CONSTRAINTS ON THE PARAMETERS OF A $U(1)_{B-L}$ GAUGE BOSON

CONSTRAINTS ON THE PARAMETERS OF A $U(1)_{B-L}$ GAUGE BOSON

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Abstract

The stability of the proton provides a strong constraint on models with baryon number-violating physics arising at a scale $M < 10^{15}$ GeV. In such cases, an additional gauge symmetry is often introduced in order to make such models phenomenologically viable. Here, we consider the particular case where the Standard Model is extended to include an additional $U(1)$ gauge boson that couples to baryon number minus lepton number $(B - L)$. Constraints are obtained on such a field's gauge coupling, mass, and kinetic mixing coefficient (which controls the strength of mixing between it and the hypercharge $U(1)$ present in the Standard Model). We derive updated bounds, relevant to the mass range from 1 keV to 10 TeV, by considering changes in primordial nucleosynthesis, in neutrino scattering, and in several precision electroweak observables. These constraints are overlaid in order to determine the best constraints on the gauge coupling, as a function of mass, at various values of the kinetic mixing parameter.

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Chapter 1

Introduction and Motivation

The goal of this report is to work out constraints on the parameters of a $U(1)$ gauge boson, similar to the Z boson in the Standard Model (SM), that couples to baryon number minus lepton number $(B-L)$. (Here, we refer to this field as the X boson.) Since we require this theory to be renormalizable, the parameters of interest are: the gauge coupling, g_X; the mass of the X boson, M_X (particularly in the range 1 keV$< M_X <$ 10 TeV); and the kinetic mixing constant χ, which controls the strength with which the X boson mixes with the hypercharge field B_μ from the SM. More specifically, we consider an effective Lagrangian (density) of the form

$$\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}_{B-L} + \mathcal{L}_{mix} + \mathcal{L}_{extra} \tag{1.1}$$

where:

- $\mathcal{L}_{SM}$ is the usual Standard Model Lagrangian;
- $\mathcal{L}_{B-L}$ describes the X boson, as well as its couplings to the SM fermions (this is written out explicitly in the following chapter);
- $\mathcal{L}_{mix}$ is of the form
$$\mathcal{L}_{mix} = \chi B_{\mu\nu} X^{\mu\nu} \tag{1.2}$$
where $V_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu$ is the curl of the gauge field V_μ, where $V_\mu = B_\mu,\ X_\mu$;
- $\mathcal{L}_{extra}$ contains contributions from a right-handed neutrino, which is required for anomaly cancellation (this is discussed in detail later on), and a singlet scalar, which is used to generate mass terms for the X boson and the right-handed neutrino.

Here, we focus on the physics pertaining to the first three terms of $\mathcal{L}$, and so we ignore the presence of $\mathcal{L}_{extra}$ wherever possible. In the end, we obtain plots of the g_X vs. M_X parameter space (over the range 1 keV$< M_X <$ 10 TeV) for various values of χ, ranging from 0 to 0.3. In the case of the $\chi = 0$ plot, we are able to compare with [1], who considers similar observables over the same parameter space.

1.1 Z' Physics

The Standard Model (SM) of particle physics is one the most accurate and best-tested models in Science. Although it has many unsatisfying characteristics (e.g. the hierarchy problem), it is in very good agreement with all experimental results, with the exception of neutrino oscillations. Nevertheless, theorists search for a framework in which the different forces and particles in Nature appear as a single, unified entity [2]. Many such extensions to the Standard Model predict the existence of extra gauge bosons. For example, the $SO(10)$ [3] and E_6 [4] Grand Unified Theories (GUTs) both predict additional $U(1)$ gauge fields. Also, GUTs which are based on the gauge group $SU_L(2) \times SU_R(2) \times U_{B-L}(1)$ [5] (usually called Left-Right Symmetric Models) propose an additional gauge boson that couples to baryon number minus lepton number.

The E_6 model is of particular interest since it has been shown [6] that certain supersymmetric string theories in 10 dimensions have an E_6 gauge theory as a low-energy effective field theory (see [7] for a more detailed discussion of this). Also, it has been shown that the low-energy effective field theories of such string models cannot contain global symmetries [8].

The SM contains four global symmetries: baryon number (B), electron lepton number (L_e), muon lepton number (L_μ), and tau lepton number (L_τ). One is then left with two possible outcomes in the context of the high-energy GUT:

1. the global symmetries are accidental, and there are some large energy scales M_i at which they are each broken;

2. the global symmetries are in fact present and, so, we would expect them to be gauged if the high-energy theory excludes global symmetries (as in string theory).

In case 1, given that the proton decay has not yet been observed, the energy at which baryon number is broken, M_B, should be at least 10^{15} GeV. Therefore, any theory that introduces baryon number-violating modes at energies less than this should necessarily fall into case 2.

If the SM global symmetries are to be gauged, we must ensure that anomalies cancel. (In the following section, we shall work out the details of this explictly.) It turns out that only the combinations $B - L$ ($L = L_e + L_\mu + L_\tau$), $L_e - L_\mu$, $L_\mu - L_\tau$, and $L_\tau - L_e$ are anomaly-free in the case where right-handed neutrinos are included in the particle spectrum. (Neutrino oscillations give evidence consistent with the existence of right-handed neutrinos, although none have been directly observed.) Only $B - L$ has the added benefit of limiting the possible proton decay products such that the predicted decay rate is phenomenologically viable.

Much work has been done to understand the phenomenological implications of a Z' field. However, most do not include a kinetic mixing term. The notion of a possible kinetic mixing term between $U(1)$ gauge fields is first introduced in [9]. Constraints on kinetic mixing that arise from precision electroweak experiments are considered in [10]; the most recent bounds are found in [11]. There are also several papers [12] that consider constraints on a mass mixing term between the Z and Z' (specifically, this is a term of the form $\mathcal{L}_{mix} = \delta m_z^2 Z_\mu X^\mu$). This type of mixing is not considered here, since we assume that the $B-L$ symmetry is broken by a SM singlet, which means that there should not be any mass mixing with the X boson (at tree-level).

Z' studies often include gauge fields coupled to other (anomaly-free) charges, depending on the model. Experimental searches, such as [13], quote bounds on the mass of the Z' while assuming charges identical to that of the Z. Others [14] derive bounds with the Z' coupling only to baryon number. A good resource for the various possible Z' models is [11].

Although most work is focused on constraining the Z' coupling(s) for masses at the GeV-TeV scale, there are several (stringent!) constraints at lower masses as well, as shown in [1]. At low masses, constraints on kinetic mixing arise from considering measurements of the cosmic microwave background, as shown in [15].

In the next section, we explain the role that anomaly cancellation plays in providing support for the existence of a $B-L$ gauge symmetry.

1.2 Anomaly Cancellation of Global Symmetries in the Standard Model

A gauged $B-L$ symmetry is partly motivated by the cancellation of the anomalies that arise for the global symmetries of the SM. In this section, we highlight the relevant anomalies present with the SM, and then show how these cancel in the $B-L$ model.

It is generally true that any anomaly is proportional to the following trace:

$$A(a,b,c) \equiv \mathrm{tr}\left(T_a \left\{T_b, T_c\right\}\right). \tag{1.3}$$

where the T_a's are the generators for some symmetry a (an example in which this is true is given in Appendix A; for more details, see [16]). We are now in a position to consider the anomaly coefficients $A(a,b,c)$ for the global symmetries of the SM. It is possible to show that all gauge anomalies cancel in the SM [17]. This is required since gauge symmetries must couple to conserved currents, and since currents are conserved only when gauge anomalies cancel.

Here, we are interested in the anomalies involving the four global symmetries of the SM: B (baryon number), L_e (lepton number, electron generation), L_μ (lepton number, muon generation), and L_τ (lepton number, tau generation). Therefore, we consider $A(a,b,c)$ for $a,b,c \in \{3,2,1,B,L_e,L_\mu,L_\tau\}$

(where numbers denote gauge symmetry generators and letters denote global symmetry generators) and where at least one of a, b, c corresponds to a global symmetry generator. When summing, we shall for now only include contributions for the SM fermions. Their representations under the relevant gauge groups are given in Table 1.1.

Table 1.1: Representations for the SM fermions (including right-handed neutrinos) under the SM and $B-L$ gauge groups. In the case of $SU(2)$, the bracketed values are the fermions' charge under τ_3.

	ν	l^-	u	d	$\overline{\nu}$	l^+	$\overline{u}$	$\overline{d}$
	L.H.							
$SU_c(3)$	1	1	3	3	1	1	$\overline{3}$	$\overline{3}$
$SU_L(2)$	2 (+1)	2 (−1)	2 (+1)	2 (−1)	1	1	1	1
$U_Y(1)$	−1/2	−1/2	+1/6	+1/6	0	+1	−2/3	+1/3
$U_{B-L}(1)$	−1	−1	+1/3	+1/3	+1	+1	−1/3	−1/3
	R.H.							
$SU_c(3)$	1	1	3	3	1	1	$\overline{3}$	$\overline{3}$
$SU_L(2)$	1	1	1	1	2 (−1)	2 (+1)	2 (−1)	2 (+1)
$U_Y(1)$	0	−1	+2/3	−1/3	+1/2	+1/2	−1/6	−1/6
$U_{B-L}(1)$	−1	−1	+1/3	+1/3	+1	+1	−1/3	−1/3

The non-zero anomaly coefficients are (recall that $\{\tau_a, \tau_b\} = 2\delta_{ab}$ where $\tau_a/2$ are the $SU_L(2)$ generators):

$$A(B,2,2) = \sum_{doublets} B = (3 \text{ gen.'s}) \times (3 \text{ colours}) \times \left(+\frac{1}{3}\right) = +3 \tag{1.4}$$

$$\begin{aligned} A(B,1,1) &= \sum_{all} BY^2 = 3 \times 3 \times \left[(2 \text{ per doublet})\left(+\frac{1}{3}\right)\left(+\frac{1}{6}\right)^2 \right. \\ &\qquad \left. + \left(-\frac{1}{3}\right)\left(-\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)\left(+\frac{1}{3}\right)^2\right] \\ &= -\frac{3}{2} \end{aligned} \tag{1.5}$$

$$A(L_e,2,2) = \sum_{doublets} L_e = +1 \tag{1.6}$$

$$\begin{aligned} A(L_e,1,1) &= \sum_{all} L_e Y^2 = (2 \text{ per doublet})(+1)\left(-\frac{1}{2}\right)^2 + (-1)(+1)^2 \\ &= -\frac{1}{2} \end{aligned} \tag{1.7}$$

$$A(L_e,L_e,L_e) = \sum_{all} L_e^3 = (2 \text{ per doublet})(+1)^3 + (-1)^3 = +1 \tag{1.8}$$

(The results for L_μ, L_τ are identical to those obtained for L_e.) Note that, with the exception of the last anomaly, cancellation occurs for the linear combination $B-L$ (where $L = L_e + L_\mu + L_\tau$), i.e.

$$A(B-L,2,2) = (3) - 3\times(1) = 0 \quad (1.9)$$

$$A(B-L,1,1) = \left(-\frac{3}{2}\right) - 3\times\left(-\frac{1}{2}\right) = 0. \quad (1.10)$$

However, we still find that

$$A(B-L,B-L,B-L) = 0 - 3\times(1) = -3. \quad (1.11)$$

This last anomaly demonstrates the need for additional fermions in the gauged $B-L$ model. Since this anomaly must vanish in order for $B-L$ to be a gauge symmetry, we introduce an additional fermion in each generation that is:

a) a SM singlet (so that it doesn't disturb the usual anomaly cancellation amongst the SM gauge groups);

b) has charge $(B-L)_{\nu R}$ such that $A(B-L,B-L,B-L) = 0$.

Satisfying requirement b) gives

$$\begin{aligned} A(B-L,B-L,B-L) &= 0 - 3\times(1) + 3\times(B-L)_{\nu R} = 0 \\ &\Rightarrow (B-L)_{\nu R} = +1 \end{aligned} \quad (1.12)$$

(Strictly speaking, the requirement is only that we have n generations of singlet fermions with $B-L$ charge $3/n$. However, since all other fermions come in three generations, consistency suggests that $n=3$.) This means that left-handed antineutrinos have lepton number -1 (i.e. right-handed neutrinos have lepton number $+1$).

With this choice, it is now possible to ensure that $A(1,B-L,B-L)$ also vanishes:

$$\begin{aligned} A(1,B-L,B-L) &= \sum_{all} Y(B-L)^2 \\ &= 3\left\{3\left[2\left(+\frac{1}{6}\right)\left(+\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)\left(-\frac{1}{3}\right)^2 + \left(+\frac{1}{3}\right)\left(-\frac{1}{3}\right)^2\right]\right. \\ &\qquad \left. +2\left(-\frac{1}{2}\right)(-1)^2 + (+1)(+1)^2 + (0)(+1)^2\right\} \\ &= 3\left[3\left(\frac{1}{27} - \frac{2}{27} + \frac{1}{27}\right) - 1 + 1\right] \\ &= 0 \end{aligned} \quad (1.13)$$

as required.

1.A Appendix: Abelian Chiral Anomaly Example

As an example [16] of an abelian chiral anomaly, consider n massless fermions coupled to a massless spin-1 field:

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \sum_{a=1}^{n} \bar{\psi}_a \not{D} \psi_a \tag{1.14}$$

where $\bar{\psi}_a \equiv \psi_a^\dagger \left(i\gamma^0\right)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\not{D} = \gamma^\mu D_\mu = \gamma^\mu \left(\partial_\mu - ieq_a A_\mu\right)$, e is the gauge coupling strength, and q_a is the charge of the ath fermion flavour. It is useful to split $\mathcal{L}_{EM}$ into free and interacting parts:

$$\mathcal{L}_{EM} = \mathcal{L}_o + \mathcal{L}_{int} \tag{1.15}$$

where

$$\begin{aligned} \mathcal{L}_o &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \sum_a \bar{\psi}_a \not{\partial} \psi_a \\ \mathcal{L}_{int} &= ieA_\mu \sum_a q_a \bar{\psi}_a \gamma^\mu \psi_a \end{aligned}$$

The total Lagrangian is invariant under a local $U(1)$ symmetry:

$$\psi_a(x) \to \psi_a'(x) = e^{i\varepsilon(x)eq_a\delta_{ab}}\psi_b(x) \tag{1.16}$$

(provided we transform A_μ according to $A_\mu \to A_\mu' = A_\mu + \partial_\mu \varepsilon$.) In terms of the notions introduced for general Lie groups in the previous section, this $U(1)$ symmetry group has only one generator, $Q \equiv \sum_b q_a \delta_{ab}$ and so $f^{abc} = 0 \; \forall a, b, c$. Because of this, the $U(1)$ group is referred to as an *Abelian* group. The Lagrangian $\mathcal{L}_{EM}$ is also invariant under a global chiral symmetry:

$$\psi(x) \to \psi'(x) = e^{i\varepsilon R\gamma_5}\psi(x). \tag{1.17}$$

Here, ψ is taken to mean a vector with the n Dirac spinors as entries and R, like Q, is diagonal and real.

Anomalies occur when a classical symmetry (i.e. a symmetry of the classical action) is broken when considering the corresponding quantum field theory (i.e. the effective action including higher-order loop corrections). Ultimately, this occurs because it is no longer just the classical action that must obey the symmetry; in a quantum field theory, observables are constructed from vacuum expectation values of time-ordered products, which can be calculated using path integrals as follows:

$$\langle T\left[O_1 \ldots O_n\right]\rangle = \frac{\int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}O_1 \ldots O_n e^{iS}}{\int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS}} \tag{1.18}$$

Here, $S = \int d^4x \mathcal{L}_{EM}$ is the classical action and

$$\mathcal{D}A = \prod_\mu \mathcal{D}A_\mu, \quad \mathcal{D}\psi = \prod_a \mathcal{D}\psi_a \ .$$

Due to the form of $\mathcal{L}_{int}$, it turns out [16] that $O_1 ... O_n$ must be composed entirely of combinations of A_μ and $\bar{\psi}_a \gamma^\mu \psi_a$. A tool of great use is the generating functional, defined as follows:

$$Z[J] = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left[i \int d^4x \left(\mathcal{L}_{EM} + J^\mu A_\mu\right)\right]. \tag{1.19}$$

From here, time-ordered products can be generated by taking successive functional derivatives with respect to the J's. For example,

$$\langle T\left[A_\mu A_\nu\right]\rangle = \left[\frac{1}{Z[J]}\left(-i\frac{\delta}{\delta J^\mu}\right)\left(-i\frac{\delta}{\delta J^\nu}\right) Z[J]\right]_{J=0}.$$

Since we are mainly interested in considering how the fermion fields transform under the symmetry groups, let's define

$$e^{iW[A]} \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{iS} \tag{1.20}$$

so that now we have

$$Z[J] = \int \mathcal{D}A \exp\left[iW[A] + \int d^4x J^\mu A_\mu\right] \tag{1.21}$$

From here, it is clear that, if we want our observables to be invariant under some symmetry amongst the fermions, we should require that $W[A]$ be invariant under the symmetry transformation. However, $W[A]$ includes not only the classical action, but also the functional measures $\mathcal{D}\psi$ and $\mathcal{D}\bar{\psi}$. It can be shown [18] that, under a transformation $\psi_a(x) \to \psi'_a(x) = U_a(x)\psi_a(x)$, the measures $\mathcal{D}\psi_a$ and $\mathcal{D}\bar{\psi}_a$ transform as

$$\mathcal{D}\psi_a \quad \to \quad \mathcal{D}\psi'_a = \left(\text{Det}\mathcal{U}_a\right)^{-1} \mathcal{D}\psi_a \tag{1.22}$$

$$\mathcal{D}\bar{\psi}_a \quad \to \quad \mathcal{D}\bar{\psi}'_a = \left(\text{Det}\overline{\mathcal{U}}_a\right)^{-1} \mathcal{D}\bar{\psi}_a \tag{1.23}$$

where the calligraphic U is used to denote a matrix over coordinates, as well as Dirac indices:

$$\langle x|\, \mathcal{U}_a \,|y\rangle = U_a(x)\,\delta^{(4)}(x-y) \tag{1.24}$$

and where a capital determinant (and, later, the capital trace) includes coordinate space.

In the case of the gauge transformation 1.16,

$$\bar{\psi} \to \bar{\psi}' = (\psi')^\dagger (i\gamma^0) = \left(e^{i\varepsilon(x)Q}\psi\right)^\dagger (i\gamma^0) = \psi^\dagger e^{-i\varepsilon(x)Q}(i\gamma^0) = \bar{\psi}e^{-i\varepsilon(x)Q}$$

and so

$$\bar{U}(x) = e^{-i\varepsilon(x)Q} = U^{-1}(x). \tag{1.25}$$

This means that

$$\left(\mathrm{Det}\overline{\mathcal{U}}_a\right)^{-1} = \mathrm{Det}\mathcal{U}_a \tag{1.26}$$

and, as a result,

$$\mathcal{D}\psi'_a\mathcal{D}\bar{\psi}'_a = \mathcal{D}\psi_a\mathcal{D}\bar{\psi}_a. \tag{1.27}$$

Therefore, the gauge symmetry is free of anomalies, since $W[A]$ is invariant under the symmetry transformation. The situation is not nearly as trivial for the chiral transformation: here, we find that

$$\bar{\psi} \to \bar{\psi}' = \left(e^{i\varepsilon R\gamma_5}\psi\right)^\dagger\left(i\gamma^0\right) = \psi^\dagger e^{-i\varepsilon R\gamma_5}\left(i\gamma^0\right) = \bar{\psi}e^{+i\varepsilon R\gamma_5}$$

(since $\{\gamma_\mu, \gamma_5\} = 0$) which gives

$$\mathrm{Det}\overline{\mathcal{U}}_a = \mathrm{Det}\mathcal{U}_a. \tag{1.28}$$

and so

$$\mathcal{D}\psi'_a\mathcal{D}\bar{\psi}'_a = \left(\mathrm{Det}\mathcal{U}_a\right)^{-2}\mathcal{D}\psi_a\mathcal{D}\bar{\psi}_a. \tag{1.29}$$

Now that we know that $W[A]$ is not invariant, we would like to use this form of the measure in order to calculate how much by which $W[A]$ has deviated. That is, we would like to re-write the factor $\left(\mathrm{Det}\mathcal{U}_a\right)^{-2}$ in an exponentiated form. This is done by recalling that $\mathrm{Det}(\,) = \exp\left(\mathrm{Tr}\left(\log(\,)\right)\right)$; since $\mathcal{U}_a$ is diagonal in coordinate space, $\langle x|\left(\log\mathcal{U}_a\right)|y\rangle = \log\left(\langle x|\mathcal{U}_a|y\rangle\right)$ which gives

$$\begin{aligned}
\mathrm{Tr}\left(\log(\mathcal{U}_a)\right) &= \int d^4x \mathrm{Tr}\langle x|\left(\log\mathcal{U}_a\right)|x\rangle = \int d^4x\delta^{(4)}(x-x)\,\mathrm{Tr}\left(\log U_a\right) \\
&= i\varepsilon\int d^4x\delta^{(4)}(0)\,\mathrm{tr}\left(r_a\gamma_5\right)
\end{aligned}$$

which is clearly divergent. Regularizing this [16] gives

$$\left(\mathrm{Det}\mathcal{U}_a\right)^{-2} = \exp\left(\frac{\varepsilon r_a}{32\pi^2}\int d^4x\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) \tag{1.30}$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the completely antisymmetric rank-4 tensor, with $\varepsilon^{0123} = +1$. The $\varepsilon^{\mu\nu\rho\sigma}$ tensor is obtained by tracing the γ_5 with four gamma matrices (one for each power of the cutoff Λ) which arose in the form of $\not{D}/\Lambda$ due to the requirements of Lorentz invariance and gauge covariance. The included gauge fields give rise to the factors of $F_{\mu\nu}$ with which the $\varepsilon^{\mu\nu\rho\sigma}$ tensor contracts. All in all, we find that $W[A] \to W'[A] = W[A] + \delta W$, where δW is given by

$$\delta W = -\left(\sum_a r_a\right)\frac{\varepsilon}{16\pi^2}\int d^4x\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{1.31}$$

From here, it is clear that (for $n \neq 1$) if the r_a were chosen carefully, their sum could be constructed to equal zero, thus preserving the chiral symmetry at the quantum level. This is what is meant by anomaly cancellation.

In quantum field theories with more sophisticated gauge groups, the $F_{\mu\nu}$ terms exist in particular representations of their respective gauge groups: $F_{\mu\nu} = F^a_{\mu\nu} T^a$, and so the final trace would not be over just R, but over the generators corresponding to each of the gauge fields as well. In fact, it is generally true that every chiral anomaly, regardless of whether the corresponding symmetry transformation is global or local, is proportional to a symmetric coefficient, $A\,(a,b,c)$, given by

$$A\,(a,b,c) \equiv \mathrm{tr}\left(T_a \left\{T_b, T_c\right\}\right). \tag{1.32}$$

The last two generators are symmetrized, as these are the generators belonging to $F_{\mu\nu}$ and $F_{\rho\sigma}$, and since δW is unchanged under exchange $F_{\mu\nu} \leftrightarrow F_{\rho\sigma}$ (since $\varepsilon^{\mu\nu\rho\sigma} = +\varepsilon^{\rho\sigma\mu\nu}$).

Chapter 2

The Kinetically Mixed $B-L$ Model

The goal of this chapter is to derive a formalism in which the effects of the extra gauge boson can be easily included when calculating observables. Here, in order to avoid confusion with the ordinary Z, we avoid the use of Z' to denote the $U_{B-L}(1)$ field and instead use X. We find that, as in the case of the photon, the X boson mediates a force between any particles charged under $B-L$. However, since additional long-range forces between macroscopic objects with $B-L$ charge are tightly constrained, we allow for a mass term for the X boson which suppresses the corresponding potential exponentially (i.e. $V(r) \propto \frac{1}{r} \rightarrow \frac{e^{-m_x r}}{r}$), similar to the case of the weak force carriers, $W^{\pm}$, Z. However, since this mass term is forbidden by gauge invariance, we introduce a SM singlet scalar field Φ, similar to the Higgs field, which is charged under $B-L$ and couples to right-handed neutrinos through Yukawa interactions. Therefore, as Φ acquires some vacuum expectation value (VEV), both the X boson and right-handed neutrinos acquire mass terms. Since the purpose here is to focus only on the parameters describing the X, we shall ignore the contributions to the total Lagrangian representing both right-handed neutrinos and the two Higgs fields.

