CODIMENSION-2 BRANES IN SUPERGRAVITY

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By

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Abstract

The goal of this thesis is to study the back-reaction of codimension-2 branes in chiral gauged supergravity for solutions which are consistent with axial symmetry in the bulk directions and maximal symmetry in the spatial directions of the brane (FRW-like branes). The mathematics involved in this is somewhat cumbersome so as a first step, which can later be used as a check, we study solutions which are consistent with axial symmetry in the bulk and maximally symmetric in all brane directions. In order to analyze the brane back-reaction we compute a set of equations relating the bulk fields at the brane position to the brane properties, known as junction conditions. These junction conditions combined with the low-energy on-brane effective action (which is also derived) provide the complete description of the brane back-reaction. This formalism is then applied to two examples and generalized for the case of FRW-like branes.

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Chapter 1

Introduction

1.1 Motivating Extra Dimensions

Two of the greatest successes physics has ever offered are Einstein's theories of relativity [1, 2, 3] and the standard model of particle physics [4]. But for over 200 years before these theories were developed, Newton's description of mechanics and gravity were the best tools we had to understand nature. Although Newtonian theory was very successful in most experimentally accessible physical domains of the day, it would turn out that there exist physical regimes in which his description was inadequate, suggesting that the theory needed revision. It is often the case that when theories are in need of modification it is due to hidden assumptions which everyone makes and which unnecessarily restrict the kinds of theories which could be considered, and this was no exception.

The idea that time was absolute and could be agreed upon by all observers was one of these assumptions. By relaxing this assumption and postulating that the speed of light, c, is a constant in all inertial reference frames (which was motivated from Maxwell's theory of electromagnetism), Einstein superceded Newtonian mechanics with his theory of special relativity. Special relativity revolutionized our view of the world by making us realize that space and time cannot be treated separately, but are intimately connected and form a single entity known as spacetime. Many interesting predictions of the theory, such as time dilation, length contraction, and the equivalence of mass and energy, have been verified experimentally to great accuracy. One consequence of particular importance is that the speed of light acts as a speed limit in our universe, meaning that no physical signal can travel faster than the speed of light, including the propagation of gravitational effects.

Newton's theory of gravity contradicted this fact, as it was assumed in his theory that the effects of gravitation were instantaneous. Reconciling these differences resulted in the formulation of a relativistically consistent theory of gravity, general relativity. General relativity accomplished this by relaxing yet another hidden assumption, that the structure of spacetime is flat, by dictating that the presence of matter warps and curves the structure of spacetime, and that this curvature determines the trajectory of objects through spacetime. Like all good theories, general relativity also made a number of exciting predictions which have all passed experimental tests. These include, but are not limited to, gravitational time dilaton, light deflection, and the precession of orbits. One prediction of particular importance is that any physical consequences that can occur due to matter, can also occur due to a source of energy, since the relationship $E = mc^2$ tells us that matter and energy are two different manifestations of the same thing.

Naturally when physicists discovered that the structure of spacetime was not as boring as Newton had once thought, they wanted to learn more about it. Since the geometry of spacetime is determined by both the matter and energy content of the universe, in order to study its properties we need to know all possible sources of matter and energy. The most obvious sources of curvature are regular matter and radiation, however another contributing source is the energy density of empty space, or the vacuum energy density. A consequence of having a positive (negative) vacuum energy density is that it is associated with negative (positive) pressures causing the universe to expand (contract) on large scales. Measurements indicate that the universe is currently expanding, corresponding to a positive vacuum energy density with an experimental value of $\rho_{\text{vac}} = v^4$, with $v \sim 3 \times 10^{-12}$ GeV (where we've used the units $c = 1$ and $\hbar = 1$) [5]. This can now be used to check the value we expect from our theory. In order to calculate the theoretical value for the vacuum energy density we turn to the other triumph of modern physics - the standard model, which describes the interactions of all known physical phenomena excluding gravity.

In the standard model, the vacuum state is defined as the state with the lowest energy density, there is nothing that necessarily implies that it must be zero, as one might naively expect. One process predicted by the standard model which contributes to the vacuum energy density is the spontaneous creation of pairs of virtual particles which interact with one another then quickly annihilate. Each pair of virtual particles consist of a particle and its antiparticle, having the same mass. The energy required for this process can be calculated and goes as m^4, where m is the mass of the virtual particle/antiparticle. For example, the mass of the electron is $m_{\text{e}} = 5 \times 10^{-4}$ GeV, and so according to the standard model contributes an amount $\delta\rho_{\text{e}} \sim m_{\text{e}}^4$ to the vacuum energy density, which is a catastrophically large amount in comparison to the measured value. Understanding why the standard model agrees with numerous experiments so well but badly predicts the value of vacuum energy density has proven to be impossible. This obstacle in our understanding has been dubbed the cosmological constant problem. This large discrepancy seems to indicate that we are missing something in our description of low energy physics. Just as in the case of generalizing the theory of gravity, this problem may be due to a hidden assumption.

Any serious attempt to solve this problem must do two things. It requires a modification of our description of low energy physics, and since the standard model has already proven itself time and time again on such energy scales, this modification is most likely a modification of our gravitational theory at low energies. The second requirement is that this modification not ruin excellent agreement with all the non-gravitational experiments performed to date. These two contradictory requirements is what has made a solution to the cosmological constant problem so elusive.

However, progress can potentially be made if we abandon the assumption that spacetime is four dimensional, and allow for the possibility of extra spatial dimensions. To satisfy the second of our requirements, we postulate that all ordinary particles and their interactions, excluding gravity, are trapped on a (3+1)-dimensional surface within this extra-dimensional space, known as a brane. Gravity on the other hand is not constrained to the brane and is free to propagate in the extra dimensions, known as the bulk, which facilitates the modification of gravity at low energies. This idea is motivated by the discovery of D-branes within string theory, which serves as a source of guidance in developing these ideas. Taking a cue from string theory, we also assume that the extra dimensions are supersymmetric and described by one of the many known supergravity theories. If extra dimensions do exist, and if gravity is allowed to propagate through them, the size of the extra dimensions are constrained by the requirement that the theory agrees with experimental results of Newtonian gravity on short length scales. Taking this into account provides an upper bound of $r < 100\mu m$ on the size of the extra dimensions [6]. Within the context of supersymmetric extra dimensions, the extra dimensions can only be this large if there are precisely two of them and if the fundamental scale, M_g, of the extra-dimensional physics is around 10 TeV, due to the relation $M_p = M_g^2 r$ which relates M_g and r to the observed Planck mass: $M_p = (8\pi G)^{-1/2} \sim 10^{18}$GeV (where G denotes Newton's constant).

So how does this help? Within this framework the process of creation/annihilation of virtual particles still contributes an amount m^4 to the vacuum energy density, but this energy density is no longer a cosmological constant term, rather it is a contribution to the tension of the brane. To fully understand this, we must study how this energy source curves the extra dimensions and how it effects the effective $4D$ cosmological constant which is observed on length scales larger than the length scale of the extra dimensions. The cosmological constant problem is not the only problem extra dimensions offers a solution to, it provides a framework for tackling the hierarchy problem, problems of conventional grand unification theories, and the flavour problem [7].

1.2 Studying Extra Dimensions

Our model for studying extra dimensions is a generalization of the $6D$ chiral gauged supergravity lagrangian [8, 9]. We first consider a solution which is consistent with maximal symmetry in the brane directions and axial symmetry in the bulk direction. The field equations subject to these ansätze are then computed, and it's noted that there exist solutions to these field equations which have singularities in the bulk.

These singularities are interpreted as the location of codimension-2 branes. However, dealing with codimension-2 branes is difficult because of the possibility of bulk fields diverging at the brane positions, and makes it useless to discuss any relationships between the bulk and brane quantities. To get around this problem we introduce a renormalization scheme which involves replacing the codimension-2 branes with small cylindrical codimension-1 branes located a small distance, Δ, away from the singularity positions.

Once codimension-1 branes have been included into the system we study how the inclusion of the brane affects the bulk fields at the brane position, this is done with a set of equations known as the junction conditions. While the study of codimension-1 branes is useful, our goal is to study the physics of codimension-2 branes. To accomplish this we use the codimension-1 formulation as motivation for generalizing the junction conditions to codimension-2 branes. Once this is complete we then study the low-energy on-brane effective theory to analyze back-reaction affects of the brane, and apply our formalism to an example.

While the study of maximally symmetric branes is useful, it has its limitations. In particular it is not useful for studying time dependence in the extra dimensions or cosmological applications. In order to study these cases we use a more general ansatz which is consistent with axial symmetry in the bulk directions and maximal symmetry in the spatial dimensions of the brane - that is the brane has a Friedmann-Robertson-Walker (FRW) like metric. The ultimate goal is to apply the same analysis as in the maximally symmetric brane case to the FRW-like brane and derive the relevant junction conditions and the low-energy on-brane effective action.

Chapter 2

Maximally Symmetric Branes

In this section we study the back-reaction of maximally symmetric codimension-2 branes in a generalized chiral gauged supergravity theory. However, in general there exists the possibility that bulk fields diverge at the codimension-2 brane position which makes analyzing the brane properties very difficult. To get around this we introduce a renormalization scheme which involves replacing the codimension-2 branes with small cylindrical codimension-1 branes a small distance, Δ, from the original position of the codimension-2 branes. We then use the ensuing codimension-1 formulation to motivate and generalize to a formulation codimension-2 branes where the problem of field divergences at the brane positions is avoided. Once this formulation is completed we apply it to two examples and show how both the higher and lower dimensional pictures agree with one another.

2.1 The Bulk

Our starting point is the following action describing the bulk physics,

$$S = S_B + S_{GH}. \tag{2.1}$$

The bulk action,

$$S_B = -\int_{\mathcal{M}} d^n x \sqrt{-g} \left[\frac{1}{2\kappa^2} g^{MN} \left(\mathcal{R}_{MN} + \partial_M \phi \partial_N \phi \right) + \frac{1}{4} f(\phi) F_{MN} F^{MN} + V(\phi) \right], \tag{2.2}$$

is an n dimensional scalar-tensor action describing the couplings between a real scalar field, ϕ, and both the extra-dimensional Einstein-frame metric, g_{MN},[1] and the Maxwell field, $F_{MN} = \partial_M A_N - \partial_N A_M$. $\mathcal{R}_{MN}$ is the n-dimensional Ricci tensor constructed from the metric g_{MN}. This action is a generalized version of the bosonic part of the $6D$ chiral gauged supergravity action. S_{GH} is the

[1] Our metric is mostly plus, with Weinberg's curvature conventions, which differ from those of MTW only by an overall sign in the definition of the Riemann tensor.

Gibbons-Hawking action [10] and must be included in any system in which the spacetime geometry includes a boundary. It is given by

$$S_{GH} = \frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^{n-1}x\sqrt{-\bar{g}}K, \tag{2.3}$$

where $\bar{g}$ is the induced metric on the boundary surface $\partial\mathcal{M}$, and K is trace of the extrinsic curvature of the boundary surface $\partial\mathcal{M}$. The field equations of (2.2) are

$$\frac{1}{2\kappa^2}(\mathcal{R}_{MN} + \partial_M\phi\partial_N\phi) + \frac{f}{2}F_M{}^P F_{NP} + \frac{1}{n-2}\left[V - \frac{f}{4}F_{PQ}F^{PQ}\right]g_{MN} = 0, \qquad \text{(Einstein)} \tag{2.4}$$

$$\Box\phi - \kappa^2\left[\frac{\partial V}{\partial\phi} + \frac{1}{4}\frac{\partial f}{\partial\phi}F_{MN}F^{MN}\right] = 0, \qquad \text{(Dilaton)} \tag{2.5}$$

$$\nabla_M(fF^{MN}) = 0, \qquad \text{(Maxwell)} \tag{2.6}$$

and are derived explicitly in Appendix A.

