INTERPOLATION ON ORLICZ AND BMO FUNCTION SPACES

INTERPOLATION OF LINEAR OPERATORS

ON

ORLICZ AND BMO FUNCTION SPACES

By

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ABSTRACT

This thesis is concerned with the study of interpolation theorems involving certain Orlicz spaces and spaces of functions of Bounded Mean Oscillation. In addition, we consider the Orlicz spaces in a weighted sense by applying weight functions satisfying certain growth conditions.

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CHAPTER I

Let T be a linear operator which maps a linear space X into a linear space Y. Suppose that X_1 and X_2 (respectively Y_1 and Y_2) are Banach subspaces of X (respectively Y), such that T is a bounded, (that is continuous), linear operator mapping X_i into Y_i for $i = 1,2$. Often, using the boundedness properties of T, one can determine other pairs (X', Y') of subspaces $(X' \subset X, Y' \subset Y)$ such that T maps X' into Y' continuously. Theorems concerned with the above are called interpolation theorems.

The first significant steps in interpolation theory were made by Marcel Riesz [15] in 1926. In his paper "Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires", Riesz considered the Banach spaces $L^p(X,M,\mu)$ of real-valued Lebesgue measurable functions, defined on X, whose p^{th} power is integrable.

In 1939, G. O. Thorin [21] extended and modified the interpolation or convexity theorem of M. Riesz. Thorin showed that a linear operator T, which maps $L^{p_i}(X,M,\mu)$ continuously into $L^{q_i}(X,M,\mu)$ $(i = 1,2)$, can be extended to a continuous linear operator (without change of norm) from $L^p(X,M,\mu)$ to $L^q(X,M,\mu)$, where $p \in [p_0,p_1]$ and $q \in [q_0,q_1]$.

To illustrate this result, consider the Fourier Transform, defined by $Tf = \hat{f}$, where, for $f \in L^1(R) \cap L^2(R)$,

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixt} f(t)\,dt.$$

From the definition of $\hat{f}$, it follows that

$$||Tf||_{\infty} \leq M||f||_1,$$

and from Plancherel's Theorem,

$$||Tf||_2 = ||f||_2.$$

An application of the Riesz-Thorin Convexity Theorem then shows that the Hausdorff-Young Inequality,

$$||Tf||_q \leq M||f||_p,$$

where $1 \leq p \leq 2$ and $q = \frac{p}{p-1}$, holds.

For many operators, the hypotheses of the Riesz-Thorin Convexity Theorem are too strong to be applicable. The question then arises whether the continuity of T at the endpoints can be replaced by some weaker condition. Also, one might ask if the operator T could be sublinear or quasi-linear.

J. Marcinkiewicz [11] obtained the first major results concerning these questions. He considered quasi-linear operators which satisfied certain "weak" boundedness conditions at the endpoints and he obtained an interpolation theorem. It should be noted that the Marcinkiewicz Interpolation Theorem does not imply the Riesz-Thorin Theorem, although for certain p and q it is more general.

For example, we look at the Hilbert Transform H, defined by

$$(Hf)(x) = \frac{(P)}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x - t}\,dt,$$

where the integral is considered in the Cauchy principal value sense.

Using Plancherel's Theorem, it can be shown that H maps L^2 onto itself continuously. H is not, however, bounded from L^1 to L^1, although it is weakly bounded in the sense of Marcinkiewicz (see again, [11]). Thus, we can apply the Marcinkiewicz Interpolation Theorem to obtain

$$||Hf||_p \le M\,||f||_p \qquad 1 < p \le 2.$$

A duality argument shows that the norm estimate holds then for all $p \in (1,\infty)$.

In 1957, Elias M. Stein and Guido Weiss [19] presented interpolation theorems for analytic families of operators. In particular, they extended the Riesz-Thorin Theorem by replacing the single operator T by an analytic family of operators $\{T_\omega\}_{\omega = x + iy}$, $0 \le x \le 1$. Yoram Sagher [16] obtained a Marcinkiewicz Theorem for analytic families of operators by working with Lorentz spaces. These results made it possible to interpolate between L^p-spaces having different measures.

The proofs of some of these interpolation theorems involve the concept of the non-increasing equimeasurable rearrangement of a measurable function. This in turn led to the introduction of the Lorentz spaces, L(p,q), and thus quite naturally to the development of an abstract theory. This work was largely pioneered by J. L. Lions, J. Peetre, A. P. Calderón and E. Gagliardo. They developed methods of constructing linear spaces which were intermediate to arbitrary Banach spaces. For characterizations of intermediate spaces and their application to boundary value problems, we refer to Gagliardo [3] and J. L. Lions and E. Magenes [10].

This thesis concerns itself with the study of interpolation theorems-specifically the Marcinkiewicz Theorem-for certain Orlicz function spaces and for spaces of functions of Bounded Mean Oscillation. The primary object is to obtain an interpolation theorem involving the spaces $L^p(\log^+ L)^s$ and the Lorentz spaces $L(p,q)$. Such a theorem was introduced by Richard O'Neil [14] in terms of the distribution of a function. Our proof is different in that the distribution function is replaced by the non-increasing equimeasurable rearrangement of the function. In addition, we introduce a weight in the spaces $L^p(\log^+ L)^s$ and prove an interpolation theorem involving these and certain weighted Lorentz spaces.

Interpolation theorems involving the function spaces L_p^λ were previously given by G. Stampacchia [17]. The spaces L_p^λ are more general than the spaces of functions of Bounded Mean Oscillation and they were extended in [4] to the spaces $L^\lambda(p,q)$ which are Lorentz spaces of functions of Bounded Mean Oscillation. In the last chapter, we give an interpolation theorem involving functions of $L^1(\log^+ L)^s$ and the spaces $L^\lambda(p,q)$.

CHAPTER II

Section A. Notations and Definitions

The following is a collection of definitions and notations to which we will adhere throughout this thesis. We will consider measure spaces (X,M,μ) which are either totally finite or σ-finite. The measures are all Lebesgue measures and the functions considered are complex-valued Lebesgue measurable functions.

We begin by defining the Lebesgue spaces, $L^p(X,M,\mu)$ $(0 < p \le \infty)$, of p-integrable functions defined almost everywhere (with respect to μ) on X. Keeping in mind that the Lebesgue integral does not distinguish functions which differ only on sets of measure zero, we will refer to functions when in fact, we are discussing equivalence classes of functions modulo a set of measure zero.

Definition 2.1: (i) For $0 < p < \infty$, $L^p(X,M,\mu)$ consists of those complex-valued Lebesgue measurable functions f, defined almost everywhere on X, for which $|f|^p$ is integrable.

(ii) For $p = \infty$, $L^\infty(X,M,\mu)$ consists of those complex-valued Lebesgue measurable functions f, defined almost everywhere on X, which are almost everywhere bounded.

Remark 1: (i) For $0 < p < \infty$, we define

$$||f||_p = \left(\int_X |f(x)|^p d\mu(x)\right)^{1/p}.$$

For $p \geq 1$, $||\cdot||_p$ is a norm and $L^p(X,M,\mu)$ is a normed linear space, which is complete by the Riesz-Fischer Theorem [5, p. 192]. Thus, $L^p(X,M,\mu)$ $(1 \leq p < \infty)$ is a Banach space. If $0 < p < 1$, then $d(f,g) = ||f - g||_p^p$ defines a metric, under which $L^p(X,M,\mu)$ is complete. Therefore, $L^p(X,M,\mu)$ $(0 < p < 1)$ is a Fréchet space.

(ii) For $p = \infty$, we define

$$||f||_\infty = \operatorname*{ess\,sup}_{x \in X} |f(x)|,$$

where

$$\operatorname*{ess\,sup}_{x \in X} |f(x)| = \inf \{a \in [0,+\infty);\ \mu(\{x \in X;\ |f(x)| > a\}) = 0\}.$$

$||\cdot||_\infty$ is a norm and, by the Riesz-Fischer Theorem, $L^\infty(X,M,\mu)$ is a Banach space.

We note here that the above definitions and remarks can be found, for example, in Hewitt and Stromberg [5].

A larger class of spaces is the class of Orlicz spaces. This class is larger in the sense that the Orlicz spaces are Banach spaces and that the L^p-spaces, for $1 \leq p < \infty$, are particular examples of Orlicz spaces. The following development of Orlicz spaces can be found in Zygmund [24, Vol. I, pp. 170 - 175].

For a non-negative function Φ defined on $[0,+\infty)$, and a measure space (X,M,μ), we denote, by $L_\Phi(X,M,\mu)$, the class of those complex-valued Lebesgue measurable functions f, defined almost everywhere on X, for which $\Phi(|f|)$ is integrable on X. That is, $f \in L_\Phi(X,M,\mu)$ if

$$\int_X \Phi(|f(x)|)d\mu(x) < +\infty.$$

Consider two functions ϕ and ψ which are defined on $[0,+\infty)$ and satisfy the following conditions:

(i) ϕ and ψ are continuous.

(ii) $\phi(0) = \psi(0) = 0$.

(iii) ϕ and ψ are strictly increasing.

(iv) $\phi(\psi(x)) = \psi(\phi(x)) = x$.

If we set

$$\Phi(x) = \int_0^x \phi(t)dt$$

and

$$\Psi(x) = \int_0^x \psi(t)dt,$$

then we have, for any x_0, $y_0 \geq 0$, Young's Inequality:

$$x_0 y_0 \leq \Phi(x_0) + \Psi(y_0).$$

Φ and Ψ are called complementary functions in the sense of Young.

We note here, without proof, that for any Φ defined on $[0,+\infty)$, with Φ being non-negative, convex, zero at the origin and such that $\lim_{x \to +\infty} \frac{\Phi(x)}{x} = +\infty$, there exists a function Ψ which is complementary to Φ in the sense of Young. The proof of this may be found in Zygmund [24, Vol. I, p. 25].

<u>Definition 2.2</u>: Let Φ be defined on $[0,+\infty)$, such that Φ is non-negative, convex, zero at the origin and satisfies $\lim_{x \to +\infty} \frac{\Phi(x)}{x} = +\infty$. We denote, by $L^*_\Phi(X,M,\mu)$, the set consisting of those complex-valued Lebesgue measurable functions f, defined almost everywhere on X, for

which $|fg|$ is integrable over X, for any $g \in L_\Psi(X,M,\mu)$.

Remark 2: (i) By setting

$$||f||_\Phi = \sup_g \left|\int_X f(x)g(x)d\mu(x)\right|,$$

where the supremum is taken over all those $g \in L_\Psi(X,M,\mu)$ with

$$\int_X \Psi(|g(x)|)d\mu(x) \le 1,$$

$L^*_\Phi(X,M,\mu)$ becomes a complete normed linear space. That is, it is a Banach space.

(ii) Clearly, $L^p(X,M,\mu)$ is an Orlicz space for $p \ge 1$, where $\Phi(x) = x^p$.

(iii) In Chapter III we will discuss the interpolation of certain operators in connection with the Orlicz spaces $L^p(\log^+L)^s(X,M,\mu)$ where $X = [0,1]$, $1 \le p < +\infty$ and $0 \le s \le 1$. We define these spaces as follows:

Definition 2.3: For $1 \le p < +\infty$ and $0 \le s \le 1$, $L^p(\log^+L)^s[0,1]$ consists of those complex-valued Lebesgue measurable functions f, defined almost everywhere on [0,1], for which

$$\int_0^1 |f(x)|^p(\log^+|f(x)|)^s dx < +\infty,$$

where $\log^+|f(x)| = \log|f(x)|$ if $|f(x)| \ge 1$ and 0 otherwise.

It is obvious, from the above definition, that if $s = 0$, then $L^p(\log^+L)^0[0,1] \equiv L^p[0,1]$.

<u>Definition 2.4</u>: If f is a complex-valued Lebesgue measurable function defined almost everywhere on X, the distribution function D_f, of f, is defined by

$$D_f(y) = \mu(\{x \in X;\ |f(x)| > y\}) \qquad y > 0.$$

We note that the distribution function D_f is non-increasing and continuous on the right. Also, for $y > 0$, we have

$$D_{f+g}(2y) \le D_f(y) + D_g(y).$$

This inequality follows from the set inclusions

$$\{x \in X;\ |f(x) + g(x)| > 2y\} \subset \{x \in X;\ |f(x)| + |g(x)| > 2y\}$$
$$\subset \{x \in X;\ |f(x)| > y\} \cup \{x \in X;\ |g(x)| > y\}.$$

Applying the measure μ to both sides of the above inclusion yields the desired inequality.

Closely connected with the distribution function is the non-increasing equimeasurable rearrangement of a measurable function.

<u>Definition 2.5</u>: If f is a complex-valued Lebesgue measurable function defined almost everywhere on X and D_f its distribution on $(0,+\infty)$, then the non-increasing equimeasurable rearrangement f^* of f onto $(0,+\infty)$ is defined by

$$f^*(x) = \inf\{y > 0;\ D_f(y) \le x\} \qquad x > 0.$$

The non-increasing equimeasurable rearrangement f^* of f is also a non-increasing function and is continuous on the right. If D_f is strictly decreasing, then f^* is the inverse of D_f and $f^*(D_f(y)) = y$. It should be noted that f^* is called the "equimeasurable" rearrangement of f because $D_f = D_{f^*}$. We will show this later on.