2.1 The Mixed Lagrangian

We begin by considering the Lagrangian obtained after spontaneous symmetry breaking, ignoring the strong and charged current (i.e. interactions mediated by $W^{\pm}$) sectors of the SM Lagrangian. The relevant fields are: the third component of the $SU(2)$ gauge field triplet, W_μ^3; the $U(1)$ field coupled to hypercharge, B_μ; the fermion fields f_i, where i labels colour, flavour, and generation, as appropriate; and, of course, the $B-L$ field, X_μ. More specifically, the Lagrangian of interest is

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{mix} + \mathcal{L}_{mass} + \mathcal{L}_f \tag{2.1}$$

where

$$\mathcal{L}_{kin} = -\frac{1}{4}\widetilde{W}^3_{\mu\nu}\widetilde{W}_3^{\mu\nu} - \frac{1}{4}\widetilde{B}_{\mu\nu}\widetilde{B}^{\mu\nu} - \frac{1}{4}\widetilde{X}_{\mu\nu}\widetilde{X}^{\mu\nu} \tag{2.2}$$

$$\mathcal{L}_{mix} = +\frac{\chi}{2}\widetilde{B}_{\mu\nu}\widetilde{X}^{\mu\nu} \tag{2.3}$$

$$\mathcal{L}_{mass} = -\frac{1}{2}\left(m_3\widetilde{W}^3_\mu - m_0\widetilde{B}_\mu\right)\left(m_3\widetilde{W}_3^\mu - m_0\widetilde{B}^\mu\right) - \frac{m_X^2}{2}\widetilde{X}_\mu\widetilde{X}^\mu \tag{2.4}$$

$$\begin{aligned}\mathcal{L}_f = & -\sum_i \overline{f}_i\left(\not{\partial} + m_i\right)f_i + i\left(g_2\sum_i \overline{f}_i\gamma^\mu T_i^3 P_L f_i\right)\widetilde{W}^3_\mu \\ & +i\left(g_1\sum_i \overline{f}_i\gamma_i^\mu\left(Y_{iL}P_L + Y_{iR}P_R\right)f_i\right)\widetilde{B}_\mu + i\left(g_X\sum_i \overline{f}_i\gamma^\mu\left(B-L\right)_i f_i\right)\widetilde{X}_\mu\end{aligned} \tag{2.5}$$

and where $V_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu$ for each $V_\mu \in \left\{\widetilde{W}^3_\mu, \widetilde{B}_\mu, \widetilde{X}_\mu\right\}$. The masses m_3 and m_0 are defined (as in [17]) in terms of the standard model gauge couplings g_1, g_2 and the Higgs VEV v as follows:

$$m_3 = \frac{g_2 v}{2},\ m_0 = \frac{g_1 v}{2}. \tag{2.6}$$

and the charges of some fermion f_i under W^3_μ, B_μ, and X_μ are represented by T_i^3, $Y_{iL(R)}$ and $(B-L)_i$, respectively. The projectors P_L and P_R are defined as

$$P_L = \frac{1}{2}\left(1+\gamma_5\right),\ P_R = \frac{1}{2}\left(1-\gamma_5\right).$$

These are used to give the left-handed and right-handed fermions different charges under B_μ and W^3_μ.

Defining the gauge field-valued vector $\widetilde{V}$ to be

$$\widetilde{V} = \begin{bmatrix}\widetilde{W}^3 \\ \widetilde{B} \\ \widetilde{X}\end{bmatrix}, \tag{2.7}$$

we can rewrite the above Lagrangian as

$$\mathcal{L} = -\frac{1}{4}\widetilde{V}^T_{\mu\nu}K\widetilde{V}^{\mu\nu} - \frac{1}{2}\widetilde{V}^T_\mu M\widetilde{V}^\mu + i\widetilde{J}^T_\mu\widetilde{V}^\mu \tag{2.8}$$

where

$$K = \begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & -\chi \\ 0 & -\chi & 1\end{bmatrix},\ M = \begin{bmatrix}m_3^2 & -m_3m_0 & 0 \\ -m_3m_0 & m_0^2 & 0 \\ 0 & 0 & m_X^2\end{bmatrix}, \tag{2.9}$$

and

$$\widetilde{J}_\mu = \begin{bmatrix} J^3_\mu \\ J^Y_\mu \\ \widetilde{J}^{B-L}_\mu \end{bmatrix} = \begin{bmatrix} g_2 \sum_i \overline{f}_i \gamma_\mu T^3_i P_L f_i \\ g_1 \sum_i \overline{f}_i \gamma_\mu \left(Y_{iL} P_L + Y_{iR} P_R\right) f_i \\ g_X \sum_i \overline{f}_i \gamma_\mu \left(B - L\right)_i f_i \end{bmatrix}. \tag{2.10}$$

2.2 Diagonalization

The goal is to redefine the gauge fields in such a way that $K = I$ (the 3×3 unit matrix) and that M is diagonal. We begin by performing the usual weak-mixing rotation to diagonalize the weak sector of M (note that this sector, by design, has a zero eigenvalue corresponding to the photon):

$$\widetilde{V} = O_1 V' \equiv \begin{bmatrix} \cos\theta_W & \sin\theta_W & 0 \\ -\sin\theta_W & \cos\theta_W & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z' \\ A' \\ X' \end{bmatrix} \tag{2.11}$$

(where $\cos\theta_W \equiv \frac{g_2}{\sqrt{g_1^2+g_2^2}} \equiv c_W$ and $\sin\theta_W \equiv \frac{g_1}{\sqrt{g_1^2+g_2^2}} \equiv s_W$) which gives

$$K' = O_1^T K O_1 = \begin{bmatrix} 1 & 0 & \chi s_W \\ 0 & 1 & -\chi c_W \\ \chi s_W & -\chi c_W & 1 \end{bmatrix}, \quad M' = O_1^T M O_1 = \begin{bmatrix} m_Z^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_X^2 \end{bmatrix} \tag{2.12}$$

where $m_Z \equiv \frac{1}{4}\left(g_1^2 + g_2^2\right) v^2$, as in the Standard Model. We also find that, under this transformation,

$\widetilde{J}_\mu$ becomes

$$J'_\mu = O_1^T \widetilde{J}_\mu = \begin{bmatrix} J^3_\mu c_W - J^Y_\mu s_W \\ J^3_\mu s_W + J^Y_\mu c_W \\ \widetilde{J}^{B-L}_\mu \end{bmatrix} \tag{2.13}$$

$$= \begin{bmatrix} \sum_i \overline{f}_i \gamma_\mu \left[\left(g_2 c_W T^3_i - g_1 s_W Y_{iL} \right) P_L - g_1 s_W Y_{iR} P_R \right] f_i \\ \sum_i \overline{f}_i \gamma_\mu \left[\left(g_2 s_W T^3_i + g_1 c_W Y_{iL} \right) P_L + g_1 c_W Y_{iR} P_R \right] f_i \\ \widetilde{J}^{B-L}_\mu \end{bmatrix} \equiv \begin{bmatrix} J'^Z_\mu \\ J'^A_\mu \\ J'^{B-L}_\mu \end{bmatrix}$$

We design the now massless field to be the photon by letting $g_2 s_W = g_1 c_W \equiv e \left(= \sqrt{4\pi\alpha}\right)$ and $Q_i \equiv T^3_i + Y_{iL} = Y_{iR}$. With these definitions, we now have

$$J'^A_\mu = e \sum_i \overline{f}_i \gamma_\mu Q_i f_i \tag{2.14}$$

$$\begin{aligned} J'^Z_\mu &= \frac{g_1}{s_W} \sum_i \overline{f}_i \gamma_\mu \left[\left(c_W^2 T^3_i - s_W^2 Y_{iL} \right) P_L - s_W^2 Y_{iR} P_R \right] f_i \\ &= \frac{g_1}{s_W} \sum_i \overline{f}_i \gamma_\mu \left[\left(s_W^2 + c_W^2 \right) T^3_i P_L - Q_i s_W^2 \left(P_L + P_R \right) \right] f_i \\ &= \frac{e}{s_W c_W} \sum_i \overline{f}_i \gamma_\mu \left(T^3_i P_L - Q_i s_W^2 \right) f_i \end{aligned} \tag{2.15}$$

Note that, in terms of a generic form

$$J'^V_\mu = e_V \sum_i \overline{f}_i \gamma_\mu \left(g^V_{Li} P_L + g^V_{Ri} P_R \right) f_i \tag{2.16}$$

we can identify

$$\begin{matrix} e_Z = \frac{e}{s_W c_W} & & e_A = e & & e_X = g_X \\ g^Z_{Li} = T^3_i - Q_i s_W^2 & , & g^A_{Li} = Q_i & , & g^X_{Li} = (B-L)_i \\ g^Z_{Ri} = -Q_i s_W^2 & & g^A_{Ri} = Q_i & & g^X_{Ri} = (B-L)_i \end{matrix} . \tag{2.17}$$

The new Lagrangian then becomes

$$\mathcal{L} = -\frac{1}{4} V'^T_{\mu\nu} K' V'^{\mu\nu} - \frac{1}{2} V'^T_\mu M' V'^\mu + i J'^T_\mu V'^\mu . \tag{2.18}$$

From here, we diagonalize the kinetic term by letting

$$V' = L V'' \equiv \begin{bmatrix} 1 & 0 & -\frac{\chi s_W}{\sqrt{1-\chi^2}} \\ 0 & 1 & \frac{\chi c_W}{\sqrt{1-\chi^2}} \\ 0 & 0 & \frac{1}{\sqrt{1-\chi^2}} \end{bmatrix} \begin{bmatrix} Z'' \\ A'' \\ X'' \end{bmatrix} \tag{2.19}$$

which gives

$$K'' = L^T K' L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.20}$$

$$M'' = L^T M' L = \begin{bmatrix} m_Z^2 & 0 & -m_Z^2 \frac{\chi s_W}{\sqrt{1-\chi^2}} \\ 0 & 0 & 0 \\ -m_Z^2 \frac{\chi s_W}{\sqrt{1-\chi^2}} & 0 & \frac{m_X^2 + m_Z^2 \chi^2 s_W^2}{1-\chi^2} \end{bmatrix} \tag{2.21}$$

$$J''_\mu = L^T J'_\mu = \begin{bmatrix} J'^Z_\mu \\ J'^A_\mu \\ -\frac{\chi s_W}{\sqrt{1-\chi^2}} J'^Z_\mu + \frac{\chi c_W}{\sqrt{1-\chi^2}} J'^A_\mu + \frac{1}{\sqrt{1-\chi^2}} J'^{B-L}_\mu \end{bmatrix} \tag{2.22}$$

(Note that, given the above expressions for L and O_1, it can be explicitly checked that $LO_1 = O_1 L$. This means that weak mixing does not influence the way in which we should diagonalize the kinetic terms.)

For later convenience, we introduce the variables η and c_X, defined to be

$$\eta \equiv \frac{\chi}{\sqrt{1-\chi^2}} \tag{2.23}$$

$$c_X \equiv \frac{m_X}{m_Z} \tag{2.24}$$

which simplifies the above expressions for M'' and J''_μ:

$$M'' = m_Z^2 \begin{bmatrix} 1 & 0 & -\eta s_W \\ 0 & 0 & 0 \\ -\eta s_W & 0 & c_X^2 \left(1+\eta^2\right) + \eta^2 s_W^2 \end{bmatrix}, \tag{2.25}$$

$$J''_\mu = \begin{bmatrix} J'^Z_\mu \\ J'^A_\mu \\ -\eta s_W J'^Z_\mu + \eta c_W J'^A_\mu + \sqrt{1+\eta^2} J'^{B-L}_\mu \end{bmatrix}. \tag{2.26}$$

(Note: c_X should not be associated with a cosine, as in the case of the weak angle, when $m_X > m_Z$.) Finally, we diagonalize the mass matrix by letting

$$V'' = O_2 V \equiv \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} Z \\ A \\ X \end{bmatrix} \tag{2.27}$$

where α is implicitly given by

$$\tan 2\alpha = \frac{-2\eta s_W}{1 - c_X^2 - \eta^2 \left(c_X^2 + s_W^2\right)}. \tag{2.28}$$

The final Lagrangian is

$$\mathcal{L} = -\frac{1}{4} V_{\mu\nu}^T V^{\mu\nu} - \frac{M_Z^2}{2} Z_\mu Z^\mu - \frac{M_X^2}{2} X_\mu X^\mu + i J_\mu^T V^\mu \tag{2.29}$$

where

$$M_Z^2,\ M_X^2 = \frac{m_Z^2}{2}\left[1 + \eta^2 s_W^2 + c_X^2\left(1+\eta^2\right) \pm \sqrt{\left(1 - \eta^2 s_W^2 - c_X^2\left(1+\eta^2\right)\right)^2 + 4\eta^2 s_W^2}\right] \tag{2.30}$$

($\pm$ signs are assigned so that $M_Z^2 \to m_Z^2$ and $M_X^2 \to m_X^2$ as $\eta \to 0$) and where

$$J_\mu = \begin{bmatrix} J_\mu'^Z \cos\alpha + \sin\alpha \left(-\eta s_W J_\mu'^Z + \eta c_W J_\mu'^A + \sqrt{1+\eta^2} J_\mu'^{B-L}\right) \\ J_\mu'^A \\ -J_\mu'^Z \sin\alpha + \cos\alpha \left(-\eta s_W J_\mu'^Z + \eta c_W J_\mu'^A + \sqrt{1+\eta^2} J_\mu'^{B-L}\right) \end{bmatrix} \equiv \begin{bmatrix} J_\mu^Z \\ J_\mu^A \\ J_\mu^{B-L} \end{bmatrix}. \tag{2.31}$$

2.3 The Small η Expansion

Given the result obtained in Equation 2.30, it is clear that, for arbitrary η, it is possible to tune the initial mass term m_Z so that the physical Z mass has the correct value (regardless of the value of m_X). However, since the SM is in good agreement with experiment [19], we expect the so-called "oblique" corrections [20] (which are discussed in more detail in the following section) to be small. In particular, the fractional correction to the Z mass, defined to be (as in [21])

$$z \equiv \frac{M_Z^2 - m_Z^2}{m_Z^2} \tag{2.32}$$

should be small. From Equation 2.30, we find that (using $K = \eta^2 s_W^2 + c_X^2\left(1+\eta^2\right)$)

$$z = \frac{1}{2}\left(K - 1 - \sqrt{\left(K-1\right)^2 + 4\eta^2 s_W^2}\right) \tag{2.33}$$

It is clear that, for arbitrary values of c_X, it is possible to choose η to be small enough so that z is also small. Since we expect z to be small (because the SM gives reliable results), it makes sense to perform an expansion in η, which will be valid for most values of c_X. It turns out that such an expansion is most unreliable near $c_X = 1$; we consider this in more detail shortly.

Expanding about $\eta = 0$ in Equation 2.30 gives

$$M_Z^2 \simeq m_Z^2\left(1+\frac{s_W^2}{1-c_X^2}\eta^2+O\left(\eta^4\right)\right) \tag{2.34a}$$

$$M_X^2 \simeq m_X^2\left(1+\frac{c_W^2-c_X^2}{1-c_X^2}\eta^2+O\left(\eta^4\right)\right) \tag{2.34b}$$

and so

$$z \simeq \frac{s_W^2}{1-c_X^2}\eta^2. \tag{2.35}$$

Since the first correction terms are of $O\left(\eta^2\right)$, we shall expand all of our expressions up to the same order. In particular, this gives

$$\cos\alpha \simeq 1-\frac{s_W^2}{2\left(1-c_X^2\right)^2}\eta^2+O\left(\eta^4\right), \quad \sin\alpha \simeq -\frac{s_W}{1-c_X^2}\eta+O\left(\eta^3\right) . \tag{2.36}$$

Before continuing, it is informative to first consider the values of m_X for which a perturbative expansion in η is appropriate. For example, consider the percent relative difference between $\cos\alpha$ and its expansion up to $O\left(\eta^2\right)$:

$$\varepsilon\left(\eta, c_X^2\right) = \left|\frac{\cos\alpha - 1 + \frac{s_W^2}{2\left(1-c_X^2\right)^2}\eta^2}{\cos\alpha}\right| \times 100\% \tag{2.37}$$

A plot of this function is shown in Figure 2.1.

This graph shows that, for small values of η (specifically, for values of $\eta \in [0, 0.7]$), the expansion up to $O\left(\eta^2\right)$ is valid to within 0.5% for most values of m_X, with the exception being those values of m_X within about 20 GeV of the Z mass. Therefore, we can trust the validity of this approximation for any combination of η and m_X, except for those with both larger η values and an m_X value between $\sim 70-110$ GeV.

It is useful now to introduce the variable

$$s_X^2 \equiv 1-c_X^2 = 1-\frac{m_X^2}{m_Z^2}, \tag{2.38}$$

which simplifies Equations 2.34 and 2.36. (Note that the use of s_X^2 is meant only as a memory tool because of the similarity in definition between it and $s_W^2 = 1-\frac{m_W^2}{m_Z^2}$, and that it should *not* be assumed that $s_X^2 \in [0,1]$.)

Before continuing, note that when considering expressions of the form η/s_X^2, the expressions $s_X^2 = 1-m_X^2/m_Z^2$ and $s_X^2 = 1-M_X^2/M_Z^2$ can be used interchangeably. This is because

$$1-m_X^2/m_Z^2 = 1-M_X^2/M_Z^2+O\left(\eta^2\right)$$

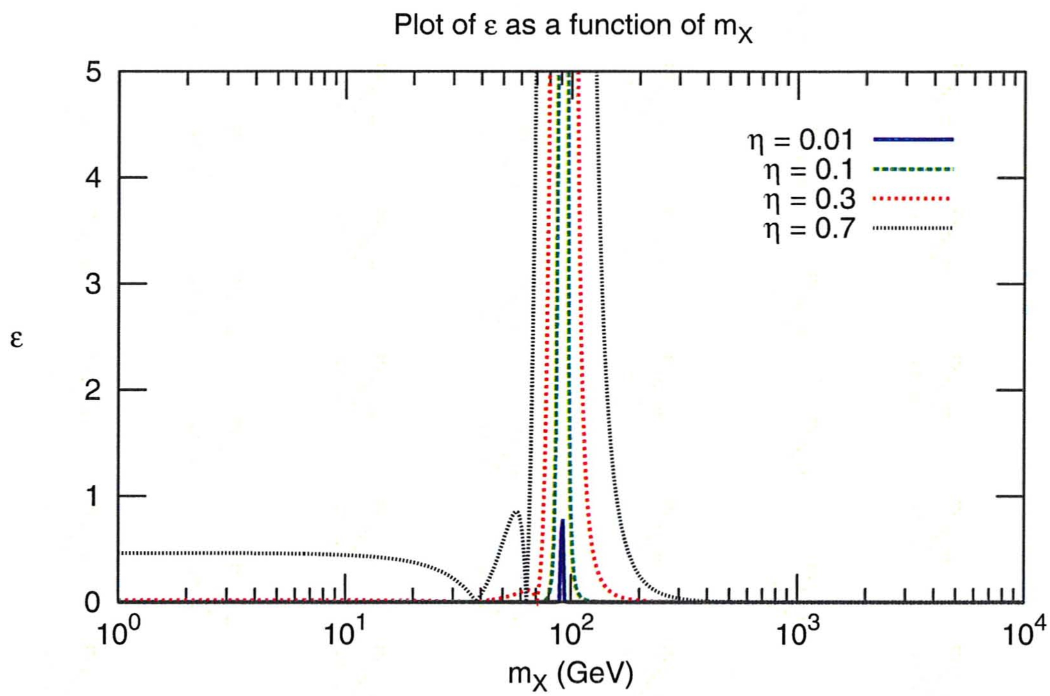

Figure 2.1: Plot of $\varepsilon\left(\eta, c_X^2\right)$ as a function of m_X.

and so corrections to η/s_X^2 arise at $O\left(\eta^3\right)$, which we are neglecting here.

With this new definition, we now have

$$M_Z^2 \simeq m_Z^2\left(1+\frac{s_W^2}{s_X^2}\eta^2\right) \tag{2.39}$$

$$M_X^2 \simeq m_X^2\left(1+\left(1-\frac{s_W^2}{s_X^2}\right)\eta^2\right) \tag{2.40}$$

$$J_\mu^Z \simeq J_\mu^{\prime Z}-\eta\frac{s_W}{s_X^2}J_\mu^{\prime B-L}-\eta^2\frac{s_W}{s_X^2}\left(-s_W J_\mu^{\prime Z}+c_W J_\mu^{\prime A}\right)-\frac{1}{2}\eta^2\frac{s_W^2}{s_X^4}J_\mu^{\prime Z} \tag{2.41}$$

$$J_\mu^{B-L} \simeq J_\mu^{\prime B-L}-\eta s_W J_\mu^{\prime Z}+\eta c_W J_\mu^{\prime A}+\eta\frac{s_W}{s_X^2}J_\mu^{\prime Z}+\frac{1}{2}\left(1-\frac{s_W^2}{s_X^4}\right)\eta^2 J_\mu^{\prime B-L} \tag{2.42}$$

Therefore, if we define

$$\Delta J_\mu^Z \equiv J_\mu^Z - J_\mu^{\prime Z} \equiv e_Z\sum_i \overline{f}_i\gamma^\mu\left(\delta g_{Li}P_L+\delta g_{Ri}P_R\right)f_i \tag{2.43}$$

$$\Delta J_\mu^X \equiv J_\mu^X - J_\mu^{\prime X} \equiv \sum_i \overline{f}_i\gamma^\mu\left(\delta k_{Li}P_L+\delta k_{Ri}P_R\right)f_i \tag{2.44}$$

(where we absorb an overall factor of g_X into the δk terms so that they have a well-defined limit as

$g_X \to 0$), then we find that (recall that $1 = P_L + P_R$)

$$\begin{aligned}(\delta g_L)_i &\simeq -\eta \frac{s_W}{s_X^2}\frac{g_X}{e_Z}(B-L)_i + \eta^2 \frac{s_W^2}{s_X^2}\left(T_i^3 - Q_i\right) - \frac{1}{2}\eta^2 \frac{s_W^2}{s_X^4}\left(g_L^{SM}\right)_i \\ (\delta g_R)_i &\simeq -\eta \frac{s_W}{s_X^2}\frac{g_X}{e_Z}(B-L)_i - \eta^2 \frac{s_W^2}{s_X^2} Q_i - \frac{1}{2}\eta^2 \frac{s_W^2}{s_X^4}\left(g_R^{SM}\right)_i .\end{aligned}$$

or

$$\left(\delta g_{L(R)}\right)_i \simeq -\eta \frac{s_W}{s_X^2}\frac{g_X}{e_Z}(B-L)_i + \eta^2 \frac{s_W^2}{s_X^2}\left[\left(g_{L(R)}^{SM}\right)_i - Q_i c_W^2\right] - \frac{1}{2}\eta^2 \frac{s_W^2}{s_X^4}\left(g_{L(R)}^{SM}\right)_i \tag{2.45}$$

(where $\left(g_L^{SM}\right)_i = T_i^3 - Q_i s_W^2$, $\left(g_R^{SM}\right)_i = -Q_i s_W^2$) and that, similarly,

$$\left(\delta k_{L(R)}\right)_i \simeq -\eta\left(1 - \frac{1}{s_X^2}\right) s_W e_Z \left(g_{L(R)}^{SM}\right)_i + \eta c_W e Q_i + \frac{1}{2}\left(1 - \frac{s_W^2}{s_X^4}\right)\eta^2 g_X (B-L)_i . \tag{2.46}$$

Here, we use the notation $\simeq$ to denote equality up to terms of $O\left(\eta^3\right)$ or higher.

From here, the next step is to consider the effect of kinetic mixing on the electroweak parameters in the SM.

2.4 The Unmixed Lagrangian and Oblique Corrections

Up until now, we have only considered a particular sector of the Standard Model Lagrangian ($\mathcal{L}_{SM}$). However, since the rest of $\mathcal{L}_{SM}$ remains unchanged by the redefinitions that we have made above, we can now write an effective Lagrangian of the form

$$\mathcal{L}_{eff} = \mathcal{L}_{SM} + \delta\mathcal{L}_{SM} + \mathcal{L}_X \tag{2.47}$$

where

$$\delta\mathcal{L}_{SM} = -\frac{z}{2} m_Z^2 Z_\mu Z^\mu + i e_Z \left(\sum_i \overline{f}_i \gamma^\mu \left(\delta g_{Li} P_L + \delta g_{Ri} P_R\right) f_i\right) Z_\mu \tag{2.48a}$$

$$\begin{aligned}\mathcal{L}_X &= -\frac{1}{4} X_{\mu\nu} X^{\mu\nu} - \frac{M_X^2}{2} X_\mu X^\mu \\ &\quad + i\left(\sum_i \overline{f}_i \gamma^\mu \left(g_X (B-L)_i + \delta k_{Li} P_L + \delta k_{Ri} P_R\right) f_i\right) X^\mu\end{aligned} \tag{2.48b}$$

Note that, in this formalism, X_μ is decoupled from the weak force gauge fields at tree-level, and that the perturbative effect of kinetic mixing on the Z mass and the neutral current couplings (up to $O\left(\eta^2\right)$) is encoded within $\delta\mathcal{L}_{SM}$. Also, in the limit of no mixing, the expressions for z, $\delta g_{L(R)i}$, and $\delta k_{L(R)i}$ reduce to 0.

From here, we can shift some parameters in the SM in order to accommodate for the presence of $\delta\mathcal{L}_{SM}$. These small changes in the constants can then be propagated to produce corresponding changes in observables.

It is practical to choose the input parameters of the theory to be the most accurately measured electroweak constants. These are: e (or $\alpha \equiv e^2/4\pi$), G_F, and M_Z (as well as the fermion masses and the CKM matrix elements).

