there was a publication in a short period of time of a number of papers which dealt with the problem of identity in modal contexts. The concern was for the substitution of 'identicals' in contexts governed by modal operators, it being claimed that truth was not preserved in such contexts. The rule of universal substitutability is based on the view that this identity means that the names of an identical object — co-referential terms — are substitutable in all contexts. However, problems arise, in e.g., belief contexts, as well as, and importantly, in modal contexts. Consider,

A: Walter Scott is identical with the author of Waverley. Necessarily, Walter Scott is identical with Walter Scott.

∴ Necessarily, Walter Scott is identical with the author of Waverley.

By some, notably Quine, the conclusion is claimed as false since, it is argued, it is plainly a contingent fact that Scott wrote Waverley. With another example this becomes more clear: is it a necessary truth that Shakespeare wrote Hamlet, or is it to be taken as contingent on the evidence that Bacon may have been the author. (The counter-argument, of course, would have to be that if Shakespeare did write

Hamlet, or Scott, Waverley, then the identity of the name and the descriptive phrase is affirmed as necessary). Quine considers this difficulty to be fatal to modal logic since he is obviously construing identity as involving substitutivity salva veritate, the non-direct referential character of definite descriptions being the stumbling block.

In order to solve problems of this type, we have to be very clear about what it means to use the term "identical" in a sentences. If in the case of a and b, either we have two things, or we have only one. That is, if a and b are identical, then there is only one thing; if they are two things, then they are not identical. What else can we mean by identity? Clearly, we do not want grades of identity, because identity turns out in the long run to be self-identity. Of course, identity statements can be used to convey new information as, for example Einsteins's "$E=MC^2$" or the (errroneous, in fact) claim that Mt. Everest= Gaurishankar. However, in such cases, the different names used are shown to name the same individual. Even with Scott and the author of Waverley, co-referentiality is one part of the problem; the conveyance of information is the other — otherwise, why would one be inclined to conjure up some nameless referent to be (1) not identical with Scott; (2) identical with the author of Waverley?

If we follow the above through a bit further and argue that identity statements are not really co-referentialaffirming statements, but express a contingency, then the most that we can mean is that it is possible that Scott is identical with the author of Waverley. With this, the conclusion (that necessarily, Scott is identical with the author of Waverley) becomes the affirmation of 'Scott is either identical with, or not identical with, the author of Waverley'; i.e., that it is necessary that it is possible that Scott is the author of Waverley, or is not the author of Waverley.

We do not wish to deny that some name identities are not necessary. Consider the example of Hesperus and Phosphorus (two "names" for the planet Venus). It is an astronomical discovery that the two names were found to refer to the same individual, viz. Venus; accordingly, we do not want to say that there are two individuals, each named by one of the two names for Venus — after all the name 'Venus' does refer to each of the individuals named above. If they named two individuals, then, of course, they could not name Venus, and if they name one individual, then they are just different names for the same thing. In any case, we are trying to avoid the case of saying that two names can be co-referring and yet refer to different things.

An argument which can be invoked here, but which has a dubious philosophical status, is based on showing that if identity statements are necessary, then their denials are contradictory. This, though, forces the modal logician to take the position which requires acceptance of some type of essentialism. In an ontological sense, this comes out as: being the author of _Waverley_ is essential to Scott. However, as Linsky notes [_Reference_, 9] quantified modal logic is committed only to the meaningfulness of essentialism, not to its truth, since it is in terms of that symbolism that one frames statements concerning essential (and accidental) property predication. Until, though, we are prepared to accept the truth of essentialism in the sense that "Necessary and contingent properties _do_ belong to objects irrespective of their modes of specification" [_Ibid._], this will remain an area of academic interest only.

In the following, we turn to a further consideration of some of the problems raised in this introduction, with consideration of Barcan Marcus, Quine and Smullyan.

§4 R. Barcan Marcus[2]

Barcan Marcus supports modal propositional logic and its quantified extension as being efficacious in understanding such intensional contexts as belief contexts, and the alethic modal ones in particular. Accordingly, she argues, we must be prepared to allow "degrees of extensionality" as she calls it (actually, it is varying degrees of substitutability) to account for the various equivalence relations

2. _Vide_ Marcus in the Bibliography.

by which terms can be related. This stratified extensionality means that, in addition to the identity relation, there are many others: similarity, congruence, and indiscernibility. From this, modal logic comes out as a fullyfledged intensional logic, with the "normal" and "absolute" extensional relation of identity being simply the strongest intensioanl equivalence relation.

Identity means <u>same object</u> -- i.e. that there is one thing and not two. For Leibniz, it meant having <u>all</u> properties in common. There is no property possessed by one and not by the other -- they are indiscernible; i.e.:

$(x)(y)[(x=y \ \& \ Fx) \supset Fy]$.

This holds, though, only in purely extensional contexts. However, as Linsky notes [<u>Reference</u>, Introduction], what is indiscernible in non-modal logic, becomes discernible with the introduction of modalities, since, then what becomes important is intensional identity, not extensional identity, as in the case of 9 and in the number of planets being identical. Barcan Marcus, though, opts for the following as being definitive:

2. x and y are indiscernible $=$ df $(\varphi)[\varphi x \equiv \varphi y]$ (in non-modal systems)

or $=$ df $(\varphi)[\varphi x \equiv \varphi y]$ (in modal systems)

Again, this is a nice way of saying that x and y are Leibniz-wise indiscernible since what applies to one applies equally to the other, though in this case, the claim is made stronger than in Leibniz' Law by using the bi-conditional.

Moreover, for her to claim 2 as adequate requires some justification. In effect, there is a choice to make concerning how we are to decide the criterion of adequacy for the calling of certain equivalence relation, the identity relation. The choice is between (a) rejecting Leibniz' Law as defining identity in intensional logic, and (b) rejecting intensional logic and its attendant contexts, and preserving Leibniz' Law. Obviously, Barcan Marcus had decided in favour of (a) because of her decision to allow varying degrees of substitutability, and hence, of intensionality. In effect, there is no one relation of identity, but many context dependent, and taking Carnapian intensional objects (concepts) as values.

However, we do not want to reject Leibniz' Law which means substitutability in all contexts, since we would be unable to assert any identities at all and end up with purely intensional names. As Quine says in Carnap's [M & N, 197] this would happen if the modalities were given free rein. We would lack even the ability to say:

The number of planets is a power of three.

i.e.

$(\exists n)$ [n is a natural number & the number of planets = $=3^n$]

since we have numbers (with an extension) as values for n. "The logical predicate 'is a natural number' . . . would have to give way to a logical predicate having the sense is a natural-number concept". [ibid.]

We would further have the difficulty of asserting identity between entities, which in an extensional logic might be at least substantive, but which in an intensional logic become pure concepts — e.g. the concept of being, say, human, vs. being a human (i.e. a member of the class of humans).

Barcan Marcus adds [Boston Studies, Discussion, 106] "that all terms may refer to objects, but that not all objects are things, where a thing is at least that about which it is appropriate to assert the identity relation." This denies application of the identity relation to things such as propositions. Consequently, identity holds between individuals only (and not individual concepts), the names of which having reference to some "concrete" thing (her "thing-reference"); however, on the level of predicate classes, attributes and propositions, there are other equivalence relations which are applicable (and which are weaker that identity between individuals). For example, 9 and the number of palnets may be related by the general equivalence relation of equality; but equality is not identity. Since 'the number of planets' is not a name, it can be interpreted by default as a predicate or a description [Ibid., 107].

This problem over the values of variables in the identity relation is rejected in the more general discussion surrounding the interpretation of functional logic. The difficulty arises in giving a clear translation of:

$(\exists x)Fx$

In light of this approach to the modalities, ~~she~~ says [Boston Studies, Modalities and Intensional Logic, 89]:

> For if quantification has to do with things and if variables for attributes or classes can be quantified upon, then in accordance with [the common reading of existential quantification as 'there is/exists at least one/some thing/person which/who . . . '] they [variables] are things.

The solution, as far as she sees it, lies in realizing that things other than individuals can be quantified (i.e., classes, attributes, etc.), and therefore, not to take the 'there exists' in a too empirical sense.

With Quine, we have the definitive objections to any quantified modal logic on the grounds that no sense can be made of identity in these so-called intensional contexts of Barcan Marcus. To this we now turn.

§5 W. V. Quine[3]

Quine's objections to Barcan Marcus in particular reflect his general reductionist standpoint, from which he

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3. Vide Quine in the Bibliography.

has shown that all constant singular terms are eliminable in favour of general terms and bound variables (reflecting his ontological maxim: to be is to be the value of a (bound) variable). Accordingly, since singular terms can be removed, the charge of referential opacity, or non-interchangeability salva veritate, must be shown to go through in the general case. For example, [Linsky, 4] from:

1. $(\exists x)$[necessarily, x is odd]

with x uniquely specified by:

2. $(\exists y)[y \neq x = yy = y + y + y]$

we can get:

3. x is odd

and hence,

$\Box$ [x is odd]

but not,

4. there are x planets.

2 and 4 each uniquely specify the same object; hence, 1 is entailed by 2, but 4 does not entail 3, (the idea being that it is incoherent to say that there is an object which is necessarily odd). This type of argument, claims Quine, is even more insidious in that the unintended result is a plea for essentialism.

Further objections lie in Barcan Marcus' type-theoretic stratification of extensions which provides for various equivalence relations obtaining between things of different logical types. Each such equivalence relation being an assertion of a species of identity in the relevant context. However, these equivalence relations are not nearly equal to the strength of the identity relation between names. As we have noted, identity either involves an assertion of identity between two entities, or it is lacking in meaning. Consequently, between Hesperus and Phosphorus, we have (according to Barcan Marcus):

$$\Box[\text{Phosphorus} = \text{Hesperus}]$$

Furthermore, the invocation of these degrees of substitutability reflects the circumstances that a logic can only be deciphered once one has determined which are predicates and which are individuals — i.e. one must become familiar with the ontology involved.

The Quinean approach is to recognize only two ontologically relevant types of expression — (bound) variables and predicates. From this standpoint, Quine is more or less required to accept the strict interpretation of identity as defined by Leibniz' Law, and hence, to accept the following:

$$(x)(y)[x = y \ \& \ \Box(x = x) \supset \Box(x = y)]$$

where s, y are variables of quantification, not names or description.