Remark 3: (i) An useful property of the non-increasing equimeasurable rearrangement f^* of f is that, for $x > 0$,

$$(f + g)^*(2x) \leq f^*(x) + g^*(x).$$

This inequality follows from the similar inequality for the distribution function D_f and the fact that f^* is the inverse of D_f if D_f^{-1} exists.

(ii) We also have that if E is a measurable subset of X ($E \subset X$) and $\mu(E) \leq t < +\infty$, then

$$\int_E |f(x)|d\mu(x) \leq \int_0^t f^*(x)dx.$$

In Chapter III we will also deal with spaces which are related to the $L^p(\log^+ L)^s$-spaces defined above. They are the $K^p(\log^+ K)^s$-spaces and are defined by:

Definition 2.6: For $0 < p < +\infty$, $0 \leq s < +\infty$, the spaces $K^p(\log^+ K)^s(X,M,\mu)$ consist of those complex-valued Lebesgue measurable functions f, defined almost everywhere on X, for which

$$\int_0^1 x^{1/p - 1} f^*(x) \left(\log 1/x\right)^{s/p} dx < +\infty.$$

Definition 2.7: If X and Y are function spaces and T maps X into Y such that $T(f + g)$ is uniquely defined whenever Tf and Tg are defined, then T is called a quasi-linear operator if there exists a constant κ (> 0), independent of f and g, such that

$$|T(f + g)| \leq \kappa(|Tf| + |Tg|).$$

If $\kappa = 1$, then T is called a sublinear operator.

When discussing the interpolation of operators on the above spaces, we will, for the sake of brevity, deal with linear operators, although the results hold also for quasi-linear and sublinear operators.

Definition 2.8: A linear operator T is said to be of strong type (p,q) $(0 < p,q \leq +\infty)$ if

$$||Tf||_q \leq A||f||_p$$

for some constant A (> 0).

Definition 2.9: A linear operator T is said to be of weak type (p,q) $(0 < p,q < +\infty)$ if there exists a constant A (> 0) such that, for any $y > 0$,

$$D_{Tf}(y) \leq \left(\frac{A||f||_p}{y} \right)^q .$$

If $0 < p < +\infty$ and $q = +\infty$, we define weak type (p,q) to be the same as strong type (p,q). That is, T is of weak type (p,∞) if

$$||Tf||_\infty \leq A||f||_p ,$$

where A is a constant (> 0).

Remark 4: (i) In terms of non-increasing equimeasurable rearrangements, a linear operator T is of weak type (p,q) $(0 < p,q < +\infty)$ if there exists a constant A (> 0) such that

$$(Tf)^*(t) \leq \frac{A||f||_p}{t^{1/q}} .$$

(ii) If T is of strong type (p,q) then it is of weak type (p,q), but not conversely. For an example that the converse does not hold, consider the operator T defined by

$$Tf(x) = \frac{1}{x}\int_0^x |f(t)|dt, \qquad x > 0.$$

This operator is of weak type (1,1) but is not of strong type (1,1).

<u>Proof</u>: We can consider $f(x) = (x + 1)^{-2}$. Then

$$\int_0^\infty |f(x)|dx = \int_0^\infty (x + 1)^{-2}dx$$

$$= -(x + 1)^{-1}\Big|_0^\infty$$

$$= 1 < \infty.$$

That is, $f \in L^1(0,+\infty)$. However,

$$||Tf||_1 = \int_0^\infty |Tf(x)|dx$$

$$= \int_0^\infty |\frac{1}{x}\int_0^x |f(t)|dt|dx$$

$$= \int_0^\infty |x^{-1}\int_0^x (t + 1)^{-2}dt|dx$$

$$= \int_0^\infty |x^{-1}(-(t + 1)^{-1}\Big|_0^x)|dx$$

$$= \int_0^\infty |x^{-1}(-(x + 1)^{-1} + 1)\, dx$$

$$= \int_0^\infty |x^{-1}(x(x + 1)^{-1})|dx$$

$$= \int_0^\infty (x + 1)^{-1} dx$$

$$= \infty.$$

Hence, there does not exist a finite positive constant A such that $||Tf||_1 \le A||f||_1$. In other words, T is not of strong type (1,1). On the other hand,

$$(Tf)^*(x) \le \frac{1}{x}\int_0^x f^*(t)dt$$

$$\le \frac{1}{x}\int_0^\infty f^*(t)dt$$

$$= \frac{1}{x}||f||_1.$$

Hence, T is of weak type (1,1).

(iii) We note here that throughout the thesis we will denote constants by the letter A. At different appearances, A may take on different values.

The last chapter will deal with the interpolation of certain sublinear operators defined on the $L^\lambda(p,q)$-spaces ($\lambda \in (-\infty,+\infty)$; $p,q > 0$) introduced by H. Heinig [4]. These spaces are defined by combining the notions of Lorentz spaces and the $L^{(p,\lambda)}$-spaces ($\lambda \in (-\infty,+\infty)$; $p \ge 1$) of G. Stampacchia [17] as follows:

<u>Definition 2.10</u>: For $0 < p,q \le +\infty$, the Lorentz spaces $L(p,q)(X,M,\mu)$ consist of those complex-valued Lebesgue measurable functions f, defined almost everywhere on X, for which $||f||^*_{p,q}$ is

finite, where

$$\|f\|^*_{p,q} = \begin{cases} \left(\frac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q t^{-1}\, dt\right)^{\frac{1}{q}} & 0 < p,q < +\infty \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) & 0 < p \le +\infty,\ q = +\infty. \end{cases}$$

We will show later that $\|f\|_p = \|f^*\|_p$ and hence $\|f\|_p = \|f\|^*_{p,p}$. From this we have that

$$L^p(X,M,\mu) \equiv L(p,p)(X,M,\mu).$$

In addition, we have that the $L(p,q)(X,M,\mu)$-spaces are, for $0 < p,q \le +\infty$, complete metric spaces (Fréchet spaces). For $1 < p \le +\infty$ and $1 \le q \le +\infty$, the spaces $L(p,q)(X,M,\mu)$ are Banach spaces under the norm $\|\cdot\|_{p,q}$ defined by

$$\|f\|_{p,q} = \begin{cases} \left(\frac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}} f^{**}(t)\right)^q t^{-1}\, dt\right)^{\frac{1}{q}} & 1 < p \le +\infty,\ 1 \le q < +\infty \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & 1 < p \le +\infty,\ q = +\infty, \end{cases}$$

where f^{**} is defined by

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)\,dt.$$

For further details consult, for example, Hunt [6].

To define the $L^{(p,\lambda)}$-spaces, we consider a cube C_0 in the n-dimensional Euclidean space R^n, where n is a positive integer. We note that $R = (-\infty,+\infty)$, the set of real numbers, and $R^n = R \times R^{n-1}$. That is, R^n is the Cartesian product of n copies of R.

For any element x of C_0 and for any positive real number ρ, we consider $C(x,\rho)$ to be the parallel subcube of C_0 with centre $x \in C_0$ and with side length ρ. Without loss of generality, we can and will assume that C_0 is centred at the origin and has side length a. We then have the following:

Definition 2.11: For $1 \leq p < +\infty$ and $\lambda \in R$, the spaces $L^{(p,\lambda)}(C_0) \equiv L^{(p,\lambda)}$ consist of those complex-valued Lebesgue measurable functions f, defined almost everywhere on C_0, such that $||f||_{L^{(p,\lambda)}}$ is finite, where

$$||f||_{L^{(p,\lambda)}} = \left(\sup_{C(x,\rho) \subset C_0} \rho^{\lambda - n} \int_{C(x,\rho)} |f(y) - f_c|^p \, dy \right)^{\frac{1}{p}}.$$

In the above integrals, f_c is the mean value of f over the subcube $C(x,\rho)$ of C_0.

Remark 5: We note here that $||\cdot||_{L^{(p,\lambda)}}$ is not a norm because $||f||_{L^{(p,\lambda)}} = 0$ implies only that $f(y) = f_c$ almost everywhere on $C(x,\rho)$. That is, only that f is constant almost everywhere on the subcube $C(x,\rho)$. The spaces $L^{(p,\lambda)}$ can be normed by setting

$$||f||_{p,\lambda} = ||f||_{L^1(C_0)} + ||f||_{L^{(p,\lambda)}}.$$

For details, see G. Stampacchia [17]. We also have two special cases for particular values of λ.

Case 1 $\lambda < 0$. If λ is negative, then a function $f \in L^{(p,\lambda)}$ is equal, almost everywhere, to a Hölder continuous function. See Meyers [12].

Case 2 $\lambda = 0$. If $\lambda = 0$ and $n = 1$, $L^{(1,0)}$ coincides with the space of functions of bounded mean oscillation which are defined below.

Definition 2.12: If f is a complex-valued Lebesgue measurable function defined almost everywhere on R^n (n a positive integer), then f is said to be of bounded mean oscillation (BMO) on R^n if there exists a constant A (> 0) such that

$$\frac{1}{\mu(C)} \int_C |f(y) - f_c| \, dy \leq A$$

for every cube C in R^n. In the above integral, f_c is the mean value of f over C.

Remark 6: From the last definition, it is clear that every bounded complex-valued Lebesgue measurable function is of BMO. However, the converse is not true. F. John and L. Nirenberg [8] provide the following example. In R^2, the function $f(x,y) = \log|x - y|$ is not bounded. However, it is of bounded mean oscillation.

The above ideas can be combined to yield the notion of the $L^\lambda(p,q)(C_0)$-spaces. If we write $F_c(t) = f(t) - f_c$ and recall the definitions of the $L(p,q)$-spaces and the $L^{(p,\lambda)}$-spaces, we have the following:

Definition 2.13: For $-\infty < \lambda < +\infty$ and $0 < p,q \leq +\infty$, the spaces

$L^{\lambda}(p,q)(C_0) \equiv L^{\lambda}(p,q)$ consist of those complex-valued Lebesgue measurable functions f, defined almost everywhere on C_0, for which $||f||_{L^{\lambda}(p,q)}$ is finite, where

$$||f||_{L^{\lambda}(p,q)} = \begin{cases} \sup\limits_{C(x,\rho) \subset C_0} \left(\rho^{\lambda - n} \int_0^{\infty} \left(t^{\frac{1}{p}} F_c^{*}(t)\right)^q t^{-1}\, dt\right)^{\frac{1}{q}} & 0 < p,q < +\infty \\ \sup\limits_{C(x,\rho) \subset C_0} \sup\limits_{t > 0} t^{\frac{1}{p}} F_c^{*}(t) & 0 < p \le +\infty,\ q = +\infty. \end{cases}$$

Remark 7: For $p = q \geq 1$, we have that the $L^{\lambda}(p,q)$-spaces reduce to the $L^{(p,\lambda)}$-spaces of G. Stampacchia [17].

Section B. Preliminary Results

This section consists of results which will be used at later stages of the thesis. The first two theorems are proved for arbitrary measure spaces even though we will deal only with measure spaces which are totally finite or σ-finite.

Theorem 2.1: (Zaanen [22, p. 127]) If $1 \le p < +\infty$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$, then for any $f \in L^p(X,M,\mu)$, where (X,M,μ) is an arbitrary measure space,

$$||f||_p = \sup_g \left|\int_X f(x)g(x)d\mu(x)\right|$$

$$= \sup_{g} \int_X |f(x)g(x)|d\mu(x),$$

where the supremum is taken over all functions $g \in L^q(X,M,\mu)$, such $||g||_q \le 1$.

<u>Proof</u>: For $1 < p < +\infty$, Hölder's inequality shows that

$$\int_X |f(x)g(x)|d\mu(x) \le ||f||_p ||g||_q$$

$$\le ||f||_p \quad \text{if } ||g||_q \le 1.$$

If $p = 1$, then the inequality follows trivially.

