RUSSELL'S NOTION OF IMPLICATION

RUSSELL'S NOTION OF IMPLICATION: AN ESSAY IN THE HISTORY OF LOGIC

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ABSTRACT: An attempt has been made in this thesis to analyse critically the notion of implication in the logical works of Bertrand Russell. An exposition of Russell's notion of implication is presented in one part, and in another part, the author discusses various substantive issues that are raised by Russell's treatment of implication. Besides the discussion of certain philosophical problems , the author has tried to understand Russell's notion of implication in terms of a developmental study in the history of logic.

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TABLE OF CONTENTS

LIST OF DIAGRAMS

A: INTRODUCTION

For the greater part of this century, the issue of entailment has created quite a flurry in the field of philosophical logic. By the "issue of entailment", I am alluding to the meaning and function of "if-then" and its metalinguistic counterpart, implication. However, in this particular paper, I am not interested in the controversy of entailment per se concerning the truth-functional or non-truth-functional aspects of "if-then". Rather than to debate this central issue of entailment, my intent here is of a more specific nature. Since various authors contend that the issue of entailment stems from the definition given to implication in Principia Mathematica and since Russell has co-authored this work, I therefore propose to give an interpretative exposition and critical evaluation of the notion of implication in the works of Bertrand Russell.[1] Such an undertaking must trace the development of Russell's position with respect to the notion of implication in its proper, historical context. Furthermore, this also means that the examination of the notion of implication in the works of Russell beginning somewaht prior to The Principles Of Mathematics and ending with the Introduction To Mathematical Philosophy must be reasonably thorough. Inasmuch as the technical works of Russell are considered to be difficult to understand, my main concern throughout this paper will be to explain just what Russell intends by the notion of implication. Although I have provided a table of contents at the beginning of the paper, I would like to make a few general comments concerning the format of the paper for the benefit of the reader. Firstly, I have dealt with Russell's notion of implication in its chronological order; this has been done in order to point out any changes in Russell's views with respect to the notion of implication. Secondly, my paper can be divided roughly into two parts - one concerning exposition and the other concerning critique. I have structured the paper in this manner in order to minimize the tendency which readers often have to confuse the opinions of a particular philosopher with the views of a philosopher's critic; as a result, I will attempt to refrain as much as possible from making any critical remarks in the expository section of the essay. Without any further preliminaries, therefore, I will move on to the exposition of Russell's notion of implication.

B: EXPOSITION

B.1: IMPLICATION PRIOR TO THE PRINCIPLES OF MATHEMATICS

There are several earlier drafts of The Principles Of Mathematics which were written before the International Congress of Philosophy in Paris in July 1900;[2] however, there does not seem to be any reference whatsoever to the notion of implication in these earlier drafts. Nevertheless, this Congress was the pivotal point of Russell's intellectual career because it was at this Congress that Russell met an Italian mathematician by the name of Giuseppe Peano.[3] By the scrutiny of Peano's work, Russell acquired a notation which, when applied to certain problems of mathematics, had the resources to dispel fundamental perplexities. It was due to Peano's notation therefore that Russell was stimulated to invent a novel approach with respect to mathematical logic: "And in the course of discovering these answers, I [Russell] was introducing a new mathematical technique, by which regions formerly abandoned to the vagueness of philosophers were conquered for the precision of exact formulae."[4]

Within a few months of his familiarity with Peano's style of proof and notation, Russell began the actual writing of <u>The Principles Of Mathematics</u>; the first draft was soon completed just before the end of 1900.[5] However, in the spring of the following year, Russell unearthed a logical antinomy with respect to the class of all classes that are not members of themselves.[6] This contradiction created such an upheaval in Russell's work, that several parts of the penultimate draft of <u>The Principles Of Mathematics</u> were revised;[7] moreover, Russell deemed the contradiction to be so overwhelming that he "decided to finish <u>The Principles Of Mathematics</u>, leaving the solution in abeyance".[8]

When Russell wrote the final draft of <u>The Principles Of Mathematics</u>, it seems that he simply inserted the pages of the penultimate draft into the pages of the final draft, and corrected the pages of the penultimate draft whenever it was required by crossing out words or lines and writing the corrected information above such deletions. However, in the final draft of <u>The Principles Of Mathematics</u>, there are no such deletions to be found in the chapter concerning the notion of implication. In the writing of the manuscript of <u>The Principles Of Mathematics</u>, Russell labelled each page in the upper left hand corner. When Russell inserted the pages of the penultimate draft into the pages of the final draft of The Principles Of Mathematics, Russell either wrote another label above the former label, or else he crossed out the former label and wrote another label above the crossed-out label.[9] If one examines the pages of the final draft of The Principles Of Mathematics, one will observe that the labels in the upper left hand corner of the pages concerning the chapter on the notion of implication have not been replaced by other labels. Nevertheless, Part I of the penultimate draft of The Principles Of Mathematics, it would seem, did have a chapter concerning the notion of implication; this is borne out by the table of contents of the penultimate draft. Unfortunately, this chapter of the penultimate draft concerning the notion of implication is not now known to be extant.

There is an unpublished article by Russell entitled "Necessity And Possibility" which probably was written sometime around 1900 or 1901; in any event, it would seem that this undated article was written after the meeting of the International Congress of Philosophy in Paris where Russell encountered Peano.[10] In his paper, "Necessity And Possibility", Russell argues against the requirement of modal operators such as "necessity" and "possibility" in formal logic. It is maintained by Russell that the modalities have only an epistemological or psychological significance; as a result, they are notions with which logic need not be concerned.[11] In the early parts of his paper, Russell discusses three positions with respect to the concept of necessity: (1) Bradley's position that if a proposition, q, is implied by a proposition, p, then q can be said to be necessary;[12] (2) Bosanquet's position that no propositions are necessary except true hypothetical or disjunctive propositions;[13] (3)Moore's position that a proposition is said to be more or less necessary according to its logical priority to other propositions (p is said to be logically prior to q if q implies p but p does not imply q).[14] Against Bradley's position, Russell states that (1) leads to the conclusion that every proposition, no matter what it may be, is necessary since any proposition can follow from some premiss or premisses. With respect to Bosanquet's position, Russell asserts that it is simply not the case that every true hypothetical or disjunctive proposition is necessary. Finally, concerning Moore's position, it is claimed firstly that it cannot accommodate an implication in which a true proposition is implied by every proposition and a false proposition implies any proposition; secondly, although true propositions are logically prior to false propositions, the criterion of logical priority does not help us to discriminate among true propositions, especially since all true propositions have the same degree of necessity thereby.

Having dismissed the positions of Bradley, Bosanquet and Moore, Russell introduces the pre-Kantian meaning given to "necessary proposition" - that is, if p is a necessary proposition, then the truth of p can be deduced from the law of non-contradiction; with respect to this view of necessity, Russell contends that it can be seen to concur with modern logic with slight modification however.[15] Furthermore, it is claimed by Russell that the customary identification of necessity with analyticity needs to be qualified in two ways: (1) if the meaning of analytic is to apply correctly to propositions, then the meaning of analytic must be altered; (2) "p is deducible from q" does not mean the same as "p is implied by q", and some other meaning must be found for "p is deducible from q".[16]

The difficulty with respect to deducibility, Russell maintains, is due to the notion of implication. According to Russell's article, "Necessity And Possibility", "p implies q" or "if p, then q" is equivalent to "p is not true or q is true". Russell does not argue this meaning of implication in detail although he does provide two means of justification for it: (1) logical arguments via the principle of exportation (if p and q implies r, then p implies that q implies r) and equivalences of the principle that a true proposition is implied by every proposition, the principle of simplification (if p and q implies p, then p implies that q implies p); (2) an argument from authority claiming that both Bradley and Shakespeare favour this interpretation of implication.

Accepting the fact that "p implies q" is equivalent to "p is not true or q is true", Russell goes on to claim that, on such grounds, "p is implied by q" cannot mean the same as "p is deducible from q". Russell's strategy is firstly, to arrive at the meaning of "p is deducible from q" and secondly, to show thereby that there is a logical difference between "p is deducible from q" and "p is implied by q". To say that q is deducible from p means that there exists a set of principles of deduction from which it can be demonstrated that p implies q. Thus, since "deducible from" is defined in terms of the principles of deduction and since these principles employ the notion of implication, it is not permissible to substitute "implied by" for "deducible from" due to the charge of circularity; it is concluded by Russell that the notion of implication is therefore more primitive than the notion of deducibility and that the latter notion is derived from the former notion.[19]

Having drawn a logical distinction between "p is implied by q" and "p is deducible from q", Russell proceeds to examine the notion of analyticity. Traditionally, Russell states, to assert that a proposition is analytic is to make two claims: (1) that in such a proposition, the notion of the subject contains the notion of the predicate; (2) that such a proposition is deducible from the law of non-contradiction or from the "laws of thought".[20] Russell dismisses (1) since he considers it not to be germane to the context of his paper. With respect to (2), however, Russell maintains that this condition is too narrow. In short, there are two points which Russell makes with regards to the second condition: (1)very few analytic propositions can be deduced from the so-called "laws of thought" (identity, non-contradiction and excluded middle); (2) the "laws of thought" must be expanded to form the "laws of logic", and it is somewhat arbitrary which "laws" we choose to form the "laws of logic" and the consequences of such "laws".[21] As a result, an analytic proposition can be defined to be a proposition which is deducible from the "laws of logic". To this extent, therefore, Russell contends, Kant was wrong in his pronouncement that the statements of pure mathematics are synthetic since this, in turn, denies the thesis that pure mathematics can be deduced solely from the "laws of logic".[22]

Having defined the notion of analyticity, Russell goes on to link this definition of analyticity with the notions of deducibility and implication. To state that "p implies q" is analytic is to state that q is an analytic consequence of p; thus, the proposition, "q is an analytic consequence of p", is equivalent to the proposition, "q is deducible from p". Consequently, if a conclusion is inferred validly from a premiss, the conclusion is an analytic consequence of the premiss; this means that the implication involved in such a case is analytic. From the definitions of the notion of implication, deducibility and analyticity, Russell returns to the topic of the modalities supporting the position that the modalities lie outside the boundaries of formal logic.

There are twelve pages of unpublished notes entitled "Lecture II: Logic of Proposition" intact in the Bertrand Russell Archives written around 1901 or 1902 when Russell was a temporary lecturer on mathematical logic at Cambridge. These rough notes contain the solution to certain problems of logic, as well as a discussion of Cantor's Theorem. Perhaps, the only relevant comment with regards to the notion of implication is a remark by Russell to the effect that for the theory of deduction, one is not concerned with mere arguments from the general to the particular; moreover, most writers of mathematics, Russell contends here, devote too much time to equations, instead of implications. In the theory of deduction, one must concentrate on implication since it is fundamental to all deduction.[25]

The last article with respect to the notion of implication prior to the publication of The Principles Of Mathematics which I wish to discuss is entitled "Recent Italian Work On The Foundation Of Mathematics"; although this paper seems to have been written around 1902 for an English journal of philosophy, it was never published.[26] The purpose of the article, Russell states, is to draw attention to the Italian school of philosophical mathematicians founded by Peano.[27] According to Russell, Peano conceived of symbolic logic along the lines of Leibniz - that is, if symbolic logic can capture "the essence of deductive reasoning", then a certain set of rules can be established to cover all forms of correct deduction.[28] Thus, symbolic logic, says Russell, must be the most fundamental part of mathematics upon which all the other demonstrations of mathematics must be grounded.[29]

In his article, "Recent Italian Work On The Foundations Of Mathematics", Russell states that the primary object studied in symbolic logic is the relation of implication between propositions. The relation of implication, however, cannot be defined, and in his logic, Peano takes implication to be one of his indefinables. As in his unpublished notes, "Lecture II: Logic Of Propositions", Russell makes the claim in this article that formal logicians are preoccupied too much with mathematical equations; logicians could spend their time more profitably if they devoted their pursuits to implications.[30]

One of the superiorities of Peano's logic, Russell claims, is that Peano differentiates between $\in$ ("$x \in a$" is equivalent to "x is an a") and $\supset$ ("$p \supset q$" is equivalent to "p implies q" or "if p, then q"). It should be noted that $\supset$ is a transitive relation whereas $\in$ is not. For example, concerning $\supset$, if all a is b and all b is c, then all a is c; however, with regards to $\in$, if x is an a and a is a b, it does not follow that x is a b.

A counter example of the transitivity of ϵ is the following: 2 is a number and number is a class, but 2 is not a class.[31] In the rest of his article, Russell summarizes some of the contributions which Peano has made to the field of symbolic logic. In the main, Russell's discussion of Peano's contributions is not germane to Russell's notion of implication. Nevertheless, there is a mention by Russell of a problem concerning propositions containing a variable. A proposition containing a term x in which x can be any member of a certain class is asserted to be true only when x holds for every member of the class in question. When x is a variable in an implication, it is written by Peano as a suffix to the sign of implication. Russell, however does not analyse the problem of implications containing a variable any further since such a discussion would digress from the importance of Peano's other work.[32]

B.2: IMPLICATION IN THE PRINCIPLES OF MATHEMATICS

According to Russell, the aims of The Principles Of Mathematics are two-fold: (1) to demonstrate that all pure mathematics follows from the principles of symbolic logic; (2) to specify exactly what the principles of symbolic logic are.[33] In the opening paragraphs of the preface to The Principles Of Mathematics, Russell elaborates on these two aims. With respect to (1), Russell's purpose is to sketch a proof in which it is shown that it is possible by means of indefinable, logical concepts and definitions to reconstruct the entirety of pure mathematics. In short, this is the thesis of logicism that all pure mathematics is derivable from symbolic logic, and it is from the point of view of logicism itself that the Russellian enterprise must be viewed. In his book on Leibniz, Russell had argued that all of Leibniz's philosophy can be reduced to a small number of logical premisses;[34] in his unpublished articles, "Necessity And Possibility" and "Recent Italian Work On The Foundations Of Mathematics", Russell had mentioned the possibility that pure mathematics may be derived from the theory of deduction or formal logic.[35] Russell, in The Principles Of Mathematics, is prepared to delineate the steps whereby the whole of pure mathematics is to be reduced to symbolic logic. To this intent, therefore, Russell devotes Parts II to VII as an outline of such a proof. Part I is reserved for Russell's second aim which is to provide an explanation of the principles of symbolic logic; such an explanation concerns the indefinable concepts upon which pure mathematics is based.[36] Thus, Russell's second task in The Principles Of Mathematics is to familiarize the reader with the indefinables of symbolic logic in such a manner that they are as clear to the mind as "redness or the taste of a pineapple".[37]

The actual text of The Principles Of Mathematics begins with a rather strange claim to the reader who is unaware of Russell's thesis of logicism. Russell contends in his opening paragraph that pure mathematics is simply the class of all propositions having the form, "p implies q", in which p and q are propositions containing the same variables, and p and q contain no constants save logical constants. We are also told that logical constants are definable in terms of certain primitives - one of which is implication.[38] In considering Euclidean geometry as a branch of pure mathematics, Russell states that in terms of pure mathematics, it can be said only that the propositions of Euclidean geometry follow from a Euclidean axiom set; Euclidean geometry is founded on implication just as much as Riemannian or Lobacheveskian geometry. In the light of pure mathematics, all geometries are equally true since they assert implications only.[39]

One finds in pure mathematics, says Russell, assertions of the form: if p is true of any entity x, then it follows that q is similarly true of that entity; neither p nor q is asserted by itself as separate from an entity. Propositions which are asserted in mathematics are of the relational form, "if p, then q", and the relation is referred to by Russell as <u>formal implication</u>.[40] At this point in Russell's presentation, the reader is confounded somewhat since the notion of implication has been introduced initially by Russell to be fundamental to all pure mathematics, and to be of the form, "if p, then q"; yet, if implication is of the form, "if p, then q", what special status or meaning has formal implication? Moreover, is formal implication also fundamental to pure mathematics or is the notion of implication equivalent to the notion of formal implication? In short, what function and meaning does formal implication have? Russell does not satisfy the reader at this juncture due to the complexity of his logical apparatus - nevertheless, he does drop hints every so often; for example, Russell states at one point in Chapter I of Part I that in all mathematical propositions, the words, "any" and "some", occur, and that these words are the characteristics of a variable and formal implication.[41]

Chapter II of Part I of The Principles Of Mathematics which contains a brief sketch of the divisions of symbolic logic makes further mention of implication. Symbolic logic, according to Russell, incorporates a small number of primitive, indefinable logical constants. For Russell, symbolic logic is "the study of the various general types of deduction";[42] it should be noted also that Russell does not distinguish between inference and deduction.[43] Logical constants are those which are indefinable and by means of which all other constants are defined; in order to arrive at the entirety of pure mathematics, Russell claims that eight or nine of such logical constants are required.[44] It is suggested by Russell that for a technical study of symbolic logic, one finds it convenient to have the indefinable notion of formal implication - that is, propositions having the form, "ϕx implies ψx for all values of x" in which ϕx and ψx, for all values of x, are propositions. Besides listing other indefinable notions of the symbolic vocabulary, Russell asserts that symbolic logic requires the indefinable notion of implication between propositions containing no variables.[45] Here, we obtain another clue to Russell's meaning of implication;

there are at least two types of implication: formal implication and another sort of implication having no variables.

Russell states that symbolic logic embodies three areas: the calculus of propositions, the calculus of classes and the calculus of relations.[46] In the calculus of propositions or propositional calculus, all propositions assert a material implication in both antecedent and consequent. The hypothesis for all propositions of the propositional calculus, Russell contends, is of the form, "p implies p"; this hypothesis means that p is a proposition in which p can only be a propositional variable - that is, if p is a proposition, we may substitute for p any proposition of the form, "Socrates is a man", but not a propositional function such as, "x is a man". Thus, Russell identifies material implication with the latter sort of implication mentioned in the above paragraph - namely, implication between propositions having no variables.

According to Russell, the purpose of the propositional calculus is to focus upon the relation of implication between propositions. One should note, Russell states, that this implication is not formal implication; formal implication deals exclusively with propositional functions in which one propositional function implies another propositional function for all values of the variable. Formal implication is not studied directly in the propositional calculus since it is characteristic of all pure mathematics; material implication, on the other hand, is studied explicitly in the propositional calculus since material implication sets the propositional calculus apart from the other subject matter of symbolic logic.[47]

Literature in the field of logic prior to the publication of The Principles Of Mathematics often has confused, Russell maintains, this distinction between material implication and formal implication. In some cases, Russell concedes that the distinction is blurred. For example, when we say, "Socrates is a man implies Socrates is a mortal", it is felt that Socrates is not an individual constant, but a variable, since no matter which man we care to choose he also will be a mortal. Hence, Russell asserts, although this implication is strictly speaking material, it can be considered to be formal. It is due to such ambiguous cases of implication that the distinction between material implication and formal implication has been confused; it is no wonder, therefore, says Russell, that logicians have not recognized this distinction in a clear-cut manner.[48] Now, as to whether formal implication is definable in terms of material implication, Russell states that this is another important issue which must be discussed in a separate chapter concerning the nature of implication itself.[49]

In the next section of the propositional calculus, Russell defends the position that implication is indefinable. The reader should be aware at this point that when Russell is talking about implication, he usually is talking about material implication, not formal implication. If the reader has not apprehended this equivalence of implication with material implication, then perhaps the reader will be led astray in thinking that there are three types of implication: material implication, formal implication, and another general sort of implication; however, this is not the case. To return to Russell's contention that material implication is indefinable, Russell asserts that if one attempts to define material implication by saying something similar to, "if p implies q, then q is true if p is true" (i.e. the truth of p implies the truth of q) or "if p implies q then q is false if p is false" (i.e. the falsehood of p implies the falsehood of q), then although the notions of truth and falsehood have been introduced, one has only defined material implication via material implication; of course, Russell states, this type of definition is open to the charge of circularity. Moreover, if one tries to employ the notions of negation and disjunction in a definition of material implication of the form, "p implies q is equivalent to p is false or q is true", then one does not escape the charge of circularity since equivalence itself involves mutual implication. Consequently, Russell concludes, material implication is indefinable.[50]

From his discussion of material implication, Russell draws three curious inferences concerning material impli- cation between propositions: (1) for any two propositions, one of these propositions must imply the other; (2) false propositions imply all propositions; (3) true propositions are implied by all propositions.[51] In his article, "Necessity And Possibility", Russell had argued against Moore's meaning of necessity since it could not accommodate an implication having (2) and (3).[52]

It has been mentioned hitherto that the formulas of the propositional calculus are prefaced by Russell with a hypothesis of the form, "p implies p". This hypothesis is to ensure that the letters contained in the formulas of the propositional calculus stand for propositions. Russell claims that although implication may be indefinable, the notion of "proposition" is definable. To begin with, says Russell, we know that a proposition implies itself; consequently, that which is not a proposition cannot imply anything. Thus, "p is a proposition" is equivalent to "p implies p".[53]

The propositional calculus, therefore, employs two indefinable notions: formal implication and material implication. In his formal attempt to axiomatize the propositional calculus, Russell proposes ten axioms and three definitions in which each axiom incorporates a main implication (formal implication) and subordinate implications (material implication); the former implications are denoted by "if-then" while the latter implications are denoted by "implies". For example, in the eighth axiom, "if p implies p and q implies q, then if p and q implies r, then p implies that q implies r", the principal implication is formal while the subordinate implications are material. What is important however in the axioms of the propositional calculus is not the stipulation that "if-then" is formal implication and "implies" is material implication since "if-then" and "implies" are interchangeable, but that formal implication is the main implication and material implication is the subordinate implication.[54]

Russell sets the stage for his major discussion of implication by examining a dispute between Schröder and MacColl. Schröder maintained that if p, q and r are propositions, "p and q implies r" is equivalent to " p implies r or q implies r". MacColl, however, claimed that while the latter implies the former, the former will not imply the latter. Russell does not view this to be a genuine dispute due to a confusion concerning implication itself. Schröder is talking about propositions and the implication between propositions - namely, material implication; on the other hand, MacColl is talking about propositional functions and formal implication. In spite of the argument put forward by Schröder in defence of the above equivalence, MacColl replies that in the case where p and q be mutually contradictory, and r is the null proposition, p and q implies r but neither p nor q implies r. Russell holds that the positions of Schröder and MacColl are compatible since both men are using different forms of implication. After substantiating the position of Schröder, Russell also attempts to justify MacColl's position. According to Russell, MacColl presupposes propositional functions and formal implication. Furthermore, a null propositional function is false for all values of x; the class of x's which satisfies the function is the null class - that is, a class having no terms. Consequently, let Λ (the class) stand for r, let ϕx stand for p and let $\sim\phi x$ stand for q where ϕx is any propositional function. Now, $\phi x \,\&\, \sim\phi x$ is false for all values of x, and therefore, $\phi x \,\&\, \sim\phi x$ implies Λ. However, Russell argues, ϕx is not always false by itself; similarly, $\sim\phi x$ is not always false by itself. As a result, it can be said that neither ϕx implies Λ nor $\sim\phi x$ implies Λ. Russell concludes on the Schröder-MacColl dispute by claiming that "p and q implies r is equivalent to p implies r or q implies r" is only logically true in the propositional calculus; in the class calculus, it happens to be false. As a result, says Russell, the Schröder-MacColl dispute rests on a confusion between material implication and formal implication.[55]

Chapter III of Part I of <u>The Principles Of Mathematics</u> is the focal point concerning the notion of implication. Russell calls this chapter, "Implication And Formal Implication", and once more, it should be assumed that by the word, "implication", Russell means material implication. To every kind of deduction, Russell maintains, two types of implication are necessary - material implication and formal implication. Russell's intent in Chapter III is to elucidate and to distinguish both forms of implication, and to propose certain methods of analysing formal implication.[56]

In the past, both philosophers and logicians had tried to introduce the notion of mind via the traditional laws of thought into the fields of logic and mathematics.[57] Here, Russell distinguishes psychology from logic, and states that psychology has no place in logic, the study of inference or "the study of the various general types of deduction".[58] When one proposition validly is inferred from another proposition, says Russell, this inference occurs due to a relation which holds between the two propositions regardless as to whether the mind is aware of the inference or not - this relation is what Russell calls material implication. Futhermore, the conditions of truth or falsehood are irrelevant to the relation of material implication: "The relation [material implication] holds, in fact, when it does hold, without any reference to the truth or falsehood of the propositions involved."[59]

Concerning material implication, Russell states that we are prone to run contrary to our intuitions in so far as a false proposition implies every proposition and a true proposition is implied by every proposition. One can make an analogy with respect to propositions and material implication. According to Russsell, propositions are akin to a set of lengths in which each length may have a different magnitude; following the analogy further, material implication may be compared to a relation such as "equal to" or "greater than" among such lengths. It would not be said ordinarily that "2 + 2 = 4" can be deduced from "Socrates is a man", or that both of these propositions are implied by "Socrates is a triangle". The reason, Russell claims, why such implications are not considered is due to a preoccupation with formal implication. There is a tendency to suppose that "Socrates is a man" is not talking about a distinct individual of a historical epoch, but rather that Socrates stands for any man. However, says Russell, Socrates is not a variable, and similarly, it is material implication, not formal implication, that holds between propositions. Indeed, although material implication may seem to be prime facie contrary to the intuitions, it does not follow that it is therefore unfounded.

Russell's fourth axiom of the propositional calculus states that if the antecedent of an implication is true, it may be dropped and the consequent added - that is, given that "p implies q" and p are true, q may be asserted to be true (p implies q, p, $\therefore$q). This principle also may be employed with formal implication in which given an instance of the variable in the antecedent, we may assert the same instance of the variable in the consequent. In other words, if ϕx implies ψx for all values of x and given ϕa, we may assert ψa. Russell's claim is that this indemonstrable principle cannot be captured strictly by formal methods, and as a result, it "points to a certain failure of formalism in general".[61]

Continuing in this line of thought with respect to the fourth axiom of the propositional calculus, Russell introduces Lewis Carroll's puzzle, "What The Tortoise Said To Achilles". The fourth axiom of the propositional calculus permits the inference that if p and q are propositions, and "p implies q" and p are true, then q is also true. Carroll's puzzle denies the possibility of such an inference: "But the puzzle in question [Lewis Carroll's puzzle] shows that this is not the case [p implies q, p, $\therefore$q], and that, until we have some new principle, we shall only be led into an endless regress of more and more complicated implications, without ever arriving at the assertion of q."[62]

According to Russell, the fourth axiom of the propositional calculus incorporates the notion of "therefore" which is unlike the notion of "implies" insofar as "therefore" is a relation holding between different sorts of entities. To this extent, Russell introduces a distinction between an asserted proposition and an unasserted proposition; grammatically speaking, this distinction is analogous to the distinction of a verb from a verbal noun as in the respective examples, "A is greater than B" and "A's being greater than B". Although it would seem that the notion of assertion is somewhat psychological, Russell expresses his concern to arrive at a non-psychological meaning of assertion. In the proposition, "p implies q", an implication is asserted but both p and q are not asserted; strictly speaking, the p and the q in "p implies q" are different from the separate propositions, p and q. For Russell, the important question is the following: how does a true proposition differ from a false proposition? In Russell's opinion, although true propositions and false propositions are quite similar, they differ in the respect that true propositions are asserted while false propositions are not asserted. Russell admits that there are difficulties with this theory insofar as it is claimed that an assertion changes a proposition; consequently, it would follow from this theory of assertion that an unasserted proposition cannot be true since if the proposition would be asserted, it would become a different proposition. On these grounds, Russell states, the theory of assertion cannot be true; for example, the p and the q in "p implies q" are unasserted, but they still may be true nevertheless. On the other hand, says Russell, we also want to maintain some logical distinction between asserted and unasserted propositions.