2.2 Metric Ansatz

Our interest is in geometries which are maximally symmetric (meaning the space is both homogeneous and isotropic) in the brane directions and axially symmetric in the bulk directions. The most general metric satisfying these symmetries is

$$ds^2 = d\rho^2 + e^{2B}d\theta^2 + e^{2W}\hat{g}_{\mu\nu}dx^\mu dx^\nu, \tag{2.7}$$

where θ labels the direction of cylindrical symmetry, the functions B and W depend only on the proper distance of the bulk, ρ, and $\hat{g}_{\mu\nu}$ denotes a maximally symmetric Minkowski-signature metric with dimension $n-2$. To see this, we first quote a theorem from the section "Spaces with Maximally Symmetric Subspaces" of the chapter "Symmetric Spaces" from [1].

Suppose that an N-dimensional spacetime has an M-dimensional maximally symmetric subspace described by the coordinates x^μ, and the other $N-M$ coordinates are given by y^a. Then it is always possible to choose the x-coordinates so that the metric of the whole space is given by

$$ds^2 = h_{ab}dy^a dy^b + e^{2W(y)}\hat{g}_{\mu\nu}(x)dx^\mu dx^\nu, \tag{2.8}$$

where $h_{ab}(y)$ and $W(y)$ are functions of the y-coordinates alone, and $\hat{g}_{\mu\nu}(x)$ is a function of the x-coordinates alone that is by itself the metric of an M-dimensional maximally symmetric space.

Using this, we can now specialize to requiring that the bulk directions (the y-coordinates) are axially symmetric. This amounts to requiring that both h_{ab} and W are independent of θ, the bulk angular direction, which leads us to (2.7), where we have also chosen our coordinates such that ρ is the proper distance of the bulk.

These choices of symmetries also have consequences for the functional dependence of the other fields in our problem, namely ϕ and A. We now show that

$$\begin{aligned} A &= A_\theta(\rho)d\theta, \\ \phi &= \phi(\rho), \end{aligned} \tag{2.9}$$

are the most general choices for ϕ and A which are consistent with maximally symmetry in the brane directions and axial symmetry in the bulk directions.

A consequence of imposing that $\hat{g}_{\mu\nu}$ is maximally symmetric is that all of the Einstein equations in the $(\mu\nu)$ directions must be identical, otherwise the space would not be homogeneous and isotropic. Thus the $(\mu\nu)$ Einstein equations must take the form of a scalar equation times the metric $\hat{g}_{\mu\nu}$. With this in mind, the fact that the term $\partial_\mu\phi\partial_\nu\phi$ in the $(\mu\nu)$ equations is not proportional to $\hat{g}_{\mu\nu}$ forces us to make ϕ independent of x^μ to ensure maximal symmetry on the brane. Axial symmetry in the bulk also dictates that ϕ must be independent of θ, thus ϕ can only depend on ρ.

A similar story holds for choosing the most general Maxwell field without breaking our given symmetries. The term $F_\mu{}^P F_{\nu P}$ in the $(\mu\nu)$ equations is also not proportional to $\hat{g}_{\mu\nu}$ and A must be chosen accordingly to make $F_\mu{}^P F_{\nu P} = 0$. This results in requiring that all the components of A must be independent of x^μ and that the components A_μ are constant. Without loss of generality we take $A_\mu = 0$, and this leaves us with the following as the most general form for A,

$$A = A_\theta(\rho)d\theta + A_\rho(\rho)d\rho, \tag{2.10}$$

where we have also made the components of A independent of θ due to axial symmetry in the bulk dimensions. We'll also set $A_\rho = 0$ since it is a pure gauge term and does not contribute to the Maxwell field, F.

In summary, our ansatz for a geometry which is maximally symmetric in the brane directions and axially symmetric in the bulk directions is

$$\begin{aligned} ds^2 &= d\rho^2 + e^{2B}d\theta^2 + e^{2W}\hat{g}_{\mu\nu}dx^\mu dx^\nu, \\ \phi &= \phi(\rho), \\ A &= A_\theta(\rho)d\theta, \end{aligned} \tag{2.11}$$

where θ labels the direction of cylindrical symmetry, the functions B and W depend only on the proper distance, ρ, and $\hat{g}_{\mu\nu}$ denotes a maximally symmetric Minkowski-signature metric with dimension $n-2$.

We now use these ansätze to find the corresponding field equations for our geometry using (2.4)-(2.6). To do this we need to dimensionally reduce the full Ricci tensor into bulk and brane quantities (see Appendix B). Doing this gives

$$
\begin{aligned}
\mathcal{R}_{\mu\nu} &= \left[\frac{1}{n-2}\hat{R} + (n-2)e^{2W}W'' + e^{2W}(W')^2\right]\hat{g}_{\mu\nu}, \\
\mathcal{R}_{\theta\theta} &= \left[B'' + (B')^2 + (n-2)W'B'\right]e^{2B}, \\
\mathcal{R}_{\rho\rho} &= (n-2)\left[W'' + (W')^2\right] + B'' + (B')^2,
\end{aligned}
\tag{2.12}
$$

where we've used $\hat{R}_{\mu\nu} = \frac{1}{n-2}\hat{R}\,\hat{g}_{\mu\nu}$ since $\hat{g}_{\mu\nu}$ is maximally symmetric with dimension $n-2$. The Einstein equations subject to this ansatz reduce to

$$\frac{1}{n-2}e^{-2W}\hat{R} + W'' + (n-2)(W')^2 + W'B' - \frac{1}{n-2}\kappa^2 e^{-2B}f(A_\theta')^2 + \frac{2\kappa^2 V}{n-2} = 0, \quad (\mu\nu) \tag{2.13}$$

$$B'' + (B')^2 + (n-2)W'B' + \frac{n-3}{n-2}\kappa^2 e^{-2B}f(A_\theta')^2 + \frac{2\kappa^2 V}{n-2} = 0, \quad (\theta\theta) \tag{2.14}$$

$$(n-2)[W'' + (W')^2] + B'' + (B')^2 + (\phi')^2 + \frac{n-3}{n-2}\kappa^2 e^{-2B}f(A_\theta')^2 + \frac{2\kappa^2 V}{n-2} = 0, \quad (\rho\rho) \tag{2.15}$$

while the dilaton and Maxwell equations become

$$\phi'' + [B' + (n-2)W']\phi' - \kappa^2\left[\frac{\partial V}{\partial\phi} + \frac{1}{2}\frac{\partial f}{\partial\phi}e^{-2B}(A_\theta')^2\right] = 0, \tag{2.16}$$

and

$$\left(e^{-B+(n-2)W}fA_\theta'\right)' = 0, \tag{2.17}$$

where prime denotes differentiation with respect to ρ, and $\hat{R}$ is the Ricci scalar constructed from $\hat{g}_{\mu\nu}$.

In some solutions to these field equations, there exist singularities in the extra dimensional metric. These singularities are interpreted as the position of branes (a lower-dimensional surface on which all standard model particles are trapped) which source the curvature of the bulk. In the cases we consider near the end of this chapter, there are two codimension-2 branes (meaning the dimension of the brane is $n-2$) located at positions of constant ρ, ρ_s with $s = 0, 1$. The back-reaction of such branes on their environment can strongly influence their low-energy properties [11], so in order to analyze the classical low energy on-brane effective action we must first study how branes back-react on their surroundings.

2.3 Boundary Conditions for Branes

The back-reaction of the codimension-2 branes sourcing the curvature of the extra dimensions is determined by a set of equations known as junction conditions. The junction conditions are the result of minimizing the boundary term arising from the variation of the full action plus the brane action at the location of the brane. They relate the functional derivatives of the codimension-2 brane action, $S_b^{(2)}$, to the bulk fields, such as ϕ, at the brane positions, ρ_s.

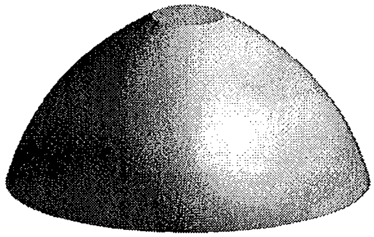

Figure 2.1: The regularized near-brane cap geometry

A problem with this approach to determining the back-reaction of the branes is that the bulk fields need not be well-defined at the brane positions (it is possible that they may diverge), in the same way that the electric field due to a point charge diverges at the location of the charge which sources the field (in 3+1 dimensions) [12]. This makes it very difficult to extract any useful information relating the bulk fields at the brane position to the brane properties. To avoid this problem we introduce a renormalization procedure in which we replace the codimension-2 branes located at positions ρ_s with very small cylindrical codimension-1 branes, $S_b^{(1)}$, situated at positions ρ_b $(b = 0, 1)$ which are a small distance, Δ, away from ρ_s. This off-set by a small distance Δ is equivalent to using the radius of the cylinder as the small parameter. We also consider exactly two branes since due to axial symmetry in the bulk, if there exist singularities in the bulk, there must be exactly two of them. Thus we have placed a codimension-1 brane at $\rho_b = \rho_s + \Delta$ with the interior geometry $(\rho < \rho_b)$ capped off with a smooth solution to the bulk field equations given by

$$ds^2_{\text{flat}} = d\rho^2 + \rho^2 d\theta^2 + e^{2W_{\text{flat}}} \hat{g}_{\mu\nu} dx^\mu dx^\nu, \tag{2.18}$$

where the W_{flat} does not depend on ρ. With this renormalization procedure we can now write down junction conditions for the codimension-1 brane, $S_b^{(1)}$, and relate the bulk and brane quantities at the brane positions, ρ_b, without fear of dealing with divergences.

2.3.1 Codimension-1 Branes

In this section we compute the junction conditions for codimension-1 branes which relate bulk and brane properties at the codimension-1 brane positions, ρ_b. Studying the junction conditions of codimension-1 branes serves as motivation to generalizing to the study of the back-reaction codimension-2 branes.