To complete the proof, we need to produce a function for which equality holds. For complex z, define

$$\text{sgn } z = \begin{cases} \frac{z}{|z|} & 0 < |z| < +\infty \\ 1 & |z| = +\infty. \end{cases}$$

Now define $h(x) = \frac{1}{\text{sgn } f(x)}$. Clearly $h \in L^\infty(X,M,\mu)$ and $||h||_\infty = 1$. For $p = 1$, we define $g(x) \equiv h(x)$. Then

$$\left|\int_X f(x)g(x)d\mu(x)\right| = \left|\int_X f(x)\left(\frac{1}{\text{sgn } f(x)}\right)d\mu(x)\right|$$

$$= \left|\int_X |f(x)|d\mu(x)\right|$$

$$= \int_X |f(x)|d\mu(x)$$

$$= ||f||_1.$$

If $1 < p < +\infty$, then we define $k(x)$ by

$$k(x) \equiv \frac{|f(x)|^{p-1}}{\operatorname{sgn} f(x)} .$$

Since

$$|k(x)|^q = |k(x)|^{\frac{p}{p-1}}$$

$$= \left|\left(\frac{|f(x)|^{p-1}}{\operatorname{sgn} f(x)}\right)\right|^{\frac{p}{p-1}}$$

$$= \frac{|f(x)|^p}{|\operatorname{sgn} f(x)|^{p/(p-1)}}$$

$$= |f(x)|^p,$$

$k \in L^q(X,M,\mu)$. Note also that $||k||_q = ||f||_p^{p/q}$. Thus, if we set

$$g(x) \equiv \frac{k(x)}{||f||_p^{p/q}},$$

where $f \not\equiv 0$, we have that $g \in L^q(X,M,\mu)$ and $||g||_q = 1$. Finally,

$$\left|\int_X f(x)g(x)d\mu(x)\right| = \left|\int_X f(x)\left(\frac{|f(x)|^{p-1}}{||f||_p^{p/q} \operatorname{sgn} f(x)}\right)d\mu(x)\right|$$

$$= \frac{1}{||f||_p^{p/q}}\left(\left|\int_X |f(x)|^p \, d\mu(x)\right|\right)$$

$$= \frac{1}{||f||_p^{p/q}}\left(\int_X |f(x)|^p \, d\mu(x)\right)$$

$$= ||f||_p^{p - p/q}$$

$$= ||f||_p.$$

Thus, the theorem is proved.

Theorem 2.2: (Zaanen [22, p. 127]) Let (X,M,μ) be a measure space such that for any set $U \in M$ with $\mu(U) < +\infty$, there exists a set V ($V \in M$) with $V \subseteqq U$ and $\mu(V) < +\infty$. Then, for any $f \in L^{\infty}(X,M,\mu)$,

$$||f||_{\infty} = \sup_{g} \left| \int_X f(x)g(x)d\mu(x) \right|$$

$$= \sup_{g} \int_X |f(x)g(x)|d\mu(x),$$

where the supremum is taken over all functions $g \in L^1(X,M,\mu)$ for which $||g||_1 \leq 1$.

Proof: As in the last theorem, we easily have

$$\left| \int_X f(x)g(x)d\mu(x) \right| \leq \int_X |f(x)g(x)|d\mu(x)$$

$$\leq ||f||_{\infty}||g||_1$$

$$\leq ||f||_{\infty}.$$

To complete the proof, we must construct a function $g \in L^1(X,M,\mu)$, for which equality holds.

For given $\varepsilon > 0$, the set $\{x \in X; |f(x)| > ||f||_{\infty} - \varepsilon\}$ contains a subset V of finite measure. Defining $g(x)$ by

$$g(x) = \begin{cases} \dfrac{1}{\mu(V)\ \mathrm{sgn}\ f(x)} & x \in V \\ 0 & x \in X\backslash V, \end{cases}$$

we have

$$\begin{aligned} ||g||_1 &= \int_X |g(x)|d\mu(x) \\ &= \int_V \left| \frac{1}{\mu(V)\ \mathrm{sgn}\ f(x)} \right| d\mu(x) \\ &= \frac{1}{\mu(V)} \int_V d\mu(x) \\ &= 1. \end{aligned}$$

That is, $g \in L^1(X,M,\mu)$ and $||g||_1 = 1$.

Also,

$$\begin{aligned} \left|\int_X f(x)g(x)d\mu(x)\right| &= \left|\int_V f(x)\left(\frac{1}{\mu(V)\ \mathrm{sgn}\ f(x)}\right)d\mu(x)\right| \\ &= \left|\int_V \left(\frac{|f(x)|}{\mu(V)}\right)d\mu(x)\right| \\ &= \int_V \left(\frac{|f(x)|}{\mu(V)}\right)d\mu(x) \\ &\geq \int_V \left(\frac{||f||_\infty - \varepsilon}{\mu(V)}\right)d\mu(x) \\ &= ||f||_\infty - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have our result.

An integral inequality which will be used later is the following theorem of G. Stampacchia [17].

Theorem 2.3: Let $f(x)$, $g(x)$, $z(x)$ be non-negative Lebesgue measurable functions defined on $(0,a)$, $(a > 0)$, with $z(x)$ increasing. For $1 \le p < +\infty$, the following integral inequalities hold:

$$\text{(i)} \quad \int_0^a f(x)\left(\int_0^{z(x)} g(y)dy\right)^p dx \le \left(\int_0^a g(y)\left(\int_{\xi(y)}^a f(x)dx\right)^{1/p} dy\right)^p$$

$$\text{(ii)} \quad \int_0^a f(x)\left(\int_{z(x)}^a g(y)dy\right)^p dx \le \left(\int_0^a g(y)\left(\int_0^{\xi(y)} f(x)dx\right)^{1/p} dy\right)^p,$$

where ξ is the inverse function of z.

Proof: For $p = 1$, the result follows from a change of order of integration.

Let $1 < p < +\infty$. To prove inequality (i), consider

$$I = \int_0^a f(x)\left(\int_0^{z(y)} g(y)dy\right)^p dx$$

$$= \int_0^a (F(x))^p dx,$$

where $F(x) = (f(x))^{1/p} \int_0^{z(x)} g(y)dy.$

Then, by Theorem 2.1, we have

$$I^{1/p} = \left|\int_0^a F(x)G(x)dx\right|$$

where $G(x) = \left(f(x)\right)^{1 - 1/p}\chi(x)$ and χ is any function which satisfies

$||G||_q = ||f^{1 - 1/p}\chi||_q \le 1.$

That is,

$$I^{1/p} = \sup_{\chi} \left|\int_0^a \left[\left(f(x)\right)^{1/p} \int_0^{z(x)} g(y)dy\right]\left[\left(f(x)\right)^{1 - 1/p}\chi(x)\right]dx\right|$$

$$= \sup_{\chi} \left|\int_0^a f(x)\left(\int_0^{z(x)} g(y)dy\right)\chi(x)dx\right|.$$

By interchanging the order of integration and the use of Hölder's inequality, we have

$$I^{1/p} \le \sup_{\chi} \int_0^a g(y)\left(\int_{\xi(y)}^a f(x)\chi(x)dx\right)dy$$

$$\le \sup_{\chi} \int_0^a g(y)\left(\int_{\xi(y)}^a f(x)dx\right)^{1/p}\left(\int_{\xi(y)}^a f(x)\left(\chi(x)\right)^q dx\right)^{1/q} dy.$$

However, we have that

$$\left(\int_{\xi(y)}^a f(x)\left(\chi(x)\right)^q dx\right)^{1/q} \le \left(\int_0^a f(x)\left(\chi(x)\right)^q dx\right)^{1/q}$$

$$\le 1,$$

so that

$$I^{1/p} \le \sup_{X} \int_0^a g(y)\left(\int_{\xi(y)}^a f(x)dx\right)^{1/p} dy$$

$$= \int_0^a g(y)\left(\int_{\xi(y)}^a f(x)dx\right)^{1/p} dy.$$

Therefore,

$$I \le \left(\int_0^a g(y)\left(\int_{\xi(y)}^a f(x)dx\right)^{1/p} dy\right)^p.$$

The proof of inequality (ii) follows along the same lines as the above. The details are therefore omitted.

Remark 8: If, in the above theorem, $z(x)$ is a decreasing function, then the right side of inequality (ii) replaces the right side of inequality (i) and vice versa.

Theorem 2.4: (Hewitt and Stromberg [5, pp. 421 - 422]) Let (X,M,μ) be a σ-finite measure space and let f be a non-negative Lebesgue measurable function defined almost everywhere on X and E a measurable subset of X ($E \in M$). If ϕ is a real-valued, non-decreasing, differentiable function defined on $[0,+\infty)$, such that $\phi(0) = 0$ and

$$\int_E \phi(f(x))d\mu(x) < +\infty,$$

then

$$\int_E \phi(f(x))d\mu(x) = \int_0^\infty \mu(E \cap E_t)\phi'(t)dt$$

where ϕ' is the derivative of ϕ and $E_t = \{x \in X;\ f(x) > t\}$.

Proof:

$$\int_E \phi(f(x))d\mu(x) = \int_X \chi_E(x)\phi(f(x))d\mu(x)$$

$$= \int_X \chi_E(x)\left(\int_0^{f(x)} \phi'(t)dt\right)d\mu(x)$$

$$= \int_X \chi_E(x)\left(\int_0^{\infty} \chi_{[0,f(x)]}(t)\phi'(t)dt\right)d\mu(x)$$

$$= \int_0^{\infty} \phi'(t)\left(\int_X \chi_E(x)\chi_{[0,f(x)]}(t)d\mu(x)\right)dt,$$

where χ_A is the characteristic function of a set A and the interchange of the order of integration is justified by the use of Fubini's Theorem.

Now,

$$\int_X \chi_E(x)\chi_{[0,f(x)]}(t)d\mu(x) = \mu(E \cap E_t).$$

Therefore,

$$\int_E \phi(f(x))d\mu(x) = \int_0^{\infty} \phi'(t)\mu(E \cap E_t)dt.$$

Corollary 1: If $p > 0$, then

$$\int_E \left(f(x)\right)^p d\mu(x) = p\int_0^{\infty} t^{p-1}\mu(E \cap E_t)dt,$$

provided the integral on the left hand side exists.

Proof: The result follows with $\phi(t) = t^p$ $(p > 0)$.

Remark 9: If we replace E by X in the above, we have

$$\int_X (f(x))^p \, d\mu(x) = p\int_0^\infty t^{p-1}\,\mu(X \cap E_t)dt$$

$$= p\int_0^\infty t^{p-1}\,\mu(E_t)dt$$

$$= p\int_0^\infty t^{p-1}\,D_f(t)dt.$$

Lemma 1: Let (X,M,μ) be a measure space and f a Lebesgue measurable function defined almost everywhere on X. Then, for all $y > 0$,

$$D_f(y) = D_{f*}(y).$$

Proof: By definition,

$$D_f(y) = \mu(\{x \in X;\ |f(x)| > y\}),$$

where $y > 0$.

Since $f*$ is monotone non-increasing, we have

$$\{x \in (0,+\infty);\ f*(x) > y\} = (0,D_f(y)).$$

The conclusion follows at once as the measure of $(0,D_f(y))$ is $D_f(y)$. That is,

$$D_{f*}(y) = D_f(y).$$

Theorem 2.5: Let (X,M,μ) be a σ-finite measure space and f a Lebesgue measurable function defined almost everywhere on X. If ϕ is a real-valued, non-decreasing, differentiable function defined on $[0,+\infty)$, such that $\phi(0) = 0$ and

$$\int_X \phi(|f(x)|)d\mu(x) < \infty,$$

then

$$\int_X \phi(|f(x)|)d\mu(x) = \int_0^\infty \phi(f^*(t))dt.$$

Proof: By Theorem 2.4 and Lemma 1, we have

$$\begin{aligned}\int_X \phi(|f(x)|)d\mu(x) &= \int_0^\infty \phi'(t)\mu(X \cap E_t)dt \\ &= \int_0^\infty \phi'(t)\mu(E_t)dt \\ &= \int_0^\infty \phi'(t)D_f(t)dt \\ &= \int_0^\infty \phi'(t)D_{f^*}(t)dt \\ &= \int_0^\infty \phi'(t)\mu((0,+\infty) \cap E_u^*)dt \\ &= \int_0^\infty \phi(f^*(t))dt,\end{aligned}$$

where $E_u^* = \{x \in X;\ f^*(t) > u\}$.

Hence, we have the desired result.

<u>Corollary 1</u>: If $p > 0$, then

$$\int_X |f(x)|^p \, d\mu(x) = \int_0^\infty (f^*(t))^p \, dt.$$

The proof follows easily from Theorem 2.5 with $\phi(t) = t^p$, for $p > 0$.

We now need a result of Colin Bennett [1].

<u>Theorem 2.6</u>: If $0 < s \le 1$ and f is any integrable function defined on [0,1], then the following statements are equivalent:

$$\text{(i)} \int_0^1 |f(x)| \left(\log^+ |f(x)|\right)^s dx < \infty$$

$$\text{(ii)} \int_0^1 f^*(t) \left(\log^+ f^*(t)\right)^s dt < \infty$$

$$\text{(iii)} \int_0^1 f^*(t) \left(\log \frac{1}{t}\right)^s dt < \infty,$$

where, as before, $\log^+|x| = \log |x|$ if $x \ge 1$ and $\log^+|x| = 0$ otherwise.