Perhaps, the initial step in the solution of Lewis Carroll's puzzle, Russell concludes, would be to claim that the relation, "therefore", holds only between asserted propositions while the relation of implication holds between unasserted propositions.[63]

In section thirty-nine of Chapter III of The Principles Of Mathematics, Russell attacks the syllogistic view that an inference must be based upon two or more premisses and a conclusion. In his unpublished article, "Necessity And Possibility", Russell had commented on the importance of the syllogism: "[It is] a subject scarcely more useful or less amusing than heraldry."[64] It is Russell's contention in The Principles Of Mathematics that the syllogistic view of inference is both complicated and unnecessary due to two reasons: (1) by the rule of exportation, axiom eight of Russell's propositional calculus, an implication containing a finite number of propositions in the antecedent by the connective of conjunction can be made equivalent to a series of single implications (i.e. if p and q implies r, then p implies that q implies r); (2) "every simultaneous assertion of a number of propositions is itself a single proposition."[65] Russell's expansion of his second reason is that if k is a class of propositions, all the propositions in k can be asserted by the single proposition;

"for all x, if x implies x, then 'x is a k' implies x" or "every k is true". Although Russell's expansion is confusing, perhaps what Russell is trying to say is that given a number of true propositions belonging to a certain class, we can always formulate these asserted propositions into one proposition (i.e. "ϕa implies ψa, ϕb implies ψb,..., ϕn implies ψn" is equivalent to "for all values of x, if x is a ϕ, then x is a ψ"). I offer this interpretation of Russell's expansion of his second reason merely as an "educated" guess; when Russell mentions the expression, "the class k", it would seem that he is talking about the calculus of classes and formal implication rather than material implication. Nevertheless, Russell's general conclusion is that material implication is a relation which holds between two propositions, not a relation between a number of finite premisses and a conclusion.[66]

Russell begins his exposition of formal implication by avoiding the notion of a propositional function and concentrating on the concept of a particular instance. However, in an attempt to discuss the concept of a particular instance, Russell offers the example of the proposition, "x is a man implies x is a mortal for all values of x". Russell states that the proposition, "x is a man implies x is a mortal for all values of x" is equivalent to "all men are mortal", "every man is mortal" and "any man is mortal". There is a curious afterthought by Russell concerning these supposed equivalences; Russell asserts that though these may be equivalences, it is doubtful aa to whether they are the same. I understand this afterthought to mean that symbolically these propositions are equivalent (i.e. "for all values of x, ϕx implies ψx" where ϕx = x is a man and ψx = x is a mortal), but ordinary English still expresses subtle distinctions among them.[67]

By giving the example, "x is a man implies x is a mortal", Russell turns his attention to the notion of a variable and as to whether the variable, x, is to be restricted to instances which are men or not. Peano seems to have held the view that the variable must be restricted to instances which are men. If this is the case, Russell states, then the antecedent in this implication is trivial since it asserts that members of the class in question are to be men only; it follows that what is being asserted in the antecedent must be contained therefore in the consequent. Thus, under Peano's view of the variable, "x is a man implies x is a mortal" is reduced to "since every man is mortal, if x denotes any man, then x is mortal". By restriction of the variable, says Russell, formal implication disappears. Is it possible, however, Russell asks, to examine the proposition, "any man is mortal", without employing the notions of formal implication and variable? Peano's view of the restriction of the variable seems to alleviate a number of difficulties. In support of Peano, Russell again offers the same example used against the syllogistic view of implication - that is, the simultaneous assertion of all the propositions of some class k. This example cannot be asserted by the proposition, "'x is a k' implies x for all values of x". In Russell's opinion, this proposition concerning the simultaneous assertion of all the propositions of some class k is incomplete, since, if x is not a proposition, "x is a k" cannot imply x; the proposition in question must be prefixed with the hypothesis, "x implies x". Thus, it is claimed by Russell that conclusions represented by a single letter in the propositional calculus must be propositions; otherwise, Russell maintains, the asserted implication will be false due to the fact that only propositions can be implied. To this intent, Russell prefixes some of the axioms of the propositional calculus with the propositional stipulation of the form, " p implies p".[68]

Russell also claims that the hypothesis for formulas of the propositional calculus - that is, that the nonlogical symbols represent propositions - is needed only for terms having a principal implication, not for terms having a subordinate implication: "It should be noted that there may be any number of subordinate implications which do not require that their terms should be propositions; it is only of the principal implication that this is required."[69] In order to prove his point, Russell takes the first axiom of the propositional calculus: if p implies q, then p implies q. According to Russell, the first axiom of the propositional calculus holds regardless of the fact that p and q be propositions or not. Russell argues in this manner: if either p or q is not a proposition, then "p implies q" will be false; however, if this is the case, then "p implies q" still will be a proposition - consequently, the antecedent will be false and the consequent will be false, but the first axiom of the propositional calculus, "if p implies q, then p implies q", will be true. Now, if the principle of importation is applied to the first axiom of the propositional calculus, we will arrive at the following formula: "'p implies q', together with p, implies q". Using the standard symbols of the propositional calculus, this is a transition from "$(p \supset q) \supset (p \supset q)$" to "$[(p \supset q) + p] \supset q$". In Russell's opinion, this latter formula can only be true when p and q are propositions, and in order to be universally true, it must be prefixed by the two hypotheses, "p implies p" and "q implies q". Thus, Russell states, principal implications require hypotheses of the form, "p implies p", but subordinate implications do not require such hypotheses.[69]

Russell's discussion of the notion of a variable is presented in terms of a dilemma. If the variable is restricted, then how is it possible to analyse the proposition, "any man is mortal", since formal implication has disappeared? Furthermore, if the variable is not restricted, then some of the axioms of the propositional calculus will be false. Russell seems to take the position of a partial restriction of the variable; the repercussions of such a position are supposed to be two-fold: (1) it allows the examination of the proposition, "any man is mortal", with the use of formal implication; (2) it preserves the truth of the axioms of the propositional calculus.

Russell returns to the implication, "x is a man implies x is a mortal" in order to comment on the relationship of the variable to formal implication. It is maintained by Russell that the variable, x, must not be restricted in this implication, since, if it is restricted, we will be forced to state that no matter what entity we choose in the universe, it will be a man. The restriction of the variable does not account for the cases in which it will be false to state that such-and-such an entity is a man; if such a view of the restriction of the variable is permitted, then one will not be able to cope with the null-class or null propositional functions. Thus, where there is a restriction on the variable, we will know that the implication in question is not a formal implication unless the said restriction is removed and prefixed to the implication. Under Russell's view of implication, therefore, an unrestricted variable is a sign of formal implication.[70]

There is also a claim by Russell that the implication, "x is a man implies x is a mortal", does not assert a relation of two propositional functions, but is itself a single propositional function which is constantly true. It is not permitted therefore to vary the x in "x is a man" independently of the x in "x is a mortal". Both variables must be varied simulataneously.[71]

In the middle of section forty-two of Chapter III, Russell finally states what formal implication is. According to Russell, formal implication is not a single implication, but is a class of implications; formal implication is a variable implication in which no member of the class of implications contains a variable. In terms of modern symbolic logic, formal implication is the implication to be found in first order predicate calculus in such formulas as "$(x)\ (\phi x \supset \psi x)$".[72]

Diagram I

Formal Implication

$(x)(\phi x \supset \psi x)$	class or variable implication:
$\phi a \supset \psi a$	members of class implication
$\phi b \supset \psi b$	contain no variables and every
$\vdots \quad \vdots$	member is true.
$\phi n \supset \psi n$	

Material Implication

$p \supset p$	implication between propositions

In his analytic treatment of implication, Russell proceeds further in order to arrive at a more fundamental explanation of formal implication. Russell compares two propositions vis à vis formal implication: (1) Socrates is a man implies Socrates is a mortal; (2) Socrates is a man implies Socrates is a philosopher. In (1), the variability of "Socrates" is not restricted while in (2), the variability of "Socrates" is restricted if the implication is to remain true. As a result, Russell states, formal implication seems to involve something else besides material implication whereby a term can be varied. In (1), the relation of inclusion between the classes of men and mortals seems to be the relevant factor in the explanation of formal implication. Yet, Russell claims, the relation of inclusion cannot meet all the cases in which formal implication is employed. In

Russell's opinion, it is the notion of assertion which can deal adequately with the other troublesome cases of formal implication.[73]

Propositions have been analysed customarily in terms of subject and predicate. This, however, does not give due weight to an important element of the proposition - namely, the verb. Every proposition, says Russell, is capable of division between the subject and that which is said about the subject - what Russell calls the "assertion". For example, in the proposition, "Socrates is a man", the subject is "Socrates" while the assertion is "is a man". The verb, according to Russell, is the "distinguishing mark of propositions", and therefore, remains within the assertion.[74] It is to be noted that the assertion is nether true nor false.

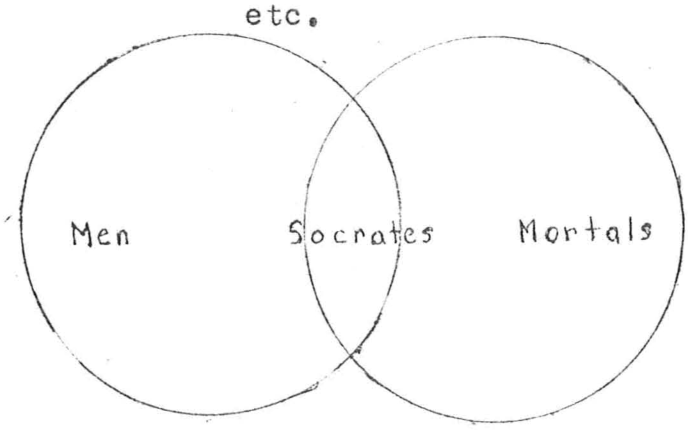
The analysis of propositions into subject and assertion is made by Russell in order to distinguish formal implications in which a variable is not restricted from those in which a variable is restricted. Russell suggests two ways of making such a distinction: (1) in the proposition, "Socrates is a man implies Socrates is a mortal", there is a relation between the two assertions, "is a man" and "is a mortal", whereby when the one holds so does the other; (2) the proposition, "Socrates is a man implies Socrates is a mortal", can be analysed into the subject, "Socrates", and an assertion about "Socrates". These two ways of distinguishing do not accommodate the proposition, "Socrates is a man implies Socrates is a mortal", into a class of material implications; they only carry the analysis of formal implication one step further. The first way of distinguishing needs to be supplemented by saying that the one same subject must be the subject of the assertions. The second way of distinguishing implications in which a term may be varied from those which cannot be varied is more objectionable than the first way. The suggested analysis of "Socrates is a man implies Socrates is a mortal" into the subject, "Socrates", and an assertion about "Socrates", and claiming that the assertion holds for all terms is not possible, says Russell. In Russell's opnion, the proposition, "Socrates is a man implies Socrates is a mortal", consists of two terms and a relation. The two terms are "Socrates is a man" and "Socrates is a mortal", and the relation is "implies". In relational propositions analysed in terms of subject and assertion, it is argued by Russell that the subject of such propositions must be one of the terms of the relation which is asserted. In this regard, Russell finds this latter way of distinguishing implications in which a term may be varied from those which cannot be varied objectionable. As a result, Russell dismisses this latter way of distinguishing in favour of the first way of distinguishing. By adopting this first way of distinguishing, Russell considers formal implication to be obtained from a relation between assertions.[75]

The relation of inclusion between classes is unsatisfactory to explain the notion of formal implication; Russell believes this to be the case due to the nature of relational propositions. If one takes a relational example of formal implication in which the assertions concern different subjects such as "A is before B implies B is after A", one can say that both propositions making up the implication have A and B as subjects - this is not to say that each proposition has the same subject, "A and B". One can analyse the proposition, "Socrates is a man implies Socrates is a mortal", by the use of class inclusion.

Diagram II

the class of mortal things: Socrates, other human beings, animals, etc.

the class of men: Adam, Socrates, Bertrand Russell, etc.



However, other relational propositions cannot be broken down in the same fashion via class inclusion, especially is this the case with asymmetrical relations. Russell therefore concludes that the notion of propositional function and assertion are more basic than the notion of class, and that the notion of class is inadequate to explain formal implication.[76]

Russell also maintains that the notion of "every term" is indefinable and primitive. Since formal implication holds for every term, it will not do to explain "every term" by means of formal implication. In fact, Russell goes further to say that if the notion of "every term" is not admitted, formal truths - that is, truths of formal implication - will not be possible.[77]

According to Russell, the importance of formal implication can be found in the fact that it is employed in all the rules of inference. Formal implication, Russell asserts, cannot be wholly defined in terms of material implication; besides material implication, some further principle must be cited in the explanation of formal implication.[78] Russell sums up his discussion on formal implication by stating that formal implication is "the affirmation of <u>every</u> material implication of a certain class".[79]

With respect to the class of material implications, one can say that it is the class of all propositions in which there is an assertion made concerning a certain subject or subjects implying another assertion having the same subject or subjects; thus, formal implication is derived from a relation between assertions. Russell acknowledges that there are logical difficulties with regards to his notion of formal implication; an examination, Russell states, of the components of a proposition is thus called for. The remainder of The Principles Of Mathematics is put forward with this aim in mind.[80]

B.3: IMPLICATION BETWEEN THE PRINCIPLES AND PRINCIPIA

In the interim period between The Principles Of Mathematics and Principia Mathematica, Russell wrote a number of articles which have some relevance to his notion of implication. At this point in my exposition of Russell's notion of implication, I would like to discuss these articles.

There is a lengthy article by Russell published in 1906 in the American Journal Of Mathematics entitled "The Theory Of Implication". Russell begins his article, "The Theory Of Implication", in the most explicit manner by stating exactly what he intends to do: "The purpose of the present article is to set forth the first chapter of the deduction of pure mathematics from its logical foundations."[81] I take this statement of purpose by Russell to be of prime importance since it reaffirms the thesis of logicism. In the first preface to The Principles Of Mathematics, Russell similarly maintained "that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical principles..."[82] This so-called "first chapter" is to focus upon deduction, and deduction involves the study of the principles which permit inferences from a set of premisses. If assumptions are to be made explicit, then the assumptions underlying deduction must be made explicit first.[83]

In The Principles Of Mathematics, Russell claimed that symbolic logic is the discipline which is concerned with deduction.[84] Traditionally, Russell states, symbolic logic is said to consist of two parts which are supposedly equally important: (1) the theory of propositions; (2) the theory of classes. Yet, in terms of deduction, Russell claims that the theory of propositions necessarily precedes the theory of classes since the latter depends upon the former in the use of principles whereby one proposition is inferred from another proposition. However, Russell shows a dissatisfaction in describing his article as an exposition of the theory of propositions; more precisely speaking, "The Theory Of Implication" is to be "the theory of how one proposition can be inferred from another".[85]

If one proposition can be inferred from another proposition, this inference is possible only on the grounds that there exists a relation which permits such an inference.[86] This relation, Russell states, is known as implication; thus, a formal system must incorporate a satisfactory number of premisses with respect to implication in order to ensure deductive technique.

Russell's article, "The Theory Of Implication", puts forward a number of propositions which act as the characteristic premisses of implication. With regards to these premisses, Russell makes three claims: (1) they are true; (2) they are sufficient to carry out all the customary forms of deduction; (3) Russell himself has not been able to diminish their number.[87] In spite of the fact that the formal rules of a system tend to act against the likelihood of any unconscious presuppositions, it is still quite possible to employ a principle unconsciously; due to such a possibility, Russell attempts to qualify his second claim. On the other hand, Russell's third claim states that the proposed set of premisses may not be necessary to the theory of deduction. Hence, Russell's general contention with respect to these premisses is that they are sufficient, but not necessary for all usual forms of inference.

In "The Theory Of Implication", Russell acknowledges the use of certain symbols borrowed from Peano - one of these borrowed symbols is the "$\supset$" for implication. In his unpublished article, "Recent Italian Work On The Foundation Of Mathematics", Russell had referred to Peano's symbol for implication: "For the relation of implication between propositions, Peano employs a capital C upside down, thus: $p \supset q$."[88] While the symbolism in "The Theory Of

Implication" is due in part to Peano, Russell concedes that the ideas in the article are mainly due to Frege. Two of the ideas in "The Theory Of Implication" adopted from Frege are: (1) the interpretation given to "p implies q"; (2) the use of implication and negation as primitive connectives or terms.[89] In _The Principles Of Mathematics_, Russell arrived at his results independently of Frege's work: "If I had become acquainted sooner with the work of Professor Frege, I should have owed a great deal to him, but as it is I arrived independently at many results which he had already established."[90]

In the first appendix to _The Principles Of Mathematics_, Russell briefly discussed the doctrines contained in Frege's logic and arithmetic. It was stated there that although Frege also took implication to be the fundamental relation in the propositional calculus, Frege's notion of implication differed somewhat from the notion of implication expounded in _The Principles Of Mathematics_. According to Frege, implication is a relation which holds between p and q whenever p is false or q is true; on the other hand, Russell in _The Principles Of Mathematics_ maintained that implication is a relation which holds between the _propositions_, p and q, whenever p is false or q is true. Thus, in contradistinction to Frege's notion of implication, the notion of implication in _The Principles Of Mathematics_ required that the nonlogical symbols be propositions. It was admitted by Russell that Frege's notion of implication had a "formal advantage" insofar as the hypothesis of the form, "p implies p", was avoided thereby.[91] Yet, Russell had his misgivings with Frege's notion of implication due to two drawbacks: (1) negation cannot be defined; (2) it is also not possible to define what is meant by a "proposition".[92] In "The Theory Of Implication", however, Russell adopts Frege's interpretation of implication in spite of the objections put forward in The Principles Of Mathematics.

According to Russell, a deductive system must contain undefined terms; otherwise, circularity will result. No doubt Euclid's Elements suffered from this defect. Euclid did not list any primitive terms, but attempted to define all his terms within the system. Such an enterprise is doomed from the outset to the charge of circularity or to the charge of an infinite regress.[93] In "The Theory Of Implication", Russell is aware that a deductive system requires primitive terms; moreover, in contrast to The Principles Of Mathematics, Russell recognizes that to a certain extent it is arbitrary which terms are to be made primitive.[94] Nevertheless, the "arbitrariness" of the primitive terms is guided by two considerations: (1) the number of primitive terms is to be as small as possible; (2) between two distinct sets of primitive terms whose number is equal, the set of primitive terms which appears to be easier and simpler will be chosen.[95]

Russell remarks that what he is about to say concerning implication may appear to be artificial at a prima facie level, but he considers the meaning which he gives to implication to be the most appropriate and convenient. It should be noted here that unlike The Principles Of Mathematics, Russell, in "The Theory Of Implication", allows the fact that there are other legitimate interpretations which may be given to implication. In The Principles Of Mathematics, Russell was more dogmatic with respect to the interpretation given to implication: "Two kinds of implication, the material and the formal, were found to be essential to every kind of deduction."[97] In "The Theory Of Implication", there seems to be a change of disposition with regards to the treatment of the topic of implication. Russell seems to be more aware of the complexity of the problem involved in the interpretation of "implies"; moreover, some of the presuppositions contained in The Principles Of Mathematics are abandoned by Russell.

The proposition which Russell considers to be essential for the notion of implication is the following: "What is implied by a true proposition is true."[98] Russell intends to say by this claim that if anything is inferred from a true proposition, then that inference itself must be true; as Russell maintains, this characteristic of implication guarantees the notion of proof, and if it is not adhered to, then whatever is meant by deduction is lost in the confusion of contradiction and <u>non sequitur</u>. Thus, Russell's interpretation - namely, material implication - simply asserts that in the implication, "p implies q", unless p be true and q be false, "p implies q" is true.[99]

According to Russell, the chief advantage of his interpretation of implication is that "it avoids hypotheses which are otherwise necessary".[100] If we wish to assert "$p \supset p$", then if implication is possible only between propositions, we will have to foreword "$p \supset p$" with the hypothesis that p is a proposition; as a result, if we are going to use "$p \supset p$" in some particular case, we must provide a proof that in this case, p is a proposition - otherwise, "$p \supset p$" will be false. What Russell is trying to say is that if we do not accpet material implication as the correct interpretation of implication between propositions, then we are left with quite an open and flexible interpretation of implication. In <u>The Principles Of Mathematics</u>, it was maintained by Russell that "'$p \supset q$' was false if p was not a proposition..."[101] It was necessary therefore in <u>The Principles</u>

<u>Of Mathematics</u> to restrict the meaning of implication by prefacing many of the axioms of the propositional calculus with a hypothesis of the form, "$p \supset p$". Russell contended in <u>The Principles Of Mathematics</u> that every proposition implies itself and that if something were not a proposition, it could not imply anything whatsoever. Thus, it was claimed that "p is a proposition" is equivalent to "p implies p".[102] Following Frege, Russell in "The Theory Of Implication" does not preface his axioms with the hypothesis of the form, "$p \supset p$"; the reasons for the abandonment of the hypothesis of the form, "$p \supset p$", are not entirely given in "The Theory Of Implication". Russell simply asserts that the hypothesis of the form, "$p \supset p$", is inconvenient, and that if we restrict the meaning of implication by claiming that the nonlogical symbols must represent propositions, then there are paradoxes which result. One of Russell's examples to illustrate the difficulties involved in the restrixtion of implication is the formula, "$p \supset (q \supset p)$"; it is Russell's contention that q is not subject to any limitation whatsoever since even if q is not a proposition, "$p \supset (q \supset p)$" would still be true.[103]

After having explained his primitive propositions in "The Theory Of Implication", Russell proceeds to give proofs of various propositions; Russell also introduces the notions of propositional product (conjunction), propositional sum (disjunction) and equivalence in terms of his primitive ideas of negation and implication.[104] In the final section of his paper, Russell introduces another primitive idea - namely, "(x)(Cx)"; this primitive idea asserts "the truth of (C)(x) for all values of x".[105]

In The Principles Of Mathematics, Russell had maintained that there are two types of implication: material implication and formal implication. Material implication, according to Russell, is "a relation holding between nothing except propositions of which either the first is false or the second true".[106] On the other hand, formal implication is not a single implication, but "a class of implications, no one of which contains a variable, and we assert that every member of this class istrue".[107] After much analysis, Russell contended that formal implication could not be defined in terms of material implication.

Although Russell does draw a distinction between material implication and formal implication in his article, "The Theory Of Implication", there is no explicit statement by Russell to the effect that both implications are unique. At the end of his article, Russell claims that the primitive propositions along with those deduced from such primitive propositions constitute "what is most important in the theory of implication";[108] propositions of the form, "$p \supset q$", concern material implication only. In terms of logical order, the theory of material implication precedes the theory of formal implication in which propositions of the form, "$(x)(\phi x \supset \psi x)$", are asserted.[109] Russell's distinction between the theory of material implication and the theory of formal implication is tantamount to the modern, standard distinction between the propositional calculus and first order predicate calculus. Unlike <u>The Principles Of Mathematics</u>, Russell in "the Theory Of Implication" does not attempt to explain the exact differences between material implication and formal implication. In the former work, it was pointed out by Russell that the notion of inclusion is unsatisfactory to explicate the notion of formal implication. To this endeavour, Russell proposed a theory of assertion which was employed in order to explain relational examples of formal implication.[110] In "The Theory Of Implication", Russell does not pursue this issue concerning the nature of formal implication. There is only the statement by Russell that formal implication involves propositions of the form, "$(x)(\phi x \supset \psi x)$", and that under Peano's notation, "$(x)(\phi x \supset \psi x)$" is equivalent to

"$\phi x \supset_x \psi x$".[111]

In "The Theory Of Implication", Russell's aim is to present the propositional calculus in such a manner that all the assumptions of the undertaking are rendered explicit. In comparison to The Principles Of Mathematics, Russell incorporates a number of novel ideas in "The Theory Of Implication": (1) a symbolic technique; (2) the assertion sign, "$\vdash$"; (3) Occam's Razor; (4) the primitive connectives, "$\supset$" and "$\sim$"; (5) the rejection of the hypothesis of the form, "$p \supset p$", for well-formed formulas.[112]

There are two articles which appeared in Mind in 1908 - one is written by MacColl and the other is a reply by Russell. MacColl's article which is entitled "'If' And 'Imply'" concerns Russell's interpretation of implication and the consequences which accrue from such an interpretation. MacColl remarks that Russell in The Principles Of Mathematics adopts the usual interpretation given to implication - that is, "A implies B" (or "if A then B") is equivalent to the disjunctive, "either A is false or B is true". From this interpretation of implication, Russell had contended in The Principles Of Mathematics that it follows that of any two propositions, one must imply the other; with respect to this contention, MacColl admits that it does follow from

Russell's interpretation of implication. However, MacColl claims, the contention is so paradoxical that it renders Russell's interpretation of implication to be suspect. In MacColl's opinion, it is a mistake to think that an implication or conditional is equivalent to a disjunctive. If we take the two statements, "He is a doctor" and "He is red-haired", each of which is a variable, it follows from Russell's interpretation of implication that either "if he is a doctor, he is red-haired" or "if he is red-haired, then he is a doctor". From these two conditionals, MacColl states, it follows that either all doctors are red-haired or all those that have red hair are doctors. According to MacColl, Russell's interpretation of implication leads to the view that if we find a person with red hair, then he must be a doctor or if we find a doctor, then he must have red hair. However, says MacColl, it is not necessarily the case that if a person is red-haired, then he is a doctor; moreover, just because a person is a doctor, it does not follow that he has to be red-haired. Thus, the contention, "of any two propositions, one must imply the other", cannot be true; furthermore, since this contention follows from Russell's interpretation of implication, Russell's interpretation cannot be in keeping with "ordinary linguistic usage".[113]

Russell's rejoinder is entitled "'If' And 'Imply', A Reply To Mr. MacColl". It is claimed by Russell that the arguments put forward by MacColl show that MacColl "has failed to grasp .. . the distinction between propositions and propositional functions".[114] The statements, "He is a doctor" and "He is red-haired", are variables in the sense that if it is undecided as to whether they are true or false; they are not propositions therefore, and cannot imply each other. Thus, Russell affirms, although the contention, "of any two propositions, one must imply the other", is true, the contention, "of any two statements, one must imply the other", is not true. A proposition such as "Mr. Smith is a doctor" cannot be a variable because Mr. Smith is either a doctor or he is not a doctor. Consequently, if we take any two propositions, we know that under Russell's interpretation of implication, namely, material implication, one proposition must imply the other:

> Of the propositions 'Mr. Smith is a doctor' and 'Mr. Smith is red-haired', it is easy to see that one must imply the other, using the word 'imply' in the sense in which I use it. (That this is not the usual sense, may be admitted; all that I affirm is that it is the sense which I most often have to speak of, and therefore for me the most convenient sense.) 115

For two propositions, there are four cases in which one proposition must materially imply the other.[116]

Diagram III

p	q	
T	T	$p \supset q$, $q \supset p$
F	T	$p \supset q$
T	F	$q \supset p$
F	F	$p \supset q$, $q \supset p$

Of any two propositions, one must imply the other.