We have already seen that the prediction for the Z mass is shifted as a result of kinetic mixing. In order to cancel this effect, we choose the m_Z parameter in $\mathcal{L}_{SM}$ such that the prediction obtained from $\mathcal{L}_{eff}$ is the measured value (up to terms $O\left(z^2\right)$):

$$m_Z \equiv M_Z \left(1 - \frac{z}{2}\right) \tag{2.49}$$

As for e, it does not change under the previous field redefinitions since $J^A_\mu = J'^A_\mu$:

$$e = \widetilde{e} \tag{2.50}$$

Here, $\widetilde{\ }$ denotes dimensionless constants found in $\mathcal{L}_{SM}$, in order to distinguish them from the corresponding constants in $\mathcal{L}_{eff}$. (For dimensionful constants, as in m_Z and M_Z, we use lower/uppercase to differentiate.) Note that the original definition of $\widetilde{e}$, $\widetilde{e} \equiv \widetilde{g}_2 \widetilde{s}_W$, still holds. However, it is important to note that $e = \widetilde{e}$ does not imply that $g_2 = \widetilde{g}_2$ and $s_W = \widetilde{s}_W$ as well.

Finally, when considering muon decay, $\mathcal{L}_{SM}$ and $\mathcal{L}_{eff}$ both predict the following expression (at tree-level) for G_F:

$$\frac{G_F}{\sqrt{2}} = \frac{\widetilde{g}_2^2}{8m_W^2} = \frac{e^2}{8\widetilde{s}_W^2 \widetilde{c}_W^2 m_Z^2} \tag{2.51}$$

To determine the z-dependence of $\widetilde{s}_W^2$, we first use the above equation to *define* the value of s_W^2 in terms of our input parameters:

$$s_W^2\left(1 - s_W^2\right) \equiv \frac{\sqrt{2}e^2}{8G_F M_Z} \tag{2.52}$$

From here, we choose the z-dependence of $\widetilde{s}_W^2$ so that Equation 2.51 reproduces the experimental value of G_F (i.e. it cancels the z-dependence already present in Equation 2.51, to linear order in z):

$$\widetilde{s}_W^2 \widetilde{c}_W^2 = s_W^2 c_W^2 \left(1 + z\right)$$

which, in turn, implies that

$$\widetilde{s}_W^2 = s_W^2 \left[1 + \frac{c_W^2}{c_W^2 - s_W^2} z\right]. \tag{2.53}$$

From here, we can calculate the total change in the coefficients g_{Li} and g_{Ri}, once these shifts of parameters are made, by considering the portion of $\mathcal{L}_{eff}$ coupled to Z_μ, $\mathcal{L}_{eff,NC}$:

$$\begin{aligned}
\mathcal{L}_{eff,NC} &= i\frac{\widetilde{e}}{\widetilde{s}_W\widetilde{c}_W}\left(\sum_i \overline{f}_i\gamma^\mu\left(T_i^3 P_L - Q_i\widetilde{s}_W^2\right)f_i\right. \qquad (2.54)\\
&\qquad \left. +\sum_i \overline{f}_i\gamma^\mu\left(\delta g_{Li}P_L + \delta g_{Ri}P_R\right)f_i\right)Z_\mu \\
&\simeq i\frac{e}{s_W c_W}\left(1-\frac{z}{2}\right)\sum_i \overline{f}_i\gamma^\mu\left[T_i^3 P_L - Q_i s_W^2\left(1+\frac{c_W^2}{c_W^2 - s_W^2}z\right)\right]f_i \qquad (2.55)\\
&\quad +i\frac{e}{s_W c_W}\sum_i \overline{f}_i\gamma^\mu\left(\delta g_{Li}P_L + \delta g_{Ri}P_R\right)f_i Z_\mu \\
\mathcal{L}_{eff,NC} &\simeq \mathcal{L}_{SM,NC} + i\frac{e}{s_W c_W} \\
&\quad \times\sum_i \overline{f}\gamma^\mu\left[\begin{array}{c}\left(-\frac{z}{2}\left(T_i^3 - Q_i s_W^2\right) - zQ_i\frac{s_W^2 c_W^2}{c_W^2 - s_W^2} + \delta g_{Li}\right)P_L \\ +\left(-\frac{z}{2}\left(-Q_i s_W^2\right) - zQ_i\frac{s_W^2 c_W^2}{c_W^2 - s_W^2} + \delta g_{Ri}\right)P_R\end{array}\right]f_i Z_\mu \qquad (2.56)
\end{aligned}$$

By referring to [21], we can find definitions for the conventional "oblique" parameters, first introduced in [20]. These give

$$\alpha S = \alpha U = 0 \qquad (2.57a)$$

$$\alpha T = -z = -\eta^2\frac{s_W^2}{s_X^2}. \qquad (2.57b)$$

In terms of these parameters, we have

$$\left(\Delta g_{L(R)}\right)_i = \frac{\alpha T}{2}g_{L(R)i}^{SM} + \alpha T\frac{s_W^2 c_W^2}{c_W^2 - s_W^2}Q_i + \delta g_{L(R)i} \qquad (2.58)$$

in agreement with [21]. Alternatively, we can write these expressions explicitly in terms of η, g_X and M_X^2:

$$\begin{aligned}
\left(\Delta g_{L(R)}\right)_i &= -\frac{1}{2}\eta^2\frac{s_W^2}{s_X^2}g_{L(R)i}^{SM} - \eta^2\frac{s_W^2}{s_X^2}\frac{s_W^2 c_W^2}{c_W^2 - s_W^2}Q_i \\
&\quad -\eta\frac{s_W}{s_X^2}\frac{g_X}{e_Z}\left(B-L\right)_i + \eta^2\frac{s_W^2}{s_X^2}\left(g_{L(R)i}^{SM} - Q_i c_W^2\right) - \frac{1}{2}\eta^2\frac{s_W^2}{s_X^4}g_{L(R)i}^{SM}
\end{aligned}$$

or

$$\left(\Delta g_{L(R)}\right)_i = -\eta\frac{s_W}{s_X^2}\frac{g_X}{e_Z}\left(B-L\right)_i - \eta^2\frac{s_W^2}{s_X^2}\left(\frac{c_X^2}{2s_X^2}g_{L(R)i}^{SM} + Q_i\frac{c_W^4}{c_W^2 - s_W^2}\right) \qquad (2.59)$$

Note that a similar procedure could be pursued to calculate additional shifts in $\delta k_{L(R)i}$ for the X boson. However, since the SM parameters all appear in terms proportional to η, corrections are

$O\left(\eta^3\right)$ and can, therefore, be neglected. Therefore, we simply have that

$$\left(\Delta k_{L(R)}\right)_i = \left(\delta k_{L(R)}\right)_i = -\eta\left(1-\frac{1}{s_X^2}\right) s_W e_Z \left(g_{L(R)}^{SM}\right)_i + \eta c_W e Q_i + \frac{1}{2}\left(1-\frac{s_W^2}{s_X^4}\right)\eta^2 g_X \left(B-L\right)_i . \quad (2.60)$$

In summary, we have used field redefinitions to show that the X boson contributes to observables through: a) shifts in the SM couplings; b) additional couplings to X^μ of the form $J_\mu X^\mu$. With this new formalism, we obtain constraints on the $B-L$ parameters by calculating predictions for experimentally observable quantities using $\mathcal{L}_{eff}$. The purpose of future chapters is to describe several such quantities and to use them to obtain bounds on η, g_X, and M_X.

Chapter 3

Constraints on Electroweak Observables

In this chapter, we consider the influence of the kinetically-mixed X boson on various electroweak observables. We begin with two observables that are unchanged by interactions with the X boson at tree-level: the W boson mass and the decay rate for the process $Z \to l^+l^-$. These constraints arise solely due to the presence of the kinetic mixing term. We also consider a process that receives tree-level contributions from X boson interactions: the cross section for electron-positron annihilation into hadrons evaluated at the Z pole.

3.1 The W Mass

Our aim is to constrain η, M_X, and g_X by calculating their contribution to the change of the W mass from its value in the Standard Model. Since current measurements of the W mass agree with the Standard Model value, the experimental uncertainty places a bound on how big these variables can be without causing the model to produce unphysical predictions.

We know that the above field redefinitions do not change the constant in front of the term $W_\mu^\dagger W^\mu$ in the SM Lagrangian, so the W mass only receives corrections due to its definition in terms of $m_Z^2\ (= M_Z^2\,[1 + \alpha T])$ and $\tilde{s}_W^2\ \left(= s_W^2\left[1 - \frac{c_W^2}{c_W^2 - s_W^2}\alpha T\right]\right)$:

$$\begin{aligned}
M_W^2 &= m_W^2 = m_Z^2 \tilde{c}_W^2 = m_Z^2\left(1 - \tilde{s}_W^2\right) \\
&= M_Z^2 (1 + \alpha T)\left[1 - s_W^2\left(1 - \frac{c_W^2}{c_W^2 - s_W^2}\alpha T\right)\right] \\
&= M_Z^2 \left(1 - s_W^2\right)(1 + \alpha T)\left(1 + \frac{s_W^2}{c_W^2 - s_W^2}\alpha T\right) \\
&\simeq M_Z^2 c_W^2\left[1 + \alpha T\left(1 + \frac{s_W^2}{c_W^2 - s_W^2}\right)\right]
\end{aligned}$$

In general, the W mass will also receive loop corrections that alter the tree-level expression $M_W = M_Z c_W$. However, since these corrections are known to be small, we can neglect their product with αT (another quantity expected to be small) in the above expression for M_W^2. Therefore, if we use the notation $(x)_{SM}$ to mean the value of x that is obtained using $\mathcal{L}_{SM}$ (including loop corrections), we find that

$$M_W^2 = \left(M_W^2\right)_{SM}\left[1 + \frac{c_W^2}{c_W^2 - s_W^2}\alpha T\right] \tag{3.1}$$

From here, we can calculate $\Delta M_W \equiv M_W - (M_W)_{SM}$:

$$\begin{aligned} \Delta M_W &\simeq M_Z c_W \left[\frac{1}{2}\frac{c_W^2}{c_W^2 - s_W^2}\left(-\eta^2 \frac{s_W^2}{s_X^2}\right)\right] = \frac{M_Z^3}{2}\frac{s_W^2 c_W^3}{c_W^2 - s_W^2}\left(\frac{\eta^2}{M_X^2 - M_Z^2}\right) \\ \Delta M_W &= 1.10 \times 10^5 \left(\frac{\eta^2}{M_X^2 - M_Z^2}\right) \ \text{GeV}^3. \end{aligned} \tag{3.2}$$

This is in agreement with the result given in [22] in the limit where $M_X \gg M_Z$; [22] quotes this result as

$$\Delta M_W = (17 \text{ MeV})\left(\frac{\eta}{0.1}\right)^2\left(\frac{250 \text{ GeV}}{M_X}\right)^2.$$

Given that, from experiment, $\Delta M_W \leq 0.025$ GeV [23] (1σ uncertainty), we can plot the implicit constraint on η as a function of M_X. This is shown in Figure 3.1.

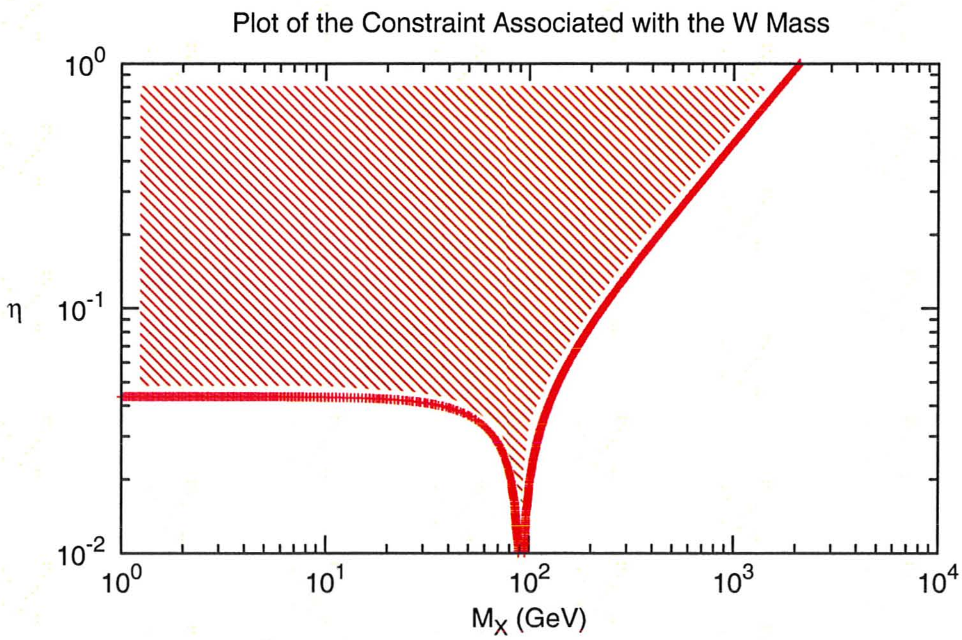

Figure 3.1: Constraint obtained from limiting the influence of kinetic mixing on the SM value of the W mass. The excluded region is shaded.

Note:

- the strongest constraints on η exist near the Z pole (i.e. when $M_X \simeq M_Z$) (note: the bound should not be trusted for M_X arbitrarily close to the Z pole - see Section 2.3 for details);

- when $M_X \ll M_Z$, the bound becomes simply $\eta \leq 4.4 \times 10^{-2}$ as the M_X-dependence vanishes. This is expected from the form of Equation 3.2;

- when $M_X \gg M_Z$, it's the ratio η/M_X that is constrained: $\eta/M_X \leq 4.8{\times}10^{-4}$ or $\left(\frac{\eta}{0.5}\right)\left(\frac{1\ \mathrm{TeV}}{M_X}\right) \lesssim 1$.

3.2 Z Decay

The Z decay rate has been measured with great accuracy at LEP and SLC (for details regarding their analysis, see [19]). The PDG [23] value for the decay $Z \to l^+l^-$, where l can be any of the charged leptons, is $\Gamma_{l^+l^-} = 83.984 \pm 0.086$ MeV. Since the SM agrees with this result (which is [23] 83.988 ± 0.016 MeV), we can use the experimental error to constrain the correction to the SM value due to the X boson. To obtain the correction, we calculate the decay rate of the Z boson into some charged fermions using $\mathcal{L}_{eff}$ given in Equations 2.47 and 2.48. This is shown explicitly in Appendix C (some of the techniques used there are first discussed in Appendix A). We find that

$$\Gamma_{l^+l^-} = \frac{M_Z e_Z^2}{24\pi}\left(g_L^2 + g_R^2\right)$$

where

$$\begin{aligned} g_i &= g_i^{SM} + \Delta g_i \\ &= -\frac{1}{2}\delta_{Li} + s_W^2 + \eta\frac{s_W}{s_X^2}\frac{g_X}{e_Z} - \eta^2\frac{s_W^2}{s_X^2}\left[\frac{c_X^2}{2s_X^2}\left(-\frac{1}{2}\delta_{Li} + s_W^2\right) - \frac{c_W^4}{c_W^2 - s_W^2}\right]. \end{aligned}$$

(δ_{iL} is a Kronecker delta function: $\delta_{LL} = 1$, $\delta_{LR} = 0$.) Substituting this expression (and keeping only terms up to $O\left(\eta^2\right)$) gives an expression of the form

$$\Gamma_{l^+l^-} = \left(\Gamma_{l^+l^-}\right)_{SM} + A\left(s_X^2\right) g_X \eta + \left[B\left(s_X^2\right) + g_X^2 C\left(s_X^2\right)\right]\eta^2.$$

Requiring that $\Gamma_{l^+l^-} - \left(\Gamma_{l^+l^-}\right)_{SM} \leq 0.086$ MeV (1σ uncertainty) then gives an implicit bound on the parameters g_X, η, and M_X. Using Maple, plots of these bounds have been generated for various values of η. These are shown in Figure 3.2. (Note that α_X, not g_X, is used as the y-coordinate.)

A few features should be highlighted:

- In all four plots, there is a small region containing allowed g_X/M_X combinations that is found, for $M_X < M_Z$, between the upper and lower limits of this bound. It is for this reason that the upper and lower limits have been distinguished in the plots;

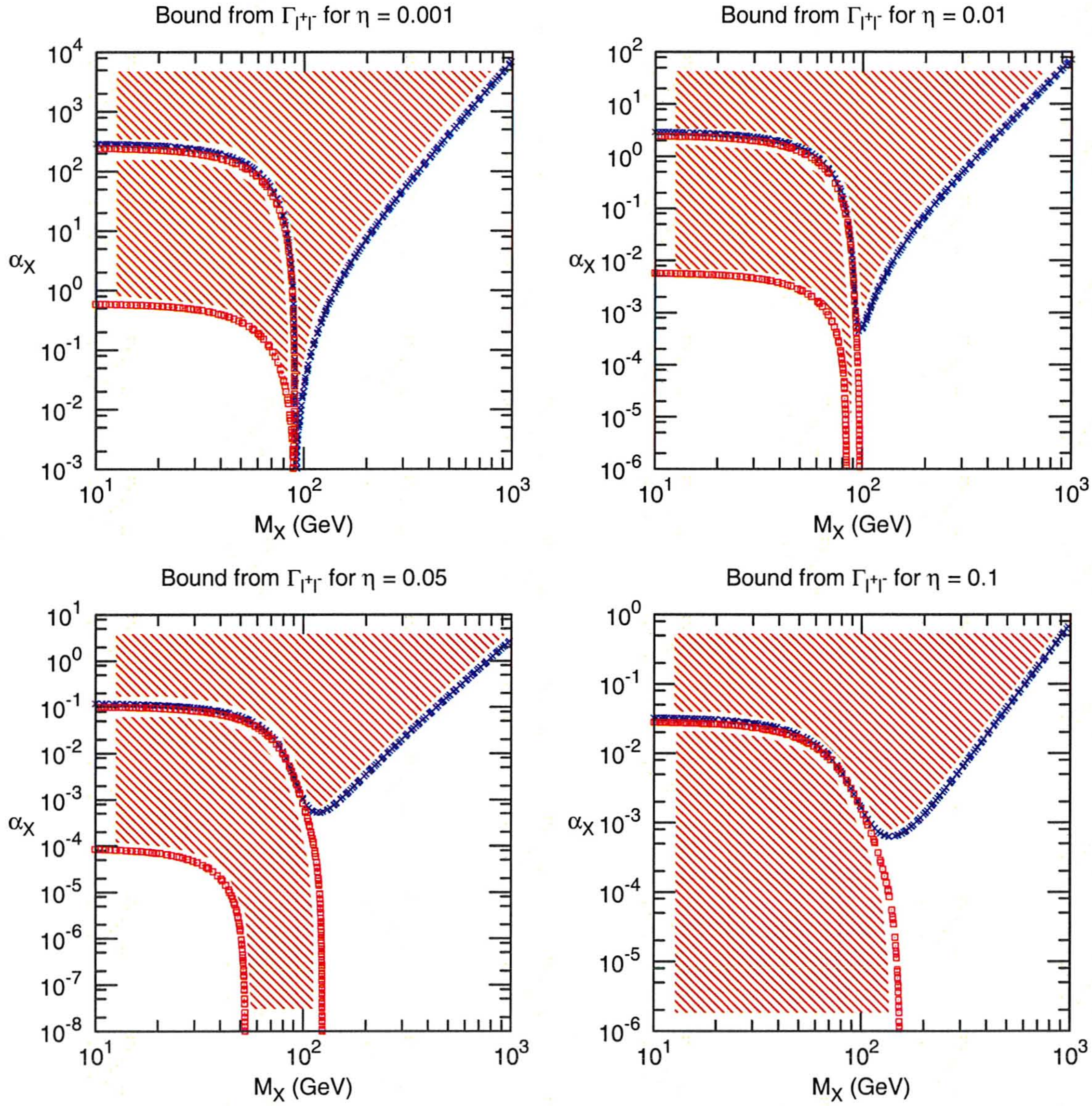

Figure 3.2: Plot of the constraint arising from considering Z decay into leptons. Here, we plot the bound on α_X as a function of M_X for various values of η. The upper bound ($\Delta\Gamma = +\Delta\Gamma_{\text{exp.}}$) is marked with blue crosses; the lower bound ($\Delta\Gamma = -\Delta\Gamma_{\text{exp.}}$) is marked with red squares. Excluded regions have been shaded out.

- At large M_X, the slope is found to correspond to the ratio g_X/M_X^2 being constrained. This is, in fact, expected from the form of the g_i, whose linear term in η is proportional to g_X/M_X^2 for large M_X;
- As η grows larger there is a point, similar to the one found when considering electron-positron annihilation, at which even very small gauge couplings are ruled out for small M_X.

The bound in the limit where $M_X \ll M_Z$ is shown in Figure 3.3.

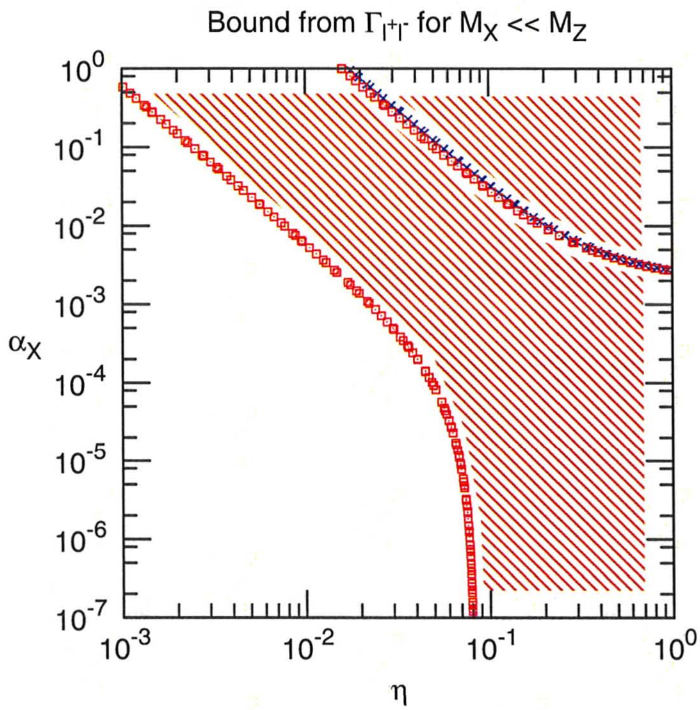

Figure 3.3: Plot of the constraint arising from considering Z decay into leptons in the limit where $M_X \ll M_Z$. The upper bound ($\Delta\Gamma = +\Delta\Gamma_{\text{exp.}}$) is marked with blue crosses; the lower bound ($\Delta\Gamma = -\Delta\Gamma_{\text{exp.}}$) is marked with red squares. Excluded regions have been shaded out.

This plot still exhibits a thin region of allowed g_X/M_X combinations, even past the cut-off for small g_X at $\eta \simeq 0.08$, although this region shrinks for larger η.

3.3 Electron-Positron Annihilation

We are now interested in calculating the cross section for the process $e^+e^- \to f\bar{f}$ in the case where the incoming electron and positron have a combined energy (in the centre of mass frame) equal to the rest mass of the Z boson. The reason for this is that very accurate measurements [19] have been performed of this cross section at this energy and so deviations from the SM are expected to be well constrained.

The relevant tree-level diagrams all contain a mediating gauge boson (either the γ, Z, or X) and are shown in Figure 3.4.

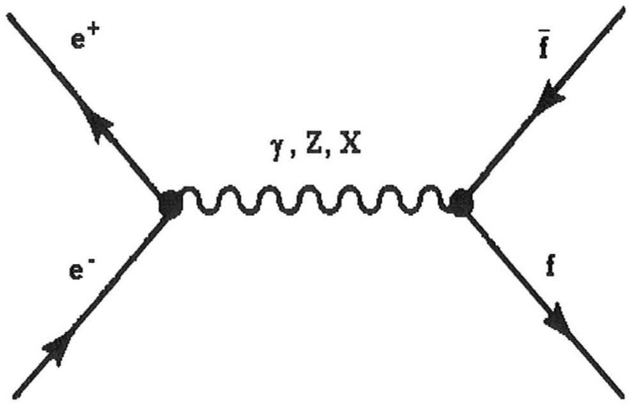

Figure 3.4: Relevant tree-level Feynman diagrams corresponding to electron-positron annihilation.

The relevant part of the Lagrangian is of the form

$$\mathcal{L} = \mathcal{L}_o + \mathcal{L}_{int} \tag{3.3}$$

where

$$\mathcal{L}_o = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}X_{\mu\nu}X^{\mu\nu} - \frac{1}{2}M_Z^2 Z_\mu Z^\mu - \frac{1}{2}M_X^2 X_\mu X^\mu \tag{3.4}$$

$$- \sum_{f=e,u,d,s,c,b} \overline{f}\left(\partial\!\!\!/ + m_f\right) f \tag{3.5}$$

and

$$\mathcal{L}_{int} = \sum_{V=A,Z,X}\ \sum_{f=e,u,d,s,c,b} ie_V V_\mu \left(\overline{f}\gamma^\mu \Gamma_f^V f\right). \tag{3.6}$$

Here, we use an abbreviated notation to represent different gauge fields as follows:

$$V_\mu = \begin{Bmatrix} A_\mu \\ Z_\mu \\ X_\mu \end{Bmatrix},\ e_V = \begin{Bmatrix} e,\ V = A \\ e_Z = \frac{e}{s_W c_W},\ V = Z \\ g_X,\ V = X \end{Bmatrix},\ \Gamma_f^V = \begin{Bmatrix} Q_f,\ V = A \\ g_{fL}P_L + g_{fR}P_R,\ V = Z \\ \frac{1}{g_X}\left(k_{fL}P_L + k_{fR}P_R\right),\ V = X \end{Bmatrix} \tag{3.7}$$

(The factor of $\frac{1}{g_X}$ in the definition of Γ_f^X is included to ensure that the k_{fi}'s have a well-defined limit as $g_X \to 0$.)

