COARSE GEOMETRY AND HOMOLOGY THEORIES

COARSE GEOMETRY AND HOMOLOGY THEORIES

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Abstract

An exposition of several homology and cohomology theories is given. Particular emphasis is placed on coarse homology and coarse analogues of the Eilenberg-Steenrod axioms. Relations between coarse homology and end homology are considered, and an isomorphism between these two theories is proved under a certain contractibility condition on the underlying space.

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Introduction

Homology and cohomology theories are useful algebraic constructions which can provide topological invariants. Originally built to capture an intuitive idea of counting holes in a space, the basic ideas have since been axiomatized and generalized to capture a wider variety of notions. Over the past century, a large number of theories have been developed, many with the intention of generalizing duality results: relations between existing theories on certain classes of manifolds. Several others have been developed to obtain invariants with a particular flavor advantageous to an area of research. Many of these theories coincide for spaces satisfying sufficiently nice properties; their differences tend to lie in how they handle spaces with pathological features.

One aim of this thesis is to give an overview of several homology and cohomology theories which have been developed, indicating their important properties (analogues of the Eilenberg-Steenrod axioms) and their relationships with one another. This overview indicates several ways in which theories can be designed or modified to alter its focus on particular features of a space. For example, use of Čech and anti-Čech systems permit a focus on small-scale or large-scale topological features, and certain limit constructions present theories which can disregard topological behavior which is confined to compact sets.

A primary goal of this thesis is a more detailed exposition of the coarse homology theory for metric spaces than is found in the literature, featuring more thorough proofs of coarse analogues of the Eilenberg-Steenrod axioms. Coarse homology is a theory developed to fit nicely with coarse geometry, an area featuring the notions of quasi-isometry and coarse equivalence, and guided by an intuition suggesting that any fixed finite distance should eventually be considered irrelevant. As an example, the space consisting of two parallel lines in $\mathbb{R}^2$ separated by a fixed distance is coarsely equivalent to the space consisting of a single line in $\mathbb{R}^2$, and so both spaces should have the same coarse homology. Coarse geometry in general has applications to areas such as geometric group theory, through constructions like the Cayley graph and word metric of a group, some properties of which are invariant under quasi-isometry.

In addition to this exposition, I investigate the relation between coarse homology and end homology, an older theory which disregards local behavior in a different way. Both measure homology "at infinity" in some sense; for example, each theory assigns the same groups to a noncompact space X and $X - C$ where $C \subseteq X$ is compact.

Chapter 1 is a review of the definitions, constructions, and algebraic theorems which are frequently used throughout the thesis. This material is standard and many references are available. In particular, use was made of [**3**], [**5**], [**14**], and [**22**]. Chapters 2 and 3 discuss the Eilenberg-Steenrod axioms in generality and follow the construction of various homology and cohomology theories, noting which axioms and other particularly interesting properties they satisfy, as well as discussing their relations to one another. Greater attention is paid to the cohomology theories based on Alexander-Spanier chains in sections 3.2 and 3.3. Chapter 4 discusses the construction of several theories which are defined in terms of other theories, indicating a few of the methods for doing so: the algebraic dual or "naive dual" method of applying the Hom functor to a chain or cochain complex, taking direct limits over suitable subspaces, and by taking quotients of the chain groups of distinct theories.

Chapter 5 represents an introduction to coarse algebraic topology on proper metric spaces. It includes an account of locally finite homology, the basic properties of coarse maps, coarse homology and cohomology, and a section on the relation between coarse homology and asymptotic dimension. Section 5.3 is the exposition of coarse homology, including more detailed proofs and discussion of the coarse analogues of the Eilenberg-Steenrod axioms than are currently found in existing literature. The coarse Mayer-Vietoris theorem is slightly generalized to allow for certain decompositions of X into subsets A and B even when $A \cup B \neq X$ and $A \cap B = \emptyset$:

THEOREM A (Coarse Mayer-Vietoris). *Let A and B be subsets of the proper metric space X. If $A \cup B$ is coarsely equivalent to X by inclusion and if A and B coarsely intersect in X, then there is a long exact sequence*

$$\cdots \to HC_p(\mathcal{I}) \to HC_p(A) \oplus HC_p(B) \to HC_p(X) \to HC_{p-1}(\mathcal{I}) \to \cdots$$

for every coarse intersection $\mathcal{I}$ of A and B in X.

The property I have called coarsely intersecting involves the notion previously identified as the property required of $A \cap B$ in X to get the original Mayer-Vietoris sequence; here it has been extended to spaces which do not necessarily have a nonempty usual intersection. Despite this greater generality, example 5.3.13 shows that it is possible to fail to have a coarse intersection even when $A \cap B$ is nonempty.

A coarse analogue of the excision axiom is also stated and proved separately:

THEOREM B (Coarse Excision). *If $A, E \subseteq X$ are such that $E \subseteq A$ and for all $R > 0$, there is some $S > 0$ such that $D_R(E) - E \subseteq D_S(A - E)$, then the inclusion map $i : (X - E, A - E) \to (X, A)$ induces an isomorphism.*

Section 5.5 is an original exposition of the relation between end homology and coarse homology. It includes results on isomorphisms between the end and coarse theories and also discusses the relation between coarse homology and the theory obtained by mimicking the construction of coarse homology using end homology instead of locally finite homology. In particular, the following theorems are proved:

THEOREM C. *Let X be a regimented metric simplicial complex with bounded coarse geometry. Then $H_n^e(X) \cong HC_n(X)$ for $n > 1$. If in addition $\mathbb{R}^+$ coarsely embeds into X, then $H_0^e(X) \cong HC_0(X)$.*

THEOREM D. *Let X be a proper metric space. Then $HC_n(X) \cong HC_n^e(X)$ for $n > 1$. If in addition $\mathbb{R}^+$ coarsely embeds into X, then $HC_0(X) \cong HC_0^e(X)$.*

The property I have called being regimented is defined in 5.5.10. It is a relaxation of uniform contractibility, which is one of the properties required for an isomorphism between locally finite and coarse homology. Regimented spaces can wildly fail uniform contractibility, but satisfy a sort of "eventual" uniform contractibility of R-balls for each $R > 0$. Examples of such spaces are given.

Chapter 6 reviews Poincaré and Alexander duality, further relating many of the theories which appear in earlier chapters.

I would like to thank Professor Ian Hambleton, who served as my advisor for this thesis, providing guidance and encouragement along the way.

CHAPTER 1

Background Material

1.1. Chain Complexes and Chain Maps

The definitions of homology groups follow a common theme. As such, the following definitions and theorems are very useful. As noted in the introduction, this and the next few sections consist of standard material available in many sources, including **[3]**, **[5]**, **[14]**, and **[22]**.

DEFINITION 1.1.1. A **chain complex** is a collection of modules C_n over a ring R and homomorphisms $\partial_n \colon C_n \to C_{n-1}$ each indexed by the set of integers, such that for every sequence $C_{n+1} \overset{\partial_{n+1}}{\to} C_n \overset{\partial_n}{\to} C_{n-1}$, the composition $\partial_n \partial_{n+1}$ is the 0 homomorphism. That is, the following diagram commutes for each n.

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[rr, bend left, "0"] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

The elements of C_n are called **n-chains**. The homomorphisms are called **boundary maps**, elements in their images are called **boundaries**, and elements in their kernels are called **cycles**. A chain complex is said to be **bounded below** if $C_n = 0$ for all n less than some integer N. It is **bounded above** if there is an N such that C_n is trivial for all n greater than N. It is **bounded** if it is bounded both above and below.

Motivation for the choice of terminology is clear after studying simplicial homology, where boundaries and cycles will have nice geometric interpretations. The chain complexes encountered when discussing homology for topological spaces are usually bounded below, and often bounded above as well, so that we will typically be considering complexes of the form $0 \to C_n \to \cdots \to C_0 \to 0$.

The requirement that $\partial_n \partial_{n+1} = 0$ in a chain complex is equivalent to $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$. As such, there is a sequence of quotient modules associated to any chain complex.

DEFINITION 1.1.2. The **n-th homology module** $H_n(C)$ of the chain complex C is the quotient module $\ker \partial_n / \operatorname{im} \partial_{n+1}$.

The goal when defining a homology theory is routinely to construct a sequence of modules which holds some topological information about a space.

This is often done by constructing chain complexes whose chain groups consist of functions into or out of the space and taking homology modules as defined above. These modules are often our end goal, but in some cases they may be further manipulated. For example, some of the theories we discuss involve constructing chain complexes for each cover in a sequence of open covers of a space, and using the resulting homology modules as objects in a direct limit, letting the open covers vary. Careful choices in construction of the chain complex can result in the homology modules being invariant under certain maps on the space we start with. For example, some homology theories will be invariant under homotopy equivalence, while others may be invariant under proper homotopy equivalences. A recurring notion in discussing how maps between spaces lead to maps between homology modules is the notion of a chain map.

DEFINITION 1.1.3. A **chain map** between chain complexes C and D is a collection f of homomorphisms $f_n\colon C_n \to D_n$ such that each square of the form

$$\begin{tikzcd} C_n \arrow[r, "\partial_n^C"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "\partial_n^D"'] & D_{n-1} \end{tikzcd}$$

commutes. That is, $f_{n-1} \circ \partial_n^C = \partial_n^D \circ f_n$.

The commutativity required of chain maps ensures that they induce homomorphisms on the homology modules of the complexes. This and several relevant properties of the induced maps follow easily from the definition above and are recorded below.

PROPOSITION 1.1.4. *Let $f\colon (C, \partial^C) \to (D, \partial^D)$ and $g\colon (D, \partial^D) \to (E, \partial^E)$ be chain maps.*

(a) *Each f_n satisfies $f_n(\ker \partial_n^C) \subseteq \ker \partial_n^D$ and $f_n(\operatorname{im} \partial_{n+1}^C) \subseteq \operatorname{im} \partial_{n+1}^D$. That is, cycles are sent to cycles and boundaries are sent to boundaries.*

(b) *Each f_n induces a homomorphism between the homology modules $H_n(C)$ and $H_n(D)$ defined by $c + \operatorname{im} \partial_{n+1}^C \mapsto f_n(c) + \operatorname{im} \partial_{n+1}^D$. Moreover, the map which sends chain complexes to the corresponding sequence of homology groups and which sends chain maps to the sequence of induced homomorphisms is a functor; that is:*
   (a) *The identity chain map induces identity homomorphisms in each dimension.*
   (b) *The homomorphisms induced by the composition of the two chain maps $f\colon C \to D$ and $g\colon D \to E$ are the same as the composition of the induced homomorphisms. That is, $(gf)_* = g_* f_*$.*

DEFINITION 1.1.5. Given chain maps f and g between the complexes C and D, a **chain homotopy** h from f to g is a collection of homomorphisms $h_n\colon C_n \to D_{n+1}$ such that $f_n - g_n = \partial^D_{n+1} h_n + h_{n+1}\partial^C_n$.

$$\begin{tikzcd}
& C_n \arrow[r, "\partial^C_n"] \arrow[dl, "h_n"'] \arrow[d, "f_n - g_n"] & C_{n-1} \arrow[dl, "h_{n-1}"] \\
D_{n+1} \arrow[r, "\partial^D_{n+1}"] & D_n &
\end{tikzcd}$$

If such an h exists, f and g are said to be **chain homotopic**, and we write $f \simeq g$.

PROPOSITION 1.1.6. *If f and g are chain homotopic, then they induce the same maps on homology.*

PROOF. We show that for cycles c that $f_n(c)$ and $g_n(c)$ differ at most by a boundary, and hence differ by 0 in homology. Let c be an element of $\ker \partial_{C,n}$. Since $f \simeq g$, there is some chain homotopy h such that $(f_n - g_n)(c) = \partial^D_{n+1} h_n(c) + 0$. We may then write $f_n(c) = g_n(c) + \partial^D_{n+1} h_n(c)$. It is then clear that when we consider the induced maps on the homology modules, we get $f_{*n}(c) = g_{*n}(c) + 0$. $\square$

DEFINITION 1.1.7. Two chain complexes C and D are **chain homotopy equivalent**, denoted $C \simeq D$ if there are chain maps $f\colon C \to D$ and $g\colon D \to C$ such that gf and fg are chain homotopic to the identity chain maps $C \to C$ and $D \to D$. If such an f and g exist, they are each said to be **chain homotopy equivalences**.

Chain homotopy equivalence is an equivalence relation on chain complexes. The following proposition is a useful tool in demonstrating the invariance of homology modules under certain induced maps. It says that chain homotopy equivalent complexes have isomorphic homology modules.

PROPOSITION 1.1.8. *If f and g are chain homotopy equivalences, then the induced maps f_{*n} and g_{*n} are isomorphisms.*

PROOF. We have that $gf \simeq \mathrm{id}_C$ and $fg \simeq \mathrm{id}_D$, so $(gf)_* = g_* f_* = \mathrm{id}_{H(C)}$ and $(fg)_* = f_* g_* = \mathrm{id}_{H(D)}$. This implies that f_* and g_* are isomorphisms. $\square$

DEFINITION 1.1.9. A chain complex C is **contractible** if it is chain homotopy equivalent to the 0 chain complex. Since the only chain map that can fit in $0 \to C \to 0$ as either arrow is the 0 map, this is equivalent to $\mathrm{id}_C \simeq 0$.

DEFINITION 1.1.10. A chain complex C is **acyclic** if its homology groups are all 0.

Given proposition 1.1.8, contractible complexes are acyclic.

DEFINITION 1.1.11. Let f be a chain map from C to D. The **mapping cone** $\mathrm{Cone}(f)$ of f is the chain complex $C \oplus \Sigma D$ where $(C \oplus \Sigma D)_n$ is $C_n \oplus D_{n+1}$ and $\partial_n^{C\oplus\Sigma D}$ is given by sending (c, d) to $(-\partial_{C,n}(c), f_n(c) + \partial_{D,n+1}(d))$ for $c \in C_n$ and $d \in D_{n+1}$.

The map ∂_n above can be viewed as left multiplication by the matrix $\begin{pmatrix} -\partial_{C,n} & 0 \\ f_n & \partial_{D,n+1} \end{pmatrix}$ on the column vector $\begin{pmatrix} c \\ d \end{pmatrix}$. Thus, the composition $\partial_{n-1}\partial_n$ corresponds to the matrix $\begin{pmatrix} \partial_{C,n-1}\partial_{C,n} & 0 \\ -f_{n-1}\partial_{C,n} + \partial_{D,n}f_n & \partial_{D,n}\partial_{D,n+1} \end{pmatrix}$ which reduces to 0.

PROPOSITION 1.1.12. *A chain map $f\colon C \to D$ is a homotopy equivalence if and only if* $\mathrm{Cone}(f)$ *is contractible.*

1.2. Cochains

Homology theories have a dual notion called cohomology. Cohomology modules are defined using cochain complexes, and in discussing them, we will talk about cocycles and coboundaries. There is essentially little difference between chain complexes and cochain complexes, as can be seen from the definitions.

DEFINITION 1.2.1. A **cochain complex** is a collection of modules C^n of a ring R and homomorphisms $\delta^n\colon C^n \to C^{n+1}$ each indexed by the set of integers, such that any composition $\delta^{n+1}\delta^n$ is the 0 homomorphism. Elements of C^n are called **n-cochains**. The homomorphisms are called **coboundary maps**. Elements in the kernel of a coboundary map are called **cocycles**, and elements in the image are called **coboundaries**.

This differs from the definition of a chain complex only in that the homomorphisms map upward in index from n to $n+1$ rather than downward. Since $\delta^{n+1}\delta^n$ is always 0, we are again able to define quotient modules.

DEFINITION 1.2.2. The **n-th cohomology module** $H^n(C)$ of the chain complex C is the quotient module $\ker\delta^n / \operatorname{im}\delta^{n-1}$.

There are corresponding notions of cochain maps, cochain homotopy equivalences, and so on. The definitions are analogous to those for chain complexes, with the necessary changes made to be compatible with the indices.

We can obtain a cochain complex from a chain complex by applying the $\mathrm{Hom}(-, R)$ functor where R is any ring. This functor sends a chain group C_n to the group $\mathrm{Hom}(C_n, R)$ of homomorphisms from C_n to R. It sends a boundary homomorphism $\partial_n\colon C_n \to C_{n-1}$ to the homomorphism $\mathrm{Hom}(\partial_n, R)$: $\mathrm{Hom}(C_{n-1}, R) \to \mathrm{Hom}(C_n, R)$ determined by $\phi \mapsto \phi \circ \partial_n$ for each ϕ in

$\mathrm{Hom}(C_{n-1}, R)$. If we take $\mathrm{Hom}(C_n, R)$ to be our cochain group C^n, and we take our coboundary maps to be $\delta_n = \mathrm{Hom}(\partial_{n+1}, R) : \mathrm{Hom}(C_n, R) \to \mathrm{Hom}(C_{n+1}, R)$, then this gives us a cochain complex. To see that $\delta_{n+1} \circ \delta_n = 0$, note that it sends any ϕ in C^n to $\phi \circ \partial_{n+1} \circ \partial_{n+2} = \phi \circ 0 = 0$ in C^{n+2}.

The following theorem states that the cohomology groups of a cochain complex obtained in the above way are determined by the homology group of the starting chain complex.

THEOREM 1.2.3 (Universal Coefficient Theorem for Cohomology). *Let C be a chain complex of free abelian groups, let $H_n(C)$ be the homology groups of C, and let G be an abelian group. Then, the cohomology groups $H^n(C; G)$ of the cochain complex $\mathrm{Hom}(C_n, G)$ are determined by the split exact sequences*

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \to 0$$

where h is the map sending a class of maps $\bar{\phi} \in H^n(C; G)$ represented by a map $\phi \colon C_n \to G$ to the map $\widehat{\phi} \colon H_n(C) \to G$ induced by ϕ.

1.3. Direct Sums and Direct Products

Let R be a ring and let $\{M_i\}$ be a collection of R-modules indexed by I.

DEFINITION 1.3.1. The **direct sum** $\bigoplus_i M_i$ is the module consisting of almost-everywhere zero sequences $(m_i)_{i \in I}$ where the i-th term m_i is an element of M_i, addition is defined componentwise, and scalars distribute across all components. By an almost-everywhere zero sequence, we mean a sequence which takes on non-trivial values on at most finitely many indices.

DEFINITION 1.3.2. A **direct system of modules** is a collection of modules M_i indexed by a partially ordered set $(I, \leq)$ and homomorphisms $p_{i,j} : M_i \to M_j$ for each $i \leq j$ in I such that

(a) for each i and j in I, there exists a k in I such that $i, j \leq k$,
(b) $p_{i,i}$ is the identity homomorphism on M_i,
(c) whenever $i \leq j \leq k$, we have $p_{j,k} p_{i,j} = p_{i,k}$.

DEFINITION 1.3.3. Given a direct system of modules $(M_i, p_{i,j})$, the **direct limit** $\varinjlim(M_i, p_{i,j})$ is defined to be the quotient module $\bigsqcup_{i \in I} M_i \big/ \sim$ where $\bigsqcup$ is the disjoint union and $\sim$ is the equivalence relation given by $m_i \sim m_j$ for $m_i \in M_i$ and $m_j \in M_j$ iff there is some $k \geq i, j$ such that $p_{i,k}(m_i) = p_{j,k}(m_j)$. Informally, we can say that the direct limit identifies elements if they are eventually equal. Addition of elements x and y in the direct limit is done by finding representatives of x and y in the same M_i, performing the addition on the representatives there, and then taking the equivalence class of the sum. Scaling is likewise done on a representative.

We can obtain the direct sum $\bigoplus_{i\in I} M_i$ as a direct limit as follows. First, let $\mathcal{F}$ be the family of finite subsets of I ordered by subset inclusion. For each set F in $\mathcal{F}$, define M_F to be the cartesian product $\times_{i\in F} M_i$. For each F, G in $\mathcal{F}$ such that $F \subseteq G$, let the homomorphism $p_{F,G}$ be the inclusion map.

DEFINITION 1.3.4. The **direct product** $\prod_i M_i$ is the module consisting of all sequences $(m_i)_{i\in I}$ where the i-th term m_i is in M_i, addition is defined componentwise, and scalars distribute across all components.

DEFINITION 1.3.5. An **inverse system of modules** is a collection of modules M_i indexed by a partially ordered set $(I, \leq)$ and homomorphisms $p_{j,i}\colon M_j \to M_i$ for each $i \leq j$ in I such that

(a) for each i and j in I, there exists a k in I such that $i, j \leq k$,
(b) $p_{i,i}$ is the identity homomorphism on M_i,
(c) whenever $i \leq j \leq k$, we have $p_{j,i}p_{k,j} = p_{k,i}$.

Note that the difference between the definition of an inverse system and the definition of a direct system is essentially that the direction of the homomorphisms is reversed.

DEFINITION 1.3.6. Given an inverse system of modules $(M_i, p_{j,i})$, the **inverse limit** $\varprojlim(M_i, p_{i,j})$ is defined to be the submodule $\{(m_i)_{i\in I} \mid m_i = p_{j,i}(m_j) \text{ for all } i \leq j\}$ of the cartesian product $\times_{i\in I} M_i$. Addition is done termwise, and scaling distributes across all terms.

We can obtain the direct product $\prod_{i\in I} M_i$ as an inverse limit as follows. Again, let $\mathcal{F}$ be the family of finite subsets of I ordered by subset inclusion, and for each set F in $\mathcal{F}$, define M_F to be the cartesian product $\times_{i\in F} M_i$. For each F, G in $\mathcal{F}$ such that $F \subseteq G$, let the homomorphism $p_{G,F}$ be the projection map onto the indices in F.

It is clear from the definitions that the direct sum is the submodule of the direct product containing only the almost-everywhere zero sequences. Direct sums and products frequently appear in chain and cochain groups. For example, the chain groups in simplicial homology consist of finite formal sums of simplices. Thus the simplicial n-chain group is isomorphic to $\bigoplus_{S_n}\mathbb{Z}$, the direct product of copies of $\mathbb{Z}$ indexed by the set S_n of n-simplices. In locally finite homology, we admit infinite formal sums. These groups are isomorphic to direct products $\prod_{S_n}\mathbb{Z}$ where S_n is the set of n-simplices. We also obtain direct products in singular cohomology when we dualize the singular chain complex using the Hom functor. This is a result of the fact that $\mathrm{Hom}(\bigoplus_{i\in I}\mathbb{Z}, R)$ is isomorphic to

$\prod_{i\in I} R$. To see this, note that since any ϕ in $\mathrm{Hom}(\bigoplus_{i\in I}\mathbb{Z}, R)$ is determined by the values it takes on the generators of $\bigoplus_{i\in I}\mathbb{Z}$, each ϕ has a unique corresponding sequence $(f(1_i))_{i\in I}$ in $\prod_{i\in I} R$ where 1_i is the element of $\bigoplus_{i\in I}\mathbb{Z}$ with 1 in the i-th position and 0 everywhere else.

PROPOSITION 1.3.7. *Let $(A_i, f_{i,j})$ and $(B_i, g_{i,j})$ be direct systems of modules. Suppose that $\{\theta_i\colon A_i \to B_i\}$ is a homomorphism of direct systems and that for each i, there is some $j_i > i$ and homomorphism $h_i\colon A_i \to B_j$ such that the diagram*

$$\begin{tikzcd} A_j \arrow[r, "\theta_j"] & B_j \arrow[dl, "h_i"', leftarrow] \\ A_i \arrow[u, "f_{i,j}"] \arrow[r, "\theta_i"'] & B_i \arrow[u, "g_{i,j}"'] \end{tikzcd}$$

commutes. Then the induced map $\theta\colon \varinjlim A_i \to \varinjlim B_i$ is an isomorphism.

PROOF. For each i we have the following exact sequence

$$0 \to \ker\theta_i \to A_i \xrightarrow{\theta_i} B_i \to \operatorname{coker}\theta_i \to 0$$

It is not hard to check that when we have $j_i > i$ such that h_i exists, then the induced maps $\ker\theta_i \to \ker\theta_{j_i}$ and $\operatorname{coker}\theta_i \to \operatorname{coker}\theta_{j_i}$ are the 0 map. So for each i we have the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & \ker\theta_j \arrow[r] & A_j \arrow[r, "\theta_j"] & B_j \arrow[r] & \operatorname{coker}\theta_j \arrow[r] & 0 \\ 0 \arrow[r] & \ker\theta_i \arrow[r] \arrow[u, "0"] & A_i \arrow[r, "\theta_i"'] \arrow[u, "f_{i,j}"] & B_i \arrow[r] \arrow[u, "g_{i,j}"'] \arrow[ul, "h_i"] & \operatorname{coker}\theta_i \arrow[r] \arrow[u, "0"'] & 0 \end{tikzcd}$$

It follows that $\varinjlim \ker\theta_i = \varinjlim \operatorname{coker}\theta_i = 0$. This implies that after taking direct limits, we get an exact sequence

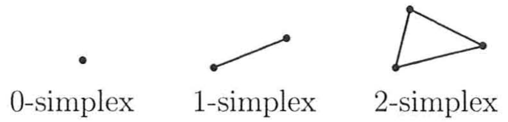
$$0 \to \varinjlim A_i \xrightarrow{\theta} \varinjlim B_i \to 0$$

and so θ is an isomorphism. □

1.4. Simplicial Complexes

Many of the theories we will discuss involve the notions of simplicial complexes and abstract simplicial complexes. These two ideas are closely related. Informally, simplicial complexes are spaces which are nicely viewed as a collection of simple geometric objects: points, lines, triangles, and their higher-dimensional generalizations. Abstract simplicial complexes are collections of finite sets which generalize the behavior of vertices in a simplicial complex.

DEFINITION 1.4.1. An n-**simplex** is the smallest convex set in $\mathbb{R}^{n+1}$ containing $n+1$ points $v_0, \ldots, v_n$ which are not all contained in a single $(n-1)$-dimension hyperplane. The points $v_0, \ldots, v_n$ are called the **vertices**, and since the set of vertices uniquely determines the particular n-simplex, we may denote an n-simplex using its vertices as $\langle v_0, \ldots, v_n \rangle$.



For example, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a filled-in triangle, and a 3-simplex is a filled-in tetrahedron. For each nonnegative integer n, there is a standard n-simplex.

DEFINITION 1.4.2. The **standard** n-**simplex** Δ_n is the n-simplex determined by the set of standard basis vectors e_i in $\mathbb{R}^{n+1}$ where the i-th component of e_i is 1 and all other components are 0.

For example, the standard 1-simplex is the line joining $(1,0)$ and $(0,1)$ in $\mathbb{R}^2$, and the standard 2-simplex is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in $\mathbb{R}^3$. We can explicitly describe Δ_n as the set $\{ \sum_{i=0,\ldots,n} t_i e_i \mid$ each t_i is in $\mathbb{R}$ and $\sum_{i=0,\ldots,n} t_i = 1\}$.

DEFINITION 1.4.3. Given an n-simplex σ with corresponding vertex set $\langle v_0, \ldots, v_n \rangle$, we define an m-**face** of σ to be an m-simplex determined by a nonempty $(m+1)$-element subset of $\{v_0, \ldots, v_n\}$.

For example, the 1-faces of a solid triangle are its edges, its 0-faces are its vertices. Simplices can be collected together into combinatorial objects which will be important in many of the homology and cohomology theories we will discuss.

DEFINITION 1.4.4. A **simplicial complex** $\mathcal{K}$ is a collection of simplices such that if $\sigma \in \mathcal{K}$, then every face of σ is in $\mathcal{K}$, and for any two simplices σ_1 and σ_2 in $\mathcal{K}$, the intersection of σ_1 and σ_2 is a face of both simplices. The **vertex set** $\mathcal{K}^0$ is the set of all 0-simplices in $\mathcal{K}$.

DEFINITION 1.4.5. An **abstract** n-**simplex** is a set containing $n+1$ elements.

DEFINITION 1.4.6. An m-**face** of an abstract n-simplex σ is a nonempty subset of σ containing $m+1$ elements. A **vertex** of σ is a 0-face.

DEFINITION 1.4.7. An **abstract simplicial complex** $\mathcal{K}$ is a collection of abstract simplices such that if σ is in $\mathcal{K}$, then every face of σ is as well. The **vertex set** $\mathcal{K}^0$ is the set of all 0-simplices in $\mathcal{K}$.

An abstract simplicial complex is exactly a collection of finite sets which is closed under taking subsets. This notion captures the combinatorial aspect of a simplicial complex without reference to any geometry.

We can easily obtain an abstract simplicial complex $\mathcal{K}'$ from any simplicial complex $\mathcal{K}$. To do so, we take advantage of the fact that each simplex in $\mathcal{K}$ corresponds uniquely to its set of vertices. As such, there is an injection v from $\mathcal{K}$ to the collection of subsets of the vertex set $\mathcal{K}^0$. The range of v is the desired abstract simplicial complex $\mathcal{K}'$.

EXAMPLE 1.4.8. Let $\mathcal{K}$ be the simplicial complex consisting of a solid triangle T determined by vertices $\langle v_0, v_1, v_2\rangle$. That is, $\mathcal{K}$ is the set containing T, $L_{0,1}$, $L_{0,2}$, $L_{1,2}$, v_0, v_1, and v_2, where $L_{i,j}$ is the edge of the triangle joining v_i and v_j. The injection v sends T to $\{v_0, v_1, v_2\}$, sends $L_{i,j}$ to $\{v_i, v_j\}$, and sends v_i to $\{v_i\}$. The abstract simplicial complex that comes from $\mathcal{K}$ is thus the set

$$\{\{v_0, v_1, v_2\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_0\}, \{v_1\}, \{v_2\}\}$$

We can also obtain from any abstract simplicial complex $\mathcal{K}$ a simplicial complex $|\mathcal{K}|$ called its **geometric realization**. The basic idea is to map the vertex set $\mathcal{K}^0$ to the standard basis vectors of a real vector space, and then attach copies of standard simplices where appropriate. First, choose an injection j from $\mathcal{K}^0$ to the set of basis vectors of the vector space. Now, for each abstract n-simplex σ, we want to include a copy of the standard n-simplex with its vertices on the appropriate basis vectors. Since $j(\sigma) = \{e_{\sigma,1}, e_{\sigma,2}, \dots, e_{\sigma,n}\}$ is the set of basis vectors we want to use, we can choose the obvious embedding of Δ_n where the point e_i in Δ_n is mapped to $e_{\sigma,i}$. Call the image of this embedding Δ_σ. Our desired simplicial complex $|\mathcal{K}|$ is the union $\bigcup_{\sigma\in\mathcal{K}} \Delta_\sigma$. We can assign a topology to $|\mathcal{K}|$ either by viewing it as a subspace of the real vector space we have constructed it in, or by using a path metric.

It is not hard to see that if we use the method above for obtaining an abstract simplicial complex from $|\mathcal{K}|$, we will essentially obtain a relabeled version of $\mathcal{K}$. That is, with the notation above, $|\mathcal{K}|' = \mathcal{K}$ for abstract simplicial complexes $\mathcal{K}$.

1.5. Čech and Anti-Čech Systems

The classical homology and cohomology theories can give the impression that homology and cohomology are defined essentially in terms of the constituent pieces of a space. However, it is possible to associate other structures to a space and work with them to indirectly gain information. This can have advantages over the more direct constructions. One method we will discuss is to approximate the space using sequences of open covers. Čech and anti-Čech systems fit nicely with this idea; they present a sequence of open covers which become progressively finer or coarser along the sequence.

DEFINITION 1.5.1. A **refinement** of a cover $\mathcal{U}$ of a space X is a cover $\mathcal{V}$ such that each V in $\mathcal{V}$ is contained in some U in $\mathcal{U}$. A **refinement map** is a map $p\colon \mathcal{V} \to \mathcal{U}$ such that $V \subseteq p(V)$ for all V in $\mathcal{V}$.

It is clear that refinement maps always exist for a refinement, though they might not be unique.

DEFINITION 1.5.2. A cover $\mathcal{U}$ of X is **locally finite** if for every $x \in X$, there are only finitely many $U \in \mathcal{U}$ such that $x \in U$.

DEFINITION 1.5.3. A **Čech system** $\{\mathcal{U}_i, p_i\}$ for a space X is a sequence of locally finite open covers $\mathcal{U}_i$ and maps p_i such that the following hold.

(a) $\mathcal{U}_{i+1}$ is a refinement of $\mathcal{U}_i$ for each i,
(b) p_i is a refinement map $p_i\colon \mathcal{U}_{i+1} \to \mathcal{U}_i$ for each i, and
(c) the limit of $\sup\{\text{diameter}(U) \mid U \in \mathcal{U}_i\}$ goes to 0 as $i \to \infty$.

DEFINITION 1.5.4. An **anti-Čech system** $\{\mathcal{U}_i, p_i\}$ for a space X is a sequence of locally finite open covers $\mathcal{U}_i$ and maps p_i such that the following hold.

(a) Each $\mathcal{U}_i$ has a diameter $\sup\{\text{diameter}(U) \mid U \in \mathcal{U}_i\}$ bounded by some positive constant R_i,
(b) the Lebesgue number L_{i+1} of $\mathcal{U}_{i+1}$ is at least as large as the upper bound R_i of the diameter of $\mathcal{U}_i$.
(c) p_i is a refinement map $p_i\colon \mathcal{U}_i \to \mathcal{U}_{i+1}$ for each i, and
(d) the diameters R_i tend to ∞ as $i \to \infty$.

Conditions (a) and (b) in the definition of an anti-Čech system guarantee that $\mathcal{U}_i$ is a refinement of $\mathcal{U}_{i+1}$. Condition (d) forces the diameters of the $\mathcal{U}_i$ to go to infinity as i increases. While both Čech and anti-Čech systems are sequences of refinements of open covers, Čech systems become finer as we go along the sequence, and anti-Čech systems become coarser. When we define theories using these systems, this behavior will allow us to capture small-scale and large-scale topological features respectively.

For future use, we include the following proposition, which is proved in [**20**].

PROPOSITION 1.5.5. *Let X be a proper metric space. There is a subset $Y \subseteq X$ such that the distance between distinct points of Y is at least $\frac{1}{2}$ and the collection of open balls of unit radius $B(y,1)$ for $y \in Y$ cover X. Moreover, for such a Y, the sequence of collections $\mathcal{U}_n = \{B(y,3^n) \mid y \in Y\}$ forms an anti-Čech system for X.*

PROOF. Let $\mathcal{F}$ be the family of subsets S of X such that the distance between distinct points in S is at least $\frac{1}{2}$. The family $\mathcal{F}$ can be partially ordered by subset inclusion, and every chain in such an ordering has an upper bound (namely the union over the chain). By Zorn's Lemma, there exists an element Y in $\mathcal{F}$ which is maximal with respect to $\subseteq$. If $\{B(y,1) \mid y \in Y\}$ did not cover X, then there would be some $x \in X$ at least distance 1 from all $y \in Y$. For such an x, we have $Y \cup \{x\} \in \mathcal{F}$ and $Y \subset Y \cup x$, contradicting maximality of Y.

We now check that the collections $\mathcal{U}_n$ are locally finite covers. Let A be a bounded subset of X. Then $D(A, 3^n) = \{x \in X \mid d(x, A) \leq 3^n\}$ is also bounded. Since X is proper, the sets in $\mathcal{U}_n$ as well as $D(A, 3^n)$ have compact closures. Any $U \in \mathcal{U}_n$ intersects A iff it is centered at some $y \in D(A, 3^n) \cap Y$. But, the closure of $D(A, 3^n)$ intersects Y at only finitely many points, so A can only intersect finitely many elements of $\mathcal{U}_n$.

Each set in $\mathcal{U}_n$ has diameter $R_n = 2 \cdot 3^n$. We need to check that the Lebesgue number of $\mathcal{U}_{n+1}$ is at least $2 \cdot 3^n$. Suppose $Z \subseteq X$ has diameter $\leq 2 \cdot 3^n$. Then Z is contained in a closed ball with radius $2 \cdot 3^n$. Let $z \in X$ be the center of this closed ball. By definition of Y, there is some $y \in Y$ such that $d(z, y) < 1$. We have

$$Z \subseteq D(z, 2 \cdot 3^n) \subseteq B(y, 2 \cdot 3^n + 1) \subseteq B(y, 2 \cdot 3^{n+1}) \in \mathcal{U}_{n+1}$$

as required. $\square$

1.6. Nerves and Vietoris-Rips Complexes

Given any relation $R \subseteq X \times Y$, there are two abstract simplicial complexes which can be defined.