<u>Proof</u>: That (i) and (ii) are equivalent follows from Theorem 2.5. Now, assume that (ii) holds. That is,

$$\int_0^1 f^*(t) \left(\log^+ f^*(t)\right)^s dt < \infty.$$

Setting $$E_1 = \{t \in [0,1];\ f^*(t) \le t^{-1/2}\}$$

and $$E_2 = \{t \in [0,1];\ f^*(t) > t^{-1/2}\},$$

we have

$$\int_0^1 f^*(t)\left(\log t^{-1}\right)^s dt$$

$$= \int_{E_1} f^*(t)\left(\log t^{-1}\right)^s dt + \int_{E_2} f^*(t)\left(\log t^{-1}\right)^s dt$$

$$\equiv I_1 + I_2, \text{ respectively.}$$

Considering I_1, we get

$$\int_{E_1} f^*(t)\left(\log t^{-1}\right)^s dt \le \int_{E_1} t^{-1/2} \left(\log t^{-1}\right)^s dt$$

$$\le \int_0^1 t^{-1/2} \left(\log t^{-1}\right)^s dt$$

$$= \Gamma(s+1)2^{s+1}$$

$$< \infty,$$

where Γ is the Gamma function.

Now consider I_2. We have

$$I_2 = \int_{E_2} f^*(t)\left(\log t^{-1}\right)^s dt.$$

Since $f^*(t) > t^{-1/2}$ on E_2 and $0 < t < 1$, we have $f^*(t) > t^{-1/2} > 1$, so $\left(f^*(t)\right)^2 > t^{-1} > 1$, and hence

$$I_2 \le \int_{E_2} f^*(t)\left(\log \left(f^*(t)\right)^2\right)^s dt$$

$$= 2^s \int_{E_2} f^*(t)\left(\log f^*(t)\right)^s \, dt$$

$$\leq 2^s \int_0^1 f^*(t)\left(\log^+ f^*(t)\right)^s \, dt$$

$< \infty$, by hypothesis.

Therefore, we have

$$\int_0^1 f^*(t)\left(\log t^{-1}\right)^s \, dt$$

$$= \int_{E_1} f^*(t)\left(\log t^{-1}\right)^s \, dt + \int_{E_2} f^*(t)\left(\log t^{-1}\right)^s \, dt$$

$$\leq \Gamma(s + 1) \;{}^{s + 1} + 2^s \int_0^1 f^*(t)\left(\log^+ f^*(t)\right)^s \, dt$$

$< \infty$.

Thus, (ii) implies (iii).

For the converse, we note that if g satisfies

$$\int_0^1 |g(x)| dx \leq 1,$$

then $tg^*(t) \leq 1$. To see this, note that for $0 < t < 1$,

$$tg^*(t) = g^*(t) \cdot \int_0^t du$$

$$\leq \int_0^t g^*(u) du$$

$$\leq \int_0^1 g^*(u)du$$

$$= \int_0^1 |g(x)|dx$$

$$\leq 1.$$

In other words, if $|g|$ is integrable, then $tg^*(t)$ is bounded.

Now consider

$$\int_0^1 f^*(t)\left(\log^+ f^*(t)\right)^s dt$$

$$= \int_0^{1/e} f^*(t)\left(\log^+ f^*(t)\right)^s dt + \int_{1/e}^1 f^*(t)\left(\log^+ f^*(t)\right)^s dt$$

$$\equiv J_1 + J_2, \text{ respectively.}$$

J_2 is clearly bounded by $f^*(1/e)\left(\log^+ f^*(1/e)\right)^s$ because the function f^* is bounded away from the origin.

For J_1, we note that in the interval $(0,1/e)$, $\log t^{-1} > 1$. In particular, $\log t^{-1} > t$. Using the fact that $f^*(t)\left(\log t^{-1}\right)^s$ is integrable, we get, from the above remarks,

$$tf^*(t)t^s \leq tf^*(t)\left(\log t^{-1}\right)^s$$

$$\leq 1,$$

or

$$f^*(t) \leq t^{-1-s}.$$

Thus,

$$\int_0^{1/e} f^*(t)\left(\log^+ f^*(t)\right)^s dt \leq \int_0^{1/e} f^*(t)\left(\log t^{-(1+s)}\right)^s dt$$

$$\leq (1+s)^s \int_0^{1/e} f^*(t)\left(\log t^{-1}\right)^s dt$$

$$\leq (1+s)^s \int_0^{1} f^*(t)\left(\log t^{-1}\right)^s dt$$

$$< \infty.$$

Therefore, (iii) implies (ii) and we are done.

Remark 10: If $1 \leq s < +\infty$ and $f \in L^1(\log^+ L)^s[0,1]$, then

$$\int_0^1 f^*(t)\left(\log t^{-1}\right)^s dt < \infty.$$

This follows by the same methods used in the last part of the proof of Theorem 2.6.

We now prove the following useful inclusion relation:

Theorem 2.7: For $0 < s \leq 1$ and $1 < p < +\infty$, the following inclusions hold:

$$L^p[0,1] \subseteq L^1(\log^+ L)^s[0,1] \subseteq L^1[0,1].$$

Proof: Assume that $f \in L^p[0,1]$ for $1 < p < +\infty$. Then, by Theorems 2.4 and 2.5,

$$\left(\int_0^1 |f(x)|^p\, dx\right)^{1/p} = \left(\int_0^1 \left(f^*(t)\right)^p dt\right)^{1/p}$$

$$= \left(\int_0^1 D_f(y)\, p y^{p-1}\, dy\right)^{1/p}$$

$$< \infty.$$

Clearly,

$$\int_0^\infty D_f(y)y^{p-1}\,dy > \int_a^\infty D_f(y)y^{p-1}\,dy$$

where a > 1.

Thus,

$$\infty > \int_a^\infty D_f(y)y^{p-1}\,dy$$

$$= \int_a^\infty D_f(y)\left(\log\ y\right)^s\ y^{p-1}\ \left(\log\ y\right)^{-s}\,dy.$$

Now consider $g(y) \equiv y^{p-1}\left(\log\ y\right)^{-s}$, where $y \geq a$ and $1 < p < +\infty$. Routine calculations yield that $y_0 = e^{s/(p-1)} > 1$ is a relative minimum for $g(y)$. Choosing $1 < a < y_0$,

$$\int_a^\infty D_f(y)\left(\log\ y\right)^s\ \left(y^{p-1}\ \left(\log\ y\right)^{-s}\right)dy$$

$$= \int_a^\infty D_f(y)\left(\log\ y\right)^s\ g(y)dy$$

$$\geq g(y_0)\int_a^\infty D_f(y)\left(\log\ y\right)^s\,dy.$$

Now,

$$\int_0^1 |f(x)|\left(\log^+|f(x)|\right)^s\,dx$$

$$= \int_{E_1} |f(x)|\left(\log^+|f(x)|\right)^s dx + \int_{E_2} |f(x)|\left(\log^+|f(x)|\right)^s dx,$$

where $E_1 = \{x \in [0,1];\ |f(x)| > a\}$

and $E_2 = \{x \in [0,1];\ |f(x)| \le a\}$,

for some constant $a > 1$. Now,

$$\int_{E_2} |f(x)|\left(\log^+|f(x)|\right)^s dx$$

$$\le \int_{E_2} a\left(\log a\right)^s dx$$

$$\le a\left(\log a\right)^s \int_0^1 dx$$

$$= a\left(\log a\right)^s$$

$$= A, \text{ say.}$$

On the other hand,

$$\int_{E_1} |f(x)|\left(\log^+|f(x)|\right)^s dx$$

$$= \int_a^\infty D_f(y)\left[\left(\log y\right)^s + s\left(\log y\right)^{s-1}\right]dy$$

$$\le \int_a^\infty D_f(y)\left(\log y\right)^s dy + s\left(\log a\right)^{s-1} \int_a^\infty D_f(y)dy.$$

Since $f \in L^p[0,1]$,

$$\int_a^\infty D_f(y)\left(\log\, y\right)^s\, dy < \infty$$

from above, and the second integral is finite since

$$\int_a^\infty D_f(y)dy \le a^{1-p}\int_a^\infty y^{p-1}\, D_f(y)dy$$

$$\le a^{1-p}\int_0^\infty y^{p-1}\, D_f(y)dy$$

$$= \frac{a^{1-p}}{p}\, ||f||_p^p$$

$$< \infty.$$

Hence, we have the desired result.

That is,

$$L^p[0,1] \subseteqq L^1(\log^+L)^s[0,1].$$

Now suppose that $f \in L^1(\log^+L)^s[0,1]$, where $0 < s \le 1$. This is equivalent to saying

$$\int_0^1 f^*(t)\left(\log^+ f^*(t)\right)^s\, dt < \infty.$$

Setting $E_1 = \{t \in [0,1];\ f^*(t) > e\}$

and $E_2 = \{t \in [0,1];\ f^*(t) \le e\}$,

we have

$$\int_0^1 f^*(t)dt = \int_{E_1} f^*(t)dt + \int_{E_2} f^*(t)dt$$

$$\equiv I_1 + I_2, \text{ respectively.}$$

For I_2,

$$\int_{E_2} f^*(t)dt \le \int_{E_2} e\,dt$$

$$\le \int_0^1 e\,dt$$

$$= e.$$

Note that, on E_1, $f^*(t) > e$. Thus $\log^+ f^*(t) = \log f^*(t) \ge 1$ and hence, $\left(\log^+ f^*(t)\right)^s \ge 1$ and $f^*(t)\left(\log^+ f^*(t)\right)^s \ge f^*(t)$.

Therefore,

$$I_1 = \int_{E_1} f^*(t)dt$$

$$\le \int_{E_1} f^*(t)\left(\log^+ f^*(t)\right)^s dt$$

$$\le \int_0^1 f^*(t)\left(\log^+ f^*(t)\right)^s dt$$

$$< \infty, \text{ by assumption.}$$

Collecting the above estimates,

$$\int_0^1 f^*(t)dt \le e + \int_0^1 f^*(t)\left(\log^+ f^*(t)\right)^s dt$$

$$< \infty.$$

That is,

$$L^1(\log^+ L)^s[0,1] \subseteqq L^1[0,1].$$

Lastly, we extend the Hardy Inequalities by considering certain weight functions. Recall that the Hardy Inequalities are given by:

Theorem 2.8: ([6, p. 256 - 257]) If $1 \le q < +\infty$, $0 < r < +\infty$ and f is a non-negative Lebesgue measurable function defined on $(0,+\infty)$, then

(i) $$\left(\int_0^\infty \left(\int_0^t f(y)dy \right)^q t^{-r-1}\, dt \right)^{1/q} \le \frac{q}{r} \left(\int_0^\infty \left(yf(y)\right)^q y^{-r-1}\, dy \right)^{1/q}$$

and

(ii) $$\left(\int_0^\infty \left(\int_t^\infty f(y)dy \right)^q t^{r-1}\, dt \right)^{1/q} \le \frac{q}{r} \left(\int_0^\infty \left(yf(y)\right)^q y^{r-1}\, dy \right)^{1/q} .$$

In order to extend the above, we state, without proof, the following inequality of Jensen, which may be found, for example, in Hewitt and Stromberg [5, p. 202].

Theorem 2.9: (Jensen's Inequality) Suppose ϕ is a convex function defined on $(0,+\infty)$ and f is a non-negative Lebesgue measurable function defined on X with

$$\int_X d\mu(x) = \mu(X) < \infty,\ \mu(X) \ne 0.$$

Then

$$\phi\left(\frac{1}{\mu(X)} \int f(x)d\mu(x) \right) \le \frac{1}{\mu(X)} \int_X \phi\left(f(x)\right)d\mu(x).$$

We now prove the following:

Theorem 2.10: If f is a non-negative Lebesgue measurable function defined on $(0,+\infty)$ and if $1 \le q < +\infty$, $0 < r < +\infty$ and ω is a non-negative non-increasing function defined on $(0,+\infty)$ with the property that for $\alpha = r/q$ there exists a constant A (> 0) such that

$$x^{-\alpha} \int_0^x t^{\alpha - 1} \omega(t)dt \le A\omega(x),$$

then

$$\int_0^\infty \omega(t)t^{r-1} \left(\int_t^\infty f(y)dy \right)^q dt \le A \int_0^\infty (tf(t))^q \, t^{r-1} \omega(t)dt.$$

Proof: We first note that

$$\int_0^\infty \omega(t)t^{r-1} \left(\int_t^\infty f(y)dy \right)^q = \int_0^\infty \omega(t)t^{-1} \left(t^{r/q} \int_t^\infty f(y)dy \right)^q dt$$

$$= \int_0^\infty \omega(t)t^{-1} \left(\frac{q}{r}\right)^q \left(\frac{r}{q} \, t^{r/q} \int_t^\infty f(y)dy\right)^q dt$$

$$= \left(\frac{q}{r}\right)^q \int_0^\infty \omega(t)t^{-1} \left(\frac{r}{q} \, t^{r/q} \int_t^\infty f(y)dy\right)^q dt.$$

If we set $d\mu(y) = y^{-r/q - 1} dy$, then

$$\int_t^\infty d\mu(y) = \mu((t,\infty))$$

$$= \frac{q}{r} t^{-r/q}$$

$$< \infty,$$

since $t \in (0,+\infty)$.

