ORDERED DESARGUESIAN AFFINE HJELMSLEV PLANES

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AFFINE HJELMSLEV PLANES

By

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ABSTRACT

The first chapter provides basic prerequisites.   In the
second chapter we demonstrate that an ordered Desarguesian
A. H. plane is coordinatized by an ordered A. H. ring.   We then
show that given an ordered A. H. ring, one can construct an
ordered Desarguesian A. H. plane.   In the remaining chapters
we give an example of such a structure and examine its
relationship to the associated ordinary affine plane, the radical
in the A. H. ring and the Archimedean axiom.

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I wish to extend my gratitude to my advisor Dr. N. D. Lane for his insight and encouragement during the preparation of this thesis.

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INTRODUCTION

An affine Hjelmslev plane (or A. H. plane) may be
described as a geometry where more than one line may pass through
two distinct points.   This is usually defined by a neighbouring
relation and eight axioms.   A Desarguesian A. H. plane may be
coordinatized by an A. H. ring.

Hjelmslev himself studied ordered A. H planes using
reflections and motions.   In this thesis we will give a more rigourous
account for Desarguesian A. H. planes using modern methods.
In his paper [11] "On Ordered Geometries", P. Scherk discussed
the equivalence of an ordering of a Desarguesian affine plane to
an ordering of its division ring.   We will define an ordered
Desarguesian A. H. plane and follow Scherk's methods.

We begin by showing that the coordinate ring of such a
plane is ordered.   Then we construct an ordered A. H. plane
from a given ordered A. H. ring.

The next section provides an example of such an ordered
A. H. ring, and hence justifies our definition.   We then briefly
examine the ordinary affine plane associated with an ordered
A. H. plane.

Next we examine the radical of an ordered A. H. ring and
its relationship to the associated ordered A. H. plane.   Finally,
we prove a suggestion of Hjelmslev's that any Archimedean ordered

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A. H. ring is a division ring, and thus any Archimedean ordered

A. H. plane is an ordinary affine plane.

CHAPTER 1

Elementary Properties of Affine Hjelmslev Planes

1.1. $\mathcal{H} = [\ \mathbb{P},\ \mathbb{L},\ I, \|\ ]$ is called an incidence structure with parallelism if:

    1. $\mathbb{P}$ and $\mathbb{L}$ are sets.

    2. $I \subseteq \mathbb{P} \times \mathbb{L}$.

    3. $\| \subseteq L \times L$ is an equivalence relation (parallelism).

The elements of $\mathbb{P}$ are called points and are denoted by P, Q, R...

The elements of $\mathbb{L}$ are called lines and are denoted by l, m, n...

(P, l) $\in$ I is written P I l and is read, "P is incident with l";

similarly, (l, m) $\in \|$ is written $l \| m$ and is read, "l is parallel

to m". P, Q I l will mean P I l and Q I l, and P I l, m will mean

P I l and P I m. We write $l \wedge m = \{P \in \mathbb{P} \mid P\ I\ l, m\}$ and

$l \vee m = \{P \in \mathbb{P} \mid P\ I\ l\ \text{or}\ P\ I\ m\}$.

1.2. Two points, P and Q, are neighbours (written $P \sim_{\mathbb{P}} Q$,

or just $P \sim Q$) if there exist l, m $\in \mathbb{L}$, $l \neq m$, such that P, Q I l, m.

Two lines, l and m, are neighbours (written $l \sim_{\mathbb{L}} m$, or just $l \sim m$)

if for any P I l, there exists a Q I m such that $P \sim Q$ and for any

Q I m, there exists a P I l such that $Q \sim P$. The non-neighbouring

relationship will be denoted by $\not\sim$.

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<u>1.3.</u>   An incidence structure with parallelism is called an

<u>affine Hjelmslev plane</u> (or an A. H. plane) if it satisfies the

following axioms.

A1. For any P, Q $\in$ $\mathbb{P}$, there exists $l \in$ $\mathbb{L}$ such that

P, Q I l.   If P $\not\sim$ Q, we write l = PQ.

A2. There exist P_1, P_2, P_3 $\in$ $\mathbb{P}$ such that $P_i P_j \not\sim P_i P_k$

where (i, j, k) is any permutation of (1, 2, 3).

$\left\{ P_1, P_2, P_3 \right\}$ is called a <u>triangle</u>.

A3. $\sim$ is transitive on $\mathbb{P}$.

A4. If P I l, m, then $l \not\sim$ m iff $|l \wedge m|$ = 1.

A5. If $l \not\sim$ m; P, R I l; Q, R I m and P $\sim$ Q, then R $\sim$ P, Q.

A6. If $l \sim$ m and n $\not\sim l$ with P I l, n and Q I m, n, then

P $\sim$ Q.

A7. If $l \parallel$ m; P I l, n and $l \not\sim$ n, then m $\not\sim$ n and there

exists a point Q such that Q I m, n.

A8. For every P $\in$ $\mathbb{P}$ and every $l \in$ $\mathbb{L}$, there exists a

unique line L(P, l) such that P I L(P, l) and

L(P, l) $\parallel$ l.

From A3 and the definition of the neighbour relation on L,

it is obvious that $\sim$ is transitive on L also.

<u>1.4.</u>   The set π_1 = $\left\{ m \in \mathbb{L} \mid m \parallel l \right\}$ is a <u>pencil</u> of L.

Two pencils, π_1 and π_2, are <u>neighbours</u> (written $\pi_1 \sim \pi_2$) if there

exists $l_1 \in \pi_1$ and $l_2 \in \pi_2$, such that $l_1 \sim l_2$.   In view of A8 it

is clear that if $l \parallel$ m then either $l \wedge$ m = $\emptyset$ or l = m.   (However, the

possibility that two non-parallel lines are disjoint is not excluded.)

1.5. Lemma. If $P \, I \, l$, then there exists $Q \, I \, l$ such that $P \not\sim Q$.

Proof. [7], Lemma 1. 1. 9.

1.6. Lemma. If $g \wedge h = \emptyset$ or $g \sim h$, then there exists j such that $j \parallel h$; $\; j \sim g$ and $j \wedge g \neq \emptyset$.

Proof. [9], Lemma 2. 1.

1.7. Lemma. Let $\{P_1, P_2, P_3\}$ be a triangle, then for any $l \in \mathbb{L}$ there exists $P \in \{P_1, P_2, P_3\}$ such that $P \not\sim X$ for each $X \, I \, l$.

Proof. [9], Lemma 2. 4.

1.8. Lemma. If $g \wedge h \neq \emptyset$, then there exists a point S such that $S \not\sim X$ for all $X \, I \, g \vee h$.

Proof. [9], Lemma 2. 9.

1.9. Lemma. Let $g_1 \parallel g_2$, then the following are equivalent:

    1. $g_1 \sim g_2$.

    2. There exist $P_i \, I \, g_i$, $i = 1, 2$, such that $P_1 \sim P_2$.

Proof. [7], Lemma 1. 1. 10.

1.10. Lemma. If $P_i = g_i \wedge j$, $i = 1$, 2, and $g_1 \| g_2$, then the following are equivalent:

1. $g_1 \sim g_2$.

2. $P_1 \sim P_2$.

Proof.　　[7], Lemma 1. 1. 11.

1.11.　A mapping $\sigma : \mathbb{P} \longrightarrow P$ is called a dilatation of $\mathcal{H}$ if P, Q I l implies σP I $L(Q, l)$, for each $l \in \mathbb{L}$.　　Let D be the set of all dilatations of $\mathcal{H}$ and let D_p be the set of all dilatations of $\mathcal{H}$ with a fixed point P.　　For $P \in \mathbb{P}$, O_p will denote the dilatation which maps each point of　P into P, and 1 will denote the identity map.

1.12.　A line l is called a trace of the dilatation σ if there exists P I l such that σP I l.　　If l is a trace of σ, then σQ I l for all Q I l.

1.13.　A dilatation $\hat{\tau}$ is called a quasitranslation if $\hat{\tau}$ has no fixed point, or $\hat{\tau} = 1$.

A quasitranslation τ is called a translation if for any trace, g, of τ , and any line h, h $\|$ g, then h is also a trace of τ .

A pencil π is called a direction of the translation τ if π is a pencil of traces of τ .

Let T be the set of all translations.

Let $D_\tau = \left\{ \pi \mid \pi \text{ is a direction of } \tau \right\}$. Let $T_\pi = \left\{ \tau \in T \mid \pi \in D_\tau \right\}$.

The set $ND = \left\{ \sigma \in D \mid \sigma P \sim P \text{ for each } P \right\}$ is called the set of

neighbouring dilatations. $N = ND \cap T$ is called the set of

neighbouring translations. It can be shown that if $\tau \in T \setminus N$,

then $P \not\sim \tau P$ for any $P \in \mathbb{P}$, cf. [10], Theorem 3.6.

1.14. Lemma. Let $\tau \in T$, then the following are equivalent:

    1. $\tau \notin N$.

    2. If g and h are traces of τ, then $h \parallel g$.

    3. $\mid D_\tau \mid = 1$.

Proof. [9], Lemma 3. 11.

1.15. An A. H. plane $\mathcal{H}$ is called a translation plane

(or a T.plane) if it satisfies the following axiom.

A9. T is a transitive group.

If T satisfies A9, then τ_{PQ} will denote the unique translation

mapping P into Q. From [10], Theorem 3.7, T is abelian.

1.16. Let $\mathcal{H}$ be a T-plane. A trace preserving endomorphism

is a map, a: $T \to T$, ($\tau \longrightarrow \tau^a$) such that:

    1. a is a group endomorphism of T.

    2. $D_\tau \subseteq D_{\tau^a}$.

H will denote the set of all these endomorphisms. Then H is a ring

with identity with respect to the operations:

$$\tau^{ab} = (\tau^b)^a, \quad \text{and} \quad \tau^{a+b} = \tau^a \tau^b.$$

The identity element, 1, and the zero, o of H are defined by $\tau^1 = \tau$ and $\tau^0 = 1 \in T$, respectively, for all $\tau \in T$.

H* will denote the multiplicative monoid of H, and η will denote the set of non-units of H.

1.17. <u>Theorem</u>. Let $\mathcal{H}$ be a translation plane, and let P be any point.

1. For each $a \in H$, there exists a unique dilatation $\sigma = \sigma(a) \in D_P$ such that $(\tau_{PS})^a = \tau_{P \sigma S}$ for each $\tau_{PS} \in T$.

2. The mapping $h_P : H^* \longrightarrow D_P$ defined by $h_P(a) = \sigma(a)$ is a monoid isomorphism.

In fact $a \in \eta$ iff $\sigma(a) \in M_P$, where $M_P = \left\{ \sigma \in D_P \mid \sigma Q \sim P, \text{ for all } Q \right\}$.