DEFINITION 1.6.1. The **nerve** of R is the abstract simplicial complex $\mathcal{K}_R$ whose n-simplices are finite subsets $\{x_0, \ldots, x_n\} \subseteq X$ such that for some $y \in Y$, we have $(x_i, y) \in R$ for all i.

The **Vietoris-Rips complex** of R is the abstract simplicial complex $\mathcal{L}_R$ whose n-simplices are the finite subsets $\{y_0, \ldots, y_n\} \subseteq Y$ such that for some $x \in X$, we have $(x, y_i) \in R$ for all i.

On page 89 of **[4]**, it is shown that certain homology groups as well as cohomology groups of the geometric realizations of the nerve and Vietoris-Rips complex of a relation coincide, and moreover, it is shown that these realizations have the same homotopy type.

We are interested in using nerves in conjunction with Čech systems and anti-Čech systems. The relation in consideration is that of set membership between a space X and an open cover $\mathcal{U}$. Since we will make frequent use of geometric realizations of such nerves, we make the following definitions. These definitions and the proposition below appear often in literature involving coarse homology, including **[19]** and **[21]**.

DEFINITION 1.6.2. Let X be a topological space and let $\mathcal{U}$ be an open cover of X. The **nerve** $\mathcal{K}$ of $\mathcal{U}$ is the geometric realization of the abstract simplicial complex whose n-simplices are finite subsets $\{U_0, \ldots, U_n\}$ of $\mathcal{U}$ such that for some $x \in X$, we have $x \in \bigcap U_i$. That is, $\mathcal{K}$ consists of a 0-simplex $\{U\}$ for each U in $\mathcal{U}$, and an n-simplex $\{U_0, \ldots, U_n\}$ whenever $U_0 \cap \cdots \cap U_n$ is nonempty.

DEFINITION 1.6.3. Let $A \subseteq X$, let $\mathcal{U}$ be an open cover of X, and let $\mathcal{K}$ be the nerve of $\mathcal{U}$. The **subnerve** $\mathcal{K} \restriction_A$ consists of those simplices $\{U_0, \ldots, U_n\}$ in $\mathcal{K}$ such that $\bigcap U_i \cap A$ is nonempty.

Given a Čech system or an anti-Čech system $\{\mathcal{U}_i, p_i\}$, there is an associated system $\{\mathcal{K}_i, p_i^*\}$ of nerves and induced maps.

PROPOSITION 1.6.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be locally finite open covers such that $\mathcal{V}$ is a refinement of $\mathcal{U}$, and let $\mathcal{K}_{\mathcal{U}}$ and $\mathcal{K}_{\mathcal{V}}$ be the nerves of $\mathcal{U}$ and $\mathcal{V}$ respectively. Let $p\colon \mathcal{V} \to \mathcal{U}$ be a refinement map. Then p induces a continuous, proper map $p^*\colon \mathcal{K}_{\mathcal{V}} \to \mathcal{K}_{\mathcal{U}}$.*

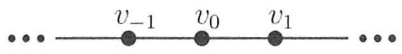
PROOF. First, define a map p' which maps finite sets $\{V_0, \ldots, V_n\} \subseteq \mathcal{V}$ to finite sets $\{p(V_0), \ldots, p(V_n)\}$. Note that p' is not necessarily injective since p is not necessarily injective. Now, since $\mathcal{K}_V$ is a geometric realization based on an abstract simplicial complex whose simplices are finite sets, each simplex in $\mathcal{K}_{\mathcal{V}}$ is an embedding of a standard simplex Δ_n with vertices corresponding to some abstract simplex $\{V_0, \ldots, V_n\}$. Similarly for $\mathcal{K}_{\mathcal{U}}$. Thus for each simplex σ in $\mathcal{K}_{\mathcal{V}}$, p' determines a map from the vertices of some Δ_n to the vertices of some Δ_m, and so it determines a continuous map c_{Δ_n,Δ_m} from $\Delta_n \to \Delta_m$. Let i_{Δ_n} and j_{Δ_m} be the embedding maps of Δ_n and Δ_m into $\mathcal{K}_{\mathcal{V}}$ and $\mathcal{K}_{\mathcal{U}}$ respectively, and note that $\sigma = i_{\Delta_n}(\Delta_n)$. We now define $p_\sigma\colon \sigma \to j_{\Delta_m}(\Delta_m)$ by $p_\sigma(x) = j_{\Delta_m} \circ c_{\Delta_n,\Delta_m} \circ i_{\Delta_n}^{-1}(x)$. This is clearly a continuous, proper map. To get p^*, we take the union over all p_σ for each $\sigma \in \mathcal{K}_{\mathcal{V}}$. Since the covers $\mathcal{U}$ and $\mathcal{V}$ are locally finite, p^* is proper. $\square$

DEFINITION 1.6.5. Let $\{\mathcal{U}_i, p_i\}$ be a Čech system or an anti-Čech system for X. The **system of nerves** $\{\mathcal{K}_i, p_i^*\}$ associated to $\{\mathcal{U}_i, p_i\}$ consists of the nerves $\mathcal{K}_i$ of $\mathcal{U}_i$ for each i and the continuous, proper maps $p_i^*\colon \mathcal{K}_{i+1} \to \mathcal{K}_i$ induced by each p_i.

Note that we have defined the nerve of an open cover to be a geometric realization rather than an abstract simplicial complex. This is because we will later define homology groups for topological spaces, and so we want to have a topology to work with, rather than just a set. However, we will regularly conflate the nerve of an open cover with the abstract simplicial complex of which it is the geometric realization so that we can more easily refer to the simplices in the nerve. This is a slight abuse of notation since the nerve $\mathcal{K}$ of a cover $\mathcal{U}$ will not actually have elements such as $\{U_1, U_2\}$. Though, these sets do unambiguously determine simplices in $\mathcal{K}$ when they intersect.

EXAMPLE 1.6.6. Let $\mathcal{U}$ be the open cover of $\mathbb{R}$ consisting of open intervals $(n, n+2)$ for each integer n. The nerve $\mathcal{K}$ of $\mathcal{U}$ has a 0-simplex $\langle v_n \rangle =$

$\{(n, n+2)\}$ for each n, and a 1-simplex $\langle v_n, v_{n+1}\rangle = \{(n, n+2), (n+1, n+3)\}$ for each intersecting pair of intervals. Part of $\mathcal{K}'$ is depicted schematically below. Note that this depiction is of a space homeomorphic to $\mathcal{K}$.



The above example happens to produce a nerve whose geometric realization is homeomorphic to the space we started with. This is a consequence of the particular open covering we used, and will not generally be the case. For contrast, consider the following examples.

EXAMPLE 1.6.7. Let $\mathcal{U}$ be the open cover of $\mathbb{R}$ containing intervals of the form $(n, n+3)$ for all integers n. The nerve of $\mathcal{U}$ will now contain 0-simplices $\langle v_n\rangle = \{(n, n+3)\}$ and 1-simplices $\langle v_n, v_{n+1}\rangle = \{(n, n+3), (n+1, n+4)\}$ similarly to the previous example, but it will also contain 1-simplices $\langle v_n, v_{n+2}\rangle = \{(n, n+3), (n+2, n+5)\}$ and even 2-simplices $\langle v_n, v_{n+1}, v_{n+2}\rangle = \{(n, n+3), (n+1, n+4), (n+2, n+5)\}$ because of the more complicated intersection behavior.

EXAMPLE 1.6.8. Consider the open cover $\mathcal{U}$ of a space X consisting solely of the whole space. That is, $\mathcal{U} = \{X\}$. The nerve of $\mathcal{U}$ is $\mathcal{K} = \{\{X\}\}$, and its geometric realization is a single point.

While very trivial, this last example shows in an exaggerated way how using a nerve to approximate a space can lead to a loss of local information.

1.7. Sheaf Theory

Sheaf theory provides a language for defining and generalizing theories which depend on open sets and coverings.

DEFINITION 1.7.1. A **presheaf** F on topological space X with values in a category $\mathcal{C}$ is a contravariant functor from the category of open subsets of X and inclusion maps to the category $\mathcal{C}$. That is, F is a function which assigns to each open $U \subseteq X$ an object $F(U)$ in $\mathcal{C}$, and assigns to each inclusion map $V \to U$ with $V \subseteq U$ a morphism $F^U_V \colon F(U) \to F(V)$ in $\mathcal{C}$ called a **restriction morphism** which satisfies the following properties.

- For each open set U, the restriction morphism F^U_U is the identity on $F(U)$.
- For $W \subseteq V \subseteq U$, the restriction morphisms satisfy $F^U_W = F^V_W \circ F^U_V$.

The object $F(U)$ is called the **sections** of F over U, and if the objects in $\mathcal{C}$ can be thought of as having elements, then each of the elements in $F(U)$ is called a **section** over U. A section over X is called a **global section**.

EXAMPLE 1.7.2. Given an R-module G, there is a presheaf called a **constant presheaf** which assigns G to every nonempty $U \subseteq X$, assigns the trivial group 0 to the empty set, and assigns the identity on G to each inclusion $V \to U$.

DEFINITION 1.7.3. Given two presheaves F_1 and F_2 on X, a **homomorphism of presheaves** $h\colon F_1 \to F_2$ is a natural transformation of functors. That is, a collection of homomorphisms $h_U\colon F_1(U) \to F_2(U)$ for $U \subseteq X$ open such that each h_U commutes with the restriction morphisms.

DEFINITION 1.7.4. A **sheaf** is a presheaf which additionally satisfies the following unique gluing property. If $\{U_i\}_{i\in I}$ is a collection of open sets with union $U = \bigcup_{i\in I} U_i$ and we have given $s_i \in F(U_i)$ for each i such that $F^{U_i}_{U_i\cap U_j}(s_i) = F^{U_j}_{U_i\cap U_j}(s_j)$ for all i, j in I, then there is a unique $s \in F(U)$ such that $F^{U}_{U_i}(s) = s_i$ for all i.

We can associate a sheaf to any presheaf using the notion of germs.

DEFINITION 1.7.5. Let F be a presheaf on X. Let M be $\{s \in F(U) \mid U \text{ is open and } x \in U \subseteq X\}$. Let $\sim$ be the equivalence relation in which $s \in F(U)$ and $t \in F(V)$ are equivalent iff there is some open W satisfying $x \in W \subseteq U \cap V$ for which $F^U_W(s) = F^V_W(t)$. The **set of germs of** F **at** x is defined to be the set $F_x = M/\sim$ of equivalence classes of M modulo $\sim$. An equivalence class in F_x containing $s \in F(U)$ is called the **germ of** s **at** $x \in U$ and is denoted by s_x.

DEFINITION 1.7.6. Given a presheaf F, the **sheaf generated by** F is the disjoint union of the sets of germs F_x over all x, with the topology generated by the open sets $\{s_x \in F_x \mid x \in U\}$ for all $s \in F(U)$ and open $U \subseteq X$.

CHAPTER 2

Homology

2.1. Eilenberg-Steenrod Axioms for Homology

Homology theories are typically described by assigning groups to certain kinds of pairs of spaces, for example pairs (X, A) where X is a topological space and $A \subseteq X$, or pairs (X, A) where X is locally compact and A is a closed subset of X. This assignment is then usually shown to behave nicely with a certain class of maps, such as continuous maps or proper maps. The properties satisfied by the different theories tend to be similar. A list of axioms was first abstracted from the properties of early theories and presented by Eilenberg and Steenrod **[6]**, and we include their development here. Apart from providing a theoretical framework for what a homology theory should be, the axioms often characterize theories on a particular category of spaces and maps up to isomorphism of the assigned groups. Also, the axioms are sometimes sufficient for identifying the groups associated to simple spaces without needing to explicitly calculate them using the definition of a particular theory.

DEFINITION 2.1.1. We say that (X, A) is a **pair of sets** if $A \subseteq X$.

DEFINITION 2.1.2. Let (X, A) and (Y, B) be pairs of sets. A function $f: X \to Y$ is said to be a **map of pairs** from (X, A) to (Y, B) is $f(A) \subseteq B$. Such a map is denoted by $f: (X, A) \to (Y, B)$.

DEFINITION 2.1.3. The **lattice** of the pair (X, A) is the collection of all pairs in the following diagram, along with their identity maps, the indicated inclusion maps of pairs, and their compositions.

$$\begin{tikzcd}
 & & (X, \emptyset) \arrow[dr] & & \\
(\emptyset, \emptyset) \arrow[r] & (A, \emptyset) \arrow[ur] \arrow[dr] & & (X, A) \arrow[r] & (X, X) \\
 & & (A, A) \arrow[ur] & &
\end{tikzcd}$$

A map of pairs $f: (X, A) \to (Y, B)$ defines maps between corresponding members of the lattices of (X, A) and (Y, B) by restriction where necessary. For example, $f \restriction_A$ is a map of pairs $(A, \emptyset) \to (B, \emptyset)$.

DEFINITION 2.1.4. A family $\mathcal{C}$ of pairs of spaces and maps is an **admissible category** if it satisfies the following properties. The spaces and maps in an admissible category are called **admissible**.

(a) If $(X, A) \in \mathcal{C}$, then the lattice of (X, A) is contained in $\mathcal{C}$.
(b) If $f: (X, A) \to (Y, B)$ is in $\mathcal{C}$, then (X, A) and (Y, B) are in $\mathcal{C}$, as well as all maps from members of the lattice of (X, A) to the corresponding members of the lattice of (Y, B) which f defines.
(c) If f and g are in $\mathcal{C}$, then if their composition fg is defined, it is in $\mathcal{C}$.
(d) Let $I = [0, 1] \subset \mathbb{R}$. If $(X, A) \in \mathcal{C}$, then $(X \times I, A \times I) \in \mathcal{C}$ and the maps $i_0, i_1: (X, A) \to (X \times I, A \times I)$ defined by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$ are in $\mathcal{C}$.
(e) $\mathcal{C}$ contains a space P_0 consisting of a single point. Also, if X and P are in $\mathcal{C}$, if $f: P \to X$, and if P is a single point, then $f \in \mathcal{C}$.

EXAMPLE 2.1.5. The following are admissible categories for homology theory.

(a) The set of all pairs of arbitrary sets (X, A) and all maps of such pairs.
(b) The set of all pairs of topological spaces (X, A) and all continuous maps of such pairs.
(c) The set of pairs (X, A) with X a locally compact space and A closed in X together with all proper maps of such pairs.

DEFINITION 2.1.6. Two maps of pairs $f_0, f_1: (X, A) \to (Y, B)$ in an admissible category $\mathcal{C}$ are said to be **$\mathcal{C}$-homotopic** if there is a map $h: (X \times I, A \times I) \to (Y, B)$ in $\mathcal{C}$ such that $f_0 = h \circ i_0 = h(x, 0)$ and $f_1 = h \circ i_1 = h(x, 1)$. The map h is called a **$\mathcal{C}$-homotopy** between f_0 and f_1.

We can now list the axioms for homology theories.

DEFINITION 2.1.7. Let $\mathcal{G}$ be a collection either of abelian groups or of R-modules for some fixed ring R. A **homology theory** H on an admissible category $\mathcal{C}$ is a collection of functions as follows.

- The first function H is defined for each admissible pair (X, A) and each integer q and assigns values in $\mathcal{G}$. The value of the function is usually written $H_q(X, A)$ and is called the **q-dimensional relative homology group** of X modulo A. If A is the empty set, then $H_q(X, A)$ is often abbreviated as $H_q(X)$.
- The second function is defined for each admissible map $f: (X, A) \to (Y, B)$ and each integer q and assigns a homomorphism $f_{*q}: H_q(X, A) \to H_q(Y, B)$ called the **homomorphism induced by** f. The homomorphism f_{*q} is typically written as f_* when it is not ambiguous to do so.
- The third function ∂ is defined for each admissible (X, A) and each integer q and assigns a homomorphism $\partial(q, X, A): H_q(X, A) \to H_{q-1}(A, \emptyset)$

called the **boundary operator**. This homomorphism is typically written as ∂ when it is not ambiguous to do so.

The first two functions above are required to be functorial:

Axiom 1: If f is the identity map $(X, A) \to (X, A)$, then f_* is the identity map $H_q(X, A) \to H_q(X, A)$ for each q.

Axiom 2: If $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (Z, C)$ are admissible, then $(gf)_* = g_* f_*\colon H_q(X, A) \to H_q(Z, C)$.

The third function must behave well with the first two:

Axiom 3: If $f\colon (X, A) \to (Y, B)$ is admissible, then the map ∂f_* is $(f \restriction_A)_* \partial\colon H_q(X, A) \to H_{q-1}(B, \emptyset)$.

Additionally, the following axioms must be satisfied:

Axiom 4 (Exactness): If (X, A) is admissible and if $i\colon (A, \emptyset) \to (X, \emptyset)$ and $j\colon (X, \emptyset) \to (X, A)$ are inclusion maps, then the following sequence is exact.

$$\cdots \xrightarrow{\partial} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \xrightarrow{i_*} \cdots$$

This sequence is called the **homology sequence** of the pair (X, A).

Axiom 5 (Homotopy Invariance): If the admissible maps f_0, f_1 from (X, A) to (Y, B) are $\mathcal{C}$-homotopic, then for each q, the homomorphisms $f_{0*}, f_{1*}\colon H_q(X, A) \to H_q(Y, B)$ are equal.

Axiom 6 (Excision): Let (X, A) be an admissible pair. If U is an open subset of X whose closure $\bar{U}$ is contained in the interior of A, and if the inclusion map $i\colon (X - U, A - U) \to (X, A)$ is admissible, then i induces an isomorphism $i_*\colon H_q(X - U, A - U) \to H_q(X, A)$ for all q. Any inclusion map i satisfying these conditions is called an **excision map**.

Axiom 7 (Dimension): If P is an admissible space consisting of a single point, then $H_q(P) = 0$ for all $q \neq 0$. The value of $H_0(P)$ is called the **coefficient group** or **coefficient module** of the homology theory, depending on whether $\mathcal{G}$ is a family of abelian groups or modules.

We'll now prove some basic results which follow from the axioms.

DEFINITION 2.1.8. Two admissible pairs (X, A) and (Y, B) are **$\mathcal{C}$-isomorphic** if there are admissible maps $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (X, A)$ such that both fg and gf are identity maps. Such a map f is called a **$\mathcal{C}$-isomorphism** and g is called the inverse of f.

PROPOSITION 2.1.9. *A $\mathcal{C}$-isomorphism $f\colon (X, A) \to (Y, B)$ induces isomorphisms $f_*\colon H_q(X, A) \to H_q(X, B)$ for all q.*

PROOF. Since f has an inverse g and fg is the identity, we have that $(fg)_* = f_* g_*$ is the identity. Similarly, gf is the identity, so $(gf)_* = g_* f_*$ is the identity. This shows that f_* has an inverse homomorphism, and so f_* is an isomorphism. $\square$

DEFINITION 2.1.10. Admissible pairs (X, A) and (Y, B) are said to be $\mathcal{C}$-**homotopy equivalent** if there are admissible maps $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (X, A)$ such that gf and fg are each $\mathcal{C}$-homotopic to the identity maps on (X, A) and (Y, B) respectively. Such a map f is called a $\mathcal{C}$-**homotopy equivalence**, and g is called the $\mathcal{C}$-**homotopy inverse** of f.

PROPOSITION 2.1.11. *If $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (X, A)$ are $\mathcal{C}$-homotopy inverses, then $f_*\colon H_q(X, A) \to H_q(Y, B)$ is an isomorphism with inverse g_*.*

PROOF. Since gf is $\mathcal{C}$-homotopic to id_X, they induce the same maps by the homotopy invariance axiom. So we have that $(gf)_* = g_*f_*$ is the identity $H_q(X, A) \to H_q(X, A)$. Similarly, fg is $\mathcal{C}$-homotopic to id_Y, so they induce the same maps, and hence $(fg)_* = f_*g_*$ is the identity $H_q(Y, B) \to H_q(Y, B)$. This implies that f_* and g_* are inverse homomorphisms. $\square$

The following properties can be proved from the axioms as well.

PROPOSITION 2.1.12 (Direct Sum Property for Homology). *Let $X = X_1 \cup \cdots \cup X_n$ be the union of disjoint sets each of which is closed (and thus open) in X. Let $A_i \subseteq X_i$ for each i and let $A = A_1 \cup \cdots \cup A_n$. Assume that all pairs formed from the sets X_i and A_i and their unions are admissible, as well as all inclusion maps between such pairs. Let $i_\alpha\colon (X_\alpha, A_\alpha) \to (X, A)$ be the inclusion map for each $\alpha = 1, \ldots, n$. Then, the induced homomorphisms $i_{\alpha *}\colon H_q(X_\alpha, A_\alpha) \to H_q(X, A)$ yield an injective representation of $H_q(X, A)$ as a direct sum. That is, each $u \in H_q(X, A)$ can be written uniquely in the form $\sum_\alpha i_{\alpha *} u_\alpha$ where $u_\alpha \in H_q(X_\alpha, A_\alpha)$.*

DEFINITION 2.1.13. A **triad** $(X; A, B)$ consists of a space X and two subsets A and B such that $X, A, B, A \cup B, A \cap B$ and all pairs formed from these are admissible, and all of their inclusion maps are admissible. A triad is called **excisive** if the inclusion maps $k_1\colon (B, A \cap B) \to (A \cup B, A)$ and $k_2\colon (A, A \cap B) \to (A \cup B, B)$ induce isomorphisms between homology groups of all dimensions.

Excisive triads give rise to some particularly useful exact sequences. Note that if $B \subseteq A \subseteq X$, then there are inclusion maps $(A, \emptyset) \xrightarrow{i} (A, B) \to (X, B) \to (X, A)$ with induced maps on homology, and a boundary map $H_q(X, A) \xrightarrow{\partial} H_{q-1}(A)$. So, when $B \subseteq A \subseteq X$, there is an associated boundary operator $\bar{\partial}\colon H_q(X, A) \to H_{q-1}(A, B)$ defined by $\bar{\partial} = i_*\partial$.

PROPOSITION 2.1.14. *Suppose $B \subseteq A \subseteq X$. Then the following is a long exact sequence.*

$$\cdots \to H_q(A,B) \to H_q(X,B) \to H_q(X,A) \xrightarrow{\bar{\partial}} H_{q-1}(A,B) \to \cdots$$

PROOF. The proof can be found in [6] on page 25. We only note here that it involves applications of axioms 1, 2, 3, and 4. □

The following proposition follows easily if we consider the long exact sequence associated to $B \subseteq A \cup B \subseteq X$.

PROPOSITION 2.1.15. *Suppose $(X; A, B)$ is an excisive triad. Then the sequence*

$$\cdots \to H_q(A, A\cap B) \to H_q(X,B) \to H_q(X, A\cup B) \xrightarrow{\partial} H_{q-1}(A, A\cap B) \to$$

is exact, where ∂ here is the composition of the map $\bar{\partial}\colon H_q(X, A\cup B) \to H_{q-1}(A\cup B, B)$ associated to $B \subseteq A\cup B \subseteq X$ and the inverse of the isomorphism induced by $k_2\colon (A, A\cap B) \to (A\cup B, B)$.

PROPOSITION 2.1.16 (Mayer-Vietoris Homology Sequence). *Let $(X; A, B)$ be an excisive triad such that $X = A\cup B$. Then the* ***Mayer-Vietoris homology sequence***

$$\cdots \to H_q(A\cap B) \xrightarrow{\psi} H_q(A)\oplus H_q(B) \xrightarrow{\phi} H_q(X) \xrightarrow{\Delta} H_{q-1}(A\cap B) \to \cdots$$

is exact, where ψ, ϕ, Δ are defined by

$$\begin{aligned} \psi &= (h_{1*}, -h_{2*}) \\ \phi(v_1, v_2) &= m_{1*}(v_1) + m_{2*}(v_2) \\ \Delta &= -\partial k_{1*}^{-1} l_{1*} \end{aligned}$$

with inclusion maps $h_1\colon A\cap B \to A$, $h_2\colon A\cap B \to B$, $m_1\colon A \to X$, $m_2\colon B \to X$, $l_1\colon X \to (X,A)$, and $k_1\colon (B, A\cap B) \to (X,A)$, and boundary operator $\partial\colon H_q(B, A\cap B) \to H_{q-1}(A\cap B)$.

The next proposition states that Mayer-Vietoris sequences exist in a relative form, when $A\cup B$ is not necessarily the entirety of X.

PROPOSITION 2.1.17 (Relative Mayer-Vietoris Sequence). *Let $(X; A, B)$ be an excisive triad. Then the* ***relative Mayer-Vietoris sequence***

$$\cdots \to H_q(X, A\cap B) \xrightarrow{\psi} H_q(X,A)\oplus H_q(X,B) \xrightarrow{\phi}$$
$$H_q(X, A\cup B) \xrightarrow{\Delta} H_{q-1}(X, A\cap B) \to \cdots$$

is exact.

It is worth noting that the ability to manually compute homology and cohomology groups is attributable to the excision axiom. In contrast, higher homotopy groups satisfy similar axioms and provide similar invariants, but lack an analogue of the excision theorem. This represents one of the main advantages of homology theories.

For many homology theories, the 0-dimensional homology groups of a single point and other very simple spaces are nontrivial. In some cases it is desirable to discount this from the theory, and so there is the notion of reduced homology groups.

DEFINITION 2.1.18. Let H be a homology theory, let X be a nonempty admissible space, let P be a one-point space, and let $c\colon X \to P$ be the unique constant map. If c is an admissible map, then the **n-th reduced homology group** $\bar{H}_n(X)$ is defined to be the kernel of the homomorphism $c^*\colon H_n(X) \to H_n(P)$ induced by c.

PROPOSITION 2.1.19. *If H is a homology theory and if the reduced homology $\bar{H}$ is defined, then $H_n(X) \cong \bar{H}_n(X) \oplus H_n(P)$ for all n and for any nonempty admissible space X and one-point space P.*

PROOF. The map $c\colon X \to P$ is surjective and so there is a map $d\colon P \to X$ such that $cd\colon P \to P$ is the identity. Since d maps from a one-point space to X, it is admissible. Thus, c and d both induce homomorphisms on homology, and their induced homomorphisms satisfy $c^*d^* = (cd)^* = (\mathrm{id}_P)^*$. So, $c^*\colon H_n(X) \to H_n(P)$ is also surjective. Since $\bar{H}_n(X)$ is the kernel of c^*, we have an exact sequence $0 \to \bar{H}_n(X) \to H_n(X) \xrightarrow{c^*} H_n(P) \to 0$ and a map d^* such that c^*d^* is the identity on $H_n(P)$. It follows from the splitting lemma that $H_n(X) \cong \bar{H}_n(X) \oplus H_n(P)$. □

2.2. Simplicial Homology

Simplicial homology is defined for topological spaces that are homeomorphic to a simplicial complex. It is easily computed and familiarity with this theory provides much of the intuition for working with other theories.

In the discussion that follows, we fix a simplicial complex X. We follow the development in [**8**].

DEFINITION 2.2.1. For each nonnegative integer n, the **n-th simplicial chain group** C_n is the group of finite formal sums of n-simplices. That is, elements of C_n are of the form $a_1\sigma_1 + \cdots + a_k\sigma_k$ where k is a nonnegative integer, each a_i is an integer, and each σ_i is an n-simplex. Elements of C_n are called **simplicial n-chains**.

There is a homomorphism from each C_n to the chain group one dimension lower C_{n-1} called the boundary homomorphism or boundary map.

DEFINITION 2.2.2. For positive n, the n**-th simplicial boundary map** $\partial_n : C_n \longrightarrow C_{n-1}$ is the homomorphism which is defined for individual simplices by

$$\partial_n(\sigma) = \sum_{i=0}^{n}(-1)^i\langle v_0, \ldots, \widehat{v_i}, \ldots v_n\rangle$$

where the hat symbol $\widehat{\ }$ denotes that the corresponding entry has been removed from the vertex set. We define ∂_n for $n \leq 0$ to be the 0 map.

Each ∂_n extends from individual simplices to all of C_n additively. That is, $\partial(\sigma_1 + \sigma_2)$ is $\partial(\sigma_1) + \partial(\sigma_2)$. Stated in the terminology of chain complexes, the set of n-boundaries is the image of ∂_{n+1}, and the set of n-cycles is the kernel of ∂_n.

REMARK. To see why ∂ is called the boundary map, note that the image of an n-simplex σ is a formal sum of the $(n-1)$-dimensional faces of σ. The alternating sign in the sum accounts for orientation. For example, consider the case where σ is a triangle with vertex set $\langle v_0, v_1, v_2\rangle$. The image of σ under ∂_2 is $\langle v_1, v_2\rangle - \langle v_0, v_2\rangle + \langle v_0, v_1\rangle$. If we think of the terms in this sum as oriented edges and treat $-\langle v_0, v_2\rangle$ as the same as $\langle v_2, v_0\rangle$, then the sum corresponds to a loop on the edges of the triangle.

It can be checked by direct computation that the boundary maps satisfy the property that $\partial_n \circ \partial_{n+1} = 0$ for every n. Equivalently, the image of ∂_{n+1} is always a subgroup of the kernel of ∂_n.

DEFINITION 2.2.3. Given a chain complex $\{C_n, \partial_n\}$ where C_n is the n-th simplicial chain group and ∂_n is the n-th simplicial boundary map for each n, we define the n**-th simplicial homology group** to be the quotient group $H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}$ for each n. That is, H_n is the set of equivalence classes of n-cycles modulo the n-boundaries.

To illustrate these concepts, we compute the simplicial homology groups of the circle and of the real line.

EXAMPLE 2.2.4. First, we consider the circle S^1. We will use the simplicial complex consisting of the vertices v_0, v_1, and v_2 of a triangle and the edges between them.

We will first see that $H_0(S^1)$ is isomorphic to $\mathbb{Z}$. To do so, we will show that every 0-cycle is in the same equivalence class as a multiple of $\langle v_0\rangle$. Let $z = a\langle v_0\rangle + b\langle v_1\rangle + c\langle v_2\rangle$ be an arbitrary 0-cycle in our complex. Let $x = b\langle v_1, v_0\rangle + c\langle v_2, v_0\rangle$. Then x is a 1-cycle with boundary

$$\partial_1(x) = b\langle v_0\rangle - b\langle v_1\rangle + c\langle v_0\rangle - c\langle v_2\rangle$$

Hence, $z + \partial(x) = (a+b+c)\langle v_0\rangle$ and thus z is in the same equivalence class as $(a+b+c)\langle v_0\rangle$. Each integer multiple of $\langle v_0\rangle$ is distinct since no boundary can have the form $b\langle v_0\rangle$ with b nonzero. Hence there is exactly one class in $H_0(S^1)$ for each integer, and so $H_0(S^1) \cong \mathbb{Z}$.

To compute $H_1(S^1)$, we compute $\ker \partial_1$ and $\operatorname{im} \partial_2$. We know that a 1-chain $a\langle v_0, v_1\rangle + b\langle v_1, v_2\rangle + c\langle v_2, v_0\rangle$ maps to $(c-a)\langle v_0\rangle + (a-b)\langle v_1\rangle + (b-c)\langle v_2\rangle$ under ∂, and so it is in the kernel if $c - a = a - b = b - c = 0$. This is equivalent to $a = b = c$. Thus, there is exactly one 1-chain in the kernel for each integer a, and so $\ker \partial_1$ is isomorphic to $\mathbb{Z}$. The chain group C_2 is trivial since there are no 2-simplices in our triangle, so the image of ∂_2 is trivial as well. It follows that $H_1(S^1)$ is also isomorphic to $\mathbb{Z}$.

All higher dimension simplicial homology groups of the triangle are trivial, since the chain groups are 0 and so the boundary homomorphisms have trivial kernels.

EXAMPLE 2.2.5. Now, we consider the real line. Consider the simplicial complex consisting of a 0-simplex $\langle n\rangle$ and 1-simplex $\langle n, n+1\rangle$ for every integer n. The chain group C_0 is generated by countably many vertices, so it is the direct sum of countably many copies of $\mathbb{Z}$. Hence, the kernel of ∂_0 is the direct sum of countably many copies of $\mathbb{Z}$. Since every 0-cycle is a finite sum, each can be written as a multiple of the 0-simplex $\langle 0\rangle$ plus finitely many 0-boundaries of the form $a\langle 0\rangle - a\langle n\rangle$ with a an integer. For example, the 0-cycle $5\langle 2\rangle - 3\langle 9\rangle$ can be written as $2\langle 0\rangle + \partial_1(5\langle 0,2\rangle - 3\langle 0,9\rangle) = 2\langle 0\rangle + 5\langle 2\rangle - 5\langle 0\rangle - 3\langle 9\rangle + 3\langle 0\rangle$. Hence there is exactly one equivalence class in $\ker \partial_0 / \mathrm{Im}\partial_1$ for each integer multiple of $\langle 0\rangle$, and these are the only classes. This gives us that $H_0(\mathbb{R})$ is isomorphic to a single copy of $\mathbb{Z}$. To compute $H_1(\mathbb{R})$ we need to determine $\ker \partial_1$. An element c of C_1 is of the form $\sum a_n\langle n, n+1\rangle$ with each a_n an integer and only finitely many a_n nonzero. Such an element c is mapped by ∂_1 to the 0-chain $\sum(a_n\langle n+1\rangle - a_n\langle n\rangle)$ or equivalently $\sum(a_{n-1} - a_n)\langle n\rangle$. Hence c is in the kernel of ∂_1 iff $a_{n-1} - a_n$ is always 0, which happens iff $a_{n-1} = a_n$ for all n. This can only happen when $a_n = 0$ for all n, since if N is the maximum index of the nonzero a_n, then the vertex $\langle N+1\rangle$ appears with coefficient $(a_N - a_{N+1})$ and a_{N+1} is necessarily 0. Thus, only the trivial 1-chain is in the kernel of ∂_1, and so $H_1(\mathbb{R})$ is the trivial group 0.

The ease of computation for simplicial homology comes at the cost of it only being defined for very nice spaces. Singular homology generalizes this theory to all topological spaces by generalizing the notion of a simplex to the image of a continuous map from some standard simplex into the space. This allows for the construction of a theory on arbitrary topological spaces, but forfeits the ease of computation. However, using singular homology, one can define an equivalent and much more computable theory for CW-complexes, a large class of spaces which includes simplicial complexes. In conjunction with excision, this enables computation of the singular homology groups for many spaces.

2.3. Singular Homology

Singular homology generalizes simplicial homology to arbitrary topological spaces, allowing us to use similar ideas even when we do not have a simplicial complex to work with. Here, instead of chain groups generated by the simplices of a space, the chain groups are generated by continuous maps from standard simplices into the space. These maps can be constant or self-intersecting; we only require continuity. There are many sources available for discussion of the singular theories; here we follow **[8]**, which includes the definitions and proofs of theorems mentioned below.

DEFINITION 2.3.1. Let G be an abelian group. A map $\sigma\colon \Delta_n \to X$ is called a **singular n-simplex**. The **n-th singular chain group** $C_n(X, G)$ of X with coefficients in G is the collection of finite formal sums with coefficients in G of singular n-simplices. Elements of $C_n(X, G)$ are called **singular n-chains**. When it is not ambiguous to do so, $C_n(X, G)$ is abbreviated by $C_n(X)$.

Note that if A is a subspace of X, then $C_n(A)$ is a subgroup of $C_n(X)$.

Before we define the boundary homomorphisms, note that any $\sigma : \Delta_n \to X$ has restrictions to the $(n-1)$-dimensional faces of Δ_n. We denote these restrictions by $\sigma \restriction \langle e_0, \dots, \widehat{e_i}, \dots, e_n\rangle$ where the hat symbol indicates that the marked entry has been removed from the list, and we view the restriction as a map from Δ_{n-1} by composing with a canonical embedding of Δ_{n-1} onto the face we are considering in Δ_n.