Also, $\mu((t,\infty))$ is non-zero since $t \in (0,+\infty)$. Hence, we can apply Theorem 2.9 to the above to get

$$\int_0^\infty \omega(t)t^{r-1} \left(\int_t^\infty f(y)dy \right)^q dt$$

$$\leq \left(\frac{q}{r}\right)^q \int_0^\infty t^{-1} \omega(t) \left(\frac{r}{q} t^{r/q}\right) \left(\int_t^\infty \left(f(y)y^{r/q+1}\right)^q y^{-r/q-1} dy\right) dt$$

$$= \left(\frac{q}{r}\right)^{q-1} \int_0^\infty t^{-1} \omega(t)t^{r/q} \left(\int_t^\infty \left(f(y)y^{r/q+1}\right)^q y^{-r/q-1} dy\right) dt$$

$$= \left(\frac{q}{r}\right)^{q-1} \int_0^\infty \left(f(y)y^{r/q+1}\right)^q y^{-r/q-1} \left(\int_0^y \omega(t)t^{r/q-1} dt\right) dy$$

$$= \left(\frac{q}{r}\right)^{q-1} \int_0^\infty \left(f(y)y^{r/q+1}\right)^q y^{-1} \left(y^{-r/q} \int_0^y \omega(t)t^{r/q-1} dt\right) dy$$

$$\leq A\left(\frac{q}{r}\right)^{q-1} \int_0^\infty \left(f(y)y^{r/q+1}\right)^q y^{-1} \omega(y)dy$$

$$= A \int_0^\infty \left(yf(y)\right)^q y^{r-1} \omega(y)dy.$$

<u>Theorem 2.11</u>: Let f be a non-negative Lebesgue measurable function defined on $(0,+\infty)$ and ω a non-negative decreasing function

defined on $(0,+\infty)$. If $1 \le q < +\infty$ and $0 < r < +\infty$, then

$$\left(\int_0^\infty \omega(t)t^{-r-1}\left(\int_0^t f(y)dy\right)^q dt\right)^{1/q} \le A\left(\int_0^\infty (yf(y))^q\, y^{-r-1}\,\omega(y)dy\right)^{1/q}.$$

Proof: We use the fact that ω is decreasing to get

$$\left(\int_0^\infty \omega(t)t^{-r-1}\left(\int_0^t f(y)dy\right)^q dt\right)^{1/q}$$

$$= \left(\int_0^\infty t^{-r-1}\left((\omega(t))^{1/q}\int_0^t f(y)dy\right)^q dt\right)^{1/q}$$

$$\le \left(\int_0^\infty t^{-r-1}\left(\int_0^t (\omega(y))^{1/q} f(y)dy\right)^q dt\right)^{1/q}.$$

We can now use Theorem 2.8, part (i), to estimate the last integral. The result is

$$\left(\int_0^\infty \omega(t)t^{-r-1}\left(\int_0^t f(y)dy\right)^q dt\right)^{1/q}$$

$$\le A\left(\int_0^\infty \left(y(\omega(y))^{1/q} f(y)\right)^q y^{-r-1}\,dy\right)^{1/q}$$

$$= \left(\int_0^\infty \omega(y)(yf(y))^q\, y^{-r-1}\,dy\right)^{1/q}.$$

This is the desired inequality.

The last proposition of this section will be an extension of Calderón's Lemma due to R. Johnson [9].

Theorem 2.12: ([9, p. 293]) If f is a non-negative decreasing Lebesgue measurable function defined on $(0,+\infty)$ and $0 < p \le q < +\infty$, then for any real α,

$$\left(\int_0^\infty \left(t^\alpha f(t)\right)^q t^{-1}\, dt\right)^{1/q} \le \left(\int_0^\infty \left(t^\alpha f(t)\right)^p t^{-1}\, dt\right)^{1/p}.$$

Remark 11: For $\alpha = 1/r$, $r > 0$, the above inequality is Calderón's Lemma.

CHAPTER III

In this chapter we will deal with the interpolation of linear operators on the Orlicz spaces $L^1(\log^+ L)^s[0,1]$ and the spaces $K^p(\log^+ K)^s(X,M,\mu)$ which have been defined previously. Richard O'Neil [14] gave the first explicit formulation of an interpolation theorem concerning these spaces. We will provide a different proof of his result by utilizing non-increasing equimeasurable rearrangements rather than distribution functions. This will correspond more closely with the definitions of the Lorentz spaces and the Orlicz spaces $L^1(\log^+ L)^s[0,1]$. After proving O'Neil's results, we will extend them further to weighted spaces with weights satisfying certain growth conditions.

The first result is:

__Theorem 3.1__: ([14]) Suppose $0 < p < r < +\infty$, $0 \le s \le 1$, $1 \le q < +\infty$ and T is a linear operator simultaneously of weak types $(1,p)$ and (q,r). Then, for any $f \in L^1(\log^+ L)^s[0,1]$, we have that $Tf \in L(p,\frac{1}{s})(X,M,\mu)$.

__Proof__: Since we are dealing with spaces which are totally finite, we may assume $\mu(X) = 1$. There is no loss of generality in making this assumption.

For any measurable function f, defined on [0,1], we define

$$f^u(x) = \begin{cases} f(x) & \text{if } |f(x)| > f^*(u) \\ 0 & \text{otherwise,} \end{cases}$$

and $f_u(x) = f(x) - f^u(x)$, where u is a function to be determined later. Then, by the definitions of f* and D_f, we have

$$f^{u*}(y) \le \begin{cases} f^*(y) & 0 < y < u \\ 0 & y \ge u \end{cases}$$

and

$$f_u^*(y) \le \begin{cases} f^*(u) & 0 < y < u \\ f^*(y) & y \ge u. \end{cases}$$

Now we consider $||Tf||_{p,1/s}$ when $0 < s \le 1$. Then, by the linearity of T, the properties of sums of rearrangements of functions, and Minkowski's inequality,

$$||Tf||_{p,1/s}$$

$$= \left(\int_0^1 \left((Tf)^*(t)t^{1/p}\right)^{1/s} t^{-1}\, dt\right)^s$$

$$= \left(\int_0^1 \left((Tf)^*(t)t^{1/p - s}\right)^{1/s}\, dt\right)^s$$

$$= \left(\int_0^1 \left(\left(T\left(f^u + f_u\right)\right)^*(t)t^{1/p - s}\right)^{1/s}\, dt\right)^s$$

$$\le \left(\int_0^1 \left((Tf^u)^*(t/2)t^{1/p - s} + (Tf_u)^*(t/2)t^{1/p - s}\right)^{1/s} dt\right)^s$$

$$\le A\left(\left(\int_0^1 \left((Tf^u)^*(t/2)t^{1/p - s}\right)^{1/s} dt\right)^s \right.$$

$$\left. + \left(\int_0^1 \left((Tf_u)^*(t/2)t^{1/p - s}\right)^{1/s} dt\right)^s \right).$$

$\equiv A(I_1 + I_2)$, respectively.

Recall now that if T is a linear operator of weak type (p,q), then there exists a constant A (> 0), such that,

$$(Tf)^*(t) \le At^{-1/q} ||f||_p.$$

For I_1, since T is of weak type (1,p),

$$I_1^{1/s} = \int_0^1 \left((Tf^u)^*(t/2)t^{1/p - s}\right)^{1/s} dt$$

$$\le A \int_0^1 ||f^u||_1^{1/s}\, t^{-1}\, dt$$

$$= A \int_0^1 \left(\int_X |f^u(y)|d\mu(y)\right)^{1/s} t^{-1}\, dt$$

$$\le A \int_0^1 \left(\int_0^{\mu(X) = 1} f^{u*}(y)dy\right)^{1/s} t^{-1}\, dt$$

$$\le A \int_0^1 \left(\int_0^u f^*(y)dy\right)^{1/s} t^{-1}\, dt.$$

Suppose that $u \equiv u(t) = t^{\lambda}$ for some $\lambda > 0$. Then, by Theorem 2.3,

$$I_1^{1/s} \le A \int_0^1 \left(\int_0^{t^\lambda} f^*(y)dy \right)^{1/s} t^{-1}\, dt$$

$$\le A \left(\int_0^1 f^*(y) \left(\int_{y^{1/\lambda}}^1 t^{-1}\, dt \right)^s dy \right)^{1/s}$$

$$= A \left(\int_0^1 f^*(y) \left(\log 1 - \log y^{1/\lambda} \right)^s dy \right)^{1/s}$$

$$= A \left(\int_0^1 f^*(y) \left(\log 1/y \right)^s dy \right)^{1/s}$$

$$< \infty,$$

by hypothesis and by Theorem 2.6.

Now consider I_2. Since T is of weak type (q,r), we have

$$I_2^{1/s} = \int_0^1 \left((Tf_u)^*(t/2) t^{1/p - s} \right)^{1/s} dt$$

$$\le A \int_0^1 t^{1/sp - 1/sr - 1} ||f_u||_q^{1/s}\, dt$$

$$= A \int_0^1 t^{1/sp - 1/sr - 1} \left(\int_X |f_u(y)|^q\, d\mu(y) \right)^{1/sq} dt$$

$$\le A \int_0^1 t^{1/sp - 1/sr - 1} \left(\int_0^1 (f_u^*(y))^q\, dy \right)^{1/sq} dt$$

$$\le A\left(\int_0^1 t^{1/sp - 1/sr - 1}\left[\int_0^u (f^*(u))^q\,dy + \int_u^1 (f^*(y))^q\,dy\right]^{1/sq} dt\right.$$

$$\le A\left\{\int_0^1 t^{1/sp - 1/sr - 1}\left(\int_0^u (f^*(u))^q\,dy\right)^{1/sq} dt\right.$$

$$\left. + \int_0^1 t^{1/sp - 1/sr - 1}\left(\int_u^1 (f^*(y))^q\,dy\right)^{1/sq} dt\right)$$

$\equiv A(J_1 + J_2)$, respectively.

Again, consider $u \equiv u(t) = t^\lambda$ where $\lambda > 0$. We estimate the above integrals by considering two cases.

<u>Case 1</u> $sq \in (0,1]$.

In this case,

$$J_1 = \int_0^1 t^{1/sp - 1/sr - 1}\left(\int_0^{t^\lambda} (f^*(t^\lambda))^q\,dy\right)^{1/sq} dt$$

$$= \int_0^1 t^{1/sp - 1/sr - 1}\,(f^*(t^\lambda))^{1/s}\,t^{\lambda/sq}\,dt.$$

Substituting $t^\lambda = x$ in the last line yields

$$J_1 = A\int_0^1 x^{(1/sp - 1/sr)/\lambda + 1/sq - 1/\lambda}\,(f^*(x))^{1/s}\,x^{1/\lambda - 1}\,dx$$

$$= A\int_0^1 x^{(1/sp - 1/sr)/\lambda + 1/sq - 1}\,(f^*(x))^{1/s - 1}\,f^*(x)dx.$$

Now $f \in L^1(\log^+ L)^s[0,1] \subseteq L^1[0,1]$ and we have seen previously (see the proof of Theorem 2.6) that for $f \in L^1[0,1]$, $f^*(x) \le Ax^{-1}$.

Substituting this inequality into the last integral we get

$$J_1 \le A \int_0^1 x^{(1/sp - 1/sr)/\lambda + 1/sq - 1/s} \left(Ax^{-1}\right)^{1/s - 1} f^*(x)dx$$

$$= A \int_0^1 x^{(1/sp - 1/sr)/\lambda + 1/sq - 1/s} f^*(x)dx.$$

By choosing $\lambda \in (0,q(r - p)/pr(q - 1)]$, the last integral is dominated by

$$A \int_0^1 f^*(x)dx < \infty.$$

To estimate J_2, we employ Theorem 2.3. Then,

$$J_2 = \int_0^1 t^{1/sp - 1/sr - 1} \left(\int_{t^\lambda}^1 \left(f^*(y)\right)^q dy\right)^{1/sq} dt$$

$$\le \left(\int_0^1 \left(f^*(y)\right)^q \left(\int_0^{y^{1/\lambda}} t^{1/sp - 1/sr - 1} dt\right)^{sq} dy\right)^{1/sq}$$

$$= A \left(\int_0^1 \left(f^*(y)\right)^q \left(t^{1/sp - 1/sr}\Big|_0^{y^{1/\lambda}}\right)^{sq} dy\right)^{1/sq}$$

$$= A \left(\int_0^1 \left(f^*(y)\right)^q y^{q(1/p - 1/r)/\lambda} dy\right)^{1/sq}$$

$$= A \left(\int_0^1 \left(f^*(y)\right)^{q - 1} f^*(y) y^{q(1/p - 1/r)/\lambda} dy\right)^{1/sq}$$

$$\leq A \left(\int_0^1 f^*(y) y^{q(1/p - 1/r)/\lambda + 1 - q}\, dy \right)^{1/sq},$$

where the last inequality follows again from $f^*(x) \leq Ax^{-1}$. Choosing λ as above yields

$$J_2 \leq A \left(\int_0^1 f^*(y)dy \right)^{1/sq}$$

$$< \infty.$$

<u>Case 2</u> $1 < sq < \infty$.

The procedure for estimating J_1 is the same as above and is therefore omitted.