Junction Conditions

The junction conditions for a codimension-1 brane, which are derived in detail in Appendix C, are

$$\frac{1}{2\kappa^2}\left[\sqrt{-\bar{g}}\,(K^{mn} - K\bar{g}^{mn})\right]_{\rho_b} + \frac{\delta S_b^{(1)}}{\delta \bar{g}_{mn}} = 0,$$

$$\frac{1}{\kappa^2}\left[\sqrt{-\bar{g}}\partial_\rho\phi\right]_{\rho_b}+\frac{\delta S_b^{(1)}}{\delta\phi}=0,$$
$$\left[\sqrt{-\bar{g}}fF^{\rho M}\right]_{\rho_b}+\frac{\delta S_b^{(1)}}{\delta A_M}=0, \tag{2.19}$$

where $\bar{g}$ is the induced metric on the codimension-1 brane, K is the extrinsic curvature, and the notation $[F]_{\rho_b}$ for a bulk quantity, F, denotes

$$[F]_{\rho_b}=\lim_{\epsilon\to0}\left[F(\rho_b+\epsilon)-F(\rho_b-\epsilon)\right]. \tag{2.20}$$

For any metric of the form $ds^2=d\rho^2+\bar{g}_{mn}dx^m dx^n$ the extrinsic curvature is $K_{mn}=\frac{1}{2}\partial_\rho\bar{g}_{mn}$ (see Appendix C), and in particular, for our ansatz, (2.11), it can be shown that

$$K_{\mu\nu}=e^{2W}W'\hat{g}_{\mu\nu},\qquad K_{\theta\theta}=e^{2B}B',\qquad K=B'+(n-2)W', \tag{2.21}$$

where $K=\bar{g}^{mn}K_{mn}$. Plugging these into our junction conditions, (2.19), and making use of the definition of $[F]_{\rho_b}$, (2.20), as well as the flat geometry, (2.18), we get

$$\frac{1}{\kappa^2}e^{B+(n-2)W}\sqrt{-\hat{g}}\phi'+\frac{\delta S_b^{(1)}}{\delta\phi}=0,$$
$$\sqrt{-\hat{g}}e^{-B+(n-2)W}fA'_\theta+\frac{\delta S_b^{(1)}}{\delta A_\theta}=0,$$
$$\frac{1}{2\kappa^2}\sqrt{-\hat{g}}e^{B+(n-2)W}g^{\theta\theta}((n-2)W')-\frac{\delta S_b^{(1)}}{\delta g_{\theta\theta}}=0,$$
$$\frac{1}{2\kappa^2}\sqrt{-\hat{g}}e^{(n-2)W}\hat{g}^{\mu\nu}[e^B((n-3)W'+B')-1]-\frac{\delta S_b^{(1)}}{\delta\hat{g}_{\mu\nu}}=0, \tag{2.22}$$

where it is now understood that all the fields are evaluated at the brane positions, ρ_b. These junction conditions can be recast in terms of the brane quantities $\mathcal{S}_\phi^{(1)},\mathcal{S}_{A_\theta}^{(1)},\mathcal{S}_\theta^{(1)},\mathcal{S}_g^{(1)}$, as follows

$$e^B\phi'=e^{-(n-2)W}\mathcal{S}_\phi^{(1)},$$
$$\kappa A'_\theta=e^{-(n-2)W+B}\frac{\mathcal{S}_{A_\theta}^{(1)}}{f},$$
$$e^BW'=e^{-(n-2)W}\mathcal{S}_\theta^{(1)},$$
$$e^BB'-1=-e^{-(n-2)W}\left[\mathcal{S}_g^{(1)}+(n-3)\mathcal{S}_\theta^{(1)}\right], \tag{2.23}$$

where we have made the following definitions

$$\begin{aligned}
\mathcal{S}_{\phi}^{(1)} &= -\kappa^2 \frac{1}{\sqrt{-\hat{g}}} \frac{\delta S_b^{(1)}}{\delta \phi}, \\
\mathcal{S}_{A_\theta}^{(1)} &= -\kappa \frac{1}{\sqrt{-\hat{g}}} \frac{\delta S_b^{(1)}}{\delta A_\theta}, \\
\mathcal{S}_{\theta}^{(1)} &= 2\kappa^2 \frac{g_{\theta\theta}}{n-2} \frac{1}{\sqrt{-\hat{g}}} \frac{\delta S_b^{(1)}}{\delta g_{\theta\theta}}, \\
\mathcal{S}_{g}^{(1)} &= -2\kappa^2 \frac{\hat{g}_{\mu\nu}}{n-2} \frac{1}{\sqrt{-\hat{g}}} \frac{\delta S_b^{(1)}}{\delta \hat{g}_{\mu\nu}}.
\end{aligned} \tag{2.24}$$

In the case of a pure tension brane, $S_b^{(1)} = -\int_{\rho_b} d^{n-1}x\sqrt{-\hat{g}}\,T_b$, we have clear physical interpretations of $\mathcal{S}_{\phi}^{(1)}, \mathcal{S}_{A_\theta}^{(1)}, \mathcal{S}_{\theta}^{(1)}, \mathcal{S}_{g}^{(1)}$. $\mathcal{S}_{g}^{(1)}$ represents the brane tension, T_b, and $\mathcal{S}_{\phi}^{(1)}$ represents the derivative of the brane tension, $\partial_\phi T_b$. $\mathcal{S}_{\theta}^{(1)}$ is related to the brane contribution to the scalar potential within the low-energy on brane effective theory, while $\mathcal{S}_{A_\theta}^{(1)}$ is interpreted as describing the microscopic axial currents within the brane, or equivalently any microscopic magnetic flux these currents encloses within the brane. These interpretations are clear once the low-energy on-brane effective action is studied, but these interpretations are also used as part of our motivation to generalizing to codimension-2 branes.

Another motivating feature of codimension-1 branes that is used to generalize to codimension-2 branes is an important constraint relating the quantities $\mathcal{S}_{\phi}^{(1)}, \mathcal{S}_{A_\theta}^{(1)}, \mathcal{S}_{\theta}^{(1)}, \mathcal{S}_{g}^{(1)}$, to each other known as the brane constraint. This constraint is obtained by combining the junction conditions, (2.24), with the bulk equations of motion, (2.13)-(2.17).

The Brane Constraint

It should be pointed out that the equations of motion, (2.13)-(2.17), consist of five differential equations for only four unknown functions. Naively this might seem as though the problem is overconstrained and not well defined, however it turns out that these equations are not all independent from one another. One way to see this is to derive an explicit constraint equation. One such constraint arises taking a linear combination of the field equations such that all second derivative terms, ∂_ρ^2, of the bulk fields are eliminated, and can be thought of as a constraint on the evolution of the fields in the ρ direction. The relevant combination of the field equations that give rise to this constraint is $(n-2)(\mu\nu) + (\theta\theta) - (\rho\rho)$ and results in[2]

$$(n-3)(n-2)(W')^2 + 2(n-2)W'B' - (\phi')^2 - \kappa^2 e^{-2B} f(A_\theta')^2 + e^{-2W}\hat{R} + 2\kappa^2 V = 0. \tag{2.25}$$

In order to turn this into a constraint on the brane properties we multiply (2.25) through by

[2] This equation follows from the $(\rho\rho)$ Einstein equation written in terms of the Einstein tensor, $G_{\rho\rho} + \kappa^2 T_{\rho\rho} = 0$, and once the constraint is imposed on initial conditions at $\rho = \rho_0$, the Bianchi identity ensures that it holds for all ρ.

$e^{2B+2(n-2)W}$, take the limit $\rho \to \rho_b$ and trade W', B', ϕ', A'_θ for $\mathcal{S}_\phi^{(1)}$, $\mathcal{S}_\theta^{(1)}$, $\mathcal{S}_g^{(1)}$, $\mathcal{S}_{A_\theta}^{(1)}$ using the junction conditions, (2.24), giving

$$(n-3)(n-2)\left(\mathcal{S}_\theta^{(1)}\right)^2 - 2(n-2)\left[e^{(n-2)W} - \mathcal{S}_g^{(1)}\right]\mathcal{S}_\theta^{(1)} + \left(\mathcal{S}_\phi^{(1)}\right)^2 + \frac{e^{2B}}{f}\left(\mathcal{S}_{A_\theta}^{(1)}\right)^2 - e^{2B+2(n-2)W}\left[e^{-2W}\hat{R} + 2\kappa^2 V\right] = 0. \tag{2.26}$$

This constraint plays an important role in generalizing the codimension-1 formulation to a codimension-2 formulation, which we now turn to.

2.3.2 Codimension-2 Branes

Recall that the singularity positions, ρ_s, in the extra dimensions are interpreted as the location of codimension-2 branes which source the curvature of the bulk. However, the possibility of bulk fields diverging at these positions makes it hard to gain useful relationships between bulk and brane properties. So we instead resort to a renormalization scheme in which we replace the codimension-2 branes with codimension-1 branes a small distance, Δ, away from the singularity positions, ρ_b, and use this setup to derive the junction conditions. In this section we use the codimension-1 formalism to guide us in generalizing and defining junction conditions for codimension-2 branes.

Motivation

One approach to constructing a codimension-2 brane action from a codimension-1 brane action is to simply integrate out θ, the angular coordinate of the extra dimensions. Since we have assumed that the extra dimensions are axially symmetric, $S_b^{(1)}$ cannot explicitly depend on θ, and we get

$$S_b^{(2)} = \oint_{\rho_b} d\theta S_b^{(1)} = 2\pi S_b^{(1)}. \tag{2.27}$$

We can then recast our junction conditions in terms of codimension-2 quantities by integrating out θ, for example we can generalize the scalar junction condition as follows,

$$\begin{aligned} \oint_{\rho_b} d\theta e^B \phi' &= \oint_{\rho_b} d\theta e^{-(n-2)W}\mathcal{S}_\phi^{(1)} \\ e^B\phi' &= e^{-(n-2)W}\mathcal{S}_\phi^{(2)} \end{aligned} \tag{2.28}$$

where

$$\mathcal{S}_\phi^{(2)} = -\frac{\kappa^2}{2\pi}\frac{1}{\sqrt{-\hat{g}}}\frac{\delta S_b^{(2)}}{\delta\phi}. \tag{2.29}$$

Doing this for all our junction conditions results in

$$
\begin{aligned}
e^B \phi' &= e^{-(n-2)W} \mathcal{S}_\phi^{(2)}, \\
\kappa A_\theta' &= e^{-(n-2)W+B} \frac{\mathcal{S}_{A_\theta}^{(2)}}{f}, \\
e^B W' &= e^{-(n-2)W} \mathcal{S}_\theta^{(2)}, \\
e^B B' - 1 &= -e^{(n-2)W} \left[\mathcal{S}_g^{(2)} + (n-3)\mathcal{S}_\theta^{(2)} \right],
\end{aligned} \tag{2.30}
$$

where

$$\mathcal{S}_\phi^{(2)} = -\frac{\kappa^2}{2\pi} \frac{1}{\sqrt{-\hat{g}}} \frac{\delta S_b^{(2)}}{\delta \phi}, \tag{2.31}$$

$$\mathcal{S}_{A_\theta}^{(2)} = -\frac{\kappa}{2\pi} \frac{1}{\sqrt{-\hat{g}}} \frac{\delta S_b^{(2)}}{\delta A_\theta}, \tag{2.32}$$

$$\mathcal{S}_\theta^{(2)} = \frac{\kappa^2}{\pi} \frac{g_{\theta\theta}}{n-2} \frac{1}{\sqrt{-\hat{g}}} \frac{\delta S_b^{(2)}}{\delta g_{\theta\theta}}, \tag{2.33}$$

$$\mathcal{S}_g^{(2)} = -\frac{\kappa^2}{\pi} \frac{\hat{g}_{\mu\nu}}{n-2} \frac{1}{\sqrt{-\hat{g}}} \frac{\delta S_b^{(2)}}{\delta \hat{g}_{\mu\nu}}. \tag{2.34}$$

With these new codimension-2 quantities, the brane constraint becomes

$$
\begin{aligned}
(n-3)(n-2) \left(\mathcal{S}_\theta^{(2)} \right)^2 - 2(n-2) \left[e^{(n-2)W} - \mathcal{S}_g^{(2)} \right] \mathcal{S}_\theta^{(2)} + \left(\mathcal{S}_\phi^{(2)} \right)^2 \\
+ \frac{e^{2B}}{f} \left(\mathcal{S}_{A_\theta}^{(2)} \right)^2 - e^{2B+2(n-2)W} \left[e^{-2W} \hat{R} + 2\kappa^2 V \right] = 0.
\end{aligned} \tag{2.35}
$$

Generalization

The problem with the above method of generalizing to a codimension-2 brane is that an honest to god codimension-2 brane action, $S_b^{(2)}$, can only depend on fields which live on the brane, and thus cannot depend explicitly on $g_{\theta\theta}$ or A_θ. So varying the brane action with respect to either of these quantities no longer makes any sense, and in particular the definitions for $\mathcal{S}_{A_\theta}^{(2)}$ and $\mathcal{S}_\theta^{(2)}$, (2.32) and (2.33), are not well defined. All is not lost though, and we still use the above method as a motivation to guide our generalization for codimension-2 branes.