<u>Proof.</u>   [9], Theorem 4.4.

1.18. <u>Corollaries</u>. [9], 4. 4. 1. - 4. 4. 6.

1. Each $a \in H$ is uniquely determined by its action on one $\tau \in T \setminus N$.

2. If $\tau \in T \setminus N$ and $a \in H$, then $\tau^a = 1$ implies $a = o$, and $\tau^a = \tau^b$ implies $a = b$.

3. $N^a \subseteq N$ for each $a \in H$.

4. Let $a \in H$, then the following are equivalent:

   a)   There exists $\tau \in T \setminus N$ such that $\tau^a \in N$.

   b) $T^a \subseteq N$.

   c) $a \in \eta$.

5. H is a local ring with unique maximal ideal η.

1.19. $\mathcal{H}$ is called <u>Desarguesian</u> if $\mathcal{H}$ is a translation plane and satisfies the following axiom.

A10. If $\tau_1 \in T \setminus N$ and $\tau_2 \in T$ and $D_{\tau_1} \subseteq D_{\tau_2}$, then

there exists a $\in$ H such that $\tau_1^a = \tau_2$.

We note that if O, A, B I l and $O \not\wedge \not X$, then $B = \tau_{OA}^b (O)$ for some $b \in H$.

1.20. Axiom A10 is connected with the following one, which applies to any point P.

A10.P. For each collinear triple P, Q, R, with $P \not\wedge Q$,

there exists a dilatation $\sigma \in D_P$ such that

$\sigma Q = R$.

1.21. Theorem. Let $\mathcal{H}$ be a translation plane, then the following are equivalent.

1. A10.

2. A10.P for every point P.

3. There exists P such that A10.P. holds.

<u>Proof.</u> [9], Theorem 5.3.

Lorimer and Lane have also shown, [9], Theorem 5.11, that A10.P is equivalent to a Desarguesian configurational condition for P.

1.22. Coordinates can be introduced in a Desarguesian affine Hjelmslev plane, in the following way.

__Theorem.__ Let $\pi_1 \not\sim \pi_2$. Let $\tau_i \in T_{\pi_i} \setminus N$; $i = 1, 2$. Then for

each $\tau \in T$ there exists a unique a, b $\in$ H such that $\tau = \tau_1{}^a \tau_2{}^b$.

__Proof.__ [9], 6.1.

__1.23. Theorem.__ Let $\mathcal{H}$ be a Desarguesian A. H. plane with the ring H, then H is a local ring with the properties:

1. η is the set of two-sided zero divisors of H.

2. If a, b $\in$ H, then a $\in$ bH or b $\in$ aH.

__Proof.__ [9], Theorem 6.9.

__1.24.__ Such a local ring is called an __affine Hjelmslev ring__ (or an __A. H. ring__). One can always construct an analytic model of a Desarguesian A. H. plane $\mathcal{H}$(H) over such a ring, H, ([8], Section 3). The A. H. plane so constructed will have as local ring of trace preserving endomorphisms an A. H. ring H' isomorphic to H; [8], 3.10.

__1.25 Lemma.__ If a $\in$ H, a $\neq$ o and O, A, B $\in$ $\mathbb{P}$ are not collinear, then, given A, B I l, $\tau_{OB}{}^a$(O) I L($\tau_{OA}{}^a$(O), l); see Figure 1.

__Proof.__ $\tau_{AB} = \tau_{OB} \tau_{AO} = \tau_{OB} \tau_{OA}{}^{-1}$.

Thus $\tau_{AB}{}^a = \tau_{OB}{}^a \tau_{OA}{}^{-a}$. Let $\tau_{OA}{}^a$(O) = C, $\tau_{OB}{}^a$(O) = D.

Then, $\tau_{AB}{}^a$(C) = $\tau_{OB}{}^a$($\tau_{OA}{}^{-a}$(C)) = $\tau_{OB}{}^a$(O) = D.

ℓ

A

$\tau_{OA}{}^a(0)$

O

B

$\tau_{OB}{}^a(0)$

Figure 1.

t

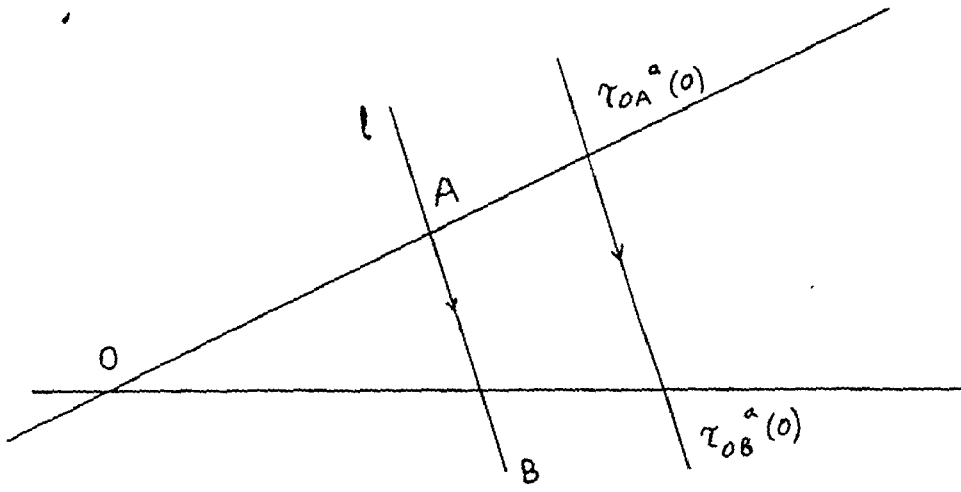$C = \tau_{OA}{}^c(0)$
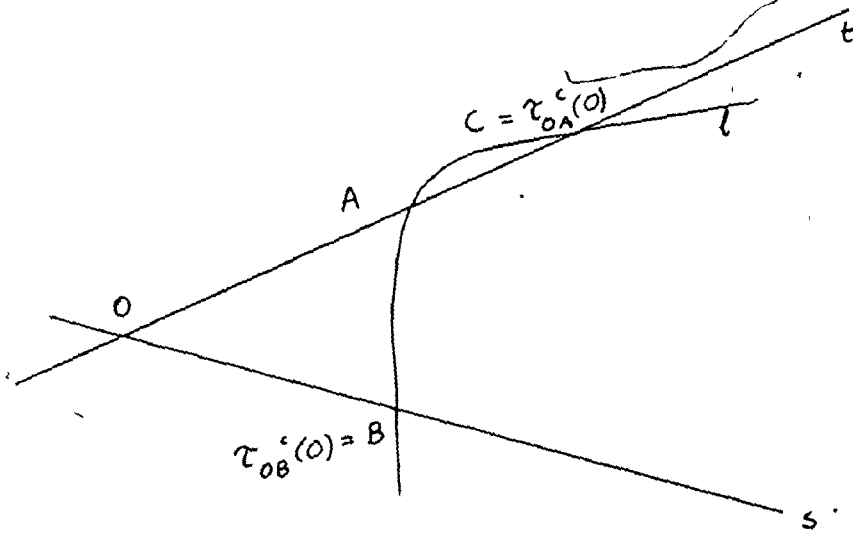
ℓ

A

O

$\tau_{OB}{}^c(0) = B$

s

Figure 2.

Since l is a τ_{AB} trace, it is a $\tau_{AB}{}^a$ trace.   Let m = L(C, l).

Then m is a $\tau_{AB}{}^a$ trace.   Thus $\tau_{AB}{}^a(C)$ I m, but $\tau_{AB}{}^a(C)$ = D = $\tau_{OB}{}^a(O)$.

1.26. Note. Since we have put no restrictions on O, A and B, it is possible that L($\tau_{OA}{}^a(O)$, l) = l even if a ≠ 1; cf. for example Figure 2. In a Desarguesian A. H. plane, suppose O, B I s and A ≁ X for any X I s. Then AB = l and l ≁ s.   Let OA = t and suppose l ∼ t.   Thus, by A4, $|1 \wedge t| \neq 1$.   Take another point, C, in l∧t; C ≠ A.   Then, since O ≁ A, $\tau_{OA} \notin$ N, and so C = $\tau_{OA}{}^c(O)$ for some c ∈.H, where c ≠ 1.

By 1.25, $\tau_{OB}{}^c(O)$ I L(C, l). However, C I l, thus $\tau_{OB}{}^c(O)$ I l and, since l ≁ s, $\tau_{OB}{}^c(O)$ = l ∧ s = B.   However, in our dicussion we shall always consider the situation where l ≁ s, t.

CHAPTER 2

The Construction of an Ordered A. H. Ring from a
Geometrically Ordered Desarguesian A. H. plane

2.1. We shall assume that $\mathcal{H}$ is a given Desarguesian affine
Hjelmslev plane.   Thus, in addition to Al to A8, $\mathcal{H}$ is a translation
plane, and satisfies A10.

Following P. Scherk, [11], we shall introduce seven axioms
of order which will enable us to say that a point on a line lies
"between" two other points on the same line.

We shall show that this geometric ordering of $\mathcal{H}$ induces an
algebraic ordering of the A. H. ring H of the trace preserving
endomorphisms of the group T of translations of $\mathcal{H}$.

2.2. An A. H. ring H will be called <u>ordered</u> if there exists
a subset, H^+, of H, such that:

    1. Every $a \in H$ satisfies exactly one, of $a \in H^+$, $-a \in H^+$, $a = 0$.

    2. If $a, b \in H^+$, then $a + b \in H^+$.

    3. If $a, b \in H^+$, and $b \notin \eta$, then $ab \in H^+$.

(Note: Chapter 4 contains an example of an A. H. ring in which
condition 3 holds but $a, b \in H^+$ and $a \notin \eta$ does not imply $ab \in H^+$.)

2.3. An ordering of the affine Hjelmslev plane, $\mathcal{H}$, is a
ternary relation on $\mathbb{P}$, which satisfies the following axioms.

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O1.   (A, B, C) implies that A, B, C are mutually

distinct and collinear.

O2.   (A, B, C) implies (C, B, A).

O3.   If A, B and C are mutually disinct and collinear,

then exactly one, of (A, B, C), (B, C, A) and (C, A, B)

holds.

O4.   If A, B, C, D are collinear, then (A, B, C) and

(B, C, D) imply (A, B, D) and (A, C, D).

O5.   If A, B, C, D, are collinear, then (A, B, C) and

(A, C, D) imply (A, B, D) and (B, C, D).

O6.   Two, of (B, A , C), (C, A, D) and (D, A, B) exclude

the third, if A, B, C, D are collinear.