Now consider J_2. We have

$$J_2 = \int_0^1 t^{1/sp - 1/sr - 1} \left(\int_{t^\lambda}^1 (f^*(y))^q\, dy \right)^{1/sq} dt$$

$$\leq A \int_0^1 t^{1/sp - 1/sr - 1} \left(f^*(t^\lambda)\right)^{1/s} \left(\int_{t^\lambda}^1 dy \right)^{1/sq} dt$$

$$\leq A \int_0^1 t^{1/sp - 1/sr - 1} \left(f^*(t^\lambda)\right)^{1/s} \left(\int_0^1 dy \right)^{1/sq} dt$$

$$= A \int_0^1 t^{1/sp - 1/sr - 1} \left(f^*(t^\lambda)\right)^{1/s} dt$$

The substitution $x = t^\lambda$ yields

$$J_2 \leq A \int_0^1 x^{(1/sp - 1/sr)/\lambda - 1} \left(f^*(x)\right)^{1/s - 1} f^*(x)dx$$

$$\le A \int_0^1 x^{(1/sp - 1/sr)/\lambda - 1/s} f^*(x)dx$$

and, choosing $\lambda \in (0,(r - p)/pr]$, it follows that

$$J_2 \le A \int_0^1 f^*(x)dx$$

$$< \infty.$$

Note that if $s = 0$, then $L^1(\log^+ L)^s[0,1] \equiv L^1[0,1]$, so that

$$||Tf||^*_{p,1/s} = ||Tf||^*_{p,\infty}$$

$$= \sup_{0 < t < 1} t^{1/p} (Tf)^*(t),$$

and since T is of weak type (1,p),

$$\sup_{0 < t < 1} t^{1/p} (Tf)^*(t) \le \sup_{0 < t < 1} t^{1/p} \left(At^{-1/p} ||f||_1\right)$$

$$= \sup_{0 < \ < 1} A ||f||_1$$

$$< \infty,$$

which completes the proof.

Theorem 3.2: Suppose $0 < p < r < +\infty$, $1 \le s < +\infty$, $1 \le q < +\infty$ and T is a linear operator simultaneously of weak types (1,p) and (q,r). Then for $f \in L^1(\log^+ L)^s[0,1]$, $Tf \in K^p(\log^+ K)^{p(s - 1)}(X,M,\mu)$.

<u>Proof</u>: If we define f^u and f_u as in Theorem 3.1, then

$$||Tf||_{K^p(\log^+K)^{p(s-1)}}$$

$$= \int_0^1 t^{1/p - 1} (Tf)^*(t)(\log 1/t)^{p(s-1)/p}\, dt$$

$$= \int_0^1 t^{1/p - 1} \left[\left(T(f^u + f_u)\right)^*(t)\right](\log 1/t)^{s-1}\, dt$$

$$\le \int_0^1 t^{1/p - 1} \left[(Tf^u)^*(t/2) + (Tf_u)^*(t/2)\right](\log 1/t)^{s-1}\, dt$$

$$= \int_0^1 t^{1/p - 1} (Tf^u)^*(t/2)(\log 1/t)^{s-1}\, dt$$

$$+ \int_0^1 t^{1/p - 1} (Tf_u)^*(t/2)(\log 1/t)^{s-1}\, dt$$

$\equiv I_1 + I_2$, respectively.

Since T is of weak type (1,p), we have

$$I_1 = \int_0^1 t^{1/p - 1} (Tf^u)^*(t/2)(\log 1/t)^{s-1}\, dt$$

$$\le \int_0^1 t^{-1} A||f^u||_1 (\log 1/t)^{s-1}\, dt$$

$$\le A \int_0^1 t^{-1} (\log 1/t)^{s-1} \left[\int_0^1 f^{u*}(y)dy\right]dt$$

$$\leq A \int_0^1 t^{-1} \left(\log 1/t\right)^{s-1} \left(\int_0^u f^*(y)dy \right) dt.$$

Setting $u \equiv u(t) = t^{\lambda}$, $\lambda > 0$, yields

$$I_1 \quad A \int_0^1 t^{-1} \left(\log 1/t\right)^{s-1} \left(\int_0^{t^{\lambda}} f^*(y)dy \right) dt$$

and, applying Theorem 2.3, we get

$$I_1 \leq A \int_0^1 f^*(y) \left(\int_{y^{1/\lambda}}^1 t^{-1} \left(\log 1/t\right)^{s-1} dt \right) dy$$

$$\leq A \int_0^1 f^*(y) \left(\log 1/y^{1/\lambda}\right)^{s-1} \left(\int_{y^{1/\lambda}}^1 t^{-1} dt \right) dy$$

$$= A \int_0^1 f^*(y) \left(\log 1/y^{1/\lambda}\right)^{s} dy$$

$$= A \int_0^1 f^*(y) \left(\log 1/y\right)^{s} dy$$

$$< \infty,$$

by hypothesis.

To estimate I_2, we use the fact that T is of weak type (q,r), to get

$$I_2 = \int_0^1 t^{1/p - 1} \left(Tf_u\right)^*(t/2) \left(\log 1/t\right)^{s-1} dt$$

$$\leq A \int_0^1 t^{1/p - 1} \left(\log 1/t\right)^{s-1} \left(t^{-1/r} ||f_u||_q\right) dt$$

$$= A \int_0^1 t^{1/p - 1/r - 1} (\log 1/t)^{s - 1} ||f_u||_q \, dt$$

$$\leq A \int_0^1 t^{1/p - 1/r - 1} (\log 1/t)^{s - 1} \left(\int_0^1 (f_u^*(y))^q \, dy \right)^{1/q} dt$$

$$\leq A \int_0^1 t^{1/p - 1/r - 1} (\log 1/t)^{s - 1} \left(\int_0^u (f^*(u))^q \, dy \right.$$

$$\left. + \int_u^1 (f^*(y))^q \, dy \right)^{1/q} dt$$

$$\leq A \left(\int_0^1 t^{1/p - 1/r - 1} (\log 1/t)^{s - 1} \left(\int_0^u (f^*(u))^q \, dy \right)^{1/q} dt \right.$$

$$\left. + \int_0^1 t^{1/p - 1/r - 1} (\log 1/t)^{s - 1} \left(\int_u^1 (f^*(y))^q \, dy \right)^{1/q} dt \right)$$

$\equiv A \left(J_1 + J_2\right)$, respectively.

With u defined as above,

$$J_1 = \int_0^1 t^{1/p - 1/r - 1} (\log 1/t)^{s - 1} \left(\int_0^{t^\lambda} (f^*(t^\lambda))^q \, dy \right)^{1/q} dt$$

$$= \int_0^1 t^{1/p - 1/r - 1} (\log 1/t)^{s - 1} f^*(t^\lambda) t^{\lambda/q} \, dt$$

$$= \int_0^1 t^{1/p - 1/r - 1 + \lambda/q} (\log 1/t)^{s - 1} f^*(t^\lambda) dt,$$

and the substitution $x = t^{\lambda}$ yields

$$J_1 = A \int_0^1 x^{(1/p - 1/r - 1)/\lambda + 1/q + 1/\lambda - 1} \left(\log 1/x^{1/\lambda}\right)^{s - 1} f^*(x)dx$$

$$= A \int_0^1 x^{(1/p - 1/r)/\lambda - 1 + 1/q} \left(\log 1/x\right)^{s - 1} f^*(x)dx$$

$$= A \left(\int_0^{1/e} x^{(1/p - 1/r)/\lambda - 1 + 1/q} \left(\log 1/x\right)^{s - 1} f^*(x)dx \right.$$

$$\left. + \int_{1/e}^1 x^{(1/p - 1/r)/\lambda - 1 + 1/q} \left(\log 1/x\right)^{s - 1} f^*(x)dx \right)$$

$$\equiv A \left(K_1 + K_2\right), \text{ respectively.}$$

Now,

$$K_1 = \int_0^{1/e} x^{(1/p - 1/r)/\lambda - 1 + 1/q} \left(\log 1/x\right)^{s - 1} f^*(x)dx$$

$$\leq \int_0^{1/e} x^{(1/p - 1/r)/\lambda - 1 + 1/q} \left(\log 1/x\right)^{s} f^*(x)dx$$

$$\leq \int_0^{1/e} \left(\log 1/x\right)^{s} f^*(x)dx$$

$$\leq \int_0^1 f^*(x) \left(\log 1/x\right)^{s} dx$$

$$< \infty,$$

provided $\lambda \in (0, q(r - p)/pr(q - 1)]$.

For the second integral, K_2, if λ belongs to the same set as above, we get

$$K_2 = \int_{1/e}^{1} x^{(1/p - 1/r)/\lambda + 1/q - 1} \left(\log 1/x\right)^{s - 1} f^*(x)dx$$

$$\leq \left(\log e\right)^{s - 1} \int_{1/e}^{1} x^{(1/p - 1/r)/\lambda + 1/q - 1} f^*(x)dx$$

$$\leq A \int_{1/e}^{1} f^*(x)dx$$

$$\leq A \int_{0}^{1} f^*(x)dx$$

$$< \infty.$$

Now consider J_2.

$$J_2 = \int_{0}^{1} t^{1/p - 1/r - 1} \left(\log 1/t\right)^{s - 1} \left(\int_{t^\lambda}^{1} \left(f^*(y)\right)^q dy\right)^{1/q} dt$$

$$\leq \int_{0}^{1} t^{1/p - 1/r - 1} \left(\log 1/t\right)^{s - 1} f^*(t^\lambda) \left(\int_{t^\lambda}^{1} dy\right)^{1/q} dt$$

$$\leq \int_{0}^{1} t^{1/p - 1/r - 1} \left(\log 1/t\right)^{s - 1} f^*(t^\lambda) \left(\int_{0}^{1} dy\right)^{1/q} dt$$

$$\leq \int_{0}^{1} t^{1/p - 1/r - 1} \left(\log 1/t\right)^{s - 1} f^*(t^\lambda)dt,$$

and, with $x = t^\lambda$,

$$J_2 \le A \int_0^1 x^{(1/p - 1/r)/\lambda - 1} (\log 1/x)^{s-1} f^*(x)dx$$

$$= A \left(\int_0^{1/e} x^{(1/p - 1/r)/\lambda - 1} (\log 1/x)^{s-1} f^*(x)dx \right.$$

$$\left. + \int_{1/e}^1 x^{(1/p - 1/r)/\lambda - 1} (\log 1/x)^{s-1} f^*(x)dx \right)$$

$$\equiv A (Q_1 + Q_2), \text{ respectively.}$$

Clearly, if $\lambda \in (0,(r - p)/pr]$,

$$Q_1 = \int_0^{1/e} x^{(1/p - 1/r)/\lambda - 1} (\log 1/x)^{s-1} f^*(x)dx$$

$$\le \int_0^{1/e} x^{(1/p - 1/r)/\lambda - 1} (\log 1/x)^{s} f^*(x)dx$$

$$\le A \int_0^{1/e} (\log 1/x)^{s} f^*(x)dx$$

$$\le A \int_0^{1} (\log 1/x)^{s} f^*(x)dx$$

$$< \infty.$$

Lastly, we obtain, for $\lambda \in (0,(r - p)/pr]$, the estimate

$$Q_2 = \int_{1/e}^1 x^{(1/p - 1/r)/\lambda - 1} (\log 1/x)^{s-1} f^*(x)dx$$

$$\leq \{\log e\}^{s-1} \int_{1/e}^{1} x^{(1/p - 1/r)/\lambda - 1} f^*(x)dx$$

$$\leq A \int_{1/e}^{1} f^*(x)dx$$

$$\leq A \int_{0}^{1} f^*(x)dx$$

$$< \infty.$$

Collecting terms, we are done since $(0,(r - p)/pr] \neq \phi$.

Remark 12: As an application of Theorem 3.1, we consider the case when $s = 1$ and we retrieve the following theorem due to A. Zygmund [23]:

Theorem 3.3: If f is a periodic function of period 2π such that $f \in L^1(\log^+ L)^1[0,2\pi]$, then $\bar{f} \in L^1[0,2\pi]$, where $\bar{f}$ is the conjugate function of f.

Remark 13: We recall that for such a function f, the conjugate function $\bar{f}$ is defined by the Cauchy principal value integral

$$\bar{f}(x) = -\frac{1}{\pi} \lim_{\varepsilon \to 0^+} \left(\int_{-\pi}^{-\varepsilon} \frac{f(x - t)}{2 \tan(t/2)}\, dt + \int_{\varepsilon}^{\pi} \frac{f(x - t)}{2 \tan(t/2)}\, dt \right).$$

The operator T which maps f into $\bar{f}$ is interesting in that it is of weak type $(1,1)$ but not of strong type $(1,1)$. T is also of strong type $(2,2)$. See, for example, Edwards [2, Vol. 2, pp. 169 - 177].