DEFINITION 2.3.2. For positive n, the **n-th singular boundary map** is the homomorphism $\partial_n\colon C_n \to C_{n-1}$ defined by $\partial_n(\sigma) = \sum_{i=0}^{n}(-1)^i\sigma \restriction \langle e_0, \dots, \widehat{e_i}, \dots, e_n\rangle$ for maps $\sigma\colon \Delta_n \to X$ and extended to all n-chains additively. We define ∂_n for $n \leq 0$ to be the 0 map.

EXAMPLE 2.3.3. Let X be any topological space. Consider a singular 2-simplex σ, which is a map from Δ_2 to X. The boundary of σ is the 1-chain

$$c = \sigma \restriction \langle e_1, e_2\rangle - \sigma \restriction \langle e_0, e_2\rangle + \sigma \restriction \langle e_0, e_1\rangle$$

and the boundary of c is

$$\sigma \restriction \langle e_2\rangle - \sigma \restriction \langle e_1\rangle - (\sigma \restriction \langle e_2\rangle - \sigma \restriction \langle e_0\rangle) + \sigma \restriction \langle e_1\rangle - \sigma \restriction \langle e_0\rangle$$

which reduces to 0 after cancellation.

EXAMPLE 2.3.4. Consider a 1-chain $\sigma_1 + \sigma_2 + \sigma_3$ where each σ_i is a distinct map $\Delta_1 \to X$. In this case we have that $\partial_1(\sigma_1 + \sigma_2 + \sigma_3)$ is equal to the 0-chain

$$\sigma_1 \restriction \langle e_1\rangle - \sigma_1 \restriction \langle e_0\rangle + \sigma_2 \restriction \langle e_1\rangle - \sigma_2 \restriction \langle e_0\rangle + \sigma_3 \restriction \langle e_1\rangle - \sigma_3 \restriction \langle e_0\rangle$$

which is zero only if the images of the maps form (possibly degenerate) loops, or if the maps are themselves loops, so that for example, $\sigma_1 \restriction \langle e_1\rangle$ is the same map as $\sigma_1 \restriction \langle e_0\rangle$.

It can be directly checked that $\partial_n \circ \partial_{n+1}$ is always 0 for the boundary maps defined above. Moreover, if A is a subspace of X, then $\partial\colon C_n(X) \to C_{n-1}(X)$ sends the subgroup $C_n(A)$ to $C_{n-1}(A)$. So, there are induced boundary maps from $C_n(X)/C_n(A)$ to $C_{n-1}(X)/C_{n-1}(A)$ which we refer to by the same notation.

DEFINITION 2.3.5. Let X be a topological space and let A be a subspace of X. The **n-th singular homology group** $H_n(X)$ is the quotient group $\ker \partial_n / \operatorname{im} \partial_{n+1}$ where $\partial_n\colon C_n(X) \to C_{n-1}(X)$. The **$n$-th relative singular homology group** $H_n(X, A)$ is the quotient group using $\partial_n\colon C_n(X)/C_n(A) \to C_{n-1}(X)/C_{n-1}(A)$.

Singular homology satisfies all of the Eilenberg-Steenrod axioms for continuous maps and pairs (X, A) with X a topological space and A a subspace. It is isomorphic to simplicial homology on simplicial complexes, cellular homology on CW-complexes. For certain spaces including CW-complexes, singular homology is isomorphic to homology with compact supports, as is recorded in 4.3.6.

2.4. Cellular Homology

Cellular homology is essentially a more convenient way of calculating the singular homology groups for a class of spaces called CW-complexes. These spaces decompose nicely into unions of spaces with easily calculated singular homology groups. The singular groups are taken as chain groups for a new, more easily computable homology theory, and then the two theories are shown to be isomorphic. In section 5.1, we record a similar construction for homology based on infinite chains. The exposition that follows is again based on that of [**8**], which includes the definitions, theorems, and proofs below.

DEFINITION 2.4.1. A **CW-complex** is a space X which can be written as a union $X = \bigcup X^n$ of spaces $(X^0, X^1, \dots)$ such that the following hold.

(a) The set X^0 is a discrete set whose points are called **0-cells**.
(b) The set X^n is formed from the set X^{n-1} by attaching open n-disks referred to as **1-cells**. That is, X^n is the quotient space of the disjoint union $X^{n-1} \bigsqcup_\alpha D^n_\alpha$ of X^{n-1} with a collection of closed n-disks under the identification $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$, where ϕ_α is a map $S^{n-1} \to X^{n-1}$ for each α.
(c) If $X \neq X^n$ for any finite n, then X is given the weak topology: $A \subseteq X$ is open iff $A \cap X^n$ is open in X^n for all n, and $B \subseteq X$ is closed iff $B \cap X^n$ is closed in X^n for all n.

If $X = X^n$ for some n, we say that X is **finite dimensional**, and the smallest such n is called the **dimension** of X.

DEFINITION 2.4.2. If X is a CW-complex and if A is a closed subset of X which can be written as a union of cells in X, then we say that A is a **subcomplex** and we call the pair (X, A) a **CW pair**.

If A is a subcomplex of X, then it is not hard to see that A is also a CW-complex. The following lemma allows us to easily compute the chain groups for this new theory and to demonstrate its equivalence with singular homology.

LEMMA 2.4.3. *If X is a CW-complex, then*

(a) *$H_k^s(X^n, X^{n-1})$ is 0 for $k \neq n$ and is free abelian for $k = n$ with basis in one-to-one correspondence with the n-cells of X.*
(b) *$H_k^s(X^n) = 0$ for $k > n$. In particular, if X has finite dimension n, then $H_k^s(X) = 0$ for $k > n$.*
(c) *The inclusion $i\colon X^n \to X$ induces an isomorphism $i_*\colon H_k^s(X^n) \to H_k^s(X)$ for $k < n$.*

DEFINITION 2.4.4. The **n-th dimensional cellular chain group** of X is $C_n^{CW}(X) = H_n^s(X^n, X^{n-1})$. Then n-th **boundary map** d_n is defined to be the composition $j_{n-1}\partial_n$, where ∂_n is the boundary operator from the long exact sequence of the pair (X^n, X^{n-1}) in singular homology and j_{n-1} is the map $H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ from the long exact sequence of the pair (X^{n-1}, X^{n-2}).

Since $H_n^s(X^n, X^{n-1})$ is a free abelian group with basis in one-to-one correspondence with the n-cells of X, the elements of $C_n^{CW}(X)$ can be thought of as linear combinations of n-cells of X. Since $d_n = j_{n-1}\partial_n$, it is clear that $d_{n-1}d_n = 0$ for all n.

DEFINITION 2.4.5. The **n-th dimensional cellular homology group** of X is $H_n^{CW}(X) = \ker d_n / \operatorname{im} d_{n+1}$.

As mentioned at the beginning of this section, the benefit of this construction is that it provides an easier method for calculating singular homology groups. This is because H^{CW} coincides with H^s.

PROPOSITION 2.4.6. *If X is a CW-complex, then $H_n^{CW}(X) \cong H_n^s(X)$ for all n.*

PROOF. By the above lemma, $H_n^s(X^{n+1}) \cong H_n^s(X)$ and $H_{n-1}^s(X^{n+1}, X^n) = 0$, so the long exact sequence of the pair (X^{n+1}, X^n) contains the exact sequence

$$H_{n+1}^s(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n^s(X^n) \to H_n^s(X) \to 0$$

Thus, $H_n^s(X) \cong H_n^s(X^n)/\operatorname{im}\partial_{n+1}$.

Also by the above lemma, $H_n^s(X^{n-1}) = 0$ and so the long exact sequence of the pair (X^n, X^{n-1}) contains the exact sequence

$$0 \to H_n^s(X^n) \xrightarrow{j_n} H_n^s(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}^s(X^{n-1})$$

Hence, j_n is injective, and so it maps $\operatorname{im}\partial_{n+1}$ isomorphically onto $\operatorname{im} j_n\partial_{n+1} = \operatorname{im} d_{n+1}$, and it maps $H_n^s(X^n)$ isomorphically onto $\operatorname{im} j_n = \ker\partial_n$.

Similarly, the long exact sequence of the pair (X^{n-1}, X^{n-2}) contains the exact sequence

$$0 \to H_{n-1}^s(X^{n-1}) \overset{j_{n-1}}{\to} H_{n-1}^s(X^{n-1}, X^{n-2})$$

Hence j_{n-1} is injective as well, and so $\ker\partial_n = \ker d_n$. By exactness, we have $\operatorname{im}\partial_{n+1} = \operatorname{im} d_{n+1}$.

It follows that j_n induces an isomorphism of $H_n^s(X^n)/\operatorname{im}\partial_{n+1}$ onto the homology group $\ker d_n/\operatorname{im} d_{n+1}$. □

2.5. Steenrod Homology

Steenrod homology was introduced in **[23]** in order to provide topological invariants which better captured the connectivity of a space than earlier theories. In particular, Steenrod notes that the earlier Vietoris homology is not satisfactory for solenoids, which are connected but neither locally connected nor path connected. Another notable example is the topologist's sine curve:

EXAMPLE 2.5.1. Let T be the subspace of $\mathbb{R}^2$ which is the disjoint union of the graph of $\sin(1/x)$ on $(0, 1]$, its limit set along on the y-axis $\{0\} \times [-1, 1]$, and a path from $(0, -1)$ to $(1, \sin(1/1)) = (1, 0)$. The space T is connected and path connected, but it is not locally path connected. The singular homology of T is trivial in the first dimension, despite the fact that T separates the plane into two connected components.

The duality results that existed at the time failed for the topologist's sine curve because they predicted that the first dimensional homology should be that of a circle, as one might expect since it encloses a disk. Correcting this shortcoming of earlier homology theories allowed Steenrod to extend duality results to a wider class of spaces.

Steenrod defined his homology groups for compact metric pairs.

DEFINITION 2.5.2. A **compact metric pair** is a pair (X, A) where X is a compact metric space and A is closed in X.

DEFINITION 2.5.3. Given a simplicial complex K, **a regular map** of K in X is defined to be a function f from the set of vertices of K to X such that for every $\varepsilon > 0$, all but finitely many simplices have vertices mapped into sets of diameter less than ε.

DEFINITION 2.5.4. A **regular n-chain** of X is a triplet (A, f, c) where A is a simplicial complex, f is a regular map of A in X, and c is a locally finite n-chain of A. If c is a locally finite n-cycle, then (A, f, c) is called a regular n-cycle.

DEFINITION 2.5.5. Two regular n-cycles (A_1, f_1, c_1) and (A_2, f_2, c_2) are defined to be **homologous** if there is a regular $(n+1)$-chain (A, f, c) such that A_1 and A_2 are closed subcomplexes of A, the regular map f agrees with f_1 on A_1 and with f_2 on A_2, and $\partial_{n+1}(c) = c_1 - c_2$.

The relation of being homologous is an equivalence relation on the set of n-cycles.

DEFINITION 2.5.6. Let X be a compact metric space. The **n-th Steenrod homology group** $H_n^{st}(X)$ is the quotient group obtained by taking the group of regular n-chains modulo the relation of being homologous.

On page 90 of **[15]**, Milnor gives a construction extending this definition to compact metric pairs (X, A). It is shown that these groups are related to Čech cohomology groups $\check{H}^q(X, A)$ by the split exact sequence

$$0 \to \mathrm{Ext}(\check{H}^{q+1}(X, A); G) \to H_n^{st}(X, A; G) \to \mathrm{Hom}(\check{H}^q(X, A); G) \to 0$$

Steenrod also gave a construction in **[23]** of his groups based on nerves of open coverings. Since X is compact, a Čech system for X can be found which consists of finite coverings. Steenrod uses the sequence of nerves $\mathcal{K}_n$ associated to such a system to define a fundamental complex $\mathcal{K}$, which is the disjoint union of the $\mathcal{K}_n$ with line segments added between points which correspond under the maps induced by refinement. He then shows that computing homology groups of $\mathcal{K}$ using infinite simplicial n-chains gives groups isomorphic to $H_n^{st}(X)$.

In **[15]**, Milnor shows that Steenrod homology satisfies all of the Eilenbger-Steenrod axioms for continuous maps and compact metric pairs (X, A). Furthermore, it is shown that H^{st} satisfies an additional two axioms: the relative homeomorphism axiom and the cluster axiom.

THEOREM 2.5.7. *Let (X, A) be a compact metric pair.*

(a) *H^{st} satisfies all Eilenberg-Steenrod axioms for continuous maps and compact metric pairs (X, A).*

(b) ***(Relative Homeomorphism Axiom)** If $f\colon (X, A) \to (Y, B)$ is a continuous map of compact metric pairs which maps $X - A$ homeomorphically onto $Y - B$, then $f_*\colon H_q^{st}(X, A) \to H_q^{st}(Y, B)$ is an isomorphism.*

(c) ***(Cluster Axiom)** Suppose that X is the union of compact subsets $X_1, X_2, \ldots$ with diameters approaching 0, and suppose that $X_i \cap X_j = \{b\}$ for all $i \neq j$. Let $r_i\colon (X, b) \to (X_i, b)$ denote the unique retraction which carries each X_j for $j \neq i$ into the base point b. Then the map $u \mapsto ((r_1)_*(u), (r_2)_*(u), \ldots)$ is an isomorphism from $H_n^{st}(X, b)$ onto the direct product of the groups $H_n^{st}(X_i, b)$.*

Furthermore, Milnor proves that these two additional axioms together with the Eilenberg-Steenrod axioms characterize Steenrod homology.

THEOREM 2.5.8. *If H and H' are two homology theories defined for continuous maps and compact metric pairs, and if both H and H' satisfy the Eilenberg-Steenrod axioms along with the relative homeomorphism axiom and cluster axiom above, then any coefficient isomorphism $H_0(b) \cong H_0'(b)$ extends to an equivalence between H and H'.*

This theorem can be applied to show that H^{st} coincides with the theories H^c and H^∞ of 4.3 and 4.2 respectively for compact metric pairs. For this reason, and since H^∞ satisfies similar duality results as H^{st}, the theory H^∞ can be viewed as a generalization of Steenrod homology.

2.6. Borel-Moore Homology

Borel-Moore homology was originally defined in [**2**] for locally compact spaces in order to obtain a Poincaré duality result (section 7 of [**2**]). It is a theory based on infinite chains with closed supports and is defined for locally compact spaces. Proper maps induce homomorphisms on the homology groups.

The Poincaré duality obtained is with cohomology with compact supports. Since homology based on infinite chains also satisfies this Poincaré duality, it is clear that these theories are isomorphic at least in some cases. This isomorphism can be shown to hold for any locally compact Hausdorff space which is second-countable by applying uniqueness results due to Milnor [**15**].

PROPOSITION 2.6.1. *If X is a locally compact, second-countable, Hausdorff space, then $H_n^{BM}(X,G) \cong H_n^\infty(X,G)$ for all n, where H^∞ is homology based on infinite chains, as defined in section 4.2 below.*

The definition of Borel-Moore homology in [**2**] is given sheaf-theoretically, but there are also standard constructions of Borel-Moore homology given in terms of generalized singular chains. This singular Borel-Moore homology construction uses infinite sums of singular simplices so long as they are locally finite.

DEFINITION 2.6.2. The **group of generalized singular n-chains** on X is $C_n(X,G) = \prod_{\sigma^n : \Delta^n \to X} G$, and its elements are called **generalized singular n-chains**.

A generalized singular n-chain $\prod_{\sigma^n} k_{\sigma^n}$ is **locally finite** if every $x \in X$ has a neighborhood which intersects only finitely many of the images $\sigma^n(\Delta^n)$ where $k_{\sigma^n} \neq 0$.

It can be checked that the locally finite generalized singular n-chains form a subgroup.

DEFINITION 2.6.3. The n-**th Borel-Moore chain group** on X is denoted $C_n^{BM}(X, G)$ and defined to be the group of locally finite generalized singular n-chains on X.

Moreover, the usual singular boundary map extends to a boundary map ∂^{BM} on Borel-Moore chains, and continues to satisfy $\partial^{BM}\partial^{BM} = 0$. Thus, we have a chain complex.

DEFINITION 2.6.4. The n-**th Borel-Moore homology group** is defined to be $H_n^{BM}(X, G) = \ker \partial_n^{BM} / \operatorname{im} \partial_{n+1}^{BM}$.

Massey notes in **[14]** that such a theory only satisfies a weak version of the excision axiom, and so is less satisfactory than homology based on infinite chains as in 4.2.

CHAPTER 3

Cohomology

3.1. Eilenberg-Steenrod Axioms for Cohomology

Now we will list the axioms for cohomology theories. The definition and axioms are very similar to those of homology theories; the difference is that cohomology theories assign homomorphisms with directions reversed from those assigned by homology theories, and the operators here increase indices rather than decrease them. Again, these axioms, definitions, and properties were stated by Eilenberg and Steenrod in **[6]**.

DEFINITION 3.1.1. Let $\mathcal{G}$ be a collection either of abelian groups or of R-modules for some fixed ring R. A **cohomology theory** H on an admissible category $\mathcal{C}$ is a collection of functions as follows.

- The first function H is defined for each admissible pair (X, A) and each integer q and assigns values in $\mathcal{G}$. The value of the function is usually written $H^q(X, A)$ and is called the **q-dimensional relative cohomology group** of X modulo A. If A is the empty set, then $H^q(X, A)$ is often abbreviated as $H^q(X)$.
- The second function is defined for each admissible map $f\colon (X, A) \to (Y, B)$ and each integer q and assigns a homomorphism $f^{*q}\colon H^q(Y, B) \to H^q(X, A)$ called the **homomorphism induced by** f. The homomorphism f^{*q} is typically written as f^* when it is not ambiguous to do so.
- The third function δ is defined for each admissible (X, A) and each integer q and assigns a homomorphism $\delta(q, X, A)$ from $H^q(A, \emptyset)$ to $H^{q+1}(X, A)$ called the **boundary operator**. This homomorphism is typically written as δ when it is not ambiguous to do so.

The first two functions above are required to be **functorial**:

Axiom 1: If f is the identity map $(X, A) \to (X, A)$, then f^* is the identity map $H_q(X, A) \to H_q(X, A)$ for each q.

Axiom 2: If $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (Z, C)$ are admissible, then $(gf)^* = f^*g^*\colon H_q(Z, C) \to H_q(X, A)$.

The third function must behave well with the first two:

Axiom 3: If $f\colon (X, A) \to (Y, B)$ is admissible, then the map $f^*\delta$ is $\delta(f\restriction_A)^*\colon H^q(B, \emptyset) \to H^{q+1}(X, A)$.

Additionally, the following axioms must be satisfied:

Axiom 4 (Exactness): If (X, A) is admissible and if $i\colon (A, \emptyset) \to (X, \emptyset)$ and $j\colon (X, \emptyset) \to (X, A)$ are inclusion maps, then the following sequence is exact.

$$\cdots \xleftarrow{\delta} H^q(A) \xleftarrow{i_*} H^q(X) \xleftarrow{j_*} H^q(X, A) \xleftarrow{\delta} H^{q-1}(A) \xleftarrow{i_*} \cdots$$

This sequence is called the **cohomology sequence** of the pair (X, A).

Axiom 5 (Homotopy Invariance): If the admissible maps f_0, f_1 from (X, A) to (Y, B) are $\mathcal{C}$-homotopic, then for each q, the homomorphisms $f_0^*, f_1^*\colon H^q(Y, B) \to H^q(X, A)$ are equal.

Axiom 6 (Excision): Let (X, A) be an admissible pair. If U is an open subset of X whose closure $\bar{U}$ is contained in the interior of A, and if the inclusion map $i\colon (X - U, A - U) \to (X, A)$ is admissible, then i induces an isomorphism $i_*\colon H^q(X, A) \to H^q(X - U, A - U)$ for all q. Any inclusion map i satisfying these conditions is called an **excision map**.

Axiom 7 (Dimension): If P is an admissible space consisting of a single point, then $H_q(P) = 0$ for all $q \neq 0$. The value of $H_0(P)$ is called the **coefficient group** or **coefficient module** of the homology theory, depending on whether $\mathcal{G}$ is a family of abelian groups or modules.

PROPOSITION 3.1.2. *A $\mathcal{C}$-isomorphism f induces an isomorphism f^* for all q.*

PROPOSITION 3.1.3. *If $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (X, Y)$ are $\mathcal{C}$-homotopy inverses, then $f^*\colon H^q(Y, B) \to H^q(X, A)$ is an isomorphism with inverse g^*.*

PROPOSITION 3.1.4 (Direct Sum Property for Cohomology). *Under the conditions above, the induced homomorphisms $i_\alpha^*\colon H^q(X, A) \to H^q(X_\alpha, A_\alpha)$ yield a projective representation of $H^q(X, A)$ as a direct sum. That is, for each sequence $(u_1, \ldots, u_n)$ in $H^q(X_1, A_1) \times \cdots \times H^q(X_n, A_n)$, there is a unique element $u \in H^q(X, A)$ such that $i_\alpha^*(u) = u_\alpha$ for each $\alpha = 1, \ldots, n$.*

PROPOSITION 3.1.5 (Mayer-Vietoris Cohomology Sequence). *Let $(X; X_1, X_2)$ be an excisive triad with $X = X_1 \cup X_2$. Then the* ***Mayer-Vietoris cohomology sequence***

$$\cdots \leftarrow H^q(X_1 \cap X_2) \xleftarrow{\psi} H^q(X_1) \oplus H^q(X_2) \xleftarrow{\phi} H^q(X) \xleftarrow{\Delta} H^{q-1}(X_1 \cap X_2) \leftarrow \cdots$$

is exact, where ψ, ϕ, Δ are defined by

$$\begin{aligned}\psi(v_1, v_2) &= h_1^*(v_1) - h_2^*(v_2)\\ \phi &= (m_1^*, m_2^*)\\ \Delta &= -l_1^* k_1^{*-1} \delta\end{aligned}$$

with the maps $h_1, h_2, m_1, m_2, l_1, k_1$ as above and coboundary operator

$$\delta\colon H^{q-1}(X_1 \cap X_2) \to H^q(X_2, X_1 \cap X_2)$$

PROPOSITION 3.1.6 (Relative Mayer-Vietoris Sequence). *Let $(X; A, B)$ be an excisive triad. Then the **relative Mayer-Vietoris sequence***

$$\xleftarrow{\phi} H^q(X, A \cup B) \xleftarrow{\Delta} H^{q-1}(X, A \cap B) \leftarrow \cdots$$
$$\cdots \leftarrow H^q(X, A \cap B) \xleftarrow{\psi} H^q(X, A) \oplus H^q(X, B)$$

is exact.

3.2. Cohomology with Compact Supports

Cohomology with compact supports is defined for locally compact Hausdorff spaces. It produces cohomology groups with induced homomorphisms defined for proper continuous maps. These groups are defined using the notion of p-functions and their supports. We follow the development in [**14**], which includes the definitions, theorems, and proofs below.

DEFINITION 3.2.1. A function $f\colon X \to Y$ is **proper** if the preimage of every compact subset of Y is compact in X.

DEFINITION 3.2.2. Let X be a space, let p be a nonnegative integer, and let G be an abelian group. A **p-function** on X with values in G is any function $\phi\colon X^{p+1} \to G$ where X^{p+1} is the cartesian product of $p+1$ copies of X. We will denote the set of all such p-functions by $\Phi^p(X, G)$ or Φ^p when it is not ambiguous to do so.

DEFINITION 3.2.3. A p-function ϕ is **finitely valued** if the image of ϕ is a finite set. We denote the set of finitely valued p-functions on X with values in G by $\Phi^p_F(X, G)$ or Φ^p_F. If X is empty, we define $\Phi^p_F(X, G) = \{0\}$ for all p.

The set $\Phi^p(X, G)$ is an abelian group under pointwise addition, and $\Phi^p_F(X, G)$ is a subgroup. There is a particular homomorphism between each Φ^p and Φ^{p+1} that is important for our purposes.

DEFINITION 3.2.4. For each p, let $d^p\colon \Phi^p(X, G) \to \Phi^{p+1}(X, G)$ be defined by

$$d^p\phi(x_0, \ldots, x_p) = \sum_{i=0}^{p+1} (-1)^i \phi(x_0, \ldots, \widehat{x_i}, \ldots, x_{p+1})$$

where the hat symbol indicates removal of the term.

The fact that $d^{p+1}d^p = 0$ can be checked directly in a similar way to that of the boundary operators of other theories. It can also be shown that d^p preserves the subgroup of finitely-valued p-functions, and so it restricts to a homomorphism $\Phi^p_F \to \Phi^{p+1}_F$.

To get the cochain groups for our current theory, we will take certain subgroups and quotient groups of the group of finitely valued p-functions. The definition of these groups uses the notion of the support of a p-function.

DEFINITION 3.2.5. The **support of a p-function** $\phi\colon X^{p+1} \to G$ is the set $|\phi|$ consisting of all $x \in X$ such that every neighborhood of x contains elements $x_0, \dots, x_p$ for which $\phi(x_0, \dots, x_p)$ is nonzero. Equivalently, x is not in $|\phi|$ iff there is a neighborhood V of x for which $\phi(x_0, \dots, x_p) = 0$ for all $x_0, \dots, x_p \in V$.

The following properties of $|\phi|$ are easy to see.

PROPOSITION 3.2.6. *Let $\phi\colon X^{p+1} \to G$ be a p-function.*

(a) *$|\phi|$ is closed in X,*
(b) *$|0|$ is empty,*
(c) *$|\phi + \psi|$ is a subset of $|\phi| \cup |\psi|$, and*
(d) *$|d^p\phi|$ is a subset of $|\phi|$.*
(e) *$|\phi|$ is empty if and only if there is an open covering $\mathcal{U}$ of X such that for all $U \in \mathcal{U}$ and all $x_0, \dots, x_p \in U$, $\phi(x_0, \dots, x_p) = 0$.*

DEFINITION 3.2.7. Let $\Phi^p_F(X, G)$ be a group of finitely valued p-functions. The **subgroup of p-functions with empty support** is $\Phi^p_{F0}(X, G) = \{\phi \in \Phi^p_F(X, G) \mid |\phi| \text{ is empty}\}$, and the **subgroup of p-functions with compact support** is $\Phi^p_{FC}(X, G) = \{\phi \in \Phi^p_F(X, G) \mid |\phi| \text{ is compact}\}$.

It follows from property (c) above that the above defined sets are actually subgroups, and that $\Phi^p_{F0}(X, G)$ is a subgroup of $\Phi^p_{FC}(X, G)$. It follows from (d) that d^p maps Φ^p_{F0} into Φ^{p+1}_{F0} and maps Φ^p_{FC} into Φ^{p+1}_{FC}. Thus, we may define quotient groups and induced homomorphisms.

DEFINITION 3.2.8. The **p-cochain group with compact support** $C^p_c(X, G)$ is the quotient group $\Phi^p_{FC}(X, G)/\Phi^p_{F0}(X, G)$. We define the **$p$-th relative cochain group with compact support** $C^p_c(X, A, G)$ for A a closed subset of X by first defining $\Phi^p_{FC}(X, A, G)$ to be the subgroup of compactly supported finitely valued p-cochains of X whose restrictions to A^{p+1} have empty support. Then $C^p_c(X, A, G) = \Phi^p_{FC}(X, A, G)/\Phi^p_{F0}(X, G)$. When it is not ambiguous to do so, we write $C^p_c(X)$ and $C^p_c(X, A)$ for these groups.

It follows from property (c) that if ϕ and ψ are two elements of the same equivalence class in $C^p_c(X)$, then they have the same support. So we may define the support of a p-cochain in the following way.

DEFINITION 3.2.9. Let $\bar{\phi} \in C^p_c(X)$ and let ϕ be a representative of $\bar{\phi}$. We define the support $|\bar{\phi}|$ of $\bar{\phi}$ by $|\bar{\phi}| = |\phi|$.

DEFINITION 3.2.10. The **p-th coboundary homomorphism** δ^p from $C^p_c(X, G)$ to $C^{p+1}_c(X, G)$ is the homomorphism induced from d^p when it is restricted to $\Phi^p_{FC}(X, G)$ and composed with the quotient homomorphism.

We have that $\delta^{p+1}\delta^p$ is 0 since $d^{p+1}d^p$ is 0, so the collection of p-cochain groups with compact supports forms a cochain complex. Moreover, if A is a closed subset of X, then each δ^p restricts to a homomorphism $C^p_c(X, A) \to C^{p+1}_c(X, A)$ satisfying the same properties, so the collection of these subgroups also forms a cochain complex.

DEFINITION 3.2.11. The **p-th cohomology group (of X with coefficients in G) with compact supports** is $H^p_c(X, G) = \ker \delta^p / \operatorname{im} \delta^{p-1}$. If A is closed in X, we define the relative group $H^p_c(X, A, G)$ similarly. When it is not ambiguous, we will write $H^p_c(X)$ and $H^p_c(X, A)$.

THEOREM 3.2.12. *If A is a closed subset of X, then the group $H^p_c(X, A, G)$ is isomorphic to the group $H^p_c(X - A, G)$.*

Continuous, proper maps $f\colon X \to Y$ induce homomorphisms $H^p_c(Y, G) \to H^p_c(X, G)$ for all p. Continuity is necessary to ensure preservation of empty supports, and properness is required to ensure preservation of compact supports.

DEFINITION 3.2.13. Let $f\colon X \to Y$ be a continuous, proper map. Then $f^{\#}\colon \Phi^p(Y, G) \to \Phi^p(X, G)$ is the homomorphism defined by sending ϕ in $\Phi^p(Y, G)$ to the map $f^{\#}\phi$ defined by $(f^{\#}\phi)(x_0, \ldots, x_p)) = \phi(f(x_0), \ldots, f(x_p))$. We define $f^*\colon H^p_c(Y, G) \to H^p_c(X, G)$ to be the map induced by $f^{\#}$.

If U is a nonempty open subset of X, then the inclusion map $i\colon U \to X$ defines a homomorphism from $H^p_c(U)$ to $H^p_c(X)$ as follows. Let $Q^p(U)$ be the set $\{\phi \in C^p(X) \mid |\phi| \subseteq U\}$ of p-cochains of X whose support is contained in U. The map i is not necessarily proper, but it induces an isomorphism $i^{\#}\colon Q^p(U) \to C^p_c(U)$. Now, in the diagram

$$\begin{tikzcd} Q^p(U) \arrow[r, "j"] \arrow[d, "i^{\#}"'] & C^p_c(X) \\ C^p_c(U) \arrow[ur, "\sigma_{U,X}"'] & \end{tikzcd}$$

where j is the inclusion map, the only map $\sigma_{U,X}$ which makes the diagram commute is given by $\sigma_{U,X} = j(i^{\#})^{-1}$. This map $\sigma_{U,X}$ commutes with the coboundary homomorphism, and so it induces a homomorphism on the cohomology groups.

DEFINITION 3.2.14. The map $\tau_{U,X}\colon H^p_c(U) \to H^p_c(X)$ is the homomorphism induced by the map $\sigma_{U,X}$ in the above diagram.

Stating the properties of of this cohomology theory requires highlighting one more homomorphism. The construction of this homomorphism uses the fact that the sequence of cochain complexes

$$0 \to C_c(X, A) \xrightarrow{j} C_c(X) \xrightarrow{i^{\#}} C_c(A) \to 0$$

is short exact, where j is inclusion and $i^{\#}$ is induced by the continuous, proper inclusion map of A into X. By identifying the cohomology groups of the complex $C_c(X, A)$ with the cohomology groups of the complex $C_c(U)$, it can be shown that this sequence leads to a long exact sequence of cohomology groups.

$$\cdots \to H_c^p(U) \xrightarrow{j^*} H_c^p(X) \xrightarrow{i^*} H_c^p(A) \xrightarrow{\delta} H_c^{p+1}(U) \to \cdots$$

The homomorphism δ is the map we wish to name.

DEFINITION 3.2.15. If U is an open subset of X and $A = X - U$, we denote by $\delta_{X,A} \colon H_c^p(A) \to H_c^p(U)$ the homomorphism which appears in the long exact sequence above.

The following theorem summarizes the properties of cohomology with compact supports.

THEOREM 3.2.16. *Let X, Y, and Z be locally compact Hausdorff spaces, and let G be an abelian group.*

(a) *H_c satisfies all Eilenberg-Steenrod axioms for continuous, proper maps and pairs (X, A) with X locally compact Hausdorff and A closed in X.*
(b) *If X is the space consisting of a single point, then $H_c^0(X, G)$ is G.*
(c) *$\tau_{X,X} \colon H_c^p(X) \to H_c^p(X)$ is the identity map.*
(d) *If U and V are open subsets of X satisfying $V \subseteq U \subseteq X$, then $\tau_{V,X} = \tau_{U,X}\tau_{V,U}$.*
(e) *If U and V are open subsets of X and Y, and if $f \colon X \to Y$ is a continuous, proper map such that $f(U) \subseteq V$ and $f(X - U) \subseteq Y - V$, then the following diagram is commutative.*

$$\begin{tikzcd}
H_c^p(Y) \arrow[r, "f^*"] & H_c^p(X) \\
H_c^p(V) \arrow[u, "\tau_{V,X}"] \arrow[r, "(f\restriction_U)^*"] & H_c^p(U) \arrow[u, "\tau_{U,X}"']
\end{tikzcd}$$

(f) *Suppose U and V are open subsets of X and Y. Let $f \colon X \to Y$ be a continuous, proper map such that $f(U) \subseteq V$ and $f(X - U) \subseteq Y - V$. Let $f_1 \colon A \to B$ and $f_2 \colon U \to V$ be the maps induced by restriction of f. Then the following diagram is commutative.*

$$\begin{tikzcd}
H_c^p(V) \arrow[r, "(f_2)^*"] & H_c^p(U) \\
H_c^{p-1}(B) \arrow[u, "\delta"] \arrow[r, "(f_1)^*"] & H_c^{p-1}(A) \arrow[u, "\delta"']
\end{tikzcd}$$

(g) *If A is a closed subset of X and $U = X - A$, then the following sequence is exact:*

$$\cdots \to H_c^p(U) \xrightarrow{\tau} H_c^p(X) \xrightarrow{i^*} H_c^p(A) \xrightarrow{\delta} H_c^{p+1}(U) \to \cdots$$

(h) *Let U and V be open subsets of X such that $V \subseteq U$. Let $A = X - U$, let $B = X - V$, and let $i\colon A \to B$ be the inclusion map. Then the following diagram is commutative.*

$$\begin{tikzcd}
H_c^p(V) \arrow[r, "\tau_{V,U}"] & H_c^p(U) \\
H_c^{p-1}(B) \arrow[u, "\delta_{X,B}"] \arrow[r, "i^*"] & H_c^{p-1}(A) \arrow[u, "\delta_{X,A}"']
\end{tikzcd}$$

(i) *Let U and V be open subsets of X such that $U \subseteq V$, and let $A = X - U$. Then the following diagram is commutative.*

$$\begin{tikzcd}
H_c^p(V \cap A) \arrow[dd, "\tau"'] \arrow[dr, "\delta"] & \\
& H_c^{p+1}(U) \\
H_c^p(A) \arrow[ur, "\delta"'] &
\end{tikzcd}$$

(j) *If X is the the disjoint union of open subsets X_i indexed by an arbitrary set I, then each $\tau_{X_i,X}\colon H_c^p(X_i) \to H_c^p(X)$ is injective, and $H_c^p(X)$ is the direct sum of their images.*

(k) *$H_c^p(X)$ is the direct limit of the groups $H_c^p(U)$ where U is an open subset of X with compact closure.*

(l) *If A and B are closed subsets of X such that $X = A \cup B$, then the Mayer-Vietoris sequence*

$$\cdots \to H_c^p(X) \to H_c^p(A) \oplus H_c^p(B) \to H_c^p(A \cap B) \to H_c^{p+1}(X) \to \cdots$$

is exact.