We generalize to a codimension-2 brane by keeping the form of our junction conditions, (2.30), and the definitions of brane properties that still make sense (ie, that are defined in terms of fields that live on the brane), which are $\mathcal{S}_\phi^{(2)}$ and $\mathcal{S}_g^{(2)}$, (2.31) and (2.34).

The two quantities that now need to be dealt with and redefined are $\mathcal{S}_{A_\theta}^{(2)}$ and $\mathcal{S}_\theta^{(2)}$. Instead of $\mathcal{S}_{A_\theta}^{(2)}$ being defined as in (2.32), we redefine it to be a tunable parameter describing any microscopic magnetic flux which enclose the brane, based on its physical interpretation discussed at the end of the codimension-1 section. To generalize $\mathcal{S}_\theta^{(2)}$ we let the brane constraint, (2.35), be its defining equation.

In summary, for a codimension-2 brane action, $S_b^{(2)}$, the junction conditions are

$$\begin{aligned}
e^B \phi' &= e^{-(n-2)W} \mathcal{S}_\phi^{(2)}, \\
\kappa A_\theta' &= e^{-(n-2)W+B} \frac{\mathcal{S}_{A_\theta}^{(2)}}{f}, \\
e^B W' &= e^{-(n-2)W} \mathcal{S}_\theta^{(2)}, \\
e^B B' - 1 &= -e^{(n-2)W} \left[\mathcal{S}_g^{(2)} + (n-3) \mathcal{S}_\theta^{(2)} \right],
\end{aligned} \tag{2.36}$$

where $\mathcal{S}_{A_\theta}^{(2)}$ is a tunable paramater describing any microscopic magnetic flux which enclose the brane,

$$\mathcal{S}_\phi^{(2)} = -\frac{\kappa^2}{2\pi} \frac{1}{\sqrt{-\hat{g}}} \frac{\delta S_b^{(2)}}{\delta \phi}, \tag{2.37}$$

$$\mathcal{S}_g^{(2)} = -\frac{\kappa^2}{\pi} \frac{\hat{g}_{\mu\nu}}{n-2} \frac{1}{\sqrt{-\hat{g}}} \frac{\delta S_b^{(2)}}{\delta \hat{g}_{\mu\nu}}, \tag{2.38}$$

$$\begin{aligned}
\mathcal{S}_\theta^{(2)} = \frac{1}{n-3} \Bigg[& \left(e^{(n-2)W} - \mathcal{S}_g^{(2)} \right) \\
& \pm \sqrt{\left(e^{(n-2)W} - \mathcal{S}_g^{(2)} \right)^2 - \frac{(n-3)}{(n-2)} \left[\left(\mathcal{S}_\phi^{(2)} \right)^2 + \frac{e^{2B}}{f} \left(\mathcal{S}_{A_\theta}^{(2)} \right)^2 - e^{2B+2(n-2)W} (e^{-2W} \hat{R} + 2\kappa^2 V) \right]} \Bigg],
\end{aligned} \tag{2.39}$$

and the $\pm$ is chosen so that $\mathcal{S}_\theta^{(2)} \to 0$ when $\mathcal{S}_\phi^{(2)} \to 0$ and $e^{2B} \to 0$. These are the generalized versions of the codimension-1 junction conditions given by (2.23).

Now that we have our description of properties of a codimension-2 brane we are ready to study how the back-reaction of codimension-2 branes affects the classical low-energy on-brane effective action.

2.4 The Classical Low-Energy On-Brane Effective Action

When considering length scales which are much longer than the size of the extra dimensions, the system and its dynamics are essentially $(n-2)$-dimensional. The action describing the physics on such length scales is the low-energy on-brane effective action, S_{eff}, and is obtained from the full action by using the equations of motion to eliminate high energy fields, in favour of their low-energy counterparts, and subsequently integrating out the extra dimensions. From this perspective the low-energy on-brane effective action is composed of the $(n-2)$-dimensional fields which survive at low energies and an effective potential, V_{eff}. The possible excitations of fields surviving at low energies are $\hat{g}_{\mu\nu}$, $g_{\mu\theta}$, $g_{\rho\rho}$, A_μ, and ϕ. The only low energy field we consider is due to $\hat{g}_{\mu\nu}$, since the others break maximal symmetry. We now calculate S_{eff} and determine V_{eff}.

To eliminate the heavy fields we make use of the (ab) Einstein equations, (2.4). These comprise of two independent equations, which we take to be the sum and difference of the $(\rho\rho)$ and $(\theta\theta)$ components. The difference, (2.15)-(2.14), gives

$$(n-2)\left(W'' + (W')^2 - W'B'\right) + (\phi')^2 = 0, \tag{2.40}$$

while the sum is equivalent to contracting the (ab) Einstein equations with the extra-dimensional metric, h^{ab}, to give

$$\frac{1}{2\kappa^2}h^{ab}\left(\mathcal{R}_{ab} + \partial_a\phi\partial_b\phi\right) = -\frac{n-3}{2(n-2)}fF^{ab}F_{ab} - \frac{2}{n-2}V. \tag{2.41}$$

We begin by explicitly separating the on-brane metric from the extra-dimensional metric in the bulk action, using (2.12),

$$\begin{aligned}
S_B &= -\int d^n x\sqrt{-g}\left[\frac{1}{2\kappa^2}g^{MN}\left(\mathcal{R}_{MN} + \partial_M\phi\partial_N\phi\right) + \frac{1}{4}f(\phi)F_{MN}F^{MN} + V(\phi)\right] \\
&= -\int d^n x\, e^{B+(n-2)W}\sqrt{-\hat{g}}\left[\frac{1}{2\kappa^2}\left(e^{-2W}\hat{g}^{\mu\nu}\hat{R}_{\mu\nu} + h^{ab}\left(\mathcal{R}_{ab} + \partial_a\phi\partial_b\phi\right)\right)\right. \\
&\qquad\qquad \left. + \frac{1}{2\kappa^2}(n-2)\left(W'' + (n-2)(W')^2 W'B'\right) + \frac{1}{4}fF_{ab}F^{ab} + V\right] \\
&= -\int d^n x\, e^{B+(n-2)W}\sqrt{-\hat{g}}\left[\frac{1}{2\kappa^2}e^{-2W_1}\hat{g}^{\mu\nu}\hat{R}_{\mu\nu} + \frac{1}{2\kappa^2}(n-2)\left(W'' + (n-2)(W')^2 W'B'\right)\right. \\
&\qquad\qquad \left. - \frac{n-4}{4(n-2)}fF_{ab}F^{ab} + \frac{n-4}{n-2}V\right] \\
&= -\int d^n x\, e^{B+(n-2)W}\sqrt{-\hat{g}}\left[\frac{1}{2\kappa^2}e^{-2W_1}\hat{g}^{\mu\nu}\hat{R}_{\mu\nu} + \frac{1}{2\kappa^2}(n-2)\left((n-3)(W')^2 + 2W'B'\right)\right. \\
&\qquad\qquad \left. -(\phi')^2 - \frac{n-4}{4(n-2)}fF_{ab}F^{ab} + \frac{n-4}{n-2}V\right],
\end{aligned} \tag{2.42}$$

where we have used both (2.40) and (2.41) in the above. The effective bulk action describing the low energy physics in the bulk is obtained by integrating out the extra dimensions, which yields

$$S_{B_{\text{eff}}} = \int d^2x\, S_B = -\int d^{n-2}x\sqrt{-\hat{g}}\left[\frac{1}{2\kappa_N^2}\hat{g}^{\mu\nu}\hat{R}_{\mu\nu} + V_B\right], \tag{2.43}$$

where $\kappa_N^2 = 8\pi G_N$ is the lower dimensional Newton's constant given by

$$\frac{1}{\kappa_N^2} = \frac{1}{\kappa^2}\int d^2x\, e^{B+(n-4)W}, \tag{2.44}$$

and V_B is the bulk potential given by

$$V_B = \int d^2x\, e^{B+(n-2)W} \left[\frac{1}{2\kappa^2}(n-2)\left((n-3)(W')^2 + 2W'B' - (\phi')^2\right) - \frac{n-4}{4(n-2)} f F_{ab}F^{ab} + \frac{n-4}{n-2}V\right]. \quad (2.45)$$

To obtain the full low-energy on-brane effective action we also need to include the boundary contributions due to the Gibbons-Hawking action, S_{GH}, as well as the brane action, $S_b^{(2)}$, itself. With two branes located at positions ρ_b, $b = \{0, 1\}$, the Gibbons-Hawking action is,

$$\begin{aligned} S_{GH} &= \sum_{b=0}^{1} \int_{\rho_b} d\theta d^{n-2}x \frac{1}{\kappa^2}\sqrt{-g}K \\ &= \frac{2\pi}{\kappa^2}\sum_{b=0}^{1}(-)^b \int_{\rho_b} d^{n-2}x\sqrt{-\hat{g}}e^{B+(n-2)W}\left[B' + (n-2)W'\right] \\ &= -\frac{2\pi}{\kappa^2}\sum_{b=0}^{1}\int_{\rho_b} d^{n-2}x\sqrt{-\hat{g}}\left[\mathcal{S}_\theta^{(2)} - \mathcal{S}_g^{(2)}\right]. \end{aligned} \quad (2.46)$$

Combining (2.43), (2.46) and the brane action,

$$S_b^{(2)} = \int d^{n-2}x\sqrt{-\hat{g}}\mathcal{L}_b^{(2)}, \quad (2.47)$$

yields the full low-energy on-brane effective action, S_{eff}, which is given by

$$S_{\text{eff}} = -\int d^{n-2}x\,\sqrt{-\hat{g}}\left[\frac{1}{2\kappa_N^2}\hat{g}^{\mu\nu}\hat{R}_{\mu\nu} + V_{\text{eff}}\right], \quad (2.48)$$

where

$$V_{\text{eff}} = V_B - \frac{2\pi}{\kappa^2}\sum_{b=0}^{1}\left[\mathcal{S}_\theta^{(2)} - \mathcal{S}_g^{(2)} - \frac{\kappa^2}{2\pi}\mathcal{L}_b^{(2)}\right]. \quad (2.49)$$

This provides us with a complete description of the low-energy on-brane theory. However, there are many cases in which we are only concerned with finding the value of the effective potential at its minimum, $V_{\text{eff}}(\phi_0)$, and we are not concerned with perturbations about it. In these cases instead of substituting only the extra-dimensional Einstein equations into the bulk action, we substitute all of the Einstein equations in, that is we evaluate the action at a classical solution. The result of doing this is

$$S_B(g^c_{MN}, \phi^c, A^c_M) = -\frac{2}{n-2}\int d^nx\sqrt{-g}\left[\frac{1}{4}fF^{ab}F_{ab} - V\right], \quad (2.50)$$

and so

$$V_B(\phi_0) = -\frac{2}{n-2} \int d^2x e^{B+(n-2)W} \left[\frac{1}{4} f F^{mn} F_{mn} - V\right]. \tag{2.51}$$

Related to this is the fact that whenever the bulk action enjoys a classical scaling symmetry, the bulk contribution to the effective potential can be written as a sum of contributions localized at the position of each brane, in the same way that was done for he Gibbons-Hawking term. More specifically, when the bulk action, $S_B = \int d^n x \mathcal{L}_B$, has the property that

$$\mathcal{L}_B[\lambda^{p_i}\psi_i] \equiv \lambda \mathcal{L}_B[\psi_i], \tag{2.52}$$

for arbitrary real constant λ, it can be shown that (by taking the derivative of (2.52) with respect to λ)

$$\begin{aligned} \mathcal{L}_B &= \sum_i p_i \left[\psi_i \frac{\partial \mathcal{L}_B}{\partial \psi_i} + \partial_M \frac{\partial \mathcal{L}_B}{\partial(\partial_M \psi_i)}\right] \\ &= \sum_i \left\{\partial_M \left[p_i \frac{\partial \mathcal{L}_B}{\partial(\partial_M \psi_i)}\right] + p_i \psi_i \left[\frac{\partial \mathcal{L}_B}{\partial \psi_i} - \partial_M \left(\frac{\partial \mathcal{L}_B}{\partial(\partial_M \psi_i)}\right)\right]\right\}, \end{aligned} \tag{2.53}$$

which shows that the action becomes a total derivative whenever it is evaluated at any classical solution. Whenever this is true the entire low-energy potential can be interpreted as the sum over brane contributions.