Remark 14: We can also discuss the linear operator T defined by

$$Tf(x) = x^{-1} \int_0^x |f(t)|dt.$$

As was shown in Chapter II, T is of weak type (1,1). T is also of strong type (p,p) for $1 < p < +\infty$, hence of weak type (p,p). Setting $p = 1/s$, where $0 < s < 1$, we can apply Theorem 3.1 so that for $f \in L^1(\log^+ L)^s[0,1]$, $Tf \in L(1,1/s)$. That is,

$$\int_0^1 x^{1/s - 1} \left[(Tf)^*(x)\right]^{1/s} dx < \infty.$$

This result was shown by Max Jodeit, Jr. [7] for the case when

$$\int_0^1 |f(x)| \left(\log^+|f(x)|\right)^s dx < \infty.$$

Theorem 3.1 will now be extended to a weighted form. We have the following:

Theorem 3.4: Suppose T is a linear operator simultaneously of weak types (1,p) and (q,r) where $0 < p < r < +\infty$ and $1 < q < +\infty$. Let ω be a non-negative non-increasing function defined on (0,1), with the property that there is a constant A (> 0) such that

$$x^{-\alpha} \int_0^x t^{\alpha - 1} \omega(t)dt \le A\omega(x)$$

for $\alpha = 1/p - 1/r$. Then, if $0 < s < 1$,

$$\left(\int_0^1 \left(\omega(t)\left(Tf\right)^*(t)\right)^{1/s} t^{1/sp - 1} \, dt\right)^s < \infty,$$

whenever

$$\int_0^1 \omega(t^{1/\sigma}) f^*(t) \left(\log 1/t\right)^s \, dt$$

is finite, where σ is the slope of the line segment between $(1,1/p)$ and $(1/q,1/r)$. That is, $\sigma = (1/p - 1/r)/(1 - 1/q)$.

<u>Proof</u>: Defining f^u and f_u as in the two previous theorems, recall that

$$f^{u*}(y) \leq \begin{cases} f^*(y) & 0 < y < u \\ 0 & y \geq u \end{cases}$$

and

$$f_u^*(y) \begin{cases} f^*(u) & 0 < y < u \\ f^*(y) & y \geq u. \end{cases}$$

Setting

$$I = \left(\int_0^1 \left(\omega(y)\left(Tf\right)^*(y)\right)^{1/s} y^{1/sp - 1} \, dy\right)^s,$$

we have

$$I^{1/s} = \int_0^1 \left(\omega(y)\left(Tf\right)^*(y)\right)^{1/s} y^{1/sp - 1} \, dy$$

$$\leq \int_0^1 \left[\omega(y)\left(\left(Tf^u\right)^*(y/2) + \left(Tf_u\right)^*(y/2)\right)\right]^{1/s} y^{1/sp - 1}\,dy$$

$$\leq A \left(\int_0^1 \left(\omega(y)\left(Tf^u\right)^*(y/2)\right)^{1/s} y^{1/sp - 1}\,dy \right.$$

$$\left. + \int_0^1 \left(\omega(y)\left(Tf_u\right)^*(y/2)\right)^{1/s} y^{1/sp - 1}\,dy\right)$$

$$\equiv A\left(I_1 + I_2\right), \text{ respectively.}$$

From the hypothesis that T is of weak type (1,p),

$$I_1 = \int_0^1 \left(\omega(y)\left(Tf^u\right)^*(y/2)\right)^{1/s} y^{1/sp - 1}\,dy$$

$$\leq A \int_0^1 \left(\omega(y)\right)^{1/s} ||f^u||_1^{1/s}\, y^{-1}\,dy$$

$$\leq A \int_0^1 \left(\omega(y)\right)^{1/s} y^{-1} \left(\int_0^1 f^{u*}(t)\,dt\right)^{1/s} dy$$

$$\leq A \int_0^1 \left(\omega(y)\right)^{1/s} y^{-1} \left(\int_0^u f^*(t)\,dt\right)^{1/s} dy.$$

If we set $u \equiv u(y) = y^\sigma$ where $\sigma = (1/p - 1/r)/(1 - 1/q)$, then application of Theorem 2.3 yields

$$I_1 \leq A \left(\int_0^1 f^*(t)\left(\int_{t^{1/\sigma}}^1 \left(\omega(y)\right)^{1/s} y^{-1}\,dy\right)^s dt\right)^{1/s}$$

$$\leq A \left(\int_0^1 f^*(t)\omega(t^{1/\sigma}) \left(\int_{t^{1/\sigma}}^1 y^{-1}\, dy \right)^s dt \right)^{1/s}$$

$$= A \left(\int_0^1 f^*(t)\omega(t^{1/\sigma}) \left(\log 1/t^{1/\sigma}\right)^s dt \right)^{1/s}$$

$$= A \left(\int_0^1 f^*(t)\omega(t^{1/\sigma}) \left(\log 1/t\right)^s dt \right)^{1/s},$$

which is finite by hypothesis.

To estimate I_2, we use the fact that T is of weak type (q,r) to get

$$I_2 = \int_0^1 \left(\omega(y)\left(Tf_u\right)^*(y/2)\right)^{1/s} y^{1/sp - 1}\, dy$$

$$\leq \int_0^1 \left(\omega(y)\right)^{1/s} \|f_u\|_q^{1/s} y^{1/sp - 1/sr - 1}\, dy$$

$$\leq A \int_0^1 \left(\omega(y)\right)^{1/s} \left(\int_0^u \left(f^*(u)\right)^q dt + \int_u^1 \left(f^*(t)\right)^q dt \right)^{1/qs} y^{1/sp - 1/sr - 1}\, dy$$

$$\leq A \left(\int_0^1 \left(\omega(y)\right)^{1/s} y^{1/sp - 1/sr - 1} \left(\int_0^u \left(f^*(u)\right)^q dt \right)^{1/qs} dy \right.$$

$$\left. + \int_0^1 \left(\omega(y)\right)^{1/s} y^{1/sp - 1/sr - 1} \left(\int_u^1 \left(f^*(t)\right)^q dt \right)^{1/qs} dy \right)$$

$\equiv A\left(J_1 + J_2\right)$, respectively.

But,

$$J_1 = \int_0^1 \left(\omega(y)\right)^{1/s} y^{1/sp - 1/sr - 1} \left(\int_0^{y^\sigma} \left(f^*(y^\sigma)\right)^q dt\right)^{1/qs} dy$$

$$= \int_0^1 \left(\omega(y)\right)^{1/s} y^{1/sp - 1/sr - 1} \left(f^*(y^\sigma)\right)^{1/s} y^{\sigma/sq} dy,$$

so that the substitution $x = y^\sigma$ yields

$$J_1 = A \int_0^1 \left(\omega(x^{1/\sigma}) f^*(x)\right)^{1/s} x^{(1/sp - 1/sr)/\sigma + 1/sq - 1} dx$$

$$= A \int_0^1 \left(\omega(x^{1/\sigma}) f^*(x) x\right)^{1/s} x^{-1} dx.$$

By Theorem 2.12,

$$J_1^s = A \left(\int_0^1 \left(\omega(x^{1/\sigma}) f^*(x) x\right)^{1/s} x^{-1} dx\right)^s$$

$$\le A \left(\int_0^1 \left(\omega(x^{1/\sigma}) f^*(x) x\right)^p x^{-1} dx\right)^{1/p}$$

for any $p \in (0,1/s]$. In particular, we can set $p = 1$, so that

$$J_1 \le A \left(\int_0^1 \omega(x^{1/\sigma}) f^*(x) dx\right)^{1/s}.$$

But, since $L^1(\log^+ L)^s[0,1] \subseteq L^1[0,1]$, J_1 is finite.

For J_2, we again make the substitution $x = y^\sigma$, to get

$$J_2 = \int_0^1 \left(\omega(y)\right)^{1/s} y^{1/sp - 1/sr - 1} \left(\int_{y^\sigma}^1 \left(f^*(t)\right)^q dt\right)^{1/sq} dy.$$

<u>Case 1</u> $sq \in (0,1]$. We apply Theorem 2.3 and then Theorem 2.12 to get

$$J_2 \le \left(\int_0^1 (f^*(t))^q \left(\int_0^{t^{1/\sigma}} (\omega(y))^{1/s} y^{1/sp - 1/sr - 1} dy \right)^{sq} dt \right)^{1/sq}$$

$$\le A \left(\int_0^1 (f^*(t))^q \left(\int_0^{t^{1/\sigma}} \omega(y) y^{1/p - 1/r - 1} dy \right)^q dt \right)^{1/sq}.$$

We now apply the growth condition satisfied by the weight function ω. Note that we are using $x = t^{1/\sigma}$ and $\alpha = 1/p - 1/r$. The result is

$$J_2 \le A \left(\int_0^1 (f^*(t))^q t^{\alpha q/\sigma} (A\omega(t^{1/\sigma}))^q dt \right)^{1/sq}$$

$$= A \left(\int_0^1 (f^*(t))^q t^{q-1} (\omega(t^{1/\sigma}))^q dt \right)^{1/sq}$$

$$\le A \left(\int_0^1 (f^*(t))^q (\omega(t^{1/\sigma}))^q dt \right)^{1/sq}$$

$$\le A \left(\int_0^1 f^*(t)\omega(t^{1/\sigma}) dt \right)^{1/s}$$

$$< \infty.$$

<u>Case 2</u> $sq \in (1,\infty)$. In this case we have

$$J_2 = \int_0^1 (\omega(y))^{1/s} y^{1/sp - 1/sr - 1} \left(\int_{y^\sigma}^1 (f^*(t))^q dt \right)^{1/sq} dy.$$

$$= \int_0^1 (\omega(y))^{1/s} y^{1/sp - 1/sr - 1} \left(\int_{y^\sigma}^1 (t^{1/q} f^*(t))^q t^{-1} dt \right)^{1/sq} dy$$

$$= \int_0^1 (\omega(y))^{1/s} y^{1/sp - 1/sr - 1} \left(\int_{y^\sigma}^1 (t^{1/sq} (f^*(t))^{1/s})^{sq} t^{-1} dt \right)^{1/sq} dy.$$

Applying Theorem 2.12 to the inner integral yields

$$J_2 \le A \int_0^1 (\omega(y))^{1/s} y^{1/sp - 1/sr - 1} \left(\int_{y^\sigma}^1 t^{1/sq - 1} (f^*(t))^{1/s} dt \right) dy.$$

We now interchange the order of integration and apply Theorem 2.12 again to get

$$J_2 \le A \int_0^1 t^{1/sq - 1} (f^*(t))^{1/s} \left(\int_0^{t^{1/\sigma}} y^{1/sp - 1/sr - 1} (\omega(y))^{1/s} dy \right) dt$$

$$= A \int_0^1 t^{1/sq - 1} (f^*(t))^{1/s} \left(\int_0^{t^{1/\sigma}} (y^{1/p - 1/r} \omega(y))^{1/s} y^{-1} dy \right) dt$$

$$\le A \int_0^1 t^{1/sq - 1} (f^*(t))^{1/s} \left(\int_0^{t^{1/\sigma}} y^{1/p - 1/r - 1} \omega(y) dy \right)^{1/s} dt$$

$$= A \int_0^1 t^{1/sq - 1} (f^*(t))^{1/s} t^{\alpha/s\sigma} \left(t^{-\alpha/\sigma} \int_0^{t^{1/\sigma}} y^{\alpha - 1} \omega(y) dy \right)^{1/s} dt,$$

where $\alpha = (1/p - 1/r)$. By hypothesis, the last line is dominated by

$$A \int_0^1 t^{1/sq - 1} (f^*(t))^{1/s} t^{(1/p - 1/r)/s\sigma} (A\omega(t^{1/\sigma}))^{1/s} dt$$

$$= A \int_0^1 t^{1/sq - 1} \left(f^*(t)\right)^{1/s} t^{(1 - 1/q)/s} \left(\omega(t^{1/\sigma})\right)^{1/s} dt$$

$$= A \int_0^1 \left(tf^*(t)\omega(t^{1/\sigma})\right)^{1/s} t^{-1} dt.$$

Once again we apply Theorem 2.12 to obtain the estimate

$$J_2 \le A \left(\int_0^1 tf^*(t)\omega(t^{1/\sigma})t^{-1} dt\right)^{1/s}$$

$$= A \left(\int_0^1 f^*(t)\omega(t^{1/\sigma})dt\right)^{1/s}.$$

By hypothesis,

$$\int_0^1 \omega(t^{1/\sigma})f^*(t)\left(\log 1/t\right)^s dt < \infty$$

and so, by Theorem 2.6, J_2 is finite.

Collecting terms, we are done.

<u>Remark 15</u>: We note here that in the case that $\omega(x) \equiv 1$, we retrieve the original theorem; that is, Theorem 3.1.

CHAPTER IV

In this chapter we consider functions of bounded mean oscillation and obtain an interpolation theorem involving the $L^{\lambda}(p,q)$-spaces. First we will state, without proof, interpolation results of H. Heinig [4] involving functions in $L(p,q)$, and then we will prove results for functions which belong to the $L^1(\log^+ L)^s$-spaces.