(m) *If A and B are open subsets of X such that $X = A \cup B$, then the Mayer-Vietoris sequence*

$$\cdots \to H_c^p(A \cap B) \to H_c^p(A) \oplus H_c^p(B) \to H_c^p(X) \to H_c^{p+1}(A \cap B) \to \cdots$$

is exact.

(n) *If X is not compact and $\dot{X}$ is the Alexandroff one-point compactification of X, then the inclusion $X \to \dot{X}$ induces an isomorphism between $H_c^q(X, G)$ and the reduced homology $\tilde{H}_c^q(\dot{X}, G)$.*

(o) *For any closed A in X, if $\{N_\alpha, i_\alpha\}$ is the collection of closed neighborhoods of A and inclusions $i_\alpha \colon A \to N_\alpha$, then $\{i^*_\alpha\}$ provides a representation of $H^q(A)$ as the direct limit of the system $\{H^q_c(N), i^*_{N_2,N_1}\}$ where the maps $i^*_{N_2,N_1}$ are induced by the inclusions $i_{N_1,N_2} \colon N_1 \to N_2$.*

PROPOSITION 3.2.17. *The cohomology groups of $\mathbb{R}^n$ for $n \geq 0$ and S^n for $n \geq 1$ are given by*

$$H^q(\mathbb{R}^n, G) = \begin{cases} G & q = n \\ 0 & q \neq n \end{cases}$$

$$H^q(S^n, G) = \begin{cases} G & q = 0, n \\ 0 & q \neq 0, n \end{cases}$$

PROOF. The proof is given by induction on the dimension of $\mathbb{R}^n$. The result holds easily for $\mathbb{R}^0 = \{0\}$, so we proceed to the inductive step.

Let $R^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$ be a half-space in $\mathbb{R}^n$. The one-point compactification $\dot{R}^n_+$ is homeomorphic to a closed n-disc, and any closed n-disc is contractible to a point. So $H^q(R^n_+, G) = 0$ for all q. Now, consider the following long exact sequence, which is the cohomology sequence of the pair (R^n_+, R^{n-1}) where $R^{n-1} \cong \mathbb{R}^{n-1}$ is $\{(0, x_2, \ldots, x_n) \in \mathbb{R}^n\}$.

$$\cdots \to H^q_c(R^n_+) \to H^q_c(R^{n-1}) \xrightarrow{\delta} H^{q+1}_c(R^n_+ - R^{n-1}) \to H^{q+1}_c(R^n_+) \cdots$$

Since $H^q(R^n_+, G) = 0$ for all q, this is

$$\cdots \to 0 \to H^q_c(R^{n-1}) \xrightarrow{\delta} H^{q+1}_c(R^n_+ - R^{n-1}) \to 0 \to \cdots$$

Thus, the coboundary operator δ is an isomorphism at each q. Now, note that $R^n_+ - R^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$ is isomorphic to $\mathbb{R}^n$. Thus $H^q_c(\mathbb{R}^{n-1}) \cong H^q_c(R^{n-1}) \cong H^{q+1}_c(\mathbb{R}^n)$. This completes the inductive step.

The result for S^n follows since S^n is homeomorphic to the one-point compactification of $\mathbb{R}^n$. $\square$

3.3. Alexander-Spanier Cohomology

The ideas involved in the definition of cohomology with compact support can be used to define cohomology groups for all spaces. This is done by using groups of arbitrary p-functions, or groups of locally finitely valued p-functions, rather than restricting ourselves to finitely valued p-functions as was done previously. Let X be a topological space and let G be an abelian group. We will use the definition and properties of arbitrary p-functions as in section 3.2 in addition to the following definitions. We include the development in Massey **[14]**, though Spanier's own text **[22]** covers this content.

DEFINITION 3.3.1. A p-function $\phi\colon X^{p+1} \to G$ is **locally finitely valued** if every $(p+1)$-tuple $(x_0, \ldots, x_p) \in X^{p+1}$ has a neighborhood U in the product topology of X^{p+1} such that $\phi(U)$ is a finite set. We denote the set of locally finite valued p-functions by $\Phi^p_L(X, G)$ or Φ^p_L.

The group of locally finitely valued p-function is a subgroup of the group of arbitrary valued ones. The homomorphisms d^p defined in section 3.2 preserve locally finitely valued p-functions; that is, d^p maps $\Phi^p_L(X, G)$ into $\Phi^{p+1}_L(X, G)$. So, we may define the following cochain groups and induced coboundary homomorphisms.

DEFINITION 3.3.2. We define $\Phi^p_0(X, G)$ to be the subgroup of p-functions with empty support, and we define $\Phi^p_{L0}(X, G) = \Phi^p_L \cap \Phi^p_0$ to be the subgroup of locally finitely valued p-functions with empty support.

DEFINITION 3.3.3. The **p-th Alexander-Spanier cochain group** $C^p_\infty(X, G)$ is the quotient group $\Phi^p(X, G)/\Phi^p_0(X, G)$. The **$p$-th Alexander-Spanier cochain group based on locally finitely valued cochains** $C^p_L(X, G)$ is the quotient group $\Phi^p_L(X, G)/\Phi^p_{L0}(X, G)$.

DEFINITION 3.3.4. The **p-th coboundary homomorphism** δ^p is the homomorphism induced by d^p. We will abuse terminology and use this same name and notation for the induced maps on C^p_∞ and C^p_L

The coboundary homomorphisms satisfy $\delta^{p+1}\delta^p = 0$ in both cases, so we may define cohomology groups.

DEFINITION 3.3.5. The **p-th Alexander-Spanier cohomology group** is $H^p_\infty(X, G) = \ker \delta^p / \operatorname{im} \delta^{p-1}$ where the δ maps are the ones in the complex of arbitrary cochain groups. The **p-th Alexander-Spanier cohomology group based on locally finitely valued cochains** is $H^p_L(X, G) = \ker \delta^p / \operatorname{im} \delta^{p-1}$ where the δ maps are those in the complex of locally finitely valued cochain groups.

PROPOSITION 3.3.6. *If X is paracompact and Hausdorff, then the inclusion maps $C^p_L \to C^p_\infty$ induce isomorphisms between the cohomology groups $H^p_L(X, G)$ and $H^p_\infty(X, G)$.*

As Massey notes, the above isomorphism indicates that for most spaces considered, there is no essential difference between the cohomology groups. The value in having the separate constructions is mostly derived from the increased ease in proving properties. In some cases, the simple generality of the arbitrary valued cochain construction provides easy groups to work with, in other cases, the locally finitely valued construction lends itself to the task more readily.

Continuous maps $f\colon X \to Y$ induce homomorphisms between cohomology groups in this theory. We use the same definition of $f^\#$ as in definition 3.2.13, and note that continuity ensures that $f^{\#}$ maps $\Phi^p_L(Y)$ into $\Phi^p_L(X)$ and $\Phi^p_0(Y)$ into $\Phi^p_0(X)$.

DEFINITION 3.3.7. Let $f\colon X \to Y$ be a continuous map. Abusing notation, we denote by f^* both of the homomorphisms $H^p_\infty(Y) \to H^p_\infty(X)$ and $H^p_L(Y) \to H^p_L(X)$ induced by $f^{\#}$.

In order to state a result about exact sequences of Alexander-Spanier cohomology groups of pairs (X, A) where A is a subspace of X, we need to first define relative cohomology groups.

DEFINITION 3.3.8. Let X be a space, let A be a subspace of X, and let i be the inclusion map $A \to X$. The **relative cochain complex** $C^*_\infty(X, A, G)$ is the cochain complex with p-th cochain group $C^p_\infty(X, A, G)$ equal to the kernel of the induced map $i^{\#}\colon C^p(X, G) \to C^p(A, G)$.

If A is closed in X, then we also define the **relative cochain complex based on locally finitely valued cochains** $C^*_L(X, A, G)$ to be the cochain complex with p-th cochain group $C^p_L(X, A, G)$ equal to the kernel of the induced map $i^{\#}_L\colon C^p_L(X, G) \to C^p_L(A, G)$.

DEFINITION 3.3.9. The **p-th relative Alexander-Spanier cohomology group** $H^p_\infty(X, A, G)$ is the p-th cohomology group of the relative cochain complex $C^*_\infty(X, A, G)$. The **p-th relative Alexander-Spanier cohomology group based on locally finitely valued cochains** $H^p_L(X, A, G)$ is the p-th cohomology group of $C^*_L(X, A, G)$.

The definitions above clearly imply the following.

PROPOSITION 3.3.10. *If A is a subspace of X, then the following sequence of cochain complexes is exact.*

$$0 \to C^*_\infty(X, A, G) \xrightarrow{j^{\#}} C^*_\infty(X, G) \xrightarrow{i^{\#}} C^*_\infty(A, G) \to 0$$

where $j^{\#}$ is the inclusion map.

If A is closed in X, then the following sequence is also exact.

$$0 \to C^*_L(X, A, G) \xrightarrow{j^{\#}_L} C^*_L(X, G) \xrightarrow{i^{\#}_L} C^*_L(A, G) \to 0$$

where $j^{\#}_L$ is the inclusion map.

If we let i^*, i^*_L, j^*, and j^*_L be the maps induced on cohomology groups by the maps in the above proposition, then we have the following.

PROPOSITION 3.3.11. *If A is a subspace of X, the following is a long exact sequence of cohomology groups.*

$$\cdots \xrightarrow{d} H^p_\infty(X, A, G) \xrightarrow{j^*} H^p_\infty(X, G) \xrightarrow{i^*} H^p_\infty(A, G) \xrightarrow{d} H^{p+1}_\infty(X, A, G) \to \cdots$$

If A is closed in X, then the following is also a long exact sequence.

$$\cdots \xrightarrow{d} H_L^p(X,A,G) \xrightarrow{j_L^*} H_L^p(X,G) \xrightarrow{i_L^*} H_L^p(A,G) \xrightarrow{d} H_L^{p+1}(X,A,G) \to \cdots$$

We now summarize the properties of Alexander-Spanier cohomology. The proofs are contained in Spanier's text, and Massey refers us to [**1**] and [**13**] in his book for characterizations of this theory.

THEOREM 3.3.12. *Let X be a space, and let G be an abelian group.*

(a) *H_∞ satisfies all Eilenberg-Steenrod axioms for continuous maps and pairs (X,A) where X is any topological space and A is any subspace of X.*
(b) *H_L satisfies all Eilenberg-Steenrod axioms for continuous maps and pairs (X,A) where X is paracompact Hausdorff and A is closed in X.*
(c) ***(Strong Excision)** If X and Y are paracompact Hausdorff spaces, if A and B are closed subsets of X and Y respectively, and if $f\colon (X,A) \to (Y,B)$ is a closed, continuous map which is bijective between $X-A$ and $Y-B$, then the induced maps $f^*\colon H_\infty^p(Y,B) \to H_\infty^p(X,A)$ and $f_L^*\colon H_L^p(Y,B) \to H_L^p(X,A)$ are isomorphisms.*
(d) *If X is paracompact Hausdorff, A is a closed subset of X, and U is an open subset of X such that $U \subseteq A$, then the inclusion map $j\colon (X-U, A-U) \to (X,A)$ induces isomorphisms $j^*\colon H_\infty^p(X,A) \to H_\infty^p(X-U,A-U)$ and $j_L^*\colon H_L^p(X,A) \to H_L^p(X-U,A-U)$.*
(e) *If A is a subspace of X and X is the union of mutually disjoint open subsets X_i as above, then there is an isomorphism $H_\infty^p(X,A) \cong \prod_i H_\infty^p(X_i, X_i \cap A)$. If A is closed, then there is also an isomorphism $H_L^p(X,A) \cong \prod_i H_L^p(X_i, X_i \cap A)$.*
(f) *If X is paracompact Hausdorff and both A and B are closed subsets of X, then the triad $(X;A,B)$ is excisive for both H_∞ and H_L.*
(g) *If X is a normal space, and if $X = A \cup B$ with A and B open subsets of X, then the triad $(X;A,B)$ is excisive for H_∞.*
(h) ***(Vietoris-Begle Theorem)** If X and Y are paracompact Hausdorff spaces, if $f\colon X \to Y$ is a closed, continuous, surjective map, and if for every $y \in Y$, the reduced homology group $\widetilde{H}_\infty^p(f^{-1}(y)) = 0$ for $p \leq n$, then the induced map $f^*\colon H_\infty^p(Y) \to H_\infty^p(X)$ is an isomorphism for $p \leq n$ and a monomorphism for $p = n+1$.*
(i) *If X is paracompact Hausdorff and if either G is countable or the topology of X is compactly generated, then the group of homotopy classes of maps of X into the Eilenberg-Mac Lane space $K(G,n)$ is isomorphic to $H_\infty^n(X,G)$.*
(j) *If X is paracompact Hausdorff, then the covering dimension of X is equal to the largest p (or ∞) such that $H_\infty^p(X,A,G) \neq 0$ for some closed $A \subseteq X$.*

Note that the hypothesis of the strong excision property is stronger than only requiring that f be a homeomorphism between $X - A$ and $Y - B$. Also note that since H_∞ and H_L satisfy the Eilenberg-Steenrod axioms, excisive triads have cohomology sequences and Mayer-Vietoris sequences which are exact, as shown in section 3.1.

There is a natural transformation from Alexander-Spanier cohomology into singular cohomology. The definition of this transformation makes use of a notion of supports for singular cochains. This notion is used in the construction of a function μ from $\Phi^p(X, G)$ to a subgroup of the singular p-cochain group, which is given by sending $\phi \in \Phi^p(X, G)$ to the singular p-cochain defined by

$$(\mu\phi)(\sigma) = \phi(\sigma(e_0), \ldots, \sigma(e_n))$$

PROPOSITION 3.3.13. *The map μ above induces a homomorphism*

$$\mu^* \colon H^p_\infty(X, A, G) \to H^p_s(X, A, G)$$

where H^p_s is the p-th singular cohomology group with respect to the abelian group G, as defined in section 4.1 below.

Under certain conditions, μ^* is an isomorphism. Two such conditions are given in [**22**]. The notable feature of these conditions is that they are requirements that X be locally nice in some way.

Moreover, Spanier showed that Alexander-Spanier cohomology is isomorphic to Čech cohomology for paracompact Hausdorff spaces.

PROPOSITION 3.3.14. *If X is a paracompact Hausdorff space, then $H^p_\infty(X, G)$, $H^p_L(X, G)$, and $\check{H}^p(X, G)$ are all isomorphic, where $\check{H}$ denotes Čech cohomology, as defined in section 3.4 below.*

This is demonstrated by giving a construction of Čech cohomology with coefficients in a presheaf, showing that the presheaf Γ of cochain complexes which assigns the Alexander-Spanier cochain complex $C(U, U \cap A; G)$ to open $U \subseteq X$ is isomorphic to its generated sheaf $\hat{\Gamma}$, and exhibiting an isomorphism between Čech cohomology and the cohomology groups of $\hat{\Gamma}$ for paracompact Hausdorff spaces.

3.4. Čech Cohomology Using Sheaves

The creation of Čech cohomology was motivated by the need for theories better at handling local pathologies than the singular theories. The topologist's sine curve of example 2.5.1 serves as a good example here as well, this time indicating a difference between singular and Čech cohomology.

There are several developments of Čech cohomology. The recurring theme in the constructions is the use of a direct system of open covers and refinement maps. In one development, nerves of each open cover are taken, singular cohomology groups are assigned to the nerves, and Čech cohomology is defined as the direct limit under refinement of these singular cohomology groups. Since cohomology reverses the direction of the refinement maps, the singular groups represent finer and finer topological information as one progresses in the sequence, and so the direct limit captures fine or local information about the space. However, the use of open covers as approximations to the space prevents certain pathological features from being detected. For example, nerves of a Čech system are unable to separate the limit set of the topologist's sine curve from the curve itself.

Below we include the construction of Čech cohomology using presheaves, as treated by Spanier **[22]**.

DEFINITION 3.4.1. Let Γ be a presheaf of modules on X and let $\mathcal{U}$ be an open cover of X. For each $q \geq 0$, define $C^q(\mathcal{U};\Gamma)$ be the module of functions $\psi\colon \mathcal{U}^{q+1} \to \Gamma$ which map $(q+1)$-tuples $(U_0,\ldots,U_q)$ to elements $\psi(U_0,\ldots,U_q) \in \Gamma(U_0 \cap \cdots \cap U_q)$.

DEFINITION 3.4.2. The coboundary operator $\delta\colon C^q(\mathcal{U};\Gamma) \to C^{q+1}(\mathcal{U};\Gamma)$ is defined by the usual alternating sum formula:

$$\delta\psi(U_0,\ldots,U_{q+1}) = \sum_{0\leq i\leq q+1} \psi(U_0,\ldots,\widehat{U_i},\ldots,U_{q+1})$$

where the hat symbol indicates that entry should be removed.

The coboundary operator satisfies $\delta\delta = 0$. So, these cochain groups and coboundary operators form a cochain complex. We denote the cohomology groups of this complex by $H^q(\mathcal{U};\Gamma)$

If $\mathcal{V}$ is a refinement of the open cover $\mathcal{U}$ and $p\colon \mathcal{V} \to \mathcal{U}$ is a refinement map, then there is a cochain map $p\colon C(\mathcal{U};\Gamma) \to C(\mathcal{V};\Gamma)$ defined by $p\psi(V_0,\ldots,V_q) = \psi(p(V_0),\ldots,p(V_q))$. Different choices of refinement maps lead to cochain homotopic cochain maps. Thus, there is a homomorphism $p^*\colon H^q(\mathcal{U};\Gamma) \to H^q(\mathcal{V};\Gamma)$ for each q which is independent of the choice of refinement map. Since the collection of open covers of X and refinement maps form a direct system, we get a direct system of cohomology groups $H^q(\mathcal{U};\Gamma)$ using homomorphisms induced by refinement.

DEFINITION 3.4.3. The **q-th dimension Čech cohomology of X with coefficients in Γ** is defined to be the direct limit

$$\check{H}^q(X;\Gamma) = \varinjlim H^q(\mathcal{U};\Gamma)$$

over open covers $\mathcal{U}$ and homomorphisms induced by refinement.

Note that if G is an abelian group, we can take Γ to be a constant presheaf G as in 1.7.2 to obtain a cohomology theory with coefficients in G. It is easy to see how the presheaf construction above corresponds to the nerve construction in this case, since 0 must be assigned to any empty intersection of open sets in a cover.

CHAPTER 4

Derived Theories

4.1. Singular Cohomology

Singular cohomology is the dual notion to singular homology. Essentially, we just use the $\mathrm{Hom}(-, R)$ functor on the singular chain complex and take quotient groups. In the following discussion, let X be any fixed topological space and let R be a ring. We include the development appearing in [**8**].

DEFINITION 4.1.1. The **n-th singular cochain group** is the group $C^n = \mathrm{Hom}(C_n(X), R)$ of homomorphisms from the singular n-chain group to the ring R. The homomorphisms in C^n are called **singular n-cochains.**

Since C_n is a free group generated by the singular n-simplices $\sigma\colon \Delta_n \to X$, any homomorphism on C_n is determined by its values on the singular n-simplices. Hence, each n-cochain corresponds uniquely to a function from the set of n-simplices in X to the ring R. We can therefore view any n-cochain as an assignment of elements in R to the n-simplices in X. Hence, there is an obvious isomorphism from C^n to the direct product $\prod_{\sigma \in S_n} R$ where S_n is the set of singular n-simplices. This differs from the singular n-chains, which could be viewed as a direct product or an assignment where only finitely many n-simplices have a corresponding nonzero element of R.

DEFINITION 4.1.2. The **n-th singular coboundary map** $\delta^n\colon C^n \to C^{n+1}$ is defined by sending each n-cochain ϕ to the $(n+1)$-cochain $\delta^n(\phi) = \phi\partial_{n+1}$ where ∂ is the singular boundary map. That is, $\delta^n\phi$ is the $(n+1)$-cochain which assigns to an $(n+1)$-simplex σ whatever the n-cochain ϕ assigns to the boundary of σ. We can explicitly write out $\delta\phi(\sigma) = \sum_{i=0}^{n+1}(-1)^i\phi(\sigma \restriction \langle e_0, \ldots, \widehat{e_i}, \ldots, e_{n+1}\rangle)$ for $(n+1)$-simplices σ.

Note that being a cocycle is equivalent to vanishing on boundaries. That is, ϕ is in $\ker \delta_n$ iff ϕ maps all boundaries of $(n+1)$-chains to 0.

EXAMPLE 4.1.3. Consider the following coboundary computation. Let X be the triangulation of $\mathbb{R}$ given above, let ϕ be the 1-cochain which assigns the element $1 \in \mathbb{Z}$ to each 1-simplex in X, and let σ be a map from Δ_2 to X. We will determine what value is assigned by the coboundary of ϕ to the singular 2-simplex σ. First, the boundary of σ is $\sigma \restriction \langle e_1, e_2\rangle - \sigma \restriction \langle e_0, e_2\rangle + \sigma \restriction \langle e_0, e_1\rangle$. Next, ϕ assigns 1 to each term in the sum. So, ϕ assigns $1 - 1 + 1 = 1$ to the boundary of σ. Since σ was an arbitrary 2-simplex, this shows that the coboundary of ϕ assigns 1 to every singular 2-simplex in X.

It is clear that $\delta^{n+1}\delta^n$ is the 0 map, since $\delta^{n+1}\delta^n\phi = \delta^{n+1}\phi\partial_{n+1} = \phi\partial_{n+1}\partial_{n+2}$ and we already know the ∂ composition is the 0 map.

DEFINITION 4.1.4. The n-**th singular cohomology group** with respect to the ring R is $H^n(X;R) = \ker\delta_n/\operatorname{im}\delta_{n-1}$.

Similarly to singular homology, singular cohomology can be more easily computed for CW-complexes. An analogous construction leads to the result that the singular cohomology groups are obtained by using cochain groups consisting of the algebraic duals of those for cellular homology. We will not trace out the analogous construction, but we will make use of this fact in the following example.

EXAMPLE 4.1.5. Let X be S^1 as a subspace of $\mathbb{R}^2$, and let $\mathbb{Z}$ be the target ring for our cochains. First we will compute $H^0(X)$, and to do so we must determine $\ker\delta_0$. A 0-cochain ϕ is in the kernel of δ_0 if its coboundary is the 0 homomorphism 1-cochain . That is, $\delta_0\phi$ must assign 0 to every singular 1-simplex. Let $\sigma\colon\Delta_1\to X$ be a singular 1-simplex. Its image under $\delta_0\phi$ is $\phi\circ\partial_1(\sigma) = \phi(\sigma\restriction\langle e_1\rangle - \sigma\restriction\langle e_0\rangle) = \phi(\sigma\restriction\langle e_1\rangle) - \phi(\sigma\restriction\langle e_0\rangle)$. Hence ϕ is in $\ker\delta_0$ iff $\phi(\sigma\restriction\langle e_1\rangle) = \phi(\sigma\restriction\langle e_0\rangle)$ for every singular 1-simplex σ. Since singular 0-simplices are determined by their image, and each σ can have arbitrary points in $X = S^1$ for $\sigma(e_0)$ and $\sigma(e_1)$, this means that ϕ must be constant on the set of 1-simplices. Hence, the kernel of δ_0 is isomorphic to $\mathbb{Z}$, since each constant cochain is uniquely paired with its integer image. For the 0-th dimension cohomology, we simply take the kernel of δ_0, so we have that $H^0(X)\cong\mathbb{Z}$. Now, we will compute $H^1(X)$, utilizing the isomorphism between the singular and simplicial theories. Consider the exact sequence of simplicial homology and chain groups $0\to H_1\xrightarrow{r} C_1\xrightarrow{\partial_1} C_0\xrightarrow{q} H_0\to 0$ where r maps the equivalence class of the cycle $a(\langle v_0,v_1\rangle+\langle v_1,v_2\rangle+\langle v_2,v_0\rangle)$ to the cycle $a(\langle v_0,v_1\rangle+\langle v_1,v_2\rangle+\langle v_2,v_0\rangle)$ in C_1 for each $a\in\mathbb{Z}$, and q is the quotient map which mods out boundaries. If we represent each 1-chain $a\langle v_0,v_1\rangle+b\langle v_1,v_2\rangle+c\langle v_2,v_0\rangle$ in C_1 as a column vector $\begin{pmatrix} a\\ b\\ c\end{pmatrix}$, then the boundary map ∂_1 can be expressed as the matrix $\begin{pmatrix} -1 & 0 & 1\\ 1 & -1 & 0\\ 0 & 1 & -1\end{pmatrix}$ acting by left multiplication. If we apply $\mathrm{Hom}(-,\mathbb{Z})$ to the above sequence, we get the exact sequence $0\to\mathrm{Hom}(H_0)\xrightarrow{q^*} C^0\xrightarrow{\delta_0} C^1$ involving simplicial cochain groups and the coboundary operator which is given by the transpose $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ of the earlier matrix. We already know what H_0, C^0, and C^1 are isomorphic to, so we can write the exact sequence $0 \to \mathbb{Z} \xrightarrow{q^*} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\delta_0} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Recall that the simplicial cohomology group H^1 is $\ker \delta_1 / \operatorname{im} \delta_0$. The matrix for δ_0 maps $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ onto a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and $\delta_1 \colon C^1 \to C^2$ has kernel $C^1 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ since there are no 2-simplices and so C^2 is trivial. Thus, we have the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\delta_0} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$ where the right-most $\mathbb{Z}$ is isomorphic to $H^1(X)$. Because the simplicial and singular theories coincide, this is the 1st cohomology group of S^1.

Singular cohomology satisfies all of the Eilenberg-Steenrod axioms for continuous maps and pairs (X, A) with X a topological space and A a subspace of X. There is a homomorphism from singular cohomology to Alexander-Spanier cohomology, which we noted in 3.3.13. This homomorphism is an isomorphism for spaces satisfying certain nice local properties.

4.2. Homology Based on Infinite Chains

In this section, we discuss the homology theory which is dual to the cohomology theory of Section 3.2. Hence, this homology theory is defined for locally compact Hausdorff spaces and has homomorphisms induced by continuous, proper maps. In the following, we fix a locally compact Hausdorff space X and an abelian group G. We include the development which appears in [**14**].

DEFINITION 4.2.1. The **group of n-chains of X with coefficient group** G is the group $C_n^\infty(X, G) = \operatorname{Hom}(C_c^n(X, \mathbb{Z}), G)$.

The boundary map is defined as the dual of the coboundary map δ defined in 3.2.10.

DEFINITION 4.2.2. The **n-th boundary homomorphism** $\partial_n \colon C_n(X, G) \to C_{n-1}(X, G)$ is defined by $\partial_n(\phi) = \phi\delta_{n-1}$ for each ϕ in $C_n(X, G)$.

The collection of n-chains and boundary maps defined above forms a chain complex, and so we can define homology groups.

DEFINITION 4.2.3. The **n-th homology group (of X with coefficients in G) based on infinite chains** is $H_p^\infty(X, G) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

DEFINITION 4.2.4. Let $f \colon X \to Y$ be a continuous, proper map between locally compact spaces X and Y. Let $f^\# \colon C_c^n(Y, \mathbb{Z}) \to C_c^n(X, \mathbb{Z})$ denote the induced map from definition 3.2.13. Then the induced map $f_\# \colon C_n^\infty(X, G) \to C_n^\infty(Y, G)$ is defined by $f_\#(\phi) = \phi f^\#$ for each ϕ in $C_n^\infty(X, G)$.

DEFINITION 4.2.5. Let U be an open subset of X. Let $\sigma = \sigma_{U,X} \colon C_c^n(U,\mathbb{Z}) \to C_c^n(X,\mathbb{Z})$ be defined as it is above in definition 3.2.14. We define the map $\sigma_\# = \sigma_{\# X,U} \colon C_n^\infty(X,G) \to C_n^\infty(U,G)$ by $\sigma_\#(\phi) = \phi\sigma$ for all ϕ in $C_n^\infty(X,G)$.

Both of the above defined induced maps commute with the boundary operator, and hence they are chain maps. Thus, they both induce homomorphisms on the homology groups.

DEFINITION 4.2.6. The map induced by the chain map $f_\#$ is denoted $f_* \colon H_n^\infty(X,G) \to H_n^\infty(Y,G)$. The map induced by the chain map $\sigma_{\# X,U}$ is denoted ρ or $\rho_{X,U} \colon H_n^\infty(X,G) \to H_n^\infty(U,G)$.

The statement of the properties of this homology theory will make use of a map $\partial_{X,A} \colon H_{p+1}^\infty(U) \to H_p^\infty(A)$ between the homology groups of an open subset U of X and its complement $A = X - U$. The definition relies on the fact that the following is a split exact sequence of cochain complexes, where $i^\#$ is induced from the inclusion map $i \colon A \to X$.

$$0 \to C_c^n(X,A,\mathbb{Z}) \to C_c^n(X) \xrightarrow{i^\#} C_c^n(A) \to 0$$

By applying $\mathrm{Hom}(-,G)$ to this sequence and identifying the groups $C_n^\infty(X,A,G)$ and $C_n^\infty(U,G)$, it is possible to show that the following is a short exact sequence of chain complexes.

$$0 \to C_n^\infty(A,G) \xrightarrow{i_\#} C_n^\infty(X,G) \xrightarrow{\sigma_\#} C_n^\infty(U,G) \to 0$$

If we take homology groups of this sequence, the following long exact sequence can be constructed.

$$\cdots \to H_p^\infty(A) \xrightarrow{i_*} H_p^\infty(X) \xrightarrow{\rho} H_p^\infty(U) \xrightarrow{\partial} H_{p-1}^\infty(A) \to \cdots$$

The homomorphism ∂ in this long exact sequence is the homomorphism we want.

DEFINITION 4.2.7. If U is an open subset of X and $A = X - U$, then we denote by $\partial_{X,A}$ the connecting homomorphism that appears in the long exact sequence above.

THEOREM 4.2.8. *Let X, Y, and Z be locally compact Hausdorff spaces, and let G be an abelian group.*

(a) *H^∞ satisfies all Eilenberg-Steenrod axioms for continuous, proper maps and pairs (X,A) where X is locally compact, Hausdorff and A is closed in X.*
(b) *If X is a single point, then $H_0^\infty(X,G)$ is G.*
(c) *$\rho_{X,X} \colon H_p^\infty(X) \to H_p^\infty(X)$ is the identity homomorphism.*
(d) *If $V \subseteq U \subseteq X$ with U and V open, then $\rho_{X,V} = \rho_{U,V}\rho_{X,U}$.*

(e) *If U and V are open subsets of X and Y respectively, and if $f\colon X \to Y$ is a continuous, proper map such that $f(U) \subseteq V$ and $f(X-U) \subseteq Y-V$, then the following diagram is commutative.*

$$\begin{tikzcd} H_p^\infty(X) \arrow[r, "f_*"] \arrow[d, "\rho"'] & H_p^\infty(Y) \arrow[d, "\rho"] \\ H_p^\infty(U) \arrow[r, "(f\restriction_U)_*"] & H_p^\infty(V) \end{tikzcd}$$

(f) *Let U and V be open subsets of X and Y respectively, and let $A = X - U$ and $B = Y - V$. Suppose $f\colon X \to Y$ is a continuous, proper map such that $f(U) \subseteq V$ and $f(A) \subseteq B$. Let $f_1\colon A \to B$ and $f_2\colon U \to V$ be the maps induced by restriction of f. Then the following diagram is commutative.*

$$\begin{tikzcd} H_p^\infty(U) \arrow[r, "(f_2)_*"] \arrow[d, "\partial"'] & H_p^\infty(V) \arrow[d, "\partial"] \\ H_{p-1}^\infty(A) \arrow[r, "(f_1)_*"] & H_{p-1}^\infty(B) \end{tikzcd}$$

(g) *Let V and U be open subsets of X such that $V \subseteq U$. Let $A = X - U$, let $B = X - V$, and let i be the inclusion map $A \to B$. Then the following diagram is commutative.*

$$\begin{tikzcd} H_p^\infty(U) \arrow[r, "\rho"] \arrow[d, "\partial"'] & H_p^\infty(V) \arrow[d, "\partial"] \\ H_{p-1}^\infty(A) \arrow[r, "i_*"] & H_{p-1}^\infty(B) \end{tikzcd}$$

(h) *Let U and V be open subsets of X such that $U \subseteq V$, and let $A = X - U$. Then the following diagram is commutative.*

$$\begin{tikzcd} & H_p^\infty(A) \arrow[dd, "\rho"] \\ H_{p+1}^\infty(U) \arrow[ur, "\partial"] \arrow[dr, "\partial"'] & \\ & H_p^\infty(V \cap A) \end{tikzcd}$$

(i) *If X is the disjoint union of open subsets X_i, then each inclusion map $j\colon X_i \to X$ induces an injective homomorphism $j_*\colon H_p^\infty(X_i) \to H_p^\infty(X)$, and $H_p^\infty(X)$ is equal to the cartesian product of the images of these homomorphisms.*

(j) *If A and B are closed subsets of X such that $X = A \cup B$, then the Mayer-Vietoris sequence*

$$\cdots \to H_p^\infty(A \cap B) \to H_p^\infty(A) \oplus H_p^\infty(B) \to H_p^\infty(X) \to H_{p-1}^\infty(A \cap B) \to \cdots$$

is exact.

(k) *If A and B are open subsets of X such that $X = A \cup B$, then the Mayer-Vietoris sequence*

$$\cdots \to H_p^\infty(A \cap B) \to H_p^\infty(A) \oplus H_p^\infty(B) \to H_p^\infty(X) \to H_{p-1}^\infty(A \cap B) \to \cdots$$

is exact.

4.3. Homology with Compact Supports

In this section we will discuss the homology theory which is dual to Alexander-Spanier cohomology. This homology theory is defined for arbitrary Hausdorff spaces. In the following, we fix a Hausdorff space X and an abelian group G. We include the development which appears in **[14]**.

DEFINITION 4.3.1. If X is Hausdorff and A is a subspace of X, then we say that (X, A) is a **Hausdorff pair**. If X is compact Hausdorff and A is a closed subset of X, then we say that (X, A) is a **compact pair**. We define subset inclusion between any kind of pairs $(X, A) \subseteq (Y, B)$ to mean that both $X \subseteq Y$ and $A \subseteq B$.

We will define homology groups of Hausdorff pairs (X, A) in this theory as the direct limit over certain homology groups associated to compact pairs contained in (X, A). Note that if (Y, B) is a compact pair, then $Y - B$ is locally compact Hausdorff and $H_p^\infty(Y - B, G)$ can be defined as in Section 4.2. Furthermore, if $(Y, B) \subseteq (X, A)$, then there is an inclusion map $i\colon (Y, B) \to (X, A)$ satisfying $i(Y) \subseteq X$ and $i(B) \subseteq A$. This inclusion map induces a map i_* between $H_p^\infty(Y - B)$ and $H_p^\infty(X - A)$ as follows. Let $U = i^{-1}(X - A)$ and let $i_1\colon U \to X - A$ be the restriction of i to U. Then U is a subset of $Y - B$ and $\rho\colon H_p^\infty(Y - B) \to H_p(U)$ is defined as in 4.2.6. The map $i_*\colon H_p^\infty(Y - B) \to H_p^\infty(X - A)$ is defined as the composition ρi_1. Now, the collection of compact pairs (Y, B) satisfying $(Y, B) \subseteq (X, A)$ is directed under this inclusion relation, and this induces a relation on the collection of homology groups $H_p^\infty(Y - B)$ which makes it also a directed set. Taken with the maps i_* induced by inclusion between the compact pairs, this gives us a direct system $\{H_p^\infty(Y - B), i_*\}$.

DEFINITION 4.3.2. Let (X, A) be a Hausdorff pair. We define the **p-th homology group with compact support of the pair (X, A) with coefficients in G** to be the direct limit

$$H_p^c(X, A) = \varinjlim H_p^\infty(Y, B) = \varinjlim H_p^\infty(Y - B)$$

over compact pairs $(Y, B) \subseteq (X, A)$ and maps i_* induced by inclusion.