A rough sketch of the calculation is as follows (the explicit calculation can be found in Appendix A): we begin by working out the matrix elements $\left\langle \overline{f}\left(\mathbf{k}',\xi\right) f\left(\mathbf{p}',\zeta\right)\right| S \left|e^+\left(\mathbf{k},\tau\right) e^-\left(\mathbf{p},\sigma\right)\right\rangle$ (assuming single-particle eigenstates of the free Hamiltonian in the initial and final states) up to leading order in perturbation theory. S is known as the scattering matrix (in the interaction picture):

$$S \equiv T\left[\exp\left(i\int_{-\infty}^{\infty} d^4x \mathcal{L}_{int}(x)\right)\right]$$

Here, T denotes the time-ordering of operators. For convenience, we define $\mathcal{M}$ such that

$$\langle \overline{f} f | S | e^+ e^- \rangle = 1 - i\mathcal{M}(2\pi)^4 \delta^{(4)}(p + k - k' - p')$$

Here, leading order in perturbation theory consists of contributions from those graphs shown in Figure 3.4. The Feynman rules for tree-level graphs are also given in Appendix A.

Using the matrix elements, we can calculate the corresponding cross section using the following relation:

$$d\sigma\left(e^+e^- \to f\overline{f}\right) = -\frac{1}{16\pi s}\left(\frac{1}{4}\sum|\mathcal{M}|^2\right)\delta(s+t+u)\,dudt \tag{3.8}$$

where s, t, and u are the Mandelstam variables, as defined in Appendix A (Equation 3.34). Here,

$$\frac{1}{4}\sum|\mathcal{M}|^2 = N_c\left[\left(|A_{LL}(s)|^2 + |A_{RR}(s)|^2\right)u^2 + \left(|A_{LR}(s)|^2 + |A_{RL}(s)|^2\right)t^2\right] \tag{3.9}$$

where the $\frac{1}{4}\sum$ denotes that we have averaged over all spins in the initial state and summed over all spins in the final state, and where

$$A_{ij}(s) \equiv e^2\frac{Q_eQ_f}{s} + e_z^2\frac{g_{ei}g_{fj}}{s - M_Z^2 + i\Gamma_Z M_Z} + \frac{k_{ei}k_{fj}}{s - M_X^2 + i\Gamma_X M_X}. \tag{3.10}$$

(Recall that factors of g_X are absorbed into the k_{fi}'s so that they have a smooth limit as $g_X \to 0$). In this expression, Γ_Z and Γ_X are the full decay widths for the Z and X boson, respectively:

$$\Gamma_Z = \frac{e_Z^2 M_Z}{24\pi}\sum_{f\neq t}\left[(g_{Lf})^2 + (g_{Rf})^2\right]N_c \tag{3.11}$$

$$\Gamma_X = \frac{M_X}{24\pi}\sum_{f\neq t}\left[(k_{Lf})^2 + (k_{Rf})^2\right]N_c \tag{3.12}$$

(A derivation of the first of these expressions is given in Appendix C.)

It is also important to note that, in deriving the above differential cross section, the assumption was made that the fermions in the initial and final states were much lighter than either the Z or the X boson (i.e. $m_f^2/M_V^2 \ll 1$ for all $V \in \{Z, X\}$, $f \in \{e, u, d, c, s, b\}$).

To find the total cross section, we simply integrate Equation 3.8 over u and t:

$$\sigma\left(e^+e^- \to f\overline{f}\right) = \frac{N_c s}{48\pi}\left(|A_{LL}|^2 + |A_{RR}|^2 + |A_{LR}|^2 + |A_{RL}|^2\right)$$

(Note: $u, t \in [-s, 0]$ in the limit of massless fermions).

3.3.1 The Hadronic Cross Section at the Z Pole

A quantity that is measured accurately at LEP and SLC [19] is the cross section into hadrons,

$$\sigma_{had}(s) \equiv \sum_{f=u,d,c,s,b} \sigma\left(e^+e^- \to f\overline{f}\right)(s),$$

evaluated at the Z pole (i.e. when the centre of mass energy, $E_{cm}\ (= \sqrt{s}) = M_z$). Note that the sum over fermions does not include the top quark; this is because its high mass excludes it as a possible final state fermion-antifermion pair due to energy conservation. Its experimental value is measured to be 41.541 ± 0.037 nb [23]. In order to obtain a bound on the X boson parameters from this quantity, we must not only include the effect of an extra mediating field, but we must also use the adjusted fermion couplings, as demonstrated in the previous chapter (see Equations 2.59 and 2.60). Specifically, we let $g_{fi} = g_{fi} + \Delta g_{fi}$ and $k_{fi} = g_X (B-L)_f + \Delta k_{fi}$, where

$$\begin{aligned}
\Delta g_{fi} &= -\eta \frac{s_W}{s_X^2}\frac{g_X}{e_Z}(B-L)_f - \eta^2 \frac{s_W^2}{s_X^2}\left(\frac{c_X^2}{2s_X^2} g_{fi} + Q_f \frac{c_W^4}{c_W^2 - s_W^2}\right) \\
\Delta k_{fi} &= -\eta\left(1 - \frac{1}{s_X^2}\right) s_W e_Z g_{fi} + \eta c_W e Q_f + \frac{1}{2}\left(1 - \frac{s_W^2}{s_X^4}\right)\eta^2 g_X (B-L)_f .
\end{aligned}$$

All in all, this results in a very non-linear form for $\Delta\sigma_{had}(g_X, M_X, \eta) = \sigma_{had}(g_X, M_X, \eta) - \sigma_{had}(0, M_X, 0)$. (This notation assumes that the cross section is being evaluated at $s = M_Z^2$.) Nevertheless, we can program this convoluted expression into a computer algebra system (Maple was used here) and expand it, for the sake of consistency, up to quadratic order in η. From here, we can produce a plot of those values of g_X and M_X, for a particular value of η, that satisfy the constraint

$$|\Delta\sigma_{had}(g_X, M_X, \eta)| = 0.037 \text{ nb}$$

where $\Delta\sigma_{had}\left(s = M_Z^2\right)|_{exp.} = 0.037$ nb (1σ uncertainty) is an estimate of the maximum amount by which the existence of the X boson is allowed to change the value of $\sigma_{had}\left(s = M_Z^2\right)$ without producing a disagreement with experiment. A plot of such values is shown in Figure 3.5.

Here, we have fixed the range of M_X values to be from 10 GeV to 1 TeV. The lower limit is a result of the quantity being considered; since the hadronic cross section includes contributions from the bottom quark (with mass $m_b \simeq 4$ GeV [23]), and since we have made the assumption that $m_b^2/M_X^2 \ll 1$, we should only trust the results of this analysis down to about $M_X \simeq 10$ GeV. The upper limit is fixed as a result of the constraint losing its practical value; for all values of η being considered here, the bound in the region $M_X > 1$ TeV eliminates only values of α_X for which the perturbative calculation considered here would not be trusted anyways.

There are certain features of these graphs that can be confirmed from the general form of the above expressions:

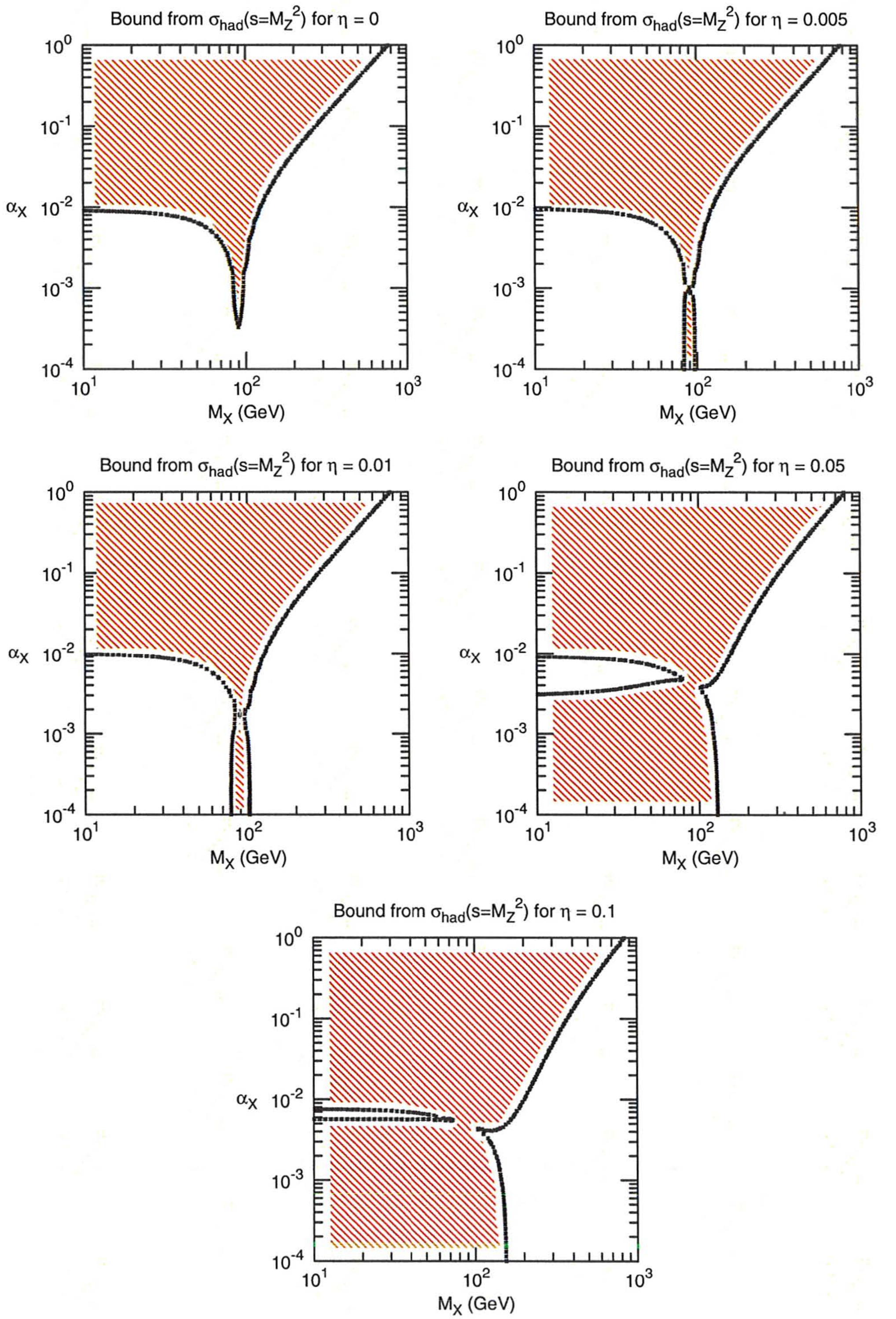

Figure 3.5: Plot of the constraint from $\sigma_{had}\left(s = M_z^2\right)$. Here, we plot the bound on α_X as a function of M_X for various values of η. Excluded regions have been shaded out.

- in the region where $M_X \ll M_Z$, we would expect the mass dependence to drop out, since M_X always appears in the above expression for $\Delta\sigma_{had}(g_X, M_X, \eta)$ in the form of $s_X^2 = 1 - M_X^2/M_Z^2$, which approaches 1 in this limit;

- in the region where $M_X \gg M_Z$, we can see (easily, in the case where $\eta = 0$) that leading order corrections to σ_{had} should be proportional to g_X^2/M_X^2, due to the form of the X term in A_{ij}. This agrees with what is found in this region. Note that this is different from the case of Z decay, where it was the ratio g_X/M_X^2 that was being constrained;

- since some terms in Δg_{fi} and Δk_{fi} do not have any dependence on g_X we expect that, for sufficiently large values of η, the region of parameter space for which $g_X \to 0$ (i.e. $\log \alpha_X \to -\infty$) is excluded. Furthermore, it is expected that this will occur more readily in the mass region $M_X \ll M_Z$, since the $1/s_X^2$ terms in Δg_{fi} and Δk_{fi} are $O(1)$, rather than some small value $\sim -M_Z^2/M_X^2$ in the region where $M_X \gg M_Z$.

This last point can be examined more directly by looking at the bound on η and α_X in the region where $M_X \ll M_Z$. In other words, we can implicitly plot the bound on α_X obtained from $|\Delta\sigma_{had}(g_X, M_X = 0, \eta)| \leq 0.037$ nb as a function of η. This is shown in Figure 3.6.

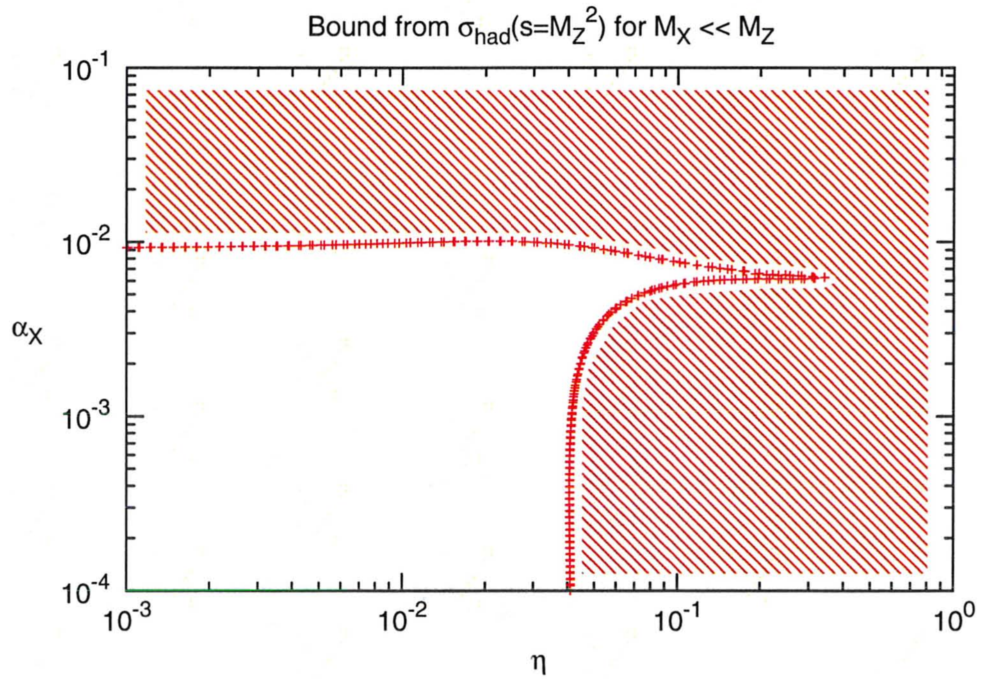

Figure 3.6: Plot of the constraint from $\sigma_{had}(s = M_Z^2)$ in the region where $M_X \ll M_Z$. The excluded region is shaded.

From this plot, we can see that for η greater than about 0.04, it is no longer possible to avoid a bound with an arbitrarily small gauge coupling. Furthermore, we see that all of the η, g_X parameter space is excluded for η greater than approximately 0.4.

It is also useful to note that, since $\left(\Delta g_{L(R)}\right)_i \sim Ag_X\eta + B\eta^2$ (A, B can take on either positive or negative values), we would expect possible cancellation between the effects of these two terms when $g_X\eta \sim \eta^2$ or, alternatively, when $\alpha_X \sim \eta^2$. This relation is, in fact, satisfied within the narrow region of allowed values found above $\eta = 0.04$.

So far, the constraints from σ_{had} and $\Gamma_{l^+l^-}$ have been considered at $E_{cm} = M_Z$. In the next chapter, we consider some constraints that arise at lower centre of mass energies.

3.A Appendix: Calculating the Cross Section $\sigma\left(e^+e^- \rightarrow f\overline{f}\right)$

3.A.1 Explicit Calculation of Scattering Matrix Elements

We are interested in calculating the scattering matrix elements in the limit where, for very early and very late times, the particles are non-interacting and so are well represented by momentum eigenstates (i.e. the states that diagonalize the Hamiltonian associated with $\mathcal{L}_o$ in Equation 3.4). In this case, these are given by

$$\left\langle \overline{f}\left(\mathbf{k}',\xi\right), f\left(\mathbf{p}',\zeta\right)\right| S \left|e^+\left(\mathbf{k},\tau\right), e^-\left(\mathbf{p},\sigma\right)\right\rangle \tag{3.13}$$

where

$$\begin{aligned} S &= T\left[\exp\left(i\int_{-\infty}^{\infty} d^4x\mathcal{L}_{int}\left(x\right)\right)\right] \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\infty} d^4x_1...d^4x_nT\left[\mathcal{L}_{int}\left(x_1\right)...\mathcal{L}_{int}\left(x_n\right)\right] \\ &= 1 + i\int_{-\infty}^{\infty} d^4xT\left[\mathcal{L}_{int}\left(x\right)\right] - \frac{1}{2}\int_{-\infty}^{\infty} d^4xd^4yT\left[\mathcal{L}_{int}\left(x\right)\mathcal{L}_{int}\left(y\right)\right] + ... \end{aligned}$$

is the time evolution operator in the interaction picture, also known as the scattering matrix. Since an overall momentum-conserving delta function is expected to factorize from any result (a particular example of which will be shown below), it is conventional to define matrix elements $\mathcal{M}$ such that

$$\left\langle \overline{f}f\right| S \left|e^+e^-\right\rangle = 1 - i\mathcal{M}\left(2\pi\right)^4\delta^{(4)}\left(p + k - k' - p'\right). \tag{3.14}$$

It is, in fact, these matrix elements that will be of most use to calculations of decay rates and cross sections.

Keeping in mind that the fields can be written in the form

$$V^{\mu}(x) = \sum_{\lambda=-1}^{1} \int \frac{d^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} \left[\varepsilon^{\mu}(\mathbf{p},\lambda)\, a_{\mathbf{p},\lambda} e^{ip\cdot x} + \varepsilon^{*\mu}(\mathbf{p},\lambda)\, a^{*}_{\mathbf{p},\lambda} e^{-ip\cdot x}\right] \tag{3.15}$$

$$f(x) = \sum_{\sigma=\pm\frac{1}{2}} \int \frac{d^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} \left[u(\mathbf{p},\sigma)\, b_{\mathbf{p},\sigma} e^{ip\cdot x} + v(\mathbf{p},\sigma)\, \overline{b}^{*}_{\mathbf{p},\sigma} e^{-ip\cdot x}\right] \tag{3.16}$$

we can see that the tree-level contribution to the matrix elements arises from the quadratic term in S:

$$\begin{aligned}\left\langle \overline{f}, f\right| S \left|e^{+}, e^{-}\right\rangle &= -\frac{1}{2}\int_{-\infty}^{\infty} d^4x d^4y \left\langle \overline{f}, f\right| T\left[\mathcal{L}_{int}(x)\,\mathcal{L}_{int}(y)\right] \left|e^{+}, e^{-}\right\rangle \\ &= \sum_{V=A,Z,X} e_V^2 \int_{-\infty}^{\infty} d^4x d^4y \left\langle \overline{f}, f\right| T\left[V_{\mu}\left(\overline{f}\gamma^{\mu}\Gamma_f^V f\right)(x)\right. \\ &\qquad \left. \times V_{\nu}\left(\overline{e}\gamma^{\nu}\Gamma_e^V e\right)(y)\right] \left|e^{+}, e^{-}\right\rangle\end{aligned} \tag{3.17}$$

To simplify the time-ordered product, we use Wick's theorem:

$$\begin{aligned}&\left\langle \overline{f} f\right| T\left[V_{\mu}\left(\overline{f}\gamma^{\mu}\Gamma_f^V f\right) V_{v}\left(\overline{e}\gamma^{v}\Gamma_e^V e\right)\right] \left|e^{+}e^{-}\right\rangle \\ &= G_{\mu\nu}^{V}(x,y) \left\langle \overline{f}, f\right| : \left(\overline{f}\gamma^{\mu}\Gamma_f^V f\right)\left(\overline{e}\gamma^{v}\Gamma_e^V e\right) : \left|e^{+}, e^{-}\right\rangle\end{aligned} \tag{3.18}$$

where : : denotes a normal-ordered product and where $G_{\mu\nu}^{V}(x,y)$ is the propagator for the gauge field V_{μ}:

$$G_{\mu\nu}^{V}(x,y) = \langle 0| T\left[V_{\mu}(x)\, V_{v}(y)\right] |0\rangle = -i \int \frac{d^4p}{(2\pi)^4} \frac{\Pi_{\mu\nu}^{V}(\mathbf{p})}{p^2 + M_V^2 - i\varepsilon} e^{ip\cdot(x-y)} \tag{3.19a}$$

$$\Pi_{\mu\nu}^{V}(\mathbf{p}) = \left\{ \begin{array}{c} \eta_{\mu\nu},\ V = A \\ \eta_{\mu\nu} + \frac{p_{\mu}p_{\nu}}{M_Z^2},\ V = Z \\ \eta_{\mu\nu} + \frac{p_{\mu}p_{\nu}}{M_X^2},\ V = X \end{array} \right\} \tag{3.19b}$$

Note: the $\Pi_{\mu\nu}^{V}(\mathbf{p})$ function is to be evaluated on-shell, e.g. $\Pi_{00}^{Z}(\mathbf{p}) = -1 + \frac{E_{\mathbf{p}}^2}{M_Z^2}$. (In the photon propagator, we are using the Feynman gauge so that a term proportional to $\frac{p_{\mu}p_{\nu}}{p^2}$ is not present.) A derivation of this propagator is presented in Appendix B.