We now have a complete description of the low-energy on-brane theory and proceed to apply it to two examples.

2.5 Examples

The above formalism is now tested by applying it to situations in which explicit solutions are known for the higher-dimensional theory. In particular we study compactifications to four dimensions of supersymmetric and nonsupersymmetric six-dimensional theories. In both examples we start off by studying the bulk solutions to the higher-dimensional theory, which is then used to relate the brane and bulk properties near the brane using the junction conditions. The examples are then concluded by demonstrating that the low-energy on-brane theory exactly agrees with the higher-dimensional theory. These examples follow very closely the analysis done in [11].

2.5.1 Brane-Axion Couplings in 6D

Our first example is a nonsupersymmetric theory describing two branes coupled to a bulk Goldstone mode (axion), ϕ, in six dimensions. Its bulk action is given by

$$S_B = -\int_{\mathcal{M}} d^6x\sqrt{-g}\left[\frac{1}{2\kappa^2}g^{MN}\left(\mathcal{R}_{MN} + \partial_M\phi\partial_N\phi\right) + \frac{1}{4}F_{MN}F^{MN} + \Lambda\right], \tag{2.54}$$

which is (2.2) with $n = 6$, $f(\phi) = 1$, and $V(\phi) = \Lambda$. These choices ensure the action has a shift symmetry, $\phi \to \phi+\xi$, that guarantees the existence of a scalar KK zero mode having a constant profile across the bulk. This is the only such classically massless scalar KK mode, because the presence of the bulk cosmological term, Λ, breaks the rigid scaling symmetry that the Einstein action normally has.

Bulk Solutions

A simple solution to the field equations of (2.54) subject to the ansatz (2.11) can be shown to be (see Appendix D)

$$\begin{aligned} ds^2 &= d\rho^2 + \alpha^2L^2\sin^2\left(\frac{\rho}{L}\right)d\theta^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu, \\ F_{\rho\theta} &= \alpha\mathcal{B}_0 L\sin\left(\frac{\rho}{L}\right), \qquad \phi = \phi_0, \text{ a constant,} \end{aligned} \tag{2.55}$$

where $\mathcal{B}_0$, L, Λ are constants related by

$$\mathcal{R}_{(2)} = -\frac{2}{L^2} = -\kappa^2\left(\frac{3\mathcal{B}_0^2}{2} + \Lambda\right), \tag{2.56}$$

and the curvature of the on-brane metric is

$$\hat{R} = 2\kappa^2\left(\frac{\mathcal{B}_0^2}{2} - \Lambda\right). \tag{2.57}$$

When $\alpha = 1$ the extra-dimensional metric describes a sphere or radius L. When $\alpha \neq 1$ the geometry would still look like a sphere if we redefine $\theta \to \alpha\theta$, although θ is then not periodic with period 2π. This indicates there are conical singularities (by definition) at both $\rho = 0$ and $\rho = \pi L$, with defect angle given by $\delta = 2\pi(1-\alpha)$. Thus by our renormalization prescription, we place two branes at positions $\rho_1 = \Delta$ and $\rho_2 = \pi L - \Delta$. However before we consider branes and junction conditions, we first show that we can solve for $\mathcal{B}_0$ and L in terms of α and Λ by considering the flux quantization condition [13].

To do this we first consider the gauge potential corresponding to the Maxwell field strength, which is

$$A^{\pm} = \alpha \mathcal{B}_0 L^2 \left[\pm 1 - \cos\left(\frac{\rho}{L}\right)\right] d\theta, \tag{2.58}$$

where the $\pm$ sign indicates the solution for the northern or southern hemisphere, since A must vanish at the corresponding pole. The difference between these two solutions near the equator must be a gauge transformation, $gA^+ - gA^- = d\Omega$, and so

$$\Omega(\theta) = 2\alpha g \mathcal{B}_0 L^2 \theta, \tag{2.59}$$

where g is the gauge coupling corresponding to the field A (corresponding to $D_\mu\psi = \partial_\mu\psi - igA_\mu\psi$). Requiring the gauge transformation $\psi \to e^{i\Omega}\psi$ be single valued requires $\Omega(\theta + 2\pi) = \Omega(\theta) + 2\pi n$, for integer n. This implies the constants $\mathcal{B}_0$ and L must be related by

$$g\mathcal{B}_0 = \frac{n}{2\alpha L^2}. \tag{2.60}$$

The equations (2.56) and (2.60) determine the constants $\mathcal{B}_0$ and L in terms of α and Λ, with solutions

$$\frac{1}{L^2} = \frac{8\alpha^2 g^2}{3n^2\kappa^2}\left[1 \pm \sqrt{1 - \left(\frac{3n^2\kappa^4\Lambda}{8\alpha^2 g^2}\right)}\right] \tag{2.61}$$

and

$$\mathcal{B}_0 = \frac{n}{2\alpha g L^2} = \frac{4\alpha g}{3n\kappa^2}\left[1 \pm \sqrt{1 - \left(\frac{3n^2\kappa^4\Lambda}{8\alpha^2 g^2}\right)}\right]. \tag{2.62}$$

With these solutions, and using (2.57), the on-brane curvature, $\hat{R}$, can be computed in terms of α and Λ. However α is also an arbitrary constant of integration which needs to be solved for, and to accomplish this we turn to our junction conditions applied to our solution.

Brane Properties

The solution (2.55) states

$$\phi' = 0, \qquad W = 0, \qquad A'_\theta = \alpha \mathcal{B}_0 L \sin\left(\frac{\rho}{L}\right), \qquad e^B B' = \alpha\cos\left(\frac{\rho}{L}\right), \tag{2.63}$$

which implies the junction conditions, (2.36), become

$$\mathcal{S}^{(2)}_\phi = 0,$$
$$\mathcal{S}^{(2)}_{A_\theta} = \kappa\mathcal{B}_0,$$

$$\mathcal{S}_\theta^{(2)} = 0,$$
$$\mathcal{S}_g^{(2)} = 1 - \alpha \cos\left(\frac{\rho_b}{L}\right). \tag{2.64}$$

To complete this picture we now need to choose an appropriate brane action. This allows us to solve for α and to see how these branes affect the $4D$ geometry. We consider a pure tension codimension-2 brane action given by

$$S_b^{(2)} = -\int d^4x \sqrt{-\hat{g}}\, T_b(\phi), \tag{2.65}$$

where T_b is a brane tension. Since the action has no terms which are of the order Δ^2, neither do $\mathcal{S}_\phi^{(2)}$ or $\mathcal{S}_g^{(2)}$, and so for consistency in our results we neglect any terms of this order in other quantities. Specifically this means the definition of $\mathcal{S}_\theta^{(2)}$, (2.39), reduces to (with $n = 6$ and $W = 0$)

$$\mathcal{S}_\theta^{(2)} = \frac{1}{3}\left[\left(1 - \mathcal{S}_g^{(2)}\right) \pm \sqrt{\left(1 - \mathcal{S}_g^{(2)}\right)^2 - \frac{3}{4}\left(\mathcal{S}_\phi^{(2)}\right)^2}\right], \tag{2.66}$$

since $e^B \to \Delta$ as $\rho \to \rho_b$. Using this brane action, (2.65), we calculate the various brane properties using the definitions (2.37), (2.38), and (2.66), which gives

$$\mathcal{S}_\phi^{(2)} = \frac{\kappa^2}{2\pi}\partial_\phi T_b,$$
$$\mathcal{S}_g^{(2)} = \frac{\kappa^2}{2\pi}T_b,$$
$$\mathcal{S}_\theta^{(2)} = \frac{1}{3}\left[\left(1 - \frac{\kappa^2}{2\pi}T_b\right) \pm \sqrt{\left(1 - \frac{\kappa^2}{2\pi}T_b\right)^2 - \frac{3}{4}\left(\frac{\kappa^2}{2\pi}\partial_\phi T_b\right)^2}\right]. \tag{2.67}$$

Combining these with the junction conditions, (2.64), results in

$$\kappa^2 T_b = 2\pi(1-\alpha),$$
$$\partial_\phi T_b = 0, \tag{2.68}$$

thus the junction conditions gives us the expression for the defect angle in terms of the brane tension. Here we've made use of the approximation $\alpha \cos\left(\frac{\rho}{L}\right) \to \alpha$ as $\rho/L \to \rho_b/L \ll 1$. The second relation states that in order for solutions to exist, the two tensions must both satisfy $T_b'(\phi_b) = 0$ at both branes. Also notice that the these two relationships, (2.68), necessarily imply that $\mathcal{S}_\theta^{(2)} = 0$, which is consistent with the junction condition for $\mathcal{S}_\theta^{(2)}$.