O7.   Parallel projections preserve order.

2.4.   In the context of affine Hjelmslev planes we shall

assume that a parallel projection, from a line l to a line m, in the

direction π is a bijective mapping from the distinct points of l

onto the distinct points of m;   i.e. X = l $\wedge$ L(X, π) is taken into

X' = m $\wedge$ L(X, π), for any X I l.   Thus, by A7, L(X, π) $\neq$ l, m.

2.5.   These axioms are not independent, since O5 can be

deduced from O1, O2, O3, O4 and O6.   However, they are in a

convenient form for application.

Theorem.   O1, O2, O3, O4 and O6 imply O5.

<u>Proof.</u> Assume A, B, C and D are collinear, and (A, B, C) and

(A, C, D). If (B, C, D) is false, then, by O3, either (C, D, B)

or (D, B, C).

<u>Case 1:</u> Suppose (C, B, D). (using O2).

Then, by O6, (A, B, C) and (C, B, D) exclude (A, B, D). Hence,

by O3, either (B, D, A) or (D, A, B).

However, by O4, (D, A, B) and (A, B, C) imply (D, A, C), which

contradicts (A, C, D), by O3. On the other hand, (A, D, B) and

(D, B, C) imply (A, D, C), which contradicts (A, C, D).

Thus (C, B, D) is false.

<u>Case 2:</u> Suppose (C, D, B).

Then (A, C, D) and (C, D, B) imply (A, C, B), by O4, which contradicts

(A, B, C), by O3. Thus (C, D, B) is false.

Hence (B, C, D).

Finally, (A, B, C) and (B, C, D) imply (A, B, D), by O4.

    <u>2.6.</u> B and C are said to <u>lie on the same side of A</u> if

exactly one, of (A, B, C), (A, C, D) and B = C $\neq$ A, holds. This will

be denoted B, C|A.

<u>2.7. Theorem.</u> If 1 is any line through A, then the property of

lying on the same side of A, on 1, is an equivalence relation on

($\mathbb{P} \smallsetminus \{A\}$) $\wedge$ 1.

<u>Proof.</u> 1. If B $\neq$ A, then B, B|A.

2. If $B,C|A$, then $C,B|A$, by definition.

3. __Claim__: If $B,C|A$ and $C,D|A$, then $B,D|A$.

If $B = C$ or $C = D$, then $B,D|A$. If $B = D$, then $B,D|A$.

Hence we may assume that B,C and D are mutually distinct.

Since they are all different from A, there remain four possible

cases.

__Case 1__: (A, D, C) and (B, C, A) imply (A, D, B), by O5.

__Case 2__: (A, B, C) and (A, C, D) imply (A, B, D).

__Case 3__: (A, D, C) and (A, B, C). Then, by O3, one, of

$B = D \neq A$, (A, B, D), (B, D, A) and (B, A, D) holds. But (B, A, D) and

(A, D, C) imply (B, A, C), by O4, which contradicts (A, B, C), by O3.

Hence $B, D|A$.

__Case 4__: (A, C, D) and (B, C, A). Then, by O3, one, of

$B = D \neq A$, (A, B, D), (B, D, A) and (D, A, B) holds. But (A, C, D) and

(D, A, B) imply (C, A, B), by O5, which contradicts (A, C, B).

Hence $B,D|A$.

Thus in every case $B,D|A$. The claim is proved.

__2.8. Lemma.__ The property of lying on the same side of A is

preserved under a parallel projection, as described in __2.4.__

__Proof.__ Let X' be the image of X under a parallel projection from

a line l to a line m. Let A, B, C I l and $B,C|A$. Thus one, of

$B = C \neq A$, (B, C, A) and (C, B, A) holds. Hence, one, of $B' = C' \neq A'$,

(B', C', A') and (C', A', B') holds, by O7 and A7. Thus $B', C'|A'$.

2.9. Lemma. If A, B, C, D are collinear, then (B, A, C) and (B, A, D) imply C,D|A.

Proof. By O6, (B, A, C) and (B, A, D) exclude (C, A, D). Hence C = D ≠ A or (C, D, A) or (D, C, A). Thus C,D|A.

2.10. Theorem. Translations preserve order, see Figure 3.

Proof. Let A, B, C I 1 and (A, B, C) and $\tau \in$ T.

Then τB, τC I L(τA, 1) = m.

Case 1: 1 is not a τ trace.

a) 1 ≁ m.

Then A ≁ X for X I m, by <u>1.9</u>, so L(A, τA) ≁ m, and τA ≁ Y for any Y I 1, so L(A, τA) ≁ 1. Thus for any Y I 1, L(Y, L(A, τA)) ≁ m, 1, by A7. Since τ is a translation, τY I L(Y, L(A, τA)) ∧ m. So τ can be considered to be generated by a parallel projection having a pencil of lines parallel to L(A, τA). Thus O7 applies and so (A, B, C) implies (τA, τB, τC).
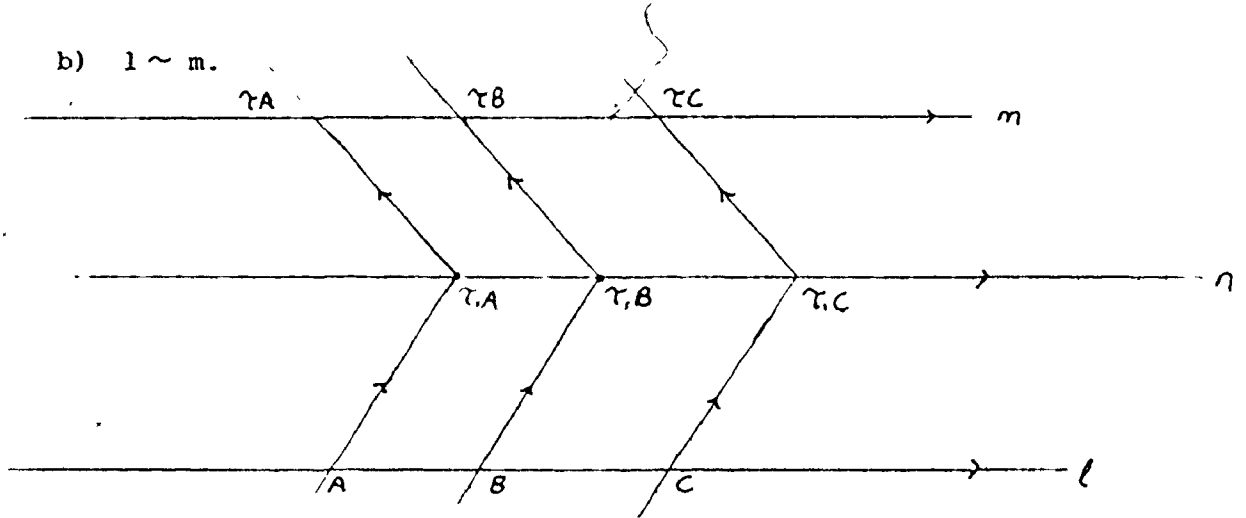
b) 1 ∼ m.

Then there exists a point Z ɤ 1 with Z ≁ Y for any Y I 1, by <u>1.7</u>. Take n = L(Z, 1). Then n ≁ 1 and so n ≁ m. Take some X I n such that neither t = AX, nor s = τAX, are τ traces. Then $\tau_1 = \tau_{AX}$ and $\tau_2 = \tau_{X \tau A}$ are as in Case 1 a) and $\tau = \tau_2 \tau_1$. Thus (A, B, C) implies (τ_1A, τ_1B, τ_1C), which

Case 1: l is not a τ trace.

  a) $l \not\sim m$.
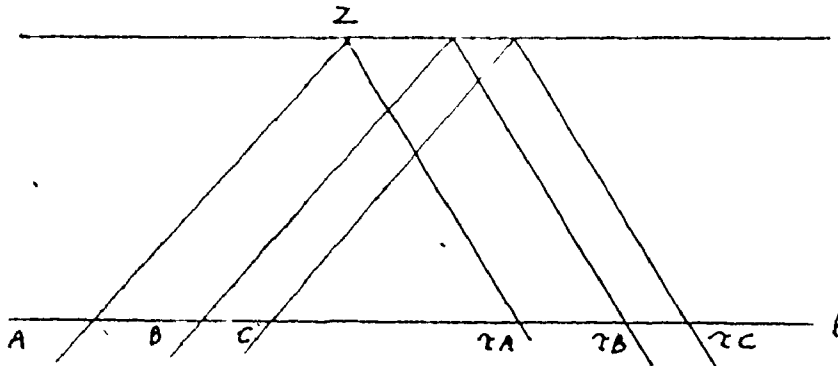

  b) $l \sim m$.
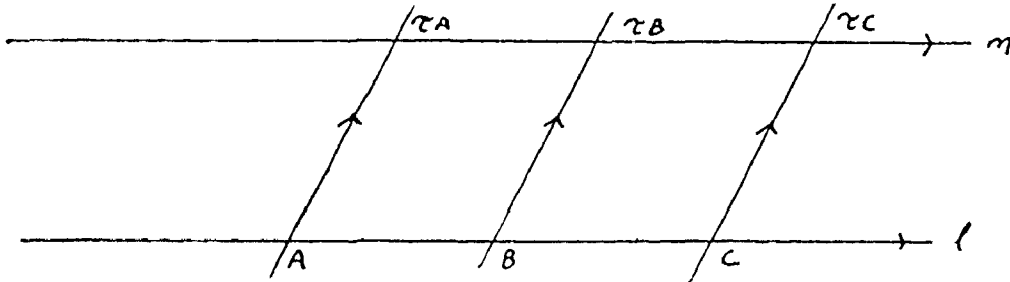
Case 2: l is a τ trace.

Figure 3.

implies $(\tau_2 \tau_1 A, \tau_2 \tau_1 B, \tau_2 \tau_1 C) = (\tau A, \tau B, \tau C)$.

Case 2: 1 is a τ trace.

Choose Z a point which is not a neighbour of any point on 1, then $Z \not{I} 1$. Take $\tau_1 = \tau_{AZ}$ and define τ_2 by $\tau_2(\tau_1 A) = \tau A$. Clearly AZ and $Z \tau A$ are not τ traces. Then by Case 1, (A, B, C) implies $(\tau_1 A, \tau_1 B, \tau_1 C)$ which implies $(\tau_2 \tau_1 A, \tau_2 \tau_1 B, \tau_2 \tau_1 C) = (\tau A, \tau B, \tau C)$.

2.11. We define $a \in H^+(O, A)$ if $\tau_{OA}{}^a(O)$, $A | O$, where $O \not{\mathbin{\sim}} A$. We say that a is __positive__ with respect to O and A.