In conclusion, Russell makes the point that MacColl's classification of statements into the classes of certain, doubtful and impossible is correct if it is to be understood that the classification is for propositional functions, not propositions. Propositions have only a twofold division governed by the claims of truth and falsity.[117]

Concerning the last article which I wish to discuss in this section, namely, "Mathematical Logic As Based On The Theory Of Types", there are two points which should be made with respect to Russell's notion of implication. In the enumeration of his primitive ideas, Russell does not take negation and implication to be his primitive connectives as in "The Theory Of Implication"; retaining negation as a primitive idea, Russell discards implication as a primitive idea in favour of disjunction:

> "In a previous article in this journal ["The Theory Of Implication"], I took implication as indefinable instead of disjunction. The choice between the two is a matter

> of taste; I now choose disjunction, because it enables us to diminish the number of primitive propositions." [118]

Of course, Russell is slightly mistaken with regards to his claim that implication was taken to be indefinable in "The Theory Of Implication". Although Russell took implication to be indefinable in The Principles Of Mathematics, he maintained that implication was undefined in "The Theory Of Implication".[119] Moreover, this distinction between "indefinable" and "undefined" was deemed to be crucial by Russell in "The Theory Of Implication".[120]

The second point to be made with respect to Russell's article, "Mathematical Logic As Based On The Theory Of Types", vis à vis Russell's notion of implication is that Russell defines "$p \supset q$" (read "p implies q") as "$\sim p \vee q$" meaning "p is false or q is true". Moreover, Russell adds a consideration to his definition of implication: "I do not mean to affirm that 'implies' can not have any other meaning, but only that this meaning is the one which it is most convenient to give to 'implies' in symbolic logic."[121]

B.4: IMPLICATION IN *PRINCIPIA MATHEMATICA*

In the preface to *The Principles Of Mathematics*, Russell wrote:

> The second volume [of *The Principles Of Mathematics*], in which I have had the great fortune to secure the collaboration of Mr. A. N. Whitehead, will be addressed exclusively to mathematicians...The present volume, which may be regarded as a commentary upon, or as an introduction to, the second volume, is addressed in equal measure to the philosopher and to the mathematician. 122

However, as Russell's work progressed, it was discerned that the subject matter had become much larger; moreover, some of the problems discussed in *The Principles Of Mathematics* which were left "doubtful and obscure" demanded new solutions.[123] To this intent, Russell and Whitehead expounded the three-volume *magnum opus* known as *Principia Mathematica*.

References to the notion of implication are contained mainly in the first volume of *Principia Mathematica*. In the first edition of that work, negation and disjunction were taken as the two indefinables of the propositional calculus; in the second edition, it was acknowledged that these two indefinables could be replaced by Sheffer's stroke, "p/q" (read "p stroke q" or "p is incompatible with q").[124] Chapter I of the introduction incorporates an explanation of the notation and ideas to be found in the rest of the text, and it is here that the reader firstly encounters the notion of implication.

According to Russell, there are four fundamental functions with respect to propositions: (1) the contradictory function; (2) the disjunctive function; (3) the conjunctive function; (4) the implicative function.[125] I will not discuss the first three functions since they are not germane to this paper. The implicative function for Russell is a function having two arguments, p and q, in which p and q are propositions; the proposition expressing the claim that "either not-p or q" is true is "$\sim p \vee q$" - this proposition in effect states that p implies q, "$p \supset q$". In his account of the notion of implication, Russell makes an interesting point: "But 'implies' as used here expresses nothing else than the connection between p and q also expressed by the disjunction 'not-p or q'."[126] It should be noted also that Russell considers the proposition, "p implies q", to be interchangeable with the proposition, "if p, then q". Moreover, in Principia Mathematica as in his earlier article, "Necessity And Possibility", Russell does not equate the notion of inference with the notion of implication.[127]

Implication in Principia Mathematica means material implication, and Russell is quite emphatic in his differentiation between material implication and formal implication. When an apparent variable of the form, ϕx, is associated with implication, this yields an extension of implication called <u>formal implication</u>; thus, formal implication, Russell states, is a "derivative idea" of material implication.[128] Russell's definition of equivalence as mutual implication also involves the counterpart of formal implication - namely, formal equivalence.[129]

As it has been said hitherto, inference for Russell is not the same as implication; rather, Russell claims, inference is "the dissolution of an implication".[130] In other words, given two assertions, p and "p implies q", one can infer q. Moreover, the means whereby one proposition is inferred from another proposition cannot be formally stated.[131]

Returning to the topic of formal implication, Russell states that when "$\phi x \supset \psi x$" always holds [i.e. "$(x)(\phi x \supset \psi x)$" holds], it is said that ϕx formally implies ψx. In many cases, we have an implication of the form, "'Socrates is a man' implies 'Socrates is mortal'"; in such a case, the said implication is often a particular instance of a formal implication, "$(x)(\phi x \supset \psi x)$". It has been thought in the past, Russell contends, that if an implication is not a particular case of a formal implication, then it is not really an implication at all. Furthermore, such implications have been ignored due to practical considerations insofar as such implications cannot be known unless the antecedent is false or the consequent is true; as a result, such implications do not help us know the consequent either since in the first case when the antecedent is false, the consequent may be true of false, and in the second case, the consequent is known to be true already. In this sense, therefore, such implications do not serve the purported purpose of implication - namely, by the rules of deduction to produce conclusions which previously were not known. On the other hand, Russell asserts, formal implications do serve this purported purpose of implication; it is often known that "$(x)(\phi x \supset \psi x)$" and ϕy are the case whereby it follows that ψy can be known via the rules of deduction.[132] An alternative notation to express formal implication has been put forward by Peano in which "$(x)(\phi x \supset \psi x)$" is equivalent to "$\phi x \supset_x \psi x$".[133]

At the beginning of Part I of the first volume of Principia Mathematica, Russell talks about implication with respect to the theory of deduction. Russell makes a number of claims concerning his notion of implication, but it would seem that most of these claims have been enunciated previously in his article, "The Theory Of Implication". There is the same point contained in The Principles Of Mathematics and "The Theory Of Implication" that if one proposition is to be inferred from another, then there must exist a relation whereby one proposition can be the consequence of the other.[134] Of course, this relation is material implication. As in his earlier works, deduction for Russell is dependent upon implication, and any deductive system must embody a sufficient number of premisses with respect to the properties of implication in order to ensure all the customary forms of inference.[135]

When a proposition q follows from a true proposition p, we know that q is true; thus, p implies q. In Russell's opinion, this is the essential property of implication - namely, "what is implied by a true proposition is true".[136] Again, Russell defines implication in terms of negation and disjunction, and he maintains that although there may be other well-grounded interpretations of implication, material implication is best suited for his purposes.[137]

In The Principles Of Mathematics, one of the prerequisites of an implication was that the nonlogical symbols be propositions; as a result, Russell prefaced most of the axioms of the propositional calculus with the hypothesis of the form, "p implies p". In his article, "The Theory Of Implication", Russell abandoned this hypothesis maintaining the view that the nonlogical symbols need not be propositions. Although Russell does not stipulate in *Principia Mathematica* that the hypothesis of the form, "p implies p", must preface well-formed formulas, it would seem that his position is not in agreement with "The Theory Of Implication". Russell does claim in *Principia Mathematica* that if implication is to occur, then the nonlogical symbols must stand for propositions; however, unlike *The Principles Of Mathematics*, there is no mention of a hypothesis of the form, "p implies p", to preface well-formed formulas.[138]

B.5: IMPLICATION AFTER PRINCIPIA

It can be maintained that after the writing of *Principia Mathematica* and its subsequent publication, Russell did not devote his study to mathematical logic as he had done in the period between 1900 and 1910. Of course, there are exceptions to this claim. In 1910, Russell became a lecturer at Cambridge in logic and the principles of mathematics.[139] When he was invited to give the Lowell lectures at Boston in 1914, Russell was also a visiting professor at Harvard where he taught a course in epistemology and a course in advanced logic.[140] No doubt there are other exceptions as well, but it would seem that this claim is well-founded. As Russell relates in his *Autobiography*, the period during which *Principia Mathematica* was written was painstaking; besides the occasion of personal tragedies, the "mathematical elaboration" involved in *Principia Mathematica* was a formidable task of much labour.[141] Having finished *Principia Mathematica*, Russell's enthusiasm for mathematical logic abated greatly: "This invitation to write a general outline of my philosophy came at a fortunate moment. I was glad to escape from the rigours of symbolic reasoning ..."[142] In fact, with the oncoming of World War I, Russell diverted his interests to the issues imminent in the contending hostilities :

It may seem curious that the War should rejuvenate

> anybody, but in fact it shook me out of my prejudices and made me think afresh on a number of fundamental questions. It also provided me with a kind of activity, for which I did not feel the staleness that beset me whenever I tried to return to mathematical logic. 143

Due to his anti-war activities, however, Russell was imprisoned, and it was there that he wrote Introduction To Mathematical Philosophy in 1918.

According to Russell, his book, Introduction To Mathematical Philosophy, was to have been a "semi-popular version" of The Principles Of Mathematics.[145] The section germane to this paper which I propose to discuss is Chapter XIV, "Incompatibility And The Theory Of Deduction". Having examined the elements of mathematics which do not require the notion of class, Russell turns his attention to those parts of mathematics which do presuppose the notion of class. To this intent, Russell's concern in Chapter XIV of Introduction To Mathematical Philosophy is to explain the theory of deuction.[146]

Since mathematics is deductive, Russell claims that it is possible to begin with certain premisses from which theorems can be deduced establishing the whole body of mathematics. It is maintained by Russell that mathematical proof does not rely upon "intuition", but rather is based upon the rules of deduction. As in The Principles Of

Mathematics, Russell argues against any psychological intrusion into the sphere of mathematics; in Russell's opinion, the fact that pure mathematics can be reduced to symbolic logic is evidence enough to destroy Kant's contention that deduction demands the support of "intuition".[147]

Having dispelled the role of "intuition" in deduction, Russell states very simply that when one proposition is inferred from another proposition, deduction is said to take place; thus, deduction involves an inference from a premiss to a conclusion. Hence, if the deduction has been carried out correctly, there must exist a relation between the premiss and the conclusion such that if the premiss is true, the conclusion must be true. It is no surprise therefore that for Russell, the main feature of the theory of deduction should lie in this relation known as "implication".[148]

Although it would seem natural to take "p implies q" as the primitive relation in the theory of deduction, it is more appropriate for technical reasons, Russell claims, to choose some other relation as the primitive idea of the theory of deduction. For Russell in the Introduction To Mathematical Philosophy, there are at least five functions:

negation, disjunction, conjunction, incompatibility and implication;[149] all these functions are similar in the sense that their truth-value depends upon the truth or falsehood of the propositions involved. Moreover, these functions are not independent of each other since it is possible to define some of them in terms of the other functions. In the Introduction To Mathematical Philosophy, the relation of incompatibility, "p/q", is taken to be the primitive idea for the theory of deduction; however, for the benefit of the reader, Russell employs five formal principles having implication and disjunction as primitive notions. Later on in the chapter in his exposition of the theory of deduction, Russell shows how Nicod by means of the relation of incompatibility reduced these five formal principles to one formal principle.[150]

As in some of his other works, implication for Russell in the Introduction To Mathematical Philosophy can be defined to mean "either p is false or q is true". With regards to this interpretation of implication, Russell makes two initial claims: (1) implication is to be interpreted in its widest sense which allows q to be true if it is inferred from the truth of p; (2) there may be other interpretations of implication, but these interpretations need not be considered since material implication is convenient to the purposes at hand.[151]

Although Russell states previously that psychology has no place in the theory of deduction, he does acknowledge that psychology inevitably creeps into the discussion: "There is always unavoidably <u>something</u> psychological about inference."[152] According to Russell, when we actually infer the truth of q from the truth of p and the truth of "$p \supset q$", the process of inference to the assertion, q, is indeed psychological. However, the task of logic is not to describe such a process; rather, the task of logic is to unfold the nature of the relation whereby it is permissible to infer correctly the truth of q from the truth of p and the truth of "$p \supset q$".[153]

It would seem, Russell states, that there is a confusion among some authors concerning the relation between propositions whereby an inference is said to be valid. When q is validly inferred from p, it is necessary that both p and "not-p or q" be true; consequently, q must also be true. However, inference can occur only if "not-p or q" is known by means other than the knowledge of not-p or the knowledge of q. If not-p is known to be true, then "not-p or q" will also be known to be true, but in this case, we could never validly infer q since such an inference necessitates the truth of p. Similarly, Russell argues, if q is known to be true, "not-p or q" is also known to be true, but inference cannot occur here either since q is already known and q therefore need not be inferred. Thus, in this example, inference can only arise when it is known which of the two disjuncts will make the disjunction, "not-p or q", true. To take another example which Russell uses, there exists a formal relation between the two well-formed formulas, "r implies not-s" and "s implies not-r", such that it is known that the first implies the second without having the knowledge that the first is false or that the second is true. It is under such examples as these, Russell claims, that "the relation of implication is practically useful for drawing inferences".[154]

Having disagreed with the definition of implication given in Principia Mathematica, C.I. Lewis maintained that "not-p or q" is too wide as a definition of "p implies q". In the Introduction To Mathematical Philosophy, Russell made an attempt to answer such a criticism. Russell's first claim is that it is quite unimportant how we define something as long as we consistently use that definition. Russell sums up the chief point of his disagreement with Lewis as follows: when q is "formally deducible" from p in which p and q are are propositions, Lewis contends that there exists a relation between p and q which he calls "strict implication" such that the relation of strict implication is not equivalent to "not-p or q" but is a more confined relation whenever there are "certain formal connections" present between p and q;[155] in response to Lewis, Russell states that mathematics does not require the notion of strict implication regardless as to whether it exists or not, and consequently, via Occam's Razor, it should not be allowed as a primitive idea in the system. It is contended by Russell that all the points which Lewis employs against the view of material implication can be countered, and moreover, the criticisms of Lewis rest upon an unconscious presupposition taken from a certain perspective of implication which Russell rejects initially. In his final conclusion with regards to his disagreement with Lewis concerning the nature of implication, Russell states that any implication which cannot be defined in terms of truth functions should not be accepted.[156]

In the Introduction To Mathematical Philosophy, it would seem that Russell is defending the positions resulting from the techniques of Principia Mathematica, especially is this the case with respect to the notion of implication. Although Russell does not comment in detail as to whether material implication can hold only between propositions, again it would seem as in Principia Mathematica that this is indeed the case in contradistinction to the position of "The Theory Of Implication". The only note in the *Introduction To Mathematical Philosophy* with respect to formal implication is that propositions having the form, "ϕx always implies ψx", are formal implications.[157]

C: CRITIQUE

In the consideration of clarity, I would like to preface this section of my paper with a few general remarks. The "critique" part of the paper is divided into a number of sub-sections. In all of these sub-sections, save the last sub-section, I have attempted in general to appraise certain issues concerning Russell's notion of implication without the aid of the views of other critics. In the last sub-section, however, I have reviewed certain comments put forward by the authors of various secondary sources in order to establish as to whether such comments are warranted or not. Moreover, due to the fact that there are myriad issues embodied in Russell's early philosophy, I have found it necessary for reasons of relevance to limit my critical remarks to those issues which have some direct bearing on his notion of implication.

C.1: THE SCHRÖDER-MACCOLL DISPUTE

In The Principles Of Mathematics, Russell discusses a disagreement between Schröder and MacColl. Schröder had maintained that on the assumption that p, q and r are propositions, "p and q implies r" is equivalent to "p implies r or q implies r". This claim by Schröder was objected to by Hugh MacColl who stated that although "p implies r or q implies r" implies "p and q implies r", "p and q implies r" does not imply "p implies r or q implies r". According to Russell, there is no genuine disagreement between Schröder and MacColl since Schröder is talking about propositions and material implication while MacColl is talking about propositional functions and formal implication. After giving an eludidative exlanation of Schröder's position, Russell divulges the manner in which MacColl presents his objection. MacColl had pointed out that if p and q are mutually contradictory and r is the null proposition, then p and q implies r but neither p nor q implies r; thus, if MacColl's criticism is apt, it would seem that there can be no equivalence between "p and q implies r" and "p implies r or q implies r". However, Russell contends that unlike Schröder, MacColl is dealing with propositional functions and formal implication, and therefore, since both positions begin with different ground rules, there can be no genuine dispute present.[158]

If we consider Schröder's perspective of propositions and material implication, we know via the logical apparatus of the propositional calculus that "p and q implies r" is equivalent to "p implies r or q implies r".

Proof: (by subordinate method)[159]

(1)	$(p \,\&\, q) \supset r$	hypothesis
(2)	$\sim(p \supset r)$	hypothesis
(3)	q	hypothesis
(4)	$\sim(p \supset r)$	reit (2)
(5)	$p \,\&\, \sim r$	$\supset$ - & -Tx (4)
(6)	p	& -elim (5)
(7)	$p \,\&\, q$	& -intro (6), (3)
(8)	$(p \,\&\, q) \supset r$	reit (1)
(9)	r	M.P. (7), (8)
(10)	$q \supset r$	$\supset$ -intro (3)-(9)
(11)	$\sim(p \supset r) \supset (q \supset r)$	$\supset$ -intro (2)-(10)
(12)	$(p \supset r) \vee (q \supset r)$	$\supset$ -v-Tx (11)
(13)	$(p \supset r) \vee (q \supset r)$	hypothesis
(14)	$p \,\&\, q$	hypothesis
(15)	$(p \supset r) \vee (q \supset r)$	reit (13)
(16)	$p \supset r$	hypothesis
(17)	$p \,\&\, q$	reit (14)
(18)	p	& -elim (17)
(19)	r	M.P. (18), (16)
(20)	$q \supset r$	hypothesis
(21)	$p \,\&\, q$	reit (14)
(22)	q	& -elim (21)
(23)	r	M.P. (22), (20)
(24	r	v-elim (15), (16)-(19), (20)-(23)
(25)	$(p \,\&\, q) \supset r$	$\supset$ -intro (14)-(24)
(26)	$[(p \,\&\, q) \supset r] \equiv [(p \supset r) \vee (q \supset r)]$	$\equiv$ -intro (1)-(12), (13)-(25)

Q.E.D.

We have been told by Russell that formal implication deals with propositional functions and that material implication deals with propositions. In the light of this distinction, the positions of Schröder and MacColl with regards to the formula, "$[(p \,\&\, q) \supset r] \equiv [(p \supset r) \vee (q \supset r)]$", are claimed to be compatible by Russell. In what sense, however, is it possible for such a distinction to remedy the Schröder-MacColl dispute? In Russell's claim that Schröder employs material implication and propositions and that MacColl employs formal implication and propositional functions true? We have shown by the method of subordinate proof that "$(p \,\&\, q) \supset r$" is equivalent to "$(p \supset r) \vee (q \supset r)$"; it is safe to conclude therefore that Schröder's position in terms of material implication and propositions is correct. Is there an alternative position, that of MacColl's, in which under the interpretation of formal implication and propositional functions, it is correct to state that "$(p \,\&\, q) \supset r$" is <u>not</u> equivalent to "$(p \supset r) \vee (q \supset r)$"? Obviously, Russell thinks that there is such an alternative position.

Russell claims that MacColl is thinking of propositional functions and formal implication. Let us grant that presupposition. As a result, p is rendered to the form, ϕx, and following MacColl's argument, q is the contradictory of ϕx (i.e. $\sim\phi x$). No matter what values we give to ϕx, we know a priori, so to speak, that "$\phi x \,\&\, \sim\phi x$"

is constantly false. Since Λ, replacing r, is the null propositional function, it also has false values; it follows that ϕx & $\sim\phi x$ implies Λ. We can agree with Russell at least to this point in the argument concerning the first part of the equivalence in question. Russell in support of MacColl goes on to argue that ϕx is not always false, and in the same fashion, we cannot say that $\sim\phi x$ is not always false. Therefore, neither always implies Λ which itself is always false. Since this is the case, Russell concludes that " $[(p \,\&\, q) \supset r] \equiv [(p \supset r) \vee (q \supset r)]$ " cannot be a legitimate equivalence in the class calculus.

It would <u>seem</u> that Russell misses the point here. Although it may be true that ϕx is not always false and $\sim\phi x$ is not always false, it does not follow that the equivalence in question does not hold. The second part of the equivalence, namely, "$(\phi x \supset \Lambda) \vee (\sim\phi x \supset \Lambda)$", is treated by Russell as an implication, but it is a disjunction. By the meaning of disjunction, in order for a disjunction to be true, one of its parts must be true. Consequently, if ϕx does not imply Λ in a specific case, it follows that $\sim\phi x$ will imply Λ in that specific case - the converse is also true. Hence, given that Λ is a null propositional function, "$(\phi x \supset \Lambda) \vee (\sim\phi x \supset \Lambda)$" is a tautology - "$(\phi x \,\&\, \sim\phi x) \supset \Lambda$" is also a tautology.

Diagram IV

ϕx	$\sim\phi x$	$\phi x \,\&\, \sim\phi x$	$\phi x \supset \Lambda$	$\sim\phi x \supset \Lambda$
T	F	F	F	T
F	T	F	T	F

$(\phi x \,\&\, \sim\phi x) \supset \Lambda$	$(\phi x \supset \Lambda)\ v\ (\sim\phi x \supset \Lambda)$
T	T
T	T

It may be argued that ϕx has no truth value since it is a propositional function; however, this is not a moot point in the discussion. No matter what truth values are posited for ϕ, $\sim\phi x$ will have opposite truth values. In any event, the "x" in ϕx is bound by a universal quantifier.

It would seem here that Russell's contention is that when " $[(p \,\&\, q) \supset r] \equiv [(p \supset r)\ v\ (q \supset r)]$ " is interpreted in terms of formal implication and propositional functions, then the equivalence does not hold. At a prima facie level, this would seem to concern the following equivalence: "$(x)[(\phi x \,\&\, \sim\phi x) \supset \Lambda] \equiv (x)[(\phi x \supset \Lambda)\ v\ (\sim\phi x \supset \Lambda)]$". However, although "$(x)[(\phi x \,\&\, \sim\phi x) \supset \Lambda]$ " is a formal implication, "$(x)[(\phi x \supset \Lambda)\ v\ (\sim\phi x \supset \Lambda)]$" is not a formal implication. It would seem therefore that the equivalence in question is <u>not</u> "$(x)[(\phi x \,\&\, \sim\phi x) \supset \Lambda] \equiv (x)[(\phi x \supset \Lambda)\ v\ (\sim\phi x \supset \Lambda)]$", but that Russell is talking about the following equivalence: "$(x)[(\phi x \,\&\, \sim\phi x) \supset \Lambda] \equiv [(x)(\phi x \supset \Lambda)\ v\ (x)(\sim\phi x \supset \Lambda)]$". There are at least two good reasons to suppose that this latter equivalence is the equivalence in question: (1) Russell argues that neither $\emptyset x$ nor $\sim \emptyset x$ is always false, and therefore, neither always implies Λ - this argument can only be accepted if the second part of the equivalence is " $(x)(\emptyset x \supset \Lambda) \vee (x)(\sim \emptyset x \supset \Lambda)$ "; (2) Russell makes the following remark with respect to his analysis of MacColl's interpretation of " $[(p \& q) \supset r] \equiv [(p \supset r) \vee (q \supset r)]$ ":

> This may be easily rendered obvious by the following considerations: let $\emptyset x$, ψx, χx be three propositional functions. Then "$\emptyset x \, . \, \psi x$ implies χx" implies, for all values of x, that either $\emptyset x$ implies χx or ψx implies χx. But it does not imply that either $\emptyset x$ implies χx for all values of x, or ψx implies χx for all values of x. 160

In this respect, therefore, if " $[(p \& q) \supset \Lambda] \equiv [(p \supset r) \vee (q \supset r)]$ " is interpreted in terms of formal implication and propositional functions, then the said equivalence cannot hold. Consequently, if Russell's interpretation of MacColl's position is correct, there can be no genuine dispute between Schröder and MacColl.

With regards to Russell's remark that " $[(p \& q) \supset \Lambda] \equiv [(p \;\; r) \vee (q \supset r)]$ " is false in the class calculus, one can say that this is not necessarily the case. If " $[(p \& q) \supset \Lambda] \equiv [(p \supset r) \vee (q \supset r)]$ " is interpreted as " $(x)[(\emptyset x \& \sim \emptyset x) \supset \Lambda] \equiv (x)[(\emptyset x \supset \Lambda) \vee (\sim \emptyset x \supset \Lambda)]$ ", then " $[(p \& q) \supset \Lambda] \equiv [(p \supset r) \vee (q \supset r)]$ " is still true in the class calculus. It is only when " $[(p \& q) \supset r] \equiv [(p \supset r) \vee (q \supset r)]$ " is interpreted as " $(x)[(\emptyset x \& \sim \emptyset x) \supset \Lambda] \equiv [(x)(\emptyset x \supset \Lambda) \vee (x)(\sim \emptyset x \supset \Lambda)]$ " that the former equivalence will be false in the class calculus. In the Schröder-MacColl dispute, therefore, one notices one of Russell's tendencies - namely, instead of talking in terms of quantification, to talk in terms of formal implication.

The discussion of the Schröder-MacColl dispute in The Principles Of Mathematics is deemed by Russell to be significant for at least two reasons: (1) it attempts to show how implications are affected by various nonlogical symbols; (2) it illustrates that for Russell, there is a definite distinction to be made between material implication and formal implication. Insofar as these two issues are connected in one way or another with Russell's notion of implication, I would like to devote the next few sections of my critique to them.

C.2: PROPOSITIONS AND IMPLICATION

In *The Principles Of Mathematics*, Russell maintains that p can imply q only on the presupposition that p and q are propositions; if p and q are not propositions, then the said implication will be false. There are also a number of other claims with respect to implication and the nonlogical symbols involved in an implication. Implication, that is, material implication, is indeed indefinable in *The Principles Of Mathematics*. Secondly, although implication is indefinable, it is possible by means of implication to define what is meant by a proposition.[161] There are many other contentions in *The Principles Of Mathematics* with regards to implication and the nonlogical symbols involved in an implication, but to begin with, I will focus upon the contentions mentioned above in order to introduce into my presentation other claims made by Russell in *The Principles Of Mathematics*.