Note that the remaining matrix element in Equation 3.18 can also be simplified by noting that only particular operators in the field operator sums will give a non-vanishing matrix element. For example, applying a particular b_{p_1,σ_1} gives

$$\begin{aligned} b_{p_1,\sigma_1}\left|e^+(\mathbf{k},\tau),e^-(\mathbf{p},\sigma)\right\rangle &= b_{p_1,\sigma_1}b^*_{p,\sigma}\left|e^+(\mathbf{k},\tau),0\right\rangle \\ &= \left\{b_{p_1,\sigma_1},b^*_{p,\sigma}\right\}\left|e^+(\mathbf{k},\tau),0\right\rangle \\ &= 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}_1)\delta_{\sigma,\sigma_1}\left|e^+(\mathbf{k},\tau),0\right\rangle \end{aligned}$$

where $\{x,y\}$ denotes anticommutation and, where our normalization of one-particle states is chosen such that $\langle \mathbf{p},\sigma|\mathbf{q},\xi\rangle = \langle 0|\left\{b_{p,\sigma},b^*_{q,\xi}\right\}|0\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{\sigma,\xi}$. Hence,

$$\sum_{\sigma=\pm\frac{1}{2}}\int\frac{d^3\mathbf{p}_1}{2E_{\mathbf{p}_1}(2\pi)^3}u(\mathbf{p}_1,\sigma_1)e^{ip_1\cdot y}b_{p_1,\sigma_1}\left|e^+(\mathbf{k},\tau),e^-(\mathbf{p},\sigma)\right\rangle = u(\mathbf{p},\sigma)e^{ip\cdot y}\left|e^+(\mathbf{k},\tau),0\right\rangle.$$

Repeating this procedure as needed gives the following expression for $\langle\overline{f},f|{:}\left(\overline{f}\gamma^\mu\Gamma^V_f f\right)(\overline{e}\gamma^\upsilon\Gamma^V_e e){:}|e^+,e^-\rangle$:

$$\begin{aligned} &\left\langle\overline{f}(\mathbf{k}',\xi),f(\mathbf{p}',\zeta)\right|:\left(\overline{f}\gamma^\mu\Gamma^V_f f\right)(x)\left(\overline{e}\gamma^\upsilon\Gamma^V_e e\right)(y):\left|e^+(\mathbf{k},\tau),e^-(\mathbf{p},\sigma)\right\rangle \\ &\qquad = e^{-i(k'+p')\cdot x}\overline{u}(\mathbf{p}',\zeta)\gamma^\mu\Gamma^V_f v(\mathbf{k}',\xi)e^{i(p+k)\cdot y}\overline{v}(\mathbf{k},\tau)\gamma^\upsilon\Gamma^V_e u(\mathbf{p},\sigma). \end{aligned}$$

We now have an expression for the time-ordered matrix element found in Equation 3.17:

$$\begin{aligned} &\left\langle\overline{f},f\right|T\left[V_\mu\left(\overline{f}\gamma^\mu\Gamma^V_f f\right)V_\upsilon\left(\overline{e}\gamma^\upsilon\Gamma^V_e e\right)\right]\left|e^+,e^-\right\rangle \\ &\qquad = e^{-i(k'+p')\cdot x}e^{i(p+k)\cdot y}\overline{u}(\mathbf{p}',\zeta)\gamma^\mu\Gamma^V_f v(\mathbf{k}',\xi)G^V_{\mu\nu}(x,y)\overline{v}(\mathbf{k},\tau)\gamma^\upsilon\Gamma^V_e u(\mathbf{p},\sigma). \end{aligned}$$

From here, we can complete the calculation of the S-matrix element:

$$\begin{aligned} \left\langle\overline{f}(\mathbf{k}')f(\mathbf{p}')\right|S\left|e^+(\mathbf{k})e^-(\mathbf{p})\right\rangle = &\sum_{V=A,Z,X} e^2_V\overline{u}(\mathbf{p}',\zeta)\gamma^\mu\Gamma^V_f v(\mathbf{k}',\xi)\overline{v}(\mathbf{k},\tau)\gamma^\upsilon\Gamma^V_e u(\mathbf{p},\sigma) \\ &\times\int_{-\infty}^{\infty}d^4x\,d^4y\,e^{-i(k'+p')\cdot x}e^{i(p+k)\cdot y}G^V_{\mu\nu}(x,y) \end{aligned}$$

where

$$\begin{aligned}\int_{-\infty}^{\infty} d^4x d^4y e^{-i(k'+p')\cdot x} e^{i(p+k)\cdot y} G_{\mu\nu}^{V}(x,y) &= -i\int \frac{d^4q}{(2\pi)^4}\frac{\Pi_{\mu\nu}^{V}(\mathbf{q})}{q^2+M_V^2-i\varepsilon}\int_{-\infty}^{\infty} d^4x e^{i(q-k'-p')\cdot x} \\ &\quad \times \int_{-\infty}^{\infty} d^4y e^{i(p+k-q)\cdot y} \\ &= -i\int \frac{d^4q}{(2\pi)^4}\frac{\Pi_{\mu\nu}^{V}(\mathbf{q})}{q^2+M_V^2-i\varepsilon}\int_{-\infty}^{\infty} d^4x e^{i(q-k'-p')\cdot x} \\ &\quad \times (2\pi)^4\delta^{(4)}(q-p-k) \\ &= -i\int_{-\infty}^{\infty} d^4x \frac{\Pi_{\mu\nu}^{V}(\mathbf{p}+\mathbf{k})}{(p+k)^2+M_V^2-i\varepsilon} e^{i(p+k-k'-p')\cdot x}\end{aligned}$$

or

$$\begin{aligned}\int_{-\infty}^{\infty} d^4x d^4y e^{-i(k'+p')\cdot x} e^{i(p+k)\cdot y} G_{\mu\nu}^{V}(x,y) &= -i\frac{\Pi_{\mu\nu}^{V}(\mathbf{p}+\mathbf{k})}{(p+k)^2+M_V^2-i\varepsilon} \\ &\quad \times (2\pi)^4\delta^{(4)}(p+k-k'-p')\end{aligned} \tag{3.20}$$

This factor of $(2\pi)^4\delta^{(4)}(p+k-k'-p')$ is precisely the delta function which is factored out in the definition of $\mathcal{M}$ as shown in Equation 3.14. Finally, we find the following expression for $\mathcal{M}$:

$$\mathcal{M} = \sum_{V=A,Z,X} e_V^2 \overline{v}(\mathbf{k},\tau)\gamma^{v}\Gamma_e^V u(\mathbf{p},\sigma)\frac{\Pi_{\mu\nu}^{V}(\mathbf{p}+\mathbf{k})}{(p+k)^2+M_V^2-i\varepsilon}\overline{u}(\mathbf{p}',\zeta)\gamma^{\mu}\Gamma_f^V v(\mathbf{k}',\xi) \tag{3.21}$$

3.A.2 Feynman Rules for Tree-level Graphs

In retrospect (and after doing several other examples), a connection can be made between the diagrams shown in Figure 3.4 and the expression for $\langle \overline{f}f|\,\mathcal{M}\,|e^+e^-\rangle$. This correlation is usually referred to as "Feynman Rules". In this case, the applicable rules are that:

- the incoming fermion line is represented by $u(\mathbf{p},\sigma)$;
- the incoming antifermion line is represented by $\overline{v}(\mathbf{k},\tau)$;
- the outgoing fermion line is represented by $\overline{u}(\mathbf{p}',\zeta)$;
- outgoing antifermion line is represented by $v(\mathbf{k}',\xi)$;
- vertices are represented by $-e_V\gamma^{\mu}\Gamma_f^V$;
- intermediate bosons are represented by $\frac{\Pi_{\mu\nu}^{V}(\mathbf{p})}{p^2+M_V^2-i\varepsilon}$, where p is the 4-momentum in the corresponding line as calculated using 4-momentum conservation at the vertices.

Applying these rules, along with the understanding that matrix multiplication occurs in reverse order along fermion lines, will give the three expressions found in $\langle \overline{f}f|\,\mathcal{M}\,|e^+e^-\rangle$, one for each mediating boson. In general, there are also rules for symmetry factors and for relative signs amongst different graphs, but these are not needed here. In other contexts herein, these rules will be applied rather than pursuing a full calculation.

3.A.3 The Cross Section: A Meaningful Quantity in the Infinite-Volume Limit

Now that the scattering matrix elements have been calculated, it is possible to work out expressions involving the probability for such a transition to occur. For example, the differential rate for an initial state i to transition into some small set of final states Δf is

$$d\Gamma\left(i \rightarrow \Delta f\right) = \frac{dP\left(i \rightarrow \Delta f\right)}{T} \tag{3.22}$$

where $dP\left(i \rightarrow \Delta f\right)$ is the differential probability for that transition to occur and T is the time required for this transition to occur. (T is assumed to be large compared to other time scales in the problem.) Since we know the amplitude $\left\langle f\right| S \left|i\right\rangle$, $dP\left(i \rightarrow \Delta f\right)$ will be simply

$$dP\left(i \rightarrow \Delta f\right) = \left|\left\langle f\right| S \left|i\right\rangle\right|^2 \Delta f \tag{3.23}$$

The main problem now is that $\left\langle f\right| S \left|i\right\rangle$ is proportional to an overall, momentum-conserving, delta function and so, $\left|\left\langle f\right| S \left|i\right\rangle\right|^2$ is ill-defined. However, if this problem is recast within a box of finite volume $V = L^3$, then we find that these expressions make much more sense. In this case, momenta would be discretized:

$$\mathbf{p} = \left(\frac{2\pi}{L}n_x, \frac{2\pi}{L}n_y, \frac{2\pi}{L}n_z\right) \tag{3.24}$$

where n_i are integers. States within a finite volume would have the normalization $\left\langle \mathbf{n}_p,\sigma | \mathbf{n}_q, \xi\right\rangle = \delta_{\mathbf{n}_p,\mathbf{n}_q}\delta_{\sigma,\xi}$, whereas before we were using the normalization $\left\langle \mathbf{p},\sigma | \mathbf{q}, \xi\right\rangle = 2E_{\mathbf{p}}\left(2\pi\right)^3 \delta^{(3)}\left(\mathbf{p}-\mathbf{q}\right)\delta_{\sigma,\xi}$. Since, using Equation 3.24, we can write $\delta_V^{(3)}\left(\mathbf{p}-\mathbf{q}\right) = \frac{V}{(2\pi)^3}\delta_{\mathbf{n}_p,\mathbf{n}_q}$, it is clear that there is a difference in normalization when going from $\left|\mathbf{n}_p,\sigma\right\rangle$ to $\left|\mathbf{p},\sigma\right\rangle$:

$$\left|\mathbf{n}_p,\sigma\right\rangle \leftrightarrow \frac{1}{\sqrt{2E_{\mathbf{p}}V}}\left|\mathbf{p},\sigma\right\rangle \tag{3.25}$$

This introduces the same difference in the normalization of creation and annihilation operators. Therefore, in order to translate our previous results into the finite volume case, whenever particles were created or annihilated we should add an extra factor of $\left(2E_{\mathbf{p}}V\right)^{-1/2}$ onto $\mathcal{M}$.

At finite volume, we find that delta functions can be written in terms of the corresponding Fourier transform:

$$\delta_{VT}^{(4)}\left(p_i - p_f\right) = \frac{1}{\left(2\pi\right)^4}\int_{-T/2}^{T/2} dt \int_V d^3x e^{i(p_i-p_f)\cdot x} \tag{3.26}$$

Using this expression, we can work out the meaning of "the square of a delta function"

$$\left|\delta_{VT}^{(4)}(p_i - p_f)\right|^2 = \delta_{VT}^{(4)}(0)\,\delta_{VT}^{(4)}(p_i - p_f) = \frac{VT}{(2\pi)^4}\delta_{VT}^{(4)}(p_i - p_f)$$

All in all, this means that the square of the scattering matrix element would be, in the finite volume limit,

$$|\langle f|\,S_{VT}\,|i\rangle|^2 = VT\,(2\pi)^4\,\delta_{VT}^{(4)}(p_i - p_f)\,|\mathcal{M}|^2\left(\prod_{k=i,f}\frac{1}{2E_{\mathbf{p}_k}V}\right) \tag{3.27}$$

and so

$$\begin{aligned} d\Gamma\,(i \to \Delta f) &= \frac{|\langle f|\,S_{VT}\,|i\rangle|^2}{T}\left(\prod_f \Delta\mathbf{n}_f\right) \\ &= V\,(2\pi)^4\,\delta_{VT}^{(4)}(p_i - p_f)\,|\mathcal{M}|^2\left(\prod_{k=i,f}\frac{1}{2E_{\mathbf{p}_k}V}\right)\left(\prod_f\frac{V}{(2\pi)^3}d^3\mathbf{p}_f\right) \\ d\Gamma\,(i \to \Delta f) &= V\,(2\pi)^4\,\delta_{VT}^{(4)}(p_i - p_f)\,|\mathcal{M}|^2\left(\prod_i\frac{1}{2E_{\mathbf{p}_i}V}\right)\left(\prod_f\frac{d^3\mathbf{p}_f}{2E_{\mathbf{p}_f}(2\pi)^3}\right) \end{aligned} \tag{3.28}$$

In the end, it is those quantities that are independent of T and V that are physically relevant. In the case of two particles in the initial state, we see that $d\Gamma\,(i \to \Delta f) \propto V^{-1}$ so this does not qualify as a good quantity. However, if we instead calculate the differential cross section, defined to be

$$d\sigma\,(i \to \Delta f) = \frac{1}{F}d\Gamma\,(i \to \Delta f) \tag{3.29}$$

where F is the incident flux, then we find that this quantity does have a well-behaved limit, since $F \propto V^{-1}$ as well. More specifically, in the rest frame of one the initial-state particles, the flux is the number density of incoming particles times their speed:

$$F\,|_{rest\ frame\ of\ 1} = n_2\,|\mathbf{v}_2|\left(= \frac{N}{AT}\right). \tag{3.30}$$

(Alternatively, the flux can be defined as the number of incident particles arriving per unit cross-sectional area, per unit time.) Note that A is a Lorentz-invariant quantity since it is always measured perpendicular to the particles' motion, and that N and $dP = d\Gamma \times T$ are also Lorentz-invariant. Therefore, we conclude that $d\sigma = AdP/N$ is a Lorentz-invariant quantity. Consequently, $d\Gamma$ and F must have the same transformation properties. This is ensured if F is proportional to a Lorentz-covariant factor of $(E_1E_2V)^{-1}$, as in $d\Gamma$. After some work, we find that the expression that reproduces Equation 3.30 in the rest frame is

$$F = \frac{(-p_1 \cdot p_2)}{E_1 E_2 V} v_{rel} \tag{3.31}$$

where $v_{rel} \equiv \sqrt{1 - \frac{m_1^2 m_2^2}{(p_1 \cdot p_2)^2}}$. To check this, note that if particle 1 is at rest then $-p_1 \cdot p_2 = E_1 E_2 = m_1 E_2$ and so F becomes

$$F = \frac{1}{V}\sqrt{1 - \frac{m_2^2}{E_2^2}} \tag{3.32}$$

which is, indeed, equal to $n_2 |\mathbf{v}_2|$ since $E_2 = \gamma m_2 = \left(1 - |\mathbf{v}_2|^2\right)^{-1/2} m_2$ and since, for a single particle in a box of volume V, $n = \frac{1}{V}$. Finally, we find that

$$d\sigma\left(i \to \Delta f\right) = \frac{|\mathcal{M}|^2}{4\left(-p_1 \cdot p_2\right) v_{rel}} \left(2\pi\right)^4 \delta^{(4)}\left(p_1 + p_2 - p_3 - p_4\right) \left(\prod_f \frac{d^3\mathbf{p}_f}{2E_{\mathbf{p}_f}\left(2\pi\right)^3}\right). \tag{3.33}$$

From here, it is useful to simplify this expression by integrating over the four delta functions present. Since we are mostly interested in situations in which $m_f \ll M_V$, it is convenient to evaluate the above expression in the ultra-relativistic limit. We also introduce the Lorentz-invariant Mandelstam variables which are defined, in terms of our original momentum variables, as

$$\begin{aligned}
s &\equiv -\left(p + k\right)^2 \simeq -2p \cdot k \\
&= -\left(p' + k'\right)^2 \simeq -2p' \cdot k' \\
t &\equiv -\left(p - p'\right)^2 \simeq 2p \cdot p' \\
&= -\left(k - k'\right)^2 \simeq 2k \cdot k' \\
u &\equiv -\left(p - k'\right)^2 \simeq 2p \cdot k' \\
&= -\left(p' - k\right)^2 \simeq 2p' \cdot k
\end{aligned} \tag{3.34}$$

These will be of use in what follows. Note that, in the ultra-relativistic limit,

$$s + t + u = 2p \cdot \left(k' + p' - k\right) = 2p \cdot p \simeq 0$$

becomes the requirement for momentum conservation.

In terms of these new variables, $d\sigma$ becomes (more details in [17])

$$
\begin{aligned}
d\sigma\left(e^+e^- \to f\bar{f}\right) &= \frac{|\mathcal{M}|^2}{2s}(2\pi)^4\delta^{(4)}\left(p+k-p'-k'\right)\frac{d^3\mathbf{p}'}{2E_{\mathbf{p}'}(2\pi)^3}\frac{d^3\mathbf{k}'}{2E_{\mathbf{k}'}(2\pi)^3} \\
&= \frac{|\mathcal{M}|^2}{2s}(2\pi)^4\delta^{(4)}\left(p+k-p'-k'\right)\frac{d^3\mathbf{p}'}{2E_{\mathbf{p}'}(2\pi)^3}\left(2\pi\delta\left(k'^2\right)\theta\left(k'^0\right)\frac{d^4k'}{(2\pi)^4}\right) \\
&\Longrightarrow \frac{\pi|\mathcal{M}|^2}{s}\delta\left(\left(p+k-p'\right)^2\right)\frac{d^3\mathbf{p}'}{2E_{\mathbf{p}'}(2\pi)^3}
\end{aligned}
$$

or, in terms of the Mandelstam variables,

$$
d\sigma\left(e^+e^- \to f\bar{f}\right) = -\frac{|\mathcal{M}|^2}{16\pi s}\delta\left(s+t+u\right)dudt \tag{3.35}
$$

Hence, once the matrix elements $\mathcal{M}$ are known, it is only a matter of integrating in order to find the cross section. In the next section, we evaluate $|\mathcal{M}|^2$ explicitly.

3.A.4 Evaluating $|\mathcal{M}|^2$: Matrix Traces

Since we will be comparing with an experiment that (ideally) has no spin polarization preference, we are interested in contributions from all possible combinations of spins. Therefore, we should sum $|\mathcal{M}|^2$ over all final spins and average over all initial spins. Specifically, the shorthand $\frac{1}{4}\sum|\mathcal{M}|^2$ means

$$
\begin{aligned}
\frac{1}{4}\sum|\mathcal{M}|^2 &= \left(\frac{1}{2}\sum_{\sigma=\pm1/2}\right)\left(\frac{1}{2}\sum_{\tau=\pm1/2}\right)\sum_{\zeta=\pm1/2}\sum_{\xi=\pm1/2}\sum_{col.}\mathcal{M}\mathcal{M}^* \qquad (3.36) \\
&= \frac{N_c}{4}\sum_{spins}\sum_{V,V'}e_V^2e_{V'}^2\left[\overline{v}\left(\mathbf{k},\tau\right)\gamma^\mu\Gamma_e^V u\left(\mathbf{p},\sigma\right)\right]\left[\overline{v}\left(\mathbf{k},\tau\right)\gamma^\upsilon\Gamma_e^{V'}u\left(\mathbf{p},\sigma\right)\right]^* \\
&\quad\times H_{\mu\alpha}^V H_{\nu\beta}^{V'}\left[\overline{u}\left(\mathbf{p}',\zeta\right)\gamma^\alpha\Gamma_f^V v\left(\mathbf{k}',\xi\right)\right]\left[\overline{u}\left(\mathbf{p}',\zeta\right)\gamma^\beta\Gamma_f^{V'}v\left(\mathbf{k}',\xi\right)\right]^*
\end{aligned}
$$

where

$$
H_{\mu\nu}^V = \frac{\Pi_{\mu\nu}^V\left(\mathbf{p}+\mathbf{k}\right)}{(p+k)^2+M_V^2-i\varepsilon} \tag{3.37}
$$

$$
N_c = \begin{cases} 1, & f = \text{lepton} \\ 3, & f = \text{quark} \end{cases} \tag{3.38}
$$

Note that

$$\begin{aligned}\left[\overline{v}(\mathbf{k},\tau)\gamma^{\nu}\Gamma_{e}^{V'}u(\mathbf{p},\sigma)\right]^{*} &= u^{\dagger}(\mathbf{p},\sigma)\Gamma_{e}^{V'}(\gamma^{\nu})^{\dagger}\beta v(\mathbf{k},\tau)\\ &= -\overline{u}(\mathbf{p},\sigma)\beta\Gamma_{e}^{V'}\beta\gamma^{\nu}v(\mathbf{k},\tau)\\ &= -\overline{u}(\mathbf{p},\sigma)\gamma^{\nu}\Gamma_{e}^{V'}v(\mathbf{k},\tau)\end{aligned}$$

(using $\beta \equiv i\gamma^0$, $\beta^2 = 1$, $\gamma_\mu^\dagger\beta = -\beta\gamma_\mu$, and $\{\beta, \gamma^5\} = \{\gamma^\mu, \gamma^5\} = 0$). Similarly,

$$\left[\overline{u}(\mathbf{p}',\zeta)\gamma^{\beta}\Gamma_{f}^{V'}v(\mathbf{k}',\xi)\right]^{*} = -\overline{v}(\mathbf{k}',\xi)\gamma^{\beta}\Gamma_{f}^{V'}u(\mathbf{p}',\zeta)$$

as well. Therefore, we can write

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{N_c}{4}\sum_{V,V'}e_V^2 e_{V'}^2 P^{\mu\nu}H_{\mu\alpha}^{V}H_{\nu\beta}^{V'}K^{\alpha\beta} \tag{3.39}$$

where we have defined

$$\begin{aligned}P^{\mu\nu} &\equiv \sum_{\sigma,\tau}\left[\overline{v}(\mathbf{k},\tau)\gamma^{\mu}\Gamma_{e}^{V}u(\mathbf{p},\sigma)\right]\left[\overline{u}(\mathbf{p},\sigma)\gamma^{\nu}\Gamma_{e}^{V'}v(\mathbf{k},\tau)\right] &(3.40)\\ K^{\mu\nu} &\equiv \sum_{\zeta,\xi}\left[\overline{u}(\mathbf{p}',\zeta)\gamma^{\mu}\Gamma_{f}^{V}v(\mathbf{k}',\xi)\right]\left[\overline{v}(\mathbf{k}',\xi)\gamma^{\nu}\Gamma_{f}^{V'}u(\mathbf{p}',\zeta)\right]. &(3.41)\end{aligned}$$

In order to simplify these expressions further, we use the spin-sum identities

$$\sum_{\sigma}u(\mathbf{p},\sigma)\overline{u}(\mathbf{p},\sigma) = -i\not{p}+m_e,\ \sum_{\tau}v(\mathbf{k},\tau)\overline{v}(\mathbf{k},\tau) = -i\not{k}-m_e$$

to rewrite $P^{\mu\nu}$ as

$$\begin{aligned}P^{\mu\nu} &= \sum_{\sigma,\tau}\mathrm{tr}\left(\overline{v}(\mathbf{k},\tau)\gamma^{\mu}\Gamma_{e}^{V}u(\mathbf{p},\sigma)\overline{u}(\mathbf{p},\sigma)\gamma^{\nu}\Gamma_{e}^{V'}v(\mathbf{k},\tau)\right)\\ &= \mathrm{tr}\left[\gamma^{\mu}\Gamma_{e}^{V}\left(\sum_{\sigma}u(\mathbf{p},\sigma)\overline{u}(\mathbf{p},\sigma)\right)\gamma^{\nu}\Gamma_{e}^{V'}\left(\sum_{\tau}v(\mathbf{k},\tau)\overline{v}(\mathbf{k},\tau)\right)\right]\\ P^{\mu\nu} &= \mathrm{tr}\left[\gamma^{\mu}\Gamma_{e}^{V}\left(-i\not{p}+m_e\right)\gamma^{\nu}\Gamma_{e}^{V'}\left(-i\not{k}-m_e\right)\right] &(3.42)\end{aligned}$$

and, similarly,

$$K^{\mu\nu} = \mathrm{tr}\left[\gamma^{\mu}\Gamma_{f}^{V}\left(-i\not{k}'-m_f\right)\gamma^{\nu}\Gamma_{f}^{V'}\left(-i\not{p}'+m_f\right)\right]. \tag{3.43}$$

From here, the trace identities

$$\mathrm{tr}(\gamma^\mu)=0,\ \mathrm{tr}(\gamma^\mu\gamma^\nu)=4\eta^{\mu\nu},\ \mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu)=0,\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\lambda)=0,\ \mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda)=0,$$
$$\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)=4i\varepsilon^{\mu\nu\lambda\sigma},\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)=4\left(\eta^{\mu\nu}\eta^{\lambda\sigma}-\eta^{\mu\lambda}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\lambda}\right)$$

$\left(\varepsilon^{0123} \equiv +1\right)$ as well as the projector identities

$$P_L^2 = P_L,\ P_R^2 = P_R,\ P_L P_R = P_R P_L = 0,\ P_L \gamma^\mu = \gamma^\mu P_R$$

give

$$\begin{aligned} P^{\mu\nu} &= \mathrm{tr}\left[\gamma^\mu \Gamma_e^V \left(-i\not{p} + m_e\right) \gamma^\nu \Gamma_e^{V'} \left(-i\not{k} - m_e\right)\right] \\ &= -p_\lambda k_\sigma \mathrm{tr}\left(\gamma^\mu \Gamma_e^V \gamma^\lambda \gamma^\nu \Gamma_e^{V'} \gamma^\sigma\right) - m_e^2 \mathrm{tr}\left(\gamma^\mu \Gamma_e^V \gamma^\nu \Gamma_e^{V'}\right) \\ &= -p_\lambda k_\sigma \mathrm{tr}\left[\left(g_{eL}^V g_{eL}^{V'} P_L + g_{eR}^V g_{eR}^{V'} P_R\right) \gamma^\lambda \gamma^\nu \gamma^\sigma \gamma^\mu\right] - 2m_e^2 \left(g_{eL}^V g_{eR}^{V'} + g_{eR}^V g_{eL}^{V'}\right) \eta^{\mu\nu} \\ &= 2\left(g_{eL}^V g_{eL}^{V'} + g_{eR}^V g_{eR}^{V'}\right) \left(\eta^{\mu\nu} p \cdot k - p^\mu k^\nu - p^\nu k^\mu\right) \\ &\quad -2i\left(g_{eL}^V g_{eL}^{V'} - g_{eR}^V g_{eR}^{V'}\right) \varepsilon^{\mu\nu\lambda\sigma} p_\lambda k_\sigma - 2m_e^2 \left(g_{eL}^V g_{eR}^{V'} + g_{eR}^V g_{eL}^{V'}\right) \eta^{\mu\nu} \end{aligned} \tag{3.44}$$

and, similarly,

$$\begin{aligned} K^{\mu\nu} &= 2\left(g_{fL}^V g_{fL}^{V'} + g_{fR}^V g_{fR}^{V'}\right) \left(\eta^{\mu\nu} p' \cdot k' - p'^\mu k'^\nu - p'^\nu k'^\mu\right) \\ &\quad +2i\left(g_{fL}^V g_{fL}^{V'} - g_{fR}^V g_{fR}^{V'}\right) \varepsilon^{\mu\nu\lambda\sigma} p'_\lambda k'_\sigma - 2m_f^2 \left(g_{fL}^V g_{fR}^{V'} + g_{fR}^V g_{fL}^{V'}\right) \eta^{\mu\nu} \end{aligned} \tag{3.45}$$