2.12. Theorem. For any choice of O and A, $O \not{\mathbin{\sim}} A$, $1 \in H^+(O, A)$.

Proof. A, $\tau_{OA}{}^1(O) = A$, lie on the same side of O.

2.13. Lemma. $H^+(O, A)$ does not depend on the choice of A; $A \not{\mathbin{\sim}} O$, see Figure 4.

Proof. Let B be any point such that $B \not{\mathbin{\sim}} A$ and $B \not{\mathbin{\sim}} O$.
Case 1: $B \not{I} OA = 1$.

Let $BO = m$. Since $1 \wedge m \neq \emptyset$, there exists a point S, such that $S \not{\mathbin{\sim}} Z$ for any $Z \, I \, 1 \vee m$, by 1.8. Let $OS = n$. Then $n \not{\mathbin{\sim}} 1, m$. Since $O \not{\mathbin{\sim}} A$, any $X \, I \, 1$ can be expressed in the form $\tau_{OA}{}^X(O)$, for a unique $x \in H$. Then, by 1.25. $\tau_{OS}{}^X(O) \, I \, L(X, AS)$. Also $\tau_{OS}{}^X(O) \, I \, n$.

If AS $\sim$ n, then, since $1 \not{\mathbin{\sim}} n$, A6 implies $O \sim A$; a contradiction.

Case 1: B ⌿ OA = 1.

$\tau_{OB}{}^x(0)$

$\tau_{OS}{}^x(0)$

$\tau_{OA}{}^x(0)$
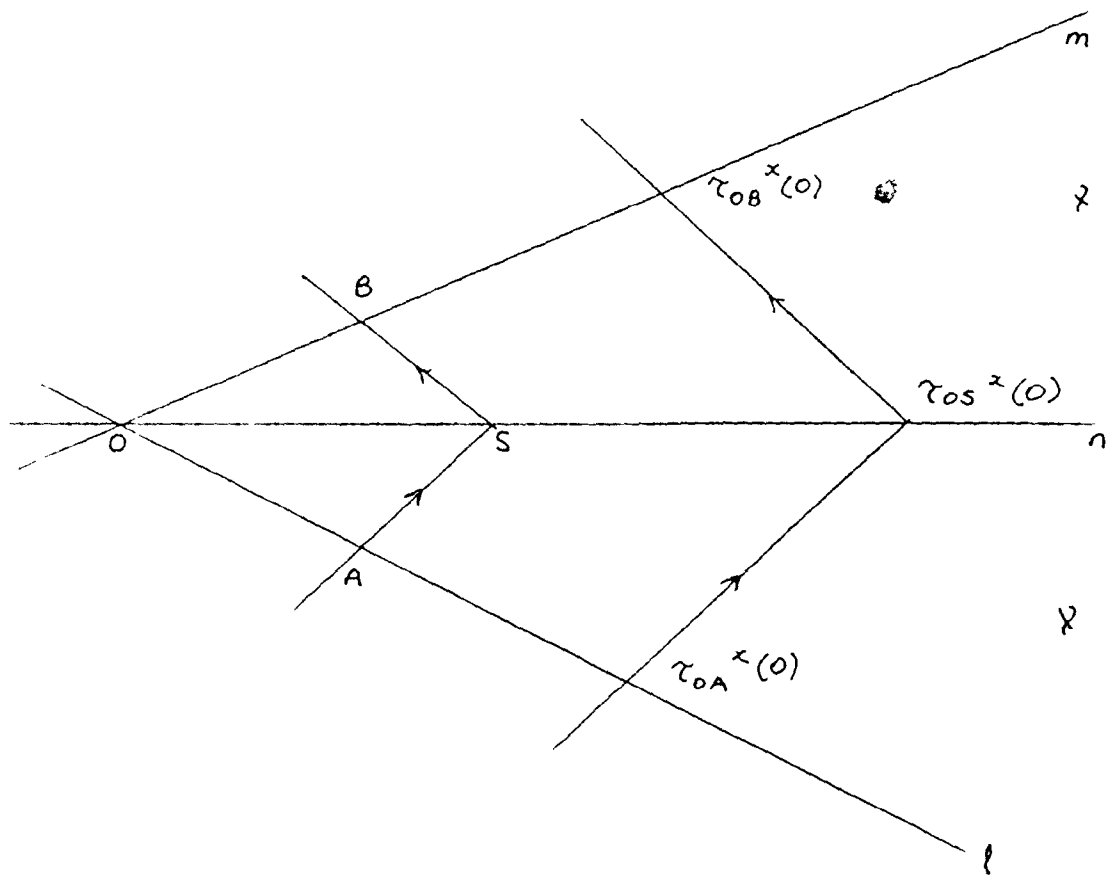
Figure 4.

Thus AS $\not\vdash$ n and also AS $\not\vdash$ 1. Then the map which takes X = $\tau_{OA}{}^x$(0) I 1

into $\tau_{OS}{}^x$(0) I n is a parallel projection, as in 2.4, and so

A, $\tau_{OA}{}^x$(0) | o implies S, $\tau_{OS}{}^x$(0) | o.

Similarly, since SB $\not\vdash$ m and SB $\not\vdash$ n, the map which takes Y = $\tau_{OS}{}^y$(0) I n

into $\tau_{OB}{}^y$(0) I m is a parallel projection. Thus S, $\tau_{OS}{}^y$(0) | o

implies B, $\tau_{OB}{}^y$(0) | o.

Thus a $\in$ H$^+$(O, A) implies a $\in$ H$^+$(O, B) and symmetrically a $\in$ H$^+$(O, B)

implies a $\in$ H$^+$(O, A).

Case 2: B I OA = 1.

Choose C, C $\not\vdash$ X for any X I 1. Then A, $\tau_{OS}{}^a$(0) | o

iff C, $\tau_{OC}{}^a$(0) | o iff B, $\tau_{OB}{}^a$(0) | o. Thus a $\in$ H$^+$(O, A) iff

a $\in$ H$^+$(O, B).

2.14. Notation. We may now write H$^+$(O, A) as H$^+$(O).

2.15. Lemma. H$^+$(O) does not depend on the choice of O, see Figure 5.

Proof. Let O' $\neq$ O; O, O' I 1.

Choose a point A which is not a neighbour of any point on 1.

Put τ_{OA}(O') = A'. Then A' I L(A, 1) = m and τ_{OA} = $\tau_{O'A'}$, thus

OA || O'A'. Since A is not a neighbour of any point on 1,

1 $\not\vdash$ m and also OA $\not\vdash$ 1. Thus, by A7, O'A' $\not\vdash$ 1.

Hence O7 applies to the projection Π , parallel to 1, taking any

$\tau_{OA}{}^x(0)$ $\tau_{O'A'}{}^x(O')$

m

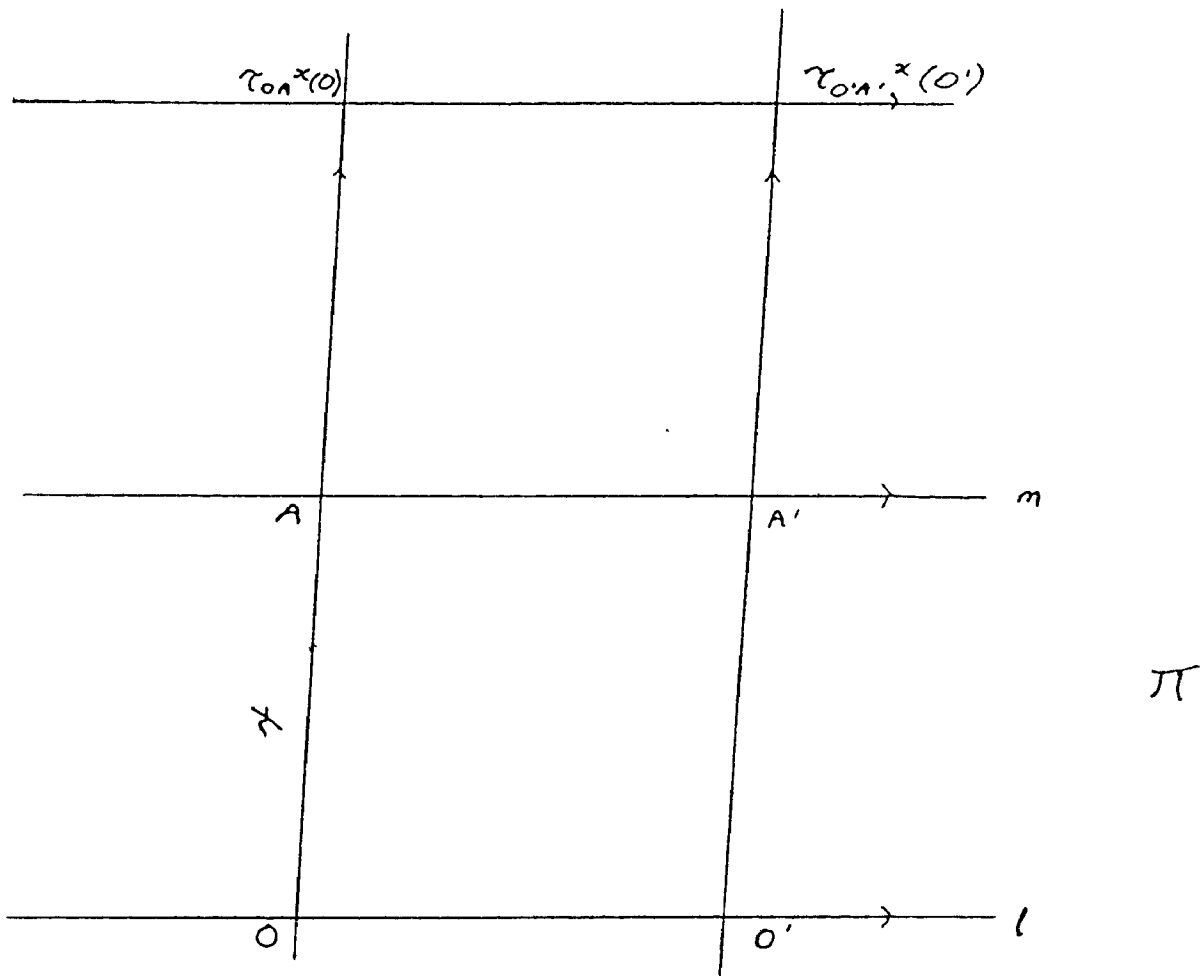A A'

π

x

O O' ℓ

Figure 5.