However, if there is to be an identity relation in modal logic, then it cannot be the 'usual' Leibnizean one, since as we noted above, what is indiscernible in non-modal logic can be made discernible in modal logic due to its intensional nature; hence the identity relation must change. Accordingly, we find Quine following in the steps of Carnap [M & N, esp. 173ff] in preferring some principle of congruence. The result of this is that the variables in quantified modal logic range over concepts (intensional objects) rather than concrete objects. With regard to Venus, Morning Star and Evening Star, we discover not one entity, but three. That is, [q.v. Quine, Problem, 271/2] (with 'C' being the relation of congruence) we have:

1. Morning Star C Evening Star & □[Morning Star C Morning Star]

Next, in order to use congruence, we must admit that existential quantification holds when there is some substitutable constant which would make the statement true; i.e.

2. (∃x)[x C Evening Star & □[x C Morning Star]] (Replacing 'Morning Star' in 1. by 'x').

As an alternate to 1, we can say:

3. Evening Star C Evening Star & −□[Evening Star C Morning Star]

on the basis that it is not necessary that the two names of Venus actually name the same individual -- it is an astronomical fact that they do. From 3., we have:

4. (∃x)[x C Evening Star & ~□[x C Morning Star].

The conclusion from 2 and 4 to this, then, is that there are two objects which satisfy the two incompatible operands.

(As Quine notes, if we introduce 'Venus', we would have three such objects.) The result is that material objects are eliminated, and replaced by intensional objects: 'the concept of being the Morning Star', 'the concept of being the Evening Star'. Congruence seems to be either meaningless, or such a weak relation between terms as to assert, if anything, conceptual divergence. The final result is that when the relation of congruence obtains between two terms, they are related by two subspecies of the relation. First, they can be said to be materially identical; by which is meant simply that they involve the same object -- in this sense Leibniz' Law suffers no undermining since what can be said of one can be said of the other. Second, they can be said to be contingently co-conceptual in so far as they do not make mutually exclusive statements about the same object. The contingency arises as the recognition that conceptual identity arises only in cases of self-identity, but when different words are used to make the same thing, differences arise from the different meanings the words have -- "many" and "few" can be the count adjective for the same number of objects as, e.g. the number of planets, yet one would hardly want to say they are strictly identical; Leibniz' Law fails in this regard then, because of the falsity of speaking of conceptual identity.

The unfortuante result of this is that we are no longer able to deal with concrete individuals -- the machinery for talking about them has been replaced by that which is concerned solely with intensional objects. With Quine we go the other way and have the elimination of concepts.

Quine notes that since we are talking about modalities in a propositional (or a functional) logic, certain constructions are possible which require making a distinction between necessary and contingent attributes — i.e., a plea for essentialism (hopefully, Aristotelean). For example, it is a mathematical requirement that 9 necessarily exceeds 7; however, the number of planets contingently exceeds 7, even though '9 = the number of planets'. To take the identity relation too seriously would be to claim that '9 $\neq$ the number of planets' is a contradiction, which it is not, so the solution is to begin equivocating about what we mean by predication, i.e., that what can be said of one cannot, without qualification, be said of the other. When we say that the number of planets is nine, we are expressing <u>at most</u> a contingent fact of astronomy, from this it follows that there could have been ten planets (the asteroid belt may be a broken up planet). Two statements follow:

1. $\Box[9 > 7]$

and

2. $\Diamond[9 \neq \text{the number of planets}]$

which means that

1'. the attribute of exceeding 7 is a necessary attribute of 9

2'. the attribute of 9 that it numbers the planets is an accidental attribute.

This distinction, reasons Quine, is enough to require of the modal logician the acceptance, in principle, of the meaningfulness of essentialism.

The distinction, though, seems to say more. The first is a mathematical truth, and it would be hoped that mathematical truths were not contingently true, but necessarily true. However, the second is a statement of empirical fact and such statements can be denied without eliciting undesirable consequences. It is the old analytic/synthetic distinction all over again, but cast in a new role, this type in terms of essentialism. If, as Quine argues, modal logic is committed to essentialism then it is similarly committed to the analytic/synthetic distinction, and hence, the proponent of modal logic must advance arguments to try to clarify the distinction although it is too much to ask for any criterion for determining synonymy. In addition to this, the modal logician has the task of making a distinction between accidental and essential attributes, since he <u>is</u> committed to its meaningfulness.

Quine's claims appeared to have some plausibility; however, in the following discussion we shall see how Smullyan tries to meet Quine's objections.

§6 A. F. Smullyan[4]

The central problem which Quine has raised with such a devastating effect concerned itself with the identity relation and the interchangeability of identicals. This substitutivity, claimed Quine, broke down in intensio l contexts and especially in cases wherein the identity relation involved a proper name and a descriptive phrase. Smullyan has investigated this question and presented what could be the definitive analysis of the question.

Smullyan, in "Modality and Description" investigates the controversy concerning substitution into modal contexts as it is affected by identity and definite descriptions. With Smullyan, we now consider the formal analogue of the argument raised above about Scott and his being the author of *Waverley*.

Let 'W' represent '① is the author of *Waverley*' and 's' represent 'Scott'; we then have:

A. $s = (\iota x)Wx$

$[s = s]$

$[s = (\iota x)Wx]$

the conclusion stating that it is necessary that Scott be the author of *Waverley*, and which is presumably false since it is quite conceivable that someone else could have written *Waverley*. It is at this point that Quine, as noted, levels

4. *Vide* Smullyan in the Bibliography.

hsi criticism of modal logic on the grounds of referential opacity (or non-interchangeability in all contexts, <u>salva veritate</u>). Quine's conclusion from this type of argunent was that individual variables in modal logic range over intensional objects, not concrete individuals. What is involved are concepts, which clearly differ according to the linguistic mode of presentation whether it is by means of proper names, or definite descriptions.

Implicit in Russell's definite descriptions is an ambiguity regarding scope, which arises in the primitive form of argument A, in regard to the placing of the modal operator. At a more elementary level let's consider this ambiguity with regard to negation and the resulting consequences. Consider the first premise of argument A:

$$S = (\imath x)(Wx)$$

this statement can be negated in two non-equivalent ways:

n1. $S \neq (\imath x)(Wx)$,

and

n2. $-[S = (\imath x)(Wx)]$.

From these we derive the following expansions into primitive notation:

n1'. $S \neq (\imath x)(Wx) = df \quad (\exists b)(Wx \equiv_x x = b \;\&\; s \neq b)$

which reads that there exists a <u>b</u>, identical to x, and satisfying Wx, but which is notidentical to Scott. On the other hand, the second one becomes

n2'. $-[(\exists b)(Wx \equiv_x x = b \;\&\; s = b)]$

and says that no b exists which meets the necessary requirements for satisfying Wx and being Scott.

Smullyan pays close attention to this distinction and offers a definitive interpretaion of a valid conclusion from the two premises in A. By construing the description as taking small scope we got the false conclusion, i.e.,

$$\text{c1.}\quad \Box[[(\iota x)(Wx)][s = (\iota x)(Wx)] =_{df} \Box[(\exists x)(Wy \equiv_y y = x \;\&\; s = x)],$$

however, the modal operator, like the negation sign in n1' and n2', can be in one of two places. Accordingly, by taking large scope we get:

$$\text{c2.}\quad [(\iota x)(Wx)]\Box[s = (\iota x)(Wx)] =_{df} (\exists x)[Wy \equiv_y y = x \;\&\; \Box(s = x)]$$

And this may be taken as true. As smullyan notes in order to obtain the conclusion c1, one of the premises must be strengthened, i.e., we need

$$\Box[s = (\iota x)(Wx)]$$

which, when conjoined with $\Box[s = s]$ readily gives the conclusion c1.

The above is obtained without the loss of Leibniz' Law (the demise of which surely results in confusion); the second premise can be paraphrased as:

$$\text{1.}\quad (\exists x)(Wy \equiv_y x = y \;\&\; s = x).$$

When conjoined with the first premise yields

$$\text{2.}\quad (\exists x)(Wy \equiv_y x = y \;\&\; s = x \;\&\; \Box[s = s])$$

This statement, reduces to (via Leibniz' Law):

3. $(\exists x)(Wy \equiv_y x = y \;\&\; \Box[x = x])$

i.e.,

4. $[(\imath x)(Wx)]\Box[s = (\imath x)(Wx)]$;

hence the above.

Smullyan appears to have met Quine's objections to modal logic, by showing how the paradox results from scope ambiguity. The result is that concrete individuals are retained, because Leibniz' Law stays, but so do modal operators, because of the resolving of the above scope-ambiguity. Smullyan's tactics neatly handle this standard case, but seem not to be so easily applied to any necessity of identity between Phosphorus and Hesperus, particularly because there is no paradox resolvable in primitive notation regarding the locating of the scope-symbol. If "Hesperus = Phosphorus", is it necessarily the case? If we stick strictly to Smullyan's use of Leibniz' Law then there should be no difficulty -- identity comes down to there being no "mentionable difference" (indiscernibility), and our faith in ontology restored by there not being two objects a and b such that a = b.

Moreover, statements such as '$s = (\imath x)(Wx)$' become, in primitive notation, contingent existence statements -- i.e. non-identity statements. The necessity of the identity statement stands, but given the individual which meets certain existential requirements laid down by the expansion of the description.

§7 Conclusion

From the preceding, we see that the initial argument against modal logic, namely Quine's objections, fail when we consider Smullyan's analysis. The idenity relation is preserved in its Leibnizean beauty, and then we do not have the loss of concrete individuals for intensional objects, of which no meaningful identity relation can be asserted.

CHAPTER II

THE SEMANTICS OF MODALITY

§1 Introduction

In the preceding chapter, the concern was to present an historical-dialectical analysis of the foundations of contemporary modal logic. Quine's objections have a firm basis, but can be met by some shifting of the very grounds upon which modal logic is founded (the purely systematic problem of substitution and Leibniz' law is decisive); Smullyan has shown that we can have both modal logic and Leibniz. Subsequent to the works referred to in Chapter I, comes the work of Saul Kripke who based his analyses of modal logic (particularly the Lewis systems) on the construction of a suitable semantic framework, the <u>model</u>. In this chapter, the approach is first to become clear about what a model is, second, to explicate examples of Kripke's semantic systems, and finally, to construct a modal semantic which reflects certain preconceived notions of what a modal semantic should involve. The system which results is S4.3, (in the S4-S5 spectrum) S4 being too weak and S5 being (to put it in unscholarly fashion) absurd.