Firstly, therefore, is implication indefinable? Although Russell maintains that implication is indefinable in *The Principles Of Mathematics*, by the time that he had written "The Theory Of Implication", it would seem that his views had altered on the matter.[162] In *Principia Mathematica*, material implication was defined as follows: "$p \supset q = \sim p \vee q$ Df".[163] It should be noted that in the works after *The Principles Of*

Mathematics, Russell does not provide any counterarguments to the effect that implication is definable; instead of producing counterarguments against the point of view taken in The Principles Of Mathematics, Russell simply abandoned the former position in his later works. I would therefore like to review the argument of The Principles Of Mathematics, the position that implication is indefinable, in order to ascertain whether it is sound or not.

Briefly, Russell's argument that implication cannot be defined is as follows: if one attempts to claim that p implies q can be defined by saying that if p is true, then q is true or by saying that if q is false, then p is false, then although the notions of truth and falsehood have been introduced, the definition cannot be accepted since we have defined implications by means of implication; furthermore, if one attempts to define implication by means of disjunction and negation by saying that "p implies q" is equivalent to "not-p or q", then one cannot accept this definition either since implication is defined by means of equivalence and equivalence is mutual implication.[164] The gist of Russell's argument is that no matter which way we choose to define implication, we will be open to the charge of circularity; as a result, implication cannot be defined.

In a strict sense, Russell is quite right when he states that implication is indefinable, but I will argue that although circularity inevitably arises in a definition of implication, it is not the type of circularity which makes the definition trivial. An argument is said to be circular when it takes for granted what it is supposed to prove. For example, if we define what is meant by the word, "sleep", by saying that sleep is a soporific state, we have taken for granted what we wanted to define since "soporific" is simply another word for "sleep"; this type of definition is circular because it does not tell us anything that we did not know previously save the fact that the word, "soporific", is interchangeable with the adjectival form of the word, "sleep". This type of argument is indeed blatantly circular, and the definition is trivial since it gives us no new information. On the other hand, however, to state that "p implies q" is equivalent to "not-p or q" is not trivial although strictly speaking, the definition is circular; in this definition of implication, we are told that implication can be broken down in terms of negation and disjunction - that is, some new information is imparted. Since the definition of implication is not blatantly circular, it is not open to a charge of circularity which is significant - consequently, implication can be defined. In <u>The Principles Of Mathematics</u>, Russell is too strict in the charge of circularity with regards to a definition of implication. If we adhered to Russell's strict view of circularity, we would have to admit that very few connectives, if any, could be defined within the boundaries of the propositional calculus. For example, if one attempted to define disjunction by saying that "p or q" is equivalent to "not(not-p and not-q)", then one would be open to the charge of circularity since equivalence is mutual implication and implication is reducible to disjunction and negation. In The Principles Of Mathematics, Russell's criteria for the circularity of a definition are in fact too narrow; it is possible therefore to define implication, and as Russell states in "The Theory Of Implication", it is to some extent a matter of choice as to which connectives are to remain undefined in a particular system.[165]

Although Russell takes implication to be indefinable in The Principles Of Mathematics, he does not take the notion of proposition to be indefinable. According to Russell, every proposition implies itself, and that which is not a proposition cannot imply anything. It is claimed by Russell therefore that "p is a proposition" is equivalent to "p implies p". The extension of Russell's argument is that if we have a formula, for example, "$p \supset q$", and we know that p is not a proposition, then we know a priori so to speak, whatever p and q may stand for, that "$p \supset q$" is false since p is not a proposition and that which is not a proposition cannot imply anything. As a result, the axioms of the propositional calculus, wherever it is necessary, must be prefaced with a hypothesis of the form, "p implies p". This hypothesis of the form, "p implies p", applies only to principal implications, not to subordinate implications. For example, in the formula, "$(p \supset q) \supset (p \supset q)$", if either p or q is not a proposition, then the subordinate implications will be false, but the principal implication will be true; it is not necessary to preface this formula, therefore, with the hypotheses, "p implies p and "q implies q", since the formula is logically true regardless as to whether p and q are propositions or not. 166

In the first appendix of <u>The Principles Of Mathematics</u> concerning Frege's logic and arithmetic, Russell states that Frege's interpretation of implication is similar to the interpretation of implication in <u>The Principles Of Mathematics</u> save for the fact that for Frege, the nonlogical symbols involved in an implication need not be propositions whereas the interpretation of implication in <u>The Principles Of Mathematics</u> stipulates that the nonlogical symbols involved in an implication must represent propositions. 166
Russell's interpretation of implication in <u>The Principles</u>

Of Mathematics stems from Peano.[168] Moreover, the interpretation of implication given by Frege is objected to by Russell on the grounds that the notions of negation and proposition cannot be defined thereby.[169]

In his article, "The Theory Of Implication", Russell discards the Peanesque interpretation of implication; in adherence of Frege's interpretation, the nonlogical symbols involved in an implication need not be propositions, and implication is simply defined as "p is not true or q is true". In support of his position, Russell cites the example of "$p \supset p$". If we wish to assert "$p \supset p$", then under the Peanesque interpretation, we must preface this formula with the hypothesis that p is a proposition; according to Russell, it is quite inconvenient to prove in each case when we wish to employ "$p \supset p$" that what it applies to is in fact a proposition. Furthermore, Russell argues in "The Theory Of Implication", there are paradoxes which accrue if implication is restricted by a hypothesis of the form, "p is a proposition". Consider the formula, "$(p \cdot q) \supset p$". If we follow the position of *The Principles Of Mathematics*, says Russell, "$(p \cdot q) \supset p$" will be false if q is not a proposition. However, "$(p \cdot q) \supset p$" means that if p and q are true, then p is true. We also know via exportation that "$(p \cdot q) \supset p$" is equivalent to "$p \supset (q \supset p)$". In this case, Russell contends that even if q is not a proposition, "$p \supset (q \supset p)$" will still hold since a true proposition must be capable of being implied by every entity q. It is the conclusion of "The Theory Of Implication" that the nonlogical symbols involved in an implication need not be propositions; in this light, the position in The Principles Of Mathematics vis à vis implication is considered to be incorrect.[170]

I wish to pause for a moment now in order to examine some of the claims made by Russell in both The Principles Of Mathematics and "The Theory Of Implication" with respect to implication and as to whether the nonlogical symbols involved in an implication must be propositions. The claims that I consider to be relevant to my analysis are the following: (1) every proposition implies itself; (2) whatever is not a proposition cannot imply anything; (3) "$p \supset q$" is false if p is not a proposition; (4) in the formula, "$p \supset (q \supset p)$", q is not subject to any limitation whatsoever since "$p \supset (q \supset p)$" would still be true even if q were not a proposition; (5) paradoxes result if we limit implication by the hypothesis of the form, "$p \supset p$".

(1) is a statement taken from The Principles Of Mathematics. I consider (1) to be both true and obvious. However, it is from (1) that Russell equates "p is a

proposition" with "p implies p". I would like to make the point that I do not accept this equivalence. While it can be maintained that if p is a proposition, then p implies p, I do not think that the converse is true. It is a presupposition of any well-formed formula, including "$p \supset p$", that the non-logical symbols within the well-formed formula are statement letters - that is, that the nonlogical symbols represent propositions. If the nonlogical symbols within the expression, "$p \supset p$", are not statement letters, then the expression is not well-formed. We can never know as to whether p implies p unless we are told beforehand that p is a statement letter.

(2) is also taken from The Principles Of Mathematics.[172] Like (1), I consider (2) to be also both true and obvious. Since implication is inextricably bound up with validity, it is impossible to have an argument in which a conclusion is validly inferred from premisses which are not propositions. A nonlogical symbol, p, which purports to be a proposition, and is in fact not a proposition, is not well-formed. Consequently, if p is not well-formed, then p cannot imply anything.

(3) is also taken from The Principles Of Mathematics.[173] (3) on the correct understanding of (2) is false. Russell

maintains in The Principles Of Mathematics that "$p \supset q$" is false if p is not a proposition; however, since p is not a proposition, "$p \supset q$" cannot be false, and it cannot be true either - it is simply not well-formed. From the perspective of The Principles Of Mathematics, Russell would argue that (1) and (2) are true, but he would maintain that (3) is also true. It is Russell's acceptance of (3) that leads to the paradoxes mentioned in "The Theory Of Implication". However, instead of rejecting (3) by itself, Russell in "The Theory Of Implication" rejected (2) as well. It is Russell's rejection of (2), I will maintain, that leads to other paradoxes which he does not seem to be aware of in "The Theory Of Implication". I will attempt to elaborate on this latter claim later on in my analysis.

(4) is a statement taken from "The Theory Of Implication".[174] In (4), it is claimed by Russell that even if q is not a proposition, "$p \supset (q \supset p)$" will still be true; (3) and (4) are similar in the respect that they both deny (2). However, if q is not a proposition, then "$p \supset (q \supset p)$" cannot be true, and it cannot be false either - on the presupposition that q is not a proposition, "$p \supset (q \supset p)$" is simply not well-formed. Hence, like (3), (4) is also false. It should be noted that Russell's discussion in "The Theory Of Implication" of the formula, "$p \supset (q \supset p)$", is very unclear,

and it would seem that rather than giving reasons, Russell argues more from the obviousness of his claim that "$p \supset (q \supset p)$" does not require that the nonlogical symbols be propositions if it is to be true. From the perspective of <u>The Principles Of Mathematics</u>, if q were not a proposition, "$p \supset (q \supset p)$" would not necessarily be false; the antecedent, "$q \supset p$", would be false, but "$p \supset (q \supset p)$" would be true when p is false and false when p is true.

Diagram V

p	$q \supset p$	$p \supset (q \supset p)$
T	F	F
F	F	T

if q is not a proposition: from the perspective of <u>The Principles Of Mathematics</u>.

(5) is also a statement from "The Theory Of Implication".[175] The hypothesis of the form, "$p \supset p$", was prefaced to several axioms of the propositional calculus in <u>The Principles Of Mathematics</u> in order to ensure that the nonlogical symbols represented propositions. Russell in "The Theory Of Implication" claims (5) to be true. As I have stated hitherto, Russell's explanation with respect to the fromula, "$p \supset (q \supset p)$", and its relationship to the hypothesis of the form, "$p \supset p$", is very unclear. However, if the assumptions of <u>The Principles Of Mathematics</u> are accepted, I do think that (5) will be true; this is not to say that (5)

is true, but only that if we accept certain premisses of *The Principles Of Mathematics*, (5) will be true. For example, if q is not a proposition, then the formula, "$[(p \supset q) \cdot p] \supset q$" will be false according to the perspective of *The Principles Of Mathematics*. We know however via importation that "$[(p \supset q) \cdot p] \supset q$" is equivalent to "$(p \supset q) \supset (p \supset q)$". Now, if q is not a proposition in the formula, "$(p \supset q) \supset (p \supset q)$", both the antecedent and the consequent will be false, but "$(p \supset q) \supset (p \supset q)$" will be true.

Diagram VI

$[(p \supset q) \cdot p] \supset q$	$p \supset q$	$(p \supset q) \supset (p \supset q)$
F	F	T

if q is not a proposition: from the perspective of *The Principles Of Mathematics*

It has been claimed that "$[(p \supset q) \cdot p] \supset q$" is equivalent to "$(p \supset q) \supset (p \supset q)$"; yet, if these two formulas are equivalent, they should have the same truth-value. However, if certain premisses of *The Principles Of Mathematics* are accepted, the said formulas differ in truth-value. Thus, if we accept certain assumptions from *The Principles Of Mathematics*, (5) will be true. It was perhaps to this type of paradox that Russell referred in "The Theory Of Implication".

Nevertheless, I would like to argue that with certain qualifications, (5) is flase. In the propositional calculus of The Principles Of Mathematics, the purpose of the hypothesis of the form, "p implies p", is to guarantee that the nonlogical symbols involved in an implication stand for propositions. To this extent, (2) was declared to be true in The Principles Of Mathematics. However, Russell considered (2) to imply (3). (2) does not imply (3), but (2) implies an amended version of (3) - namely, (3)', "$p \supset q$" is not a well-formed formula if p is not a proposition. If we accept (3)' rather than (3), we will not generate the so-called paradoxes that Russell loosely talks about in "The Theory Of Implication". In order for (5) to be false, we require one other qualification - that is, instead of the hypothesis of the form, "$p \supset p$", we require a hypothesis of the form, "p is a proposition". This latter qualification, as I have argued previously, indicates that "$p \supset p$" is not equivalent to "p is a proposition".[176] Now, I am not asserting here that we should in fact preface every well-formed formula with a hypothesis of the form, "p is a proposition". As Russell states in "The Theory Of Implication", the hypothesis of the form, "p is a proposition", would prove to be inconvenient if it had to be placed in front of every well-formed formula; however, such a hypothesis is only inconvenient because it need not be placed in front of every well-formed formula. In other words, in setting up the propositional calculus, we only need to state that every nonlogical symbol is a statement letter; this rids us of the inconvenience of stating this hypothesis in every case in which a well-formed formula appears. Russell's reason for the inconvenience of the hypothesis of the form, "p is a proposition", is due to the fact that he considers this to be a restriction upon implication resulting in paradoxes; consequently, Russell denies (2) and (3). If we accept (2) and (3)', I do not think that there are any paradoxes which result from the hypothesis of the form, "p is a proposition". My reason for stating that it is inconvenient to place the hypothesis of the form, "p is a proposition", in front of every well-formed formula is made only in the consideration of elegance- that is, it is more elegant to state initially when we are setting up the propositional calculus that all the nonlogical symbols are statement letters.

In the first appendix to The Principles Of Mathematics, Russell gives two reasons why Frege's interpretation of implication is unsatisfactory: (1) the notion of negation cannot be defined; (2) the notion of proposition cannot be defined. Russell defines negation as follows in The Principles Of Mathematics: "not-p is equivalent to the assertion that p implies all propositions, i.e. that 'r implies r' implies 'p implies r' whatever r may be."[177] Although this is a viable definition of negation, it should be noted that it is couched in the propositional calculus of The Principles Of Mathematics, a propositional calculus which is quite cumbersome. In "The Theory Of Implication", Russell abandons this definition of negation, and negation is taken to be a primitive idea.[178] Russell's first objection to Frege's interpretation of implication cannot count as a criticism since negation is usually considered to be undefined. With regards to Russell's second objection to Frege's interpretation of implication, we have shown hitherto that Russell's definition of proposition in The Principles Of Mathematics is inadequate. In his later book, Introduction To Mathematical Philosophy, although Russell differentiates between proposition and propositional function, he also remarks that "the word proposition cannot be formally defined".[179]

There is an interesting passage in The Principles Of Mathematics with respect to the role of nonlogical symbols involved in an implication: "Thus, for example, the proposition, 'x and y are numbers implies $(x+y)^2 = x^2+2xy+y^2$' holds equally if for x and y we substitute Socrates and Plato: both hypothesis and consequent, in this case, will be false, but the implication will still be true."[180] Although it is put in the context of elementary algebra, this is simply in effect Russell's contention that if a nonlogical symbol in an implication is not a proposition, then the implication in question will be false; it just so happens that in this particular example, we have subordinate implications embedded in a principal implication in which as a result, the principal implication is true. Not only does Russell abandon this contention in "The Theory Of Implication", but Russell in "The Theory Of Implication" also is willing to allow anything whatsoever to appear as a nonlogical symbol within an implication. I will now consider this position in an attempt to show its paradoxical nature.

Let us take a formula - for example, "$p \supset (q \supset p)$". If we substitute "Socrates" for "p" and "Plato" for "q" in the said formula, then we arrive at the expression, "Socrates $\supset$ (Plato $\supset$ Socrates)". In what sense can it be said that this expression is true? Certainly, we would not want to say that "Socrates" is true or false. If we consider once again Russell's algebraic example from <u>The Principles Of Mathematics</u>, $(x+y)^2 = x^2 + 2xy + y^2$, we can arrive at similar results by substituting "Socrates" for "x" and "Plato" for "y" - namely, $(\text{Socrates} + \text{Plato})^2 = \text{Socrates}^2 + 2\text{Socrates Plato} + \text{Plato}^2$. Consequently, it can be said that when a term is substituted for a variable within an expression, that term must fall within the variable class; if the term does not fall within the variable class, then the expression will not be well-formed. In the case of the propositional calculus, nonlogical symbols are required to represent propositions if they are to imply anything. It just so happens that propositions have that curious property of being either true or false, and it is in virtue of this characteristic of truth or falsity that propositions can be said to imply in Russell's sense of the word, "imply". I therefore consider the position taken in "The Theory Of Implication" with respect to propositions and implication to be fundamentally wrong.

In his later works, Principia Mathematica and Introduction To Mathematical Philosophy, Russell's interpretation of implication stipulates that the nonlogical symbols involved in an implication must represent propositions. In these works, there does not seem to be a continuation of the controversy as to whether the nonlogical symbols involved in an implication must be propositions or not; in effect, the problem for Russell ceases to be a problem. Moreover, it should be noted that this final position differs from the viewpoint of The Principles Of Mathematics and "The Theory Of Implication"; in this respect, one can discern a definite improvement of Russell's insight into the relationship between an implication and its nonlogical symbols.

C.3: FORMAL IMPLICATION

In The Principles Of Mathematics, Russell maintains that there are two types of implication "essential to every kind of deduction": formal implication and material implication.[181] In this present section, I propose to focus upon the former sort of implication in order to arrive at some understanding of it in the light of contemporary logic.

As Russell conceives of it, is formal implication a special sort of implication? It would seem at a prima facie level that the answer to this question must be in the affirmative. After all, Russell differentiates between formal implication and material implication, and just from this alone, it would seem that formal implication must be characterized by some distinctive feature that is not captured by material implication. There is much evidence in The Principles Of Mathematics that this is in fact the case: "This shows that we cannot hope wholly to define it [formal implication] in terms of material implication, but that some further element or elements must be involved."[182] However, in other parts of The Principles Of Mathematics, it would seem as if Russell is stating that formal implication can be derived from material implication: "A formal implication appears to be the assertion of a whole class of material implications."[183] In what follows, I will argue that although the answer to this question is not entirely clear-cut in the context of The Principles Of Mathematics, it would seem that Russell does in fact conceive of formal implication as a special sort of implication, distinct from material implication. However, before I pursue the answer to this question, I would like to discuss another issue with respect to formal implication - that is, the relationship of formal implication to the propositional calculus.

According to Russell's conception of pure mathematics, if a mathematical proposition, p, is asserted to be true of an entity, x, or a certain set of entities, x, y, z,..., then another proposition, q, can also be asserted to be true of the same entity or set of entities; neither p nor q is asserted separately of such entities. What is asserted, Russell claims, is a relation between p and q; it is this relation which Russell calls formal implication.[184] For Russell, mathematical propositions are characterized by the fact that they assert implications containing variables. It may appear at first sight, says Russell, that this is not the case - for example, the proposition of elementary arithmetic, "1+1=2", does not seem to be of this form; however, Russell maintains that the actual meaning of this proposition is the following: "if x is only one and y is one, and x differs from y, then x and y are two".[185]

In The Principles Of Mathematics, symbolic logic is said to require a number of indefinables; in Russell's opinion, one of these indefinables is formal implication having the form, "ϕx implies ψx for all values of x" where ϕx and ψx, for all values of x, are propositions. There is another sort of implication which is considered by Russell to be indefinable; it is an implication between propositions containing no variables - this, of course, is material implication.[186] At the outset, therefore, it would seem that the essential difference between these two types of implication is that formal implication contains variables whereas material implication does not.

It should be noted that both these types of implication are claimed to be relations by Russell. The propositional calculus as a branch of symbolic logic focuses upon the relation of material implication; it is a relation which can hold only between propositions. Yet, for Russell, formal implication somehow or other is involved in the propositional calculus; formal implication is a relation between propositional functions when one propositional function implies another propositional function for all values of the variable.[187] The question which I would like to pose at this juncture in my critique of formal implication is the following: if formal implication deals exclusively with propositional functions, then how is it possible for formal implication to play a role in the propositional calculus in which material implication is the type of implication holding between propositions?

In order to answer this question, it is necessary to understand the structure which Russell gives to the propositional calculus of The Principles Of Mathematics. According to Russell, the nonlogical symbols of the propositional calculus are not propositions; rather, they are variables. However, Russell also claims that p can imply q only on the presupposition that p and q are propositions containing no variables. If implication is therefore to occur at all in the propositional calculus, the variables must be restricted such that their values are those of propositions. Since material implication cannot handle variables, there must be some other means whereby a variable is allowed to take on the value of a proposition. It is here where formal implication is invoked in order that some sense can be made of the propositional calculus.[188]

Let us briefly look at one of Russell's axioms of the propositional calculus - the fifth axiom, that of simplification: if p implies p and q implies q, then p and q implies p; as in his other axioms which contain a hypothesis of the form, "p implies p", the main implication is said to be formal while the subordinate implications are said to be material. Russell's rationale for saying that the main implication is formal and that the subordinate implications are material is that the main implication concerns variables whereas the subordinate implications concern propositions which do not contain variables. In other words, the fifth axiom is simply a string of symbols divided into logical symbols and nonlogical symbols. The logical symbols have been given a meaning supposedly prior to the construction of any axiom whatsoever; as a result, they are constants: "We can now understand why the constants in mathematics are to be restricted to logical constants..."[189] Thus, the nonlogical symbols of the propositional calculus are variables. If certain formulas of the propositional calculus are to form the axioms of the system, then the variables within these formulas must be restricted if the implications in question are to be true. Since formal implication is the implication dealing with variables, it is the task of formal implication in the propositional calculus to restrict certain variables via the hypothesis of the form, "p implies p". When the said variables are restricted to take on the value of propositions, then these propositions are involved in subordinate implications; these subordinate implications are material. As a result, the principal implication of the fifth axiom is formal rendering the variables to take on values of propositions which in turn contain no variables. Although certain formulas may contain formal implications, this does not mean that the formula will necessarily be true - for example, if p implies p, then p implies p implies that p and not-p. Following the interpretation of The Principles Of Mathematics, the "if-then" is a formal implication and the first "p implies p" designates that in what follows the nonlogical symbol, p, is a proposition; if this formula did not have to be prefaced with the hypothesis, "p implies p", then the actual formula would read, " 'p implies p' implies 'p and not-p'" i.e. $(p \supset p) \supset (p \cdot \sim p)$; in the actual formula itself, the implications are material. In so far as the actual formula contains material implications, rather than formal implications, it is this relation of material implication that is of prime significance to the propositional calculus:

> Our calculus studies the relation of implication between propositions. This relation must be distinguished from the relation of formal implication, which holds between propositional functions when one implies the other for all values of the variable. Formal implication is also involved in this calculus, but is not explicitly studied. 190

Thus far, I have tried to explain the role which formal implication plays in the propositional calculus of The Principles Of Mathematics. Hitherto, I have argued that the hypothesis of the form, "p implies p", is not equivalent to "p is a proposition"; I therefore will not go into detail once again to the effect that the hypothesis of the form, "p implies p", cannot restrict a variable to the value of a proposition. With respect to the role that Russell gives to formal implication in the propositional calculus of The Principles Of Mathematics, I will contend firstly that the usage of formal implication is unnecessary in the propositional calculus.

In setting up the propositional calculus, one of Russell's main difficulties is that he allows the nonlogical symbols to be variables. Since Russell also holds that the expression, "$p \supset q$", is false, if either p or q is not a proposition, then if the truth of certain intuitive axioms is to be preserved, it would become necessary to restrict certain variables to the values of propositions; to this intent, Russell employs formal implication. However, all of these technical adjustments need not occur if one simply stipulates that the nonlogical symbols are propositional letters. If one is presented with an expression in which one of the nonlogical symbols is not a propositional letter, then the expression is not well-formed. This manner of setting up the propositional calculus is to be preferred in comparison to the propositional calculus of The Principles Of Mathematics on general grounds of economy. Consequently, formal implication need not be employed in the propositional calculus.

Furthermore, I also would like to contend that even if one accepts Russell's presupposition that the nonlogical symbols are variables, it follows by no means that the usage of formal implication is warranted. According to Russell, formal implications have the form, "ϕx implies ψx for all values of x" where ϕx and ψx, for all values of x, are propositions. However, in the propositional calculus,

formal implication is the principal implication of expressions having the following form: if p implies p and q implies q, then p and q implies p. Although it can be conceded that in both cases formal implication concerns variables, the interpretation of formal implication does not coincide with its usage in the propositional calculus. Following the interpretation of formal implication given by Russell in The Principles Of Mathematics, one can say that formal implication is characterized by propositional functions and the word, "every"; moreover, a propositional function has the form, "x is a ϕ", or if we interpret propositional functions extensionally, "x is in ϕ". On the other hand, formal implication as it is employed in the propositional calculus by Russell is not characterized by the word, "all"; it is also the case that the nonlogical symbols of the propositional calculus are single letters which when they are interpreted may have the form, "x is a ϕ", but need not have this form at all. I conclude therefore that firstly, Russell's usage of formal implication in the propositional calculus is unnecessary, and secondly, that even if the presupposition that the nonlogical symbols of the propositional calculus are variables is accepted, Russell's usage of formal implication in the propositional calculus is still unwarranted.

Having discussed the role which Russell gives to formal implication in the propositional calculus of The Principles Of Mathematics, I now would like to return directly to the issue as to whether Russell conceives of formal implication as a special sort of implication. In the first and second chapters of Part I of The Principles Of Mathematics, Russell in a number of places states that formal implication is a relation; at one point, Russell says that formal implication is a relation between propositional functions for all values of the variable, and at another point, Russell simply says that formal implication is a relation between assertions. In any event, it is safe to claim that for Russell in The Principles Of Mathematics, formal implications concern expressions having the form, "ϕx implies ψx for all values of x" where ϕx and ψx, for all values of x, are propositions.[191]

It would seem however in virtue of his analysis of formal implication in chapter three of Part I of The Principles Of Mathematics that Russell is not entirely satisfied with the mere contention that formal implication is a relation. Consequently, Russell's twofold purpose in chapter three of Part I of The Principles Of Mathematics is "to examine and distinguish these two kinds [of implication], and to discuss some methods of attempting to analyze the second of them [formal implication] ".[192]

In his attempt to explain formal implication more fully, Russell is led to consider the function of the variable with respect to the expression, "x is a man implies x is a mortal". Dismissing the Peanesque view of the variable in which the variable is allowed to vary only to members which belong to the class of men, Russell opts for the view in which the variable is allowed to take on all possible values.[193]

There is a claim by Russell that the implication, "x is a man implies x is a mortal", does not assert a relation of two propositional functions, but is itself a single propositional function which is constantly true.