$X = \tau_{OA}{}^{x}(O)$ I OA,for a unique x $\in$ H, into $\tau_{O'A'}{}^{x}(O')$. Thus

A, $\tau_{OA}{}^{a}(O)$ | O implies A', $\tau_{O'A'}{}^{a}(O')$ | O' i.e., a $\in$ $H^{+}(O)$

implies a $\in$ $H^{+}(O')$. Symmetrically a $\in$ $H^{+}(O')$ implies a $\in$ $H^{+}(O)$.

2.16. Notation. We may now write $H^{+}(O)$ as H^{+}.

2.17. Theorem. If a, b $\in$ H^{+} and b $\notin \eta$, then ab $\in$ H^{+}.

Proof. Choose O, A $\in$ $\mathbb{P}$ such that O $\nsim$ A.

Then b $\in$ H^{+} implies A, $\tau_{OA}{}^{b}(O)$ | O. Put B = $\tau_{OA}{}^{b}(O)$.

Then B $\neq$ O and since b $\notin \eta$, B $\nsim$ O. (If B $\sim$ O then $\tau_{OB} \in$ N,

but $\tau_{OB} = \tau_{OA}{}^{b}$ so, by 1.18, b $\in \eta$, since $\tau_{OA} \notin$ N; a contradiction.)

Then a $\in$ H^{+} implies B, $\tau_{OB}{}^{a}(O)$ | O. But $\tau_{OB}{}^{a}(O) = (\tau_{OA}{}^{b})^{a}(O) =$

$\tau_{OA}{}^{ab}(O)$. A,B|O and B, $\tau_{OB}{}^{a}(O)$|O imply A, $\tau_{OB}{}^{a}(O)$ | O, i.e,

A, $\tau_{OA}{}^{ab}(O)$ | O, by 2.7. Thus ab $\in$ H^{+}.

2.18. We may assume that every line of $\mathcal{H}$ is incident with at least three points, or O1 to O7 are satisfied trivially, and H has only two elements with no ordering. In fact, for every proper affine Hjelmslev plane, H has at least four elements and each line is incident with at least four points; cf. [3], Chapter 2.

2.19. Lemma. The characteristic of H is not equal to two.

Proof. Assume (O, A, B).

If we apply τ_{OA} to O, A and B, then 2.10 implies

(τ_{OA}(O), τ_{OA}(A), τ_{OA}(B)), which equals (A, τ_{OA}^{2}(O), $\tau_{OA}\tau_{OB}$(O)).

Put $\tau_{OA}\tau_{OB}$(O) = C = $\tau_{OB}\tau_{OA}$(O). Now if we apply τ_{OB} to

(O, A, B), then 2.10 implies (τ_{OB}(O), τ_{OB}(A), τ_{OB}(B)), which

equals (B, C, τ_{OB}^{2}(O)).

If the characteristic of H is two, then τ_{OA}^{2}(O) = τ_{OA}^{O}(O) = O

and τ_{OB}^{2}(O) = O. So (A, O, C) and (B, C, O) hold and these imply

(A, O, B), by O4; a contradiction.

2.20. If C, B I l, we define O I l to be the midpoint of

C and B on l if τ_{CO} = τ_{OB}.

2.21. Lemma. If Q, C, O, B, P I l and (P, B, C) and O is the

midpoint on l of both Q and P, and C and B, then (P, B, Q).

Proof. Take T I l so that O $\neq$ T.

Then τ_{OB} = τ_{CO} = τ_{OT}^{a} for a unique a $\in$ H, and τ_{OP} = τ_{QO} = τ_{OT}^{b}

for a unique b $\in$ H, i.e., B = τ_{OT}^{a}(O), C = τ_{OT}^{-a}(O), P = τ_{OT}^{b}(O),

Q = τ_{OT}^{-b}(O). Now choose T' $\neq$ X for any X I l, by 1.7.

Then T' $\nmid$ l (see Figure 6). Construct a parallel projection from

l to OT' = n, in the direction of TT' = m. Then n $\neq$ m, or by A6, O $\sim$ T.
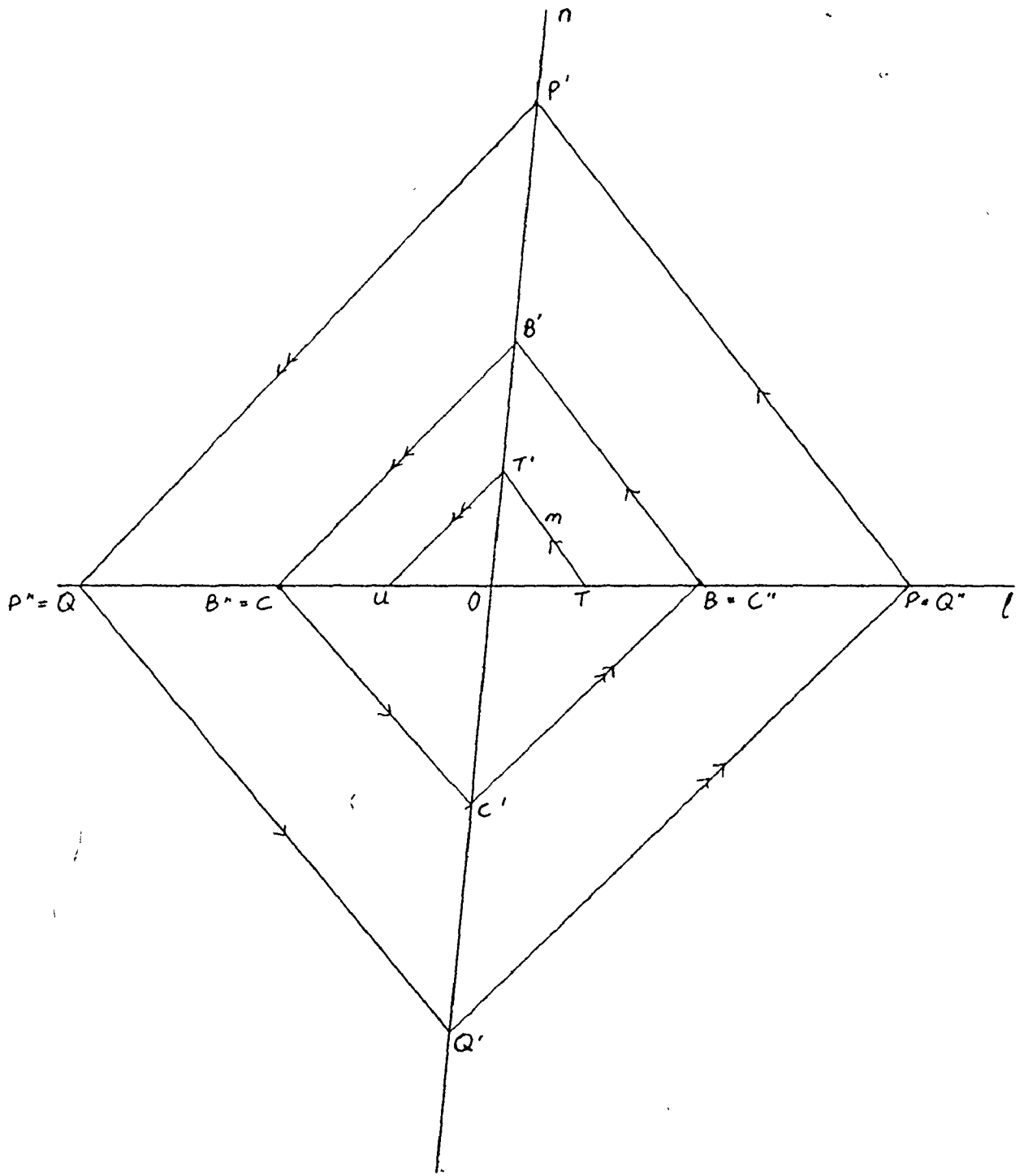
Figure 6.

Thus $L(B, m) \not\sim n$ and meets n at a unique point, say B', by A7.

Similarly, C', P' and Q' are obtained uniquely. By 1.25,

$\tau_{OT'}{}^{a}(0)$ I L($\tau_{OT}{}^{a}(0)$, m). Thus, since B', C', P' and Q'

are unique, B' = $\tau_{OT'}{}^{a}(0)$, C' = $\tau_{OT'}{}^{-a}(0)$, P' = $\tau_{OT'}{}^{b}(0)$

and Q' = $\tau_{OT'}{}^{-b}(0)$.

Now construct a second parallel projection from n to 1, taking

T' into U = $\tau_{OT}{}^{-1}(0)$. T'U $\not\sim$ 1, by our choice of T', so A7

uniquely determines B", C", P", Q" and, as before, B" = $\tau_{OU}{}^{a}(0)$ etc.

These parallel projections are as described in 2.4, since n $\not\sim$ m and

n $\not\sim$ 1 and also T'U $\not\sim$ 1 and T'U $\not\sim$ m (if T'U $\sim$ m then U $\sim$ 0, but

$\tau_{OT} \notin$ N and $\tau_{OT} = \tau_{UO}$, thus U $\not\sim$ 0). Thus 07 applies to

both parallel projections. So (P, B, C) implies (P', B', C') which

implies (P", B", C"). However, P" = $\tau_{OU}{}^{b}(0)$ = ($\tau_{OT}{}^{-1}$)${}^{b}(0)$ = Q

and B" = $\tau_{OU}{}^{a}(0)$ = C, C" = $\tau_{OU}{}^{-a}(0)$ = B. Thus we have (Q, C, B)

which with (P, B, C) implies (Q, B, P).

2.22. Theorem. If a, b $\in$ H^{+}, a $\not\sim$ b, then a + b $\in$ H^{+}.

Proof. Choose 0, A; 0 $\not\sim$ A.

Then a, b $\in$ H^{+} imply A, $\tau_{OA}{}^{a}(0)$ $|$ 0 and A, $\tau_{OA}{}^{b}(0)$ $|$ 0.

Put $\tau_{OA}{}^{a}(0)$ = B, $\tau_{OA}{}^{b}(0)$ = C. Thus A, B $|$ 0 and A, C $|$ 0, so B, C $|$ 0.

Also $a \neq b$ implies that $B \neq C$. Hence (O, B, C) or (O, C, B).

<u>Case 1:</u> (O, B, C).

O, A, B I l. Let E be the midpoint of B, C on l.

Set $Q = \tau_{OE}(E) = \tau_{OE}^2(O)$. Thus E is also the midpoint of

Q and O on l. Then $\tau_{OA}^{a+b} = \tau_{OA}^a \tau_{OA}^b = \tau_{OB} \tau_{OC} = \tau_{EB} \tau_{OE} \tau_{EC} \tau_{OE} = \tau_{OE}^2 = \tau_{OQ}$. Hence $Q = \tau_{OA}^{a+b}(O)$.