A consequence of the modal semantic constructed herein is that, unlike Kripke, modalized truth is not the goal, but simply an explication of "true". That there can be, to use Russell's words, ". . . no one fundamental logical notion of necessity, not consequently of possibility. [and] . . . no such comparative and superlative of truth as is implied by the notions of contingency and necessity" [Necessity & Possibility] results from the semantic. The modalities do not express a type of truth but are, instead, ways of talking about statements in the language. That is, on the one hand, to say a statement S is true in language L, means, in a limited Tarskian sense, that S is satisfied by every sequence of the domain of L; on the other hand, that S is necessarily true in L means that L is structured with respect to the elements of its domain in such a way that S is true and is unfalsifiable (after all, a necessary truth, in the model theoretic sense, has no counter-model).

§2 Concept of a Model

The concept of a model is related to the concepts of satisfiability and validity. If we consider some arbitrary wff, A, a model of this wff is any valuation V such that V(A) = T(rue). That is, if A is the formula

$(\exists x)(Fx \,\&\, \sim Gx)$,

then any value assignment which renders A true is a model of that mormula. A valuation function performs this task by being a function from formulas to truth values. Included within a valuation function is an interpretation function which is a mapping from predicate letters onto classes and constants to individuals. The model is fully specified with reference to the domain of individuals and the valuation function which assigns predicates an expension from among the members of the domain. From this, two definitions follow. <u>A</u> is said to be <u>satisfiable</u> if it comes out true under at least one value assignment, and <u>valid</u> if it comes out true under all value assignments.

We can extend the above to in clude consideration of varying domains of individuals. In this case, <u>A</u> is satisfiable in a domain $\mathcal{D}$ if $\mathcal{D}$ is the universe of discourse of model of <u>A</u>, and <u>A</u> is valid in a domain $\mathcal{D}$ if every interpretation of <u>A</u> in $\mathcal{D}$ is a model of <u>A</u>. In all cases, $\mathcal{D}$ is a non-empty.

For example:

$$\sim(\exists x)Gxx \; \& \; (x)(\exists y)Gyx \; \& \; (x)(y)(z)[Gxy \; \& \; Gyz \supset Gxz]$$

is satisfiable in the domain of positive integers, where 'G' means '①>②' (but if $\mathcal{D}$ is finite, obviously it is not satisfiable since there would be a great est [or least] member).

The basis of these models is a "possible world". Each possible world can be said to be "inhabited" by the set of individuals which is the domain of individuals which exist in that world, and each world is described by a set of propositions. For example some wff, (x)Fx, say, is true in some possible world if "F" is true of every individual a in the possible world.

Obtaining among the members of $\mathcal{D}$ will be various relations $R_1, \ldots, R_n$ represented by the set of predicates $\{P_1, \ldots, P_n\}$. A statement will be true, or false, according as the individual involved is, or is not, a member of the set which is the extension of the involved predicate. For example, let R_1 be the monadic property of being red, represented by the predicate P_1: ① is red; then, letting 'my copy of Little Red Book' refer to some member of x of $\mathcal{D}$, the sentence "my copy of Little Red Book is red" is true if x is a member of the set of all red things represented by R_1, i.e., if $x \in R_1$. Polyadic predicates are treated similarly, except that we say that the ordered n-tuple $\langle x_1, \ldots, x_n \rangle$ is in the extension of the appropriate relation, i.e. $\langle x_1, \ldots, x_n \rangle \in R_n$.

In addition to the above description of a model we need to be able to get at all the true sentences which the model represents. In that case, we need what has alternately been called a "possible world"; that is, a world which supplies us the facts which the model purports to model. The test of the model is its consistency with the facts of the world (and, of course, not the other way around). In general, this "world" is a set of propositions describing a world, or some part of it. Finally, we need a valuation, or interpretation, function on the set of predicates onto the domain.

A model, then, becomes an ordered triple $\langle \mathcal{D}, f, \mathcal{W} \rangle$, where $\mathcal{D}$ is a non-empty set of individuals, f is the valuation function assigning to each predicate P a relation R among $\mathcal{D}$, and $\mathcal{W}$ is a non-empty set of propositions.

Each sentence in this model is truth-functionally determinable. The following specifies the conditions when a given formula, <u>A</u> is true in a model M. We abbreviate thus;

$M \models A =$ df A is satisfied in M; i.e. $f(A) = T$

for $M = \langle \mathcal{D}, f, \mathcal{W} \rangle$, we have:

1. $M \models Pa_1 \ldots a_n$ iff $\langle f(a_1), f(a_2), \ldots, f(a_n) \rangle \in f(P)$ in $\mathcal{W}$.
2. $M \models \sim A$ iff not $[M \models A]$
3. $M \models [A \& B]$ iff $M \models A$ and $M \models B$
4. $M \models [A \vee B]$ iff either $M \models A$ or $M \models B$ or both
5. $M \models (x)A$ iff $M \models (a/x)A$ for every constant $a \in \mathcal{D}$.

Other formulations follow. A. Robinson construes a model structure M as a set-theoretic construction consisting of [Klibansky, <u>Philosophy</u> 61-73]:

(a) a set of individuals,

(b) a quantity of relations, $R_1, \ldots, R_n$

(c) a function, φ , from $\mathcal{D}$ into $\mathcal{D}$.

From these three, we find that an interpretation of some sentences x in M "presupposes a correspondence which assigns to each relational symbol occurring in x a relation of M and to each function symbol of x a function of M" [Klibansky, Philosophy 64]. The assumption is that for every individual, $x \in \mathcal{D}$, there is some corresponding constant a in the language (which the model models) which denotes x. We have a correspondence between the set of terms, T, built up out of the individual constants which denote the members of $\mathcal{D}$ by means of function symbols which denote functions in M. That is, letting '$\rightarrow_D$' represent the correspondence between the abovementioned terms and $\mathcal{D}$, we have, for example,

$$\text{if } a \in T \rightarrow_D x_1 \in \mathcal{D} \text{ and } b \in T \rightarrow_D x_2 \in \mathcal{D} \text{ and}$$
$$f \rightarrow_D \varphi, \text{ then } f(a,b) \rightarrow_D \varphi(x_1, x_2).$$

In this way we arrive at Robinson's characterization of truth conditions. For R an n-ary relation symbol which denotes the relation ρ in M, and $a_1, \ldots, a_n$ terms in T (as above) denoting $x_1, \ldots, x_n$ in $\mathcal{D}$, respectively,

$$M \models R(a_1 \ldots a_n) \text{ iff } \langle x_1, \ldots, x_n \rangle \in \rho$$

(and so on, again as above). [q.v. Bell & Slamson, Models, esp. 50-57; 72-79]

§3 Kripke's Semantics

In commenting on Kripke's semantics, Makinson notes [Meaningful, 332]:

> At the centre of Kripke's approach is a concept of a collection of all (possible) sets of all possible worlds, which is tiself rather unclear and in need of explanation, and which one may argue has only a tenuous and heuristic link with the formal details of modelling. In certain of Kripke's modellings (for proper sub-systems of S5) we are also faced with a relation of "accessibility" between possible worlds, whose nature and properties are even less clear.

Makinson protests justifiably. In the following we will present examples of Kripke's semantics and in so doing clarify the nature of the relation of accessibility. According to the way we construe this relation, the modal semantics will change, with the optimal choice being made manifest in our construction presented herein.

Kripke began by assuming in [Completeness] that whatever exists in any possible world exists in every possible world. That is, for $\mathcal{D}$ some non-empty set, $\mathcal{D}$ is the domain of individuals that exist in some possible world w, (which may be the actual world). What constitutes an individual, and hence, a member of $\mathcal{D}$, is left as indefinite, because it is claimed that one can easily imagine an individual having different properties and standing in different relations to the other members of the possible world.

Since we are dealing with a variety of possible worlds, or situations involving a variety of individuals, one must be able unambiguously to specify the reference of any arbitrary n-ary predicate, say ψ, not only in <u>any</u> possible world, but in <u>every</u> possible world. With this latter requirement, a proposition is necessary if and only if it is true in all possible worlds; that is, for formula A:

' A' is true in $\mathcal{W}$ iff A is true in every possible world. The truth of A in $\mathcal{W}$ depends on its truth as determined by reference to its extension in each possible world.

This semantics is developed more fully as follows. $\mathcal{D}$, a non-empty domain of individuals, is interpreted with reference to a system of possible worlds, and the ordered pair $\langle S, \mathcal{D} \rangle$, where S is the set of all possible worlds, and $\mathcal{D}$ as above. An interpretation I on this ordered pair is a function on all the ordered pairs $\langle v, \mathcal{W} \rangle$, where v is some variable, and $\mathcal{W} \in S$ such that :

(i) if v is an individual variable, $I(v, \mathcal{W}) \in \mathcal{D}$, and for any arbitrary $W_1, W_2, \in S$, $I(v, W_1) = I(v, W_2)$;

(ii) if v is an n-ary predicate letter, $I(v, \mathcal{W})$ is a set of n-tuples of $\mathcal{D}$; that is, I assigns an extension to v.

A wff <u>A</u> is satisfiable in $\langle S, \mathcal{D} \rangle$ if and only if it comes out true in some $W_n \in S$ under some I on $\langle S, \mathcal{D} \rangle$; a wff

A is valid in $\langle S, \mathscr{D} \rangle$ if and only if it comes out true in every $W_n \epsilon S$ under every I on $\langle S, \mathscr{D} \rangle$, and finally, A is universally valid if and only if it is valid in every non-empty domain.

We get in this model the following assignment of truth values [with 'T' representing 'true'; 'F' representing 'false']:

1. if A is a sentential variable, then $V(A) = I(A,W)$ under I on $\langle S, \mathscr{D} \rangle$;
2. if A is an atomic wff $\psi x_1 \ldots x_n$, then $V(A) = T$ under I on $\langle S, \mathscr{D} \rangle$, if the n-tuple $\langle I(x_1,W), \ldots, I(x_n,W) \rangle$ is a member of the set $I(\psi, W)$ of the n-tuples of individuals; otherwise $V(A) = F$.
3. if A is $\sim B$, then $V(A) = T(F)$ under I on $\langle S, \mathscr{D} \rangle$ if and only if $V(B) = F(T)$ under I on $\langle S, \mathscr{D} \rangle$;
4. if A is $B \vee C$, then $V(A) = T$ under I on $\langle S, \mathscr{D} \rangle$ if either $V(B) = T$ or $V(C) = T$ or $V(B) = T$ and $V(C) = T$ under I on $\langle S, \mathscr{D} \rangle$; otherwise, $V(A) = F$.
5. if A is $\Box B$, then $V(A) = T$ under I on $\langle S, \mathscr{D} \rangle$ if for every $W' \epsilon S$, $V(B,W') = T$ under I on $\langle S, \mathscr{D} \rangle$; otherwise, $V(A) = F$.
6. if A is $(\exists a)B$, then $V(A) = T$ under I on $\langle S, \mathscr{D} \rangle$ if for at least one interpretation I'

on $\langle S, \mathcal{D} \rangle$ which is a variant of I in the I' assigns different values to A, V(B) = T under I' on $\langle S, \mathcal{D} \rangle$; otherwise V(A) = F.