Note that if (X, A) is a compact pair, then $H_p^c(X, A)$ as defined above coincides with $H_p^\infty(X - A)$. If X is only locally compact, however, the groups defined in these two sections can be distinct; for example, $H_1^\infty(\mathbb{R}) = G$ but $H_1^c(\mathbb{R}) = 0$.

We will now define the homomorphisms on these new homology groups induced from maps of Hausdorff pairs and inclusions maps.

Let $f\colon (X, A) \to (Y, B)$ be a map of Hausdorff pairs. That is, f is a continuous map from X to Y such that $f(A) \subseteq B$. Then, for each compact pair $(P, Q) \subseteq (X, A)$, we have that $(f(P), f(Q))$ is a compact pair in (Y, B). Thus, f induces an order-preserving map from the direct system of compact pairs in (X, A) to that of (Y, B). In turn, this induces a map between the direct systems of homology groups $H_p^\infty(P-Q)$ and $H_p^\infty(f(P)-f(Q))$ of the compact pairs. This map finally induces the map f_* we are interested in between the direct limits.

Definition 4.3.3. If $f\colon (X, A) \to (Y, B)$ is a map of Hausdorff pairs, then we define the induced homomorphism $f_*\colon H_p^c(X, A) \to H_p^c(Y, B)$ to be the homomorphism described above.

We can obtain the exact sequence

$$\cdots \to H_n^c(A) \xrightarrow{i_*} H_n^c(X) \xrightarrow{j_*} H_n^c(X, A) \xrightarrow{\partial} H_{n-1}^c(A) \to \cdots$$

for this homology theory for a Hausdorff pair (X, A) from the exact sequence of the theory in Section 4.2, since direct limits preserve exactness.

Theorem 4.3.4. *Let (X, A) and (Y, B) be Hausdorff pairs, and let G be an abelian group.*

(a) *H^c satisfies all Eilenberg-Steenrod axioms for continuous maps and pairs (X, A) where X is Hausdorff and A is any subspace of X.*
(b) *If A and B are closed subsets of X such that $X = A \cup B$, then the triad $(X; A, B)$ is excisive for H^c.*
(c) *Let U and V be open subsets of X such that $X = U \cup V$, then the triad $(X; U, V)$ is excisive for H^c.*
(d) ***(Strong Excision 1)*** *Suppose that X is paracompact Hausdorff and A is a closed subset of X. Let B be a closed subset of Y, and let $f\colon (X, A) \to (Y, B)$ be a closed, continuous function which maps $X - A$ homeomorphically onto $Y - B$. Then f induces an isomorphism $f_*\colon H_p^c(X, A) \to H_p^c(Y, B)$ for all p.*
(e) ***(Strong Excision 2)*** *Let A and B be closed subsets of X and Y respectively. Let $f\colon (X, A) \to (Y, B)$ be a proper map of pairs which maps $X - A$ bijectively onto $Y - B$. Then f induces an isomorphism $f_*\colon H_p^c(X, A) \to H_p^c(Y, B)$.*

(f) *If X is the union of mutually disjoint subsets X_i with inclusion maps $j\colon X_i \to X$, then the induced maps $j_*\colon H_p^c(X_i, X_i \cap A) \to H_p^c(X, A)$ are injective, and $H_p^c(X, A)$ is the direct sum of their images.*

(g) *Let (X, A) be a Hausdorff pair with A closed in X, then $H_p^c(X, A)$ is the direct limit of the homology groups $H_p^\infty(P, P \cap A)$ where P ranges over compact subsets of X such that $P - A$ is dense in P.*

Note that excisive triads have exact homology sequences and Mayer-Vietoris sequences as in 2.1 since this homology theory satisfies the Eilenberg-Steenrod axioms.

There is a natural homomorphism μ from singular homology into homology with compact supports. In the following, let $H_p^s(X, A, G)$ be the singular homology of the pair (X, A). The definition of μ relies on the following.

LEMMA 4.3.5. *Let (X, A) be a pair of spaces and let G be an abelian group. Then, for any $u \in H_p^s(X, A, G)$, there is a pair (K, L) with K a finite CW-complex and L a subcomplex, a continuous map $f\colon (K, L) \to (X, A)$, and an element $u' \in H_p^s(K, L)$ such that $f_*^s(u') = u$ where f_*^s is the induced map on singular homology groups.*

One consequence of the Eilenberg-Steenrod axioms is that if two homology theories satisfying the axioms have the same coefficient group G, then their homology groups are isomorphic on any CW-complex. Now, let u be an element of $H_p^s(X, A, G)$. By the above lemma, there is a pair (K, L) of finite CW-complexes, a map $f\colon (K, L) \to (X, A)$, and some u' in $H_p^s(K, L, G)$ such that $f_*^s(u') = u$. By the uniqueness theorem just mentioned, $H_p^s(K, L, G)$ and $H_p^c(K, L, G)$ are isomorphic, so there is a unique u'' in $H_p^c(K, L, G)$ corresponding to u'.

PROPOSITION 4.3.6. *The map $\mu = f_*\colon H_p^s(X, A, G) \to H_p^c(X, A, G)$ defined by $\mu(u) = f_*(u'')$ is a homomorphism. It is an isomorphism for manifolds and spaces homotopy equivalent to a CW-complex.*

The above definition is independent of the particular choices made in selecting (K, L), f, or the elements u and u'.

4.4. End Cohomology

End cohomology and homology are so named because of their relation to the ends of a space, which can be found, for example, in **[11]**. We will not discuss these relations, but for consideration we give a definition of ends and a few examples. For consideration, the end homology groups of these examples are given in 4.5.6. We include the development of **[14]**, but the reader is also referred to **[12]**.

DEFINITION 4.4.1. Let $K_0 \subset K_1 \subset \cdots$ be a nested sequence of compact sets whose interiors cover X. An **end** of X is a sequence $U_0 \supset U_1 \supset \cdots$ where U_i is a connected component of $X - K_i$.

Different choices of sequences $\{K_i\}$ lead to sets of ends which are in bijection with each other. This can be generalized to spaces which do not admit an exhaustion by compact sets using an inverse limit over $\pi_0(X - K_i)$ for a direct system of compact sets. There is a topology which can be placed on the ends of a space, and there is a method of compactification where a point at infinity is added for each end.

EXAMPLE 4.4.2.

(a) If $X \subseteq \mathbb{R}^2$ is the union of n distinct rays originating at the origin, then X has n ends.
(b) $\mathbb{R}^2$ has one end.
(c) Let $X \subseteq \mathbb{R}^2$ be an "infinite ladder": for example, the union of the rays $\{(0, y) \mid y \geq 0\}$ and $\{(1, y) \mid y \geq 0\}$ along with line segments $\{(x, n) \mid 0 \leq x \leq 1\}$ joining these rays for each $n \in \{0, 1, 2, \dots\}$. Then X has one end, despite $\pi_0(X - K)$ being infinite for any compact $K \subseteq X$.
(d) If X is compact, then X has no ends.

End cohomology is defined for locally compact Hausdorff spaces. Its definition is stated in terms of the cochain groups C^p_c and C^p_L of sections 3.2 and 3.3 respectively.

It is clear from their definitions that the group C^p_c is a subgroup of C^p_L. We will call the inclusion homomorphism $\gamma_p \colon C^p_c \to C^p_L$. Then the following is a short exact sequence

$$0 \to C^p_c \xrightarrow{\gamma_p} C^p_L \xrightarrow{\pi} C^p_L / C^p_c \to 0$$

where π is the quotient homomorphism.

DEFINITION 4.4.3. The group C^p_e is defined to be the quotient group C^p_L / C^p_c.

DEFINITION 4.4.4. The coboundary homomorphism $\delta^p_e \colon C^p_e \to C^{p+1}_e$ is defined to be the map induced by the coboundary map $\delta^p \colon C^p_L \to C^{p+1}_L$.

These coboundary maps clearly satisfy $\delta^{p+1}_e \delta^p_e = 0$, so we have a new cochain complex.

DEFINITION 4.4.5. The **p-th end cohomology group** is $H^p_e(X, G) = \ker \delta^p_e / \operatorname{im} \delta^{p-1}_e$.

We can fit the groups $H^p_e(X)$ into a long exact sequence with the groups $H^p_c(X)$ and $H^p_L(X)$ by noting that the following is a short exact sequence of cochain complexes.

$$0 \to C_c^*(X) \xrightarrow{\gamma} C_L^*(X) \to C_e^* \to 0$$

Hence, we have a long exact sequence of cohomology groups

$$\cdots \to H_c^p(X) \xrightarrow{\gamma^*} H_L^p(X) \to H_e^p(X) \xrightarrow{\delta} H_c^{p+1}(X) \to \cdots$$

THEOREM 4.4.6. *The cohomology group $H_e^p(X)$ is isomorphic to the direct limit of the groups $H_L^p(X-U)$ where U ranges over all open subsets of X with compact closure. The maps used for this direct limit are the maps $i_*\colon H_L^p(X-U_1) \to H_L^p(X-U_2)$ induced by the inclusion maps $i\colon X-U_2 \to X-U_1$ when $U_1 \subseteq U_2$.*

Moreover, the exact sequence

$$\cdots \to H_c^p(X) \xrightarrow{\gamma^*} H_L^p(X) \to H_e^p(X) \xrightarrow{\delta} H_c^{p+1}(X) \to \cdots$$

is the direct limit of the sequences

$$\cdots \xrightarrow{\delta} H_\infty^q(X, X-U) \xrightarrow{j^*} H_\infty^q(X) \xrightarrow{i_*} H_\infty^q(X-U) \xrightarrow{\delta} H_\infty^{q+1}(X, X-U) \to \cdots$$

as U ranges over all open subsets of X with compact closure.

PROPOSITION 4.4.7. *Let X be a compact Hausdorff space, let A be closed in X, and let $U = X - A$. Then the following sequence is long exact.*

$$\cdots \to H_\infty^p(X,U) \xrightarrow{i_*} H_L^p(A) \to H_e^p(U) \xrightarrow{\delta} H_\infty^{p+1}(X,U) \to \cdots$$

Here $H_c^p(A) = H_L^p(A)$ since A is compact.

4.5. End Homology

End homology is defined for locally compact Hausdorff spaces. Similarly to end cohomology, it is defined using chain groups from other theories. However, the homology theory in Section 4.3 was defined as a direct limit, without first constructing a chain complex. So, we will be concerned with the chain groups $C_p^\infty(X)$ of 4.2 and the chain groups $C_p^c(X, A)$, which we define to be the direct limit of the groups $C_p^\infty(P, Q)$ such that (P, Q) is a compact pair in (X, A). The homology groups associated to these $C_p^c(X, A)$ are the groups defined in 4.3. Note that the literature contains different constructions of end homology, some of which (e.g., **[12]**) have a dimension shift and may have nontrivial groups with negative indices. Here we have included the presentation which appears in **[14]**.

First we will see that $C_p^c(X)$ is isomorphic to a subgroup of $C_p^\infty(X)$. Let P be a compact subset of X. Then the inclusion map $i\colon P \to X$ induces a monomorphism $i_\#\colon C_p^\infty(P) \to C_p^\infty(X)$. Thus, if Q is another compact subset of X such that $P \subseteq Q$, then the following diagram is a commutative diagram of monomorphisms.

$$\begin{tikzcd} C_*^\infty(P) \arrow[rr] \arrow[dr] & & C_*^\infty(Q) \arrow[dl] \\ & C_*^\infty(X) & \end{tikzcd}$$

Hence, there is a monomorphism γ_p from the direct limit $C_p^c(X)$ into the group $C_c^\infty(X)$. Abusing notation, we will denote the quotient of $C_p^\infty(X)$ modulo the image of γ_p by $C_p^\infty(X)/C_p^c(X)$. Then the following is a short exact sequence.

$$0 \to C_p^c(X) \xrightarrow{\gamma_p} C_p^\infty(X) \to C_p^\infty(X)/C_p^c(X) \to 0$$

DEFINITION 4.5.1. The group $C_p^e(X)$ is defined to be the quotient group $C_p^\infty(X)/C_p^c(X)$.

DEFINITION 4.5.2. The boundary homomorphism $\partial_p^e \colon C_p^e(X) \to C_{p-1}^e(X)$ is defined to be the map induced by the boundary map $\partial_p \colon C_p^\infty(X) \to C_{p-1}^\infty(X)$.

Together, these homology groups and boundary maps form a chain complex, so we may define homology groups.

DEFINITION 4.5.3. The p**-th end homology group** is defined to be the quotient $H_p^e(X, G) = \ker \partial_p^e / \operatorname{im} \partial_{p+1}^e$.

Like the end cohomology groups, these homology groups fit into a long exact sequence with the other homology groups. Since

$$0 \to C_*^c(X) \xrightarrow{\gamma} C_*^\infty(X) \to C_*^e(X) \to 0$$

is a short exact sequence of chain complexes, we get the long exact sequence of homology groups

$$\cdots \to H_p^c(X) \xrightarrow{\gamma_*} H_p^\infty(X) \to H_p^e(X) \xrightarrow{\partial} H_{p-1}^c(X) \to \cdots$$

There is also a theorem expressing $H_p^e(X)$ as a direct limit of homology groups.

THEOREM 4.5.4. *The homology group $H_p^e(X)$ is isomorphic to $\varinjlim H_p^\infty(X - P)$ where P ranges over all compact subsets of X. The maps used for this direct limit are the maps $\rho \colon H_p^\infty(X - P_1) \to H_p^\infty(X - P_2)$ as defined in 4.2.6 when $P_1 \subseteq P_2$ and hence $X - P_2$ is an open subset of $X - P_1$.*

Moreover, the exact sequence

$$\cdots \to H_p^c(X) \xrightarrow{\gamma_*} H_p^\infty(X) \to H_p^e(X) \xrightarrow{\partial} H_{p-1}^c(X) \to \cdots$$

is the direct limit of the sequences

$$\cdots \to H_q^c(P) \xrightarrow{i_*} H_q^\infty(P) \xrightarrow{\rho} H_q^\infty(X - P) \xrightarrow{\partial} H_{q-1}(P) \to \cdots$$

as P ranges over all compact subsets of X.

PROPOSITION 4.5.5. *Let X be a compact Hausdorff space, let A be closed in X, and let $U = X - A$. Then the following sequence is long exact*

$$\cdots \to H_p^c(A) \xrightarrow{i_*} H_p^c(X,U) \to H_p^e(U) \xrightarrow{\partial} H_{p-1}^c(A) \to \cdots$$

Here $H_p^c(A) = H_p^\infty(A)$ since A is compact.

EXAMPLE 4.5.6.

(a) Let $X \subseteq \mathbb{R}^2$ be the union of n distinct rays originating at the origin. Then, $H_0^e(X,\mathbb{Z}) \cong 0$ and $H_1^e(X,\mathbb{Z}) \cong \mathbb{Z}^2$.
(b) $H_0^e(\mathbb{R}^2,\mathbb{Z}) \cong 0$, $H_1^e(\mathbb{R}^2,\mathbb{Z}) \cong 0$, and $H_1^e(\mathbb{R}^2,\mathbb{Z}) \cong \mathbb{Z}$.
(c) Let X be an infinite ladder, as in 4.4.2. Then $H_0^e(X,\mathbb{Z}) \cong 0$ and $H_1^e(X,\mathbb{Z})$ is infinite.
(d) If X is compact, then $H_p^e(X,\mathbb{Z}) \cong 0$ for all p.

CHAPTER 5

Coarse Algebraic Topology

5.1. Locally Finite Homology

When we define coarse homology later, we will make use of homology groups of nerves. The homology theory we will use is that of section 4.2, but we can make use of the fact that nerves are simplicial complexes in order to compute these homology groups more efficiently. The simplification follows from the more general result that when X is a locally compact Hausdorff space which can be nicely decomposed into cells, then the cohomology groups of section 3.2 and homology groups of section 4.2 can be computed directly in terms of the cells.

It is worth noting that locally finite homology is typically defined for simplicial complexes by generalizing simplicial homology to allow chain groups consisting of possibly infinite sums rather than just finite sums of simplices. However, the following development, which appears in chapter 4 of Massey's book **[14]**, constructs both cohomology and homology groups for the more general class of spaces with cellular decompositions. This is done as a special case of cohomology with compact supports and homology based on infinite chains, making use of the cellular structure of the space. The end result is the same as the previously mentioned development when restricted to simplicial complexes, which can be seen easily from the propositions below. Since we will only be using locally finite homology for simplicial complexes, the development below may seem to be a less direct way of reaching the same goal, but it lets us immediately inherit properties as special cases of previous theories.

DEFINITION 5.1.1. Let X be a Hausdorff space. A **cellular decomposition** of X is a nested sequence of closed subspaces of X

$$K^0 \subseteq K^1 \subseteq \cdots \subseteq K^q \subseteq \cdots$$

such that $\bigcup_i K^i = X$ and the following hold:

(a) K^0 is a possibly empty discrete subspace of X. The elements of K^0 are called the **vertices** or **0-cells**.

(b) For $q \geq 1$, $K^q - K^{q-1}$ is the disjoint union of open subsets, each of which is homeomorphic to the open unit q-ball in $\mathbb{R}^q$. The elements of K^q are called the q-**cells** and K^q is called the q-**skeleton**.

A cellular decomposition is **finite** if it consists of only finitely many cells. It is **finite dimensional** if the dimension of the cells is bounded above. That is, there is some q such that $K^q = K^{q'}$ for all $q' > q$.

Note that this definition of cellular decompositions is more general than that of CW-complexes in 2.4.1, in that it does not require the use of attaching maps.

DEFINITION 5.1.2. Let X be a locally compact Hausdorff space, and let $K = \{K^q\}$ be a cellular decomposition of X. The **q-cochains of K with coefficient group** G is denoted by $C^q(K, G)$ and is defined to be $H_c^q(K^q - K^{q-1}, G)$. The **q-chains of K with coefficient group** G is denoted $C_q(K, G)$ and defined to be $\mathrm{Hom}(C^q(K, \mathbb{Z}), G)$.

The following proposition states that the cochain and chain groups are isomorphic to direct sums and direct products of the coefficient group indexed by the cells of the decomposition.

PROPOSITION 5.1.3. *Let X and K be as above. If $\{e_i\}$ is the collection of q-cells in K, then $C^q(K, G)$ is isomorphic to the direct sum $\bigoplus_i H_c^q(e_i) = \bigoplus_i G$ and $C_q(K, G)$ is isomorphic to the direct product $\prod_i H_c^q(e_i) = \prod_i G$.*

PROOF. That $H_c^q(K^q - K^{q-1})$ is isomorphic to the direct sum $\bigoplus_i H_c^q(e_i)$ follows from the direct sum theorem for cohomology with compact supports. The equality of $\bigoplus_i H_c^q(e_i)$ and $\bigoplus_i G$ follows from

$$H_c^n(e_i) = \begin{cases} 0 & n \neq q \\ G & n = q \end{cases}$$

The rest of the proposition follows from properties of Hom. □

DEFINITION 5.1.4. The coboundary operator $\delta_q \colon C^q(K, G) \to C^{q+1}(K, G)$ is taken to be the operator $\delta_{(K^{q+1}-K^{q-1}, K^q - K^{q-1})} \colon H_c^q(K^q - K^{q-1}) \to H_c^{q+1}(K^{q+1} - K^q)$. The boundary operator $\partial_q \colon C_q(K, G) \to C_{q-1}(K, G)$ is defined by $\partial(\phi) = \phi\delta_{q-1}$.

The operator used in the above definition exists for cohomology with compact supports since $K^q - K^{q-1}$ is a closed subset of $K^{q+1} - K^{q-1}$ with open complement $K^{q+1} - K^q$.

PROPOSITION 5.1.5. *Let $K = \{K^i\}$ be a cellular decomposition of a locally compact Hausdorff space X. If $H^n(K, G)$ is the n-th cohomology group of the cochain complex $\{C^q(K, G), \delta^q\}$, then $H^n(K, G)$ is isomorphic to $H_c^n(X, G)$. Similarly, if $H_n(K, G)$ is the n-th homology group of the chain complex $\{C_q(K, G), \partial_q\}$, then $H_n(K, G)$ is isomorphic to $H_n^\infty(X, G)$.*

In combination with proposition 5.1.3, this last proposition tells us that we can compute the homology and cohomology of our space X directly from the cells when we have a cellular decomposition. The final observation we should make is that when X is a simplicial complex, these homology groups can be computed analogously to the simplicial homology groups of X, with the only difference being that here we allow infinite chains of simplices, since our chain groups are direct products over simplices. That the boundary operator behaves as it does for simplicial homology can be seen from the definitions. In the context of simplicial complexes, these homology groups are typically denoted by $H_n^{lf}(X)$ and called the locally-finite homology groups of X.

5.2. Coarse Maps

In this section, we discuss the sorts of maps and notions from coarse geometry that will be needed later. The reader is referred to **[21]** and **[20]** for these definitions, theorems, and proofs, and for further discussion of coarse geometry.

DEFINITION 5.2.1. Let X and Y be proper metric spaces. A function $f\colon X \to Y$ is **Lipschitz** if there is a constant C such that $d_Y(f(x), f(x')) \leq C \cdot d_X(x, x')$ for all $x, x' \in X$. We say that f is C-Lipschitz when we know the constant to be C.

DEFINITION 5.2.2. Let X and Y be proper metric spaces. A function $f\colon X \to Y$ is called a **coarse map** if

(a) for any bounded subset $B \subseteq Y$, the inverse image $f^{-1}(B)$ is bounded in X, and
(b) for all $R > 0$, there is an $S > 0$ such that $d(x_1, x_2) < R$ implies $d(f(x_1), f(x_2)) < S$ for all $x_1, x_2 \in X$.

DEFINITION 5.2.3. Let X and Y be proper metric spaces. Two maps $f\colon X \to Y$ and $g\colon X \to Y$ are called **close** if $\sup\limits_{x\in X}\{d(f(x), g(x))\}$ is finite.

DEFINITION 5.2.4. A coarse map $f\colon X \to Y$ is a **coarse equivalence** if there is a coarse map $g\colon Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are close to the identity maps 1_X and 1_Y respectively. The map g is called a **coarse inverse** of f.

EXAMPLE 5.2.5. Let $i\colon \mathbb{Z} \to \mathbb{R}$ be the inclusion map, and let $j\colon \mathbb{R} \to \mathbb{Z}$ be the map which sends $r \in \mathbb{R}$ to the greatest integer less than or equal to r. Both i and j are coarse maps and each is a coarse equivalence.

EXAMPLE 5.2.6. Let $a\colon \mathbb{R} \to \mathbb{R}^+ = [0, \infty)$ be defined by $f(x) = |x|$, and let $i\colon \mathbb{R}^+ \to \mathbb{R}$ be the inclusion map. Both a and i are coarse maps, but they are not coarse equivalences. In fact, there is no coarse equivalence between $\mathbb{R}$ and $\mathbb{R}^+$.

DEFINITION 5.2.7. Let $A \subseteq X$. We say that A is **coarsely dense** in X if there is some fixed finite distance $R > 0$ such that for all $x \in X$, there is some $a \in A$ with $d(x, a) < R$.

LEMMA 5.2.8. *Let $A \subseteq X$. The inclusion map $i\colon A \to X$ is a coarse equivalence if and only if A is coarsely dense in X.*

PROOF. Suppose i is a coarse equivalence. Then, i has a coarse inverse $g\colon X \to A$. We have that $i \circ g$ must be close to the identity on X, and hence there is some uniform bound S such that $d(g(x), x) = d(i \circ g(x), x) < S$ for all $x \in X$. But this implies that every $x \in X$ is at most S away from A. So, A is coarsely dense in X.

Conversely, suppose A is coarsely dense in X. Then we can construct a coarse inverse for i as follows. Since A is coarsely dense in X, there is some $R > 0$ such that for each $x \in X$, there is some $a_x \in A$ with $d(x, a_x) < R$. We select one such a_x for each $x \in X - A$. We then define the map $g\colon X \to A$ by

$$g(x) = \begin{cases} x & \text{if } x \in A \\ y_x & \text{if } x \in X - A \end{cases}$$

It is easily seen that g is coarse. We have that $g \circ i = \mathrm{id}_A$ already, so they are close. To see that $i \circ g$ is close to id_X, note that $d(x, g(x)) < R$ for all $x \in X$. Hence, g is a coarse inverse for i, and so i is a coarse equivalence. □

DEFINITION 5.2.9. Let X and Y be proper metric spaces, and let $f, g\colon X \to Y$ be coarse maps. A **Lipschitz homotopy** from f to g is a coarse map $H\colon X \times \mathbb{R}^+ \to Y \times \mathbb{R}^+$ of the form $H(x,t) = (h(x,t), t)$ for some map h, such that

(a) $f = h(x, 0)$ and $g = \lim_{t\to\infty} h(x,t) = h(x, \infty)$,
(b) for each bounded $B \subseteq X$, there is some $t_B \in \mathbb{R}^+$ such that $h(x,t)$ is constantly $g(x)$ for $t \geq t_B$ and $x \in B$, and
(c) for each bounded $B \subseteq Y$, the set $\{x \in X \mid h(x,t) \in B \text{ for some } t \in \mathbb{R}^+\}$ is bounded.

If f and g can be linked by a chain of Lipschitz homotopies, we say that f and g are **Lipschitz homotopic**.

A straightforward translation of the usual notion of homotopy to a coarse setting suggests the following definition.

DEFINITION 5.2.10. A **coarse homotopy** between two coarse maps $f, g\colon X \to Y$ between proper metric spaces is a coarse map $H\colon X \times [0,1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

However, it is not hard to see that such a coarse homotopy exists iff f and g are close. In particular, the coarseness of H implies that the distance between $H(x,0) = f(x)$ and $H(x,1) = g(x)$ is uniformly bounded since the distance from $(x,0)$ to $(x,1)$ is always 1.

5.3. Coarse Homology

In this section, we will define coarse homology for proper metric spaces. Its definition requires the use of a homology theory which is defined on the category of locally compact spaces and proper maps. Here we have chosen to use homology based on infinite chains, which is locally finite homology on simplicial complexes. The exposition that follows (except for the discussion of the more general Mayer-Vietoris sequence and the coarse excision theorem) is found in several lecture notes and texts by Roe [**20, 19, 21**], an article by Roe and Higson [**10**], and an article and its correction by Mitchener [**16, 17**]. The sources variously work in metric spaces and the more general class of coarse spaces, and so some notions have been developed below in less generality than that in which they can be treated.

DEFINITION 5.3.1. Let X be a proper metric space, and let $\{\mathcal{U}_i, p_i\}$ be an anti-Čech system for X. Let $\{\mathcal{K}_i, p_i^*\}$ be the associated system of nerves. The **n-th coarse homology group** $HC_n(X)$ of X based on locally finite homology is defined by the following direct limit, where i varies and the maps between homology groups are induced by the maps p_i^*.

$$HC_n(X) = \varinjlim H_n^{lf}(\mathcal{K}(\mathcal{U}_i))$$

If $A \subseteq X$, then the **n-th relative coarse homology group** $HC_n(X, A)$ is defined to be the following direct limit.

$$HC_n(X, A) = \varinjlim H_n^{lf}(\mathcal{K}_i, \mathcal{K}_i \restriction_A) = \varinjlim H_n^{\infty}(\mathcal{K}_i - \mathcal{K}_i \restriction_A)$$

We will see below that this definition does not depend on the choice of anti-Čech system. The fact that coarse homology satisfies the Eilenberg-Steenrod exactness and dimension axioms follows easily from the corresponding properties for homology based on infinite chains.

PROPOSITION 5.3.2. *Let $f\colon X \to Y$ be a coarse map. Then f induces a homomorphism $f_*\colon HC_n(X) \to HC_n(Y)$ for all n.*

PROOF. Let $\{\mathcal{U}_i, p_i\}$ and $\{\mathcal{V}_j, q_j\}$ be anti-Čech systems for X and Y respectively. Since the Lebesgue numbers of the $\mathcal{V}_j$ tend to infinity, f is coarse, and the open sets in each $\mathcal{U}_i$ are uniformly bounded, we have that for each i there is a j_i such that if $U \in \mathcal{U}_i$, then $f(U)$ is contained in some open set V in $\mathcal{V}_j$. Hence, for each i there is a proper map $f_i\colon \mathcal{K}(\mathcal{U}_i) \to \mathcal{K}(\mathcal{V}_{j_i})$ for some j_i.

We obtain the following commutative diagram for each i

$$\begin{tikzcd}
\mathcal{K}(\mathcal{U}_i) \arrow[r, "p_i^*"] \arrow[d, "f_i"'] & \mathcal{K}(\mathcal{U}_{i+1}) \arrow[d, "f_{i+1}"] \\
\mathcal{K}(\mathcal{V}_{j_i}) \arrow[r, "q^*"] & \mathcal{K}(\mathcal{V}_{j_{i+1}})
\end{tikzcd}$$

where q^* is induced by a composition of refinement maps starting from $\mathcal{V}_{j_i}$ and ending at $\mathcal{V}_{j_{i+1}}$. After taking locally finite homology groups, induced maps on locally finite homology, and direct limits, we obtain the desired induced map f_*. If f_i and f_i' are different choices of maps, then $f_i(U)$ and $f_i'(U)$ will eventually be contained in a common element of some $\mathcal{U}_n$ with $n \geq i$, so this f_* is independent of the choices made in defining the f_i. $\square$

It can be seen from the construction of the induced maps above and the analogous properties of induced maps for homology based on infinite chains that this assignment of induced maps is functorial:

COROLLARY 5.3.3. *For coarse maps $f\colon X \to Y$ and $g\colon Y \to Z$, we have $(gf)_* = g_* f_*$.*
The identity map $\mathrm{id}_X\colon X \to X$ induces the identity homomorphism.

COROLLARY 5.3.4. *The definition of $HC_n(X)$ is independent up to isomorphism of the choice of anti-Čech system for X.*

PROOF. Let $\{\mathcal{U}_i, p_i\}$ and $\{\mathcal{V}_i, q_i\}$ be two anti-Čech systems for X. The identity map $\mathrm{id}_X\colon X \to X$ induces a homomorphism $f_{\mathcal{U}}$ from the coarse homology group defined using $\{\mathcal{U}_i, p_i\}$ to the one defined using $\{\mathcal{V}_i, q_i\}$ and a homomorphism $f_{\mathcal{V}}$ the reverse way. Since the construction of these induced homomorphisms does not depend on the choice of maps $\mathcal{K}(\mathcal{U}_i) \to \mathcal{K}(\mathcal{V}_{j_i}) \to \mathcal{K}(\mathcal{U}_{k_{j_i}})$, we know that after taking direct limits, their composition must yield the identity map. Hence, the composition $f_{\mathcal{V}} \circ f_{\mathcal{U}}$ is the identity map on the coarse homology group defined using $\{\mathcal{U}_i, p_i\}$. Similarly, $f_{\mathcal{U}} \circ f_{\mathcal{V}}$ is the identity map. So, the maps $f_{\mathcal{U}}$ and $f_{\mathcal{V}}$ are isomorphisms between the coarse homology groups defined using the different anti-Čech systems. $\square$

COROLLARY 5.3.5. *If f and g are close maps from X to Y, then the induced homomorphisms f_* and g_* are the same.*

PROOF. Since f and g are close and the open sets in any $\mathcal{U}_i$ are uniformally bounded by some R_i, there is some $S_i > 0$ such that for every $U \in \mathcal{U}_i$, the union of the images $f(U) \cup g(U)$ is contained with a ball of radius S_i. Since the Lebesgue numbers of the coverings in an anti-Čech sequence must eventually exceed S_i, we have that for each i, there is a j_i such that f_i and g_i in the construction of the induced maps can be selected as the same map $\mathcal{K}(\mathcal{U}_i) \to \mathcal{K}(\mathcal{V}_{j_i})$. Thus, the induced maps on coarse homology groups are the same. $\square$

In particular, if X and Y are coarsely equivalent spaces, then there are maps f and g such that $f \circ g$ and $g \circ f$ are close to the respective identity maps, and hence induce identity maps on coarse homology groups. Thus, X and Y have isomorphic coarse homology.

Before we state and prove the existence of a Mayer-Vietoris sequence for coarse homology, we need to define a property that helps to identify the intersection term in the sequence.

DEFINITION 5.3.6. Let A and B be subsets of a proper metric space X. We say that $A \cap B$ is **coarsely dense in thickened intersections of A and B in X** if for all $R > 0$, $A \cap B$ is coarsely dense in $D_R(A) \cap D_R(B)$, where $D_R(A) = \{x \in X \mid d(x, A) < R\}$ and similarly for $D_R(B)$.

DEFINITION 5.3.7. Let A and B be subsets of a proper metric space X. We say that $(X; A, B)$ is a **coarsely excisive decomposition of X** if

(a) $A \cup B = X$, and
(b) $A \cap B$ is coarsely dense in thickened intersections of A and B in X.

The following lemma will be used.

LEMMA 5.3.8. *Let $A \subseteq X$, let $\mathcal{U}$ be an open cover of X and let $\mathcal{K}$ be the nerve of $\mathcal{U}$. Let $\mathcal{U} \cap A$ be the open cover $\{U \cap A \mid U \in \mathcal{U}\}$ of A, and let $\mathcal{H}$ be its nerve. Then $\mathcal{K} \restriction_A$ and $\mathcal{H}$ are isomorphic as simplicial complexes.*

PROOF. We have that $\{U_0, \ldots, U_n\}$ is a simplex in $\mathcal{K} \restriction_A$ iff $(\bigcap U_i) \cap A = \bigcap(U_i \cap A)$ is nonempty. So, $\{U_0, \ldots, U_n\} \in \mathcal{K} \restriction_A$ if and only if $\{U_0 \cap A, \ldots, U_n \cap A\} \in \mathcal{H}$. The assignment $\{U_0, \ldots, U_n\} \mapsto \{U_0 \cap A, \ldots, U_n \cap A\}$ thus gives a bijective map which sends simplices to simplices. $\square$

THEOREM 5.3.9 (Coarse Mayer-Vietoris Sequence). *Let A and B be subsets of the proper metric space X. If $(X; A, B)$ is a coarsely excisive decomposition of X, then the following is a long exact sequence.*

$$\cdots \to HC_p(A \cap B) \to HC_p(A) \oplus HC_p(B) \to HC_p(X) \to HC_{p-1}(A \cap B) \to \cdots$$

PROOF. The following proof is based on the one appearing in [**16**], which seems to have overlooked some details. The idea of the proof is to use the Mayer-Vietoris sequence for locally finite homology and take a direct limit.

Let $\{\mathcal{U}_i p_i\}$ be an anti-Čech system for X, and let $\{\mathcal{K}_i, p_i^*\}$ be the associated system of nerves. For each $i \in I$, let $\mathcal{A}_i = \mathcal{K} \restriction_A$, and let $\mathcal{B} = \mathcal{K} \restriction_B$.

We first verify that $\mathcal{K}_i = \mathcal{A}_i \cup \mathcal{B}_i$ for each i. If this were not the case, there would exist a simplex $\{U_0, \ldots, U_n\} \in \mathcal{K}_i$ such that $U_0 \cap \cdots \cap U_n$ is nonempty, yet both $U_0 \cap \cdots \cap U_n \cap A$ and $U_0 \cap \cdots \cap U_n \cap B$ are empty. Let x be an element in $U_0 \cap \cdots \cap U_n$. Then x is not in $A \cup B$, which contradicts x being an element of $X = A \cup B$.

Now, for each $i \in I$, we have $\mathcal{K}_i = \mathcal{A}_i \cup \mathcal{B}_i$. Since $\mathcal{A}_i$ and $\mathcal{B}_i$ are closed subsets of $\mathcal{K}_i$, by (j), we have a Mayer-Vietoris sequence of locally-finite homology groups for each $i \in I$.

$\cdots \to H_p^{lf}(\mathcal{A}_i \cap \mathcal{B}_i) \to H_p^{lf}(\mathcal{A}_i) \oplus H_p^{lf}(\mathcal{B}_i) \to H_p^{lf}(\mathcal{K}_i) \to H_{p-1}^{lf}(\mathcal{A}_i \cap \mathcal{B}_i) \to \cdots$

When we take the direct limit of these sequences, we obtain a corresponding sequence for coarse homology. All that is left to show is that the direct limit of the terms in these sequences give the claimed coarse homology groups.