(O, B, C) implies (O, B, Q), by <u>2.21.</u> Thus B, $Q \mid O$ and, since

A, $B \mid O$, A, $Q \mid O$, i.e., A, $\tau_{OA}^{a+b}(O)$ O, i.e., $a + b \in H^+$.

<u>Case 2:</u> (O, C, B).

The proof follows symmetrically.

<u>2.23.</u> Let H^- be the complement of $H^+ \cup \{o\}$ in H.

Thus $H = H^+ \cup \{o\} \cup H^-$. The elements of H^- are called <u>negative</u> elements of H.

<u>2.24. Theorem.</u> $a \in H^+$ implies $-a \in H^-$.

<u>Proof.</u> $a \in H^+$ implies $a \neq o$, hence $-a \neq o$. Since characteristic

of H is not two, $a \neq -a$. If $-a \in H^+$, then $a+(-a) \in H^+$, by <u>2.22;</u>

a contradiction.

<u>2.25. Lemma.</u> $a \in H^-$ iff $(\tau_{OA}^a(O), O, A)$, where $O \neq A$.

Proof. If $a \in H$, then $a \in H^+ \cup H^-$ iff $a \neq o$ iff $\tau_{OA}{}^a \neq 1$

iff $\tau_{OA}{}^a(O) \neq 0$ iff exactly one, of $\tau_{OA}{}^a(O) = A$, $(\tau_{OA}{}^a(O), A, O)$,

$(A, \tau_{OA}{}^a(O), O)$, $(A, O, \tau_{OA}{}^a(O))$ holds. $a \in H^+$ is equivalent to the

assertion that exactly one of the first three relations holds.

Thus, the last relation is equivalent to the $a \in H^-$.

2.26. Theorem. If $a, b \in H^-$, $a \neq b$, then $a + b \in H^-$.

Proof. Choose $O \neq A$.

Then $a, b \in H^-$ imply $(\tau_{OA}{}^a(O), O, A)$ and $(\tau_{OA}{}^b(O), O, A)$,

which exclude $(\tau_{OA}{}^a(O), O, \tau_{OA}{}^b(O))$, by O6. Hence

$\tau_{OA}{}^a(O), \tau_{OA}{}^b(O) \mid O$. As in 2.22, let $\tau_{OA}{}^a(O) = B$, $\tau_{OA}{}^b(O) = C$

so (B, C, O) or (O, B, C).

Case 1: (O, B, C).

Letting $Q = \tau_{OE}(E)$, where E is the midpoint on 1 of B and C,

we have, as in 2.22, (O, B, Q). Since (B, O, A) we have $(Q, O, A) =$

$(\tau_{OA}{}^{a+b}(O), O, A)$, i.e., $a + b \in H^-$.

Case 2: (O, C, B).

The proof follows symmetrically.

2.27. Theorem. $a \in H^-$ implies $-a \in H^+$.

Proof. $a \in H^-$ implies $a \neq o$, hence $-a \neq o$. Since characteristic of

H is not two, $a \neq -a$. If $-a \in H^-$, then $-a + a \in H^-$, by 2.26; a contradiction.

2.28. Theorem. For all $a \in H$, exactly one, of $a \in H^+$, $a = o$, $-a \in H^+$ holds.

2.29. Theorem. $-1 \in H^-$.

Proof. By 2.12, $1 \in H^+$.   Hence, by 2.24, $-1 \in H^-$.

2.30. Theorem. If $a \in H^+$, then $2a \in H^+$.

Proof. Choose $0 \not/ A$.

Claim. $\tau_{OA}^a(O)$, $\tau_{OA}^{2a}(O) \mid O$.

Since $0 \not/ A$, $\tau_{OA}^a(O) \neq O$, by 1.18, and since characteristic of H

is not equal to two $\tau_{OA}^{2a}(O) \neq O$.

If $\tau_{OA}^{2a}(O) = \tau_{OA}^a(O) \neq O$, then $\tau_{OA}^a(O)$, $\tau_{OA}^{2a}(O) \mid O$.

If O, $\tau_{OA}^a(O)$ and $\tau_{OA}^{2a}(O)$ are mutually distinct (and collinear),

then, by O3, exactly one, of $(\tau_{OA}^a(O), \tau_{OA}^{2a}(O), O)$, $(\tau_{OA}^{2a}(O), \tau_{OA}^a(O), O)$

and $(\tau_{OA}^{2a}(O), O, \tau_{OA}^a(O))$ holds.

If $(\tau_{OA}^{2a}(O), O, \tau_{OA}^a(O))$, then, applying τ_{OA}^{-a}, we get

$(\tau_{OA}^a(O), \tau_{OA}^{-a}(O), O)$.   But, since $-a \in H^-$, $(\tau_{OA}^{-a}(O), O, A)$.

Thus, by O4, $(\tau_{OA}^a(O), O, A)$;   a contradiction to $a \in H^+$.

Thus $\tau_{OA}^a(O)$, $\tau_{OA}^{2a}(O) \mid O$ in all cases.

Since A, $\tau_{OA}^a(O) \mid O$, we have A, $\tau_{OA}^{2a}(O) \mid O$, i.e., $2a \in H^+$.

2.31. Thus, by 2.28; 2.22 and 2.30; 2.17, H is an ordered A. H. ring.

The Construction of a Geometrically Ordered
Desarguesian A. H. Plane from an Ordered A. H. Ring

3.1. Now, given an algebraically ordered A. H. ring H, we wish to construct a geometrically ordered Desarguesian affine Hjelmslev plane. Since Lorimer and Lane ([8], 3) have constructed a Desarguesian affine Hjelmslev plane $\mathcal{H}$ · from an A. H. ring, it remains to show that the given ordering of H induces a geometric ordering of $\mathcal{H}(H)$. Since the A. H. ring of $\mathcal{H}$ is isomorphic to H, we may identify the two A. H. rings and assume that the A. H. ring of $\mathcal{H}$ is the given A. H. ring H. Using [8], 3.11, we can verify that H and H' are order isomorphic; cf. Appendix 1.

3.2. If A, B, C are mutually distinct and collinear with a line 1, then there exist points O and E on 1 such that $O \neq E$. We define $(A, B, C)_{O,E}$ if $a < b < c$ or $c < b < a$, where $A = \tau_{OE}^{a}(O)$, $B = \tau_{OE}^{b}(O)$, $C = \tau_{OE}^{c}(O)$. B is said to lie between A and C.

3.3. Lemma. Order on a line 1 is independent of the choice of O, E on 1, where $O \neq E$.

Proof. Let $O \neq E$ and $(A, B, C)_{O,E}$.

Let O', E' be any two points such that $O' \not\sim E'$.

Then $A = \tau_{OE}{}^a(O) = \tau_{O'E'}{}^{a'}(O')$, $B = \tau_{OE}{}^b(O) = \tau_{O'E'}{}^{b'}(O')$,

$C = \tau_{OE}{}^c(O) = \tau_{O'E'}{}^{c'}(O')$. Since $O \not\sim E$, there exists

$x \in H$, $x \neq o$ such that $\tau_{O'E'} = \tau_{OE}{}^x$ and $y \in H$ such that

$\tau_{OO'} = \tau_{OE}{}^y$. Then $x \notin \eta$ since $\tau_{O'E'} \notin N$.

$A = \tau_{OE}{}^a(O) = \tau_{O'E'}{}^{a'}(O') = \tau_{O'E'}{}^{a'}\tau_{OO'}(O) = (\tau_{OE}{}^x)^{a'}(\tau_{OE})^y(O)$

$= \tau_{OE}{}^{a'x+y}(O)$. Thus $a = a'x+y$, since $\tau_{OE} \notin N$.

Assume $a' < b' < c'$, i.e., $b'-a'$, $c'-b' \in H^+$.

If $x \in H^+$, then $(b'-a')x$, $(c'-b')x \in H^+$ since $x \notin \eta$.

Hence $(b'x+y)-(a'x+y)$, $(c'x+y)-(b'x+y) \in H^+$, ie., $b-a$, $c-b \in H^+$,

i.e., $a < b < c$.

If $x \in H^-$, then $-x \in H^+$ and so $c < b < a$.

Similarly, one can deal with the case when $a' < b' < c'$.

Thus $(A, B, C)_{O,E}$ implies $(A, B, C)_{O',E'}$ for any $O' \not\sim E'$.

3.4. We are now in a position to prove O1 to O7, as in

Chapter 2. We may now write $(A, B, C)_{O,E}$ as (A, B, C).

3.5. O1. (A, B, C) implies that A, B, C are mutually distinct

and collinear, by definition.

3.6. O2. Clearly (A, B, C) implies (C, B, A).

3.7. O3. A, B, C, mutually distinct and collinear implies exactly
one, of (A, B, C), (B, C, A) and (C, A, B) holds.

Proof. Assume A, B, C I l.   If $O \not\sim E$ and O, E I l and $A = \tau_{OE}{}^{a}(O)$,
$B = \tau_{OE}{}^{b}(O)$, $C = \tau_{OE}{}^{c}(O)$, then exactly one of:

    1. $a < b < c$,
    2. $b < c < a$,
    3. $c < a < b$,
    4. $c < b < a$,
    5. $b < a < c$,
    6. $a < c < b$,   holds.

(A, B, C) is equivalent to 1 and 4.

(C, A, B) is equivalent to 3 and 5.

(B, C, A) is equivalent to 2 and 6.

Thus exactly one, of (A, B, C), (B, C, A) and (C, A, B) holds.

3.8. O4. (A, B, C) and (B, C, D) imply (A, B, D) and (A, C, D),
if A, B, C, D are collinear.

Proof. If $O \not\sim E$ and $A = \tau_{OE}{}^{a}(O)$, $B = \tau_{OE}{}^{b}(O)$, $C = \tau_{OE}{}^{c}(O)$,
$D = \tau_{OE}{}^{d}(O)$, then (A, B, C) is equivalent to $a < b < c$ or $c < b < a$.

(B, C, D) is equivalent to $b < c < d$ or $d < c < b$.   Hence (A, B, C) and

(B, C, D) are equivalent to $a < b < c < d$ or $d < c < b < a$ which implies

$a < b < d$ or $d < b < a$, i.e., (A, B, D).   Similarly we obtain (A, C, D).

3.9. O5. (A, B, C) and (A, C, D) imply (B, C, D) and (A, B, D),

if A, B, C, D are collinear.

Proof. The proof is similar to that of O4.

3.10. O6. Two, of (B, A, C), (C, A, D) and (D, A, B), exclude

the third, if A, B, C and D are collinear.

Proof. Without loss of generality assume (B, A, C) and (C, A, D).

This is equivalent to $b < a < c$ and $d < a < c$ or $c < a < b$ and $c < a < d$.