For example, for any wff A, A is valid iff $\Box A$ is valid, as are $\Box(A \supset B) \supset (\Box A \supset \Box B)$; and the so-called Barcan formula $(x)\Box Fx \supset \Box(x)Fx$, and its converse, giving $(x)\Box Fx \equiv \Box(x)Fx$. This tends to collapse any distinctions which one might like to make between ontological necessity and linguistic necessity; i.e. between necessity which attaches to individuals -- *a* is necessarily F -- and necessity which language attaches to propositions -- it is necessary that *a* is F.

In the preceding, Kripke made use of the assumption that what exists in any possible world exists in every possible world. This assumption, though, can be replaced by one that claims that the different possible worlds may have differing domains. Cpnsequently, there may be some predicate $\psi x_1 \ldots x_n$ with a (non-empty) extension in W_1, but not in W_2, though $\psi x_1 \ldots x_n$, nonetheless, may be true on W_2. This opens the door to expressions which have sense, but no reference, e.g., 'Pegasus'. We are not committed to super-domains; that is, the intensional object, dubbed 'Pegasus', is not a member of an auxiliary domain which we conjoin to the domain of individuals for W_1, producing the domain for W_2. There is only one domain of extensionally determined objects. Nevertheless, Pegasus <u>is</u>

a member of <u>this</u> universal domain.

The above model is modified by allowing the domains of different possible worlds to be different, thus making our system of worlds an ordered triple, $\langle S, \mathcal{D}, f\rangle$, where S is a non-empty set (of possible worlds), $\mathcal{D}$ is a non-empty set (of things that exist in one possible world), f a function from members of S to subsets of $\mathcal{D}$, such that $\mathcal{D}$ is the union of all sets f(W), $W \in S$; that is, $f(W_1) = D_1 \subseteq \mathcal{D}$; $f(W_2) = D_2 \subseteq \mathcal{D}$; and so on such that $D_1 \cap D_2 \cap \ldots = \mathcal{D}$.

Clause 6 is changed in the following way in order to accommodate the changes in the model. The changes reflect the model's relational element, f, which "picks out" a possible world, W, and assigns a member of f(W) to the variable of quantification, so that a predicate may have a different extension from world to world. I.e.:

6'. if A is $(\exists a)B$, then V(A,W) = T under I on $\langle S, \mathcal{D}, f\rangle$, if for at least one interpretation I' which is a variant of I and which assigns a member of f(W) to <u>a</u>, V(B,E) = T; otherwise, V(A,W) = F.

In this semantics, the Barcan formula is false, Pegasus is handled in the following way. Since the domains may differ, Pegasus may exist in some possible world (and not in every), but the predicate, '① is a winged horse', may be true in some possible world of which Pegasus is not a member of corresponding domain.

Instead of the above two systems one might replace the assumption that a proposition is necessarily true if it is true in all possible worlds with the assumption that a proposition is necessarily true if it is true in all possible worlds <u>relative to</u> some base world, or the actual world. [Kripke, <u>Considerations</u>] That is, the set of all possible worlds contains all and only those worlds "possible relative to" this base world. The assumption is that what exists in any possible world exists in every possible world relevant to a base world.

This model is an ordered relational triple, $\langle S, \mathcal{P}, R \rangle$, where $S, \mathcal{P}$ as above, R is reflexive relation on S, such that W_1RW_2 means that W_2 is <u>possible relative to</u> W_1, or that what is true in W_2 is possible in W_1. Every world is possible relative to itself because every true proposition is possible. By R being at least reflexive, we ensure that possible worlds are self-relevant.

To turn this model structure into a model, one modifies clause 5. The modification reflects the change from one world, to a system of worlds (possible worlds) related to each other by the relation R so that, given a base world -- the actual world -- the truth value of the same statement can be determined in some possible world. The change we make takes into account this diversity of worlds; consequently, $\Box A$ is true in W if and only if A is true in all possible worlds (truth-wise) related to W. I.e.:

5'. if A is $\Box$B, then V(A,W) = T under I on $\langle$S,$\mathcal{D}$,R$\rangle$ if for every possible world W' that is relevant to W (alternately: for every world W' that is possible relative to W; or, for every W'$\in$ S such that WRW', in each case W being the actual world, or some base world), V(B,W') = T under I on $\langle$S,$\mathcal{D}$,R$\rangle$; otherwise, B(A,W) = F.

If we construe the relation, R, as reflexive and transitive, then it represents Lewis' S4, and if R is an equivalence relation, then it represents S5. If the relation is reflexive and symmetrical then it represents the Brouwersche system. The axioms for these systems can be summarised as follows [following van Fraassen, Formal, 145]:

R0. If A is a theorem of the sentential calculus, $\vdash$A;

R1. If $\vdash$A and $\vdash$ A $\supset$ B, then $\vdash$ B;

R2. If $\vdash$A, then $\vdash\Box$A;

A1. $\vdash\Box$A $\supset$ A;

A2. $\vdash\Box$(A$\supset$B) $\supset$ ($\Box$A $\supset\Box$B);

A3. $\vdash$A $\supset\Box\Diamond$A;

A4. $\vdash\Box$A $\supset\Box\Box$A.

Each system has R0, R1, A1, A2; the Brouwesche system has in addition A3; S4 adds A4, but lacks A3; S5 adds both A3 and A4. Below, we shall return to this and relate the modal semantics developed to the above systems.

In addition to the above systems, others can easily be obtained. For example, one need only to restrict the domain $\mathscr{D}$ so that domains of the other different possible worlds can be mutually disjoint; i.e.:

$$\sim(\exists x)[D_m \subseteq \mathscr{D} \ \& \ D_n \subseteq \mathscr{D} \ \& \ [x \in D_m \ \& \ x \in D_n]].$$

The above modal semantics all seem to indicate a general lack of concern for the nature of the relation of relevance, or accessibility. It is trotted out almost with impudence as though it were intuitively clear. Mere meta-logical interest in consistence proofs seems for some to be sufficient justification for making whim-like changes in the relation, without trying to ensure that it serves to explain the ordinary language usage of modalized statements. It is central to this thesis that we must more fully understand this relation in its philosophically illuminating sense -- as explaining something rather than nothing.

The "possible worlds" metaphor appeals to our semantic imagination by allowing almost uninhibited generation of worlds into a logical cosmos; we are, of course, constrained at the meta-level by demands for completeness. Aside from the purely logical demands which, in terms of pure algebraic tinkering, will admit the legitimacy of such alternative models, there remains the basic question which demands of the modal logician that he provide some justification for his field of study and that what he is doing genuinely illuminates the problem. The author is not arguing that logical problems are not philosophical problems (whatever philosophy is, if anything more than scholarly parasitism), but just as K. Popper says, [Conjectures and Refutations, 72], "genuine philosophical problems are always rooted in urgent problems outside philosophy. . . ." and their solution demands a "sensitivity to problems".

In the following section, we try to be sensitive to the problem of truth by constructing a suitable semantic framework.

§4 The General Semantics of Modal Systems

In this discussion we will be using the notion related to Hintikka's "model set", which is just like a model except that it models a finite sub-portion of the actual world, rather than of the whole of creation — for Hintikka a set of model sets comprises a Kripkean model structure. The use of these model sets was initiated by Hintikka who viewed them as a "piece" of the actual world, thus requiring domains of individuals which would be subsets of the domain of individuals of the actual world. Each such "possible partial world" (as we shall call them) is to be understood as an extension, taken in isolation, of what part of the actual world would be like if such-and-such were to happen, or not to happen. Epistemic and causal modalities come creeping in here as qualifiers of logical modality. We do not engage in ontological importation or exportation; that is, we do not have an uninhibited rabbit-like breeding of all manner of possible or necessary objects. The individuals in any subdomain for some possible partial world are all members of the domain for the actual world -- there is no individual which is a member of a subdomain for some possible partial world and not a member of the domain for the actual world. Similarly for predicates, though not necessarily for predication, since some individuals may be in the extension of different predicates in different subdomains of different possible partial worlds (as in involved with genuine possibility, we can view imagination as making controlled category mistakes).

The semantics for modal logic to be developed requires that we begin with the notion of normal model structure. A model structure in this case is an ordered triple, $\langle A, \Lambda, R \rangle$, where A is "the actual world", or some base world, Λ is the set of all possible partial worlds and R is the relation of accessibility between A and members of Λ, the members of Λ are $\lambda_1, \lambda_2, \ldots$. The relation, R, is construed as reflexive, transitive and anti-symmetric, for reasons made clear below. We specify a function, f, taking as arguments members of Λ and therefore specifying the domain of individuals in the possible partial world. I.e.:

$$f(\lambda_1) = \Delta_1 \subseteq \mathcal{D} \quad ; \; f(\lambda_2) = \Delta_2 \subseteq \mathcal{D} \quad ; \; \&c.$$

We allow that the domains Δ_1, &c. may or may not overlap;

however, we do require that:

$$(x)[x \in \Delta_m \supset x \in \mathcal{D}] \text{ and}$$

$$(x)[x \in \mathcal{D} \supset (\exists\Delta)(x \in \Delta \ \& \ \Delta \subseteq \mathcal{D})].$$

To turn this into a model, we require a valuation function, V, such that $V(P^m_n, L)$ and $V(P^m_n, A)$ return truth values, where P ranges over m-adic predicate letters, A is "the actual world" and L ranges over the members of the set Λ (i.e. ranges over possible partial worlds).