The 4D Perspective

With the bulk solutions and the relationships provided by the junction conditions we are now ready to study the low-energy on-brane effective action,

$$S_{\text{eff}} = -\int d^{n-2}x\,\sqrt{-\hat{g}}\left[\frac{1}{2\kappa_N^2}\hat{g}^{\mu\nu}\hat{R}_{\mu\nu} + V_{\text{eff}}\right], \tag{2.69}$$

and the first step in doing so is to calculate the $4D$ effective potential, V_{eff}, given by

$$V_{\text{eff}} = V_B - \frac{2\pi}{\kappa^2}\sum_{b=0}^{1}\left[\mathcal{S}_\theta^{(2)} - \mathcal{S}_g^{(2)} - \frac{\kappa^2}{2\pi}\mathcal{L}_b^{(2)}\right], \tag{2.70}$$

where

$$\begin{aligned} V_B = \int d^2x\, e^{B+(n-2)W}\Bigg[&\frac{1}{2\kappa^2}(n-2)\left((n-3)(W')^2 + 2W'B'\right) - (\phi')^2 \\ &-\frac{n-4}{4(n-2)}fF_{ab}F^{ab} + \frac{n-4}{n-2}V\Bigg]. \end{aligned} \tag{2.71}$$

Using $n=6$, $W=0$, $\phi'=0$, $f=1$, $V=\Lambda$, $F_{ab}F^{ab} = 2\mathcal{B}_0^2$, and $e^B = \alpha L\sin\left(\frac{\rho}{L}\right)$, the expression for V_B becomes

$$\begin{aligned} V_B &= -\int_0^{\pi L} d^2x\,\alpha L\sin\left(\frac{\rho}{L}\right)\left(\frac{1}{4}\mathcal{B}_0^2 - \frac{1}{2}\Lambda\right) \\ &= 2\pi\alpha L^2\left(\Lambda - \frac{\mathcal{B}_0^2}{2}\right). \end{aligned} \tag{2.72}$$

The other piece of V_{eff} is computed using the junction conditions, (2.64), and the brane properties, (2.67), yielding

$$\sum_{b=0}^{1}\left[\mathcal{S}_\theta^{(2)} - \mathcal{S}_g^{(2)} - \frac{\kappa^2}{2\pi}\mathcal{L}_b^{(2)}\right] = 0, \tag{2.73}$$

since $\mathcal{L}_b^{(2)} = -T_b$ and exactly cancels the contribution due to $\mathcal{S}_g^{(2)} = \frac{\kappa^2}{2\pi}T_b$, and $\mathcal{S}_\theta^{(2)} = 0$ from our junction conditions. Thus the effective action is given by

$$V_{\text{eff}} = 2\pi\alpha L^2\left(\Lambda - \frac{\mathcal{B}_0^2}{2}\right). \tag{2.74}$$

The on-brane curvature in the effective theory is determined by the equations of motion of (2.69) which are

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu} - \kappa_N^2 V_{\text{eff}}\,\hat{g}_{\mu\nu} = 0 \tag{2.75}$$

where the $4D$ Newton coupling is

$$\frac{1}{\kappa_N^2} = \frac{2\pi}{\kappa^2}\int_0^{\pi L} d\rho e^B = \frac{4\pi\alpha L^2}{\kappa^2}, \tag{2.76}$$

and so

$$\hat{R} = -4\kappa_N^2 V_{\rm eff} = 2\kappa^2\left(\frac{\mathcal{B}_0^2}{2} - \Lambda\right), \tag{2.77}$$

in agreement with the higher-dimensional result, (2.57). Using (2.61), (2.62), and (2.68), the on-brane curvature an be expressed in terms of the known quantities T_b and Λ.

2.5.2 6D Gauged Chiral Supergravity

The second example we consider is a supersymmetric theory, also in six dimensions, whose bulk action is given by

$$S_B = -\int_{\mathcal{M}} dx^6\sqrt{-g}\left[\frac{1}{2\kappa^2}g^{MN}\left(\mathcal{R}_{MN} + \partial_M\phi\partial_N\phi\right) + \frac{1}{4}e^{-\phi}F_{MN}F^{MN} + \frac{2g^2}{\kappa^4}e^{\phi}\right], \tag{2.78}$$

which is (2.2) with $n = 6$, $f = e^{-\phi}$ and $V = 2g^2e^{\phi}/\kappa^4$. Here g denotes the $6D$ gauge coupling constant for the Maxwell field. It can easily be shown that this lagrangian enjoys the scaling symmetry discussed at the end of section 2.4. Specifically it has the property that $\mathcal{L}_B \to \lambda^2\mathcal{L}_B$ when $e^{\phi} \to \lambda^{-1}e^{\phi}$ and $g_{MN} \to \lambda g_{MN}$. By the arguments discussed in section 2.4 the lagrangian becomes a total derivative when evaluated at any classical solution and is given by

$$\mathcal{L}_B(g^c_{MN}, A^c_M, \phi^c) = \frac{1}{2\kappa^2}\sqrt{-g^c}\Box\phi^c. \tag{2.79}$$

Bulk Solutions

For this particular example, instead of using the ansatz (2.11) as is done throughout this chapter until now, it is useful to choose a slightly different ansatz,

$$\begin{aligned} ds^2 &= a^2\left(\mathcal{W}^8 d\eta^2 + d\theta^2\right) + \mathcal{W}^2\hat{g}_{\mu\nu}dx^\mu dx^\nu, \\ \phi &= \phi(\eta), \\ A'_\theta(\eta) &= Qa^2e^{\phi}, \end{aligned} \tag{2.80}$$

where $a = a(\eta)$, $\mathcal{W} = \mathcal{W}(\eta)$, $Q = Q(\eta)$, are functions of η only, and $\hat{g}_{\mu\nu}$ is a maximally symmetric $4D$ metric with Ricci scalar $\hat{R}$. In this section we use the prime notation to denote differentiation with respect to η as opposed to previous sections where it denoted differentiation with respect to ρ. Although we are using these ansätze, we can still use our equations of motion and junction conditions, which are expressed in terms of ρ, by changing coordinates using $d\rho = a\mathcal{W}^4 d\eta$.

Plugging $A'_\theta(\eta)$ into the Maxwell equation, (2.17), yields $Q' = 0$, and hence Q must be a constant. The dilaton and Einstein equations subject to these ansätze become

$$\phi'' = \frac{2g^2}{\kappa^2} a^2 \mathcal{W}^8 e^{\phi} - \frac{\kappa^2 Q^2}{2} a^2 e^{\phi}, \qquad \text{(Dilaton)} \qquad (2.81)$$

and

$$\left(\frac{\mathcal{W}'}{\mathcal{W}} + \frac{1}{2}\phi'\right)' = -\frac{1}{4}\hat{R} a^2 \mathcal{W}^6, \qquad (\mu\nu) \qquad (2.82)$$

$$\left(\frac{a'}{a} + \frac{1}{2}\phi'\right)' = -\kappa^2 Q^2 a^2 e^{\phi}, \qquad (\theta\theta) \qquad (2.83)$$

$$\frac{1}{2}(\phi')^2 - \frac{4a'\mathcal{W}'}{a\mathcal{W}} - \frac{6(\mathcal{W}')^2}{\mathcal{W}^2} = \frac{2g^2}{\kappa^2} a^2 \mathcal{W}^8 e^{\phi} + \frac{1}{2}\hat{R} a^2 \mathcal{W}^6 - \frac{\kappa^2}{2} Q^2 a^2 e^{\phi}, \qquad (\eta\eta) \qquad (2.84)$$

where the $(\eta\eta)$ equation is the constraint equation.

The case $\hat{R} = 0$ provides us with the only known closed-form solutions to these equations, which are [14, 15]

$$\begin{aligned} e^{\phi} &= \mathcal{W}^{-2} e^{\varphi_0 - \lambda_3 \eta}, \\ \mathcal{W}^4 &= \left(\frac{\kappa^2 Q \lambda_2}{2g\lambda_1}\right) \frac{\cosh[\lambda_1(\eta - \eta_1)]}{\cosh[\lambda_2(\eta - \eta_2)]}, \\ \text{and} \quad a^{-4} &= \left(\frac{2g\kappa^2 Q^3}{\lambda_1^3 \lambda_2}\right) e^{2(\varphi_0 - \lambda_3 \eta)} \cosh^3[\lambda_1(\eta - \eta_1)] \cosh[\lambda_2(\eta - \eta_2)], \end{aligned} \qquad (2.85)$$

where η_i and λ_i are integration constants. Notice that these solutions diverge at $\eta = \pm\infty$ indicating that these are the locations of branes. Since the terms involving $\hat{R}$ in the general field equations, (2.81)-(2.84), become negligible in the near-brane limit, we can conclude that all solutions to these equations possess branes at $\eta = \pm\infty$.

In the near brane limit the field equations simplify to (since $a \to 0$ in these regions)

$$\phi'' \simeq \left(\frac{\mathcal{W}'}{\mathcal{W}}\right)' \simeq \left(\frac{a'}{a}\right)' \simeq 0. \qquad (2.86)$$

Letting $b = \{0, 1\}$ for the branes at $\eta = \{-\infty, +\infty\}$ respectively then gives the following solutions,

$$\begin{aligned} \phi &\simeq (-)^b q_b \eta, \\ \mathcal{W} &\simeq \mathcal{W}_b e^{(-)^b \omega_b \eta}, \\ a &\simeq a_b e^{(-)^b \alpha_b \eta}, \end{aligned} \qquad (2.87)$$

with different choices for the constants α_b, ω_b, and q_b applying for the two limits, $\eta \to \pm\infty$. For both asymptotic regions these are related by the constraint (2.84) so that

$$q_b^2 = 4\omega_b(2\alpha_b + 3\omega_b). \tag{2.88}$$

Notice that it is only consistent in the near-brane limit to ignore the quantities $a^2\mathcal{W}^6$, $a^2\mathcal{W}^8 e^{\phi}$, and $a^2\mathcal{W}^8$ in (2.81)-(2.83) if

$$2\alpha_b + 6\omega_b > 0,\ \ 2\alpha_b + q_b > 0,\ \text{ and } 2\alpha_b + 8\omega_b + q_b > 0. \tag{2.89}$$

The first of these guarantees that the $4D$ gravitational constant, (2.44), converges,

$$\frac{1}{\kappa_N^2} = \frac{1}{\kappa^2}\int d\theta d\rho\, e^{B+2W} = \frac{2\pi}{\kappa^2}\int_{-\infty}^{\infty} d\eta\, a^2\mathcal{W}^6. \tag{2.90}$$

Since we are interested in solutions where $a \to 0$ as $\eta \to \pm\infty$, and $a \simeq a_b e^{(-)^b \alpha_b \eta}$, we must demand that $\alpha_b > 0$. This ensures that the circumference of small circles encircling the branes vanish in the limit that the branes are approached. This combined with the constraint equation, (2.88), also implies that $\omega_b > 0$. To see this, assume that $\omega_b < 0$, then this implies that $-2\alpha_b - 3\omega_b > 0$, and adding this to the first equation of (2.89) gives $\omega_b > 0$, which contradicts the initial assumption. So in summary we have that

$$\alpha_b \geq 0, \quad \omega_b \geq 0, \quad 2\alpha_b + q_b > 0, \tag{2.91}$$

but if $\alpha_b = 0$, then $\omega_b q_b \neq 0$ and vice versa.

The Ricci scalar, $\hat{R}$, can be calculated by integrating both sides of (2.82) which yields

$$\begin{aligned} -\frac{1}{4}\hat{R}\int_{-\infty}^{\infty} a^2\mathcal{W}^6 &= -\frac{\kappa^2\hat{R}}{8\pi\kappa_N^2}, \\ \int_{-\infty}^{\infty}\left(\frac{\mathcal{W}'}{\mathcal{W}} + \frac{1}{2}\phi'\right)' = \left[\left(\ln\mathcal{W} + \frac{\phi}{2}\right)\right]_{\eta=-\infty}^{\eta=+\infty} &= -\sum_b\left(\frac{q_b}{2} + \omega_b\right), \end{aligned} \tag{2.92}$$

where we have made use of (2.90) and (2.87). Equating these two gives the Hubble constant,

$$\hat{R} = \frac{8\pi\kappa_N^2}{\kappa^2}\sum_b\left(\frac{q_b}{2} + \omega_b\right). \tag{2.93}$$

This completes the discussion of the bulk solutions, which we now relate to the brane properties using the junction conditions.