One could argue with Russell concerning this claim since the variables in "x is a man" and "x is a mortal" are not bound. The implication, "x is a man implies x is a mortal", is symbolized as follows in the first order predicate calculus: $\phi x \supset \psi x$ (where ϕx = x is a man, ψx = x is a mortal). Here, Russell has forgotten to include the expression, "for all values of x". Indeed, the implication, "x is a man implies x is a mortal", asserts a relation between two propositional functions. Even if the expression, "for all values of x", were included in the said implication, I do not think that we would describe it to be a single propositional function; rather we would say that it is a universal statement (i.e. all men are mortal). However, on Russell's own premisses, if "x is a man implies x is a mortal" is a single propositional function, then it cannot be true as Russell claims it to be since a propositional function is neither true nor false.[194]

Nevertheless, the gist of Russell's argument with respect to the expression, "x is a man implies x is a mortal", is that it is not permissible to vary one variable and then to vary the other; if the variables are to be varied, then they must be varied simultaneously. Formal implication, Russell states, is not a single implication therefore, but is a class of implications; similarly, in a formal implication, we do not have an implication containing a variable, but we have a variable implication.[195]

One of Russell's main concerns with respect to formal implication is to pinpoint its differentia, that type of characteristic which sets formal implication apart from all other kinds of implication. In order to show that formal implication is not reducible to material implication, Russell offers two examples of formal implication: (1) x is a man implies x is a mortal for all values of x; (2) x is a man implies x is a philosopher for all values of x. It is possible in (1) to vary x for all possible terms; however, in (2), the variability of x is restricted if (2) is to remain true. As a result, Russell concludes that formal implication cannot be a disguised material implication: "This seems to show that formal implication involves something over and above the relation of implication, and that some additional relation must hold where a term can be varied."[196]

Since formal implication accommodates a definitive relation besides that of material implication, Russell considers it his task to specify just what this relation may be. It would seem that in many cases what we are trying to say in a formal implication as in the example, "x is a man implies x is a mortal for all x", is that the class of men is included in the class of mortals. However, Russell contends, there are other cases in which the relation of inclusion is not involved in a formal implication - for example, formal implications containing asymmetrical relations. To this endeavour, Russell proposes the notion of assertion as the discriminating factor of formal implication.[197]

I have discussed this theory of assertion hitherto in my exposition; as a result, although I will not give an explanation of it, I will state the conclusions which Russell infers from such a theory with regards to formal implication. It is claimed by Russell that formal implication is "derived from a relation between assertions";[198] it should be noted here that Russell does not say that formal implication is a relation. Furthermore, the notions of propositional function and assertion are deemed to be more primitive than the notion of class. One can discern the importance of formal implication, Russell claims, insofar as "it is involved in all the rules of inference".[199] In the final paragraph of chapter three of Part I of <u>The Principles Of Mathematics</u>, Russell sums up his discussion of formal implication by stating that formal implication is the assertion of a class of material implications in which no member itself contains a variable. The class of material implications is the class of all propositions where an assertion made concerning a certain subject or subjects implies another assertion having the same subject or subjects. In short, for Russell, formal implication holds due to some relation between the assertions in question.[200]

We have observed that in chapters one and two of Part I of The Principles Of Mathematics, Russell claimed that formal implication is a relation; sometimes, it is said that formal implication is a relation between propositional functions for all values of the variable, and at other times, it is simply said that formal implication is a relation between assertions. In chapter three, however, Russell does not state that formal implication is a relation, but rather that it is derived from a relation. Although it can be seen that this latter claim of chapter three has shifted from the earlier claim of chapters one and two, this latter claim does not seem to deny the earlier claim that formal implication is a relation. However, in his summary of Part I of The Principles Of Mathematics contained in chapter ten entitled "The Contradiction", Russell does in fact deny, consciously or unconsciously as the case may be, the claim put forward in chapters one and two that formal implication is a relation:

> In Chapter III we distinguished implication and formal implication. The former holds between any two propositions provided the first be false or the second true. The latter is not [italics mine] a relation, but the assertion, for every value of the variable or variables, of a

> propositional function which, for every value of the variable or variables, asserts an implication. 201

In this respect, Russell has at least three claims in <u>The Principles Of Mathematics</u> as to whether formal implication is a relation or not: (1) formal implication is a relation; (2) formal implication is derived from a relation; (3) formal implication is not a relation.

If (1) is true, then one can ask the following question: in what sense is the implication in formal implication different from the implication to be found in material implication? Russell does maintain unambiguously that material implication is a relation which holds between propositions. It can be said therefore that material implication applies to those cases in the propositional calculus in which the nonlogical symbols stand for propositions; for example, if p and q are propositions in the formula, "$p \supset q$", then the implication is material. On the other hand, formal implications are of the form, "$(x)(\phi x \supset \Psi x)$". My claim here with respect to formal implication is that in the expression, "$(x)(\phi x \supset \Psi x)$", the implication is a material one. The occurrences of the binary connective, "$\supset$", in the expressions, "$p \supset q$" and "$(x)(\phi x \supset \Psi x)$", are functionally equivalent. By the phrase, "functionally equivalent", I mean to say that the implications in both expressions function in the same manner, and as a result, both implications are material. In spite of the fact that "$\supset$" occurs in different contexts (that is, the one in the propositional calculus and the other, in the first order predicate calculus), it will not do to say that the "$\supset$" functions differently. If we maintain that the "$\supset$" functions differently in the first order predicate calculus than in the propositional calculus, then we should claim in a likewise manner that such logical operators or connectives as "v" also function differently in the first order predicate calculus than in the propositional calculus. Indeed, the binary connective, "$\supset$", functions in the same manner in both contexts. The difference between the two expressions, "$p \supset q$" and "$(x)(\phi x \supset \psi x)$", is not to be delineated by simply saying, as Russell does, that the former is a material implication while the latter is a formal implication. If we are to detect any difference between the two expressions, "$p \supset q$" and "$(x)(\phi x \supset \psi x)$", then we must look at the contexts in which such expressions occur, not at their implications. By this latter statement, I mean to say that there are independent notions in the first order predicate calculus which cannot be found in the propositional calculus. Some of these notions are: variable, propositional function and "every" or "some". The implication in the propositional calculus and the first order predicate calculus is the same-namely, material implication. It is not warranted to argue, as Russell implicitly does, that in virtue of the new notions present in expressions of the form, "$(x)(\phi x \supset \psi x)$", that therefore, the implication must be of a special sort. In setting up the first order predicate calculus, the approach is basically the same as that of the propositional calculus. One merely lists the symbols (the undefined connectives, the punctuation marks, individual variables, and predicate letters), the vocabulary, the axiom schemata and the rules of inference; it is permissible also to interdefine other connectives and to introduce quantifiers. It should be noted that the connectives are distinct from the variables and the quantifiers - that is, we can explain the role of the connectives without employing other subsequent notions such as the notion of the variable.

Thus, if the implication in formal implication is material, how is it possible for formal implication to be a relation? Although Russell states in chapters one and two of Part I of The Principles Of Mathematics that formal implication is a relation, he does not state therein of what this relation consists. It would seem therefore that formal implication cannot be a relation unless it has some hidden characteristics which transform it into a relation.

Since formal implications have the form, "$(x)(\phi x \supset \psi x)$", it would be more apt to describe such expressions as universal propositions. In a sense, this description of "$(x)(\phi x \supset \psi x)$" as a universal proposition is analogous to what Russell does by first stating that formal implication is derived from a relation between assertions and secondly, that formal implication is not a relation, but the assertion of a class of material implications. However, in his denial of the thesis that formal implication is a relation, Russell leaves himself open to the charge that formal implication in this respect cannot be considered to be an implication at all.

As Russell conceives of it in The Principles Of Mathematics, formal implication is indeed a special sort of implication. I have argued hitherto that the implication involved in formal implication is a material one, and as a result, formal implication cannot be considered to be a special sort of implication. One can attempt a reply by stating that it is nonsensical to ask what kind of implication is involved in formal implication since formal implication *is* exactly the kind of implication involved in formal implication, and to ask such a question concerning formal implication is like asking what sort of implication is involved in material implication to which the answer must be that it is material implication itself which is involved in material implication. I think, however, that this reply is untenable; it presupposes initially that formal implication is a special sort of implication and that it is impossible to break up formal implication into its constituent parts.

Russell need not have talked about formal implication at all. Formal implication is an unnecessary and unduly complicated notion if one is attempting to explain the role of a variable or of a propositional function. If Russell required some explanation of the function of a variable in a first order predicate calculus. for example, he could have introduced the existential quantifier, "$(\exists x)$", and for convenience sake, he could have defined the universal quantifier, "(x)", in terms of the existential quantifier - that is, "$(\exists x)(\phi x) = \mathrm{Df.} \sim (x) \sim (\phi x)$". In this way, one is not led astray in thinking that there are two types of implication - material implication and formal implication. Having accepted the notion of formal implication as a special sort of implication, it is no wonder therefore that one finds this type of statement in The Principles Of Mathematics: "The fundamental importance of formal implication is brought out by the consideration that is involved in all the rules of inference."[202]

Although Russell does talk about formal implication in his later works, it would seem, in spite of the fact that he refers to expressions of the form, "$(x)(\phi x \supset \psi x)$", as formal implications, that he does not think that formal implication is a special sort of implication, distinct from material implication. There is evidence for this claim insofar as formal implication is not involved in the propositional calculus of his later works. Moreover, in the first volume of Principia Mathematica, Russell defines formal equivalence by means of formal implication, and it would seem that he does not mean to say that this equivalence is of a special sort.[203]

C.4: MATERIAL IMPLICATION

In an inquiry concerning Russell's notion of implication, it is inevitable to a certain degree that the discussion should turn sooner or later to what have been labelled the "paradoxes of material implication". My analysis, however, of these so-called "paradoxes" does not purport to be novel in any way. What I propose to do is simply to examine the relevant material with respect to these "paradoxes" in an attempt to dispel some of the customary confusions which have been made. I will present therefore Russell's views concerning material implication contrasting such views with the objections raised by C.I. Lewis; eventually, I hope to clarify the entire issue by the employment of Quine's distinction of use versus mention.

In <u>The Principles Of Mathematics</u>, Russell takes implication to be indefinable; in fact, in one particular passage, Russell claims that material implication holds between propositions regardless of the truth conditions: "The relation [material implication] holds, in fact, when it does hold, without any reference to the truth or falsehood of the propositions involved."[204] Although I have examined this position and considered it to be erroneous, it should be noted that Russell abandoned this point of view in his later works. There is, of course, the other variation with regards to the notion of material implication - that is, as to whether the nonlogical symbols involved in an implication must represent propositions or not; with the exception of "The Theory Of Implication", Russell in general deems it necessary that the nonlogical symbols involved in an implication must represent propositions. I have also discussed this latter controversy hitherto concluding that if the nonlogical symbols of the propositional calculus are not propositional letters, then they are not well-formed and cannot imply anything whatsoever in Russell's sense of the word, "imply".

Nevertheless, in all his logical works, the interpretation given to implication, that of material implication, is fundamentally the same. For example, although Russell claims implication to be indefinable in The Principles Of Mathematics, he does state that it is a relation which holds between two propositions of which either the first is false or the second true.[205] In his article, "The Theory Of Implication", Russell states quite explicitly that the interpretation of implication is material implication: "Hence, 'p implies q' will be a relation which holds between any two entities p and q unless p is true and q is not true, i.e. whenever p is not true or q is true."[206] Similarly, in Principia Mathematica and Introduction To Mathematical

<u>Philosophy</u>, Russell defines implication, "$p \supset q$", as p is false or q is true.[207] It should be noted also that for Russell the formula, "$p \supset q$", can be read in a number of ways: (1) p implies q; (2) if p, then q; (3) if p is true, then q is true; (4) the truth of p implies the truth of q; (5) not-p or q; (6) either p is false or q is true; (7) unless p is false, q is true.[208]

In his early, unpublished article, "Necessity And Possibility", Russell argues against the introduction of the modalities such as "necessity" and "possibility" into the framework of mathematical logic; it would seem that Russell's main argument against the modalities is that they have only an epistemological or psychological significance; consequently, Russell states, they "are not notions which logic need take account of".[209] By giving the interpretation of material implication for implication, Russell also distinguishes the notion of deducibility from the notion of implication. According to Russell, "deducible from" is not to be equated with "implied by" since "deducible from" is defined in terms of the laws of deduction, and the laws of deduction employ the notion of implication; the substitution of "implied by" for "deducible from" is not permissible therefore on the grounds of circularity. In this respect, the notion of implication is more primitive than the notion of deducibility.[210]

It would seem that Russell's chief justification for the distinction between deducibility and implication is the following: if it can be shown by means of the laws of deduction that p implies q, then we know that q is deducible from p.[211]

I would like to add that Russell is not exactly clear in what he intends to say concerning this distinction between deducibility and implication. It would seem in virtue of the statements of the above paragraph that when one proposition implies another proposition, then it can be said that the latter proposition is deducible from the former proposition. We know that Russell often reads "$p \supset q$" as "p implies q"; however, if we know that "$p \supset q$" is true, this does not mean that q is deducible from p - that is, given the truth of "$p \supset q$" does not mean "$p \therefore q$". One would expect Russell to state that when q is deducible from p, it follows that p implies q; instead of this statement, the converse seems to be found in "Necessity And Possibility". Nevertheless, it is Russell's contention that inference involves something more than implication, and it is due to this fact that he concludes by saying that there is a logical distinction to be made between deduction and implication.[212]

In "Necessity And Possibility", Russell also argues briefly for the case of material implication as the correct view of implication: "This view of implication [that is, material implication] is rendered unavoidable by various considerations..."[213] Besides adducing the fact that Shakespeare and Bradley are in favour of material implication, Russell maintains that if "$(p \cdot q) \supset r$" is true, then "$p \supset (q \supset r)$" is also true; it follows from this reasoning therefore, says Russell, that if "$(p \cdot q) \supset p$" is true, then "$p \supset (q \supset p)$" must also be true. Russell considers these arguments to be somewhat cogent to support the case of material implication as the correct view of implication. Moreover, since "$p \supset (q \supset p)$" is true as a formula of material implication, we know, Russell asserts, that this means that a true proposition is implied by every proposition.[214] This latter claim - namely, "a true proposition is implied by every proposition" - has been referred to by C.I.Lewis as one of the "paradoxes of material implication".[215] It should be noted that in his article, "Necessity And Possibility", Russell does not suggest by any means that "$p \supset (q \supset p)$" is paradoxical. On the contrary, Russell deems the claim, "a true proposition is implied by every proposition", to be an essential trait of material implication. In fact, Russell's principal criticism of G.E.Moore's interpretation of the concept of necessity is that Moore's interpretation of the concept of necessity does not lead to an implication having what Lewis regards as the two "paradoxes of material implication". "This theory [concerning the concept of necessity by G.E.Moore] is not available with a doctrine of implication which holds that true propositions are implied by all propositions and false propositions imply all propositions."[216]

I would like to say just a few words concerning Russell's main argument in "Necessity And Possibility" presented for the case of material implication. Although it can be acknowledged that "$p \supset (q \supset p)$" does follow from "$(p \cdot q) \supset p$", I cannot see how this type of argument can be used as evidence for the case of material implication since material implication is tacitly presupposed by the interpretation given to "$\supset$".

In The Principles Of Mathematics, Russell takes material implication as the correct view of implication for the proper construction of pure mathematics. However, it would seem that Russell's claim with respect to the correctness of material implication extends beyond the realm of pure mathematics. Having discarded the psychological element involved in inference, Russell states that when we validly infer one proposition from another proposition, this inference is only possible on the grounds that there exists a relation between the two propositions regardless as to whether we discern such a relation or not; it is this relation which Russell refers to as material implication. Here, it would seem that Russell is maintaining that material implication is the implication to be found in ordinary language. It should be noted therefore that these are two separate claims: (1) material implication is the implication to be found in the construction of pure mathematics; (2) material implication is the implication to be found in ordinary language.[217]

Although Russell in The Principles Of Mathematics is not willing to admit that there may be other legitimate interpretations of implication, he does concede that our intuitions concerning implication probably do not coincide with that of material implication since material implication holds that a false proposition implies any proposition and a true proposition is implied by any proposition. Moreover, Russell acknowledges that it would not be claimed ordinarily that "2 + 2 = 4" can be inferred from "Socrates is a man", or that both these propositions can be implied by "Socrates is a triangle". However, it is Russell's contention that the only reason why we usually are not willing to admit such implications is due to a "preoccupation with formal implication".[218] Furthermore, Russell states, the customary unfamiliarity concerning the relation of material implication is insufficient evidence to prove that such a relation is unwarranted and specious.[219]

Before I move on to examine the notion of material implication in "The Theory Of Implication", I would like to make two brief remarks with regards to some of Russell's claims. Firstly, it is not due to a "preoccupation with formal implication" that people are unwilling to admit some of the implications accruing from the interpretation of material implication. Insofar as the implication involved in formal implication is a material one, a "preoccupation with formal implication" boils down to a preoccupation with material implication; in other words, if people are in fact preoccupied with formal implication and material implication is involved in formal implication, then since people are willing to admit the consequences of formal implication, they should also be willing to admit the implication resulting from material implication. Secondly, although Russell is quite right when he asserts that unfamiliarity with material implication does not prove that the relation is illusory, unfamiliarity with material implication does not help its credibility, especially if paradoxes or peculiarities can be generated from such a relation.

Unlike The Principles Of Mathematics, Russell in "The Theory Of Implication" admits that there are other legitimate interpretations of implication, besides that of material implication. However, although this acknowledgement is made by Russell, it is still maintained that material implication is "very much more convenient than any of its rivals".[220] This type of position with respect to the correctness of material implication is similar to the view expounded in Russell's article, "Mathematical Logic As Based On The Theory Of Types" in which Russell states that he does not mean to say that there can be no other interpretations given to implication, but rather that material implication is the most convenient interpretation of implication in the context of symbolic logic.[221] According to Russell in "The theory Of Implication", the underlying property of implication can be expressed in the following proposition: "What is implied by a true proposition is true."[222] By claiming this proposition to be the essential property of implication, one should notice that Russell is attempting to link up the notion of implication with the notion of validity since if one proposition is validly inferred from a true proposition, then that inferred proposition must be true.

Hugh MacColl's article of 1908 entitled "'If' And 'Imply'" attempts to maintain that in adopting "the usual view among logicians" concerning implication, Russell is led to paradoxical conclusions;[223] the main paradox which MacColl talks about can be found in The Principles Of Mathematics - namely, "of any two propositions, one must imply the other".[224] Since I have discussed the details of MacColl's reasoning in my exposition, I will proceed directly to Russell's reply to MacColl. Russell remarks that the reason why MacColl thinks that there are paradoxes resulting from material implication is due to a confusion between proposition and propositional function. Although Russell defends his point of view quite successfully against the objections of MacColl, Russell does admit in his reply to MacColl that material implication is not the customary meaning attributed to implication; as in his other works, Russell's justification of material implication hinges on the fact that material implication is more convenient than any other interpretation of implication in the field of mathematical logic.[225] It is interesting to contrast here MacColl's comment that material implication is the standard interpretation given to implication by logicians with Russell's comment that material implication is not the ordinary meaning which people assign to implication. In contradistinction to the perspective of The Principles Of Mathematics, it would seem that in his reply to MacColl, Russell is committing himself to the view that material implication is not the implication to be found in ordinary language.

The position taken in Principia Mathematica with respect to material implication is much akin to the position voiced in "The Theory Of Implication" insofar as it is claimed in Principia Mathematica that material implication is the most convenient interpretation of implication for the logical enterprise at hand.[226] In Principia Mathematica, however, as in "Necessity And Possibility", Russell differentiates between implication and inference; yet, it would seem that in Principia Mathematica, Russell is much clearer in what he intends to say concerning such a distinction. As Russell explains it in Principia Mathematica, the word, "implies", in the formula, "p implies q", is nothing more than a connection between p and q which can be expressed commensurately by the formula, "not-p or q". Inference, however, is not a connective although it involves implication in an implicit way - that is, if p and "p implies q" are asserted to be true, then q can be inferred to be true; for Russell in Principia Mathematica, inference involved the detachment of a true premiss - more specifically speaking, inference is "the dissolution of an implication".[227] I would like to contend nevertheless that if we adhere to Russell's usage of the word, "implies", inference is in fact related to implication, but not in the exact manner which

Russell suggests that it is. Although some inferences are of the form, "p, p implies q, $\therefore$q", there are other forms of inference which do not involve an implication - for example, "p v q, $\sim$p, $\therefore$q". As Russell uses the word, "implies", how is it possible therefore for implication to be associated with inference at all? The link between inference and implication can be expressed as follows in Russellian terminology: if a proposition, q, is inferred validly from a true proposition, p, not only do we know that q must be true, but we also know that p implies q (i.e. $p \supset q$).

In his later book, <u>Introduction To Mathematical Philosophy</u>, Russell was able to express the relationship between inference and implication in a more lucid manner:

> In order to be able validly to infer the truth of a proposition, we must know that some other proposition is true, and that there is between the two a relation of the sort called "implication", i.e. that (as we say) the premiss 'implies' the conclusion. 228

Again, one could argue with Russell's notion of inference with respect to his clause that it must be known that there is a relation of implication holding between the premiss and the conclusion. It is often the case that we can make an inference from a premiss without knowing that there is a relation of implication between the premiss and the conclusion. It would seem however that this question concerns epistemological priority. For example, given both $\sim p$ and "p v q" it is permissible to infer q. Does one infer q on the knowledge that "$[\sim p \cdot (p \vee q)] \supset q$" is true? Although we do know that "$[\sim p \cdot (p \vee q)] \supset q$" is true once we have made the inference to q, it would seem rather that we make the inference to q on a rule of inference - in this case, the rule of inference governing disjunctive arguments. Since this is an epistemological question, not a logical question, I will not pursue the issue any further.

In the Introduction To Mathematical Philosophy, Russell acknowledges that there are other interpretations of implication, besides that of material implication, but in virtue of its convenience, material implication is to be preferred to all other possible interpretations of implications. For Russell in the Introduction To Mathematical Philosophy, implication is to have that characteristic such that it is permissible to infer the truth of q given the truth of p - that is, having the true implication, "p implies q", and given the truth of p, the truth of q can be inferred.[229]

In his discussion of C.I.Lewis' repudiation of material implication Russell states in the Introduction To Mathematical Philosophy that mathematics does not require

Lewis' fundamental notions of strict implication and formal deducibility; consequently, on grounds of parsimony, Russell argues that Lewis' interpretation of implication should not be accepted. Furthermore, Russell points out, Lewis' logical system is not truth-functional, and this alone provides sufficient reason for the rejection of Lewis' interpretation of implication.[230]

Lewis' objections concerning material implication are at least twofold: (1) material implication is a relation only of truth-values, "not of content or logical significance";[231] (2) material implication is a relation which generates peculiar properties.[232] With respect to (1), Lewis maintains that Russell's interpretation of implication leads to the view that one proposition is equivalent to another proposition if the said propositions have the same truth-value; it would seem therefore that Lewis is trying to say that propositions are not necessarily equivalent if they have the same truth-value. With respect to (2), Lewis cites the two "paradoxes of material implication": (1) a true proposition is implied by any proposition i.e. $p \supset (q \supset p)$; (2) a false proposition implies any proposition i.e. $\sim p \supset (p \supset q)$. According to Lewis, there are other paradoxical results stemming from material implication, but specifically, these two paradoxes are the most blatant. In Lewis' opinion, these are due to the interpretation of "p implies q" as "not-p or q" in a two-valued system.[233]

Lewis' alternative system of strict implication interprets "p implies q" to mean "p strictly implies q", i.e. "p ⥽ q" ("p fishhook q"). In the system of *Principia Mathematica*, "p ⊃ q" is defined in terms of negation and disjunction as " ∼p v q"; in turn, " ∼p v q" is equivalent by DeMorgan's law to " ∼(p ⋅ ∼q)". In the definition of "p ⥽ q", we are introduced to the modal notions of possibility and necessity such that "p ⥽ q" is equivalent to "∼◊(p ⋅ ∼q)" ("◊" or diamond is the sign for possibility); since "◊p" can be defined in terms of the necessity operator such that "◊p = Df.∼□∼p", strict implication can also be defined as follows: "p ⥽ q = Df. □∼(p ⋅ ∼q)".[234]

It would seem that Russell does not deny Lewis' contention that there are certain paradoxes resulting from material implication. In fact, in his works on logic, Russell considers the "paradoxes of material implication" to be important definitive traits of material implication so much so that if an implication cannot generate such paradoxes, then it is deemed to be logically deficient from Russell's point of view.[235] However, this would seem to be very strange. If the "paradoxes of material implication" are indeed genuine, then material implication is rendered suspect since it is supposedly the cause of such paradoxes. Now, Russell does not state in any of his works that what Lewis refers to as the "paradoxes of material implication" are paradoxical although it is mentioned in The Principles Of Mathematics that these result "do not by any means agree with what is commonly held concerning implication..."[236] If Lewis is correct with respect to the "paradoxes of material implication", then it would seem that Russell's logical system is built upon a foundation which wrongly construes the notion of implication. In an attempt to elucidate this issue concerning such "paradoxes", I will embark upon an analysis of the meaning and function of "$\supset$" and "materially implies".

In order to distinguish between "$\supset$" and "materially implies", it would no doubt serve my purpose best if I worked in a roundabout way via other logical distinctions. Let us consider these two propositions: (1) The Bible is a book inspired by divine revelation; (2) Bible is a five letter word. A confusion often results in thinking that the employment of the word, "Bible", is contextually the same in both propositions. Indeed, if the word, "Bible", were employed in the same manner in both the above propositions, then the following argument would be valid.

P A book inspired by divine revelation is the Bible.
P Bible is a five letter word.
C A book inspired by divine revelation is a five letter word.

This argument seems to be valid, and yet, we know only too well that "a book inspired by divine revelation" is not "a five letter word". Nevertheless, if the argument is valid and both premisses are true, then the conclusion must also be true. So, where does the fallacy reside? Medieval logicians considered such arguments to suffer from a fallacy of ambiguity insofar as "Bible" is taken as a word in one place and as a thing in another; thus, in medieval terminology, the above argument commits the fallacy of <u>quaternio terminorum</u>, the fallacy of terms.[237] More specifically speaking, we can say that the above argument commits a fallacy in not distinguishing between a thing and the name of a thing; the first premiss talks about a thing - namely, the Bible - whereas the second premiss is talking about the name of a thing - namely, "Bible".