In particular either $b < a$ and $d < a$ or $a < d$ and $a < b$.   (D, A, B)

is equivalent to $d < a < b$ or $b < a < d$, which is clearly excluded.

3.11. O7. If π is a parallel projection, then it preserves order,

see Figure 7.

Proof. Assume A, B, C I l and (A, B, C) and π is a parallel

projection which takes any X I l into X' I m such that $X = l \wedge L(X, \pi)$,

$X' = m \wedge L(X, \pi)$.

Case 1: $l \quad m \neq \emptyset$.

   Say O I l, m.   Now take E I l such that $O \not\sim E$.

Then $A = \tau_{OE}{}^{a}(O)$, $B = \tau_{OE}{}^{b}(O)$, $C = \tau_{OE}{}^{c}(O)$ for some a, b, c $\in$ H

and $a < b < c$ or $c < b < a$.   By 1.25, $\tau_{OE}{}^{a}(O)$ I L($\tau_{OE}{}^{a}(O)$, L(E, π)).

Thus $A' = \tau_{OE'}{}^{a}(O)$ and, since $O \not\sim E$, L(O, π) $\not\sim$ L(E, π), by 1.10,
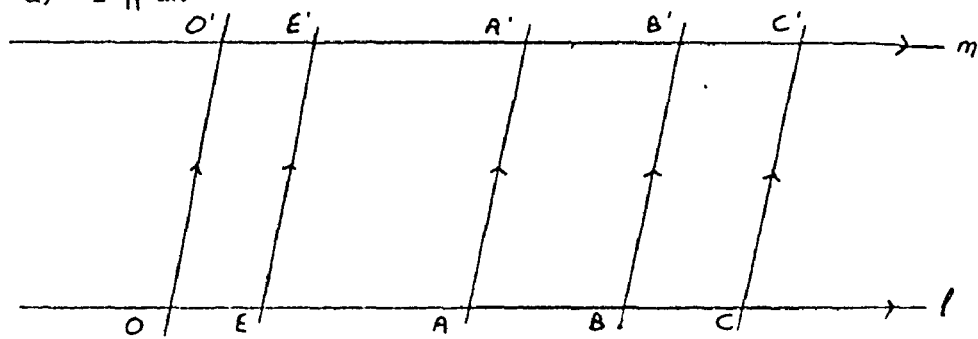
Case 1: $l \wedge m \neq \emptyset$.
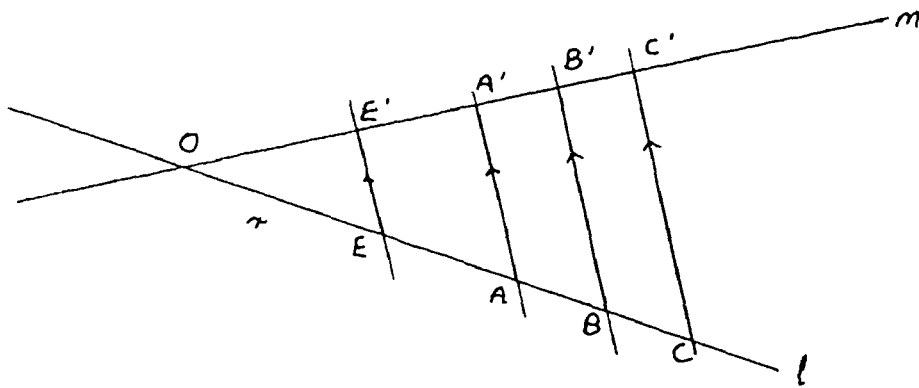
Case 2: $l \wedge m = \emptyset$.
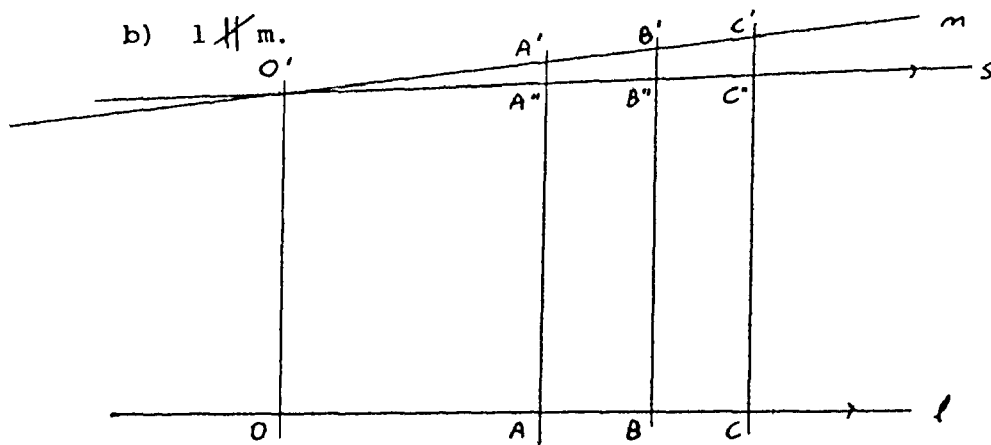
a) $l \parallel m$.

b) $l \not\parallel m$.

Figure 7.

so $O \not\vdash E'$. Since $a < b < c$ or $c < b < a$ <u>3.2</u> implies that (A', B', C').

<u>Case 2</u>: $1 \wedge m = \emptyset$.

    a) $1 \| m$.

    Take $E \ I \ 1$, $O \not\vdash E$. Then $\tau_{OE}(O') \ I \ L(E, \pi) \wedge m$, thus

$E' = \tau_{OE}(O')$ and $O' \not\vdash E'$. Since $A = \tau_{OE}^{\ a}(O)$ for some $a \in H$ and

$\tau_{OE}^{\ a}(O') \ I \ L(A, \pi) \wedge m$, $A' = \tau_{OE}^{\ a}(O')$, $B' = \tau_{OE}^{\ b}(O')$, $C' = \tau_{OE}^{\ c}(O')$.

Thus, as before, (A, B, C) implies (A', B', C').

    b) $1 \not\| m$.

    Then, by <u>1.6</u>, there exists s, $s \| 1$, $s \wedge m \neq \emptyset$ and $s \sim m$.
Say $O' \ I \ s$, m. $L(X, \pi) \not\vdash m$ for any $X \ I \ 1$, thus $L(X, \pi) \not\vdash s$.
$L(O', \pi)$ meets 1 at a unique point O, by the definition of a
parallel projection, and $L(O', \pi) \not\vdash 1$. However, $O' \ I \ s$ and
$L(O', \pi) \not\vdash s$, so by A7, for any $X \ I \ 1 \ L(X, \pi)$ meets s at a
unique point X''.
Thus, (A, B, C) implies (A'', B'', C''), by Case 2(a), which implies
(A', B', C'), by Case 1.

    <u>3.12.</u> By <u>3.5</u> to <u>3.11</u>, $\mathcal{H}$ is a geometrically ordered
Desarguesian affine Hjelmslev plane.

CHAPTER 4

An Example of an Ordered A. H. Ring

4.1. A projective Hjelmslev ring or P. H. ring

is an A. H. ring which also satisfies:

3. If a, b ∈ H, then a ∈ Hb or b ∈ Ha.

The following example of an A. H. ring which is not a P. H. ring

is originally due to R. Baer, [2], and is examined in [9], 5.

4.2. Let F be a field, F' a proper subfield of F, and

ϕ an isomorphism from F into F'. Take H = F x F and define addition

and multiplication as follows:

$$(a, b) + (c, d) = (a + c, b + d),$$

$$(a, b) \cdot (c, d) = (ac, \phi(a)d + bc).$$

Then H is easily seen to be an A. H. ring, but not a P. H. ring,

where $\eta = \left\{ (o, a) \mid a \in F \right\}$.

4.3. We can take F = Q(x), a simple transcendental

extension of the rational numbers Q, F' = Q(x^2) and ϕ the map from

F into F' which takes x into x^2.

Q(x) can be regarded as the field of real-valued rational

functions with rational coefficients, and can be made into an

ordered field by defining:

$$Q(x)^+ = \left\{ \frac{f(x)}{g(x)} \in Q(x) \middle| \; \exists \; x_* \in Q \quad \frac{f(z)}{g(z)} > 0, \; \forall z < x \right\}.$$

Then clearly,

1. For any $a \in Q(x)$, either $a \in Q(x)^+$ or, $-a \in Q(x)^+$

   or, $a = 0$.

2. $a \in Q(x)^+$, $b \in Q(x)^+$ implies $a + b \in Q(x)^+$.

3. $a \in Q(x)^+$, $b \in Q(x)^+$ implies $a \cdot b \in Q(x)^+$.

Now consider $H = Q(x) \times Q(x)$ and define a lexicographic order on H,

i.e., $H^+ = \left\{ \alpha = (a, b) \in H \middle| \; \text{either } a > 0 \text{ or } a = 0 \text{ and } b > 0 \right\}$.

Then clearly,

1. For any $\alpha \in H$, either $\alpha \in H^+$ or, $-\alpha \in H^+$ or, $\alpha = 0$.

2. $\alpha \in H^+$, $\beta \in H^+$ implies $\alpha + \beta \in H^+$.

Also,   3. $\alpha, \beta \in H^+$, $\beta \notin \eta$ implies $\alpha\beta \in H^+$.

Proof of 3.

Case 1:

Let $\alpha = (a, b)$, $\beta = (c, d)$.   Then $a > 0$ and $c > 0$.

Thus, $\alpha\beta = (ac, \phi(a)d + bc) \in H^+$, since $ac > 0$.

Case 2:

Let $\alpha = (0, b)$, $\beta = (c, d)$.   Then $b, c > 0$.

Thus, $\alpha\beta = (0 \cdot c, \phi(0)d + bc)$.   But, $\phi(0) = 0$, thus

$\alpha\beta = (0, bc) \in H^+$, since $bc > 0$.

Thus H is an ordered A. H. ring.

4.4. It is interesting to note that $\alpha, \beta \in H^+$,

$\alpha \notin \eta$, $\beta \in \eta$ does not imply that $\alpha\beta \in H^+$.

Assume $\alpha = (a, b)$, $\beta = (0, d)$, $a, d > 0$.   Thus $\alpha, \beta \in H^+$.

Then, $\alpha\beta = (a\cdot 0, \ \phi(a)d + b\cdot 0) = (0, \ \phi(a)d)$. However, $\phi(a)$ is not necessarily greater than zero in $Q(x^2)$. For example, take $a = f(x)$, where $f(x) = -x$. Then $\phi(a) = f(x^2) = -x^2$ and clearly $\phi(a) < 0$. Thus $\phi(a)d < 0$, and so $\alpha\beta \in H^-$.