A difficulty encountered in any modal semantics concerns the nature of the "actual world". We can do one of a number of things: (i) leave it undefined, but specified in the semantics; (ii) take the naïve approach and try rigidly to specify it -- in the sense of saying that there is <u>the</u> actual world as an objective, not perspective-relevant, entity; (iii) specify it in a relative way, which is our tack.

The problem with the actual world is quite complicated. On one hand we know what it is -- in a least a naïve empiricist sense -- yet what that means eludes formalization. The question centres on what "actuality" is. Leibniz' God knew which, of all the possible worlds, was the actual one. We mortals are not so blessed with this omniscience. If one of us mortals could know the identity of the actual world in comparison to any other possible alternative, they should be able to "determine the actual truth value of any given proposition... by simply taking the value of the propositon at that world" [Tichy, <u>What</u>, 91] whose actuality is known. The conclusion is that one cannot know which is the actual world.

Tichy's point bears important fruit for this discussion. He says that being "the actual world is something that this, that, or the other world may be: it is a status which worlds may enjoy" [ibid]. Hence, we cannot leave it undefined; there is something which has this status, but it cannot be determined which world has this status. The position that emerges is the final position above which is complete relativity. What the actual world is becomes determined by reference to some point of view. That is, "my world" is the actual world, and we get full play out of the token-reflexivity of "my".

This does not restrict knowledge to personal experience as is the case in solipsism; however, what is known by me does constitute my world and in a true sense I expand my horizons through learning. It does help answer the question of actuality in a weak way -- what is actual is what is actual for me. The objective world takes on a tinge of conventionalism when, on this model, it turns out to be a world which everyone can agree about -- or the world of which we have common perception.

We conclude in agreement with the above, that the closest we can come to the actual world is to consider it as a "base" world, grounded in some point of view, all such base worlds being equal contenders as "the actual world". All these base worlds appear to comprise the set of all possible worlds, and for some semanticists this is the case; however, the concern here is to narrow the members of the set of all possible worlds to what can be conceived as possible from any one point of view in which case we get the set of all possible partial worlds.

We allow for a certain amount of lee-way in one's consideration of actuality and possibility, without compromising any analysis.

The possible partial worlds we hinge our semantics on have their genesis in the actual world are not a "complete" world by being negation-complete, unlike the actual world. The partial worlds capture both epistemological and logical reasoning. First, epistemologically, then involve possibility which is always with regard to what we know, and as such, possibility is thus firmly rooted in knowledge and actuality -- and not the other way 'round, and hence we see the need for anti-symmetry in the accessibility relation. Second, logically, we capture the notion of "the actual world", A, as a base world thus:

$$A =df (\imath w)(p)[p \in w \equiv p]$$

where w is a world, and p is a proposition in w. A becomes in jargon "my world", or, for logical arguments, a "given" world.

A possible partial world contains at least one fewer individual than the base world, along with a suitably restricted set of propositions. Therefore, for arbitrary world,

λ_n, the domain of that world is a subset of the domain of the actual world.

The set of possible partial worlds, Λ, is the set of λ's:

$$\Lambda =\text{df} \ \{\lambda_1, \lambda_2, \ldots, \lambda_m, \ldots\}$$

where the arbitrary possible partial world is:

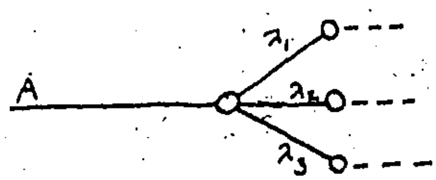
$$\lambda_n =\text{df} \ (\imath\, w_n)(\exists p_n)[p_n \in w_n \equiv p_n] \ \& \ Rw_n w$$

where p_n is a proposition, R is the relation of accessibility such that w_n is accessible from w, and $f(\lambda_n) \subset f(A)$. There is no λ_ℓ such that $f(\lambda_\ell) = f(A)$; this can never be the case because of the restricted view taken as to the nature of possible worlds as possible partial worlds. Finally, for $f(\lambda_m) = \Delta_m$ and $f(A) = \mathcal{D}$,

$$(x)[x \in \Delta_n \supset x \in \mathcal{D}].$$

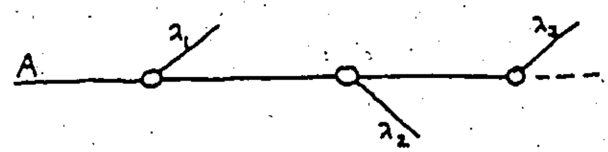
The next thing to do is to show how A and Λ are related. There are a number of possibilities.

(1) Λ actually represents "new" possible futures for A. Diagramatically:



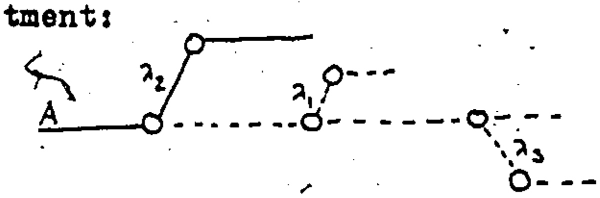
Whichever member of Λ is realized becomes the future course of A and consequently, our concern in this matter must go beyond λ_1, λ_2, λ_3 as logically possible futures, to treat them as causally possible ones as well.

(2) Λ may involve some determined aspect of A:



in which case λ_1, λ_2, λ_3 becomes simply logically possible alternatives, with no causal commitment as in (1).

(3) Here we take (2) above and introduce the notion of causal commitment:



in which case λ_1, λ_3 (as the dotted lines) are not causally possible alternatives (as the case in (1) — except λ_2 which is) since they are alternatives to λ_2 not A and hence, have logical status primarily, and epistemological status incidentally.

When we talk about some formula, F, being possibly true, what we mean is that there can logically exist some possible world in which F is true; and similarly for a necessary truth, we mean that there is no possible world in which F is false (note that F is then true in all possible and impossible worlds). This is the back-door way of getting at necessary truths, by asserting that their denials are logically (and causally and epistemologically) "unrealizable", since contradictions just are not. And since we (as self-proclaimed philosophers) ought to have a passion for reality, we really do not want to clutter up our universe with logically contradictory objects.

In the above diagrams we have represented A and members of Λ as being in some "spatial" relationship. The question that now arises concerns this relationship's logical status. It does not matter which diagram is taken as the one, except one ought to be clear about the commitments one makes in choosing any particular one. The question is this: What is the nature of the accessibility relation between A and members of Λ?

Let 'R' be the relation of accessibility between A and the members of Λ.

Let R①② =df ① is accessible from ② ,

then 'R' is characterised in the following way:

1. 'R' is reflexive in that RAA and $R\lambda_n\lambda_n$, $\lambda_n \in \Lambda$;
2. 'R' is transitive in that if $R\lambda_1 A$ and $R\lambda_2\lambda_1$ then $R\lambda_2 A$, $\lambda_1, \lambda_2 \in \Lambda$;
3. 'R' is anti-symmetric in that if $R\lambda_2\lambda_1$ and $R\lambda_1\lambda_2$ then $\lambda_1 = \lambda_2$, $\lambda_1, \lambda_2 \in \Lambda$.

These are the conditions that 'R' must meet and which we shall now justify. For now, what is meant by an accessibility relation is left undefined, but can be understood more clearly when 'R' has been more fully explained, thus demonstrating that our system is the more philosophically appealing and that our model is the correct representation of the philosophical requirements.

The requirement for reflexiveness reflects the claim that what is true need not be, (it is possible that . . .). But, we ask, are not true statements merely true and nothing else? Well, in this case, we appeal to something as yet uncriticised, i.e., our powers of conceiving. A true statement can be conceived under other circumstances as being false; with appeal to empiricism we say that what makes a statement factually true need not have occurred, in which case we get factual truth implying factual contingency.

To get closer to what we mean by reflexiveness, and consequently, the other two conditions for 'R', we must understand how the logical modalities are related. If a statement is impossible, then one would say that it is unrealizable, inconceivable, cannot be the case, etc. If a statement is not impossible, then it is called possible. We can, though, get at the logical modalities in a more factual way. Impossible statements can never be realized in a factual sense. Possible statements are always true in some possible world, though, factually they can be contingent, or can be unfalsifiable, in which latter case, they are termed necessary truths. Necessary truths are falsifiable only by impossible statements by which we then understand a necessary statement as one which can never be false, except statements about impossible objects which can never <u>be</u> (in any ontological sense). Now we can see why we want reflexiveness; factually contingent statements are possible; therefore, factually true statements are possible. The simple counterpart of factual truth is formal truth which in terms of its relation to factual truth yields modal truth. To deny reflexiveness seems pointless.

Transitivity follows from this discussion, though not as a requirement but as a desideratum; it also follows as a symmetrical result of accepting reflexiveness. Transitivity would lead us from a statement being possibly true to the possibility of that statement, i.e., possibility is implied by the possibility of possibility. Similarly, necessity implies the necessity of necessity; i.e., necessary statements are necessary. It should be noted that we can have reflexiveness without transitivity, but because of the way in which the logical modalities are to be construed herein, it is needed. It is a simple matter to reject transitivity, but more difficult to justify the rejection. In this case, the rejection of transitivity is entailed by the fact that there is some λ whose domain of individuals is not a subset of the domain of individuals of A.

Finally, the decision that 'R' be anti-symmetric is based purely on philosophical grounds.

Before we get to that, what does it mean to have an anti-symmetric relation? If transitivity holds, and symmetry is not considered, then transitivity is commutative -- the terms are ordered one way by the relation and in the other by the converse of the relation. Such a condition opens the way for symmetry. I.e., if we allow:

If $R\lambda_2\lambda_1$ and $R\lambda_3\lambda_1$ and $R\lambda_2\lambda_3$ and $R\lambda_1\lambda_2$ then $R\lambda_1\lambda_3$

then we allow:

If $R\lambda_2\lambda_1$ then $R\lambda_1\lambda_2$

given the additional requirement that λ_1 is identical to λ_3. However, the relation, 'R', does not say that if, for example, a statement 'p', is possible with respect to λ_1 and possibly possible with respect to λ_2, that p is possible with respect to λ_2 and possibly possible with respect to λ_1. Transitivity, as above asserted, allows the iteration of modalities only in one "direction" because, in each case, we have a theory of nested domains of possible partial worlds with the actual world, A, the "largest". By going to more restricted possible partial worlds we can say that we are taking increasingly more <u>microscopic</u> views of the world in terms of what propositions constitute these "smaller" possible partial worlds. Smaller worlds mean limiting the more extraneous effects of parts of the actual world by seeking the ultimate "<u>ceteris paribus</u> world" wherein only <u>primary</u> features -- logical, causal and epistemological -- are considered. Now, why do we want anti-symmetry? Mainly to avoid a logical absurdity. The condition for anti-symmetry asserts that <u>if</u> we have what appears to be a case of symmetry, we really have the identity of the relevant possible partial worlds -- they are the same world.