We want to show that the direct limit of $H_p^{lf}(\mathcal{A}_i \cap \mathcal{B}_i)$ is the coarse homology of $A \cap B$. We do so by showing both that simplices in each $\mathcal{A}_i \cap \mathcal{B}_i$ eventually map to simplices in one of the nerves we obtain if we view each $\mathcal{U}_j$ as a covering of $A \cap B$, and by showing that the converse of this holds. The elements of $\mathcal{A}_i \cap \mathcal{B}_i$ are simplices $\{U_0, \ldots, U_n\}$ of $\mathcal{K}_i$ such that $\bigcap U_\alpha$ has a nontrivial intersection with each of A and B. Thus, if $\bigcap U_\alpha \cap A \cap B$ is nonempty, then $\{U_0, \ldots, U_n\} \in \mathcal{A}_i \cap \mathcal{B}_i$ already. Conversely, we will show that the refinement maps eventually send every simplex of $\mathcal{A}_i \cap \mathcal{B}_i$ to a simplex $\{V_0, \ldots, V_m\}$ in some $\mathcal{K}_j$ which satisfies $\bigcap V_\beta \cap A \cap B \neq \emptyset$. Let $a \in \bigcap U_\alpha \cap A$ and let $b \in \bigcap U_\alpha \cap B$. The diameters of the U_α are universally bounded by some $Z_i > 0$ which depends only on the index i of the nerve, so the distance between a and b is bounded by Z_i. Thus,

$$\left(\bigcap U_\alpha \cap A\right) \cup \left(\bigcap U_\alpha \cap B\right) \subseteq D_{Z_i}(A) \cap D_{Z_i}(B)$$

Since $A \cap B$ is a coarse intersection, $D_{Z_i}(A) \cap D_{Z_i}(B) \subseteq D_{S_i}(A \cap B)$ for some $S_i > 0$. This establishes that for every simplex $\sigma = \{U_0, \ldots, U_n\}$ in $\mathcal{A}_i \cap \mathcal{B}_i$, there is some $x \in \bigcap U_\alpha$ which is at most S_i far from $A \cap B$. It follows that for any simplex σ in $\mathcal{A}_i \cap \mathcal{B}_i$, there is a $j > i$ for which the $p_{j-1}^* \cdots p_i^*(\sigma) \in \mathcal{K}_j$ is a simplex $\{V_0, \ldots, V_m\}$ such that $\bigcap V_i$ nontrivially intersects $A \cap B$. This ensures that the direct limit of $H_p^{lf}(\mathcal{A}_i \cap \mathcal{B}_i)$ is the coarse homology of $A \cap B$.

It follows from lemma 5.3.8 that $\varinjlim H_p^{lf}(\mathcal{A}_i) = HC_p(A)$ and $\varinjlim H_p^{lf}(\mathcal{B}_i) = HC_p(B)$. □

In the coarse category, Mayer-Vietoris sequences exist for slightly more general choices of $A, B \subseteq X$. It is not necessary that $A \cup B = X$; we only need that the inclusion of $A \cup B$ into X is a coarse equivalence. It is also not necessary that A and B actually intersect; we only need that some thickenings of A and B have an intersection which is dense in thicker intersections. This leads us to the following definition.

Definition 5.3.10. Let A and B be subsets of a proper metric space X. We say that A and B **coarsely intersect** if there exists some $r \geq 0$ such that for all $R > r$, $D_r(A) \cap D_r(B)$ is coarsely dense in $D_R(A) \cap D_R(B)$. Such a set $D_r(A) \cap D_r(B)$ is called a **coarse intersection** for A and B in X.

Note that the above definition is equivalent to the existence of thickenings $A' = D_{r_1}(A)$ of A and $B' = D_{r_2}(B)$ such that $A' \cap B'$ is coarsely dense in thickened intersections of A' and B' in X.

The defining property of a usual intersection $A \cap B$ is that it is the smallest set $\mathcal{I}$ such that if $x \in A$ and $x \in B$, then $x \in \mathcal{I}$. Here we have defined coarse intersections of A and B to be sets $\mathcal{I}$ where for every R there is an S such that if $x \in D_R(A)$ and $x \in D_R(B)$, then $x \in D_S(\mathcal{I})$. This property is the key feature that allows us to identify the intersection term in the coarse Mayer-Vietoris sequence. From the perspective of coarse homology, it appears that the correct notion of intersection is obtained by relaxing set membership to uniform closeness.

It is easy to see that if $\mathcal{I}(r) = D_r(A) \cap D_r(B)$ is a coarse intersection, then so is $\mathcal{I}(r') = D_{r'}(A) \cap D_{r'}(B)$ for every $r' > r$, since $\mathcal{I}(r) \subseteq \mathcal{I}(r')$. The following proposition implies that all of these coarse intersections are unique up to coarse equivalence.

PROPOSITION 5.3.11. *Let A and B be subsets of a space X, and let $r \geq 0$. Then the following are equivalent:*

(i) *$D_r(A) \cap D_r(B)$ is a coarse intersection for A and B in X.*
(ii) *For every $r' > r$, $D_r(A) \cap D_r(B)$ is coarsely dense in $D_{r'}(A) \cap D_{r'}(B)$.*
(iii) *For every $r' > r$, the inclusion map $i\colon D_r(A) \cap D_r(B) \to D_{r'}(A) \cap D_{r'}(B)$ is a coarse equivalence.*

PROOF. We have $(i) \iff (ii)$ trivially from the definition of coarse intersection. A simple application of lemma 5.2.8 gives us $(ii) \iff (iii)$. $\square$

In the following, we will exhibit examples and non-examples of coarse intersections. We will also give examples of sets $\mathcal{I}$ which satisfy $\forall R > 0, \exists S > 0, D_R(A) \cap D_R(B) \subseteq D_S(\mathcal{I})$ but are not coarse intersections and are not coarsely equivalent to coarse intersections of A and B. Satisfying $\exists S > 0, D_R(A) \cap D_R(B) \subseteq D_S(\mathcal{I})$ is the definition of being coarsely dense if $\mathcal{I} \subseteq D_R(A) \cap D_R(B)$. Hence, these examples will show that we only have coarse equivalence of coarse intersections because we have required them to be of the form $D_r(A) \cap D_r(B)$.

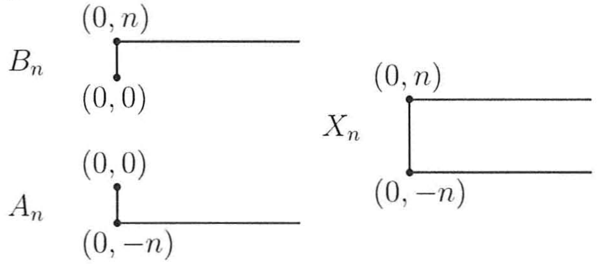
EXAMPLE 5.3.12. Consider the subsets $A = (-\infty, -1]$ and $B = [1, \infty)$ of $\mathbb{R}$.

(a) $D_0(A) \cap D_0(B) = A \cap B = \emptyset$ is not a coarse intersection of A and B; for example, $\emptyset$ is not coarsely equivalent to $D_2(A) \cap D_2(B) = [-1, 1]$.
(b) $D_1(A) \cap D_1(B) = \{0\}$ is a coarse intersection. Intersections of any further thickenings of A and B are intervals, all of which contain $\{0\}$ as a coarsely dense subset.

(c) For every $r \geq 1$, $D_r(A) \cap D_r(B) = [-r+1, r-1]$ is a coarse intersection and the inclusion map $i\colon \{0\} \to [-r+1, r-1]$ is a coarse equivalence with coarse inverse the 0 map.
(d) The set $\mathbb{R}$ satisfies $\forall R > 0, \exists S > 0, D_R(A) \cap D_R(B) \subseteq D_S(\mathbb{R})$, but it is not of the form $D_r(A) \cap D_r(B)$, and is not coarsely equivalent to sets of the form $D_r(A) \cap D_r(B)$.
(e) Every nonempty subset Y of $\mathbb{R}$ satisfies $\forall R > 0, \exists S > 0, D_R(A) \cap D_R(B) \subseteq D_S(Y)$.

Note that in the previous example, $A \cap B = \emptyset$, and yet A and B coarsely intersect. The following is an example where there is no coarse intersection of A and B, despite the fact that $A \cap B \neq \emptyset$. This example is essentially the same as that given as example 4.6 in **[7]**, where it serves as a space lacking a bounded fixed set. Both there and here, the space fails the desired property because a sequence of subsets fails to stabilize under coarse equivalence.

EXAMPLE 5.3.13. For each positive integer n, let $A_n = ([0,\infty) \times \{-n\}) \cup (\{0\} \times [-n, 0])$ and let $B_n = ([0,\infty) \times \{n\}) \cup (\{0\} \times [0, n])$, and define $X_n \subseteq \mathbb{R}^2$ to be $A_n \cup B_n$.



Let $X' = \bigcup (X_n \times \{n\}) \subseteq \mathbb{R}^3$. Finally, construct X by rotating each embedded X_n in X' by n radians about the point $(0, 0, n)$. Note that if $n \neq m$, then $n - m$ cannot be an integer multiple of 2π. Hence each embedded X_n is rotated by a unique angle. Let $A \subseteq X$ be the union over all of the rotated A_n. Define $B \subseteq X$ similarly. This construction ensures that no $D_r(A) \cap D_r(B)$ is a coarse intersection, even though $A \cap B = (0,0) \times \mathbb{Z}^+$. To see this, note that if h is the greatest integer less than $r+1$, then $D_{r+1}(A) \cap D_{r+1}(B)$ contains elements in the plane $z = h$ which are arbitrarily far away from $D_r(A) \cap D_r(B)$.

We now give version of the Mayer-Vietoris sequence for the more general case.

THEOREM A (Coarse Mayer-Vietoris). *Let A and B be subsets of the proper metric space X. If $A \cup B$ is coarsely equivalent to X by inclusion and if A and B coarsely intersect in X, then there is a long exact sequence*

$$\cdots \to HC_p(\mathcal{I}) \to HC_p(A) \oplus HC_p(B) \to HC_p(X) \to HC_{p-1}(\mathcal{I}) \to \cdots$$

for every coarse intersection $\mathcal{I}$ of A and B in X.

PROOF. Since $A \cup B$ is coarsely equivalent to X by inclusion, it is coarsely dense in X and we have that for some r_1, $X = D_{r_1}(A \cup B) = D_{r_1}(A) \cup D_{r_1}(B)$. Since A and B coarsely intersect, there is some r_2 such that $D_{r_2}(A) \cap D_{r_2}(B)$ is coarsely dense in thickened intersections. Hence, taking r to be the larger of r_1 and r_2, we have that $(X; D_r(A), D_r(B))$ is a coarsely excisive decomposition. So, we have that the Mayer-Vietoris sequence

$$\cdots \to HC_p(D_r(A) \cap D_r(B)) \to HC_p(D_r(A)) \oplus HC_p(D_r(B)) \to \\ HC_p(X) \to HC_{p-1}(A \cap B) \to \cdots$$

is exact. Now, since A is coarsely equivalent to $D_r(A)$, we have that $HC_p(D_r(A))$ is isomorphic to $HC_p(A)$. Similarly, $HC_p(D_r(B))$ is isomorphic to $HC_p(B)$. Finally, $D_r(A) \cap D_r(B)$ is coarsely equivalent to every other coarse intersection $\mathcal{I}$ of A and B in X, so $HC_p(D_r(A) \cap D_r(B))$ is isomorphic to $HC_p(\mathcal{I})$. □

The following example shows how coarse homology can fail to satisfy the Eilenberg-Steenrod excision axiom.

EXAMPLE 5.3.14 (Failure of the Eilenberg-Steenrod Excision Axiom). Let $X \subseteq \mathbb{R}^2$ be the union of $\{0,1\} \times [0,\infty)$ and $\{0\} \times [0,1]$. Let $A \subseteq X$ be the union of $\{1\} \times [0,\infty)$ and $\{0\} \times [0,1]$, and let E be the subset $\{1\} \times [0,\infty)$. We have that the closure of E is contained in the interior of A. We will show that $HC_1(X - E, A - E)$ is not isomorphic to $HC_1(X, A)$ by calculation.

First we will construct a sequence of open covers for X. For each integer $i \geq 2$, define $\mathcal{F}_i$ to be the collection $\{B_i(r,0) \mid r \in i\mathbb{Z}\}$ of open balls in $\mathbb{R}^2$. For example, $\mathcal{F}_2$ contains open balls of radius 2 which are centered at $(2n, 0)$ for integers n. Note that each ball $B_i(i \cdot n, 0)$ in $\mathcal{F}_i$ intersects only with $B_i(i \cdot (n-1), 0)$ and $B_i(i \cdot (n+1), 0)$ and that their intersections contain points in A. All three-way intersection $\bigcap B_i(i \cdot z, 0)$ is empty. Finally, each ball in $\mathcal{F}_i$ is contained in a ball in $\mathcal{F}_{2i}$.

For each $i \geq 1$, define $\mathcal{U}_i$ to be $\{B \cap X \mid B \in \mathcal{F}_{2^i}\}$. Each $\mathcal{U}_i$ is a locally finite open cover of X, and it is clear that refinement maps p_i can be chosen so that $\{\mathcal{U}_i, p_i\}$ is an anti-Čech system. Let $\{\mathcal{K}_i, p_i^*\}$ be the associated system of nerves.

Now we construct a sequence of open covers for $X - E$. Simply take $\mathcal{V}_i$ to be $\{B \cap (X - E) \mid B \in \mathcal{F}_{2i}\}$. Again, refinement maps q_i can be chosen so that $\{\mathcal{V}_i, q_i\}$ is an anti-Čech system for $X - E$. Let $\{\mathcal{H}_i, q_i^*\}$ be the associated system of nerves. Note that each $\mathcal{H}_i$ is homeomorphic to a single ray $[0, \infty)$.

First we calculate $HC_1(X, A)$. By definition, $HC_1(X, A)$ is the direct limit $\varinjlim H_1^{inf}(\mathcal{K}_i - \mathcal{K}_i \restriction_A)$. But, every simplex $\{U_0, U_1\}$ in $\mathcal{K}_i$ is in $\mathcal{K}_i \restriction_A$ since $U_0 \cap U_1$ always contains a point in A. Thus, $\mathcal{K}_i - \mathcal{K}_i \restriction_A$ is empty, and $HC_1(X, A)$ is trivial.

Now we calculate $HC_1(X - E, A - E)$. By definition, $HC_1(X - E, A - E)$ is the direct limit $\varinjlim H_1^{lf}(\mathcal{H}_i, \mathcal{H}_i \restriction_{A-E})$. The nerve $\mathcal{L}_i$ consists of a vertex

$\langle n \rangle$ for each $n \in \{0, 1, 2, \dots\}$ since $B_i(i \cdot n, 0) \cap (X - E)$ is nonempty, and a 2-simplex $\langle n, n+1 \rangle$ for each $n \in \{0, 1, 2, \dots\}$ since $B_i(i \cdot n, 0) \cap B_i(i \cdot (n+1), 0) \cap (X - E)$ is nonempty. The subnerve $\mathcal{H}_i \restriction_{A-E}$ contains a vertex $\langle n \rangle$ iff $B_i(i \cdot n, 0) \cap (X - E) \cap (A - E) \neq \emptyset$. But, this only happens when $i \cdot n = 0$, since all other $B_i(i \cdot n, 0)$ are too far away from A. Hence $\mathcal{H}_i \restriction_{A-E}$ contains only $\langle 0 \rangle$. Now, consider the locally finite 1-chain $c = \sum_{n \in \{0,1,2,\dots\}} \langle n, n+1 \rangle$. Taken modulo $\mathcal{H}_i \restriction_{A-E}$, the chain c has no boundary. Since there are no 2-simplices in $\mathcal{H}_i$, this means that c represents a locally finite homology class, and in fact it is a representative of the generator of $H_1^{lf}(\mathcal{H}_i, \mathcal{H}_i \restriction_{A-E}) \cong \mathbb{Z}$. It can be checked that the refinement map p_i sends c to the a 1-cycle in $\mathcal{H}_{i+1}$ satisfying analogous properties. Thus, the nontrivial homology persists in the direct limit and $HC_1(X - E, A - E) \cong \mathbb{Z}$.

The failure of the previous example is caused by the existence of elements in $X - E$ which are uniformly close to E but arbitrarily far away from $A - E$. The next proposition shows that we do have excision when such cases are prevented.

THEOREM B (Coarse Excision). *If $A, E \subseteq X$ are such that $E \subseteq A$ and for all $R > 0$, there is some $S > 0$ such that $D_R(E) - E \subseteq D_S(A - E)$, then the inclusion map $i : (X - E, A - E) \to (X, A)$ induces an isomorphism.*

PROOF. Let $\{\mathcal{U}_\alpha, p_\alpha\}$ be an anti-Čech system for X, and let $\{\mathcal{K}_\alpha, p_\alpha^*\}$ be the associated system of nerves. For each α, define $\mathcal{V}_\alpha$ to be $\{U \cap (X - E) \mid U \in \mathcal{U}_\alpha\}$, and let q_α be the refinement map $\mathcal{V}_\alpha \to \mathcal{V}_{\alpha+1}$ induced by p_α. We have that $\{\mathcal{V}_\alpha, q_\alpha\}$ is an anti-Čech system for $X - E$. Let $\{\mathcal{H}_\alpha, q_\alpha^*\}$ be the system of nerves associated to $\{\mathcal{V}_\alpha, q_\alpha\}$.

We want to apply proposition 1.3.7. So, we must check that for each α, there is some $\beta > \alpha$ and homomorphism $h_{\alpha,\beta}$ which makes the following diagram commute

$$\begin{tikzcd} H_n^{lf}(\mathcal{H}_\beta, \mathcal{H}_\beta \restriction_{A-E}) \arrow[r, "i^*"] & H_n^{lf}(\mathcal{K}_\beta, \mathcal{K}_\beta \restriction_A) \\ H_n^{lf}(\mathcal{H}_\alpha, \mathcal{H}_\alpha \restriction_{A-E}) \arrow[u, "q_{\alpha,\beta}"] \arrow[r, "i^*"] & H_n^{lf}(\mathcal{K}_\alpha, \mathcal{K}_\alpha \restriction_A) \arrow[u, "p_{\alpha,\beta}"'] \arrow[ul, "h_{\alpha,\beta}"'] \end{tikzcd}$$

where $p_{\alpha,\beta}$ is the map induced on relative locally finite homology groups by the composition $p_{\beta-1} \cdots p_\alpha$ of refinement maps, and similarly for $q_{\alpha,\beta}$.

Let c be a representative of a relative locally finite homology class $\bar{c} \in H_n^{lf}(\mathcal{K}_\alpha, \mathcal{K}_\alpha \restriction_A)$. Then c is of the form $(\sum \sigma_\nu) + (\sum \tau_\mu)$ with each $\sigma_\nu \in \mathcal{K}_\alpha - \mathcal{K}_\alpha \restriction_A$ and each $\tau_\mu \in \mathcal{K}_\alpha \restriction_A$. A simplex $\sigma_\nu \in \mathcal{K}_\alpha - \mathcal{K}_\alpha \restriction_A$ is a finite set $\{U_0, \dots, U_n\}$ such that $\bigcap U_j \cap X \neq \emptyset$ and $\bigcap U_j \cap A = \emptyset$. This means that $\bigcap U_j \cap (A - E) = \emptyset$.

If for every simplex $\sigma_v = \{U_0, \ldots, U_n\} \in \mathcal{K}_\alpha - \mathcal{K}_\alpha \restriction_A$, we have $U_j \cap E = \emptyset$ for each j, then we can let $V_j = U_j \cap (X - E) = U_j \in \mathcal{V}_\alpha$ for each j, and we will have that each V_j is distinct, $\bigcap V_j \cap (X - E) \neq \emptyset$, and $\bigcap V_j \cap (A - E) = \emptyset$. Thus, $\{V_0, \ldots, V_n\} = \{U_0, \ldots, U_n\}$ is a simplex in $\mathcal{H}_\alpha - \mathcal{H}_\alpha \restriction_{A-E}$. In this case, we can define $h_{\alpha,\alpha}$ by linear extension of the inclusion map which sends a simplex σ_ν to itself as a simplex in $\mathcal{H}_\alpha - \mathcal{H}_\alpha \restriction_{A-E}$. The diagram above is obviously commutative, since we have taken $\beta = \alpha$.

Suppose that there is a simplex $\sigma_\nu = \{U_0, \ldots, U_n\} \in \mathcal{K}_\alpha - \mathcal{K}_\alpha \restriction_A$ such that $U_j \cap E \neq \emptyset$ for some j. We have that $\bigcap U_j \cap (X - E)$ is nonempty, since $\bigcap U_j \cap X \neq \emptyset$ but $\sigma_\nu \notin \mathcal{K}_\alpha \restriction_A$. Since the diameters of the U_j are uniformly bounded by some R_α, each of the $U_j \in \sigma_\nu$ is contained in $D_{2R_\alpha}(E)$. It follows from the hypotheses of the proposition that there is some fixed $S_\alpha > 0$ such that each $U_j \in \sigma_\nu$ will satisfy $U_j \subseteq D_{S_\alpha}(A - E)$. Again, since the diameters of the U_j are uniformly bounded, it is not hard to see that this implies that there is a fixed $x_\nu \in A - E$ such that $d(x_\nu, U_j) < S_\alpha + R_\alpha$ for each $U_j \in \sigma_\nu$. Thus, for some fixed $\beta_\alpha > \alpha$, the composition of refinement maps $p_{\beta-1} \cdots p_\alpha$ will send each U_j to a vertex $p_{\beta-1} \cdots p_\alpha(U_j)$ which contains $x_\nu \in A - E$. But this means that $x_\nu \in \bigcap p_{\beta-1} \cdots p_\alpha(U_j)$. Hence, if $\{p_{\beta-1} \cdots p_\alpha(U_0), \ldots, p_{\beta-1} \cdots p_\alpha(U_j)\}$ is still an n-simplex in $\mathcal{K}_\beta$, then it is in $\mathcal{K}_\beta \restriction_A$. Thus, $p_{\alpha,\beta}(\sigma_\nu)$ is $0 \in H_n^{lf}(\mathcal{K}_\beta, \mathcal{K}_\beta \restriction_A)$ either because $p_{\beta-1} \cdots p_\alpha(\sigma_\nu)$ collapses to an m-simplex of $\mathcal{K}_\beta$ with $m \leq n$, or because $p_{\beta-1} \cdots p_\alpha(\sigma_\nu)$ is an element of $\mathcal{K}_\beta \restriction_A$. We can now define $h_{\alpha,\beta}$. Note again that if no $U_j \in \sigma_\nu$ intersects E nontrivially, then σ_ν is already a simplex in $\mathcal{H}_\alpha - \mathcal{H}_\alpha \restriction_{A-E}$. We define $h_{\alpha,\beta}$ by linear extension of the map given by

$$\sigma_\nu = \{U_0, \ldots, U_n\} \mapsto \begin{cases} 0 & \text{if } U_j \cap E \neq \emptyset \text{ for any } j \\ q_{\alpha,\beta}(\sigma_\nu) & \text{otherwise} \end{cases}$$

We will check commutativity of the diagram for relative locally finite homology classes represented by a single simplex. Commutativity is clear when $h_{\alpha,\beta}$ coincides with $q_{\alpha,\beta}$. Suppose some U_j does intersect E nontrivially. Then both $h_{\alpha,\beta}(\sigma_\nu) = 0$ and $p_{\alpha,\beta}(\sigma_\nu) = 0$, so $i^* h_{\alpha,\beta} = p_{\alpha,\beta}$ trivially. If $h_{\alpha,\beta} i_*(c) = 0$ for some $c \in H_n^{lf}(\mathcal{H}_\alpha, \mathcal{H}_\alpha \restriction_{A-E})$ represented by a simplex σ, then σ is of the form $\{U_0 \cap (X - E), \ldots, U_n \cap (X - E)\}$ with each $U_j \in \mathcal{U}_\alpha$ and $\bigcap p_{\beta-1} \cdots p_\alpha(U_j) \cap (A - E) \neq \emptyset$. It follows that $\bigcap q_{\beta-1} \cdots q_\alpha(U_j \cap (X - E)) \cap (A - E) \neq 0$, and hence $q_{\alpha,\beta}$ maps c to 0. So, $h_{\alpha,\beta} i^*(c) = q_{\alpha,\beta}(c)$. Commutativity of the diagram for other homology classes follows. □

The following property is stated as an axiom of relative coarse homology in Mitchener's article, but he gives no construction of relative coarse homology nor any verification that the axiom holds anywhere.

COROLLARY 5.3.15. *If $(X; A, B)$ is a coarsely excisive decomposition of X, then the inclusion maps $k_1 \colon (B, A \cap B) \to (X, A)$ and $k_2 \colon (A, A \cap B) \to (X, B)$ induce isomorphisms.*

PROOF. Let $E = X - B$. Then $X - E = B$ and $A - E = A \cap B$. The result will follow from coarse excision if we can verify that $\forall R > 0, \exists S > 0, (X-E) \cap D_R(E) \subseteq D_S(A-E)$. Let $R > 0$, and suppose that $x \in (X-E) \cap D_R(E)$. Since $X = A \cup B$, we have that $E = X - B \subseteq A$. So, $x \in D_R(E)$ implies that $x \in D_R(A)$. Thus, $x \in B \cap D_R(A)$. Since $A \cap B$ is a coarse intersection, there is some fixed S such that $x \in A \cap B = A - E$. So, coarse excision applies, and we have that the map induced by the inclusion $k_1\colon (B, A\cap B) \to (X, A)$ is an isomorphism.

That k_2 induces an isomorphism can be checked similarly, letting $E = X - A$. □

Coarse homology trivially satisfies the Eilenberg-Steenrod homotopy invariance axiom, since there is a coarse map $h\colon X \times [0,1] \to Y$ exactly when $h(x,0)$ and $h(x,1)$ are coarsely equivalent. It is apparent that taking a product with a compact space is not appropriate for coarse homology. However, when we use the more general notion of Lipschitz homotopy, we see that coarse homology satisfies a stronger property.

DEFINITION 5.3.16. A proper metric space X is **flasque** if it admits a self-map $s\colon X \to X$ such that

(a) s is coarsely equivalent to the identity map,
(b) for each compact $K \subseteq X$, there is some n_K such that for all $n \geq n_k$, $s^n(X) \cap K = \emptyset$, and
(c) s is an isometry of X into itself.

The following is proven by a technique known as the Eilenberg swindle; similar statements and proofs appear variously in the context of K-theory and C^*-algebras, as in **[9]** page 233. The general idea is that the conditions required to be flasque ensure that any cycles can be pushed out to infinity by telescoping sums.

LEMMA 5.3.17. *If X is flasque, then $HC_n(X) = 0$ for all n.*

PROPOSITION 5.3.18. *If $f, g\colon X \to Y$ are Lipschitz homotopic, then the induced maps f_* and g_* are the same map $HC_n(X) \to HC_n(Y)$ for all n.*

PROOF. Let $H(x,t) = (h(x,t), t)$ be the Lipschitz homotopy between f and g. Let $Z = \{(x,t) \mid 0 \leq t \leq t_{B_1(x)}\}$ where $t_{B_1(x_0)} \in \mathbb{R}^+$ is the value after which $h(x,t)$ is constant in t on the bounded set $B_1(x_0) \subseteq X$. Let $Z_0 = \{(x,t) \in Z \mid t = 0\}$ and let $Z_\infty = \{(x,t) \in Z \mid t = t_{B_1(x)}\}$ be the boundary pieces of Z. Projection to the first coordinate gives coarse maps π_0 and π_∞ from Z_0 and Z_∞ respectively to X. Restriction of h gives coarse maps which factor through f and g respectively. Consider the diagram

$$\begin{tikzcd}
Z_0 \arrow[r, "\pi_0"] \arrow[dr, "i_0"] \arrow[ddr, "h\restriction_{Z_0}"'] & X \arrow[dd, bend left, "f"] \\
& Z \arrow[u, "p"] \arrow[d, "h"] \\
& Y
\end{tikzcd}$$

where i_0 is inclusion and p is projection such that $\pi_0 = pi_0$. We also have an analogous map i_∞ and diagram for Z_∞ such that $\pi_\infty = pi_\infty$. We now argue that it is sufficient to show that i_0 and i_∞ induce isomorphisms on coarse homology groups. Since π_0 is a coarse equivalence, it induces an isomorphism. If i_0^* is an isomorphism, then it follows that p^* is an isomorphism, and $p^* = \pi_0^* i_0^{*-1}$. If i_∞^* is also an isomorphism, then we have that $p^* = \pi_\infty^* i_\infty^{*-1}$ and π_∞^* is an isomorphism as well. Now, p does not have an inverse, but we can define an inclusion map $j_0 \colon X \to Z$ by $j_0 = i_0\pi_0^{-1}$ and it follows that $j_0^* = p^{*-1}$. From this we get that $j_0^* = (\pi_\infty^* i_\infty^{*-1})^{-1} = i_\infty^* \pi_\infty^{*-1}$. Hence,

$$g^* = (h\restriction_{Z_\infty})^* \pi_\infty^{*-1} = h^* i_\infty^* \pi_\infty^{*-1} = h^* j_0^* = h^* i_0^* \pi_0^{*-1} = (h\restriction_{Z_0})^* \pi_0^{*-1} = f^*$$

Now we will verify that i_0^* is an isomorphism; the argument for i_∞^* is similar. Note that Z can be viewed as a subspace of $X \times \mathbb{R}$. Let $W_- = \{(x,t) \mid t \leq 0\}$ and $W_+ = \{(x,t) \mid t \geq T_{B_1(x)}\}$ be the parts of $X \times \mathbb{R}$ to the left and right of Z. Then, $(X; W_-, Z \cup W_+)$ and $(X; W_- \cup Z, Z \cup W_+)$ are both coarsely excisive decompositions of X. The Mayer-Vietoris sequences for these decompositions fit into the diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
HC((W_- \cup Z) \cap (Z \cup W_+)) \arrow[d] & HC(W_- \cap (Z \cup W_+)) \arrow[l] \arrow[d] \\
HC((W_- \cup Z)) \oplus HC(Z \cup W_+) \arrow[d] & HC(W_-) \oplus HC(Z \cup W_+) \arrow[l] \arrow[d] \\
HC(X \times R) \arrow[d] & HC(X \times R) \arrow[l] \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

where the horizontal arrows are induced by inclusion maps. Now, W_-, $W_- \cup Z$, and $Z \cup W_+$ are all flasque by translation to the left, left, and right respectively. So, their coarse homology groups are trivial. It follows that the middle and right vertical arrows in the diagram above are isomorphisms. By the five lemma, we have that the left vertical arrow is an isomorphism as well. But, $W_- \cap (Z \cup W_+) = W_- \cap Z = Z_0$, and $(W_- \cup Z) \cap (Z \cup W_+) = Z$, so we have that $i_0^* : HC_n(Z_0) \cong HC_n(Z)$ for all n. $\square$

Thus, coarse homology satisfies Eilenberg-Steenrod axioms 1, 2, 3, 4, 5, and 7 for proper metric spaces and coarse maps. It inherits the exactness and dimension axioms from homology based on infinite chains. The one axiom it fails to satisfy is the Eilenberg-Steenrod version of the excision axiom; the hypothesis requiring that the subset we intend to excise is "well contained" is not appropriate for the coarse category. However, we obtain an coarse analogue of excision when we appropriately modify the hypothesis. The Eilenberg-Steenrod version of homotopy invariance for coarse homology (where a "homotopy" would be a map from $X \times [0, 1]$) is trivially satisfied as a result of close maps inducing equal maps. Additionally, coarse homology satisfies invariance under Lipschitz homotopy, which is a more appropriate notion for the coarse category.

5.4. The Map from H^∞ to HC

There is a map from homology based on infinite chains into coarse homology which is an isomorphism when the space is locally nice. In order to define this map, we will need the following definition. The development of this material appears in [**10**].

Definition 5.4.1. Given a topological space X and an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X, a **partition of unity of X subordinate to $\mathcal{U}$** is a collection of continuous functions $\{\phi_i\}_{i \in I}$ such that the following hold:

(a) $\phi_i \colon X \to [0, 1]$ for all i,
(b) for all $x \in X$, there is a neighborhood of x where all but finitely many of the ϕ_i are 0,
(c) for all $x \in X$, the sum of the functions satisfies $\sum\limits_{i \in I} \phi_i(x) = 1$, and
(d) for all $i \in I$, the support of ϕ_i is contained in U_i.

Given a proper metric space X and a sequence of nerves $(\mathcal{K}_i)_{i \in I}$ coming from an anti-Čech sequence $(\mathcal{U}_i)_{i \in I}$, we can define proper continuous maps $\gamma_i \colon X \to \mathcal{K}_i$ as follows. A theorem of A. H. Stone states that any metric space is paracompact, and it follows that it admits partitions of unity. For each i, select a partition of unity $\{\phi_\alpha\}$ subordinate to $\mathcal{U}_i = \{U_\alpha\}$. Define the map γ_i by $\gamma_i(x) = \sum\limits_{\alpha} \phi_\alpha(x) \langle U_\alpha \rangle$ for all $x \in X$. The finitely many U_α for which

$\phi_\alpha(x)$ is nonzero define a simplex in $\mathcal{K}_i$, and so the right-hand side of the equation specifies a point in the nerve. Different choices of partitions of unity subordinate to $\mathcal{U}_i$ give rise to properly homotopic maps γ_i and γ_i'. Thus, these maps uniquely induce maps on homology groups $\gamma_{i,*}\colon H_p^\infty(X) \to H_p^\infty(\mathcal{K}_i) \cong H_p^{lf}(\mathcal{K}_i)$ for every p.

DEFINITION 5.4.2. Given a proper metric space X, the **coarsening map** $c\colon H_p^\infty(X) \to HC_p(X)$ is the map received by taking the direct limit of the above constructed $\gamma_{i,*}\colon H_p^\infty(X) \to H_p^{lf}(\mathcal{K}_i)$ over all i.

Stating when the map c is known to be an isomorphism requires the following definitions.

DEFINITION 5.4.3. A metric space X is **uniformly contractible** if for each $R > 0$ there is some $S > 0$ such that $B_R(x)$ is contractible within $B_S(x)$ for all $x \in X$.

DEFINITION 5.4.4. A path metric space X is a **metric simplicial complex** if it is a simplicial complex and its metric coincides on each simplex with the usual spherical metric obtained by regarding Δ^n as the set of points of $S^n \subseteq \mathbb{R}^{n+1}$ with nonnegative coordinates.

Note that any locally finite simplicial complex can be given a complete metric which makes it a metric simplicial complex.

DEFINITION 5.4.5. A proper metric space X has **bounded coarse geometry** if there is some $\varepsilon > 0$ such that for all $R > 0$ there is a $C > 0$ such that the maximum number of points in an ε-separated subset of any ball of radius R is at most C.

PROPOSITION 5.4.6. *Let X be a complete path metric space. Let $\mathcal{U} = \{U_\alpha\}$ be a cover of X with positive Lebesgue number and consisting of sets with bounded diameter, and let $\mathcal{K}$ be the nerve of $\mathcal{U}$. Let $\{\phi_\alpha\}$ be a partition of unity subordinate to $\mathcal{U}$. Then the map $\gamma\colon X \to \mathcal{K}$ defined by $\gamma(x) = \sum_\alpha \phi_\alpha(x)\langle U_\alpha\rangle$ is a coarse equivalence.*

LEMMA 5.4.7. *Let X be a finite-dimensional metric simplicial complex and let Y be uniformly contractible. Let $f\colon X \to Y$ be a coarse map. Then there is a continuous coarse map $g\colon X \to Y$ which is close to f. Moreover, if f is continuous on a subcomplex X' of X, then we can take $g = f$ on X'.*

PROOF. The map g is constructed by induction on the n-skeletons X^n of X. Let $g_0\colon X' \cup X^0 \to Y$ be defined by $g_0(x) = f(x)$. Suppose we have constructed $g_i\colon X' \cup X^i \to Y$. To construct $g_{i+1} : X' \cup X^{i+1}$, we have to continuously extend g_i to the $(n+1)$-simplices of $X - X'$. But, g_i is already defined on the boundary of each $(n+1)$-simplex Δ. Since f is coarse and each simplex in a metric simplicial complex has diameter at most 2, there is a fixed upper bound $R > 0$ on the diameter of $f(\Delta)$ for every simplex Δ in X. Since Y is uniformly contractible, there is some $S > 0$ such that each $f(\Delta)$ contracts in an S-ball. Hence, there is some constant $C_f > 0$ such that for each $(n+1)$-simplex Δ, $g_i \restriction_{\partial\Delta}$ can be continuously extended across Δ to a map whose image lies within a radius C ball containing $f(\partial\Delta)$.