For the situations in which we are interested, it is both useful and appropriate to assume that $m_e,\ m_f \ll M_Z,\ M_X$. In this case, we can neglect the last terms in the expressions for $P^{\mu\nu}$ and $K^{\mu\nu}$, and after some work, the expression for $\frac{1}{4}\sum |\mathcal{M}|^2$ reduces to

$$\frac{1}{4}\sum |\mathcal{M}|^2 = \frac{N_c}{4} \sum_{V,V'} e_V^2 e_{V'}^2 \frac{P^{\mu\nu} K_{\mu\nu}}{\left((p+k)^2 + M_V^2\right)\left((p+k)^2 + M_{V'}^2\right)}$$

From here, the $P^{\mu\nu} K_{\mu\nu}$ term can be computed by noting that: 1) the imaginary (i.e. cross) terms vanish because $\varepsilon^{\mu\nu\lambda\sigma} p_\lambda k_\sigma$ is anti-symmetric, whereas $\left(\eta^{\mu\nu} p \cdot k - p^\mu k^\nu - p^\nu k^\mu\right)$ is symmetric; 2) $\varepsilon^{\mu\nu\lambda\sigma} \varepsilon_{\mu\nu\alpha\beta} = 2\left(\delta_\beta^\lambda \delta_\alpha^\sigma - \delta_\alpha^\lambda \delta_\beta^\sigma\right)$. Hence, if we temporarily define $g_\pm^f = g_{fL}^V g_{fL}^{V'} \pm g_{fR}^V g_{fR}^{V'}$, we find that

$$\begin{aligned} P^{\mu\nu} K_{\mu\nu} &= 4\Big[g_+^e g_+^f \left(\eta^{\mu\nu} p \cdot k - p^\mu k^\nu - p^\nu k^\mu\right)\left(\eta_{\mu\nu} p' \cdot k' - p'_\mu k'_\nu - p'_\nu k'_\mu\right) \\ &\qquad + 2g_-^e g_-^f \left(\delta_\beta^\lambda \delta_\alpha^\sigma - \delta_\alpha^\lambda \delta_\beta^\sigma\right) p_\lambda k_\sigma p'^\alpha k'^\beta\Big] \\ &= 4\Big[2g_+^e g_+^f \left(\left(p \cdot p'\right)\left(k \cdot k'\right) + \left(p \cdot k'\right)\left(p' \cdot k\right)\right) \\ &\qquad + 2g_-^e g_-^f \left(\left(p \cdot k'\right)\left(p' \cdot k\right) - \left(p \cdot p'\right)\left(k \cdot k'\right)\right)\Big] \\ P^{\mu\nu} K_{\mu\nu} &= 8\left[\left(g_+^e g_+^f + g_-^e g_-^f\right)\left(p \cdot k'\right)\left(p' \cdot k\right) + \left(g_+^e g_+^f - g_-^e g_-^f\right)\left(p \cdot p'\right)\left(k \cdot k'\right)\right] \end{aligned} \tag{3.46}$$

where

$$\begin{aligned}\left(g_+^e g_+^f + g_-^e g_-^f\right) &= 2\left(g_{eL}^V g_{eL}^{V'} g_{fL}^V g_{fL}^{V'} + g_{eR}^V g_{eR}^{V'} g_{fR}^V g_{fR}^{V'}\right)\\ \left(g_+^e g_+^f - g_-^e g_-^f\right) &= 2\left(g_{eL}^V g_{eL}^{V'} g_{fR}^V g_{fR}^{V'} + g_{eR}^V g_{eR}^{V'} g_{fL}^V g_{fL}^{V'}\right).\end{aligned}$$

Finally, in terms of the Mandelstam variables, we find that

$$\frac{1}{4}\sum|\mathcal{M}|^2 = N_c \sum_{V,V'} e_V^2 e_{V'}^2 \frac{C(u,t)}{\left(s-M_V^2\right)\left(s-M_{V'}^2\right)}$$

where $C(u,t) \equiv \left(g_{eL}^V g_{fL}^V g_{eL}^{V'} g_{fL}^{V'} + g_{eR}^V g_{fR}^V g_{eR}^{V'} g_{fR}^{V'}\right) u^2 + \left(g_{eL}^V g_{fR}^V g_{eL}^{V'} g_{fR}^{V'} + g_{eR}^V g_{fL}^V g_{eR}^{V'} g_{fL}^{V'}\right) t^2$, or

$$\frac{1}{4}\sum|\mathcal{M}|^2 = N_c\left[\left(|A_{LL}|^2 + |A_{RR}|^2\right)u^2 + \left(|A_{LR}|^2 + |A_{RL}|^2\right)t^2\right] \tag{3.47}$$

where we define

$$A_{ij}(s) \equiv e^2\frac{Q_e Q_f}{s} + e_Z^2\frac{g_{ei}g_{fj}}{s-M_Z^2} + \frac{k_{ei}k_{fj}}{s-M_X^2}. \tag{3.48}$$

3.A.5 Avoiding Poles

A glaring issue with Equations 3.47 and 3.48 are the divergences as $s \to M_Z^2, M_X^2$. These occur as a result of the mediating gauge boson being "on-shell", which means that, if we represent the 4-momentum of the V boson as P^μ, the relation $P^2 = -M_V^2$ is satisfied (which is not always necessarily the case for virtual particles). However, if a calculation were performed to a higher order in perturbation theory (e.g. to the 1-loop level, rather than only to tree level), it is possible to show that the expression in the denominator is corrected:

$$\frac{1}{s-M_V^2} \to \frac{1}{s-\left[M_V(1+\delta)\right]^2}. \tag{3.49}$$

It is possible to work through the 1-loop calculation in order to determine δ, but there is a simpler way: consider the time-dependent wave function for the free gauge boson in its rest frame,

$$|\psi_V(t)\rangle = e^{-iM_V t}|\psi_V(0)\rangle. \tag{3.50}$$

This wavefunction yields a time-independent probability distribution:

$$P_V(t) \equiv |\langle\psi_V(t)|\psi_V(t)\rangle|^2 = |\langle\psi_V(0)|\psi_V(0)\rangle|^2.$$

However, we know this not to be the case; experimentally, we know that massive gauge bosons (specifically the Z) decay. The Z decay rate is measured to be [23] $\Gamma_Z = 2.4952 \pm 0.0023$ GeV, in agreement with the SM prediction, which is (at tree-level)

$$\Gamma_Z = \frac{\alpha_Z M_Z}{6} \sum_{f \neq t} \left(g_{Lf}^2 + g_{Rf}^2\right) N_c$$

where $\alpha_Z = e_Z^2/4\pi$. (A derivation of this expression is given in Appendix C.) Therefore, the corresponding time-*dependent* probability distribution should be

$$P_Z(t) \equiv \left|\langle\psi_Z(t)|\psi_Z(t)\rangle\right|^2 = e^{-\Gamma_Z t}\left|\langle\psi_Z(0)|\psi_Z(0)\rangle\right|^2. \tag{3.51}$$

Allowing $M_Z \to M_Z(1+\delta)$ in Equation 3.50 yields Equation 3.51 in the case of the Z, so long as

$$\delta = -\frac{i}{2}\frac{\Gamma_Z}{M_Z}.$$

This agrees with the detailed result obtained from the 1-loop calculation except for one difference: since every additional loop involves additional powers of α_Z and since $\delta \propto \alpha_Z$, the 1-loop calculation gives the δ-linearized version of the denominator in 3.49,

$$\frac{1}{s - M_Z^2 - 2M_Z^2\delta} = \frac{1}{s - M_Z^2 + i\Gamma_Z M_Z}.$$

Therefore, the expression that takes into account the non-zero decay widths of the Z and X bosons is

$$A_{ij}(s) \equiv e^2\frac{Q_e Q_f}{s} + e_Z^2\frac{g_{ei}g_{fj}}{s - M_Z^2 + i\Gamma_Z M_Z} + \frac{k_{ei}k_{fj}}{s - M_X^2 + i\Gamma_X M_X}$$

thus avoiding the issue of divergences when $s = M_Z^2, M_X^2$.

3.B Appendix: Determining the Gauge Boson Propagator

The propagator $G_{\mu\nu}^V(x,y)$ is defined formally as

$$G_{\mu\nu}^V(x,y) = \langle 0|T\left[V_\mu(x)V_\nu(y)\right]|0\rangle \tag{3.52a}$$

$$= \langle 0|V_\mu(x)V_\nu(y)|0\rangle\theta\left(x^0-y^0\right) + \langle 0|V_\mu(y)V_\nu(x)|0\rangle\theta\left(y^0-x^0\right) \tag{3.52b}$$

where $\theta(x) \equiv \begin{Bmatrix} 1, & x>0 \\ 0, & x<0 \end{Bmatrix}$ is the Heaviside step function.

In order to simplify this expression, we can insert the following mode expansion for the spin-one field V_μ:

$$V_\mu(x) = \sum_{\lambda=-1}^{1}\int\frac{d^3\mathbf{p}}{2E_\mathbf{p}(2\pi)^3}\left[\varepsilon_\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{ip\cdot x} + \varepsilon_\mu^*(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}^*e^{-ip\cdot x}\right] \tag{3.53}$$

This gives

$$\begin{aligned}\langle 0|\, V_\mu (x)\, V_\nu (y)\, |0\rangle &= \sum_{\lambda=-1}^{1} \int \frac{d^3\mathbf{p}}{2E_\mathbf{p} (2\pi)^3} \varepsilon_\mu (\mathbf{p}, \lambda)\, \varepsilon^*_\nu (\mathbf{p}, \lambda)\, e^{ip\cdot(x-y)} \\ &= \int \frac{d^3\mathbf{p}}{2E_\mathbf{p} (2\pi)^3} \Pi^V_{\mu\nu} (\mathbf{p})\, e^{ip\cdot(x-y)}\end{aligned}$$

where $\Pi^V_{\mu\nu} (\mathbf{p}) \equiv \sum_{\lambda=-1}^{1} \varepsilon_\mu (\mathbf{p}, \lambda)\, \varepsilon^*_\nu (\mathbf{p}, \lambda)$, and so

$$G^V_{\mu\nu} (x, y) = \int \frac{d^3\mathbf{p}}{2E_\mathbf{p} (2\pi)^3} \Pi^V_{\mu\nu} (\mathbf{p}) \left[e^{ip\cdot(x-y)} \theta \left(x^0 - y^0\right) + e^{-ip\cdot(x-y)} \theta \left(y^0 - x^0\right)\right]. \tag{3.54}$$

To further simplify this expression, it is useful to now insert the following integral representation of the Heaviside step function:

$$\theta (x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{ix\omega}}{\omega - i\varepsilon} \quad (\varepsilon > 0)$$

which gives

$$\begin{aligned}G^V_{\mu\nu} (x, y) &= -i \int \frac{d^3\mathbf{p}}{2E_\mathbf{p} (2\pi)^3} \int \frac{d\omega}{2\pi} \frac{1}{\omega - i\varepsilon} \Pi^V_{\mu\nu} (\mathbf{p}) \left[e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{-i\left(x^0 - y^0\right)(E_\mathbf{p} - \omega)} \right. \\ &\qquad \left. + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{+i\left(x^0 - y^0\right)(E_\mathbf{p} - \omega)} \right].\end{aligned} \tag{3.55a}$$

Substituting $p^0 = E_\mathbf{p} - \omega$ gives

$$\begin{aligned}G^V_{\mu\nu} (x, y) &= -i \int \frac{d^4p}{(2\pi)^4} \frac{1}{2E_\mathbf{p}} \frac{1}{E_\mathbf{p} - p^0 - i\varepsilon} \Pi^V_{\mu\nu} (\mathbf{p}) \left[e^{ip\cdot(x-y)} + e^{-ip\cdot(x-y)}\right] \\ &= -i \int \frac{d^4p}{(2\pi)^4} \Pi^V_{\mu\nu} (\mathbf{p})\, e^{ip\cdot(x-y)} \frac{1}{2E_\mathbf{p}} \left[\frac{1}{E_\mathbf{p} - p^0 - i\varepsilon} + \frac{1}{E_\mathbf{p} + p^0 - i\varepsilon}\right] \\ G^V_{\mu\nu} (x, y) &= -i \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \frac{\Pi^V_{\mu\nu} (\mathbf{p})}{p^2 + m^2 - i\varepsilon}\end{aligned} \tag{3.56}$$

where, in the last line, ε is implicitly rescaled: $\varepsilon \to \varepsilon / 2E_\mathbf{p}$. Note that the 4-vector p in the exponential is no longer assumed to be on-shell, i.e. $p^0 \neq E_\mathbf{p}$ necessarily.

Finally, we obtain a simplified expression for $\Pi^V_{\mu\nu} (\mathbf{p})$. Since $\Pi^V_{\mu\nu} (\mathbf{p}) \equiv \sum_\lambda \varepsilon_\mu (\mathbf{p}, \lambda)\, \varepsilon^*_\nu (\mathbf{p}, \lambda)$ is a rank-2 tensor, we can assume that the sum will have the final form

$$\Pi^V_{\mu\nu} (\mathbf{p}) = A\eta_{\mu\nu} + Bp_\mu p_\nu \tag{3.57}$$

for some constants A and B. We can evaluate this sum explicitly in the rest frame with the following choice of basis polarization 4-vectors:

$$\varepsilon_\mu(\mathbf{0},-1) = (0,1,0,0) \tag{3.58a}$$
$$\varepsilon_\mu(\mathbf{0},0) = (0,0,1,0) \tag{3.58b}$$
$$\varepsilon_\mu(\mathbf{0},+1) = (0,0,0,1) \tag{3.58c}$$

This gives

$$\sum_{\lambda=-1}^{1} \varepsilon_\mu(\mathbf{p},\lambda)\,\varepsilon^*_\nu(\mathbf{p},\lambda) = \delta^1_\mu\delta^1_\nu + \delta^2_\mu\delta^2_\nu + \delta^3_\mu\delta^3_\nu. \tag{3.59}$$

To solve for A and B, recall that $\eta_{\mu\nu} \equiv -\delta^0_\mu\delta^0_\nu + \delta^1_\mu\delta^1_\nu + \delta^3_\mu\delta^3_\nu + \delta^3_\mu\delta^3_\nu$ and that $p_\mu p_\nu = M_V^2\delta^0_\mu\delta^0_\nu$ in the rest frame. After some rearranging, this gives $A = 1,\ B = 1/M_V^2$. Therefore, in a general reference frame, we have that

$$\Pi^V_{\mu\nu}(\mathbf{p}) = \eta_{\mu\nu} + \frac{p_\mu p_\nu}{M_V^2}. \tag{3.60}$$

3.C Appendix: Z Decay Calculation

The rate at which the Z boson decays can be calculated by considering Equation 3.33 from Appendix A, in the case where there is only one particle in the initial state. (Note that, in this case, the decay rate is a well-defined quantity as $V, T \to \infty$.) In particular, the differential decay rate in the rest frame of the Z boson (where we take $p = (M_Z, \mathbf{0})$, $p' = (E_{\mathbf{p}'}, \mathbf{p}')$, $k' = (E_{\mathbf{p}'}, -\mathbf{p}')$ is

$$d\Gamma\left(Z \to f\overline{f}\right) = \frac{|\mathcal{M}|^2}{2M_Z}2\pi\delta\left(M_Z - 2p'^0\right)\frac{d^3\mathbf{p}'}{4E^2_{\mathbf{p}'}(2\pi)^3}$$
$$\Rightarrow d\Gamma\left(Z \to f\overline{f}\right) = \frac{|\mathcal{M}|^2}{32\pi M_Z}\sin\theta d\theta \tag{3.61}$$

where we have eliminated the second angular variable by taking advantage of the azimuthal symmetry.

Now to calculate $|\mathcal{M}|^2$: using the Feynman rules described in Appendix A, we find that for the decay $Z \to f\overline{f}$,

$$\mathcal{M} = -e_Z\varepsilon_\mu(\mathbf{p},\lambda)\left[\overline{u}(\mathbf{p}',\sigma)\gamma^\mu\Gamma^Z_f v(\mathbf{k}',\tau)\right]$$

where $\Gamma^Z_f = g_{fL}P_L + g_{fR}P_R$ with the $g_{L(R)i}$ defined by $J^Z_\mu = ie_Z\sum_i \overline{f}_i\gamma_\mu(g_{Li}P_L + g_{Ri}P_R)f_i$.

Averaging $|\mathcal{M}|^2$ over initial polarizations and summing over final spins gives

$$\begin{aligned}
\frac{1}{3}\sum|\mathcal{M}|^2 &= \frac{e_Z^2}{3}\sum_{pol,spins}\varepsilon_\mu(\mathbf{p},\lambda)\varepsilon_\nu^*(\mathbf{p},\lambda)\left[\overline{u}(\mathbf{p}',\sigma)\gamma^\mu\Gamma_f^Z v(\mathbf{k}',\tau)\right]\left[\overline{u}(\mathbf{p}',\sigma)\gamma^\nu\Gamma_f^Z v(\mathbf{k}',\tau)\right]^* \\
&= -\frac{e_Z^2}{3}\left(\eta_{\mu\nu}+\frac{p_\mu p_\nu}{M_Z^2}\right)\mathrm{tr}\left[\left(-i\not{p}'+m\right)\gamma^\mu\Gamma_f^Z\left(-i\not{k}'-m\right)\gamma^\nu\Gamma_f^Z\right] \\
&\simeq \frac{e_Z^2}{3}\eta_{\mu\nu}p'_\alpha k'_\beta\mathrm{tr}\left[\gamma^\alpha\gamma^\mu\Gamma_f^Z\gamma^\beta\gamma^\nu\Gamma_f^Z\right]
\end{aligned}$$

where the approximation assumes, as before, that the fermions are very light compared to other energy scales in the problem. Evaluating the trace and contracting gives

$$\frac{1}{3}\sum|\mathcal{M}|^2 = -\frac{4}{3}e_Z^2\left(g_{Lf}^2+g_{Rf}^2\right)p'\cdot k'$$

where, in the rest frame of the initial Z, $p'\cdot k' = -2E_{\mathbf{p}'}^2, = -M_Z^2/2$. Note that this expression does not depend on the polar angle, θ. Finally, we find that the decay rate in the lab frame is

$$\begin{aligned}
\Gamma\left(Z\to f\overline{f}\right) &= \frac{e_Z^2}{16\pi M_Z}\frac{4}{3}\left(g_{Lf}^2+g_{Rf}^2\right)\left(\frac{M_Z^2}{2}\right) \\
&= \frac{e_Z^2 M_Z}{24\pi}\left(g_{Lf}^2+g_{Rf}^2\right)
\end{aligned}$$

Chapter 4

Neutrino Scattering

The purpose of this section is to consider the influence of the X boson on well-measured results from neutrino scattering. In particular, two quantities will be of interest:

- $R \equiv \frac{\sigma(\nu_\mu e^- \to \nu_\mu e^-)}{\sigma(\bar{\nu}_\mu e^- \to \bar{\nu}_\mu e^-)}$ in the case neutrino-electron scattering;
- $R^- \equiv \frac{\sigma(\nu_\mu N \to \nu_\mu X) - \sigma(\bar{\nu}_\mu N \to \bar{\nu}_\mu X)}{\sigma(\nu_\mu N \to \mu^- X) - \sigma(\bar{\nu}_\mu N \to \mu^+ X)}$ (known as the Paschos-Wolfenstein Ratio [24]) in the case of neutrino-nucleon scattering.

These ratios are useful because they allow for the cancellation of systematic errors that arise in the measurement of the individual cross sections. (This will be discussed in more detail later on.) We shall first derive the $\nu_\mu e^- \to \nu_\mu e^-$ cross section using the $e^+e^- \to f\bar{f}$ result from Chapter 3. From here, we consider the specific case of zero mixing (i.e. $\eta = 0$) in order to demonstrate how measurements of R relate to the weak mixing angle. The $\eta \neq 0$ case is also considered. In considering neutrino-nucleon scattering, we begin by showing the significance of the Paschos-Wolfenstein ratio. However, rather than using it directly to produce a bound on the X parameters, we use the fit done in [25], which gives experimental values for the combinations $\varepsilon^2_{L(R)} = g^2_{uL(R)} + g^2_{dL(R)}$.

4.1 Neutrino-Electron Scattering

4.1.1 Crossing Symmetry

Crossing symmetry can be used to exploit the result obtained previously for $\sigma\left(e^+e^- \to f\bar{f}\right)$ in order to find an expression for $\sigma\left(\nu_\mu e^- \to \nu_\mu e^-\right)$. In Chapter 3, we found that, in the ultrarelativistic limit,

$$\frac{d\sigma}{dt}\left(e^+e^- \to \nu\bar{\nu}\right) = -\frac{1}{16\pi s^2}\left(\frac{1}{2}\sum|\mathcal{M}|^2\right) \tag{4.1a}$$

$$\frac{1}{2}\sum|\mathcal{M}|^2 = 2\left(|A_{LL}(s)|^2(s+t)^2 + |A_{RL}(s)|^2 t^2\right) \tag{4.1b}$$

(Note: the $|A_{LR}|^2$ and $|A_{RR}|^2$ terms are dropped since, in this case, only left-handed neutrinos and right-handed anti-neutrinos contribute.) where

$$A_{ij}(s) = e_Z^2 \frac{g_{ei} g_{\nu j}}{s - M_Z^2} + g_X^2 \frac{k_{ei} k_{\nu j}}{s - M_X^2} \tag{4.2}$$

and where

$$s = -2p \cdot k, \quad t = 2p \cdot p', \quad u = 2p \cdot k'. \tag{4.3}$$

By comparing the diagrams in Figure 4.1, we find that the scattering matrix element for $\nu e^- \rightarrow \nu e^-$ would be the exact same as what would be obtained if the original $e^+e^- \rightarrow f\bar{f}$ calculation were repeated with the following 4-momentum changes: $k \rightarrow -p'$, $k' \rightarrow -k$, $p' \rightarrow k'$.

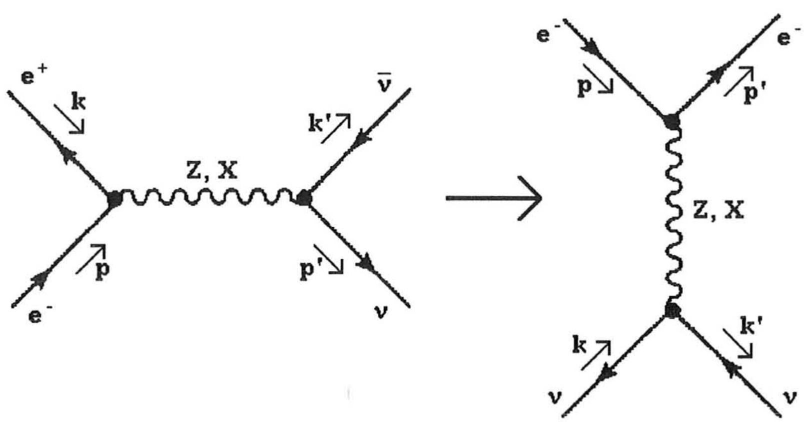

Figure 4.1: Illustration of the crossing symmetry between the process $e^+e^- \rightarrow \nu\bar{\nu}$ and $e^-\nu \rightarrow e^-\nu$ in terms of Feynman diagrams.

Using the definitions in Equation 4.3, we deduce the following changes for the s, t, and u variables:

$$s \rightarrow t, \quad t \rightarrow u, \quad u \rightarrow s. \tag{4.4}$$

However, it is important to note that this trick using crossing symmetry only works for ν_f with $f \neq e$. This is because, in the case of $e^-\nu_e$ scattering, the incoming particles can scatter through an additional s-channel W boson, which was not considered originally.

With these replacements, the cross section for the process $\nu_\mu e^- \rightarrow \nu_\mu e^-$ is

$$\begin{aligned} \frac{d\sigma}{dt}\left(\nu_\mu e^- \rightarrow \nu_\mu e^-\right) &= -\frac{1}{8\pi s^2}\left[|A_{LL}(t)|^2 s^2 + |A_{RL}(t)|^2 (s+t)^2\right] \\ &= -\frac{1}{8\pi}\left[|A_{LL}(t)|^2 + |A_{RL}(t)|^2 \left(1 + \frac{t}{s}\right)^2\right] \end{aligned} \tag{4.5}$$

Finally, it is customary to consider this cross section in the rest frame of the incoming electron, as a function of the energy of the incoming neutrino, E_ν, and the fractional neutrino energy loss, $y \equiv T/E_\nu$, where T is the kinetic energy of the outgoing electron (i.e. $T = p'^0 - m_e$). In terms of these new variables, $s = 2m_e E_\nu$ and $t = -2m_e(yE_\nu + m_e)$ and so

$$\frac{d\sigma}{dy}\left(\nu_\mu e^- \to \nu_\mu e^-\right) = \frac{m_e E_\nu}{4\pi}\left[\left|A_{LL}(t)\right|^2 + \left|A_{RL}(t)\right|^2 (1-y)^2\right] \tag{4.6}$$

where we have dropped terms of order $\frac{m_e}{E_\nu}$ (which are equivalent to terms of order $\frac{m_e^2}{s}$ that have been dropped previously).