We can argue further for the imposition of anti-symmetry as follows. We introduce the notion of "seniority"[5]. Seniority is an aspect of the relation of accessibility in that it specifies, given two possible partial worlds and where one is accessible to the other (in this case R being reflexive, transitive, and anti-symmetric), that one is "smaller" than the other by containing at least one fewer individual, the smaller such partial world being a possible aspect of a more embracing state of affairs. The most senior world is the base world which, in an epistemological sense, is a known state of affairs. Less senior partial worlds represent possible states of affairs viewed in isolation from circumstances which do not change (hence the negation non-completeness). For example, consider the present world in which we have a particular occurrence, <u>viz</u>. the resignation of Richard Nixon. Then, there can exist a possible partial world which is Nixon not resigning; it is this state of affairs that we consider possible, not the whole world in which this takes place. We need not specify exactly how big this possible partial world is except to say that (1) it is possible with respect to the base world; (2) it involves at least one

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5. This term was suggested by Prof. Wilson during a discussion on the accessibility relation.

less individual than the base world and therefore, in this case (3) it is immediately subordinate (i.e., non-senior) to the base world. Therefore, we do not specify exactly how large the possible partial world is, but we know how small it is.

One can understand the relation between knowledge and possibility as that between knowledge and imagination. One is the converse of the other, such that, for an "ideal all-knower" one says that he has no imagination, holds no beliefs. As one becomes less than perfect, one knows less and believes more, hence invokes imagination more. Thus, possible states of affairs can arise only from a context of knowledge of incomplete knowledge. For this reason, anti-symmetry is the logical characterization of the epistemological generation of possible states of affairs as possible partial worlds.

We can see that the anti-symmetry of the accessibility relation makes sense not only logically, but usefully, epistemologically, by characterizing our ordinary language use of modal idioms -- logically because truth in a possible partial world is always with respect to the base world, or in some cases to the immediately senior possible partial world, and epistemologically because possibility is always in terms of what is known.

To revert to a metaphysical stance, 'R' is presented in this way so as to reflect our concern that we view possibles from the firm ground of actuality, this being made most explicit in regard to the condition of anti-symmetry in our system's accessibility relation. Cases of symmetry really mean world-identity, the reasons for justifying the imposition of symmetry in any other being more imaginary than apparent.

Concerning other competitive semantic systems, we rule out Brouwersche, S4 and S5. Brouwersche is reflexive and symmetrical, the latter condition being fatal because of the reasons above, which apply equally to S5, which has an equivalence accessibility relation. S4 is reflexive and transitive, making it too weak, it being unspecified in terms of symmetry. The condition for anti-symmetry makes explicit how pairs of worlds are to be connected. However, of course (and as an academic disclaimer or dogmatism), if it can be shown that symmetry can be introduced without doing violence to important philosophical underpinnings (which I seriously doubt), then there is no reason for 'R' not to be an equivalence relation, or for that matter, any justifiable alternative.

The semantical truth-conditions for the non-modal connectives and the quantifiers are formulated as follows, for the actual world A in the model $\langle A, \Lambda, R\rangle$:

Cl. $V(P^n x_1 . . . x_n, A) = T$ with respect to an assignment of $a_1, . . . , a_n$ to $x_1, . . . , x_n$ iff $\langle a_1, . . . , a_n\rangle \in V(P^n, A)$; otherwise, $V(P^n x_1 . . . x_n, A) = F$.

C2. $V(-p, A) = T$ with respect to an assignment $a_1, \ldots, a_n$ iff $V(p, A) = F$ with respect to the same assignment; otherwise it is false.

C3. $V(p \,\&\, q, A) = T$ with respect the assignments $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ iff both $V(p, A) = T$ and $V(q, A) = T$, with respect to the same assignments; otherwise it is false.

C4. $V((\exists x)p, A) = T$ with respect to an assignment $a_1, \ldots, a_n$ iff there is some $a \in f(A)$ such that $V(p, A) = T$ with respect to an assignment $a, a_1, \ldots, a_n$; otherwise it is false.

and for some arbitrary possible world $\lambda \in \Lambda$ in the model $\langle A, \Lambda, R\rangle$:

C5. $V(P^n x_1 \ldots x_n, \quad) = T$ with respect to an assignment $a_1, \ldots, a_n$ iff $\langle a_1, \ldots, a_n\rangle \in V(P^n, \lambda)$ and $R \quad A$ and $V(P^n, \lambda) \subseteq f(A)$; otherwise $V(P^n x_1 \ldots x_n, \lambda) = F$.

C6. $V(-p, \lambda) = T$ with respect to an assignment $a_1, \ldots, a_n$ iff $V(p, \lambda) = F$ and $R \lambda A$ and $f(\lambda) \subseteq f(A)$; otherwise it is false.

C7. $V(p \,\&\, q, \lambda) = T$ with respect to assignments $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ iff both $V(p, \lambda) = T$ and $V(q, \lambda) = T$ and $R \lambda A$ and $f(\lambda) \subseteq f(A)$; otherwise it is false.

C8. $V((\exists x)p, \lambda) = T$ with respect to an assignment

$a_1, \ldots, a_n$ iff there is some $\underset{\sim}{a} \in f(\lambda)$ such that $V(p, \lambda) = T$ with respect to an assignment $a, a_1, \ldots, a_n$ and $R\lambda A$ and $f(\lambda) \leq f(A)$; otherwise it is false.

The semantical truth-condition for the modal operator '$\Box$', has the following formulation:

C9. $V(\Box p, A) = T$ with respect to a given assignment iff, for every $\lambda \in \Lambda$, $V(p, \lambda) = T$ and $R\lambda A$ and $f(\lambda) \subseteq f(A)$, and $V(p, A) = T$; otherwise it is false.

One aspect of possibility alone is the notion of a counterfactual conditional; i.e., statements of the form:

if A were the case, B would be [i.e. $A \rightarrow B$].

It is important to note that statements of the form 'A B' are non-truth-functional, in a purely logical sense. If they were truth-functional, then they would be analysed as (perverse?) conditionals simply by putting the logically clear 'if-then' into the strictly subjunctive form. Conditionals do not force us to accept any connexion between antecedent and consequent, since "truth" is our only concern. On the other hand, counterfactuals commit us to the consequent, given the truth (actually the realization) of the antecedent. If the statement is: "if the moon were made of green cheese, then it would be raining", then if we accept that the antecedent is true, we are committed to rain.

§5 Counterfactuals

One further aspect of modal semantics must now be presented. Counterfactual conditionals seem to be saying something true about a state of affairs in some unactualized system -- i.e., in some possible world. Stalnaker in [Conditionals] has presented the definitive analysis of such statements by resorting to a possible worlds explanation in a Kripke-type semantic framework. As he says [*ibid.*, 102]:

> ..."How are we to decide whether or not we believe a conditional statement?"...[T]he problem is to make the transition from belief conditions to truth conditions; that is, to find a set of truth conditions for statements having conditional form which explains why we use the method we do use to evalute them. The concept of a *possible world* is just what we need to make this transition, since a possible world is the ontological analogue of a stock of hypothetical beliefs.

Stalnaker proceeds to analyse such conditionals along the following lines. For the conditional 'M → N', find circumstances for which the antecedent is true. The conditional, then, is true, just in case N is true in those same circumstances. That is, M → N is true in some possible world if N is true in the same possible world that makes M true. Stalnaker embellishes the typical semantics with a selection-function, s, which selects

> for each antecedent A, a particular possible world in which A is true. The *assertion* which the conditional makes, then, is that the consequent is true in the world selected. A conditional is true in the actual world when its consequent is true in the selected world. [*ibid.*, 103]

In addition, he adds the "absurd world" (called 'X') to the system to account for the cases in which M is impossible.

We can now formulate Stalnaker's account in our terms:

C10. '$M \rightarrow N$' is true in some possible partial world λ, [or the actual world A] if $(\exists\lambda)(s(M) = T$ & $s(N) = T)$, and $R\lambda A$.

The following conditions must all be met by s:

(i) for all antecedents, M, and worlds λ, [world A], M must be true in $s(M, \lambda)$, [$s(M,A)$], and $R\lambda A$;

(ii) for all antecedents, M, and worlds λ, [world A], $s(M, \lambda) = X$, [$s(M,A) = X$], only if there exists no λ' such that $R\lambda\lambda'$ and $R\lambda' A$ and in which A is true;

(iii) for all antecedents, M, and worlds λ, [world A], if M is true in λ, [in A], then $s(M, \lambda) = \lambda$, [$s(M,A) = A$];

(iv) for all antecedents M and M' and worlds λ, [world A], if M is true in $s(M', \lambda)$, [$s(M',A)$], and M' is true in $s(M, \lambda)$, [$s(M,A)$], then $s(M, \lambda) = s(M', \lambda)$, [$s(M,A) = s(M',A)$].

These conditions on $M \rightarrow N$ allow us to analyse such conditionals in terms of what truth conditions must obtain (in a possible partial world, or in the actual world) before we can assert the conditional, since, given the possible partial world which satisfies M, it must also satisfy N, for $M \rightarrow N$ to be meaningfully assertable of the actual world (in particular).

§6 General Considerations for System QS4.3*

Before we continue with the presentation of our modal semantics for system QS4.3*, a number of general points should be made.

First of all, the domains of the various possible partial worlds could be disjoint, i.e., for the ordered n-tuple of domains $\langle D_1, \ldots, D_m, \ldots, D_n \rangle$, no domains have any common member:

$$(x)(y)[x \in D_m \,\&\, y \in D_n . \supset . x \neq y].$$

In effect, then, there may be "actual" (relative to a given domain) individuals which are in principle lacking any counterparts (to use a term of D. Lewis) in, say, the actual world, A. Kripke-type semantics do not generally give A a preferred or particular status. The semantics here gives the actual world preferred status, embodied in the egocentric claim, "It is my world". However, the crucial difference is that once this actual world is declared, only partial worlds follow and are accessible, with domains suitable nested.