Since X is finite-dimensional, we eventually obtain a map $g\colon X \to Y$ which is continuous everywhere, and as can be seen by its construction, it coincides with f on $X' \cup X^0$ and there is a $C_f > 0$ such that $d(g(x), g(x')) < C_f$ whenever $x \in X^0$ is a vertex of a simplex containing x'. Since X^0 is coarsely dense in a metric simplicial complex, this means that g is close to f. Since g is close to a coarse map, it is also coarse. □

LEMMA 5.4.8. *Let X be a finite-dimensional metric simplicial complex and let Y be a uniformly contractible space. Then if $f, g\colon X \to Y$ are coarse, continuous, and close, then they are properly homotopic.*

PROOF. As noted before, since f and g are close, there is a coarse map $h\colon X \times [0,1] \to Y$ with $h \restriction_{X\times\{0\}} = f$ and $h \restriction_{X\times\{1\}} = g$. We have assumed any such h is continuous on the subcomplex $X \times \{0,1\}$ of $X \times [0,1]$. So, if we apply the previous lemma to h, we obtain a continuous coarse map $H\colon X \times [0,1] \to Y$ which is close to h, equal to f on $X \times \{0\}$, and equal to g on $X \times \{1\}$. Hence, H is a proper homotopy between f and g. □

COROLLARY 5.4.9. *If two uniformly contractible, finite-dimensional metric simplicial complexes are coarsely equivalent, then they are proper homotopy equivalent.*

PROOF. Suppose X and Y are two such spaces, and suppose $f\colon X \to Y$ is a coarse equivalence with coarse inverse g. Then f and g are each close to continuous, coarse maps f' and g' respectively. Since gf is close to the identity, so is $g'f'$. Hence, $g'f'$ and id_X are coarse, continuous, and close, so they are properly homotopic. Similarly, $f'g'$ and id_Y are properly homotopic. □

Bounded coarse geometry has the following consequences. These allow us to build an anti-Čech sequence whose nerves never become infinite dimensional.

PROPOSITION 5.4.10. *Let X be a space with bounded coarse geometry. Then X is finite dimensional, and for any $R > 0$ there is $S > 0$ such that X has an open cover $\mathcal{U}$ satisfying the following properties.*

(a) *The Lebesgue number of $\mathcal{U}$ is at least R.*
(b) *$\mathcal{U}$ is of finite order. That is, the nerve of $\mathcal{U}$ is finite dimensional.*
(c) *The sets of $\mathcal{U}$ have diameter less than S.*

We can now prove that locally finite homology and coarse homology coincide for certain spaces.

PROPOSITION 5.4.11. *Let X be a uniformly contractible metric simplicial complex with bounded coarse geometry. Then the map $c\colon H_p^{lf}(X) \to HC_p(X)$ is an isomorphism for all p.*

PROOF. The idea is to construct an anti-Čech system and use proper homotopy invariance of H^{∞}. We begin with any finite-dimensional cover $\mathcal{U}_1$ with positive Lebesgue number and universally bounded diameter, and let $\mathcal{K}_1$ be the nerve of $\mathcal{U}_1$. Define $f_1\colon X \to \mathcal{K}_1$ by choosing a partition of unity subordinate to $\mathcal{U}_1$ and using the formula $f_1(x) = \sum\limits_{\alpha} \phi_\alpha(x)\langle U_\alpha \rangle$. Then, f_1 is a proper continuous map and a coarse equivalence. As a coarse equivalence, it has a coarse inverse $g_1\colon \mathcal{K}_1 \to X$. Since $\mathcal{K}_1$ is a finite dimensional metric simplicial complex and X is uniformly contractible, we can assume g_1 is continuous as well. Then $g_1 f_1$ is a coarse, continuous map which is close to id_X, and so $g_1 f_1$ and id_X are proper homotopic. So, g_1 is a left proper homotopy inverse for f_1. Similarly, $f_1 g_1$ is close to $\mathrm{id}_{\mathcal{K}_1}$. We do not know whether $\mathcal{K}_1$ is uniformly contractible, so we do not know whether g_1 is a right inverse. However, since $f_1 g_1$ does not send points arbitrarily far away, we can find a second finite-dimensional cover $\mathcal{U}_2$ with Lebesgue number exceeding the diameter of sets in $\mathcal{U}_1$ and with universally bounded sets, and a refinement map can be selected so that its induced map $f_2\colon \mathcal{K}_2 \to \mathcal{K}_1$ is properly homotopic to $f_2 f_1 g_1$ by a linear homotopy. Continuing on constructing an anti-Čech system and nerves in this fashion leads to the following diagram.

$$\begin{tikzcd}
X \arrow[d, "f_1"'] \arrow[dr] \arrow[drrr] & & & & \\
\mathcal{K}_1 \arrow[r, "f_2"'] & \mathcal{K}_2 \arrow[r, "f_3"'] & \cdots \arrow[r, "f_{i-1}"'] & \mathcal{K}_i \arrow[r, "f_i"'] & \cdots
\end{tikzcd}$$

The diagram clearly commutes if we let the diagonal lines be $h_i\colon X \to \mathcal{K}_i$ defined by $h_i = f_i \cdots f_1$. Moreover, for each i we have that f_i is continuous, h_i has a left proper homotopy inverse g_i, and f_{i+1} and $h_{i+1} g_i$ are properly homotopic. It follows from the proper homotopy invariance of H^{∞} that the direct limit of the induced maps $(h_i)_*\colon H^{lf}(X) \to H^{lf}(\mathcal{K}_i)$ is an isomorphism $H^{lf}(X) \to HC(X)$. $\square$

Noting that $\mathbb{R}^n$ is coarsely equivalent to a uniformly contractible metric simplicial complex with bounded coarse geometry, we have $HC_p(\mathbb{R}^n) \cong H_p^{lf}(\mathbb{R}^n)$.

COROLLARY 5.4.12. *The coarse homology of $\mathbb{R}^n$ for $n \geq 0$ is given by*

$$HC_p(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & p = n \\ 0 & p \neq n \end{cases}$$

5.5. Relations between Coarse Homology and End Homology

End homology is essentially a version of homology based on infinite chains which disregards local features of the space. Since coarse homology is also defined using homology based on infinite chains (as locally finite homology) and also disregards local features, one might suspect that there is a relation between coarse and end homology. In this section, we explore this relation.

Recall that we have the following long exact sequence

$$\cdots \to H_n^c(X) \to H_n^\infty(X) \to H_n^e(X) \to H_{n-1}^c(X) \to \cdots$$

The proposition below shows that in dimensions greater than 0, homology with compact supports cannot distinguish between a uniformly contractible proper metric space and a contractible space. That is, H_n^c is 0 for $n > 0$ and so, given the exact sequence above, H_n^∞ and H_n^e coincide for $n > 1$.

PROPOSITION 5.5.1. *If X is a uniformly contractible proper metric space, then*

$$H_n^c(X, G) = \begin{cases} 0 & n > 0 \\ G & n = 0 \end{cases}$$

PROOF. By definition, $H_n^c(X) = \varinjlim H_n^\infty(Y)$ where Y is compact in X. Let $(Y_\alpha)_{\alpha \in I}$ be the collection of compact sets in X indexed by some set I. Let $\bar{c}$ be an element of $H_n^c(X)$ represented by some $c \in H_n^\infty(Y_\alpha)$.

Since X is a metric space and Y_α is compact in X, there is some $R > 0$ such that Y_α is contained in some ball B_R of radius R. Since X is uniformly contractible, there is some $S > 0$ such that B_R is contractible within a ball B_S of radius S. Since X is proper, the closure of B_S is a compact set Y_β. Now, $Y_\alpha \subseteq B_R \subseteq B_S \subseteq Y_\beta$ and B_R is contractible in B_S, so Y_α is contractible in Y_β. If $n > 0$, it follows that the image of c is in the same homology class as 0 in $H_n^\infty(X)$. Hence $c \sim 0$ in the direct limit, and so $\bar{c} = \bar{0}$ in $H_n^c(X)$. If $n = 0$, we can select a base point x_0 and use uniform contractibility on the ball containing x_0 and the support of c to show that c is in the same homology class as a p-function defined by $p(x_0) = g \in G$ and $p(x) = 0$ for all $x \neq x_0$ in X. In this case, it is apparent that the direct limit gives the same homology group as that of a point. □

COROLLARY 5.5.2. *If X is a uniformly contractible proper metric space, then $H_n^\infty(X) \cong H_n^e(X)$ for $n > 1$.*

EXAMPLE 5.5.3. As a counterexample for the case $n = 1$, consider the ray $\mathbb{R}^+ = [0, \infty)$. This is a uniformly contractible proper noncompact metric space. We have $H_1^\infty(\mathbb{R}^+) = 0$, but $H_1^e(\mathbb{R}^+) = \mathbb{Z}$ since we can exhaust $\mathbb{R}^+$ by the sequence of compact subsets $[0, n]$ and each of $H_0^\infty(\mathbb{R}^+ - [0, n])$ is $\mathbb{Z}$. The problem here is that the boundary of an infinite 1-chain can be compact despite the nice properties of our space, so H^e may treat them as cycles when H^∞ does not.

As a counterexample for the case $n = 0$, take any compact space. We have $H_0^\infty(X) = H_0^c(X)$ nontrivial, but $H_0^e(X)$ trivial.

LEMMA 5.5.4. *If X is a simplicial complex and every vertex in X is the boundary of some infinite 1-chain, then $H_0^{lf}(X) = H_0^e(X) = 0$.*

PROOF. A cycle in $H_0^{lf}(X)$ is a potentially infinite sum of vertices v_i. Each v_i is the boundary of some w_i. So, $\sum v_i$ is the boundary of $\sum w_i$. Hence, all cycles in $H_0^{lf}(X)$ are in the same class as 0. The map induced by projection on chain complexes is a surjection $H_0^{lf}(X) \to H_0^e(X)$ in dimension 0, so we have $H_0^e(X)$ as well. □

LEMMA 5.5.5. *If X is a uniformly contractible proper metric simplicial complex which is noncompact, then every vertex in X is the boundary of some infinite 1-chain.*

PROOF. Let v_0 be a vertex in X. Since X is noncompact, we can find an infinite sequence $(v_0, v_1, \dots)$ such that the distance $d(v_0, v_i) \geq i$. Since X is uniformly contractible there is a finite 1-chain w_i with boundary $v_{i+1} - v_i$ for each i. Take w to be the 1-chain which is the infinite sum over the w_i. Then ∂w is a telescoping sum in which all vertices cancel except v_0. □

As a result, we get isomorphism in dimension 0, but this is not very interesting, since both groups are trivial.

Given 5.4.11, we now easily get the following.

COROLLARY 5.5.6. *If X is a uniformly contractible metric simplicial complex with bounded coarse geometry, then $H_n^e(X) \cong HC_n(X)$ for $n > 1$. If X is also noncompact, then $H_0^e(X) = HC_0(X) = 0$.*

The following example indicates that while the possible failure of H_1^e and H_1^{lf} to coincide precludes an isomorphism between H_1^e and HC_1, it is clearly not the only obstruction; contractibility issues are still relevant.

EXAMPLE 5.5.7. Let $X \subseteq \mathbb{R}^2$ be the union of the lines $y = 0$ and $y = 1$. Then $H_1^e(X) \cong H_1^{lf}(X) \cong \mathbb{Z}^2$ but $HC_1(X) \cong \mathbb{Z}$ since X is coarsely equivalent to $\mathbb{R}$.

We have inherited our isomorphism from the isomorphism $HC \cong H^{lf}$ when it exists. Since end homology disregards homology classes coming from cycles with compact support, and so disregards much more local information than locally finite homology, one may wonder whether $H^e \cong HC$ holds in cases where the isomorphism $HC \cong H^{lf}$ fails, in particular, for spaces failing to be uniformly contractible. We obviously cannot simply drop all contractibility requirements, as the above example and higher-dimension analogues show.

However, as the next example shows, HC and H^e can coincide in some otherwise sufficiently nice spaces where uniform contractibility and $HC \cong H^{lf}$ both fail.

EXAMPLE 5.5.8. Let X be the subspace of $\mathbb{R}^3$ consisting of the plane $z = 0$ and the top half of a sphere of radius 1 centered at the origin. We have that $H_2^{lf}(X) \cong \mathbb{Z}^2$; locally finite homology detects both the hole at the origin enclosed by part of the plane and the half-sphere, as well as the hole "at infinity" enclosed by the entire plane. However, $H_2^e(X) \cong \mathbb{Z} \cong HC_2(X)$.

We keep our isomorphism despite the lack of uniform contractibility in this example because the problematic compactly supported chains are confined to a single compact set. The next sequence of examples provides some general sense as to what is needed.

EXAMPLE 5.5.9.

(a) Let X be the infinite ladder as in examples 4.4.2 and 4.5.6. We have already noted that $H_1^e(X)$ is infinitely generated. However, $HC_1(X) \cong 0$ since X is coarsely equivalent to $\mathbb{R}^+$.

(b) In general, for any dimension $n \geq 1$, a space $L_n \subseteq \mathbb{R}^{n+1}$ analogous to the infinite ladder can be constructed by isometrically embedding $S^{n-1} \times \mathbb{R}^+$ into $\mathbb{R}^{n+1}$ so that the ray $\{0\}^n \times \mathbb{R}^+$ passes through the centers of each sphere, and attaching isometric copies of D^n at each $S^{n-1} \times \{k\}$ for $k \in \{1, 2, \dots\}$. For each L_n, $H_n^e(L_n)$ is infinitely generated and $HC_n(L_n) \cong 0$.

(c) For each $n \geq 1$, define $Q_n \subseteq \mathbb{R}^{n+1}$ similarly to L_n as above, except only attach an isometric copy of D^n at each $S^{n-1} \times \{k^2\}$ for $k \in \{1, 2, \dots\}$. For example, Q_1 is the union of the rays $x = -1$ and $x = 1$ for $y \geq 0$ and the horizontal line segments joining these rays at each $y = k^2$ for $k \in \{1, 2, \dots\}$. Then $H_n^e(Q_n) \cong H_n^e(L_n)$ and is infinitely generated, and $HC_1(Q_n) \cong 0$, since Q_n is coarsely equivalent to $\mathbb{R}^+$.

(d) For each $n \geq 1$, define $E_n \subseteq \mathbb{R}^{n+1}$ to be $\{(r \cdot x_1, r \cdot x_2, \dots, r \cdot x_n, r - 1) \mid (x_1, \dots, x_n, r - 1) \in Q_n\}$. Informally, E_n is obtained by rescaling Q_n so that it dilates perpendicularly to the axis $\{0\}^n \times \mathbb{R}^+$ as one goes farther in the space. The $r - 1$ shift is to avoid collapsing $S^{n-1} \times \{0\}$ to a point. For example, E_1 is the union of the rays $y = x - 1$ and $y = -x - 1$ for $y \geq 0$, and the horizontal line segments joining these rays at each $y = n^2$ for $n \in \{0, 1, 2, \cdots\}$. Then, for each E_n, $H_n^e(E_n) \cong H_n^e(Q_n) \cong H_n^e(L_n)$ and is infinitely generated. For E_n with $n \neq 1$, we obtain $HC_n(E_n) \cong H_n^e(E_n)$ in a natural way. Informally, since an anti-Čech sequence for E_n must consist of covers whose sets are uniformly bounded, the nerves for E_n always have infinitely many holes, and these holes correspond in an obvious way to the holes outside of some compact set in E_n. So, there is an obvious mapping of cycles in $C_n^e(E_n)$

to corresponding locally finite chains of each nerve. For the case of E_1 both H_1^e and HC_1 are infinitely generated, but the issued noted in 5.5.3 prevents a natural isomorphism. As a side note, $H_n^{lf}(E_n)$ is also infinite for all of these spaces, but contains "extra" compactly supported cycles which do not map to unique cycles on the nerves.

(e) The above comments apply equally well to the space obtained by attaching E_n to its mirror reflection in the direction of $\{0\}^n \times \mathbb{R}^-$.

These last examples are of particular interest since they indicate that H^e and HC can coincide in a natural way despite a wild failure of uniform contractibility. This strongly suggests that uniform contractibility is an unnecessarily stringent condition for the isomorphism $H^e \cong HC$.

Informally, we maintain the isomorphism $H_n^e(E_n) \cong HC_n(E_n)$ for $n \neq 1$ not because all of the holes of the space were contained in a single compact set, as in 5.5.8, but because the holes are controlled in a way that lets end homology keep up with coarse homology. Any collection of holes that the nerves on X collapse away after a finite number of steps is eventually swallowed by a single compact set in the direct limit definition of end homology. The idea we want to capture is that for each $R > 0$, there is some compact set containing every R-ball that witnesses failure of uniform contractibility. This is formalized in the following definition.

DEFINITION 5.5.10. X is **regimented** if for all $R > 0$, there is some compact $K \subseteq X$ and some $S > R$ such that for all $x \in X$, the ball of radius R centered at x is either contained in K or contracts in a the ball of radius S centered at x.

It is not hard to see the following; just take $K = \emptyset$ for each $R > 0$.

PROPOSITION 5.5.11. *Let X be a proper metric space which is uniformly contractible, then X is regimented.*

EXAMPLE 5.5.12.

(a) The spaces L_n and Q_n above are not regimented: there are too many balls of radius 2 which do not contract in larger balls.
(b) The spaces E_n are regimented.
(c) $\mathbb{R}^n$ is regimented.
(d) $\mathbb{R}^2$ with the upper half of S^2 attached along $x^2 + y^2 = 1$ is regimented.
(e) If $X \subseteq \mathbb{R}^2$ is the union of the lines $y = 0$ and $y = 1$, then X is not regimented.
(f) If $X \subseteq \mathbb{R}^2$ is the union of concentric circles of radius n^2 for $n \in \{1, 2, 3, \dots\}$, then X is regimented.

Discussion of the proof of the following theorem is delayed until the end of the next section.

THEOREM C. *Let X be a regimented metric simplicial complex with bounded coarse geometry. Then $H_n^e(X) \cong HC_n(X)$ for $n > 1$. If in addition $\mathbb{R}^+$ coarsely embeds into X, then $H_0^e(X) \cong HC_0(X)$.*

5.6. Coarse End Homology

When we defined coarse homology, we chose to use locally finite homology because we wanted a homology theory defined for the category of locally compact spaces and proper maps. End homology is also such a theory, and so we can ask what happens if we repeat the construction of coarse homology using end homology instead.

DEFINITION 5.6.1. The **coarse end homology** of X is the direct limit $HC_n^e(X) = \varinjlim H_n^e(\mathcal{K}_i)$ where $\{\mathcal{K}_i, p_i^*\}$ is a system of nerves associated to an anti-Čech system for X.

A reasonable expectation is that $HC^e(X)$ and $HC(X)$ coincide. If the nerves associated to an anti-Čech system for X eventually become uniformly contractible at some $\mathcal{K}_i$, then we can apply 5.5.2 and have this isomorphism on the level of $H_n^{lf}(\mathcal{K}_i)$ and $H_n^e(\mathcal{K}_i)$, at least for $n > 1$, and it will persist in the limit. However, even if the nerves do not become uniformly contractible, we should still expect an isomorphism. End homology is essentially only different from locally finite homology in that it ignores compact subsets. Since any compact subset of X is eventually covered by an individual open set in any anti-Čech system for X and thus becomes trivial from the perspective of coarse homology, the distinction between H^{lf} and H^e seems irrelevant in the limit.

PROPOSITION 5.6.2. *Let X be a proper metric space. Then $HC_n(X) \cong HC_n^e(X)$ for $n > 1$.*

PROOF. Let $\{\mathcal{K}_i, p_i^*\}$ be a system of nerves associated to an anti-Čech system for X. For each i, we have a long exact sequence $\cdots \to H_n^c(\mathcal{K}_i) \to H_n^{lf}(\mathcal{K}_i) \to H_n^e(\mathcal{K}_i) \to H_{n-1}^c(\mathcal{K}_i) \to \cdots$. We argue that $\varinjlim H_n^c(\mathcal{K}_i) = 0$ for $n > 0$. Since direct limits preserve exact sequences, the result will follow for $n > 1$.

Let $\bar{c} \in H_n^c(\mathcal{K}_i)$. Then, since $H_n^c(\mathcal{K}_i) = \varinjlim H_n^\infty(P)$ over $P \subseteq \mathcal{K}_i$ compact, we have $\bar{c}$ represented by some $c \in H_n^\infty(P)$ for some $P \subseteq K$ compact. Now, since X is a metric space, P is bounded, and so eventually is reduced to a vertex in some $\mathcal{K}_j$ with $j > i$. It follows that $\bar{c}$ is sent to 0 in $H_n^c(\mathcal{K}_j)$. Since no $\bar{c}$ survives the limit, we have $\varinjlim H_n^c(\mathcal{K}_i) = 0$ for $n > 0$. □

We fail to have an isomorphism in this general of a setting in dimensions 0 and 1. The spaces in 5.5.3 are counterexamples for similar reasons as to why the isomorphism between H^e and H^∞ fails. The following example shows that requiring X to be noncompact does not ensure that the map we have been working with is an isomorphism in dimension 0.

EXAMPLE 5.6.3. Let $X = \{n^2 \mid n = 1, 2, 3 \ldots\}$ as a subspace of $\mathbb{R}$, and let $\mathcal{K}_i$ be a system of nerves for X. We have that X is a proper metric space which is noncompact. The group $H_0^{lf}(\mathcal{K}_i)$ contains a nonzero homology class $\bar{c}$ which is the 0-simplex corresponding to an open set containing $1 \in X$. The image of $\bar{c}$ in any $H_0^{lf}(\mathcal{K}_j)$ continues to be nonzero, since there is never an infinite 1-chain in any $\mathcal{K}_j$. However, the map $H_0^{lf}(X) \to H_0^e(X)$ sends $\bar{c}$ to 0. Hence there is a nontrivial kernel of $H_0^{lf}(X) \to H_0^e(X)$ which persists in the direct limit.

This makes it clear that if we want isomorphism in dimension 0, we need some sort of requirement ensuring that we can push vertices out to infinity. For these coarse theories, we can do this without forcing the 0-dimensional groups to be trivial.

PROPOSITION 5.6.4. *Let X be a proper metric space and suppose that $\mathbb{R}^+$ coarsely embeds into X. Then $HC_0(X) \cong HC_0^e(X)$.*

PROOF. Let $\mathcal{K}_i$ be nerves for X. We know that the map $H_0^{lf}(\mathcal{K}_i) \to H_0^e(\mathcal{K}_i)$ is surjective, so the map $HC_0(X) \to HC_0^e(X)$ obtained in the direct limit is surjective as well. We verify that it is injective.

Let $\bar{c} \in HC_0(X)$ be such that $\bar{c} \mapsto 0$. We will show that $\bar{c}$ is 0. We have that $\bar{c}$ is represented by some $c \in H_0^{lf}(\mathcal{K}_i)$ for some nerve $\mathcal{K}_i$. Furthermore, c is represented by a cycle z which can be identified with an element of $C_0^\infty(X)$ written as $z_\infty + z_c$ where $z_\infty \in C_0^\infty - C_0^c$ and $z_c \in C_0^c$. The image of this element in $C_0^e(X)$ is represented by z_∞. Since $\bar{c} \mapsto 0$, it follows that z_∞ must eventually be in the same equivalence class as 0 in some $H_0^e(\mathcal{K}_j)$. That is, the image of z_∞ is eventually the boundary of some C_1^e chain. This chain can be pulled back to an element y of C_1^∞, where its boundary in C_0^∞ will be of the form $z_\infty + z_c'$ with $z_c' \in C_0^c$. We now argue that $z_c - z_c'$ is eventually the boundary of some element w of C_1^∞ as well. This will imply that $\partial(y + w) = z_\infty + z_c$ and hence that z is equivalent to 0.

Let f be the coarse embedding of $\mathbb{R}^+$ into X. The support of $z_c - z_c'$ is a compact set, and so it is contained inside some ball around $f(0)$. Hence, if we choose j large enough, we will have that in $\mathcal{K}_j$, the support of $z_c - z_c'$ maps into a single vertex corresponding to an open set containing $f(0)$. Furthermore, if j is large enough, $\mathcal{K}_j$ contains a 1-chain consisting of a sequence of open sets along $f(0), f(1), \ldots$. respectively. So, the vertex is the boundary of an infinite chain. □

We combine the above two propositions into a single theorem.

THEOREM D. *Let X be a proper metric space. Then $HC_n(X) \cong HC_n^e(X)$ for $n > 1$. If in addition $\mathbb{R}^+$ coarsely embeds into X, then $HC_0(X) \cong HC_0^e(X)$.*

EXAMPLE 5.6.5. We amend the earlier example of the subset $X = \{n^2 \mid n = 1, 2, 3, \dots\}$ by embedding it in $\mathbb{R}^2$ and adding a coarse embedding of $\mathbb{R}^+$. Let $Y = \{1, 2, 3, \dots\}$ and set $W = (X \times \{0\}) \cup (\{0\} \times Y)$. If we take nerves associated to open covers with large enough open sets, we can clearly write the vertex corresponding to an open set containing $(0,0)$ as the boundary of a chain corresponding to the positive y-axis. However, we still have nontrivial 0-dimensional homology coming from the x-axis.

So, HC^e differs from HC on proper metric spaces only in dimensions 0 and 1. This difference is manifested even in simple spaces: for example $HC_1(\mathbb{R}^+) = 0$ while $HC_1^e(\mathbb{R}^+) = \mathbb{Z}$ and $HC_0(X) = \mathbb{Z}$ while $HC_0^e(X) = 0$ for compact X.

We now return to Theorem C. We will use the following lemmas.

LEMMA 5.6.6. *Suppose X and Y are a proper metric spaces and that $A \subseteq X$ is compact. If $f, g\colon X \to Y$ are maps which are continuous and proper on $X - A$ and are properly homotopic on $X - A$, then they induce the same map $H^e(X) \to H^e(Y)$.*

PROOF. The restrictions of f and g to $X - A$ clearly induce the same map $H^e(X - A) \to H^e(Y)$, but $H^e(X - A)$ is naturally isomorphic to $H^e(X)$. □

LEMMA 5.6.7. *Suppose X and Y are proper metric spaces, and that $A \subseteq X$ and $B \subseteq Y$ are compact subsets. If $X - A$ is proper homotopy equivalent to $Y - B$, then $H_p^e(X) \cong H_p^e(Y)$ for all p.*

PROOF. H^e is proper homotopy invariant, so $H^e(X - A) \cong H^e(Y - B)$, but $H^e(X) \cong H^e(X - A)$ and $H^e(Y) \cong H^e(Y - B)$. □

The method of proof for Theorem C is essentially the same as that of 5.4.11. The following are analogous to the lemmas involved in proving the $H^{lf} \cong HC$ result.

LEMMA 5.6.8. *Let X be a finite-dimensional metric simplicial complex and let Y be regimented. Let $f\colon X \to Y$ be a coarse map. Then there is a coarse map $g\colon X \to Y$ which is close to f and is continuous except on some compact set. Moreover, if f is continuous on a subcomplex X' of X, then we can take $g = f$ on X'.*

PROOF. The construction is similar to that of 5.4.7, except that we only know $f(\Delta)$ contracts in an S-ball if $f(\Delta)$ is not contained in a fixed compact subset $K \subseteq Y$. So, we can only extend g_i continuously on $(i+1)$-simplices Δ which are not contained in the preimage under f of K. Since f is coarse, it is proper, and so $f^{-1}(K)$ is a compact set in X. □

LEMMA 5.6.9. *Let X be a finite-dimensional metric simplicial complex and let Y be regimented. Then if $f, g\colon X \to Y$ are coarse, continuous, close maps, then they restrict to properly homotopic maps f' and g' on $X - K$ for some compact $K \subseteq X$.*

PROOF. Let R_f be the supremum over the diameters of $f(\Delta)$ for all simplices in X, which exists since X is a metric simplicial complex and f is coarse. Similarly define R_g. Since Y is regimented, we can choose $C \subseteq Y$ and $S > 0$ to be the compact set and the radius such that every ball of radius $\max\{R_f, R_g\}$ not contained in C is contractible in a ball of radius S. Let X' be the subcomplex of X which is the complement of the interior of $(f^{-1}(C) \cup g^{-1}(C))$. Let f' and g' be the restrictions of f and g to X'. Then f' and g' are coarse, continuous, close maps, and there is a map $h\colon X' \times [0,1] \to Y$ which restricts to f' and g' on its ends. Since all images of simplices under f' and g' contract in a ball of radius S, we can repeat the construction of a continuous map $H\colon X' \times [0,1] \to Y$ close to h unchanged from the proof of 5.4.7. The map H is a proper homotopy between f' and g' on X'. These are not quite the maps claimed to exist. To finish, we need to restrict X' to the complement of the compact set $f^{-1}(C) \cup g^{-1}(C)$ rather than just its interior, and restrict f', g', and H as well. □

COROLLARY 5.6.10. *If X and Y are two regimented finite-dimensional metric simplicial complexes which are coarsely equivalent, then there are compact subsets $A \subseteq X$ and $B \subseteq Y$ such that $X - A$ and $Y - B$ are proper homotopy equivalent.*

PROOF. The proof is similar to the corollary of 5.4.8, but care has to be taken to restrict to complements of compact sets. If $f\colon X \to Y$ is a coarse equivalence with coarse inverse g, then f and g are close to coarse maps f' and g' which are continuous on $X - K_1$ and $Y - K_2$ respectively. Hence, $g'f'$ is only continuous on $X' = X - (K_1 \cup f'^{-1}(K_2))$ and $f'g'$ is only continuous on $Y' = Y - (K_2 \cup g'^{-1}(K_1))$. Since gf is close to id_X, we get that $g'f'$ is close to $\mathrm{id}_{X'}$. Similarly, $f'g'$ is close to $\mathrm{id}_{Y'}$. The preceding lemma then tells us that for some other compact subsets $K_3 \subseteq X'$ and $K_4 \subseteq Y'$, the restriction of $g'f'$ is proper homotopic to $\mathrm{id}_{X'-K_3}$ and the restriction of $f'g'$ is properly homotopic to $\mathrm{id}_{Y'-K_4}$. This gives a proper homotopy equivalence between $X - (K_1 \cup f'^{-1}(K_2) \cup K_3)$ and $Y - (K_2 \cup g'^{-1}(K_1) \cup K_4)$. □

The proof of Theorem C was delayed until this section because it is easier when the isomorphism between HC^e and HC is used.

PROOF OF THEOREM C. The proof is analogous to the proof of 5.4.11. The difference is that in this case, we can only assume that the maps g_i are continuous on the complement of a compact subset of $\mathcal{K}_i$. As a result, we only obtain a proper homotopy between a restriction of $g_i h_i$ to some $X - A_i$ with $A_i \subseteq X$ compact. So, only some restriction of g_i is a left proper homotopy inverse. Likewise, the maps f_{i+1} and $h_{i+1}g_i$ are not necessarily properly homotopic, only some restrictions of them are. However, because of 5.6.6, this is sufficient. We can make use of this more general proper homotopy invariance of H^e to get that the direct limit of the induced maps $(h_i)_*\colon H^e(X) \to H^e(\mathcal{K}_i)$ is an isomorphism $H^e(X) \to HC^e(X)$ to coarse end homology. Then, composition with the isomorphism $HC^e(X) \cong HC(X)$ gives us the claimed isomorphism. $\square$

5.7. Coarse Cohomology

There is a corresponding cohomology theory for the coarse category. Similarly to coarse homology, it can be developed for general spaces admitting coarse structures, but we focus here on metric spaces. The material here can be found in [**19**] and [**21**].

DEFINITION 5.7.1. Let E be a subset of X^{p+1}. We say that E is **controlled** if the coordinate projections $\pi_0, \ldots, \pi_p\colon E \to X$ are all close to each other.

It is easy to see that if $E \subseteq X^{p+1}$ is bounded, then it is controlled.

PROPOSITION 5.7.2. *Let $f\colon X \to Y$ be a coarse map. Let E be a controlled subset of X^{p+1} and let B be a bounded subset of Y^{p+1}.*

(a) *The image $\bar{f}(E) = \{(f(x_0), \ldots, f(x_p)) \mid (x_0, \ldots, x_p) \in E\}$ is controlled in Y^{p+1}.*
(b) *The preimage $\bar{f}^{-1}(B) = \{(x_0, \ldots, x_p) \in E \mid (f(x_0), \ldots, f(x_p)) \in B\}$ is bounded in X^{p+1}.*

DEFINITION 5.7.3. A subset $D \subseteq X^{p+1}$ is **cocontrolled** if, for every controlled $E \subseteq X^{p+1}$, the intersection $D \cap E$ is bounded.

PROPOSITION 5.7.4. *Let $f\colon X \to Y$ be a coarse map. Let $D \subseteq Y^{p+1}$ be cocontrolled. Then, the preimage $\bar{f}^{-1}(D)$ is cocontrolled in X^{p+1}.*

PROOF. Let $E \subseteq X^{p+1}$ be controlled, and let $B = \bar{f}^{-1}(D) \cap E$. We show that B is bounded. Note that $\bar{f}(B) \subseteq D \cap \bar{f}(E)$. Since E is controlled, its image $\bar{f}(E)$ is controlled. So, $D \cap \bar{f}(E)$ is bounded. Thus, $\bar{f}(B)$ is bounded. It follows that the preimage $\bar{f}^{-1}\bar{f}(B)$ is bounded as well. Since $B \subseteq \bar{f}^{-1}\bar{f}(B)$, this completes the proof. $\square$

DEFINITION 5.7.5. Let G be an abelian group. The **p-th coarse cochain group of X with coefficients in G** is denoted $C(X, G)$ and is defined to be the collection of p-functions $\phi\colon X^{p+1} \to G$ with cocontrolled support.

Using the usual coboundary map when dealing with p-functions as in the Alexander-Spanier theories, the collection of coarse cochain groups forms a cochain complex.

DEFINITION 5.7.6. The n-th **coarse cohomology of X with coefficients in G** is denoted $HC^n(X, G)$ and defined to be the n-th cohomology group of the coarse cochain complex of X

DEFINITION 5.7.7. The **character map** $c\colon HC^n(X,G) \to H_c^n(X,G)$ is defined by sending each equivalence class of a cocycle ϕ to a restriction of ϕ to some controlled neighborhood of the diagonal $\{(x,\dots,x) \in X^{n+1} \mid x \in X\}$.

The following theorem is stated in more generality than that in which we have been working. Just note that metric spaces are coarse spaces.

THEOREM 5.7.8. *If X is a uniformly contractible proper coarse space, then the character map $c\colon HC^n(X) \to H_c^n(X)$ is an isomorphism for all n.*

Coarse cohomology can also be expressed using a direct limit construction involving nerves, but it is not as easily obtained as coarse homology in this way.

PROPOSITION 5.7.9. *Let X be a proper metric space and let $\{\mathcal{U}_j\}$ be a cofinal sequence in an anti-Čech system for X. Let $\{\mathcal{K}_j\}$ be the associated sequence of nerves. Then we have the following **Milnor exact sequence** for X.*

$$0 \to \varprojlim{}^1 H_c^{q-1}(\mathcal{K}_j) \to HC^q(X) \to \varprojlim H_c^q(\mathcal{K}_j) \to 0$$

5.8. Asymptotic Dimension

Asymptotic dimension is the coarse analog of topological covering dimension. In this section, we describe how it relates to coarse homology. The definitions, theorems, and proofs below can be found in [**21**].