Now, this distinction between a thing and the name of a thing is but a preliminary to Quine's logical distinction of use and mention. It is this distinction of use and mention which will later serve to illuminate and to differentiate between " $\supset$ " and "materially implies". We can easily correct (2) by the insertion of quotation marks as follow: (3) "Bible" is a five letter word. Thus, (3) contains the name of a five letter word, but (1) contains the name of a book. Furthermore, (3) is talking about a word that it is contained in (1), and (1) is talking about a book, not a word at all. The name of a book is used in (1); consequently, the book is mentioned in (1). The quotation of the name of a book is used in (3); consequently, the name of a book is mentioned in (3). Bible is mentioned in (1), and "Bible" is used in (1); "Bible" is mentioned in (3), and "Bible" is used in (3). We may quote Quine to summarize the distinction between use and mention: "We mention x by using a name of x."[238]

Let us now consider five more propositions: (4) Willard is winsome; (5) "Willard is winsome" is true; (6) Willard is winsome is true; (7) ~Willard is winsome; (8) ~"Willard is winsome".[239] (4) is about Willard while (5) is about proposition (4). (6), in spite of its grammatical defect, is not about proposition (4) since it does not employ quotation marks appropriately. All that we can say about (6) is that it is not well-formed; we do not want to say that (6) is of the same form as (4) because it leads to the view that the suffix, "is true", is vacuous, and furthermore, that the suffix, "is false", is of the same logical status as the prefix, "~". The connective, "~", is in the object language whereas the predicates, "is true" and "is false", are in the meta-language. (5) employs the suffix, "is true", as a predicate which is attached to the name of a proposition - that is, "is true" is a predicate which is employed to speak about propositions. The connective, "~", is attached to a proposition to form a proposition, but the predicate, "is true", is attached only to the name of a proposition. Hence, (7) is similar to (4) since (7) is talking about Willard. However, (8) treats the unary connective, "~", to be of the same species as the predicate, "is false"; as a result, (8) commits a mistake in placing the name of a proposition after the prefix, "~", since the prefix, "~", can only be flanked by propositions.[240]

We now adequately have drawn enough distinctions in order to differentiate between "⊃" and "materially implies". The connective, "⊃", is of the same logical status as the other propositional connectives, "~", "&", "v", and "≡". By the phrase, "the same logical status", I mean to say that the "⊃" is a connective, and like other connectives, the "⊃" is flanked by propositions, not by the names of propositions; the connective, "⊃", belongs to the object language, not to the meta-language. Thus, we are allowed to write: (9) roses are red ⊃ roses are coloured; we are not allowed to write: (10) "roses are red" ⊃ "roses are coloured".

(9) rightly shows that the "⊃" is a binary connective flanked by propositions. (10) wrongly shows that the "⊃" is a two-place predicate or a predicate of degree two flanked by the name of propositions. The "⊃" (horseshoe or hook) is read as "if-then", and we are allowed to write: (11) if roses are red, then roses are coloured. Since the connective, "⊃", is equivalent to the connective, "if-then", we know that "if-then" is in the object language, and furthermore, that (11) is equivalent to (9). (9) and (11) do not talk about propositions, but talk about roses. In summary, the "⊃" has the same meaning as the connective, "if-then". The "⊃" functions as a closed, binary connective flanked by propositions, not by the names of propositions.

When we come to the word, "implies", we must be careful not to confuse its logical status with that of the binary connective, "⊃" or "if-then". The verb, "implies", belongs to the meta-language, not to the object language, and it is not a binary connective, but a two-place predicate or predicate of degree two, which is flanked not by propositions, but by the names of propositions. Thus, we are permitted to write: (12) "Socrates is a man" implies "Socrates is a mortal"; we are never permitted to write as Russell does: (13) Socrates is a man implies Socrates is a mortal. (12) correctly shows that the predicate, "implies", is flanked by the names of propositions whereas (13) wrongly construes "implies" to be a binary connective flanked by propositions.[241]

In like manner, we must not regard "materially implies" as the reading of " $\supset$ ". To do so would render "materially implies" as a binary connective. It can be said therefore that "materially implies" is a two-place predicate; as a result, material implication is a dyadic relation.[242] One proposition materially implies another when the truth-functional hypothetical, in which one proposition is the antecedent and the other proposition is the consequent, is true. The only condition under which one proposition will not materially imply another is when the antecedent is true and the consequent is false - that is, in all other cases, one proposition is said to materially imply another.[243] In contrast to the " $\supset$ ", "materially implies" belongs to the meta-language.

It may be asked at this juncture as to what bearing this analysis might have with respect to the issue concerning the "paradoxes of material implication". I should reply to this type of query by saying that without such an analysis of " $\supset$ " and "materially implies", it is almost inevitable that certain confusions arise resulting in supposed "paradoxes of material implication". Consequently, I do not regard the "paradoxes of material implication" to be genuine. In what follows, I will attempt to show by means of the light and labour of the foregoing analysis that the so-called "paradoxes of material implication" are merely specious.

The "paradoxes of material implication", as Lewis calls them, are essentially twofold: (1) "$p \supset (q \supset p)$" - a true proposition is implied by any proposition; (2) "$\sim p \supset (p \supset q)$" - a false proposition implies any proposition. It is claimed by Lewis that such "paradoxes of material implication" stem from Russell's interpretation of implication - namely, material implication. However, I will argue that these "paradoxes" are not the consequence of material implication, but rather are generated by Russell's and Lewis' employment of certain logical words.

Let us take for example the conditional: if it rains, then the streets are wet.[244] It would seem in this conditional that there is some sort of causal or factual connection of content insofar as the consequent seems to be factually or causally dependent upon the antecedent. The logical interpretation of material implication does not consider a conditional to embody some sort of causal connection; the only sort of connection that is said to be asserted between the antecedent and the consequent of a conditional is a truth-functional connection. In other words, the truth of a conditional depends upon the truth-values assigned to its parts. Thus, a conditional is no more than a "truth-functional compound", and no other relation can be said to exist between the antecedent and the consequent of a conditional under the context of material implication.[245]

It follows from the definition of "$p \supset q$" that if p is false, "$p \supset q$" will be true regardless of the truth-value of q, and if q is true, "$p \supset q$" will be true regardless of the truth-value of p. The consequences of the definition of "$p \supset q$" may seem odd, but they are in no way paradoxical. How do the "paradoxes of material implication" therefore come about? It is possible to explain the presence of such "paradoxes" in a number of steps. When the proposition, p, is asserted, it is considered to be true - that is, the proposition, p, is read as "a true proposition"; likewise, $\sim p$ is read as "a false proposition". Secondly, the "$\supset$" is construed to be a two-placed predicate; in other words, instead of reading "$p \supset q$" as "if p then q", "$p \supset q$" is read as "p materially implies q" or "p implies q". From these two steps, we arrive at the two "paradoxes of material implication": (1) a *true* proposition *is implied* by any proposition; (2) a *false* proposition *implies* any proposition. Now, it is possible to regard these two claims as unparadoxical if "implies" is taken as a binary connective; however, it is at this point that a shift in meaning occurs. In order to arrive at the "paradoxes of material implication", the word, "implies", is not equated with "if-then", but is equated with the predicate "follows logically from"; in this manner, the relation of implication is associated with the relation of deducibility or logical sequence. If this shift of meaning has taken place, then (1) and (2) are indeed paradoxical since they respectively mean the following: (3) a true proposition <u>follows logically</u> from any proposition; (4) any proposition <u>follows logically</u> from a false proposition. These "paradoxes of material implication" do not follow from the definition of "$p \supset q$"; on the contrary, these "paradoxes" are the result of subtle confusions of use with mention committed both by Russell and by Lewis. Basically speaking, there is a confusion between "$p \supset q$" and "$p, \therefore, q$".[246]

One can generalize with respect to the "paradoxes of material implication" by stating that the entire issue is constructed by means of a failure to carefully observe the distinctions of object language and meta- language. Some of the confusions made by Russell surrounding the issue of the "paradoxes of material implication" are the following: (1) the reading of "$p \supset q$" as "p implies q"; (2) the reading of p as "a true proposition"; (3) the reading of "$\sim$" as "is false".

In The Principles Of Mathematics, there is at least one other blatant example of a confusion between use and mention; I am referring to Russell's doctrine that any term can be a logical subject. Russell's chief argument in The Principles Of Mathematics for the claim that every constituent of a proposition can serve as a logical subject runs as follows: if we try to contend that it is not the case that every constituent of any proposition is possible of being a logical subject, then we are forced to utter propositions in defence of such a denial such as - "the is not a logical subject"; however, in such a proposition, the is employed as a logical subject, and therefore, it is possible for any constituent of a proposition to act as a logical subject.[247] This argument, of course, is fallacious; in such circumstances, one should draw the distinction between use and mention since the proposition, "the is not a logical subject", fails to employ quotation marks appropriately. It is unfortunate that Russell was unaware of the distinctions of use and mention and object language and meta-language; otherwise, he would not have arrived at some of his perplexing results, especially is the case with respect to the propositional calculus of The Principles Of Mathematics discussed hitherto.[248]

Even if the "paradoxes of material implication" were genuine, Lewis' objection to Russell's notion of material implication would be self-defeating since there are corresponding "paradoxes" in Lewis' system of strict implication: (1) $\sim\Diamond\sim p \prec (q \prec p)$ - if q is necessary, then any proposition p strictly implies q; (2) $\sim\Diamond p \prec (p \prec q)$ - if p is impossible, then p strictly implies any proposition q.[249] Russell's major criticisms of Lewis' system of strict implication are twofold: (1) it is not truth-functional; (2) the notion of strict implication along with the other notions associated with strict implication are ideas which pure mathematics does not require.[250] To be truth-functional not only means that the truth-value of a formula is a function of the truth-values of the components of the formula, but also that one formula is truth-functionally equivalent if both formulas have the same truth-value. For example, in the conditional, "Socrates is a man $\supset$ Socrates is a mortal", it is permissible to substitute a proposition for the antecedent or the consequent only if the conditional has the same truth-value throughout. Lewis' system of strict implication denies the equivalence of the former conditional with the conditional having the substituted proposition in spite of the fact that both conditionals have the same truth-value. Oddly enough, Lewis attacks Russell on the grounds that his system is only truth-functional. It would seem that here we have two interpretations of what logic should incorporate. If Russell is to contend that Lewis's system is too wide, then Lewis would only respond by saying that Russell's conception of logic is too narrow. With respect to (2), however, it would seem that Lewis' system of strict implication is not needed for the construction of pure mathematics; insofar as (2) is the case, Russell's system of material implication, on the grounds of economy, is to be preferred to Lewis' system of strict implication from the perspective of pure mathematics.

Although material implication is to be preferred with respect to pure mathematics, it may be wondered as to whether it is satisfactory with respect to ordinary language. In The Principles Of Mathematics, it would seem that Russell is stating that material implication is the implication to be found in ordinary language;[251] in his reply to MacColl, Russell explicitly denies such a claim.[252] It would seem however that there are at least two other relevant aspects to this question: (1) Russell's general belief that ordinary language is logically inadequate; (2) the fact that Russell was intent upon providing a logical framework not for ordinary language, but for the construction of pure mathematics. I do not propose to settle this question by any means, but I will attempt to show that in spite of lingering doubts, material implication would seem to be the best interpretation of the conditional.

Let us consider once more the conditional: if it rains, then the streets are wet. Now, if it just so happens that it does rain and the streets are wet, we would claim the conditional to be true, and furthermore, if it rains and the streets are not wet, we would claim the conditional to be false. Generally speaking, it is in these two cases alone that the "ordinary understanding" of the conditional coincides with that of material implication. In the other two remaining cases, when the antecedent is false and the consequent may be either true or false, there seems to be a divergence with respect to the "ordinary understanding" of the conditional. If we take the "ordinary understanding" of the conditional in these two remaining cases, I think the tendency would be to say that the truth-value of the conditional is undetermined. However, if the antecedent is false in the context of material implication, the conditional is defined to be true. I am claiming that the conditional is undetermined in these two remaining cases from the point of view of the "ordinary understanding" of the conditional because there would seem to be no reason to justify the conditional as being true or false. I offer the following diagram as a comparison of the "ordinary understanding" of the conditional with the interpretation of the conditional under the context of material implication; I take "p→q" as the representation of the "ordinary conditional". 254

Diagram VII

p	q	p ⊃ q	p → q
T	T	T	T
F	T	T	?
T	F	F	F
F	F	T	?

Although "p→q" would seem to be a more precise interpretation of the conditional with respect to ordinary language, it would seem that there are drawbacks to such an interpretation. For example, consider the conditional, "p→p". We would like to say, in order to preserve identity, that "p→p" is true, but when the antecedent of "p→p" is false, "p→p" is undetermined. Moreover, on this so-called "ordinary understanding" of the conditional, mathematical logic becomes more cumbersome due to the existence of undetermined values.

In recent years, some have maintained that formal logic is not as elastic as ordinary language, and therefore, formal logic cannot hope to cater to the complex examples of ordinary language; 255 others have replied by suggesting that a formal language has its own specific purpose to accomplish, and in the first place, formal logic does not pretend to be equivalent in breadth to ordinary language.[256] Perhaps, what can be said with respect to material implication and ordinary language is that although material implication cannot capture all the nuances expressed by a conditional, it is the best interpretation formally speaking that has been devised thus far.

C.5: OTHER CRITICS

I would like to say at the beginning of this section that there has not been a great amount of secondary literature concerning Russell's notion of implication; no doubt the most popular issue to date has concerned the "paradoxes of material implication", and insofar as I have just discussed this issue, it will not be necessary to re-examine the customary criticisms that have been put forward. Following a chronological sequence, I will endeavour to assess the views of various authors on two grounds: (1) as to whether their expositions are correct; (2) as to whether their criticisms are sound.

The first article which I wish to discuss was written in 1905 by G.E. Moore entitled "Russell's Principles Of Mathematics"; Moore's article is quite lengthy in its exhaustive treatment of The Principles Of Mathematics, and there are a number of pages devoted to Russell's notion of implication.[257] In an attempt to explain what Russell means by the phrase, "can be deduced from", in the proposition, "mathematics can be deduced from logic", Moore directly undertakes a critical investigation of Russell's notion of implication.[258]

As Moore understands The Principles Of Mathematics, implication is a simple concept, but it would seem that implication can still be defined. According to Moore, Russell in The Principles Of Mathematics defines "p implies q" as "'p is true' and 'q is false' are not both true". Moreover, Moore considers it quite obvious that two important consequences follow from this definition of implication: (1) every true proposition is implied by every other proposition; (2) every false proposition implies every other proposition. There are also other paradoxical consequences, Moore maintains, which stem from this definition of implication: (3) "Socrates is a triangle" implies "2+2=4"; (4) "2+2=4" is deducible from "Socrates was a man".[259] Before I proceed any further in the exposition of Moore's article, I would like to comment briefly on some of Moore's claims concerning The Principles Of Mathematics. Firstly, Russell does state in The Principles Of Mathematics that implication is a simple concept, but Russell contends that implication is indefinable; to state that "p implies q" is equivalent to "x" is unwarranted in Russell's opinion since equivalence involves mutual implication.[260] I regard the rest of Moore's claims to be basically correct since Russell actually does hold that (1) through (4) follow from the definition of implication; that Russell in The Principles Of Mathematics states that these consequences follow from the definition of implication is true, but this does not mean to say that these claims are in fact true. It may be questioned as to whether (4) is correct within the context of The Principles Of Mathematics. As Russell employs the word, "implies", and the phrase, "deducible from", (4) is not correct strictly speaking. "2+2=4" is not deducible from "Socrates was a man", but since "Socrates was a man" implies "2+2=4" and since "Socrates was a man" is true, it follows that "2+2=4" is deducible from these two propositions. Thus, (4) is partially correct.

Moore is not particularly upset by the paradoxical consequences which occur from the definition given to implication in The Principles Of Mathematics insofar as Russell himself concedes that such consequences are paradoxical, but Moore is concerned about another issue - that being, that Russell supposedly employs a different interpretation of implication than the one defined:

> But what he [Russell] does not point out is that he himself constantly, in his most important propositions, uses both 'implication' and 'deduction' in a sense which is different from that which he 'defines' by this assertion of equivalence - in a manner in which 'p implies q' is not equivalent to 'p is true and q is false are not both true'. 261

In an attempt to justify his claim, Moore provides three instances of the inconsistent usage of the word, "implies": (1) Russell contends that two propositions are equivalent when they mutually imply each other; it is also true, Moore states, that from Russell's definition of implication, if two propo-

sitions are true, then they mutually imply one another. However, Moore asserts, it would seem that Russell does not mean to say that any two true propositions are equivalent, but rather that only certain true propositions are equivalent. '262
(2) Russell declares that certain true propositions are "indemonstrable", and the word, "demonstration", is equated with the word, "deduction". However, if any proposition, q, is both true and indemostrable, then it can be implied by another true proposition, p, such that p implies q. Since p is true and "p implies q" is true, q can be demonstrated. It would seem, Moore claims, that on Russell's definition of deduction, any true proposition is demonstrable contrary to Russell's other statement that certain true propositions are indemonstrable. (3) Russell's major contention in <u>The Principles Of Mathematics</u> is that pure mathematics can be "deduced from" logic. However, in Russell's sense of implication and deduction, any true proposition whatsoever can be deduced from logic. Since the principles of logic are true and since on Russell's definition of implication, the principles of logic will imply any true proposition, any true proposition can be deduced from the two above propositions i.e. given that the principles of logic are true, and that the principles of logic imply any true proposition, any true proposition can be inferred. If we adhere to Russell's definition of implication and deduction, Moore states, then

Russell's claim that pure mathematics is derived from logic is "profoundly unimportant".[263]

It would seem, Moore suggests, that Russell's definition of implication as "'p is true' and 'q is false' are not both true" is different from the definition of implication involved in the notion of deduction. Concerning this other form of implication which Russell tacitly presupposes, Moore makes a number of points: (1) there is a sense of the word, "implication", in which the "paradoxes of material implication" (although Moore considers what Lewis refers to as the "paradoxes of material implication" to be paradoxical, Moore does not call them the "paradoxes of material implication") do not hold; (2) this other form of implication is philosophically more important; (3) as to the nature of this form of implication, it is difficult to say. In order to distinguish this other form of implication from Russell's definition of implication, Moore calls this other form of implication, "implication in the ordinary sense".[264]

What are we to make of Moore's major criticism supported by his examples taken from The Principles Of Mathematics? Since Moore employs his examples as justification of his position that Russell is inconsistent with respect to his usage of the word, "implication", I would like to examine briefly each of these examples.

I would like to point out that in each of the examples, Moore is aware of some sort of inconsistency, but Moore is unable to state the source of the inconsistency. Moore's first example shows that on Russell's interpretation of implication, any two true propositions are equivalent. This first example can only come about if we read "if p then q" as "p implies q" and "p if and only if q" as "p is equivalent to q". It is Moore's claim in his second example that Russell's thesis concerning the indemonstrability of certain true propositions is contrary to Russell's meaning of deduction. Moore's argument is as follows: if we have a proposition, q, which is both indemonstrable and true, then we know that any proposition, p, will imply q such that "p implies q" is true; consequently, since p and "p implies q" are true, q can be demonstrated. To this, I think, Russell could argue that if q is known to be true, then the demonstration of q is rather trivial since the premiss, "p implies q", can only be known to be true in this particular case if q is true. This type of rebuttal is put forward in the <u>Introduction To Mathematical Philosophy</u>: "Whenever q is already known to be true, "not-p or q" is also known to be true, but is again useless for inference, since q is already known, and therefore does not need to be inferred."[265] It would seem however that Moore's second objection stems from the claim, "a true proposition is implied by any proposition". - Since Moore's third example is similar to his second example, I will not delve into it.

Moore's major criticism with respect to Russell's inconsistent usage of the word, "implies", is basically sound. Russell interprets "if p then q" as equivalent to "p implies q", with the result that a binary connective is soon transformed into a two-place predicate. However, it would seem that Moore also treats "if p then q" as "p implies q". In this sense, Moore is not aware of the source of Russell's inconsistent usage of the word, "implies".

I now would like to deal with a criticism to be found in Wittgenstein's Tractatus. With regards to the hypothesis of the form, "p implies p", Wittgenstein states:

> Thus in Russell's Principles Of Mathematics 'p is a proposition' - which is nonsense - was given the symbolic rendering '$p \supset p$' and placed as an hypothesis in front of certain propositions in order to exclude from their argument-places everything but propositions.
>
> (It is nonsense to place the hypothesis '$p \supset p$' in front of a proposition, in order to ensure that its arguments shall have the right form, if only because with a non-proposition as argument the hypothesis becomes not false but nonsensical, and because arguments of the wrong kind make the proposition itself nonsensical, so that it preserves itself from wrong arguments just as well, or as badly, as the hypothesis without sense that was appended for

that purpose.)[266]

In the above passage, Wittgenstein maintains that the hypothesis of the form, "p implies p", is not required in the propositional calculus since an expression will be nonsensical if it is not a proposition. This is analogous to the position that I have taken in the second subsection of my critique; I have said in that particular subsection that if a nonlogical expression of the propositional calculus does not stand for a proposition, then it is not well-formed. Although Wittgenstein does not give an appropriate reason as to why such expressions would be considered to be "nonsensical", it has been argued hitherto that they are "nonsensical" because the semantics of the propositional calculus dictate that the nonlogical symbols must be propositional letters.

Reichenbach's article entitled "Bertrand Russell's Logic" defends Russell's notion of implication on a number of points: (1) although Russell's notion of implication is prima facie contrary to the intuition, Russell was able to give a definition of implication by means of which a reasonable logical calculus was constructed; (2) the notion of formal implication shows that Russell attempted to express a notion of implication which seems closer to what is ordinarily meant by implication; (3) Russell saw that there was a distinction to be made between inference and implication.[267]

I would like to answer each of Reichenbach's points in turn. With regards to (1), it can be said that Russell did arrive at a satisfactory definition of implication, but it took Russell quite a number of years to formulate such a definition. In The Principles Of Mathematics, Russell maintained that implication could not be defined; in this period of The Principles Of Mathematics, it is quite true to say that Russell considered all mathematical and logical formulas to be implications. In "The Theory Of Implication", however, Russell conceded that it is possible to construct a logical calculus in which the " ⊃ " is not a primitive idea. There is also the further difficulty of Russell's notion of implication insofar as the distinction between a connective and a predicate is blurred.[268] With regards to (2), I cannot say as to whether Reichenbach's remark is warranted or unwarranted. In The Principles Of Mathematics, Russell contends that the chief reason why people are skeptical of material implication is that there is a widespread preoccupation with formal implication.[269] However, this does not mean to say that Russell proposed the notion of formal implication in order to accommodate the belief that most people have in it; rather it would seem more likely that Russell had to explain expressions of a certain logical form, and to this endeavour, Russell employed the notion of formal implication. Since I have discussed Russell's usage of formal implication in a previous subsection, I will not concern myself here as to whether Russell's usage of formal implication is logically gainful. With regards to (3), I agree with Reichenbach that in his works, Russell distinguished inference from implication. In his article, Reichenbach makes an attempt to state just how Russell differentiated between inference and implication: "...Russell clearly saw that inference and implication are of a different logical nature. Whereas implication is an operation connecting propositions and leading to a new proposition, inference represents a procedure, performed on propositions."[270] To this kind of distinction between inference and implication, Reichenbach acquiesces. In reply to this kind of distinction, I can only voice certain misgivings which have been stated previously with respect to the "paradoxes of material implication" - that is, from Russell's point of view, "implies" functions as a binary connective; it is almost inevitable therefore that certain fallacies occur as a result of the confusion of use with mention. It is more serviceable to claim that "$p \supset q$" be read "if p then q". It may be asked therefore as to whether there is any distinction to be made between inference and implication on this new interpretation. I think that it would be appropriate to say that when one proposition is inferred from another proposition, then the latter proposition implies the former proposition - that is, "'Socrates is a man' is inferred from

'Socrates is a mortal'" is equivalent to "'Socrates is a mortal' implies 'Socrates is a man'". In this respect, the notions of inference and implication are quite similar insofar as "inferred from" is roughly equivalent to "implies"; thus, the notions of inference and implication both belong to the meta-language. I say that the notions of inference and implication are roughly equivalent because inference has the connotation of a psychological process whereas implication does not have this connotation.

H.N.Lee's article entitled "Note On '$\supset$' and '$\vdash$' In Whitehead And Russell's Principia Mathematica" strikes me as a rather puzzling paper. Although I am not exactly sure as to Lee's general intent, I do know that I am not in agreement with some of his claims. Lee makes the comment that the authors of Principia Mathematica do not acknowledge any modal notions; it would seem that part of Lee's intent is to show that at least one modal notion is contained implicitly in Principia Mathematica: "Thus, one modal consideration, is symbolically recognized (but obscurely) in Principia."[271]

Lee's major claim, it would seem, is that the usage of the turnstile in Principia Mathematica can be interpreted as a sign of logical necessity; thus, any formula which follows "$\vdash$" must hold necessarily. As a result, Lee maintains, if a horseshoe, ("$\supset$") is the main "relation" of a formula, it will differ in logical import from a "$\supset$" which is a secondary "relation" of the same formula.[272] At this point, I would like to state that I disagree with Lee on at least two claims. Firstly, Lee throughout his whole discussion presupposes that "logical necessity" is a plausible interpretation of the turnstile in the context of Principia Mathematica; furthermore, Lee does not cite any documentary proof to the effect that Whitehead and Russell considered "logical necessity" as a possible meaning for "$\vdash$". There would seem to be no passages whatsoever in Principia Mathematica to support the allegation that "logical necessity" is to be the interpretation for "$\vdash$". In Principia Mathematica, "$\vdash$" is given a definite meaning disassociated with the notion of logical necessity: "The sign '$\vdash$' is called the assertion-sign; it may be read 'it is true that' (although philosophically this is not exactly what it means)."[273] In recent years, the metalinguistic sign, "$\vdash$", has been used to indicate that a particular formula is a theorem in a particular system; in this way, logicians have distinguished the metalinguistic sign for theoremhood ("$\vdash$") from the metalinguistic sign for truth ("$\models$"). In the expression of the form, "$\emptyset_1 \ldots \emptyset_n \vdash \psi$", the turnstile also indicates that the expression to the right of the turnstile is derivable from the expression to the left of the turnstile.[274] Secondly, even if "$\vdash$" does mean "it is logically necessary that", I cannot see how a "$\supset$" which is the main "relation" in a formula can have a different logical import than a "$\supset$" which is a secondary "relation" in the same formula. Let us consider the expression, "$\vdash (p \supset q) \supset (p \supset q)$", on the presupposition that "$\vdash$" means "it is logically necessary that"; we will permit "$\vdash$" to be interchangeable with "$\Box$". The "$\supset$" which is the main "relation" in "$\Box[(p \supset q) \supset (p \supset q)]$", Lee contends, has a different logical import than either of the "$\supset$"'s to be found in the antecedent or the consequent. In response to Lee's contention, I would like to maintain that all the "$\supset$"'s in the expression, "$\Box[(p \supset q) \supset (p \supset q)]$" have the same logical import. "$\Box[(p \supset q) \supset (p \supset q)]$" is a necessary conditional, but this does not mean to say that the "$\supset$" is the main "relation" of "$\Box[(p \supset q) \supset (p \supset q)]$". It would seem that Lee associates "$\Box[(p \supset q) \supset (p \supset q)]$" with "$(p \supset q) \prec (p \supset q)$";[275] while "$\prec$" is the main "relation" of "$(p \supset q) \prec (p \supset q)$", "$\supset$" is not the main "relation" of "$\Box[(p \supset q) \supset (p \supset q)]$". Although it may be true that the "$\prec$" has a different logical import than the "$\supset$"'s in "$(p \supset q) \prec (p \supset q)$", it <u>cannot</u> be maintained, as Lee would like to contend, that the second "$\supset$" of "$\Box[(p \supset q) \supset (p \supset q)]$" has a different logical import than the first or third "$\supset$"'s. I might also add for reasons expounded hitherto that I do not agree with Lee in calling "$\supset$" a relation.