Brane Properties

Using $e^W = \mathcal{W}$, $a = e^B$, and $d\rho = a\mathcal{W}^4 d\eta$, the junction conditions, (2.36), become

$$\begin{aligned}
\mathcal{S}_\phi &= q_b, \\
\mathcal{S}_{A_\theta} &= \kappa Q, \\
\mathcal{S}_\theta &= \omega_b, \\
\mathcal{S}_g &= \mathcal{W}^4(\eta_b) - 3\omega_b - \alpha_b.
\end{aligned} \tag{2.94}$$

Now consider a pure tension codimension-2 brane action given by

$$S_b^{(2)} = -\int d^4x \sqrt{-\hat{g}}\, T_b(\phi). \tag{2.95}$$

Just as in the brane-axion coupling case, this action has no terms which are of order Δ^2 and so our definition of $\mathcal{S}_\theta^{(2)}$ becomes

$$\mathcal{S}_\theta^{(2)} = \frac{1}{3}\left[\left(e^{4W} - \mathcal{S}_g^{(2)}\right) \pm \sqrt{\left(e^{4W} - \mathcal{S}_g^{(2)}\right)^2 - \frac{3}{4}\left(\mathcal{S}_\phi^{(2)}\right)^2}\,\right]. \tag{2.96}$$

Using this brane action, (2.95), we calculate the various brane properties using the definitions (2.37), (2.38), and (2.96), which gives

$$\begin{aligned}
\mathcal{S}_\phi^{(2)} &= \frac{\kappa^2}{2\pi}\partial_\phi T_b, \\
\mathcal{S}_g^{(2)} &= \frac{\kappa^2}{2\pi}T_b, \\
\mathcal{S}_\theta^{(2)} &= \frac{1}{3}\left[\left(\mathcal{W}^4(\eta_b) - \frac{\kappa^2}{2\pi}T_b\right) \pm \sqrt{\left(\mathcal{W}^4(\eta_b) - \frac{\kappa^2}{2\pi}T_b\right)^2 - \frac{3}{4}\left(\frac{\kappa^2}{2\pi}\partial_\phi T_b\right)^2}\,\right].
\end{aligned} \tag{2.97}$$

Combining these with the junction conditions, (2.94), yields

$$\begin{aligned}
q_b &= \frac{\kappa^2}{2\pi}\frac{\partial T_b}{\partial\phi}, \\
3\omega_b + \alpha_b - \mathcal{W}^4 &= -\frac{\kappa^2}{2\pi}T_b, \\
\omega_b &= \frac{1}{3}\left[\left(\mathcal{W}^4(\eta_b) - \frac{\kappa^2}{2\pi}T_b\right) - \sqrt{\left(\mathcal{W}^4(\eta_b) - \frac{\kappa^2}{2\pi}T_b\right)^2 - \frac{3}{4}\left(\frac{\kappa^2}{2\pi}\partial_\phi T_b\right)^2}\,\right].
\end{aligned} \tag{2.98}$$

There are two cases here which we now consider, $\omega_b = 0$ and $\omega_b > 0$.

Case 1: $\omega_b = 0$

The constraint equation, (2.88), implies that if $\omega_b = 0$ then $q_b = 0$ as well, thus both ϕ and $\mathcal{W}$ asymptote to constants near the brane, $\phi \simeq \phi_b$, and $\mathcal{W} \simeq \mathcal{W}_b$, where ϕ_b and $\mathcal{W}_b$ are constants. In this case, the extra-dimensional metric becomes

$$e^{2\alpha_b\eta}(\mathcal{W}_b^8 d\eta^2 + d\theta^2) = d\rho^2 + \left(\frac{\alpha_b\rho}{\mathcal{W}_b^4}\right)^2 d\theta^2, \tag{2.99}$$

showing that it has a conical singularity (as was the case for the brane-axion extra-dimensional metric in the previous example) at the brane position, with defect angle $\delta_b = 2\pi(1 - \alpha_b/\mathcal{W}_b^4)$. In this case the junction conditions become

$$\frac{\partial T_b}{\partial\phi} = 0, \qquad \frac{\kappa^2 T_b}{\mathcal{W}_b^4} = 2\pi(1 - \alpha_b/\mathcal{W}_b^4) = \delta_b, \tag{2.100}$$

and the equation for ω_b in (2.98) is automatically satisifed ($\omega_b = 0$ is predicted when the other junction conditions are used). The first relation states that in order for solutions to exist, the two tensions must both satisfy $T_b'(\phi_b) = 0$ at both branes, where $\phi_b = \lim \phi(\eta)$ as $\eta \to -(-)^b\infty$. The second equation relates the tension to the size of the conical angle defect in the usual way.

Case 2: $\omega_b > 0$

If $\omega_b > 0$ then $e^W = \mathcal{W} \to 0$ in the near-brane regime. In this case the junction conditions become

$$\begin{aligned} 3\omega_b + \alpha_b &= -\frac{\kappa^2}{2\pi}T_b, \\ \omega_b &= -\frac{1}{3}\left[\frac{\kappa^2}{2\pi}T_b + \sqrt{\left(\frac{\kappa^2}{2\pi}T_b\right)^2 - \frac{3}{4}\left(\frac{\kappa^2}{2\pi}\partial_\phi T_b\right)^2}\right], \end{aligned} \tag{2.101}$$

which implies that $T_b < 0$.

The 4D Perspective

With the bulk solutions and the relationships provided by the junction conditions we are now ready to study the low-energy on-brane effective action,

$$S_{\text{eff}} = -\int d^{n-2}x\,\sqrt{-\hat{g}}\left[\frac{1}{2\kappa_N^2}\hat{g}^{\mu\nu}\hat{R}_{\mu\nu} + V_{\text{eff}}\right], \tag{2.102}$$

and the first step in doing so is to calculate the $4D$ effective potential, V_{eff}, given by

$$V_{\text{eff}} = V_B - \frac{2\pi}{\kappa^2}\sum_{b=0}^{1}\left[\mathcal{S}_\theta^{(2)} - \mathcal{S}_g^{(2)} - \frac{\kappa^2}{2\pi}\mathcal{L}_b^{(2)}\right]. \tag{2.103}$$

We are only interested in finding the value of $V_{\rm eff}$ at its minimum, ϕ_0, and thus we need not use V_B as is given by (2.45). Recall (2.45) was obtained by evaluating the action at the extra-dimensional Einstein equations, and this was done to keep track of perturbations about the minimum of V_B. If we are only interested in the value at the minimum, it suffices to evaluate the action at the full classical solution - i.e. all the Einstien equations. As a consequnce of the scaling symmetry that our lagrangian has, we know the action evaluated at any classical solution is given by (2.79), and hence the bulk potential is given by

$$S_B(g^c_{MN}, A^c_M, \phi^c) = -\int d^4x\, V_B. \tag{2.104}$$

We know that

$$S_B(g^c_{MN}, A^c_M, \phi^c) = \frac{1}{2\kappa^2}\int d^6x\sqrt{-g}\,\Box\phi = \frac{\pi}{\kappa^2}\sqrt{-\hat{g}}\,[\partial_\eta\phi]^{\infty}_{-\infty} = -\frac{\pi}{\kappa^2}\sum_b\int d^4x\, \mathcal{S}^{(2)}_\phi, \tag{2.105}$$

thus

$$V_B = -\frac{\pi}{\kappa^2}\sum_b \mathcal{S}^{(2)}_\phi. \tag{2.106}$$

The $4D$ effective potential is then given by

$$V_{\rm eff}(\phi_0) = -\frac{2\pi}{\kappa^2}\sum_b\left(\mathcal{S}^{(2)}_\theta + \frac{1}{2}\mathcal{S}^{(2)}_\phi\right) = -\frac{2\pi}{\kappa^2}\sum_b\left(\omega_b + \frac{q_b}{2}\right), \tag{2.107}$$

using (2.103), (2.94), and the fact that $\mathcal{S}^{(2)}_g + \frac{\kappa^2}{2\pi}\mathcal{L}^{(2)}_b = 0$ since $\mathcal{L}^{(2)}_b = -T_b$. Using this in the four-dimensional Einstein equations gives the $4D$ curvature

$$\hat{R} = -4\kappa_N^2 V_{\rm eff}(\phi_0) = \frac{8\pi\kappa_N^2}{\kappa^2}\sum_b\left(\omega_b + \frac{q_b}{2}\right). \tag{2.108}$$

This result is in agreement with (2.93), and shows that the $4D$ and $6D$ pictures agree with one another.

This is the end of our analysis of maximally symmetric codimension-2 branes and their back-reaction on the bulk. Although studying this is an important step to understanding the dynamics of higher dimensional branes, all the above analysis has been completely time independent. In the next section we generalize our allowed ansatz to be consistent with axial symmetry in the extra dimensions and maximal symmetry in only the spatial directions of the brane. This allows us to consider solutions wihch have non-trivial time dependence in the extra dimensions. This metric ansatz also corresponds to Friedmann-Robertson-Walker-like (FRW) branes which has cosmological applications.

Chapter 3

Friedmann-Robertson-Walker-like Branes

This chapter is concerned with studying the back-reaction of FRW-like codimension-2 branes in a generalized chiral gauged supergravity theory. The line of thought is almost identical to that which was just studied in the case of maximally symmetric branes, but we now use a more general ansatz. This ansatz describes FRW-like branes and is consistent with axial symmetry in the extra dimensions and maximal symmetry in only the spatial directions of the brane. We use the same renormalization procedure discussed in the previous chapter to avoid divergences and to generalize from codimension-1 to codimension-2 branes.

3.1 The Bulk

Our starting point is the same as in the case of maximally symmetric branes and for a complete explanation of all notation and assumptions, section 2.1 should be consulted. The action describing the bulk physics is given by $S = S_B + S_{GH}$, where S_B and S_{GH} are the bulk and Gibbons-Hawking action which are given by

$$S_B = -\int_{\mathcal{M}} d^n x\sqrt{-g}\left[\frac{1}{2\kappa^2}g^{MN}\left(\mathcal{R}_{MN} + \partial_M\phi\partial_N\phi\right) + \frac{1}{4}f(\phi)F_{MN}F^{MN} + V(\phi)\right], \tag{3.1}$$

$$S_{GH} = \frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^{n-1}x\sqrt{-\bar{g}}K. \tag{3.2}$$

The field equations of (3.1) are

$$\frac{1}{2\kappa^2}(\mathcal{R}_{MN} + \partial_M\phi\partial_N\phi) + \frac{f}{2}F_M{}^P F_{NP} + \frac{1}{n-2}\left[V - \frac{f}{4}F_{PQ}F^{PQ}\right]g_{MN} = 0, \qquad \text{(Einstein)} \quad (3.3)$$

$$\Box\phi - \kappa^2 \left[\frac{\partial V}{\partial \phi} + \frac{1}{4}\frac{\partial f}{\partial \phi} F_{MN}F^{MN}\right] = 0, \qquad \text{(Dilaton)} \tag{3.4}$$

$$\nabla_M (f F^{MN}) = 0. \qquad \text{(Maxwell)} \tag{3.5}$$

3.2 Metric Ansatz

For FRW-like branes, we need to choose a metric which is maximally symmetric in the spatial directions on the brane, and axially symmetric in the bulk directions. The ansatz we choose (note: this is not the most general ansatz) that satisifies these symmetries is

$$ds^2 = e^{2B_1} d\rho^2 + e^{2B_2} d\theta^2 - e^{2W_1} dt^2 + e^{2W_2} \check{g}_{ij} dx^i dx^j, \tag{3.6}$$

where θ labels the direction of cylindrical symmetry, the functions B_1, B_2, W_1, and W_2 depend only on the bulk radial coordinate, ρ, and time, t. Also, $\check{g}_{ij}$ denotes a maximally symmetric Euclidean-signature metric with dimension $n-3$.