Theorem 4.1: Suppose T is a quasi-linear operator defined on Lebesgue measurable functions f which are defined almost everywhere on $C_0 = C(0,a)$. If

$$||Tf||_{L^{\mu_i}(r_i,q_i)} \leq A_i \, ||f||_{p_i}$$

where $i = 0,1$; $p_0 < p_1$, $r_0 \neq r_1$ and $0 < q_i < +\infty$, then

$$||Tf||_{L^{\mu}(r,q)} \leq A \, ||f||^{*}_{p,s} \qquad s \leq q$$

where

$$\frac{\mu}{q} = \frac{\mu_0 t}{q_0} + \frac{\mu_1(1 - t)}{q_1}$$

$$\frac{1}{q} = \frac{t}{q_0} + \frac{1 - t}{q_1} \qquad 0 < t < 1$$

$$\frac{1}{p} = \frac{t}{p_0} + \frac{1 - t}{p_1}$$

$$\frac{1}{r} = \frac{t}{r_0} + \frac{1 - t}{r_1} .$$

As a corollary to the above theorem, we have:

Corollary 1: If T is a quasi-linear operator and

$$||Tf||_{L^{(q_i,\mu_i)}} \leq A_i \; ||f||_{p_i}$$

where $i = 0,1$; $p_0 < p_1$, then

$$||Tf||_{L^{(q,\mu)}} \leq A \; ||f||^*_{p,q}$$

where

$$\frac{\mu}{q} = \frac{\mu_0 t}{q_0} + \frac{\mu_1(1 - t)}{q_1}$$

$$\frac{1}{q} = \frac{t}{q_0} + \frac{1 - t}{q_1} \qquad 0 < t < 1$$

$$\frac{1}{p} = \frac{t}{p_0} + \frac{1 - t}{p_1} .$$

If the parameters are restricted to being greater than 1, we have:

Theorem 4.2: If T is a quasi-linear operator and

$$||Tf||_{L^{\mu_i}(r_i,q_i)} \leq A_i \; ||f||^*_{p_i,s_i}$$

where $i = 0,1$; $q_i > s_i > p_i > 1$, $p_0 \neq p_1$, $r_0 \neq r_1$, then

$$||Tf||_{L^{\mu}(r,q)} \leq A \; ||f||^*_{p,q}$$

where

$$\frac{\mu}{q} = \frac{\mu_0 t}{q_0} + \frac{\mu_1(1 - t)}{q_1}$$

$$\frac{1}{q} = \frac{t}{q_0} + \frac{1 - t}{q_1} \qquad 0 < t < 1$$

$$\frac{1}{p} = \frac{t}{p_0} + \frac{1 - t}{p_1}$$

$$\frac{1}{r} = \frac{t}{r_0} + \frac{1 - t}{r_1} \quad .$$

We now prove the following:

Theorem 4.3: Suppose T is a quasi-linear operator defined on Lebesgue measurable functions f which are defined almost everywhere on C_0. If

$$\text{(i)} \qquad (TF_c)^*(t) \le A\, \rho^{(n - \mu_0)/p}\, t^{-1/p}\, ||f||_1$$

and

$$\text{(ii)} \qquad (TF_c)^*(t) \le A\, \rho^{(n - \mu_1)/r}\, t^{-1/r}\, ||f||_q,$$

where $0 < p \le 1$, $0 < p < r < +\infty$, $0 < s \le 1 \le q < +\infty$, $1/p - 1/r > 1/q' = q/(q - 1)$ and $p/r \ge \mu_0/\mu_1$, then for λ satisfying $\lambda > \mu_1/rs + \max\{n(1 - 1/rs), 1 - 1/rs\}$,

$$||Tf||_{L^\lambda(p,1/s)} \le A\, ||f||_{L^1(\log^+ L)^s} + C.$$

Proof: We note here that, without loss of generality, we can consider C_0 to be a cube of side length 1, because $C_0 = C(0,a)$ has finite side length by assumption. We consider $u \equiv u(t)$ to be a non-negative function of one variable and define, for f defined almost everywhere on C_0,

$$f^u(x) = \begin{cases} f(x) & \text{if } |f(x)| > f^*(u) \\ 0 & \text{otherwise} \end{cases}$$

and $f_u(x) = f(x) - f^u(x)$. As before, we apply the definitions of f^* and D_f to obtain

$$f^{u*}(y) \le \begin{cases} f^*(y) & 0 < y < u \\ 0 & y \ge u \end{cases}$$

and

$$f_u^*(y) \le \begin{cases} f^*(u) & 0 < y < u \\ f^*(y) & y \ge u. \end{cases}$$

By setting $F_c^0(x) = f^u(x) - f_c^u$, where f_c^u is the mean value of f^u over $C(x,\rho)$, and $F_c^1(x) = f_u(x) - f_{uc}$, where f_{uc} is the mean value of f_u over $C(x,\rho)$, it follows that $F_c(x) = F_c^0(x) + F_c^1(x)$.

By Minkowski's inequality, we then have

$$||Tf||_{L^\lambda(p,1/s)} \equiv \sup_{\rho \le 1} \left[\rho^{\lambda - n} \int_0^{\rho^n} \left(t^{1/p} \left(TF_c\right)^*(t)\right)^{1/s} t^{-1}\, dt\right]^s$$

$$\le \sup_{\rho \le 1} \left(\left[\rho^{\lambda - n} \int_0^{\rho^n} \left(t^{1/p} \left(TF_c^0\right)^*(t/2)\right)^{1/s} t^{-1}\, dt\right]^s \right.$$

$$+ \left(\rho^{\lambda - n} \int_0^{\rho^n} \left(t^{1/p} \left(TF_c^1\right)^*(t/2)\right)^{1/s} t^{-1}\, dt\right]^s\Big)$$

$$\equiv A \sup_{\rho \le 1} \left(I_1 + I_2\right), \text{ respectively.}$$

To estimate I_1, we note that, by (i),

$$I_1^{1/s} \le \rho^{\lambda - n} \int_0^{\rho^n} \left(t^{1/p} A\, \rho^{(n - \mu_0)/p}\, t^{-1/p}\, ||f^u||_1\right)^{1/s} t^{-1}\, dt$$

$$= A\, \rho^{\lambda - n + (n - \mu_0)/ps} \int_0^{\rho^n} ||f^u||_1^{1/s}\, t^{-1}\, dt$$

and, setting $\rho^n x = t$,

$$I_1^{1/s} \le A\, \rho^{\lambda - n + (n - \mu_0)/ps} \int_0^1 ||f^u||_1^{1/s}\, x^{-1}\, dx$$

$$= A\, \rho^{\lambda - n + (n - \mu_0)/ps} \int_0^1 \left(\int_0^1 f^{u*}(y)dy\right)^{1/s} x^{-1}\, dx$$

$$\le A\, \rho^{\lambda - n + (n - \mu_0)/ps} \int_0^1 \left(\int_0^{u(x)} f^*(y)dy\right)^{1/s} x^{-1}\, dx.$$

Choose $u(x) = x^{\alpha}$, where $\alpha = \rho^{\gamma}$ and $\gamma > 0$. Then by Theorem 2.3,

$$I_1^{1/s} \le A\, \rho^{\lambda - n + (n - \mu_0)/ps} \left(\int_0^1 f^*(y) \left(\int_{y^{1/\alpha}}^1 x^{-1}\, dx\right)^s dy\right)^{1/s}$$

$$= A\, \rho^{\lambda - n + (n - \mu_0)/ps} \left(\int_0^1 f^*(y) \left(\log 1/y^{1/\alpha}\right)^s dy\right)^{1/s}$$

$$= A\,\rho^{\lambda - n + (n - \mu_0)/ps - \gamma} \left(\int_0^1 f^*(y) \left(\log 1/y\right)^s dy \right)^{1/s}.$$

If $\lambda - n + (n - \mu_0)/ps \geq \gamma > 0$, then from the fact that $0 < \rho \leq 1$, we have $\rho^{\lambda - n + (n - \mu_0)/ps - \gamma} \leq 1$ and so

$$I_1^{1/s} \leq A \left(\int_0^1 f^*(y) \left(\log 1/y\right)^s dy \right)^{1/s}$$

which is finite if $f \in L^1(\log^+ L)^s[0,1]$.

Now consider I_2. From (ii) we have

$$I_2^{1/s} = \rho^{\lambda - n} \int_0^{\rho^n} \left(t^{1/p} \left(TF_c^1\right)^*(t/2)\right)^{1/s} t^{-1}\,dt$$

$$\leq A\,\rho^{\lambda - n + (n - \mu_1)/rs} \int_0^{\rho^n} \left(t^{1/p - 1/r} \, ||f_u||_q\right)^{1/s} t^{-1}\,dt$$

$$= A\,\rho^{\lambda - n + (n - \mu_1)/rs + n/ps - n/rs} \int_0^1 x^{1/ps - 1/sr} \, ||f_u||_q^{1/s} \, x^{-1}\,dx$$

by setting $\rho^n x = t$. Therefore,

$I_2^{1/s}$

$$\leq A\,\rho^{\lambda - n + n/sp - \mu_1/rs} \int_0^1 x^{1/ps - 1/rs} \left(\int_0^{x^\alpha} \left(f^*(x^\alpha)\right)^q dy \right.$$

$$\left. + \int_{x^\alpha}^1 \left(f^*(y)\right)^q dy \right)^{1/sq} x^{-1}\,dx$$

$$\le A \left(\rho^{\lambda - n + n/sp - \mu_1/rs} \int_0^1 x^{1/sp - 1/sr - 1} \left(\int_0^{x^\alpha} \left(f^*(x^\alpha)\right)^q dy \right)^{1/sq} dx \right.$$

$$\left. + \rho^{\lambda - n + n/sp - \mu_1/rs} \int_0^1 x^{1/sp - 1/sr - 1} \left(\int_{x^\alpha}^1 \left(f^*(y)\right)^q dy \right)^{1/sq} dx \right)$$

$$\equiv A \left(J_1 + J_2\right).$$

Substituting $x^\alpha = t$ into J_1 yields

$$J_1 = \rho^{\lambda - n + n/ps - \mu_1/rs - \gamma} \int_0^1 t^{1/\alpha sp - 1/\alpha sr + 1/sq - 1} \left(f^*(t)\right)^{1/s} dt.$$

Since $\gamma > 0$ and $0 < \alpha \le 1$, we have $1/\alpha s \ge 1/s$ and

$$J_1 \le \rho^{\lambda - n + n/ps - \mu_1/rs - \gamma} \int_0^1 t^{1/sp - 1/sr + 1/sq - 1} \left(f^*(t)\right)^{1/s} dt.$$

If $\lambda - n + n/ps - \mu_1/rs \ge \gamma > 0$, then $0 < \rho^{\lambda - n + n/ps - \mu_1/rs - \gamma}$ and this number is bounded above by 1, so that by Theorem 2.12,

$$J_1 \le \int_0^1 t^{(1/p - 1/r + 1/q)/s - 1} \left(f^*(t)\right)^{1/s} dt$$

$$= \int_0^1 \left(t^{1/p - 1/r + 1/q} f^*(t)\right)^{1/s} t^{-1} dt$$

$$\le A \left(\int_0^1 t^{1/p - 1/r + 1/q - 1} f^*(t) dt \right)^{1/s}.$$

However, if $1/p - 1/r > 1/q' = 1 - 1/q$, the last integral is dominated

$$A\left(\int_0^1 f^*(t)dt\right)^{1/s},$$

which is finite by Theorem 2.7 and the hypothesis that $f \in L^1(\log^+ L)^s[0,1]$.

Considering J_2, we set $x^\alpha = t$ to obtain

$$J_2 = \rho^{\theta - \gamma}\int_0^1 t^{1/\alpha sp - 1/\alpha sr - 1}\left(\int_t^1 \left(f^*(y)\right)^q dy\right)^{1/sq},$$

where $\theta = \lambda - n + n/ps - \mu_1/rs$. We note that if γ is determined by the restrictions indicated above, then

$$J_2 \le A\int_0^1 t^{(1/p - 1/r)/s - 1}\left(\int_t^1 \left(f^*(y)\right)^q dy\right)^{1/sq} dt$$

$$\le A\int_0^1 t^{(1/p - 1/r)/s - 1}\left(t\left(f^*(t)\right)^q\right)^{1/sq} dt$$

$$= A\int_0^1 \left(t^{1/p - 1/r + 1/q} f^*(t)\right)^{1/s} t^{-1} dt$$

$$\le A\left(\int_0^1 t^{1/p - 1/r + 1/q - 1} f^*(t)dt\right)^{1/s}$$

$$\le A\left(\int_0^1 f^*(t)dt\right)^{1/s}$$

$$< \infty,$$

as above.

Hence, collecting terms, we have the desired result.

Lastly, we have two corollaries of the above theorem.

Corollary 1: If $p = s = 1$, then

$$||Tf||_{L^{(\lambda,1)}} = ||Tf||_{L^{\lambda}(1,1)} \le A\, ||f||_{L^1(\log^+ L)^1} + C.$$

Corollary 2: If $1 \le p = 1/s < r < +\infty$, then

$$||Tf||_{L^{(\lambda,p)}} = ||Tf||_{L^{\lambda}(p,p)} \le A\, ||f||_{L^1(\log^+ L)^s} + C.$$

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