Lee further states that "... it is inaccurate to say that in the system of material implication a false proposition implies any proposition".[276] I agree with this claim, but Lee goes on to say that this "paradox of material implication " does not hold because a false proposition does not imply its own contradictory; according to Lee, only a necessarily false proposition can imply its own contradictory, and therefore, it is a necessary false proposition which implies any proposition.[277] It should be noted that Lee in reference to this "paradox of material implication" employs the word, "implies", as a predicate in the sense of "logically deducible from"; however, Lee does not specify how he employs the word, "implies", or how Russell, for that matter, employs the word, "implies". In one passage of his article, Lee states:

> Thus, in the formula $\vdash : p \supset p . \supset . p \supset p$, all horseshoes may be asserted even though only the major one is; that is, all are assertable and state necessary relations. When the antecedent is negated, it becomes necessarily false and implies its own contradictory: $\sim (p \supset p) . \supset . p \supset p$ holds and is assertable. 278

It would seem that in the above passage, Lee reads "$\supset$" as "implies". The whole point of avoiding the "paradoxes of material implication" is to distinguish between a connective and a predicate, and it would seem that Lee has not grasped the significance of this distinction.

Lee ends his article by making a number of claims with which I cannot help but disagree. For example, there is the contention by Lee that in Principia Mathematica, the "$\supset$" functions sometimes as a predicate and sometimes as a connective; the "$\supset$" functions as a predicate when it is the major "relation" of an asserted formula, and the "$\supset$" functions as a connective when it is in a secondary position. I must confess that I do not know exactly how Lee arrives at this latter startling conclusion, but it would seem that Lee differentiates an asserted expression from an unasserted expression.[279] I have stated hitherto that I find Lee's article to be puzzling; one of the reasons for my puzzlement is that Lee initially states that "$\vdash$" is the sign of logical necessity in Principia Mathematica, and later on, in the context of Principia Mathematica, Lee interprets "$\vdash$" as a sign of assertion. It is due to this distinction between an asserted formula and an unasserted formula that Lee makes the following remark:

> Take the theorem $\vdash : p \supset q . \supset . \sim q \supset \sim p$, for example. Instead of using 'implies' for every occurrence of the horseshoe, we would better interpret the structure of the theorem by the reading 'p implying q implies not-q implying not-p'. To be meticulous in reading the symbol '$\supset$' always by the words 'if ___, then ___' does not solve the problem of the

> verbal rendering, for 'if ___, then ___' also makes an assertion, and if the antecedent itself is a conditional, the antecedent is read as an assertion. To preserve the logical distinction between the asserted and the unasserted relations while using the words 'if ___, then ___', we would have to invent some such grammatical monstrosity as "If p if-then-ing q, then not-q if-then-ing not-p". 280

As to what truth there may be in the above passage, I am not prepared to embark upon an analysis of it. Although Russell discussed the problem of "asserted versus unasserted" in <u>The Principles Of Mathematics</u>, it would seem that the issue has not been debated to any great extent in recent years.[281] With respect to Lee's article, I would like to conclude by saying that it provides little illumination, if at all, into the role of the "$\supset$" in <u>Principia Mathematica</u>.

There is a criticism of Russell's notion of implication in <u>The Development Of Logic</u> by Kneale and Kneale which may appear to be novel at first glance. According to Kneale and Kneale, when Whitehead and Russell referred to "$p \supset q$" as a proposition of material implication, Whitehead and Russell confused two questions: (1) "What justifies inference from the proposition that - P to the proposition that - Q?"; (2) "What is the weakest additional premiss which in conjunction with the premiss that - P suffices for inference to the conclusion that - Q?".[282] It would seem that this criticism breaks down to the same criticism presented in the subsection concerning the "paradoxes of material implication" - namely, Russell along with Lewis employed the word, "implies", as both a predicate and connective.

According to A.N. Prior, logic for Russell is fundamentally "the theory of implication". Prior recognizes that Russell always distinguished inference from implication; from Russell's perspective, Prior states, logic is connected with inference insofar as "logic is concerned with that in the real world which makes inference justified, and this is implication".[283] However, Prior relates, implication in the Russellian sense is ambiguous since implication is also a relation between propositions holding whenever the antecedent or the consequent is true. It follows from the definition of material implication that a false proposition materially implies any proposition and that a true proposition is materially implied by any proposition. Yet, says Prior, if implication is to justify inference, then the false proposition, "the moon is made ofgreen cheese", should justify us to infer the proposition, "the moon is made of yellow cheese", and further to this, the proposition, "the moon is made of red cheese" should justify us to infer the true proposition, "the moon is not made of cheese at all"; it would seem that in both these cases the inference in question is not justified. How does Russell therefore explain this quandary? Prior maintains in his article that inorder to escape this quandary, Russell argued in the following manner: in the first case, the inference cannot be performed because the premiss is false, and in the second case, the inference is justified but we cannot know the inference to be justified unless we know the conclusion in which case the inference is not needed; in other words, Russell would argue that although the principle, "infer a true proposition from any proposition", has no "practical use", this does not mean that it is "logically wrong".[284]

Prior is quite right when he states that Russell's usage of the word, "implies", is ambiguous in its two connotations: (1) the justification of inference; (2) a propositional connective. For example, with respect to (1), we find this type of claim in The Principles Of Mathematics: "The relation in virtue of which it is possible for us validly to infer is what I call material implication."[285] Since Russell understands logic to be the study of the various customary forms of inference, I also must agree with Prior concerning his claim that from Russell's perspective, logic is essentially "the theory of implication". With respect to the explanation which Prior attributes to Russell concerning the "paradoxes of material implication", one can admit quite readily that there is a passage in the Introduction

<u>To Mathematical Philosophy</u> which seems to support this interpretation:

> Whenever p is false, "not-p or q" is true, but it is useless for inference, which requires that p should be true. When q is already known to be true, "not-p or q" is of course also known to be true, but is again useless for inference, since q is already known, and therefore does not need to be inferred. 286

One can ask as to whether Russell's explanation is satisfactory. As Prior points out, Russell's justification of the "paradoxes of material implication" is that although the "paradoxes" do not have a practical use, this does not mean that they are wrong thereby. This kind of rationalization on Russell's part with regards to the "paradoxes of material implication" seems to me to be analogous to the following kind of argumentation: the rule, "infer a proposition from anything whatsoever", is not logically wrong because it cannot be put into effect - that is, it has no practical use - since it is not possible to infer a proposition from anything whatsoever. In short, Russell's justification of the "paradoxes of material implication" is not acceptable. The "paradoxes of material implication" are simply false; they confuse use with mention.

Besides the notion of material implication, Prior also comments on Russell's notion of formal implication.

On this subject of formal implication, Prior is quite lucid as to what Russell intends by such a notion. However, I think that in certain places of his exposition, Prior tends to oversimplify Russell's analysis of formal implication. For example, Prior states that for Russell, formal implication is a relation.[287] I have presented material in the third subsection of my critique to suggest that the issue as to whether Russell considers formal implication to be a relation is not as simple as it may appear to be.

I would like to speak very briefly with respect to Jules Vuillemin's excellent book entitled Leçons Sur La Première Philosophie De Russell. In his book, Viullemin attempts to discuss both the logical and philosophical issues raised in Russell's The Principles Of Mathematics. I must confess that I find little fault with what Vuillemin writes concerning Russell's notion of implication. Although I have done a more in depth analysis of Russell's notion of implication, it would seem that Vuillemin has given a concise and perceptive account of Russell's notion of implication having at the same time avoided the hindrance of detail without incurring the expense of the omission of that which is essential. Since I concur in general with what Vuillemin has to say concerning Russell's notion of implication, I will not forward any serious objections against Vuillemin's point of view.

I come now to an article by Frank J. Leavitt entitled "On An Unpublished Remark Of Russell's On 'If...Then'". In this short article, Leavitt tries to argue firstly that in Russell's unpublished paper, "Necessity And Possibility", there is an argument which proves that material implication is the correct interpretation of "if-then" and secondly, that a similar argument can be constructed to confirm the result of Russell's argument.[288] The particular argument of Russell's which Leavitt adduces to show that material implication is the correct interpretation of "if-then" is the following:

> This view of implication is rendered unavoidable by various considerations... Suppose p, q,r to be such that if p and q are true, then r is true. It follows that if p is true, then q is true, r is true... Now, if p and q are true, then p is true. Hence, by the above principle, if p is true, then if q is true, p is true; that is, if p is true, then q implies q; that is, a true proposition (p) is implied by every proposition (q). 289

It is from this argument contained in "Necessity And Possibility" that Leavitt attempts to construct another argument in which it is supposedly shown that if p is false, then "if p, then q" must be true. Leavitt's argument runs as follows: if p is false and q is false, then p is false; thus, if p is false, then if q is false, then p is false;

in turn, this means that if p is false, then if p is not false then q is not false - that is, if p is false, then if p is true then q is true; therefore, if p is false, then "if p then q" is true. Having presented both these arguments, Leavitt maintains that one of the following alternatives is correct: (1) both the above arguments are not sound; (2) the above arguments do not employ "if-then" in their ordinary usage; (3) "$\supset$" is "if-then". In Leavitt's opinion, since (1) and (2) are not the case, (3) must be the case. Leavitt concludes by saying that although there may be conditionals which have a tendency to shock native speakers, this does not mean to say that such conditionals are wrong; rather, it shows that native speakers are not aware of some of the ramifications of their daily usage of "if-then".[290]

In response to Leavitt's article, I would like to investigate firstly whether the passage that Leavitt cites from Russell's "Necessity And Possibility" does in fact prove that material implication is the correct interpretation of "if-then". The conclusion of Russell's argument is the following: a true proposition is implied by every proposition. Now, if this is the conclusion of Russell's argument and it would seem that it is, then how does such a conclusion establish the correctness of material implication as the correct interpretation of "if-then"? As I have pointed out previously, the proposition, "a true proposition is implied by every proposition", does not follow from the truth-functional interpretation of "if-then"; it can only follow if we allow "implies" to be a binary connective in which case the employment of "implies" will sin against use and mention. In order to prove that material implication is the correct interpretation of "if-then", we would have to prove that when the antecedent is false, the conditional is true and that when the consequent is true, the conditional is true; moreover, we would also have to show that in the remaining case, that is, when the antecedent is true and the consequent is false, the conditional is false. Insofar as Russell's argument falls short of these criteria, I cannot see how it can be understood as a proof that material implication is the correct interpretation of "if-then". In this respect, Leavitt's claim with regards to Russell's argument in "Necessity And Possibility" is mistaken. The argument in question that Russell presents in "Necessity And Possibility" is an attempt to prove that a true proposition is implied by every proposition. Since Russell considered the proposition, "a true proposition is implied by every proposition", to be an essential characteristic of implication, he wanted to show that this proposition is a consequence of the view of material implication. In spite of the fact that Russell's purpose in presenting the argument in question is to show that his interpretation of "p implies q" is correct, I do not think that Russell himself deemed such an argument to be definitive in any logical sense as a proof of material implication as the correct interpretation of "if-then"; otherwise, Russell would not have added the following lines to the argument in question: "I shall not pursue the arguments in favour of this view of implication; I shall content myself by pointing out that it is accepted (tho' without a full realization of its consequences) by Shakespeare and Mr. Bradley..."[291]

Although Russell's argument does not prove that material implication is the correct interpretation of "if-then", it may be wondered as to whether Leavitt's argument does. The outcome of Leavitt's is that if p is false, then "if p then q" is true. If Leavitt's argument is sound, then a similar argument can be produced to show that if q is true, then "if p then q" is true: (1) if q is true and p is true, then q is true; (2) thus, if q is true, then if p is true then q is true; (3) therefore, if q is true, then "if p then q" is true. In Leavitt's proof, as in the above proof, (1) to (2) is carried out by means of exportation. In both proofs, also, the inference is made from "if p is true then q is true" to "'if p then q' is true"; one may have qualms concerning this inference, but it would seem that such qualms are unfounded. For example, fom the molecular proposition, "p is true and q is true", one may infer "'p and q' are true"; one should be able to argue similarly that if p is true and q is true, then "if p then q" is true. It would seem therefore that Leavitt's argument is sound.

As to whether Leavitt's argument employs "if-then" in its ordinary usage however is another matter, especially since many people contend that there is no ordinary usage of "if-then". It should be noted that Leavitt employs propositional letters instead of actual propositions; consequently, Leavitt presupposes that there need be no connection, other than a truth-functional connection, between the antecedent and the consequent; it has been mentioned by some authors that the "if-then" of ordinary language only makes sense when there is an actual connection between the antecedent and the consequent. Although I do not know how much force this criticism may have, it would seem nevertheless that it does detract somewhat from Leavitt's argument. Whether this objection is strong enough to dismiss Leavitt's argument entirely, I cannot venture to say. Despite the succinct nature of Leavitt's argument, it would seem that the argument itself is quite subtle; a final verdict with respect to Leavitt's argument would no doubt have to take into account the whole controversy concerning the issue of entailment.

The last secondary source which I wish to appeal to is Ronald Jager's forthcoming book, The Development Of Bertrand Russell's Philosophy. According to Jager, Russell's logical system is founded upon the notion of material implication; Jager considers Russell's usage of the phrase, "material implication", to be unfortunate, especially since Russell was led to expound such a notion in a very controversial way.[292] Some authors, Jager states, try to maintain that material implication permits any inference except those inferences in which the premisses are true and the conclusion is false. In the opinion of these authors, it is not important for logic how truth is obtained; what is important for logic is that falsity should not be derived from truth, and as a result, "the paradoxes of material implication are a small price to pay in unfamiliarity".[293] Jager rejects this kind of argumentation; properly understood, Jager states, there are no "paradoxes of material implication": "It is therefore confusing and entirely unnecessary to speak of the 'paradoxes of material implication'. Russell is, however, directly responsible for having generated this tempest by supporting an ill-chosen terminology with confused arguments."[294]

In Jager's opinion, Russell along with Whitehead attempted to alleviate some of their misgivings about material implication by introducing the new notion of formal implication.

However, Jager maintains, formal implication is a juxtaposed medley of three ideas "confused by Russell's philosophical preconceptions".[295] In The Principles Of Mathematics, Russell states that formal implication "asserts a class of material implications", but in saying this, Jager contends, Russell did not differentiate how inference may be related to the following: (1) a general rule; (2) universal quantification; (3) a connection among meanings.[296]

Jager provides a number of illustrations to show how Russell interpreted formal implication in these three ways. The reader may recall that in The Principles Of Mathematics, Russell had claimed that the chief reason why we are reluctant to acknowledge such implications as "'2+2=4' implies 'Socrates is a man'" is a "preoccupation with formal implication";[297] in the same paragraph where he makes this claim, Russell goes on to say that formal implication is a more familiar notion than material implication, and even in the cases in which material implication is specifically mentioned, formal implication "is really before the mind, as a rule" [Jager's italics].[298] If we take the rule of modus ponens, Jager asserts, then this rule can be understood to say that for any propositions, if p and q are related in a particular manner, then the rule holds; according to Russell, Jager maintains, the rule of modus ponens can also be viewed as "a particular instance of some formal implication".[299] However, Jager states, not only does formal implication apply to propositions, but it applies to terms as well.

This leads us to consider formal implication as a universal quantification. Jager explains the evolution of formal implication as a universal quantification in the following manner. Firstly, we have the material implication, "'Socrates is a man' materially implies 'Socrates is a mortal'", and then Russell decides to vary "Socrates" in the material implication to the expression, "'x is a man' materially implies 'x is a mortal' for all values of x". From this latter expression, Jager claims, Russell arrives at the formal implication, "'x is a man' formally implies 'x is a mortal'", symbolically represented as "$(x)(\phi x \supset \psi x)$" or in the Peano notation, "$\phi x \supset_x \psi x$". This is formal implication as considered as universal quantification.[300]

However, Jager points out, Russell sees that there is a problem in the notion of formal implication. For Russell, the problem is the following: if formal implication is simply a matter of material implication, then how does it come about that in the implication, "'Socrates is a man' implies 'Socrates is a mortal'", "Socrates" can be varied completely whereas in the implication, "'Socrates is a man' implies 'Socrates is a philosopher'", the variability of "Socrates" is restricted? In the face of this problem, Russell concludes that there must be some other relation, besides that of material implication, that is involved in formal implication.[301] In Jager's opinion, Russell conceives the expressions, "is a man" and "is a mortal", in the formal implication, "'x is a man' formally implies 'x is a mortal'", to be of the same logical category as "p" and "q" when "p materially implies q" such that the first expression is never present without the second expression. Unfortunately, Jager states, Russell does not realize that a formal implication is a generality which can be true or false. Simply because the variability of a term is restricted in a formal implication does not mean that there must be another relation, other than material implication, to be found in formal implication.[302]

Russell's so-called "problem" with formal implication, Jager affirms, leads him to the view that there is something else involved in formal implication; Jager refers to this other component of formal implication as "meaning connection": "He [Russell] does not see that this notion - call it a 'meaning connection' - is tailor-made for his particular example, and has no general validity at for formal implication in sense (b) [universal quantification]."[303]

With regards to these three ideas implicit in formal implication, Jager states that in the sense of a general rule, formal implication is the general rule of modus ponens; formal implication in the sense of universal quantification belongs to first order predicate calculus, but "has nothing specifically to do with implication";[304] finally, the element in formal implication in which there is a connection among meanings is completely alien to Russell's logic since Russell's logic is truth-functional, not intensional.[305]

Jager's treatment of Russell's notion of implication shows a keen understanding not only of the notion of implication itself, but also of the context in which the notion of implication is to be found in Russell's logic. When it comes to the "paradoxes of material implication", Jager does not engage in making superficial remarks to the effect that perhaps these "paradoxes" are the price that one has to pay in the acceptance of material implication; Jager recognizes at the outset that there are no "paradoxes of material implication" as such. I shall not add to Jager's comments concerning the "paradoxes of material implication"; in the subsection of my critique, I have only gone into more detail with respect to the reasons why there are no "paradoxes of material implication".

Jager's analysis of the notion of formal implication is a bold and determined appraisal; I say this because there would seem to be very few accounts of Russell's notion of formal implication in the literature which are as critical and penetrating as the one that Jager presents in his forthcoming book. For example, in Reichenbach's article in the Schilpp volume of Russell's philosophy, Reichenbach writes the following with regards to formal implication:

> He [Russell] deliberately postponed the construction of concepts better fitting conversational usage, in the hope that this might be possible within the frame of an extensional logic, by the introduction of more complicated relations. His formal implication represents a stepping stone on this path. 306

It would seem that in the context of what Reichenbach writes about Russell's notion of implication, Reichenbach is more intent upon extolling the virtues of formal implication than in pointing out its defects. This, however, is not the case with what Jager says concerning formal implication.

In the main, I must agree with Jager's analysis of formal implication; this does not mean to say that Jager has written all that can be said with respect to formal implication. As Russell conceives of it, at least in The Principles Of Mathematics, formal implication is a special type of implication; Jager does not consider formal implication to be a special type of implication at all, and it seems to me that on this point, Jager cannot be contested. I do think however that Jager's division of the notion of formal implication can be contested.

Firstly, as to whether Russell's notion of formal implication involves the idea of a general rule, I can argree only partially with what Jager has to say on this subject. In The Principles Of Mathematics, Russell makes the following cryptic remark: "The fundamental importance of formal implication is brought out by the consideration that it is involved in all the rules of inference."[307] I must confess a mild consternation with regards to this claim, but it would seem that this does not mean to say that formal implication is itself a rule of inference; it is involved in the rules of inference, but it itself is not a rule. It may be asked therefore how is it possible in the light of certain quotations from The Principles Of Mathematics to deny the claim that formal implication can be considered as a rule? I would like to quote the first passage taken from The Principles Of Mathematics which Jager employs in connection with this issue:

> It would certainly not be commonly maintained that "2+2=4" can be deduced from "Socrates is a man", or that both are implied by "Socrates is a triangle". But the reluctance to admit such implications is chiefly due, I think, to preoccupation with formal implication, which is a much more familiar notion, and is really before the mind, as a rule, even where material implication is what is explicitly mentioned. 308

In Jager's book, the phrase, "as a rule", is italicized, and Jager interprets the word, "rule", in this phrase to mean "a rule of inference". I do think that this is a misinterpretation of what Russell means by the phrase, "as a rule"; it would seem to me that this phrase should be construed more naturally to mean something like "generally speaking" or "usually". The second short quotation that Jager employs from The Principles Of Mathematics is taken from a longer passage which I will quote:

> Nevertheless, wherever, as in Euclid, one particular proposition is deduced from another, material implication is involved, though as a rule the material implication may be regarded as a particular instance of some formal implication, obtained by giving some constant value to the variable or variables involved in the said formal implication. 309

Again, in this quotation, we have the phrase, "as a rule", and I would interpret this phrase to mean "usually", not "as a rule of inference". Jager also commits, if I am not mistaken, another misinterpretation when he states the following:

> What he [Russell] means, the context makes clear, is that a perfectly general rule, say modus ponens, serves as warrant for any particular deduction 'q' from 'p' and 'p ⊃ q'. The rule in effect says that for any propositions, p, q, so related, the deduction holds. He refers to this, very misleadingly, as "a particular instance of some formal implication" (Principles, p. 34). 310

In the above passage, Jager considers the rule (of inference) to be "a particular instance of some formal implication"; if we examine the original passage, we will discover that it is material implication which "may be regarded as a particular instance of some formal implication".[311] This does not mean to say that modus ponens cannot occur in the context of formal implication; as Russell states in The Principles Of Mathematics, modus ponens can be stated not only in terms of propositions, but also in terms of propositional functions:

> Another form in which the principle [modus ponens] is constantly employed is the substitution of a constant, satisfying the hypothesis, in the consequent of a formal implication. If ϕx implies ψx for all values of x, and if a is a constant satisfying ϕx, we can assert ψa, dropping the true hypothesis ϕa. 312

Thus, formal implication, according to Russell, is involved in all the rules of inference, but it itself is not a rule of inference.

How is it possible therefore from Russell's perspective for formal implication to be involved in all the rules of inference? I must admit that I am not certain as to how this can come about from Russell's point of view, but I think that I can hazard a guess. In the propositional calculus of The Principles Of Mathematics, the nonlogical letters are variables; formal implication is involved in the propositional calculus in order to restrict such variables to the value of propositions; the next logical level, that of the first order predicate calculus, has formal implication directly involved in its manipulations. Consequently, if any deductions are to be made, these deductions must incorporate formal implication in some manner or other.

Concerning Jager's other divisions of formal implication, I tend to agree with his analysis, especially in regards to what he calls "meaning connection". In The Principles Of Mathematics, Russell claims that there must be some other relation to be found in formal implication besides that of material implication; Russell's reason stems from the fact that in certain formal implications, the variability of the terms is restricted while in other formal implications, variability is not restricted. Although Russell does not state in The Principles Of Mathematics as to what this relation may consist of, it would seem that it has something to do with the connection of meaning between the assertions; in other words, if we take a formal implication such as "'Socrates is a man' implies 'Socrates is a mortal'", there is a relation between "is a man" and "is a mortal". A relation, however, which concerns the meaning of nonlogical words has no place in Russell's truth-functional enterprise.

I would like to cite two other criticisms of Jager's account of formal implication. Firstly, Jager does not seem to be aware, at least in his presentation, that formal implication is involved in the propositional calculus of The

Principles Of Mathematics; if formal implication is to be explained fully, then its role in the propositional calculus must be explained; in Russell's article of 1906, "The Theory Of Implication", the usage of formal implication in the propositional calculus was abandoned. Secondly, Jager seems to presuppose that Russell's understanding of formal implication was the same throughout the entire period in which Russell wrote on logic; Russell's interpretation of formal implication changed with the publication of "The Theory Of Implication", and it would seem that it underwent even more alteration in Principia Mathematica. For example, in Principia Mathematica, Russell writes the following with respect to formal implication: "The association of material implication with the use of an apparent variable produces an extension called 'formal implication'. This is explained later; it is an idea derivative from 'implication' here defined."[313] This kind of explanation with respect to formal implication seems to be a distant cry from some of the material that is contained in The Principles Of Mathematics.

D: CONCLUSION

In summation, I would like to make a few general comments with respect to what has been said thus far concerning Russell's notion of implication. Russell's logic is infested with several confusions generated by the nonobservance of certain metalinguistic distinctions, especially is this the case with regards to his notion of implication. For Russell, unlike many of his predecessors, logic is not a psychological investigation of the working operations of the mind; as he explains in The Principles Of Mathematics, logic is simply the study of certain acceptable forms of inference or deduction.[314] Following his skeptical approach, Russell implicitly poses the following question: if logic is to be concerned with inference, then what is the justification of inference? To this query, Russell responds that it is the relation of implication which makes inference possible.[315] Yet, at the same time, in the construction of his logical system, Russell also incorporates the notion of implication on a different, logical plane than the meaning given to the kind of implication associated with inference; in this latter employment, implication for Russell is a logical connective. In this respect, Quine is quite right in his contention that Lewis' system of strict implication is a direct consequence of Russell's ambiguous usage of the word, "implies":

> It is doubtful that Lewis would have even started this [modal logic] if Whitehead and Russell, who followed Frege in defending Philo of Megara's version of 'If p then q' as 'Not(p and not q)', had not made the mistake of calling the Philonian construction "material implication" instead of the material conditional. 316

In his unpublished article, "Recent Italian Work On The Foundations Of Mathematics", Russell makes the following statement: "Formal logic is concerned in the main with the relation of implication between propositions." [317] It is somewhat debatable as to what type of implication Russell is talking about in this passage; nevertheless, this does not impede us in making the following general claim: in Russell's work, implication is involved in a lot of issues in which it need not be involved at all. I will try to cite two examples which tend to support this claim. In the propositional calculus of The Principles Of Mathematics, Russell maintains that an implication will be false if it is not prefaced by a hypothesis of the form, "p implies p". Here, Russell did not have to talk about an implication being false. It would seem that the problem which Russell is worrying about here is the following: if something is asserted and this something is not a proposition, then how is it possible for this something to be true? Russell's explanation of this problem however is couched in terms of implications, and it is no wonder therefore that the problem had not been properly handled in Russell's new solution proposed in "The Theory Of Implication" of 1906. The second example which I wish to focus upon in support of my general claim is Russell's notion of formal implication. Although it has been pointed out hitherto that the explanation of formal implication is quite involved, I would like to draw attention to one particular facet of formal implication. We are told in The Principles Of Mathematics and elsewhere that a formal implication is of the form, "$(x)(\phi x \supset \psi x)$"; according to Russell, formal implications differ from material implications insofar as formal implications hold between propositional functions in which one propositional function implies another propositional function for all values of the variable.[318] One is tempted to ask here why the notion of formal implication is invoked by Russell. The notions of propositional function, variable and quantification all separately can be explained in an intelligible manner, and yet Russell calls upon the notion of formal implication - a notion which inextricably involves some sort of implication. This is rectified somewhat in Principia Mathematica where we find the following definition, "$(\exists x). \phi x = . \sim (x) \sim \phi x$";[319] nevertheless, in Principia Mathematica, the notion of formal implication still lingers on.