The advantage of this lies in the fact that, like logical atomism, there is only one set of individuals, the possible partial worlds involving various sub-domains of that domain which comprises the individuals in the actual world.

With the modal semantics developed herein, there is no quantification of domains, no disjoint domains, no overlapping domains (though all these are quite possible as special cases) since there is only one domain, f(A). Although the truth definition C9 exhibits the usual form for necessity there is an important and fundamental difference. Rather than this "planetary" quantification leading to the definitions of truth, we approach the problem from the other end. That is, a statement F is possibly true if there exists a λ such that (1) $\lambda \models F$ and (2) $R\lambda A$. The first condition specifies λ contain the necessary facts for the determination of F's truth, or falsity; the second condition, though, requires that λ be such that it meets the requirements of genuine possibility in accordance with what we said above concerning counterfactual conditionals -- we seek not only possibility and accessibility, but also compatibility.

With modal operators as quantifiers, we say that if some statement is possibly true, then there exists a λ which exemplifies the truth of F, and also, if some statement is necessarily true then every λ which exists exemplifies the truth of F.

We shall find ourselves concluding as Russell did that there is no superlative sense of true. It is not truth which we modalise; it is really the nature of the world. A necessarily true statement is no more than a true contingent statement; it makes a "meta" statement about the domain $\mathcal{D}$, by asserting that if a statement F has its truth conditions in the actual world, falsifying conditions are not also present;

similarly, a possibly true statement makes a meta-claim about the domain ϑ, by asserting that if F has its truth conditions in the actual world, falsifying conditions may be present. We say "may" because necessarily true statements imply possibly true statements. Of course, we are not saying that contradictions are true in the actual world; no logical contradiction is physically realizable.

The following characterises the modalities at the meta-level.

C11. '$\Diamond F$' is true =df for some $\lambda' \in \Lambda$, $\lambda' \models F$ and $R\lambda' A$.

C12. '$\Box F$' is true =df $A \models F$ and for every $\lambda \in \Lambda$, $\lambda \models F$ and there is no $\lambda' \in \Lambda$ such that $\lambda' \models -F$ and $R\lambda A$ and $-R\lambda' A$.

The first clause, C11, says simply that possible statements are true in some possible partial world. We require $-R\lambda' A$ to ensure that the claims of possibility expressed by F are related to the actual world as being a claim of possibility in the actual world. If A is necessarily true, it is true in the actual world, and also true in any λ you care to choose. In addition, it is <u>not</u> <u>false</u> in any other λ' you care to choose. The λ' may be made accessible just in case it is (1) an impossible world and (2) we decide to make it a λ (by fiat), reflecting a concern that λ' may not be relevant to A, for contextual reasons, for causal, and/or epistemological reasons; it may, though, be relevant in another context for better reasons.

In addition to that approach, we can say that we do not need the statement, "all bachelors are unmarried", as a necessary truth (apart from linguistic convention), but there are contexts in which we have no use for the statement.

Since the modal semantics developed above is concerned with domains, it seems obvious that statements which say things about the domain should go into the meta-language. After all, we could not really mean that there is such a thing as necessary truth, in itself; a necessary truth is a statement about (1) the kind of world we live in; (2) alterations in the actual world which can be made -- the possible partial worlds result; (3) the statements that are true of (1) and (2).

§7 QS4.3*

Our modal semantics is analogous to S4.3 [see Dummett and Lemmon, Logics]. Following is the axiomatization of the system with the addition of quantifiers -- i.e. QS4.3*. We present some interesting and important results, in a context of comparison with other systems which have purported to do the same thing. The logistic system is systemized as follows:

Rules of Inference:

R1. Rule of Substitution: from $\vdash A$ infer $\vdash A^{p}/_{B}$;

R2. Modus Ponens: If $\vdash A$ and $\vdash (A \supset B)$ then $\vdash B$;

R3. Rule of Necessitation: If $\vdash A$ then $\vdash \Box A$.

Axioms:

A1. $[p \vee p] \supset p$

A2. $q \supset [q \vee p]$

A3. $[p \vee q] \supset [q \vee p]$

A4. $[q \supset r] \supset [(p \vee q) \supset (p \vee r)]$

A5. $\Box p \supset p$

A6. $\Box [p \supset q] \supset [\Box p \supset \Box q]$

A7. $\Box p \supset \Box \Box p$

A8. $\Box [\Box p \supset \Box q] \vee \Box [\Box q \supset \Box p]$

Axiom A8 is the defining axiom of the system, and is equivalent to the set theoretic $[a \subseteq b] \vee [b \subseteq a]$.

Since S4.3* is a quantificational model by the introduction of the binary function, V, we, therefore, must add the following rule and axioms:

R4. Rule of Universal Generalization: If $\vdash A$ then $\vdash (a)A$.

For A and B any wff, we have:

A9. $(a)A \supset B$, where *a* does not occur free in A

A10. $(a)[A \supset B] \supset [(a)A \supset (a)B]$

A11. $(a)[A \supset B] \supset [A \supset (a)B]$, where *a* does not occur free in A.

In the next section we present some of the properties of the system.

§8 QS4.3* Properties and Comparisons

We prove that the Barcan formula and its converse are invalid in QS4.3*.

Consider first the Barcan formula: $(x)\Box Fx \supset \Box(x)Fx$. We have two partial worlds λ_1 and λ_2 such that $R\lambda_2\lambda_1$ (λ_1 may be A) and

$$f(\lambda_1) = \{a, b\} \quad a \neq b$$

$$f(\lambda_2) = \{a\}$$

therefore

(i) $V(Fx, \lambda_2) = T$

(ii) $V(Fx, \lambda_1) = T$

(iii) $V(Fy, \lambda_2) =$ undefined

(iv) $V(Fy, \lambda_1) = F$

given that

$$V(x) = a$$

and for any individual variable, y, other than x, $V(v) = b$ and $V(F) = \{\langle a, \lambda_2\rangle, \langle a, \lambda_1\rangle\}$

By (i) and (ii) $V(\Box Fx, \lambda_2) = T$

and therefore $V((x)\Box Fx, \lambda_2) = T$

For the consequent we have from (iv)

$$V((x)Fx, \lambda_1) = F$$

and given (iii) and that $R\lambda_2\lambda_1$ (and not $R\lambda_1\lambda_2$)

$$V(\Box(x)Fx, \lambda_2) = F$$

therefore

$$V((x)\Box Fx \supset \Box(x)Fx, \lambda_2) = F$$

and the Barcan formula is invalid in QS4.3*. Similarly, the converse, $\Box(x)Fx \supset (x)\Box Fx$, can be shown to be invalid.

The distinctive axiom for the system is A8, which we now show to be valid. To show that A8 is QS4.3*-valid, we need to show that A8 is true of all possible partial worlds accessible to any given base world — given the specified restrictions on the accessibility relation.

For the valuation function, V, and distinct possible partial worlds $\lambda_2, \lambda_3, \lambda_4, \lambda_5$, and the base world λ_1, (possibly A), we have:

$$V(\Box(\Box p \supset \Box q) \vee \Box(\Box q \supset \Box p), \lambda_1) = F$$

From this, for the first disjunct we have, given $R\lambda_2\lambda_1$:

$$V(\Box p \supset \Box q, \lambda_2) = F$$

i.e. $V(\Box p, \lambda_2) = T$

and $V(\Box q, \lambda_2) = F$

For the second disjunct, we have, given $R\lambda_3\lambda_1$:

$$V(\Box q \supset \Box p, \lambda_3) = F$$

i.e. $V(\Box q, \lambda_3) = T$

and $V(\Box p, \lambda_3) = F$

We still have to determine the truth values of both p and q in λ_4 and λ_5. Given that R is anti-symmetric, we have one of: $R\lambda_4\lambda_5$ or $R\lambda_5\lambda_4$. We now determine the truth values of p and q.

If $R\lambda_5\lambda_4$, then $R\lambda_5\lambda_2$

therefore $V(p, \lambda_5) = T$

which is inconsistent with

$$V(p, \lambda_4) = F, \text{ given } R\lambda_4\lambda_2.$$

If $R\lambda_4\lambda_5$, then $R\lambda_4\lambda_3$

therefore $V(q, \lambda_4) = T$

which is inconsistent with

$$V(q, \lambda_5) = F, \text{ given } R\lambda_5\lambda_3.$$

For either case of anti-symmetry, we have an inconsistency; therefore, the original assumption is false and A8 is valid in QS4.3*.

According to Zeman [Modal, 235ff] the following results are true of S4.3*.

1. If a formula A is valid in the model for S4.3*, then it is a theorem of S4.3*;
2. The model for S4.3* provides a decision procedure for S4.3*.

In comparison, Kripke's axiomatization of S5 results in the Barcan formula and the converse being invalid. We demonstrate as follows [after Hughes and Cresswell, Introduction, 181]:

For an S5 model we have two distinct possible worlds w_1 and w_2 such that:

$$f(w_1) = \{a,b\}\ ,\ a \neq b$$

$$f(w_2) = \{a\}$$

$$V(F) = \{\langle a,w_1\rangle\ ,\langle a,w_2\rangle\}$$

and that Rw_1w_2, Rw_2w_1, Rw_1w_1 and Rw_2w_2.

For the Barcan formula we have:

$$V(Fx,w_1) = T$$

and $V(Fx,w_2) = T$

therefore $V(\Box Fx,w_2) = T$

and $V((x)\Box Fx,w_2) = T$

but $V((x)Fx,w_1) = F$ since $\langle b,w_2\rangle \bar{\in} V(F)$

and therefore $V(\Box(x)Fx,w_2) = F$.

Finally $V((x)\Box Fx \supset \Box(x)Fx, w_2) = F$

i.e. the Barcan formula is invalid. In a similar way, the converse formula is proved invalid in S5, as well as S4.

One property of this system that we have insisted upon is that whenever $R\lambda_1 A$, $f(\lambda_1) \subseteq f(A)$, i.e. that every member of the domain of λ_1 is a member of A. In fact there is no individual which is a member of a possible partial world, λ_1, which is not also a member of A. It may, in these circumstances, be the case that the valuation function, V, may not be able to assign a value to a wff in a given possible partial world :

$V(F,\lambda_1)$ = T if the value assigned to any individual variable in F is in $f(\lambda_1)$ and

$V(F,\lambda_1)$ = F if it is but does not apply to F and

$V(F,\lambda_1)$ = undefined when the needed value for the individual variable is not in $f(\lambda_1)$.