4.1.2 Effective Interactions

A case of interest when dealing with interactions mediated by heavy gauge bosons is when the boson masses, M_Z and M_X, are much greater than any other energy scale in the process of interest (i.e. $\sqrt{t}$). If this condition holds, then the A_{iL}'s can be simplified:

$$\begin{aligned} A_{iL} &\simeq -\frac{e_Z^2}{M_Z^2} g_{ei} g_{\nu L} - \frac{g_X^2}{M_X^2} k_{ei} k_{\nu L} \\ &= -\frac{e_Z^2}{M_Z^2} g_{\nu L}\left(g_{ei} + \frac{M_Z^2}{e_Z^2}\frac{g_X^2}{M_X^2}\frac{k_{\nu L}}{g_{\nu L}} k_{ei}\right) \end{aligned} \tag{4.7}$$

Hence, when $t \ll M_Z^2,\ M_X^2$, the presence of the second X boson term in the A_{iL}'s can be interpreted as an additional shift in the neutral current couplings g_{ei}.

4.1.3 Special Case: $\eta = 0$

In the case of no kinetic mixing, we can substitute the usual values $g_{ei} = -\frac{1}{2}\delta_{iL} + s_W^2$, $g_{\nu L} = \frac{1}{2}$, $k_{ei} = k_{\nu j} = -1$ and obtain

$$A_{iL} = -2\sqrt{2}G_F\left(-\frac{1}{2}\delta_{iL} + s_W^2 + \frac{1}{2\sqrt{2}G_F}\frac{g_X^2}{M_X^2}\right) \tag{4.8}$$

where $2\sqrt{2}G_F \equiv e_W^2/M_W^2 \equiv e_Z^2/2M_Z^2$. ($\delta_{iL}$ is a Kronecker delta function: $\delta_{LL} = 1$, $\delta_{RL} = 0$.) In this sense, the existence of the X boson manifests itself as an additional contribution to s_W^2. Hence, measurements of s_W^2 obtained using neutrino-electron scattering constrain the ratio g_X^2/M_X^2. Specifically, this is done as follows: using the SM only, we obtain

$$\frac{d\sigma}{dy}\left(\nu_\mu e^- \to \nu_\mu e^-\right) = \frac{2G_F^2 m_e E_\nu}{\pi}\left[g_{eL}^2 + g_{eR}^2(1-y)^2\right] \tag{4.9}$$

which, when integrated from $y = 0$ to $y = 1$, gives

$$\sigma\left(\nu_\mu e^- \to \nu_\mu e^-\right) = \frac{2G_F^2 m_e E_\nu}{\pi}\left(g_{eL}^2 + \frac{1}{3}g_{eR}^2\right). \tag{4.10}$$

Conversely, if we had instead considered the process $\overline{\nu}_\mu e^- \to \overline{\nu}_\mu e^-$, we would have kept the $|A_{RR}|^2$ and $|A_{LR}|^2$, which would have given the same expression as in Equation 4.9, except that $g_{eL}^2 \leftrightarrow g_{eR}^2$:

$$\sigma\left(\overline{\nu}_\mu e^- \to \overline{\nu}_\mu e^-\right) = \frac{2G_F^2 m_e E_\nu}{\pi}\left(\frac{1}{3}g_{eL}^2 + g_{eR}^2\right). \tag{4.11}$$

From here, the dependence of $R \equiv \sigma\left(\nu_\mu e^- \to \nu_\mu e^-\right)/\sigma\left(\overline{\nu}_\mu e^- \to \overline{\nu}_\mu e^-\right)$ on s_W^2 can be seen explicitly:

$$R\left(s_W^2\right) = \frac{3g_{eL}^2 + g_{eR}^2}{g_{eL}^2 + 3g_{eR}^2} = \frac{3 - 12s_W^2 + 16s_W^4}{1 - 4s_W^2 + 16s_W^4}. \tag{4.12}$$

This is often (e.g. in [26]) rewritten as

$$R = \frac{1 + \kappa + \kappa^2}{1 - \kappa + \kappa^2} \tag{4.13}$$

where $\kappa \equiv 1 - 4s_W^2$. Experimentalists can measure R and then deduce a corresponding measurement of s_W^2. Thus, the additional contribution to s_W^2 is limited in size to the experimental error in s_W^2, i.e. $\Delta s_W^2 = \frac{1}{2\sqrt{2}G_F}\frac{g_X^2}{M_X^2}$. In [27], we find that such experiments yield $\frac{\Delta s_W^2}{s_W^2} = 0.82\%$ which gives ($G_F = 1.1664 \times 10^{-5}$ GeV^{-2} [23])

$$\frac{M_X}{g_X} \geq 4 \text{ TeV} \tag{4.14}$$

However, this bound should not be trusted down to arbitrarily low M_X; [27] obtains this value from an experiment in which $3 < T < 5$ MeV. Therefore, requiring that our assumption regarding effective interactions, $\left|\frac{t}{M_X^2}\right| \lesssim 1\%$, remain valid gives ($t = -2m_e\left(T + m_e\right)$)

$$M_X \gtrsim 10 \text{ MeV} \tag{4.15}$$

In the general case with $\eta \neq 0$, the situation is significantly more complicated. This is considered below.

4.1.4 General Case: $\eta \neq 0$

In the general mixed case, we calculate the relevant cross sections using Equation 4.7, along with the mixed expressions for $g_{fL(R)}$, $k_{fL(R)}$ as found in Chapter 2. We then determine R (up to quadratic order in η) and compare this result with the experimental value found in [27]. Since, in [27], the bound is quoted in terms of s_W^2, we translate this bound into one on R using

$$\Delta R = \left|\frac{dR}{d\left(s_W^2\right)}\right|\Delta s_W^2. \tag{4.16}$$

As in previous chapters, this bound is not linear and must be plotted numerically. This is shown in Figure 4.2.

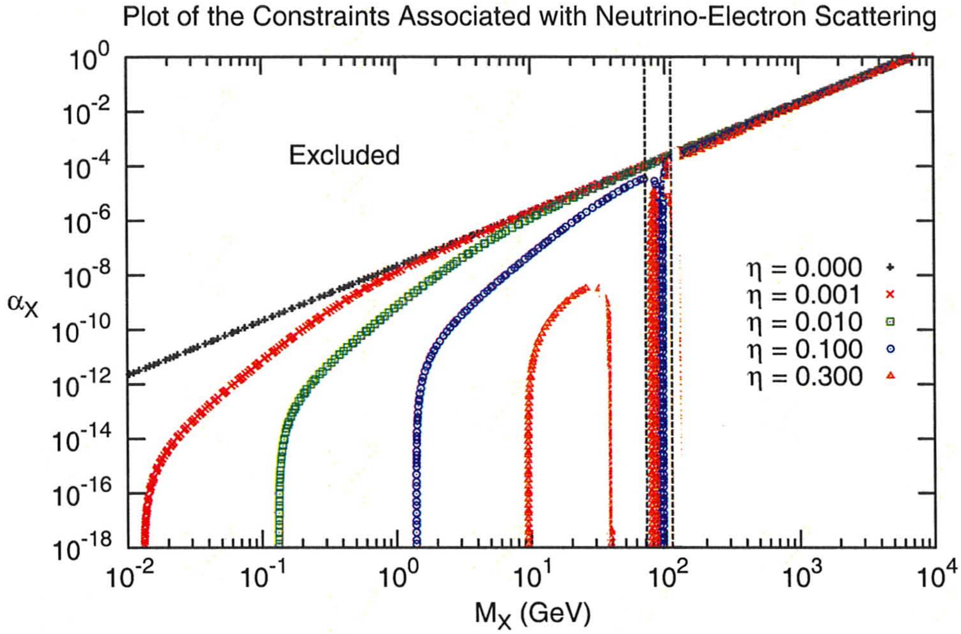

Figure 4.2: Bound obtained on g_X, by limiting the influence of the X boson on the cross section ratio R, as a function of M_X. The dotted vertical lines indicate the region in which the small-η expansion is questionable (see Section 2.3 for details).

Here, we plot the bound on g_X as a function of M_X for the mixing values $\eta = 0,\ 0.001,\ 0.01,\ 0.1,$ and 0.3. A few comments:

- For the larger values of η (i.e. for $\eta = 0.1,\ 0.3$), the result obtained in the region $M_X \in [80, 100]$ (marked by vertical dashed lines in Figure 4.2) should not be trusted, since the perturbative expansion in η is not valid here (as shown in Figure 2.1).

- For all non-zero values of η, there is a mass cutoff past which all values of the gauge coupling are excluded. This is similar to what was found in Figure 3.5 for small M_X, although here the effect is present even for small η.

- For large M_X, we see that the bound is essentially the same for all η. This is expected since $\Delta g,\ \Delta k \propto 1/s_X^2$, which is ~ 1 for $M_X \ll M_z$, but only $\sim M_z^2/M_X^2$ when $M_X \gg M_z$, which means that the term g_X^2/M_X^2 found in the second term of the A_{ij}'s dictate the bound, as in the unmixed case.

In the next section, we consider a similar process, $\nu N \to \nu N$, where N is any nucleon.

4.2 Neutrino-Nucleon Scattering

4.2.1 The Paschos-Wolfenstein (PW) Ratio

To begin, we give some motivation for the PW Ratio: consider the cross sections for charged and neutral current scattering of muon neutrinos with nucleons. Borrowing from the previous section, we find that the quark-level expressions are

$$\sigma\left(\nu_\mu u \to \nu_\mu u\right) = N_c K \left(g_{uL}^2 + \frac{1}{3} g_{uR}^2\right) \tag{4.17a}$$

$$\sigma\left(\nu_\mu d \to \nu_\mu d\right) = N_c K \left(g_{dL}^2 + \frac{1}{3} g_{dR}^2\right) \tag{4.17b}$$

$$\sigma\left(\bar{\nu}_\mu u \to \bar{\nu}_\mu u\right) = N_c K \left(\frac{1}{3} g_{uL}^2 + g_{uR}^2\right) \tag{4.17c}$$

$$\sigma\left(\bar{\nu}_\mu d \to \bar{\nu}_\mu d\right) = N_c K \left(\frac{1}{3} g_{dL}^2 + g_{dR}^2\right) \tag{4.17d}$$

for neutral currents, and

$$\sigma\left(\nu_\mu d \to \mu^- u\right) = N_c K \tag{4.18a}$$

$$\sigma\left(\bar{\nu}_\mu u \to \mu^+ d\right) = \frac{N_c K}{3} \tag{4.18b}$$

for charged currents, where $K \equiv 2G_F^2 m_e E_\nu / \pi$ and $N_c = 3$. (Note that the charged current expressions are obtained from the neutral current ones by simply setting $g_L = 1,\ g_R = 0$.) From these expressions, it is clear that the neutral current expressions can be written as linear combinations of the charged current ones. However, we are interested in the cross section for neutrino-nucleon scattering in the limit of deep inelastic scattering; from the form of Equations 4.17 and 4.18, we can derive a relationship relating charged and neutral current cross sections (valid in the deep inelastic limit):

$$\sigma\left(\nu_\mu N \to \nu_\mu X\right) = \varepsilon_L^2 \sigma\left(\nu_\mu N \to \mu^- X\right) + \varepsilon_R^2 \sigma\left(\bar{\nu}_\mu N \to \mu^+ X\right)$$

where

$$\varepsilon_{L(R)}^2 \equiv g_{uL(R)}^2 + g_{dL(R)}^2$$

Similarly, for antineutrino scattering,

$$\sigma\left(\bar{\nu}_\mu N \to \bar{\nu}_\mu X\right) = \varepsilon_L^2 \sigma\left(\bar{\nu}_\mu N \to \mu^+ X\right) + \varepsilon_R^2 \sigma\left(\nu_\mu N \to \mu^- X\right)$$

As before, we would like to consider ratios of cross sections in order to cancel systematic errors that are present in individual cross sections measurements. The following ratios make convenient choices:

$$\begin{aligned} R^{\nu} &\equiv \frac{\sigma\left(\nu_\mu N \to \nu_\mu X\right)}{\sigma\left(\nu_\mu N \to \mu^- X\right)} = \varepsilon_L^2 + r\varepsilon_R^2 \\ R^{\bar{\nu}} &= \frac{\sigma\left(\bar{\nu}_\mu N \to \bar{\nu}_\mu X\right)}{\sigma\left(\bar{\nu}_\mu N \to \mu^+ X\right)} = \varepsilon_L^2 + \frac{1}{r}\varepsilon_R^2 \end{aligned}$$

where $r \equiv \sigma\left(\bar{\nu}_\mu N \to \mu^+ X\right)/\sigma\left(\nu_\mu N \to \mu^- X\right)$. Now, everything is expressed in terms of cross section ratios, but there is an additional problem: it turns out that the ratio r is difficult to measure experimentally, due mainly to processes contributing only to $\sigma\left(\nu_\mu N \to \mu^- X\right)$ in which a (much heavier) charm quark is produced from a down or strange quark in the initial state [25]. Since the charm quark is heavy, the deep inelastic limit is no longer valid, and this ratio cannot be measured very accurately.

In order to avoid this issue, we take the linear combination of R^{ν} and $R^{\bar{\nu}}$ such that the r-dependence vanishes. That is, we use what is known as the Paschos-Wolfenstein Ratio [24]:

$$\begin{aligned} R^- &\equiv \frac{R^{\nu} - rR^{\bar{\nu}}}{1-r} = \frac{\sigma\left(\nu_\mu N \to \nu_\mu X\right) - \sigma\left(\bar{\nu}_\mu N \to \bar{\nu}_\mu X\right)}{\sigma\left(\nu_\mu N \to \mu^- X\right) - \sigma\left(\bar{\nu}_\mu N \to \mu^+ X\right)} \\ &= \varepsilon_L^2 - \varepsilon_R^2 \end{aligned}$$

Considering this particular combination of cross sections gives a result that is insensitive to systematic errors related to the ratio r.

Following the procedure from the previous section, the next step would be to calculate an expression for R^- that includes the effects of the X boson. However, since [25] has used their data to perform a fit to determine ε_L^2 and ε_R^2, we shall instead use effective expressions for these to constrain our parameters. Specifically, [25] finds that

$$\begin{aligned} \varepsilon_L^2 &= 0.30005 \pm 0.00137 \\ \varepsilon_R^2 &= 0.03076 \pm 0.00110 \end{aligned}$$

with $t \approx -20$ GeV^2. Since similar results are obtained for each of these, we shall focus only on the bound arising from ε_L^2. Assuming, as in the case of neutrino-electron scattering, that $t \ll M_X^2$, we find that

$$A_{LL}^{u(d)} \simeq -\frac{e_Z^2}{M_Z^2} g_{\nu L}\left(g_{u(d)L} + \frac{M_Z^2}{e_Z^2}\frac{g_X^2}{M_X^2}\frac{k_{\nu L}}{g_{\nu L}}k_{u(d)L}\right) \tag{4.19}$$

and so, when calculating $\varepsilon_L^2 = \left(g_{uL}^{SM} + \Delta g_{uL}\right)^2 + \left(g_{dL}^{SM} + \Delta g_{dL}\right)^2$ (up to quadratic order in η) we include an additional term in $\Delta g_{u(d)L}$ to account for the X boson term in the A_{ij}'s:

$$\begin{aligned}\Delta g_{u(d)L} &= -\eta \frac{s_W}{s_X^2}\frac{g_X}{e_Z}(B-L)_{u(d)} - \eta^2 \frac{s_W^2}{s_X^2}\left(\frac{c_X^2}{2s_X^2} g_{u(d)L}^{SM} + Q_{u(d)}\frac{c_W^4}{c_W^2 - s_W^2}\right) \\ &\quad + \frac{M_Z^2}{e_Z^2}\frac{g_X^2}{M_X^2}\frac{k_{\nu L}}{g_{\nu L}} k_{u(d)L}\end{aligned} \tag{4.20}$$

Requiring that $\Delta\varepsilon_L^2 \leq 0.00137$ gives the plots shown in Figure 4.3. Here, we cut off at low mass

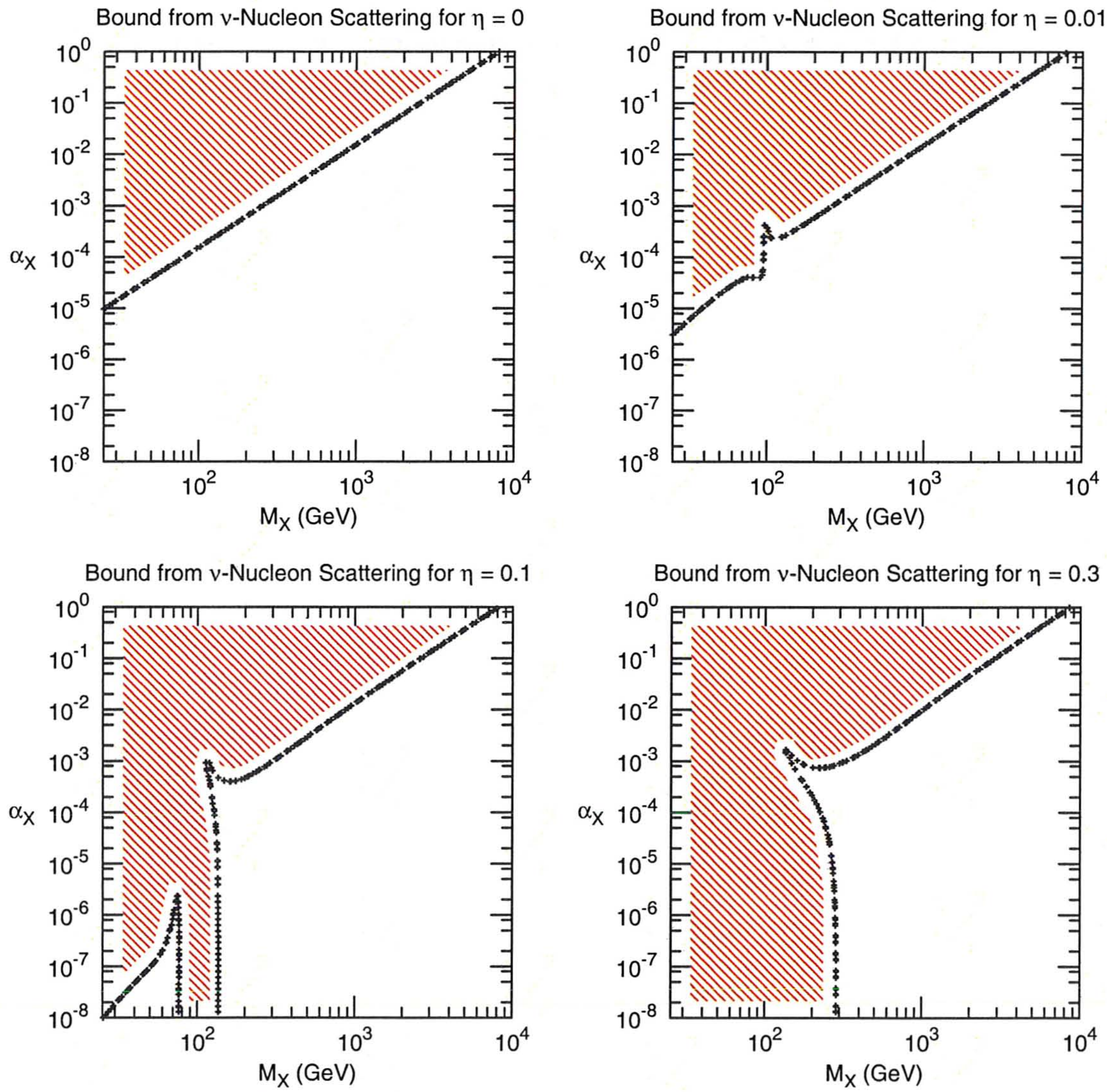

Figure 4.3: Plot of the constraint from R^- (neutrino-nucleon scattering). Here, we plot the bound on α_X as a function of M_X for various values of η. Excluded regions have been shaded out.

according to the condition $|t/M_X^2| \leq 1\%$, in order to ensure that assumption of effective interactions is valid.

Note that, for $\eta = 0$, the bound is very similar to what was found in the case of neutrino-electron scattering. However, as η increases, there is a point past which no values of g_X are allowed for small M_X.

Chapter 5

Constraint from Primordial Nucleosynthesis

5.1 Motivation

We turn to nucleosynthesis to constrain the X boson parameters since, as we shall see, very strong constraints exist in the mass range $M_X < 1$ MeV. In essence, the argument goes as follows: as the early universe cooled, high energy protons and neutrons combined to form the light nuclei that we observe today through a process known as nucleosynthesis. Present-day measurements of the abundances of these nuclei in the universe can be obtained from the observation of stellar spectra. One of the great successes of cosmology is that these abundances can be worked out from our knowledge of the SM and General Relativity. In fact, the detailed agreement between the predictions of theory and the observed abundances allow for constraints to be put on the conditions in the universe at the time of nucleosynthesis. In particular, the number of additional relativistic degrees of freedom present during nucleosynthesis can be limited using measurements of the relative abundance of ^{4}He [28]. Since the X boson allows for extra relativistic degrees of freedom, it can be constrained by considering its effect on nucleosynthesis. In this chapter, we shall first consider X boson decay, and then give an overview of the relevant cosmological concepts requisite for understanding nucleosynthesis, as well as the X boson's effect on this process. Finally, we derive a bound on the X boson parameters using the result of this analysis.

5.2 X Decay

The rate at which the X boson decays can be calculated in a way similar to that presented in Appendix C from Chapter 3. Such a calculation gives the following result in the lab frame:

$$\Gamma\left(X \to f\bar{f}\right) = \frac{M_X}{24\pi}\left(k_{Lf}^2 + k_{Rf}^2\right) \tag{5.1}$$

As found previously, $k_{L(R)i}$ are given (up to quadratic order in η) by

$$k_{L(R)i} = g_X\left(B-L\right)_i - \eta\left(1-\frac{1}{s_X^2}\right)s_W e_Z g_{L(R)i} + \eta c_W e Q_i + \frac{1}{2}\left(1-\frac{s_W^2}{s_X^4}\right)\eta^2 g_X\left(B-L\right)_i . \tag{5.2}$$

This expression simplifies greatly for the situation of interest: since we are interested in the regime where $M_X \sim T_F \simeq 0.7$ MeV (the freeze-out temperature; to be discussed later), then $s_X^2 \equiv 1-\frac{m_X^2}{m_Z^2} \simeq 1$; also, since at these energies the X boson can only decay into neutrinos, $Q_i = 0$. (Note that the assumption that $m_f \ll M_X$ is indeed valid for neutrinos.)

This gives

$$\Gamma_\nu = \frac{g_X^2 M_X}{12\pi}\left(1+\frac{1}{2}c_W^2\eta^2\right)^2 . \tag{5.3}$$

(Here, we have substituted the value $(B-L)_\nu = -1$.) This means that the total decay rate into all three generations is

$$\begin{aligned}\Gamma &= 3\Gamma_\nu \\ &= \alpha_X M_X\left(1+\frac{1}{2}c_W^2\eta^2\right)^2\end{aligned} \tag{5.4}$$

where $\alpha_X \equiv g_X^2/4\pi$, as before.

5.3 A Quick Review of Cosmology

In order to have a general understanding of primordial nucleosynthesis, it is important to understand the cosmological foundation upon which it is built.

Within the approximation of an isotropic and homogeneous universe, the most general metric $g_{\mu\nu}$ that can be written is represented by the following line element [29]:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2/R^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{5.5}$$

where $\kappa \in \{0,+1,-1\}$, corresponding to flat, closed, or open geometries. Since the universe as we observe it appears to have a flat geometry, we shall only consider the case where $\kappa = 0$.

Similarly, if we model the large-scale structure of the universe as a perfect fluid, we find that the stress-energy tensor for the universe in its rest frame is [29]

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \tag{5.6}$$

where ρ is the energy density and p is the pressure. To determine the time-dependence of the function $a(t)$, we use Einstein's Equations, given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{M_P^2} T_{\mu\nu} \tag{5.7}$$

where $M_P = 2.43 \times 10^{18}$ GeV and where $R_{\mu\nu}$, R are the Ricci tensor and Ricci scalar, respectively. By substituting Equations 5.5 and 5.6 into Einstein's Equations, we find the following two coupled differential equations:

$$H^2 = \frac{\rho}{3M_P^2} \tag{5.8}$$

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0 \tag{5.9}$$

where $H \equiv \frac{1}{a}\frac{da}{dt}$. Note that Equation 5.9 can be understood as a continuity equation for the energy within a co-expanding box of volume $V = a(t)^3$, i.e. an equation of the form

$$\frac{dE}{dt} = -p\frac{dV}{dt} \tag{5.10}$$

where $E = \rho V$.

These equations, on their own, cannot be solved completely unless a relation is known between the energy density and the pressure of the system. Such an equation is referred to as an "Equation of State", and is often of the form [29]

$$p = w\rho \tag{5.11}$$

for some constant w. For example, for a gas of photons, $w = 1/3$ whereas, for a gas of non-relativistic particles, $w \simeq 0$. Solving these three equations gives

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)} \tag{5.12a}$$

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \tag{5.12b}$$

From these we can see that, for early times, the expansion of the universe was dominated by radiation, whereas, at later times, energy sources with lower w values, such as non-relativistic matter would dominate.