In 1937 in the introduction to the second edition of The Principles Of Mathematics, Russell was to make the following remark with respect to his employment of the notion of implication in _The Principles Of Mathematics_:

> This brings me to the definition of mathematics which forms the first sentence of the "Principles". In this definition various changes are necessary. To begin with, the form "p implies q" is only one of the many logical forms that mathematical propositions may take. I was originally led to emphasise this form by the consideration of Geometry. It was clear that Euclidean and non-Euclidean systems alike must be included in pure mathematics, and must not be regarded as mutually inconsistent; we must, therefore, only assert that the axioms imply the propositions, not that the axioms are true and therefore the propositions are true. Such instances led me to lay undue stress on implication, which is only one among truth-functions, and no more important than the others. 320

Although Russell in this passage still employs the word, "implies", as a binary connective, the meaning of the passage is quite clear. Here, Russell himself admits that he had overestimated the importance of the notion of implication. In _The Principles Of Mathematics_, material implication and formal implication are indefinables. By the time of _Principia Mathematica_, material implication is neither indefinable nor primitive, but is defined in terms of disjunction and negation; likewise, in _Principia Mathematica_, formal implication is not a primitive notion although Russell does state that in expressions in which _all_ values are concerned, the form of such propositions most frequently occurs in formal implications.[321] In spite of the somewhat subdued character of implication in _Principia Mathematica_ in comparison to its primacy in _The Principles Of Mathematics_, implication still retains a basic importnace for Russell due to it intimate association with inference. The importance of implication is also borne out in the Introduction To Mathematical Philosophy with respect to Russell's defence of his logical system against the objections raised by C.I. Lewis.

In the main, Russell's notion of implication seems to be derived from the works of Peano and Frege. In The Principles Of Mathematics, the interpretation of material implication stems from Peano; following Peano, Russell maintains that material implication is indefinable and that an implication will be false if its nonlogical symbols do not stand for propositions. Having allied himself with Frege, Russell, in "The Theory Of Implication" abandons the Peanesque interpretation of material implication. It would seem also that Russell's notion of formal implication was obtained from Peano. In Peano's system, the notation cannot accomodate the formalization of "(x) Øx"; a formal implication however is formalizable.[322]

Although Russell's notion of implication is largely derived from these two sources, Peano and Frege, this does not mean to say that Russell was a mere imitator. Russell did borrow ideas concerning implication from Peano and Frege, but Russell also shaped these ideas by the new technique of mathematical logic. Furthermore, Russell's notion of implication was subject to other influences as well: (1) common sense and ordinary language; (2) philosophical and metaphysical presuppositions; (3) the thesis of logicism. No doubt there are confusions in Russell's notion of implication, but Russell was an innovator, and for an innovator, it would seem that there are always difficulties - difficulties concerning the understanding of a concept and difficulties with respect to the systematization of that concept into the body of knowledge. It is a wonder therefore that Russell accomplished as much as he did with regards to the notion of implication. On the other hand, we are only able to act as critics because we have had enough time to digest some of the repercussions inherent in Russell's logic.

FOOTNOTES

[1] Whitehead and Russell, Principia Mathematica, I, 94.

[2] Feinberg (ed.), A Detailed Catalogue Of The Archives Of Bertrand Russell, 98-99. See also: Russell, The Autobiography Of Bertrand Russell: 1872-1814, 192.

[3] Russell, The Autobiography Of Bertrand Russell: 1872-1914, 191. See also: Russell, "My Mental Development" in Schilpp (ed.), The Philosophy Of Bertrand Russell, 12.

[4] Ibid., 192.

[5] Ibid., 193. See also: Russell, My Philosophical Development, 73.

[6] Russell, My Philosophical Development, 75. See also: Russell, The Autobiography Of Bertrand Russell: 1872-1914, 194-195. (Here, Russell states that he discovered the contradiction in May of 1901). Russell, "My Mental Development", 13. (Here, Russell states that he discovered the contradiction in June of 1901).

[7] Blackwell, "The Text Of Russell's Principles Of Mathematics", 5-6. Although Russell does not state that the discovery of the contradiction caused him to rewrite certain parts of the penultimate draft of The Principles Of Mathematics (See: My Philosophical Development, 73 and The Autobiography Of Bertrand Russell: 1872-1914, 193), the penultimate draft seems to have been written before the discovery of the contradiction.

[8] Russell, The Autobiography Of Bertrand Russell: 1872-1914, 195.

[9] The pages of Part I of the final draft of The Principles Of Mathematics have "IM" as their label whereas the pages of Part I of the penultimate draft of The Principles Of Mathematics have "V" as their label. "IM" no doubt stands for the title of Part I of the final draft of The Principles Of Mathematics - namely, "The Indefinables Of Mathematics"; likewise, "V" stands for the title of Part I of the penultimate draft of The Principles Of Mathematics - namely, "The Variable".

[10] Feinberg (ed.), _op. cit._, 68. See also: Blackwell, "'Necessity And Possibility': Textual Notes", 1. Blackwell points out in his gude notes that the manuscript cannot be dated earlier than July 1900 due to a reference by Russell to an article in _Mind_ by G.E. Moore.

[11] Russell, "Necessity And Possibility", 8.

[12] _Ibid._, 11.

[13] _Ibid._, 13.

[14] _Ibid._, 15.

[15] _Ibid._, 16.

[16] _Ibid._, 16.

[17] _Ibid._, 17. In the original manuscript, Russell underlines his propositional letters (eg. _p_), and at other times, he refrains from such a practice. It would seem however that such a practice has no special significance within the context of "Necessity And Possibility".

[18] _Ibid._, 17. See also: Bradley, _The Principles Of Logic_, I, 128. Shakespeare, _The Two Gentlemen Of Verona_, (Act II, Scene v).

[19] _Ibid._, 18-20.

[20] _Ibid._, 21.

[21] _Ibid._, 21-22.

[22] _Ibid._, 23. It should be noted that in this unpublished article, Russell has changed his position with regards to the analytic-synthetic distinction. See: Russell, _A Critical Exposition Of The Philosophy Of Leibniz_, 21. ("And hence the propositions of Arithmetic, as Kant discovered, are one and all synthetic.")

[23] _Ibid._, 23a.

[24] Feinberg (ed.), _op. cit._, 68. See also: Russell, _The Autobiography Of Bertrand Russell: 1872-1914_, 195.

[25] Russell, "Lecture II: Logic Of Propositions", 1.

[26] Feinberg (ed.), _op. cit._, 68.

[27] Russell, "Recent Italian Work On The Foundations Of Mathematics", 2.

[28] _Ibid._, 3.

[29] _Ibid._, 3.

[30] _Ibid._, 4-5.

[31] _Ibid._, 7-8. In an earlier paper, Russell had pointed out that Pierce and Schröder had not distinguished between ϵ and $\supset$ in their work in the logic of relations. See: Russell, "The Logic Of Relations" in Marsh (ed.), _Logic And Knowledge: Essays 1901-1950_, 3.

[32] _Ibid._, 13.

[33] Russell, _The Principles Of Mathematics_, 9.

[34] Russell, _A Critical Exposition Of The Philosophy Of Leibniz_, 4.

[35] Russell, "Necessity And Possibility", 23. See also: Russell, "Recent Italian Work On The Foundations Of Mathematics", 3. The idea of logicism is also contained in Russell's article, "Recent Work In The Philosophy Of Mathematics" of 1901 - renamed "Mathematics And The Metaphysicians" and reprinted in _Mysticism And Logic_. See: Russell, "Mathematics And The Metaphysicians" in _Mysticism And Logic And Other Essays_, 59-60.

[36] Russell, _The Principles Of Mathematics_, xv.

[37] _Ibid._, xv.

[38] _Ibid._, 3.

[39] _Ibid._, 5.

[40] _Ibid._, 5-6.

[41] _Ibid._, 10.

[42] _Ibid._, 11.

[43] _Ibid._, 11.

[44] Russell, *The Principles Of Mathematics*, 11.

[45] *Ibid*., 11.

[46] *Ibid*., 13.

[47] *Ibid*., 14.

[48] *Ibid*., 14.

[49] *Ibid*., 14.

[50] *Ibid*., 14-15.

[51] *Ibid*., 15.

[52] Russell, "Necessity And Possibility", 15.

[53] Russell, *The Principles Of Mathematics*, 15.

[54] *Ibid*., 15-18.

[55] *Ibid*., 22.

[56] *Ibid*., 33.

[57] *Ibid*., 4, 33.

[58] *Ibid*., 10.

[59] *Ibid*., 33.

[60] *Ibid*., 33-34.

[61] *Ibid*., 34.

[62] *Ibid*., 35.

[63] *Ibid*., 35.

[64] Russell, "Necessity And Possibility", 21.

[65] Russell, *The Principles Of Mathematics*, 36.

[66] *Ibid*., 36.

[67] *Ibid*., 36.

68 Russell, _The Principles Of Mathematics_, 36-37.

69 _Ibid._, 37.

70 _Ibid._, 38.

71 _Ibid._, 38.

72 _Ibid._, 38.

73 _Ibid._, 38-39.

74 _Ibid._, 39.

75 _Ibid._, 39-40.

76 _Ibid._, 40.

77 _Ibid._, 40.

78 _Ibid._, 40.

79 _Ibid._, 41.

80 _Ibid._, 41.

81 Russell, "The Theory Of Implication" in _American Journal Of Mathematics_, 159.

82 Russell, _The Principles Of Mathematics_, xv.

83 Russell, "The Theory Of Implication", 159.

84 Russell, _The Principles Of Mathematics_, 11.

85 Russell, "The Theory Of Implication", 159.

86 Compare this statement with that made in _The Principles Of Mathematics_: "But it is plain that where we validly infer one proposition from another, we do so in virtue of a relation which holds between the two propositions whether we perceive it or not..." See: Russell, _The Principles Of Mathematics_, 33.

87 Russell, "The Theory Of Implication", 159-160.

88 Russell, "Recent Italian Work On The Foundations Of mathematics", 5.

89 Russell, "The Theory Of Implication", 160.

90 Russell, The Principles Of Mathematics, xviii. See also: Russell, The Principles Of Mathematics, 501.

91 Ibid., 519.

92 Ibid., 518-519.

93 Barker, Philosophy Of Mathematics, 22-23.

94 In The Principles Of Mathematics, Russell had claimed: "We require, then, in the propositional calculus, no indefinables except the two kinds of implication..." See: Russell, The Principles Of Mathematics, 15.

95 Russell, "The Theory Of Implication", 160. It should be noted that these guidelines are more or less the principles of Occam's Razor. Although this statement of Occam's Razor seems to be passed over rather quickly by Russell, one should not underestimate its importance for Russell. In a later article in 1914, Russell was to state: "Occam's Razor ... I should regard as the supreme methodological maxim in philosophizing ..." See: Russell, "On The Nature Of Acquaintance" in Marsh (ed.), Logic And Knowledge: Essays 1901-1905, 145.

96 Ibid., 161.

97 Russell, The Principles Of Mathematics, 33. See also: Russell, The Principles Of Mathematics, 34. "Of the various equivalent relations satisfying these conditions [of Section C of Chapter II of Part I] , one is to be called implication, and if such a notion seems unfamiliar, that does not suffice to prove that it is illusory."

98 Russell, "The Theory Of Implication". 161.

99 Ibid., 162.

100 Ibid., 162.

101 Ibid., 162. See also: Russell, The Principles Of Mathematics, 37. "This remark [i.e. prefixing formulas with the hypothesis of the form, 'p implies p'] applies generally, throughout the propositional calculus, to all uses where the conclusion is represented by a single letter; unless the letter does actually represent a proposition, the implication asserted will be false, since only propositions can be implied."

102
Russell, _The Principles Of Mathematics_, 15.
103
Russell, "The Theory Of Implication", 162.
104
Ibid., 164-176.
105
Ibid., 194.
106
Russell, _The Principles Of Mathematics_, 34.
107
Ibid., 38.
108
Russell, "The Theory Of Implication", 201.
109
Ibid., 202.
110
Russell, _The Principles Of Mathematics_, 40.
111
Russell, "The Theory Of Implication", 194.
112
Ibid., 160-164.
113
MacColl, "'If' And 'Imply'" in _Mind_, 151-152.
114
Russell, "'If' And 'Imply', A Reply To Mr. MacColl" in _Mind_, 300.
115
Ibid., 301.
116
Ibid., 301. Russell enumerates these four cases, and states which proposition implies the other in each case.
117
Ibid., 301.
118
Russell, "Mathematical Logic As Based On The Theory Of Types" in Marsh (ed.), _Logic And Knowledge: Essays 1901-1950_, 83.
119
Russell, _The Principles Of Mathematics_, 11-15. See also: Russell, "The Theory Of Implication", 160.
120
Russell, "The Theory Of Implication", 160-161. "I know no way of proving that such-and-such a system of undefined ideas contains as few as will give such-and-such results. Hence, we can only say that such-and-such ideas are undefined in such-and-such a system, not that they are indefinable."

121
Russell, "Mathematical Logic As Base On The Theory Of Types", 84.

122
Russell, The Principles Of Mathematics, xvi.

123
Whitehead and Russell, *op. cit.*, v.

124
Ibid., xiii, xvi.

125
Ibid., 6. Although Whitehead and Russell are the authors of Principia Mathematica, I will be employing Russell's name only in my paper since I am concerned with Russell's notion of implication.

126
Ibid., 7.

127
Ibid., 7. See also: Russell, "Necessity And Possibility", 18.

128
Ibid., 7.

129
Ibid., 7.

130
Ibid., 9.

131
Ibid., 9.

132
Ibid., 20-21.

133
Ibid., 22.

134
Ibid., 90. See also: Russell, "The Theory Of Implication", 159. Russell, The Principles Of Mathematics, 33. If one compares pages 91 to 94 of Principia Mathematica with pages 159 to 162 of "The Theory Of Implication", one will discover that the statements are quite similar; in fact, in most cases, the statements are verbatim.

135
Ibid., 90. See also: Russell, "The Theory Of Implication", 159.

136
Ibid., 94. See also: Russell, "The Theory Of Implication", 161.

137
Ibid., 94. See also: Russell, "The Theory Of Implication", 161.

138
Ibid., 94.

139 Russell, *The Autobiography Of Bertrand Russell: 1872-1914*, 166, 272.

140 *Ibid.*, 283. See also: *Official Register Of Harvard University*, 16-17, 21. Lenzen, "Bertrand Russell At Harvard, 1914", in *Russell: The Journal Of The Bertrand Russell Archives*, 4-5.

141 *Ibid.*, 200, 202, 205-206.

142 Russell, *My Philosophical Development*, 102.

143 Russell, *The Autobiography Of Bertrand Russell: 1914-1944*, 3. See also page 36 of the same volume.

144 *Ibid.*, 29.

145 *Ibid.*, 29.

146 Russell, *Introduction To Mathematical Philosophy*, 144.

147 *Ibid.*, 144-145.

148 *Ibid.*, 145-146.

149 *Ibid.*, 146-147.

150 *Ibid.*, 147-152. Nicod's one formal principle is "$P/\pi/Q$" where "$P = p/(q/r)$", "$\pi = t/(t/t)$", and "$Q = (s/q)|\sim(p/s)$".

151 *Ibid.*, 147.

152 *Ibid.*, 149.

153 *Ibid.*, 149.

154 *Ibid.*, 152-153.

155 *Ibid.*, 153-154.

156 *Ibid.*, 154.

157 *Ibid.*, 163.

158 Russell, *The Principles Of Mathematics*, 22-23.

159 My style of proof is taken from Fitch's *Symbolic Logic: An Introduction*.

160
Russell, _The Principles Of Mathematics_, 22.
161
Ibid., 13-15, 33.
162
Russell, "The Theory Of Implication", 162.
163
Whitehead and Russell, _op. cit._, 94.
164
Russell, _The Principles Of Mathematics_, 14-15.
165
Russell, "The Theory Of Implication", 160.
166
Russell, _The Principles Of Mathematics_, 15, 37.
167
Ibid., 518-519.
168
Russell, "Recent Italian Work On The Foundations Of Mathematics", 4-5. See also: Russell, _The Principles Of Mathematics_, 519.
169
Russell, _The Principles Of Mathematics_, 519.
170
Russell, "The Theory Of Implication", 162.
171
Russell, _The Principles Of Mathematics_, 15.
172
Ibid., 15.
173
Ibid., 37.
174
Russell, "The Theory Of Implication", 162.
175
Ibid., 162.
176
In "The Theory Of Implication", it would seem that Russell is aware that "$p \supset p$" is not equivalent to "p is a proposition". See: Russell, "The Theory Of Implication", 162. "We wish, for example, to assert '$p \supset p$'. If implication can only hold between propositions, it is necessary to preface '$p \supset p$' by the hypothesis 'p is a proposition'.
177
Russell, _The Principles Of Mathematics_, 18.
178
Russell, "The Theory Of Implication", 164.
179
Russell, _Introduction To Mathematical Philosophy_, 155.
180
Russell, _The Principles Of Mathematics_, 7.

181 Russell, <u>The Principles Of Mathematics</u>, 33.
182 *Ibid.*, 40.
183 *Ibid.*, 28.
184 *Ibid.*, 5.
185 *Ibid.*, 4-5.
186 *Ibid.*, 11.
187 *Ibid.*, 14.
188 *Ibid.*, 13.
189 *Ibid.*, 7.
190 *Ibid.*, 14.
191 *Ibid.*, 5, 11, 14.
192 *Ibid.*, 33.
193 *Ibid.*, 36.
194 *Ibid.*, 13, 38.
195 *Ibid.*, 38.
196 *Ibid.*, 38-39.
197 *Ibid.*, 39.
198 *Ibid.*, 40.
199 Ibid., 40.
200 *Ibid.*, 41.
201 *Ibid.*, 106.
202 *Ibid.*, 40.

203
Whitehead and Russell, *op. cit.*, 20-21. There is an important passage on page 7 of the first volume of *Principia Mathematica* which tends to show that Russell does not consider formal implication to be a special sort of implication: "The association of implication with the use of an apparent variable produces an extension called 'formal implication'. This is later explained: it is an idea derivative from 'implication' as here defined." I say that it tends to show rather than it shows because there are other passages which seem to claim that formal implication is a special sort of implication - for example, page 87: "We show also that *formal implication*, i.e. '$(x) . \phi x \supset \psi x$' considered as a relation of $\phi \hat{x}$ to $\psi \hat{x}$, has many properties analogous to those of *material implication*, i.e. '$p \supset q$' considered as a relation of p and q."

204
Russell, *The Principles Of Mathematics*, 33.

205
Ibid., 34, 518-519.

206
Russell, "The Theory Of Implication", 162.

207
Whitehead and Russell, *op. cit.*, 94. See also: Russell, *Introduction To Mathematical Philosophy*, 147.

208
Quine, Mathematical Logic, 31. See also: Russell, *Introduction To Mathematical Philosophy*, 147. Whitehead and Russell, *op. cit.*, 94.

209
Russell, "Necessity And Possibility", 8.

210
Ibid., 20.

211
Ibid., 18.

212
Ibid., 18.

213
Ibid., 17.

214
Ibid., 17.

215
Lewis and Langford, *Symbolic Logic*, 88.

216
Russell, "Necessity And Possibility", 15.

217
Russell, *The Principles Of Mathematics*, 33.

218
Ibid., 34.

219
Russell, _The Principles Of Mathematics_, 34.
220
Russell, "The Theory Of Implication", 161.
221
Russell, "Mathematical Logic As Based On The Theory Of Types", 84.
222
Russell, "The Theory Of Implication", 161.
223
MacColl, _op_. _cit_., 151.
224
Ibid., 151. See also: Russell, _The Principles Of Mathematics_, 15.
225
Russell, "'If' And 'Imply', A Reply To Mr. MacColl", 300-301.
226
Whitehead and Russell, _op_. _cit_., 94.
227
Ibid., 7-9.
228
Russell, _Introduction To Mathematical Philosophy_, 146.
229
Ibid., 147.
230
Ibid., 153-154.
231
Lewis and Langford, _op_. _cit_., 89.
232
Ibid., 88-89.
233
Ibid., 88.
234
Ibid., 123-124, 153. Although Lewis employs the notion of necessity, he does not have a specific symbol for necessity.
235
Russell, "Necessity And Possibility", 15.
236
Russell, _The Principles Of Mathematics_, 33-34.
237
Runes (ed.), _The Dictionary Of Philosophy_, 261-262.
238
Quine, _op_. _cit_., 23.
239
Ibid., 27.
240
Ibid., 27-28.

241
Quine, *op. cit.*, 28.

242
The modern conventions with respect to "$\supset$" and "implies" are: (1) "$\supset$" is a binary connective; (2) "implies" is a two-place predicate or a predicate of degree two; (3) implication is a dyadic relation.

243
Quine, *op. cit.*, 29.

244
I am excluding such cases for convenience sake whereby it is possible that it rains and the street will not be wet i.e. abnormal cases in which for example, the street is covered with a tarpaulin.

245
Blumberg, "Modern Logic" in Edwards (ed.), *The Encyclopedia Of Philosophy*, V, 15.

246
Ibid., 16.

247
Russell, *The Principles Of Mathematics*, 45-48.

248
In discussing Russell's notion of implication, I have employed Russell's terminology; some of this terminology involves a confusion between use and mention. When I have employed some of these expressions involving a confusion between use and mention, I have done so only in order to explain Russell in the context of Russell himself.

249
Lewis and Langford, *op. cit.*, 248.

250
Russell, Introduction To Mathematical Philosophy, 154.

251
Russell, *The Principles Of Mathematics*, 33-34.

252
Russell, "'If' And 'Imply', A Reply To Mr. MacColl", 301.

253
Russell, "Logical Atomism" in Marsh (ed.), *Logic And Knowledge: Essays 1901-1950*, 338.

254
Blumberg, *op. cit.*, 15-16. Diagram VII is also adopted from Blumberg's article.

255
Strawson, *Introduction To Logical Theory*, 214-215.

256
Quine, *Word And Object*, 159-160.

[257] Moore's article can be dated in 1905 due to a reference to it in one of his letters to Russell. See: Letter: Moore to Russell, October 23, 1905. "I have at last finished all I can say about your book for the 'Archiv'..." It should be noted that the article was not published. See: Moore, "An Autobiography" in Schilpp (ed.), *The Philosophy Of G.E. Moore*, 15, 27.

[258] Moore, "Russell's *Principles Of Mathematics*", 13.

[259] *Ibid.*, 14-16.

[260] Russell, *The Principles Of Mathematics*, 15.

[261] Moore, "Russell's *Principles Of Mathematics*", 16.

[262] *Ibid.*, 16-17.

[263] *Ibid.*, 17-18. See also: Jager, *The Development Of Bertrand Russell's Philosophy*, 114.

[264] *Ibid.*, 18-19.

[265] Russell, *Introduction To Mathematical Philosophy*, 153.

[266] Wittgenstein, *Tractatus Logico-Philosophicus*, 107-108.

[267] Reichenbach, "Bertrand Russell's Logic", in Schilpp (ed.), *The Philosophy Of Bertrand Russell*, 26-27.

[268] Quine, *Mathematical Logic*, 31.

[269] Russell, *The Principles Of Mathematics*, 34.

[270] Reichenbach, *op.cit.*, 27.

[271] Lee, "Note On '$\supset$' And '$\vdash$' In Whitehead And Russell's *Principia Mathematica*", in *Mind*, 250.

[272] *Ibid.*, 250.

[273] Whitehead and Russell, *op. cit.*, 92. See also: Russell, "The Theory Of Implication", 161.

[274] Feys (ed.) and Fitch (ed.), *Dictionary Of Symbols Of Mathematical Logic*, 153.

[275] I say this because "$\Box[(p \supset q) \supset (p \supset q)]$" and

"$(p \supset q) -\!\!\!3\ (p \supset q)$" are equivalent expresions.

276

Lee, *op.cit.*, 250.

277

Ibid., 250.

278

Ibid., 251.

279

Ibid., 252.

280

Ibid., 252.

281

Jager, "Truth And Assertion" in *Mind*, 161. "The link therefore between asserting, or statement making, and truth is an intimate one and obscure... I have said the matter is obscure, as is partly shown by its remarkable resistance in the literature." See also: Russell, *The Principles Of Mathematics*, 47-49.

282

Kneale and Kneale, *The Development Of Logic*, 553-554. Although Deveaux does not attribute his criticism of Russell's notion of implication to Kneale and Kneale, it would seem that his criticism is exactly the same as that contained in *The Development Of Logic*. See: Deveaux, *Bertrand Russell Ou La Paix Dans La Vérité*, 62. "En interprétant p *implique* q comme une implication matérielle, Russell et Whitehead ont, semble-t-il, confondu deux questions, celle de la justification de l'inférence de p à q et celle du minimum suffisant pour que de la prémisse p on puisse inférer q."

283

Prior, "Logic And Mathematics" in Edwards (ed.), *The Encyclopedia Of Philosophy*, VII, 246. This article by Prior is part of a larger article in the same volume by Edwards, Alston and Prior entitled "Bertrand Arthur William Russell".

284

Ibid., 246-247. See also: Jager, *The Development Of Bertrand Russell's Philosophy*, 112-113.

285

Russell, *The Principles Of Mathematics*, 33.

286

Russell, *Introduction To Mathematical Philosophy*, 153.

287

Prior, *op.cit.*, 247.

288

Leavitt, "On An Unpublished Remark Of Russell's On 'If...Then'", 1.

289
Russell, "Necessity And Possibility", 17.
290
Leavitt, _op.cit._, 1-2.
291
Russell, "Necessity And Possibility", 17.
292
Jager, _The Development Of Bertrand Russell's Philosophy_, 110.
293
Ibid., 112.
294
Ibid., 112.
295
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Russell, _The Principles Of Mathematics_, 38. See also: Jager, _The Development Of Bertrand Russell's Philosophy_, 115.
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Ibid., 34. See also: Jager, _The Development Of Bertrand Russell's Philosophy_, 116.
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307
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309 Russell, _The Principles Of Mathematics_, 34.

310 Jager, _The Development Of Bertrand Russell's Philosophy_, 116.

311 Russell, _The Principles Of Mathematics_, 34.

312 _Ibid._, 34.

313 Whitehead and Russell, _op.cit._, 7.

314 Russell, _The Principles Of Mathematics_, 10-11.

315 _Ibid._, 33.

316 Quine, _Word And Object_, 196.

317 Russell, "Recent Italian Work On The Foundations Of Mathematics", 4.

318 Russell, _The Principles Of Mathematics_, 14.

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320 Russell, _The Principles Of Mathematics_, vii.

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322 _Ibid._, 139.

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