CHAPTER 5

The Ordinary Affine Plane Associated

with an Ordered Desarguesian A. H. Plane

5.1. With every affine Hjelmslev plane, $\mathcal{H}$, we may

associate a structure $\overline{\mathcal{H}} = \{ \ \overline{\mathbb{P}}, \ \overline{\mathbb{L}}, \ \overline{I}, \ \overline{\|} \ \}$ by the quotient maps

$\varkappa_P$ and $\varkappa_L$ of $\sim_P$ and $\sim_L$. We define $\overline{P} \ \overline{I} \ \overline{l}$ iff there exists

$S \in \mathbb{P}$ such that $P \sim S$ and $S I l$. If $\overline{\|}$ is the parallelism

relation for ordinary affine planes, then $\overline{\mathcal{H}}$ is an ordinary

affine plane; cf. [8], 1.2. In fact, A. H. planes may be defined

in terms of their associated ordinary affine plane; cf. [9], 2.2.

5.2. Once a Desarguesian A. H. plane has been

coordinatized, we see that if $\underline{P} = (a, b)$, $\underline{Q} = (c, d)$, then $P \sim Q$

iff $a-c \in \eta$ and $b-d \in \eta$; cf. [8], 6.2,3.

5.3. If $\mathcal{H}$ is a Desarguesian A. H. plane, then it is clear

that $\overline{\mathcal{H}}$ is the ordinary Desarguesian affine plane coordinatized by

the division ring H/η . If H is an ordered A. H. ring, then

H/η is an ordered division ring and thus $\overline{\mathcal{H}}$ is an ordered affine

plane; cf. [11].

CHAPTER 6

The Radical of an Ordered A. H. Ring

6.1. Theorem. * Let H be an ordered A. H ring and η be the radical (maximal ideal) of H.

   1. If $r \in \eta$, then $-1 < r < 1$.

   2. If $c \in \eta$, then $-c < b < c$ implies $b \in \eta$.

Proof. 1. Suppose $r \in H^+$ and $r > 1$. Then $r - 1 \notin \eta$; cf. [6],pg.75. Since $r \in \eta$, there exists r' such that $r'r = o$.   We may assume, without loss of generality, that $r' \in H^+$.   Thus $r'(r - 1) \in H^+$. However, $r'(r - 1) = -r'$; a contradiction.   Thus $r < 1$. If $r < -1$, then $-r > 1$; again a contradiction.

   2. Consider, for instance, $o < b < c$ and suppose $b \notin \eta$. Then $c - b \in H^+$ and $c - b \notin \eta$.   Since $c \in \eta$, there exists c' such that $c'c = o$.   We may assume, without loss of generality, that $c' \in H^+$.   Then $c'b \in H^+$.   But $-c'b = c'(c - b) \in H^+$; a contradiction.   Thus $b \in \eta$. If $-c < b < o$, then $o < -b < c$;   again a contradiction.

---

* Dr. J. W. Lorimer brought this result to my attention.

$$E = \tau_{OE}{}^{1}(0)$$

$$A = \tau_{OE}{}^{a}(0)$$
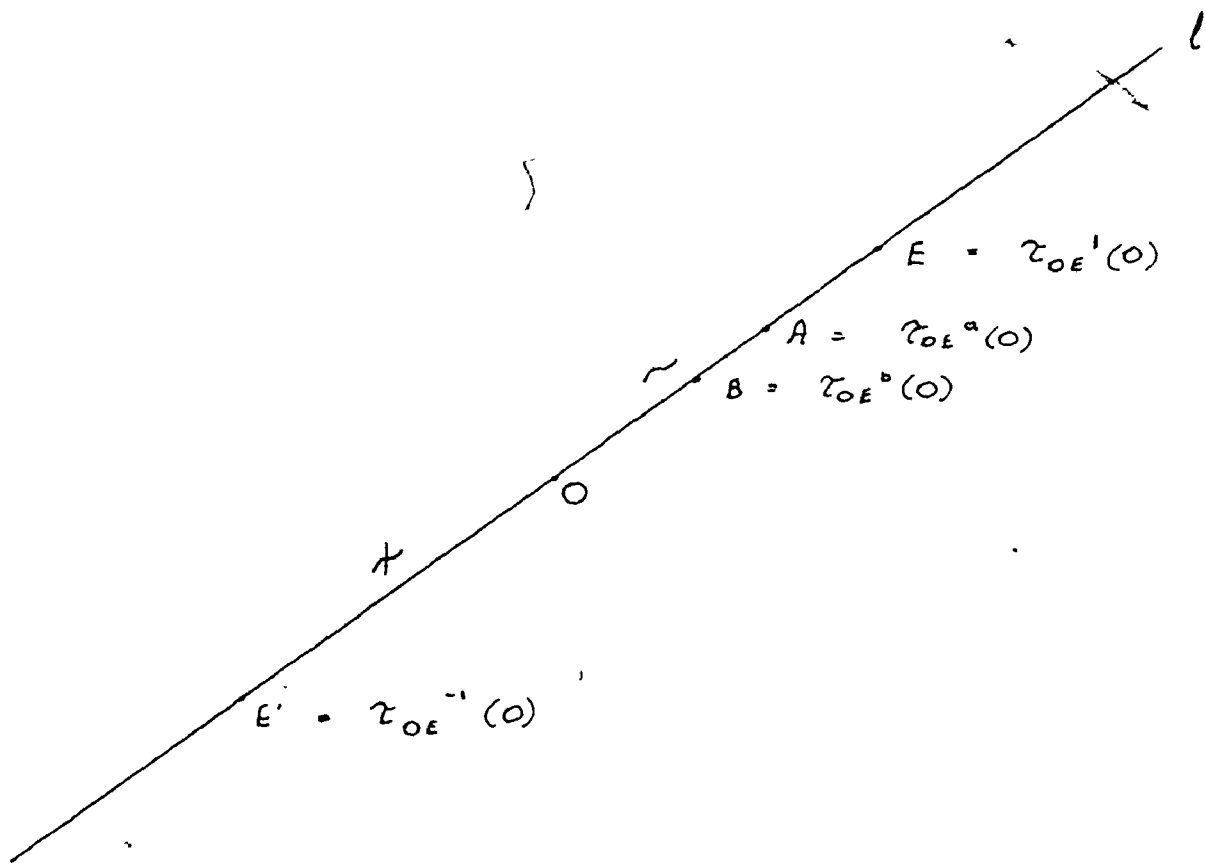
$$B = \tau_{OE}{}^{b}(0)$$

$$E' = \tau_{OE}{}^{-1}(0)$$

Figure 8.

6.2. Geometrically, this means that if O is any point on a line l, in an ordered Desarguesian A. H. plane, then all the neighbouring points of O lie between E and E', where E is any non-neighbour of O and $E' = \tau_{OE}^{-1}(O)$. If A is any neighbour of O, then all points between A and O are also neighbours of O. See Figure 8.

The Archimedean Axiom and Desarguesian A. H. Planes

7.1. An ordered A. H. ring H is called <u>Archimedean</u> if for any a, b $\in$ H$^+$ there exists n $\in$ Z such that na $>$ b. This is clearly equivalent to the Archimedean ordering of the plane $\mathcal{H}$(H), i.e., if (O, A, B) then there exists n $\in$ Z such that (O, B, τ_{OA}^n(O)). Since if O, A, B I 1, then O $\not|$ T for some T I 1 where A = τ_{OT}^a(O) and B = τ_{OT}^b(O) and A, T $|$ O and B, T $|$ O; cf. [1], pg.78.

7.2. Hjelmslev and Klingenberg suggested, [4], pg.17 and [5], pg.406, that any Archimedean ordered A. H. ring is automatically a division ring. This is indeed true. Thus any Archimedean ordered Desarguesian A. H. plane is an ordinary Desarguesian affine plane.

7.3. Theorem. Let H be an Archimedean ordered A. H. ring. Then H is a division ring, i.e., η = $\{o\}$.

<u>Proof.</u> Let H be Archimedean and assume H is not a division ring. Then, for any a, b $\in$ H$^+$ there exists n $\in$ Z such that na $>$ b, i.e., na - b $\in$ H$^+$. Take a, b $\in$ H$^+$, a $\in$ η , b $\notin$ η . Thus there exists a' $\in$ H such that a'a = o. We may assume that a' $\in$ H$^+$.

Hence $a'b \in H^+$.   Since $na - b \notin \eta$, $-a'b = a'(na - b) \in H^+$,

a contradiction.

Note.   By means of ternary rings, Dr. J. W. Lorimer has shown

that any non-Desarguesian proper A. H. plane cannot be Archimedean

ordered.

The A. H. ring, H', of trace preserving endomorphisms of an A. H. plane $\mathcal{H}$ (H) is order isomorphic to H. This is clear when we consider the following theorem, originally due to Klingenberg.

Theorem. Let H' be the A. H. ring of trace preserving endomorphisms of $\mathcal{H}$ (H) and let η be its unique maximal ideal. Then; cf.[8], 3.10.

1. $c' \in H'$ iff there exists $c \in H$ such that

$$\tau_{0 \, (a,b)}^{c'} = \tau_{0 \, (ca,cb)} \quad \text{for all } (a,b) \quad H \times H.$$

Thus $\tau_{0 \, (1,0)}^{c'} = \tau_{0 \, (c,0)}$.

2. The mapping $H \to H'$ $(c \to c')$ is a ring isomorphism.

3. $c' \in \eta'$ iff $c \in \eta$.

If $c \in H^+$ then consider the points $O = (0,0)$, $E = (1,0)$, $C = (c,0)$. $c \in H^+$ implies that $c = 1$, or $0 < 1 < c$, or $0 < c < 1$, i.e. $C = E$, or $(0, E, C)$, or $(0, C, E)$. So $E, C \mid O$, but $C = (c,0)$ and $(c,0) = \tau_{0 \, (c,0)}(0) = \tau_{0 \, (1,0)}^{c'}(0) = \tau_{OE}^{c'}(0)$. Thus $c' \in H'^+$.

If $c' \in H'^+$ then $E, C \mid O$ where $C = \tau_{OE}^{c'}(0) = \tau_{0 \, (c,0)}(0)$. Thus $0 < 1 < c$, or $0 < c < 1$, or $c = 1$, i.e. $c \in H^+$.

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ABSTRACT:

A method for the determination of partial radiation cross sections based on the energy dispersion of the capture process was undertaken. A description of the experimental reactor facility and details of the method has been presented. Measurements of gamma rays following neutron capture in isotopes of silicon, chromium and nickel revealed resonances which were analyzed for resonance parameters. The characteristics of the resonance decay properties for the different isotopes were discussed.