We are also interested in the temperature dependence of ρ, which , in turn, determines the temperature dependence of H. This can be derived by recalling the generalized Boltzmann distribution functions for bosons $(-)$ and fermions $(+)$:

$$\rho(T) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} \mathcal{N}(E_{\mathbf{p}}, T) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{E_{\mathbf{p}}}{e^{E_{\mathbf{p}}/T} \pm 1} \tag{5.13}$$

(Here, the Boltzmann constant, k_B, is set to unity, implying that temperatures are identified by their corresponding Boltzmann energy: $E = k_B T$.) For relativistic matter, $p\,(= |\mathbf{p}|) \gg m$ and so $E_{\mathbf{p}} \left(= \sqrt{p^2 + m^2}\right) \simeq p$, which gives

$$\begin{aligned} \rho(T) &\simeq \int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} \frac{p}{e^{p/T} \pm 1} \\ &= \frac{T^4}{2\pi^2} \int_0^\infty dx \frac{x^3}{e^x \pm 1} \\ \rho(T) &= g a_B T^4 \end{aligned} \tag{5.14}$$

where $a_B = \pi^2/30$ and where

$$g = \begin{cases} 1, & \text{bosons} \\ 7/8, & \text{fermions} \end{cases} . \tag{5.15}$$

Each spin or polarization degree of freedom contributes to ρ independently by an amount as given above.

Another quantity of interest is the number density of non-relativistic particles: in this case, we find that $p \ll m$ and so $E_{\mathbf{p}} \simeq m + p^2/2m$, which gives

$$\begin{aligned} n(T) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathcal{N}(E_{\mathbf{p}}, T) \\ &= \frac{T^3}{2\pi^2} \int_0^\infty dx \frac{x^2}{e^{\left(\frac{m}{T} + \frac{x^2 T}{2m}\right)} \pm 1} \end{aligned}$$

Now, for $T \ll m$, we find that

$$\begin{aligned} n(T) &= \frac{T^3}{2\pi^2} e^{-m/T} \int_0^\infty x^2 \exp\left(-\frac{x^2 T}{2m}\right) dx \\ &= \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \end{aligned} \tag{5.16}$$

This is of use when considering free protons and neutrons before nucleosynthesis.

5.4 Equilibrium and Nucleosynthesis

While the universe is still sufficiently warm, the weak interaction keeps neutrons and protons in equilibrium with the bath of relativistic neutrinos, electrons, and photons. However, as the universe expands and cools, we find that the weak interaction is no longer capable of sustaining equilibrium. The temperature at which this occurs can be estimated by examining the generalized Boltzmann equation for the relevant processes [29]:

$$\frac{d}{dt}\left(n_i a^3\right) = \sum_j \left[\kappa\left(j \to i\right) - \kappa\left(i \to j\right)\right]$$

or

$$\frac{dn_i}{dt} + 3Hn_i = \frac{1}{a^3}\sum_j \left[\kappa\left(j \to i\right) - \kappa\left(i \to j\right)\right] \tag{5.17}$$

where $\kappa\left(i \to j\right)$ is the total macroscopic scattering rate from i particles into j particles.

Equation 5.17 introduces some confusion to the notion of being in equilibrium. Normally, (stable) equilibrium is a condition that does not change over time, as well as a balance between forwards and backwards reactions. Clearly, both of these notions cannot simultaneously arise from Equation 5.17. However, in the limit where each κ is much greater than H, the second term on the left-hand side of Equation 5.17 can be neglected, yielding the usual Boltzmann Equation. Therefore, the condition for equilibrium is:

$$H \lesssim \frac{1}{N_i}\kappa \equiv \Gamma \tag{5.18}$$

where $N_i = n_i a^3$ is the total number of i particles in some volume $V = a^3$, and Γ is now the usual scattering rate per particle. From here, we can estimate the freeze-out temperature, T_F, by determining the temperature at which Relation 5.18 no longer holds. For weak interactions, $\Gamma \sim G_F^2 T^5$ [29] and $H \sim \frac{1}{M_P}T^2$. Solving using these estimations gives $T_F \simeq 0.7$ MeV.

At this temperature, the number density of neutrons and protons "freeze out", meaning that the ratio n_n/n_p remains fixed at its present value (almost - see below):

$$f \equiv \frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} e^{-(m_n - m_p)/T_F}$$

This is not exactly right since the neutron can still undergo β decay into a proton and other particles. Including the effects of this process instead gives [29]

$$f \simeq e^{-(m_n - m_p)/T_F}\left(\frac{e^{-t/\tau_n}}{2 - e^{-t/\tau_n}}\right)$$

where $\tau_n = 890$ s is the mean lifetime of the neutron and t is the amount of time before the universe has cooled enough to allow deuterium to form without being photo-dissociated immediately. This occurs when $T = 2.23$ MeV ($\to t = 89$ s).

This is a useful quantity because it turns out that the relative abundance of ^{4}He, Y_p, depends simply on f [29]:

$$Y_p \equiv \frac{\rho_{^4He}}{\rho_B} \simeq \frac{2f}{1+f}. \tag{5.19}$$

(ρ_B is the energy density of all baryons.) Therefore, any change to f results in a variation of Y_p, which has been measured experimentally [28]: $Y_p = 0.249 \pm 0.009$. Since Y_p is a ratio of energy densities, it should not vary as the universe expands. This means that the present-day measured value should match what is produced during nucleosynthesis.

This measurement of Y_p can be used to place a limit on the number of extra relativistic degrees of freedom that can be in equilibrium, since extra degrees of freedom increase H, which increases T_F and t, which increases f. Cyburt et. al. [28] find the limit to be

$$\delta N_\nu \leq 1.44 \text{ (95\% C.L.)} \tag{5.20}$$

where $\delta N_\nu = N_\nu - 3$ is often interpreted as the change in the number of neutrino generations present at freeze-out.

In order to apply this to the case in which we are interested (i.e. extra bosonic degrees of freedom), consider the corresponding maximum allowable change to the variable $g = N_b + (7/8)\, N_f$:

$$\begin{aligned} \delta g_{\max} &= \frac{7}{8}\delta N_\nu \times 2 \,(\text{spins per } \nu) \\ &= 2.52 \end{aligned} \tag{5.21}$$

From here, we can re-interpret this as a maximum allowable number of massive spin-1 bosons, N_X:

$$\begin{aligned} \delta g_{\max} &= N_X \times 3 \,(\text{pol.'s per } X) = 2.52 \\ \Rightarrow & \quad N_X = 0.84 \end{aligned}$$

Therefore, adding just one additional massive spin-1 boson into relativistic equilibrium is excluded at the 95% confidence level! From here, there are only two viable options (other than simply not including the spin-1 boson...):

1. ensure that the X boson decays into neutrinos before the freeze-out temperature, T_F, by choosing appropriate values for its mass, and gauge coupling;

2. choose the X boson to have a mass, M_X, such that $M_X > T_F$. If this were so, then the density of X bosons would be Boltzmann-suppressed, and so would not interfere with the value of Y_p.

Hence, for $M_X > T_F$, there is no constraint. However, if $M_X < T_F$, the constraint comes from requiring that $H > \Gamma$ when $T = T_F$ [1]. This ensures that the reaction $X \to \nu\overline{\nu}$ is not in equilibrium with the reverse reaction $\nu\overline{\nu} \to X$, and so those X bosons that decay are not reproduced from neutrino collisions. We can calculate H explicitly using the Friedmann equation:

$$H = \frac{1}{M_P}\sqrt{\frac{\rho(T)}{3}} = \frac{1}{M_P}\sqrt{\frac{g a_B T^4}{3}} \tag{5.22}$$

where $M_P = 1/\sqrt{8\pi G} = 2.43 \times 10^{21}$ MeV, $a_B = \pi^2/30$, and where $g = N_b + (7/8)\, N_f$ counts the effective number of degrees of freedom. In this case, we have $N_b = 2\,(\gamma) + 3\,(X) = 5$ and $N_f = 3 \times 2\,(\nu)$ $+2 \times 2\,(e^{\pm}) = 10$, which gives $g = 55/4$ $(= 13.75)$. From here, we find that, at $T = T_F \simeq 0.7$ MeV,

$$H = \frac{\pi}{6}\sqrt{\frac{11}{2}\frac{T_F^2}{M_P}} \simeq 2.48 \times 10^{-22} \text{ MeV}. \tag{5.23}$$

Now, in order to use the result we obtained previously for the X decay rate, we must re-write it in the co-expanding frame. To do so, we simply multiply by a factor of $\gamma^{-1} = M_X/E_X$ (since $\Gamma = \tau^{-1}$, i.e. the decay rate is the inverse of the e-folding time, which transforms under Lorentz transformations as $\tau \to \gamma\tau$), where $E_X \simeq (3+3)\,\frac{1}{2}T_F$ when freeze-out occurs. This gives

$$\Gamma_{CE} = \Gamma \times \frac{M_X}{3T_F} = \frac{\alpha_X M_X^2}{3T_F}\left(1 + \frac{1}{2}c_W^2\eta^2\right)^2 \tag{5.24}$$

Finally, requiring that $H > \Gamma_{CE}$ at freeze-out gives

$$\frac{\pi}{6}\sqrt{\frac{11}{2}\frac{T_F^2}{M_P}} > \frac{\alpha_X M_X^2}{3T_F}\left(1 + \frac{1}{2}c_W^2\eta^2\right)^2 \tag{5.25}$$

or

$$\alpha_X M_X^2 < \frac{\pi}{2}\sqrt{\frac{11}{2}\frac{T_F^3}{M_P}}\left(1 + \frac{1}{2}c_W^2\eta^2\right)^{-2} = 5.20 \times 10^{-22}\ \text{MeV}^2\left(1 + \frac{1}{2}c_W^2\eta^2\right)^{-2} \tag{5.26}$$

Clearly, for the values of η in which we are interested, this additional factor does little to change the overall graph; this is illustrated in Figure 5.1.

In the case where $\eta = 0$, this is off by a little less than an order of magnitude from the result quoted in [1]:

$$\alpha_X M_X^2 < 4.2 \times 10^{-21}\ \text{MeV}^2. \tag{5.27}$$

However, there are some differences between what is presented here and what is found in [1]: 1) for the total decay rate, they use $\Gamma = \frac{1}{6}\alpha_X M_X$ (for reasons unknown); 2) to go to the co-expanding frame, they multiply by $M_X/2T_F$, instead of $M_X/3T_F$ (a minor difference); 3) they consider only one generation of neutrinos; 4) they use the condition $\Gamma_{CE} = 1/t$, where t is the age of the universe, to obtain their bound (recall that $H = 1/2t$). Making these changes in the above calculation reproduces the value obtained in [1].

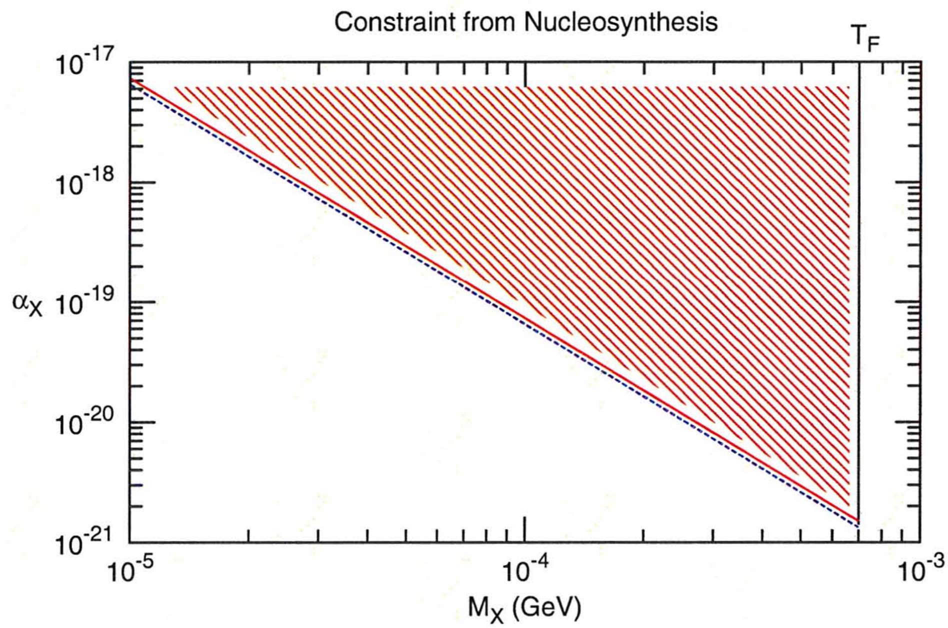

Figure 5.1: Constraint obtained from nucleosynthesis considerations - the solid (red) line corresponds to $\eta = 0$; the dashed (blue) line corresponds to $\eta = 0.4$.

Chapter 6

Summary and Conclusion

In this chapter, we compile the results obtained throughout the previous chapters, in order to obtain a global perspective on the relevant bounds at various values of the kinetic mixing parameter, η (specifically, $\eta = 0,\ 0.001,\ 0.01,\ 0.1,\ 0.3$). First, we focus on the bounds near the Z pole in order to prioritize the bounds in this mass region. From here, we consider the relevant bounds over the mass range from $M_X = 1$ keV to $M_X = 10$ TeV. Some concluding remarks are then presented.

6.1 Near the Z pole

In order to better understand which bounds dominate when the X boson mass is near the Z pole, we superpose those bounds obtained throughout the previous chapters which are relevant in the mass range $M_X \in [10, 10^4]$ GeV. These plots are presented in Figure 6.1.

For $\eta = 0$, note that there is no bound from Z decay and that the bound from σ_{had} is not very useful since the bounds from ν-scattering are stronger at every M_X. The $\nu - N$ bound dominates above the Z pole but, since it is only reliable down to $M_X \simeq 25$ GeV, the $\nu - e^-$ bound dominates for smaller M_X.

As η increases, the Z decay bound is consistently superseded by stronger constraints. Although difficult to denote graphically, the thin allowed regions in the Z decay bound for $M_X < M_Z$ are also ruled out by the stronger constraints. At (and near) $\eta = 0.01$, the σ_{had} bound develops a feature that excludes M_X values near the Z pole (recall that bounds near the Z pole for this value of η are indeed reliable; see Section 2.3 for details). For $\eta > 0.1$, the $\nu - N$, $\Gamma_{l^+l^-}$ and σ_{had} bounds develop an exclusion of all $M_X \lesssim 110$ GeV when α_X is small.

In the next section, we present all of the bounds obtained here over the range from 1 keV up to 10 TeV. For the purpose of clarity, we shall only plot the dominant bounds near the Z pole. These are

- for $\eta = 0, 0.001$: $\nu - e^-$, $\nu - N$;

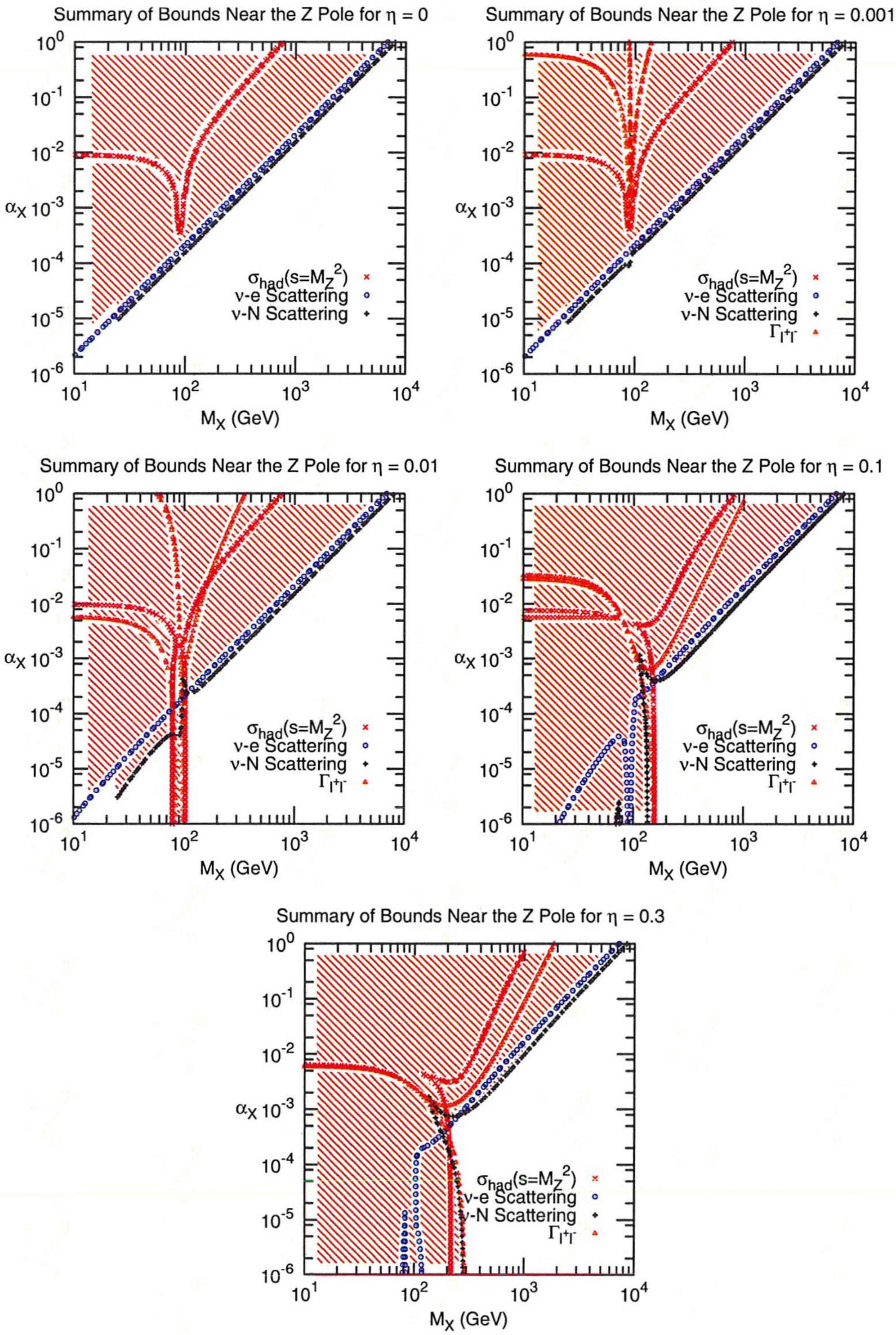

Figure 6.1: Summary of the constraints relevant near the Z pole. Here, we plot the bound on α_X as a function of M_X for various values of η. Excluded regions have been shaded out.

- for $\eta = 0.01$: $\nu - e^-$, $\nu - N$, σ_{had};
- for $\eta = 0.1, 0.3$: $\nu - N$, σ_{had}.

6.2 Summary of Constraints

Given the above dominant constraints near the Z pole, we are now in a position to present the dominant constraints over the entire mass range of interest here. This is shown in Figure 6.2.

For $\eta = 0$, the dominant bounds are simply the bounds from neutrino scattering, along with the nucleosynthesis constraint. As η increases, the $\nu - e^-$ bound develops an exclusion of all M_X below some η-dependent value, in the limit of small α_X. However, these exclusions only apply in the mass range for which we trust our bound, and so in the plots for $\eta = 0.001$ and $\eta = 0.01$, the region 1 MeV $< M_X <$ 10 MeV, as well as the region below the nucleosynthesis bound, are *not* excluded.

In the graph for $\eta = 0.01$, the σ_{had} bound excludes a region about the Z pole for small α_X. In situations such as these, the bounds are extended below the region in which they were originally considered using dotted lines. It has been checked that these bounds do, in fact, follow this behaviour.

For $\eta > 0.04$, the bound derived in Section 3.1 related to the W mass (roughly given by requiring $(\eta/0.5) \times (1 \text{ TeV}/M_X) \leq 1$, as in [22]) excludes below the corresponding mass *for all values of* α_X. Thus, for $\eta = 0.1$ and $\eta = 0.3$, this bound is the dominant one for $M_X <$ 250 GeV and $M_X <$ 635 GeV, respectively. For M_X outside of these regions, the bound appears to be dominated by $\nu - N$ bound, which roughly constrains the ratio α_X/M_X^2. For completeness, we have included the sub-dominant bounds in the region $M_X \ll M_Z$ (i.e. the $\nu - e^-$ and nucleosynthesis bounds) in the plots for $\eta = 0.1$, 0.3.

6.3 Concluding Remarks

In this report, we have presented updated bounds on the gauge coupling of a $U(1)$ field X^μ coupled to $B - L$, as a function of its mass over the range 1 keV$< M_X <$ 10 TeV. We have also considered the changes that occur in these bounds as the strength of a kinetic mixing term (between the X^μ field and the hypercharge B^μ) is varied.

We find that, for small values of the kinetic mixing parameter η, the strong bound from nucleosynthesis at $M_X < T_F \simeq 0.7$ MeV is replicated at $M_X \gtrsim 10$ MeV by a bound from neutrino-electron scattering that excludes arbitrarily small gauge couplings. For $\eta > 0.04$, the bound obtained by considering the effect of the X boson on the W mass is dominant and masses $M_X <$ 100 GeV are excluded (independent of α_X). As η increases, this bound improves and the only other relevant bound (in the mass region $M_X = O(1 \text{ TeV})$)is found to be the constraint obtained from neutrino-nucleon scattering.

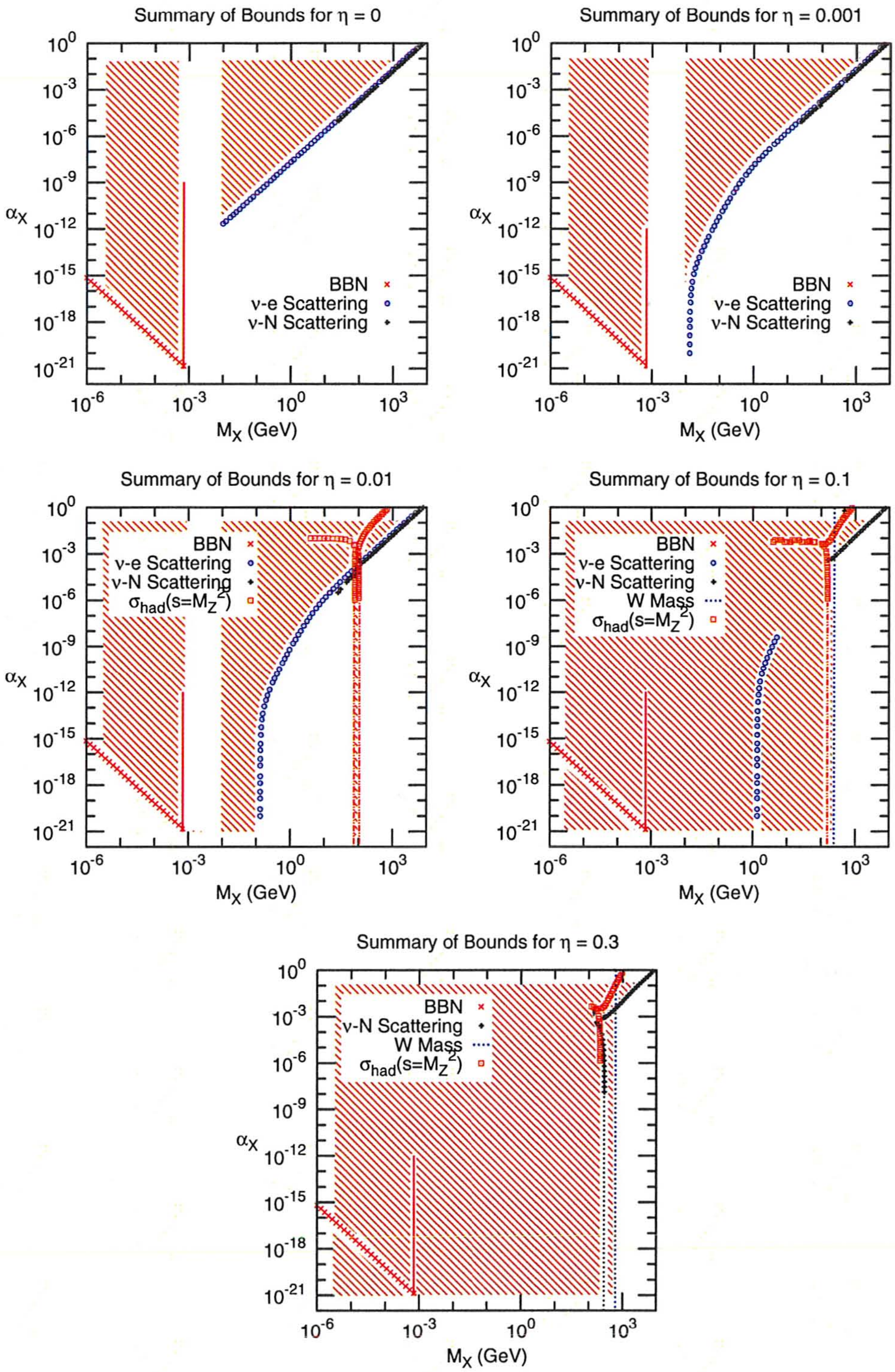

Figure 6.2: Summary of the constraints considered in this report. Here, we plot the bound on α_X as a function of M_X for various values of η. Excluded regions have been shaded out. Note that, near the Z pole, only the dominant constraints are plotted (see Figure 6.1 for details).

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