Strictly speaking, the semantical truth conditions (given above pp 74-76) ought to take into account cases where the formula is simply not significant in some possible partial world, and hence be undefined -- not either true or false.

Formulas are either evaluated, as true or false, or left undefined, a recognition of their <u>non-significance</u> in some possible partial world. If it is not possible to verify or falsify a formula in some possible partial world, then the formula is neither true nor false. The absence of an extension is not a falsifying condition, but is merely recognition of the corresponding formula's insignificance.

CHAPTER III

CONCLUDING REMARKS

It should be conceded that a philosophical analysis of the nature of necessity and possibility is a pre-requisite, if not at least a concomitant of formalisation, by means of which one ought to construe the modalities within confines which owe their existence to a concept of reality. We must, of course, recognize that there are various types of necessity and possibility which are affected by contextual considerations. It is not felt that modality be taken in the narrow sense dictated by contexts, but only that there are contexts within which logical modality does not reflect what we may intend.

In the semantic constructions currently in vogue, the construction relies on a relation of "accessibility" or "alternativeness", which has been mentioned above. This relation is central to any modal semantics, as it is in terms of this relation that we have made sense of truth "in all/ some possible partial worlds". With D. Lewis we have the quite vague concept of counterparts which supposedly are "very much like" things in the actual world. This type of thinking can only lead to more problems, of a different sort, than were claimed as solved. For Lewis a possible world is a world very much like the actual one, whatever that means —

the identification problems alone seem formidably complex: for example consider the classical question of personal identity through time.

The counterpart relation refers to individuals which are different possible partial worlds which are themselves related by the relation of accessibility. It is notorious how vague semanticists become when talk of this relation crops up. There does, though, seem to be some agreement about its rough outline. Hughes and Cresswell in [*Introduction*] have an interesting discussion of this relation. They note that when one said some statement was necessarily (or was possibly) true one meant that the statement was true in all (in some) possible worlds; there are no possible worlds, though, only possible partial worlds. Hughes and Cresswell acknowledge this; they say that what would convey the idea better would be "conceivable or envisagable state of affairs" [*Introduction*, 75] which is quite close to our position. Pursuing this claim, a possible partial world becomes a segment of the actual world which differs *in conceivable ways* from the actual world. They then make an important qualification [*Ibid.*, 77]:

> . . . our ability [to conceive of a possible (partial) world] is at least partly governed by the kind of world we live in: the constitution of the human mind and the human body, the languages which exist or do not exist, and many other things, set certain limits to our powers of conceiving.

Wilson has shed some light on difficulites here. We can say that a possible partial world is limited by our linguistic apparatus for describing or conceiving of worlds [Color, 151]:

> Our usable grasp of the descriptive primitives of our language involves factual knowledge of the entities signified by those descriptive primitives.

The point can be put another way. In order for us to know whether a statement is true in some partial world, we must be able to connect our language to the objects in that partial world. Wilson's point is that there may be a possible partial world in which objects have a completely incomprehensible colour-scheme. Accordingly, a language user of this partial world cannot determine the truth value of statements about that world -- being unable to link up his colour words with the colours that he finds an object having. I.e. "[one's] grasp of the significance of [the] word 'green' requires [one] to have factual knowledge about the entity, green -- about its exemplifications." [Ibid.]

These provide some notion of what counts as a possible partial world and hence what meaning can be given to the accessibility relation. Anyway, as has been made abundantly clear, we really do not want to conceive of a whole possible world, just the parts of it that we know anything about; the logical modalities do say something about the nature of such a partial world; hence, the most we can hope for is an understanding of alternate states of affairs.

The human mind's ability to conceive does not function in a vacuum -- it requires input, in the way of a rich environment, (a reason for education?) which can provide "raw" material and set certain limits on the extent to which the mind can enjoy a free rein. This is also recognised by Hughes and Cresswell, who note [op. cit., 78] that in order to conceive of a certain state of affairs one must have some insight into what it would be like to experience that state of affairs.

Within the logical framework, many possible partial worlds can be conceived, none of which need violate our logical presuppositions (i.e., as we understand logic to be -- particularly as explaining the relationships between objects in the world -- but bivalent logic won't help us, for example, in an analysis of the world of quantum mechanics). Accordingly, one can distinguish two types of conceiving; one is purely descriptive, the other involves a conceiver: i.e. [Ibid.]:

> The difference is that between knowing what a certain state of affairs would be like [the first] and knowing what it would be like to live in that state of affairs.

The requirement of involving the agent puts constraints on the modal system which logicians would not be prepared to accept. However, we can relax the requirement, leave the agent out, and only deem it necessary that the state of ~~affairs~~ be expressible in language -- i.e., require coherent describability. It could be the case that nothing be known about the state of affairs, it only being possible to describe soem state of affairs without being assured even of its existence. This latter, though, can admit of difficulites; being purely descriptive, it may nonetheless be unsound in some way -- for example: the golden mountain, the son of a barren woman.

There are a number of questions which must now be answered. As we have shown, the accessibility relation is an ordering of possible partial worlds, primarily in terms of containment of domains of individuals or of relation of seniority of worlds. The fact that there is only one domain of individuals has a number of philosophical consequences in our understanding of modal logic. If the relation is one of containment, in the logic, what can be said of the relation in a non-logical sense? We have characterised it by the term epistemic, which means that we must consider the logical relation as having consequences for epistemology and one's logical orientation to knowledge.

A statement of modality is merely a statement about the kind of world we live in, or the kind of world we could be living in; it is a statement, based on what is the case, which purports to embody knowledge of what might be the case. In the case of the modalities it is not sufficient to treat them as purely logical operators, because of their close connection to questions of essential and accidental properties. In effect, then, the modalities have a distinctive ontological nature not completely like that of non-modal logics. Treating the modalities in a purely logical sense, one allows that only contradictory objects be excluded from reality

(Meinong might not approve), and in some sense one can say that "anything is possible", but surely not logically possible? We must try to understand the accessibility relation as one which takes into account the various epistemological/ontological questions one can ask of modal statements: (1) How does one know that the possible statement could express the actual? (2) What kind of ontology do we need which is the grounding for anything except the logically contradictory (*Pace* Meinong), (3) How are we to interpret the modalities' *apparent* commitment to the meaningfulness of essentialism?

In terms of knowledge the difference between this view and any other is this: the other view holds that what we know constitutes a whole, and statements of possibility are extensions of our knowledge, (hence, the commitment to talk of possible worlds each of which can form the basis of the whole of knowledge -- this appears to be Kripke's position; our view holds that knowledge is a fixed whole (however, undefineable that may be) of which we can be said to possess a part. Knowledge is of what is actual; coming to know what is possible is the extension of the frontier of the known with the knowable -- the basis of all this being firmly rooted in reality. The system rejects the view that possible worlds are on a par with the actual world; there is one world, the actual world, of which we know a part and then there are the partial worlds. The coming-to-know is the extending of what we know, and the relation between this and the what-is known is expressed by the accessibility relation being defined as above, particularly, its being anti-symmetric, since we always move from the known to the unknown, or as Aristotle variously puts it, from the more familiar to the less. (This is, of course, to pass only but the most general comments on epistemology.)

The connection we have made above, then, is between the purely logical nature of the accessibility relation as "containment" of domains and its epistemological limitations dictated by the nature of modalised statements.

Under the above interpretation, the modalities take on a rather interesting interpretation. We do not need the modalities in ordinary discourse about things in the world. In the epistemological sense delineated above, the modalities concern certain aspects of knowledge, which are made explicit in modalised statements -- for example, necessary property predication. The modal semantics presented in Chapter II showed that the modalities, granted, of course, the interpretation of the accessibility relation as expounded in this chapter, logically are statements of the nature of the world, and epistemologically characteristic of the relation between the world, and our knowledge of it.

The question of possible worlds arises again: are there any? -- No, just possible partial worlds. But depending on how good one's imagination is, one could conjure all manner of possible worlds, each describable by a suitable set of statements embodying the modalities. In our modal semantics, we have given ontological priority to the actual world as epistemologically represented by knowledge, and logically by a set of propositions. The actual world is given priority by not allowing the accessibility relation to be symmetric. Kripke allows for a symmetrical relation (in his semantics for S5) with the result that anything is possible relative to anything else; that is, accessibility can proceed in either direction between any two possible worlds. Hence, for Kripke, we admit either that there are no possible worlds at all or that they are unordered. Each such possible world in the latter case being capable of priority of accessibility.

If on the other hand, we grant one world priority, in our case A, possible partial worlds are invoked and are partially ordered by the relation of containment, as well as the relation of epistemic accessibility. The members of possible partial worlds are the individuals we talk about in language, or express in non-modal quantification logic, of first order at least. This reflects our ordinary language reference to possibiles, which when added to metaphysical considerations introduces the notion of possible partial world as an extra-linguistic construct.

Finally, let us say that the system provides a holistic account of modal statements. The original position of of Lewis is abandoned as not being representative of the use of the modalities -- though it still remains a problem to characterise deducibility. Necessity of deduction seems to need a "link" between antecedent and consequent, a link which can be understood as an appeal to a meaning relation. But then do we really have necessity of deduction, or just an analytic consequent? In the latter case, it is just a property of the meanings of the statements in the language and does not seem to require some additional logical machinery (*viz.* modalities) to account for the deduction. Absence of such a link would seem to deny any necessity of deduction, and hence restrict deducibility to meaning-related terms -- i.e., to language-based terms. Quine's problem is certainly interesting and to Kripke, who utilizes a possible worlds framework, it also is appealing. The problem is "solved" by the notion of "transworld identity" in the modal semantics; however, Smullyan's analysis seems to show that a solution is available which does not resort to Kripke's semantic structures. In our system, with only one domain and a plethora of possible partial worlds whose domains are sub-domains of this domain, the problem is brought back down to earth. It is restricted to the solution of the problem of co-referentiality between proper names and definite descriptions (Scott and the author of *Waverley*) or two proper names (Morning Star and Evening Star).

Finally, the characterization of the modalities as in Chapter II allows for the elimination of this "pseudo-problem" of reference. It also provides a framework for recognising the importance of understanding essentialism and also the notion of potentiality.

Purged is the notion of possible worlds with weird and wonderful domains in favour of the more manageable domain $\mathcal{D}$ of A and possible partial worlds as reflecting a more satisfactory analysis of possibility.

The old problems remain, but the new ones are gone.

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