DEFINITION 5.8.1. Let $D \subseteq X$ and $r > 0$. We say that D is r-disconnected if D is a disjoint union $D = \bigsqcup_{\alpha=0}^{\infty} D_i$ where there is a uniform bound on the diameter of each D_α and each D_α is at least distance r from each D_β for $\alpha \neq \beta$.

DEFINITION 5.8.2. We say that X has **asymptotic dimension** $\leq n$ if for every $r > 0$, X can be written as the union of at most $n+1$ many r-disconnected subsets. If X has asymptotic dimension $\leq n$ but not $\leq n-1$, we say that X has asymptotic dimension n and write $\operatorname{asdim} X = n$.

PROPOSITION 5.8.3. *If X and Y are coarsely equivalent, then* $\operatorname{asdim} X \leq n$ *iff* $\operatorname{asdim} Y \leq n$.

PROOF. Let f be a coarse equivalence between X and Y. Let $r' > 0$. Then, since f is a coarse equivalence, there is some $r > 0$ such that if $d(x_1,x_2) > r$, then $d(f(x_1), f(x_2)) > r'$ for every $x_1, x_2 \in X$. Now, if $\operatorname{asdim} X \leq n$, X can be written as the union of at most $n+1$ many r-disconnected subsets $D_1, \dots, D_n$, each being a disjoint union of sets which are at least r away from each other and with diameters uniformly bounded by some R_i. Since there are only finitely many R_i, we can select the largest and call it R. Since f is a coarse map, there is some R' such that $d(x_1,x_2) < R$ implies $d(f(x_1), f(x_2)) < R'$

for all $x_1, x_2 \in X$. It follows that the images $f(D_0), \dots, f(D_n)$ are each r'-disconnected, since the sets $D_{i,\alpha}$ are each at least r' apart and have diameter uniformly bounded by R'.

We will prove that $\operatorname{asdim} Y \leq n$ implies $\operatorname{asdim} X \leq n$. The converse is similar. Let $f\colon X \to Y$ be a coarse equivalence, and let $r > 0$. Since f is coarse, there is some $r' > 0$ such that $d(x_1, x_2) < r$ implies $d(f(x_1), f(x_2)) < r'$; that is, $d(f(x_1), f(x_2)) \geq r'$ implies $d(x_1, x_2) \geq r$. It is easily checked that since $f(X) \subseteq Y$, we have $\operatorname{asdim} f(X) \leq n$ as well. Thus, $f(X)$ can be written as the disjoint union of at most $n+1$ many r'-disconnected subsets $D_1, \dots, D_n$, each being a disjoint union of sets which are at least r' away from each other and with diameters uniformly bounded by some R'_i. Since there are only finitely many R'_i, we can select the largest and call it R'. Since f is a coarse equivalence, there is some R such that $d(f(x_1), f(x_2)) < R'$ implies $d(x_1, x_2) < R$. Thus, $f^{-1}(D_1), \dots, f^{-1}(D_n)$ are each r-disconnected, since each is a disjoint union of sets $D_{i,\alpha}$ which are at least r apart from each other and uniformly bounded by R. □

The following appears as Theorem 9.9 on page 131 of [**21**].

PROPOSITION 5.8.4. *Let X be a proper metric space. Then, the following are equivalent.*

(a) *X has asymptotic dimension $\leq n$.*
(b) *For each $d > 0$, X admits a bounded covering $\mathcal{U}$ such that no more than $n+1$ members of $\mathcal{U}$ meet any ball of radius d.*
(c) *X admits an anti-Čech sequence made up of coverings of degree $\leq n+1$.*
(d) *For each $\varepsilon > 0$ there is an ε-Lipschitz and uniformly cobounded map from X to an n-dimensional nerve of some simplicial complex, with the affine metric.*

If X is a geodesic space, then these are also equivalent to

(e) *For each $\varepsilon > 0$ there is an ε-Lipschitz and effectively proper map from X to an n-dimensional nerve of some simplicial complex, with the intrinsic geodesic metric.*

In the above, uniformly cobounded means that there is some $R > 0$ such that the preimage of any open star of a vertex in the simplicial complex is contained in an R-ball. A map is effectively proper if for all $R > 0$ there is an $S > 0$ such that the preimage of any S-ball is contained in some R-ball.

COROLLARY 5.8.5. *If $q \geq \operatorname{asdim} X + 2$, then $HC^q(X) = HC_q(X) = 0$. If $\varprojlim^1 H_c^{q-1}(\mathcal{K}_j) = 0$, then $q \geq \operatorname{asdim} X + 1$ is sufficient.*

PROOF. Consider the Milnor exact sequence

$$0 \to \varprojlim{}^1 H_c^{q-1}(\mathcal{K}_j) \to HC^q(X) \to \varprojlim H_c^q(\mathcal{K}_j) \to 0$$

Whenever $\operatorname{asdim} X \leq q-2$, there is an anti-Čech sequence for X made up of coverings of degree $\leq q-1$. The nerve of any such cover has simplices only of dimension $\leq q-2$. Hence $H_c^n(\mathcal{K}_j) = 0$ for $n > q-2$.

If $\varprojlim^1 H_c^{q-1}(\mathcal{K}_j) = 0$, then $\operatorname{asdim} X \leq q-1$ is sufficient since we still have $\varprojlim H_c^q(\mathcal{K}_j) = 0$ by the above argument.

In both cases, the Milnor exact sequence reduces to

$$0 \to 0 \to HC^q(X) \to 0 \to 0$$

and the result follows. $\square$

CHAPTER 6

Duality

6.1. Cup and Cap Products

Cohomology with compact supports and Alexander-Spanier cohomology have a multiplicative structure called the cup product. This extra structure is lacking from the homology theories we have discussed, and it makes these cohomology theories more valuable in some cases. For example, by computing certain cup products, one can distinguish between the homotopy types of some spaces despite their cohomology and homology groups each being identical. Here we include the development which appears in [**14**].

The existence of the following homomorphism accounts for the extra structure in the cohomology theories.

DEFINITION 6.1.1. If K and L are cochain complexes of modules over a commutative ring R, then we define the natural homomorphism $\alpha : H^p(K) \otimes H^q(L) \to H^{p+q}(K \otimes L)$ by sending $u \otimes v$ to the cohomology class in $K \otimes L$ which contains the cocycle $u' \otimes v'$ in for representatives u' of u and v' of v.

This homomorphism α can be exploited to develop a product map on the level of cohomology groups, for example $\times : H_c^p(X, G_1) \otimes H_c^q(X, G_2) \to H_c^{p+q}(X \times Y, G_1 \otimes G_2)$. By further composition with Δ^*, where $\Delta\colon X \to X \times X$ is the diagonal map $x \mapsto (x, x)$, a product map $\smile\colon H_c^p(X, G_1) \otimes H_c^q(X, G_2) \to H_c^{p+q}(X, G_1 \otimes G_2)$ can be obtained. This map can be shown to result from the following map on the level of cochains. This allows us to use the following direct definition rather than the above construction.

DEFINITION 6.1.2. We define the **cup product of cochains** $\smile\colon \Phi^p(X) \otimes \Phi^q(X) \to \Phi^{p+q}(X)$ by $(f \smile g)(x_0, \ldots, x_{p+q}) = f(x_0, \ldots, x_p) \cdot g(x_p, \ldots, x_{p+q})$.

PROPOSITION 6.1.3. *The cup product of cochains is a homomorphism and satisfies the following properties for any $\phi \in \Phi^p(X, G_1)$ and $\psi \in \Phi^q(X, G_2)$.*

(a) *The support $|\phi \smile \psi|$ is contained in the intersection $|\phi| \cap |\psi|$.*
(b) *It satisfies the coboundary formula $\delta(\phi \smile \psi) = (\delta\phi) \smile \psi + (-1)^p \phi \smile (\delta\psi)$.*

The above properties allow one to show that $\smile$ induces cochain maps on the appropriate cochain complexes for the different theories, and hence that it induces homomorphisms on cohomology groups for both theories. The first property implies that if at least one of ϕ or ψ has compact support, then so does $\phi \smile \psi$. Hence, $\smile$ also induces a homomorphism between "mixed" cohomology groups, which we note for future reference.

DEFINITION 6.1.4. Let X be a locally compact Hausdorff space, and let G_1 and G_2 be R-modules for a commutative ring R. Then the **mixed cup product** is the induced homomorphism $\smile: H^p_\infty(X, G_1) \otimes H^q_c(X, G_1) \to H^{p+q}_c(X, G_1 \otimes G_2)$ discussed above.

We now turn our attention to cap products. A cap product is a homomorphism which joins a $(p+q)$-chain and a p-cochain to form a q-chain. A cap product induces a map which mixes homology and cohomology classes. We will focus in particular on a cap product $\cap : C^\infty_{p+q}(X, G_1) \otimes C^p_c(X, G_2) \to C^\infty_q(X, G_1 \otimes G_2)$ on the chain and cochain groups from homology based on infinite chains and cohomology with compact supports respectively. This is the product that is used to obtain the Poincaré duality theorems we discuss below.

DEFINITION 6.1.5. The **cap product on chains and cochains** is the map $\frown: C^\infty_{p+q}(X, G_1) \otimes C^p_c(X, G_2) \to C^\infty_q(X, G_1 \otimes G_2)$ which sends (f, u) to the homomorphism $f \frown u \in C^\infty_q(X, G_1 \otimes G_2) = \mathrm{Hom}(C^q_c(X, \mathbb{Z}), G_1 \otimes G_2)$ defined by $(f \frown u)(v) = (f \otimes \mathrm{id}_{G_2})(u \smile v)$ where $\smile$ is the mixed cup product.

PROPOSITION 6.1.6. *The cap product on chains and cochains is a homomorphism and satisfies the following properties.*

(a) *It satisfies the boundary formula* $\partial(f \frown u) = (-1)^p\left((\partial f \frown u - f \frown \delta u\right)$.
(b) *If either* $f \in C^\infty_{p+q}(X, G_1)$ *or* $u \in C^p_\infty(X, G_2)$ *has compact support, then* $f \frown u$ *does as well; that is, if* $f \in C^c_{p+q}(X, G_1)$ *or* $u \in C^p_c(X, G_2)$, *then* $f \frown u \in C^c_q(X, G_1 \otimes G_2)$.

It follows that $\frown$ induces four different homomorphisms on the level of homology and cohomology groups. On compact X, all of these cap products coincide.

DEFINITION 6.1.7. The **cap products on homology and cohomology** are defined are the homomorphisms induced by the cap product on chains and cochains. We give each type of cap product a name here to distinguish them.

Type A: $H^\infty_{p+q}(X, G_1) \otimes H^p_\infty(X, G_2) \to H^\infty_q(X, G_1 \otimes G_2)$
Type B: $H^\infty_{p+q}(X, G_1) \otimes H^p_c(X, G_2) \to H^c_q(X, G_1 \otimes G_2)$
Type C: $H^c_{p+q}(X, G_1) \otimes H^p_\infty(X, G_2) \to H^c_q(X, G_1 \otimes G_2)$
Type D: $H^c_{p+q}(X, G_1) \otimes H^p_c(X, G_2) \to H^c_q(X, G_1 \otimes G_2)$

We are specifically concerned with types A and B, as these are the ones which are important for Poincaré duality. We summarize the main properties of these types of cap products below. First, we extend the definition of the type A cap product.

DEFINITION 6.1.8. Let R and S be subsets of a Hausdorff space such that S is locally compact, and $R \cap S$ is a closed subset of S. Let A be a closed subset of R. Then we define the **extended type A cap product** $\frown\colon H^{\infty}_{p+q}((R \cap S) - (A \cap S)) \otimes H^{p}_{\infty}(R, A) \to H^{\infty}_{q}(S)$ by the formula $u \frown v = i_*(u \frown j^* v)$ where $i\colon R \cap S \to S$ and $j\colon (R \cap S, A \cap S) \to (R, A)$ are inclusion maps.

Note that the hypotheses imply that $R \cap S$ is locally compact and $A \cap S$ is a closed subset of $R \cap S$. This extension is chosen so that the following diagram is commutative.

$$\begin{tikzcd}
H^{\infty}_{p+q}((R \cap S) - (A \cap S)) \otimes H^{p}_{\infty}(R, A) \arrow[r, "\frown"] \arrow[d, "\mathrm{id} \otimes j^*"'] & H^{\infty}_{q}(S) \\
H^{\infty}_{p+q}((R \cap S) - (A \cap S)) \otimes H^{p}_{\infty}(R \cap S, A \cap S) \arrow[r, "\frown"] & H^{\infty}_{q}(R \cap S) \arrow[u, "i_*"']
\end{tikzcd}$$

This cap product can be shown to result from a generalized cap product on the chain-cochain level as well.

PROPOSITION 6.1.9. *The type A cap product satisfies the following properties.*

(A1) *Naturality under proper continuous maps. Let $f\colon (X, A) \to (Y, B)$ be a proper continuous map of locally compact pairs. Then the following diagram is commutative.*

$$\begin{tikzcd}
C^{\infty}_{p+q}(X, A) \otimes C^{p}_{\infty}(X, A) \arrow[r, "\frown"] \arrow[d, "f_\#"'] & C^{\infty}_{q}(X) \arrow[d, "f_\#"] \\
C^{\infty}_{p+q}(Y, B) \otimes C^{p}_{\infty}(Y, B) \arrow[r, "\frown"] \arrow[u, shift left=8ex, "f_\#"'] & C^{\infty}_{q}(Y)
\end{tikzcd}$$

Similarly, the above diagram with homology and cohomology groups is also commutative.

(A2) *Naturality under inclusion of open subsets. Let U be an open subset and A a closed subset of X. Then the following diagram is commutative.*

$$\begin{tikzcd}
C^{\infty}_{p+q}(X, A) \otimes C^{p}_{\infty}(X, A) \arrow[r, "\frown"] \arrow[d, "\sigma_1^\# \qquad j^\#"'] & C^{\infty}_{q}(X) \arrow[d, "\sigma^\#"] \\
C^{\infty}_{p+q}(U, U \cap A) \otimes C^{p}_{\infty}(U, U \cap A) \arrow[r, "\frown"] & C^{\infty}_{q}(U)
\end{tikzcd}$$

Here, σ_1 is induced by σ, and $\sigma_1^\#$ and $\sigma^\#$ are their algebraic dual maps. The above diagram with homology and cohomology groups is also commutative.

(A3) *Existence of a unit.* *There is a unit* $1 \in C^0_\infty(X, \mathbb{Z})$ *such that for any* $u \in C^\infty_q(X, G)$, $u \frown 1 = u$. *The class of this unit is a unit* $1 \in H^0_\infty(X, \mathbb{Z})$.

(A4) *Relationship with the boundary operator.* *Let* B *be a closed subset of* X, *and* $U = X - B$ *its open complement.* *Then* $\partial(u \frown v) = (-1)^p(\partial u) \frown v$.

(A5) *Relation with the boundary and coboundary operators. Let* A *and* B *be closed subsets of* X. *Then we have the following commutative diagram.*

$$\begin{tikzcd}
0 \arrow[d] & 0 & \\
C^\infty_{p+q}(A \cup B, A) \otimes & C^p_\infty(A \cup B, A) \arrow[u] \arrow[r, "\frown"] & C^\infty_q(A \cup B) \arrow[d, "i_\#"] \\
C^\infty_{p+q}(X, A) \otimes \arrow[u, leftarrow, "i_\#"'] & C^p_\infty(X, A) \arrow[u, "i^\#"] \arrow[r, "\frown"] & C^\infty_q(X) \arrow[d, "\cong"] \\
C^\infty_{p+q}(X, A \cup B) \otimes \arrow[u, leftarrow, "j_\#"'] \arrow[d] & C^p_\infty(X, A \cup B) \arrow[u, "j^\#"] \arrow[r, "\frown"] & C^\infty_q(X) \\
0 & 0 \arrow[u] &
\end{tikzcd}$$

Passing to homology yields the following commutative diagram.

$$\begin{tikzcd}
H^\infty_{p+q}(X - (A \cup B)) \otimes \arrow[d, "\partial"'] & H^p_\infty(X, A \cup B) \arrow[r, "\frown"] & H^\infty_q(X) \\
H^\infty_{p+q-1}(B - A) \otimes & H^{p-1}_\infty(A \cup B, A) \arrow[u, "\delta"] \arrow[r, "\frown"] & H^\infty_q(A \cup B) \arrow[u, "i_*"']
\end{tikzcd}$$

If the triad $(A \cup B; A, B)$ *is excisive for Alexander-Spanier cohomology, we also obtain the following commutative diagram.*

$$\begin{tikzcd}
H^\infty_{p+q}(X - (A \cup B)) \otimes \arrow[d, "\partial"'] & H^p_\infty(X, A \cup B) \arrow[r, "\frown"] & H^\infty_q(X) \\
H^\infty_{p+q-1}(B - A) \otimes & H^{p-1}_\infty(B, A \cap B) \arrow[u, "\delta"] \arrow[r, "\frown"] & H^\infty_q(B) \arrow[u, "i_*"']
\end{tikzcd}$$

(A6) *Duality. Let* (X, A) *be a locally compact pair,* $u \in H^\infty_{p+q}(X - A, G_1)$, $x \in H^p_\infty(X, A, G_2)$, *and* $y \in H^q_c(X, G_3)$. *Then* $u(x \smile y) = (u \frown x)(y) \in G_1 \otimes G_2 \otimes G_3$.

PROPOSITION 6.1.10. *The type B cap product satisfies the following properties.*

(B1) *Naturality under proper continuous maps. Let $f\colon (X,A) \to (Y,B)$ be a continuous proper map of locally compact pairs. Let V be an open subset of Y and let $U = f^{-1}(V)$. Then the following diagram is commutative.*

$$\begin{tikzcd}
C^{\infty}_{p+q}(X,A) \arrow[d, "f_\#"'] & \otimes\, \frac{C^p_c(X)}{C^p_c(U)} \arrow[r, "\frown"] & C^c_q(X, U\cup A) \arrow[d, "f_\#"] \\
C^{\infty}_{p+q}(Y,B) & \otimes\, \frac{C^p_c(Y)}{C^p_c(V)} \arrow[u, "f_\#"'] \arrow[r, "\frown"] & C^c_q(Y, V\cup B)
\end{tikzcd}$$

The above diagram using homology and cohomology groups is commutative as well.

(B2) *Naturality under inclusion of open subsets. Let U and V be open subsets of X, and let A be closed in X. Then the following diagram is commutative.*

$$\begin{tikzcd}
C^{\infty}_{p+q}(U, U\cap A) & \otimes\, \frac{C^p_c(U)}{C^p_c(U\cap V)} \arrow[d, "\sigma"] \arrow[r, "\frown"] & C^c_q(U, U\cap(V\cup A)) \arrow[d, "i_\#"] \\
C^{\infty}_{p+q}(X,A) \arrow[u, "\sigma^\#"] & \otimes\, \frac{C^p_c(X)}{C^p_c(V)} \arrow[r, "\frown"] & C^c_q(X, V\cup A)
\end{tikzcd}$$

(B3) *Existence of a unit. If X is compact, then there is a unit $1 \in C^0(X,\mathbb{Z})$ such that for all $u \in C^{\infty}_q(X,A,G) = C^c_q(X,A,G)$, $u \frown 1 = u$. The class of this unit is a unit $1 \in H^0(X,\mathbb{Z})$.*

(B4) *Relationship with the boundary operator. Let A and B be closed subsets of X. Let U be an open subset of X. Then the following diagram is commutative.*

$$\begin{tikzcd}
0 \arrow[d] & & & 0 \arrow[d] \\
C^{\infty}_{p+q}(A\cup B, B) \arrow[d] & \otimes\, \frac{C^p_c(A\cup B)}{C^p_c(U\cap(A\cup B))} \arrow[r, "\frown"] & W \arrow[r, "j_\#"] & C^c_q(U\cup A\cup B, U\cup B) \arrow[d] \\
C^{\infty}_{p+q}(X,B) \arrow[d] & \otimes\, \frac{C^p_c(X)}{C^p_c(U)} \arrow[u, "i^\#"'] \arrow[d, equal] \arrow[rr, "\frown"] & & C^c_q(X, U\cup B) \arrow[d] \\
C^{\infty}_{p+q}(X, A\cup B) \arrow[d] & \otimes\, \frac{C^p_c(X)}{C^p_c(U)} \arrow[rr, "\frown"] & & C^c_q(X, U\cup A\cup B) \arrow[d] \\
0 & & & 0
\end{tikzcd}$$

Where W is $C_q^c(A \cup B, B \cup (U \cap A))$. The two columns are short exact sequences. Passing to homology yields the following commutative diagram.

$$\begin{tikzcd}
H_{p+q}^{\infty}(X-(A\cup B)) \arrow[d, "\partial"'] & \otimes & H_c^p(X-U) \arrow[d, "(-1)^p i^*"] \arrow[rr, "\frown"] & & H_q^c(X, U\cup A\cup B) \arrow[d, "\partial"] \\
H_{p+q-1}^{\infty}(A-B) & \otimes & H_c^p((A\cup B)-U) \arrow[r, "\frown"] & W' \arrow[r, "j_*"] & H_{q-1}^c(U\cup A\cup B, U\cup B)
\end{tikzcd}$$

Where W' is $H_{q-1}^c(A \cup B, B \cup (U \cap A))$. When $(U \cup A \cup B; A \cup B, U \cup B)$ is an excisive triad, the diagram can be further simplified.

(B5) *A relation with the boundary and coboundary operators. If $u \in H_{p+q}^{\infty}(U)$ and $v \in H_c^{p-1}(X - U)$ for U open in X, then $i_*(u \frown \delta v) = j_*(\partial u \frown v)$, where $i_*\colon H_q^c(U) \to H_q^c(X)$ and $j_*\colon H_q^c(X - U) \to H_q^c(X)$ are induced by inclusion, and ∂ and δ are the boundary and coboundary operators for the pair $(X, X - U)$.*

(B6) *Another relation with the boundary and coboundary operators. Let U be open in X and let A be closed in X. Then the following diagram is commutative.*

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] \\
C_{p+q}^{\infty}(U, U\cap A) & \otimes & C_c^p(U) \arrow[r, "\frown"] & C_q^c(U, U\cap A) \arrow[r, "j_\#"] & C_q^c(U\cup A, A) \arrow[d] \\
C_{p+q}^{\infty}(X, A) \arrow[u, "\sigma^\#"] \arrow[d, equal] & \otimes & C_c^p(X) \arrow[u, "\sigma"'] \arrow[rr, "\frown"] \arrow[d] & & C_q^c(X, A) \arrow[d] \\
C_{p+q}^{\infty}(X, A) & \otimes & \frac{C_c^p(X)}{C_c^p(U)} \arrow[rr, "\frown"] \arrow[d] & & C_q^c(X, U\cup A) \arrow[d] \\
 & & 0 & & 0
\end{tikzcd}$$

The two columns are short exact sequences. Passing to homology yields the following commutative diagram.

$$\begin{tikzcd}
H_{p+q}^{\infty}(X-A) \arrow[d, "\rho"'] & \otimes & H_c^p(X-U) \arrow[d, "(-1)^{p+1}\delta"] \arrow[rr, "\frown"] & & H_q^c(X, U\cup A) \arrow[d, "\partial"] \\
H_{p+q}^{\infty}(U-A) & \otimes & H_c^{p+1}(U) \arrow[r, "\frown"] & H_{q-1}^c(U, U\cap A) \arrow[r, "j_*"] & H_{q-1}^c(U\cup A, A)
\end{tikzcd}$$

When $(U \cup A; U, A)$ is excisive, the diagram can be further simplified.

(B7) *Another naturality condition. Let U be open in X and let A be closed in X. Then the following diagram is commutative.*

$$\begin{tikzcd} H^\infty_{p+q}(X,A) \otimes H^p_c(X) \arrow[r, "\frown"] \arrow[d, equal] & H^c_q(X,A) \arrow[d, "j_*"] \\ H^\infty_{p+q}(X,A) \otimes H^p_c(X-U) \arrow[r, "\frown"] \arrow[u, phantom, "\uparrow i^*"] & H^c_q(X, U \cup A) \end{tikzcd}$$

(B8) *Duality. Let (X,A) be a locally compact pair, $u \in H^\infty_{p+q}(X-A,G_1)$, $x \in H^p_c(X,G_2)$, and $y \in H^q_\infty(X,A,G_3)$. Then $u(x \smile y) = (u \frown x)(y) \in G_1 \otimes G_2 \otimes G_3$.*

6.2. Poincaré Duality

The Poincaré duality theorems assert that for certain pairs of homology and cohomology theories, the q-th homology group of an orientable n-manifold M is isomorphic to the $(n-q)$-th cohomology group. The isomorphisms we consider are defined in terms of cap products and a chosen generator μ_M of the group $H^\infty_n(M,\mathbb{Z}) \cong \mathbb{Z}$. The generator μ_M is called the fundamental homology class of M. Also note that if M is an oriented n-manifold and U is a nonempty open subset, then $\rho_{M,U}(\mu_M)$ is a fundamental homology class of U.

In the statements below, the homology theory H^∞ is that of Section 4.2, H^c is that of Section 4.3, the cohomology theory H_c is that of Section 3.2, and H_∞ is that of Section 3.3. We include the development which appears in [**14**].

THEOREM 6.2.1 (Poincaré Duality). *Let M be an oriented n-manifold without boundary, and let G be an abelian group. Then the homomorphism $H^q_c(M,G) \to H^c_{n-q}(M,G)$ given by $x \to \mu_M \frown x$ is an isomorphism for all q.*

If M is paracompact, then the homomorphism $H^q_\infty(M,G) \to H^\infty_{n-q}(M,G)$ given by $x \to \mu_M \frown x$ is an isomorphism.

PROOF. We include the proof of the first part of the theorem. The proof is broken down into 6 cases depending on the structure of M; the last is the general case. We will make use of the properties (e.g., B2) of cap products listed in the previous section.

Case 1: Let $M = S^n$. In this case, it suffices to show that the homomorphisms $\mu_M \frown\colon H^n_c(S^n,G) \to H^c_0(S^n,G)$ and $\mu_m \frown\colon H^0_c(S^n,G) \to H^c_n(S^n,G)$ are isomorphisms. It can be shown that the type B cap product commutes with the homomorphisms involved in the universal coefficient theorem and expressing H^∞ in terms of integral cochain groups, from which the result follows.

Case 2: Let $M = \mathbb{R}^n$. In this case, it suffices to show that $\mu_M \frown\colon H^n_c(\mathbb{R}^n,G) \to H^c_0(\mathbb{R}^n,G)$ is an isomorphism. Consider $\mathbb{R}^n$ as an open subset of its 1-point compactification, which is homeomorphic to S^n. Applying property B2 with $X = S^n$, $U = \mathbb{R}^n$, $p = n$, $q = 0$, and $A = V = \emptyset$ and passing to homology groups yields a commutative diagrams where the vertical arrows are the isomorphisms $H_c^\infty(S^n) \to H_n^\infty(\mathbb{R}^n)$, $H_c^n(\mathbb{R}^n) \to H_c^n(S^n)$, and $H_0^c(\mathbb{R}^n) \to H_0^c(S^n)$. This reduces this case to case 1.

Case 3: $M = U \cup V$ where U and V are open sets, and suppose that we have already proved the theorem for U, V, and $U \cap V$. The triad $(M; U, V)$ is excisive for both H_c and H^c. There are long exact Mayer-Vietoris sequences for this triad in both H_c and H^c. These sequences are joined by the homomorphisms obtained by capping with fundamental classes; for example, there is $\mu_M \frown : H_c^p(M) \to H_{n-p}^c(M)$. Commutativity of the capping homomorphisms with the homomorphisms appearing in the Mayer-Vietoris sequences apart from the homomorphisms Δ follows from properties B2. The commutativity with Δ can only be verified up to sign, which can be done by considering the definition of Δ along with properties B2 for commuting with equality and e_*, B7 for commuting with inclusion maps, and B6 for commuting with connecting homomorphisms ∂ and δ. The result for this case then follows from the Five Lemma applied to the Mayer-Vietoris sequences and the capping homomorphisms, which are assumed to be isomorphisms for all but the $H_c^p(M) \to H_{n-p}^c(M)$ terms.

Case 4: Let M be the union of a nested family of open subsets $\{U_\alpha\}$, and suppose that we have already proved the theorem for each U_α. Then $H_c^p(M) = \varinjlim H_c^p(U_\alpha)$ and $H_{n-p}^c(M) = \varinjlim H_{n-p}^c(U_\alpha)$. Furthermore, for each $U_\alpha \subseteq U_\beta$ we have the following diagram.

$$\begin{tikzcd}
H_c^p(U_\alpha) \arrow[r, "\tau"] \arrow[d] & H_c^p(U_\beta) \arrow[r, "\tau"] \arrow[d] & H_c^p(M) \arrow[d] \\
H_{n-p}^c(U_\alpha) \arrow[r, "i_*"] & H_{n-p}^c(U_\beta) \arrow[r, "i_*"] & H_{n-p}^c(M)
\end{tikzcd}$$

Here, the maps τ and i_* are induced by inclusion, and the vertical arrows are the maps obtained by capping with fundamental classes. We have assumed these vertical arrows are isomorphisms except for $H_c^p(M) \to H_{n-p}^c(M)$. The diagram is commutative by property B2. The result follows after taking direct limits.

Case 5: Let M be an open subset of $\mathbb{R}^n$. First, cover M with a sequence of convex open sets U_i. Since each U_i is homeomorphic to $\mathbb{R}^n$, it follows from case 2 that the theorem holds for each U_i. It follows from case 3 and an induction argument that the theorem holds for each finite union $U_1 \cup U_2 \cup \cdots \cup U_k$. It then follows from case 4 that the theorem holds for the direct limit of these unions, which is M.

Case 6: Let M be any manifold satisfying the hypotheses of the first part of the theorem. Let $\mathcal{F} = \{U_\alpha\}$ be the family of open subsets of M for which the theorem holds. The family $\mathcal{F}$ is partially ordered by subset inclusion, and every chain has its union as an upper bound. By case 4 and Zorn's lemma, there is a maximal open subset $U \subseteq M$ for which the theorem holds. Suppose $U \neq M$. Let V be an open subset of M which is not contained in U but which is homeomorphic to an open subset of $\mathbb{R}^n$. By cases 3 and 5, the theorem holds for $U \cup V$, and so $U \cup V \in \mathcal{F}$. But $U \subsetneq U \cup V$, so this contradicts the maximality of U. □

THEOREM 6.2.2 (Poincaré-Lefschetz Duality). *Let M be an oriented n-manifold with boundary B, and let G be an abelian group. Then the homomorphism $H_c^q(M, G) \to H_{n-q}^c(M, B, G)$ given by $x \to \mu_M \frown x$ and the homomorphism $H_c^q(M - B, G) \to H_{n-q}^c(M, G)$ given by $y \to i_*(\mu_M \frown y)$ are isomorphisms, where $i_*\colon H_{n-q}^c(M - B, G) \to H_{n-q}^c(M, G)$ is induced by inclusion.*

If M is paracompact, then the homomorphism $H_\infty^q(M, B, G) \to H_{n-q}^\infty(M, G)$ given by $x \to \mu_M \frown x$ and the homomorphism $H_\infty^q(M, G) \to H_{n-q}^\infty(M - B, G)$ given by $y \to \mu_M \frown i^*(y)$ are isomorphisms, where $i^*\colon H_\infty^q(M, G) \to H_\infty^q(M - B, G)$ is induced by inclusion.

The end homology and end cohomology theories also satisfy Poincaré-Lefschetz duality. This is shown in **[12]** for piecewise linear manifolds (note Laitinen's dimension shift). Laitinen references **[18]** for general topological manifolds.

THEOREM 6.2.3. *Let M be an oriented n-manifold without boundary. Then the following is a commutative diagram with exact rows, where the vertical arrows are isomorphisms induced by taking the cap product with a fundamental class in $H_n^\infty(M)$.*

$$\begin{tikzcd}
\cdots \arrow[r] & H_c^q(M) \arrow[r] \arrow[d] & H_\infty^q(M) \arrow[r] \arrow[d] & H_e^q(M) \arrow[r, "\delta"] \arrow[d] & H_c^{q+1}(M) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n-q}^c(M) \arrow[r] & H_{n-q}^\infty(M) \arrow[r] & H_{n-q}^e(M) \arrow[r, "\partial"] & H_{n-q-1}^c(M) \arrow[r] & \cdots
\end{tikzcd}$$

The results of Chapter 5 allow us to substitute coarse homology groups for H_{n-q}^∞ or H_{n-q}^e in some conditions. In particular, this implies an isomorphism between $H_e^q(M)$ and $HC_{n-q}(M)$ under the additional hypotheses of Theorem C. Investigation into the relations between coarse cohomology and end cohomology may yield results complementing these.

6.3. Alexander Duality

An important consequence of Poincaré duality is Alexander duality. The Alexander duality theorems provide isomorphisms between cohomology groups of open subsets U of an oriented n-manifold M and homology groups of the complement $M - U$. We include the development which appears in **[14]**.

THEOREM 6.3.1 (Alexander Duality for homology and cohomology with compact support). *Let M be an oriented n-manifold, let A be a closed subset of M, let $U = M - A$, and let G be an abelian group. If $H^q_c(M,G) = H^{q+1}_c(M,G) = 0$, then $H^q_c(A,G) \cong H^c_{n-q-1}(U,G)$.*

PROOF. The proof establishes a slightly stronger result. Recall from sections 3.2 and 4.3 that there is an exact cohomology sequence for the pair $(M, M-U)$ and an exact homology sequence for the pair (M,U). By the properties of the cap product, the following diagram containing these sequences is commutative.

$$\begin{tikzcd}
\cdots \arrow[r] & H^q_c(M) \arrow[r] \arrow[d, "\mu_M\frown"] & H^q_c(M-U) \arrow[r] \arrow[d, "\mu_M\frown"] & H^{q+1}_c(U) \arrow[r] \arrow[d, "\mu_U\frown"] & H^{q+1}_c(M) \arrow[r] \arrow[d, "\mu_M\cap"] & \cdots \\
\cdots \arrow[r] & H^c_{n-q}(M) \arrow[r] & H^c_{n-q}(M,U) \arrow[r] & H^c_{n-q-1}(U) \arrow[r] & H^c_{n-q-1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

The maps $\mu_M \frown: H^q_c(M) \to H^c_{n-q}(M)$ and $\mu_U \frown: H^q_c(U) \to H^c_{n-q}(U)$ are isomorphisms for each q by the Poincaré duality theorem. It follows from the five lemma that the remaining vertical arrow $H^q_c(M-U) \to H^c_{n-q}(M,U)$ is an isomorphism as well. This establishes that the two sequences are isomorphic. It is then easy to see that if $H^q_c(M)$ and $H^{q+1}_c(M)$ are both 0 as in the hypothesis of the theorem, then we have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & H^q_c(M-U) \arrow[r, dotted, "\delta"] \arrow[d, dotted, "\mu_M\frown"] \arrow[dr, dotted, "\cong"] & H^{q+1}_c(U) \arrow[r] \arrow[d, dotted, "\mu_U\frown"] & 0 \\
0 \arrow[r] & H^c_{n-q}(M,U) \arrow[r, dotted, "\partial"] & H^c_{n-q-1}(U) \arrow[r] & 0
\end{tikzcd}$$

Here, every dotted arrow is an isomorphism, and the diagonal arrow gives the isomorphism asserted in the statement of the theorem. □

PROPOSITION 6.3.2 (Alexander Duality for homology and cohomology with arbitrary support). *Let M be a paracompact, oriented n-manifold, let A be a closed subset of M, let $U = M - A$, and let G be an abelian group. If $H^q_\infty(M,G) = H^{q+1}_\infty(M,G) = 0$, then $H^q_\infty(U,G) \cong H^\infty_{n-q-1}(A,G)$.*

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