To see that this metric is consistent with our symmetries, we again use the theorem from is described in section 2.2. Since we are requiring that the spatial directions of the brane be maximally symmetric, the metric must be of the form

$$ds^2 = h_{ab} dy^a dy^b + e^{2W_2(y)} \check{g}_{ij}(x) dx^i dx^j, \tag{3.7}$$

where the x-coordinates to be the spatial coordinates of the brane, and the y-coordinates are the bulk coordinates and time. Axial symmetry in the bulk directions also dictate that h_{ab} and W_2 must be independent of θ. We can make a coordinate transformation that eliminates the cross terms $h_{\rho t}$ and $h_{\rho\theta}$ from h_{ab}. In general we cannot eliminate all cross terms in the metric, and are left over with at least one, in this case $h_{t\theta}$, however we are only going to concern ourselves with metrics which are completely diagonal, so we set $h_{t\theta} = 0$ by hand. It should be noted that we are not working with the most general metric satisifying our symmetries, thus this choice leads us to (3.6).

These symmetries also have consequences for the functional dependence of the other fields in our problem, namely ϕ and A. We now show that

$$\begin{aligned} A &= A_\theta(\rho, t) d\theta + A_t(\rho, t) dt, \\ \phi &= \phi(\rho, t), \end{aligned} \tag{3.8}$$

are the most general choices for ϕ and A which are consistent with maximally symmetry in the spatial directions on the brane and axial symmetry in the bulk directions.

Following the same logic as in section 2.2, for consistency with $\check{g}_{ij}$ being a maximally symmetric metric and axial symmetry in the bulk, ϕ must be independent of the coordinates x^i and θ. Thus

$\phi = \phi(\rho, t)$ can only depend on ρ and t. This is because a consequence of maximal symmetry is that all terms in the (ij) Einstein equations must be proportional to $\breve{g}_{ij}$, and the term $\partial_i\phi\partial_j\phi$ is not and so must be zero. Similarly, all the components of A must be independent of θ and x^i due to axial symmetry in the bulk and because the term $F_i{}^P F_{jP}$ in the (ij) Einstein equations is not proportional to $\breve{g}_{ij}$. Another consequence of this is that the components A_i must be constants, which without loss of generality we take to be zero. Thus the most general form for A is

$$A = A_\theta(\rho,t)d\theta + A_\rho(\rho,t)d\rho + A_t(\rho,t)dt, \tag{3.9}$$

But notice that we can perform a gauge transformation on A, by the addition of a total derivative,

$$\begin{aligned} A' &= A + d\omega(\rho,t) \\ &= A_\theta(\rho,t)d\theta + A_\rho(\rho,t)d\rho + A_t(\rho,t)dt + \partial_\rho\omega d\rho + \partial_t\omega dt \\ &= A_\theta(\rho,t)d\theta + \left(A_\rho(\rho,t) + \partial_\rho\omega\right)d\rho + \left(A_t(\rho,t) + \partial_t\omega\right)dt. \end{aligned} \tag{3.10}$$

Using this, we can choose ω appropriately and choose a gauge in which the only non-zero components of A are A_θ and A_t. In summary, our ansatz for a geometry which is maximally symmetric in the spatial directions on the brane and axially symmetric in the bulk directions is

$$\begin{aligned} ds^2 &= e^{2B_1}d\rho^2 + e^{2B_2}d\theta^2 - e^{2W_1}dt^2 + e^{2W_2}\breve{g}_{ij}dx^i dx^j, \\ \phi &= \phi(\rho,t), \\ A &= A_\theta(\rho,t)d\theta + A_t(\rho,t)dt, \end{aligned} \tag{3.11}$$

where θ labels the direction of cylindrical symmetry, the functions B_1, B_2, W_1, and W_2 depend only on the bulk radial coordinate, ρ, and time, t, and $\breve{g}_{ij}$ denotes a maximally symmetric Euclidean-signature metric with dimension $n-3$.

Before writing down the field equations subject to these ansätze, we must first dimensionally reduce the full Ricci tensor into bulk and brane quantities. Doing this gives

$$\begin{aligned} \mathcal{R}_{ij} &= \left\{\frac{1}{n-3}\breve{R} + e^{-2(B_1-W_2)}\left[W_2'' + (n-3)(W_2')^2 - B_1'W_2' + B_2'W_2' + W_1'W_2'\right]\right. \\ &\qquad \left. - e^{-2(W_1-W_2)}\left[\ddot{W}_2 + (n-3)(\dot{W}_2)^2 + \dot{B}_1\dot{W}_2 + \dot{B}_2\dot{W}_2 - \dot{W}_1\dot{W}_2\right]\right\}\breve{g}_{ij}, \\ \mathcal{R}_{tt} &= \ddot{B}_1 + \ddot{B}_2 + (n-3)\ddot{W}_2 + (\dot{B}_1)^2 + (\dot{B}_2)^2 + (n-3)(\dot{W}_2)^2 - \dot{B}_1\dot{W}_1 - \dot{B}_2\dot{W}_1 \\ &\qquad - (n-3)\dot{W}_1\dot{W}_2 - e^{-2(B_1-W_1)}\left[W_1'' + (W_1')^2 - B_1'W_1' + B_2'W_1' + (n-3)W_1'W_2'\right], \\ \mathcal{R}_{\theta\theta} &= e^{-2(B_1-B_2)}\left[B_2'' + (B_2')^2 - B_1'B_2' + B_2'W_1' + (n-3)B_2'W_2'\right] \\ &\qquad - e^{2(B_2-W_1)}\left[\ddot{B}_2 + (\dot{B}_2)^2 + \dot{B}_1\dot{B}_2 - \dot{B}_2\dot{W}_1 + (n-3)\dot{B}_2\dot{W}_2\right], \end{aligned}$$

$$\begin{aligned}\mathcal{R}_{\rho\rho} &= B_2'' + W_1'' + (n-3)W_2'' + (B_2')^2 + (W_1')^2 + (n-3)(W_2')^2 - B_1'B_2' - B_1'W_1' \\ &\quad - (n-3)B_1'W_2' - e^{2(B_1-W_1)}\left[\ddot{B}_1 + (\dot{B}_1)^2 + \dot{B}_1\dot{B}_2 - \dot{B}_1\dot{W}_1 + (n-3)\dot{B}_1\dot{W}_2\right], \\ \mathcal{R}_{\rho t} &= \dot{B}_2' + (n-3)\dot{W}_2' + B_2'\dot{B}_2 - B_2'\dot{B}_1 - W_1'\dot{B}_2 - (n-3)W_1'\dot{W}_2 - (n-3)W_2'\dot{B}_1 \\ &\quad + (n-3)W_2'\dot{W}_2 \end{aligned} \tag{3.12}$$

where we've used $\breve{R}_{ij} = \frac{1}{n-3}\breve{R}\,\breve{g}_{ij}$ since $\breve{g}_{ij}$ is maximally symmetric with dimension $n-3$.

The full field equations subject to the ansätze (3.3)-(3.5) can now be written down. We first consider the $(\rho\theta)$ and (θt) Einstein equations, which are

$$(\rho\theta) : \dot{A}_\theta A_t' = 0, \qquad (\theta t) : A_\theta' A_t' = 0. \tag{3.13}$$

These equations can be considered as a constraint on our choices for the components of A. This may seem a little odd since our choice for A, (3.11), is supposed to be the most general form consistent with our symmetries. But recall that our metric was not the most general choice consistent with our symmetries, and in fact this constraint on the components of A is a direct result of this. The $(\rho\theta)$ and (θt) equations tell us that for consistency with our metric choice, either A_θ must be a constant or A_t must be independent of ρ. To make contact with the examples in the previous chapters we choose A_t to be independent of ρ, which means that it only depends on t and is a pure gauge term which can be ignored completely. Thus the only non-trivial component of A is A_θ.

The other Einstein equations subject to these ansätze reduce to

$$\begin{aligned}&\frac{1}{n-3}e^{2(B_1-W_2)}\breve{R} + \left[W_2'' + (n-3)(W_2')^2 - B_1'W_2' + B_2'W_2' + W_1'W_2'\right] \\ &\quad - e^{2(B_1-W_1)}\left[\ddot{W}_2 + (n-3)(\dot{W}_2)^2 + \dot{B}_1\dot{W}_2 + \dot{B}_2\dot{W}_2 - \dot{W}_1\dot{W}_2\right] + \frac{2\kappa^2}{n-2}e^{2B_1}V \\ &\quad - \frac{\kappa^2 f}{n-2}e^{-2B_2}(A_\theta')^2 + \frac{\kappa^2 f}{n-2}e^{2(B_1-B_2-W_1)}(\dot{A}_\theta)^2 = 0, \qquad (ij) \quad (3.14)\\ &W_1'' + (W_1')^2 - B_1'W_1' + B_2'W_1' + (n-3)W_1'W_2' - e^{2(B_1-W_1)}\Big[\ddot{B}_1 + \ddot{B}_2 + (n-3)\ddot{W}_2 \\ &\quad + (\dot{B}_1)^2 + (\dot{B}_2)^2 + (n-3)(\dot{W}_2)^2 + (\dot{\phi})^2 - \dot{B}_1\dot{W}_1 - \dot{B}_2\dot{W}_1 - (n-3)\dot{W}_1\dot{W}_2\Big] \\ &\quad + \frac{2\kappa^2}{n-2}e^{2B_1}V - \frac{\kappa^2 f}{n-2}e^{-2B_2}(A_\theta')^2 - \frac{n-3}{n-2}\kappa^2 f e^{2(B_1-B_2-W_1)}(\dot{A}_\theta)^2 = 0, \qquad (tt) \quad (3.15)\\ &B_2'' + (B_2')^2 - B_1'B_2' + B_2'W_1' + (n-3)B_2'W_2' - e^{2(B_1-W_1)}\Big[\ddot{B}_2 + (\dot{B}_2)^2 + \dot{B}_1\dot{B}_2 \\ &\quad - \dot{B}_2\dot{W}_1 + (n-3)\dot{B}_2\dot{W}_2\Big] + \frac{2\kappa^2}{n-2}e^{2B_1}V + \frac{n-3}{n-2}\kappa^2 f e^{-2B_2}(A_\theta')^2 \\ &\quad - \frac{n-3}{n-2}\kappa^2 f e^{2(B_1-B_2-W_1)}(\dot{A}_\theta)^2 = 0, \qquad (\theta\theta) \quad (3.16)\end{aligned}$$

$$B_2'' + W_1'' + (n-3)W_2'' + (B_2')^2 + (W_1')^2 + (n-3)(W_2')^2 + (\phi')^2 - B_1'B_2' - B_1'W_1'$$
$$- (n-3)B_1'W_2' - e^{2(B_1-W_1)}\left[\ddot{B}_1 + (\dot{B}_1)^2 + \dot{B}_1\dot{B}_2 - \dot{B}_1\dot{W}_1 + (n-3)\dot{B}_1\dot{W}_2\right]$$
$$+ \frac{2\kappa^2}{n-2}e^{2B_1}V + \frac{n-3}{n-2}\kappa^2 f e^{-2B_2}(A_\theta')^2 + \frac{\kappa^2 f}{n-2}e^{2(B_1-B_2-W_1)}(\dot{A}_\theta)^2 = 0, \qquad (\rho\rho) \quad (3.17)$$
$$\dot{B}_2' + (n-3)\dot{W}_2' + B_2'\dot{B}_2 - B_2'\dot{B}_1 - W_1'\dot{B}_2 - (n-3)W_1'\dot{W}_2 - (n-3)W_2'\dot{B}_1$$
$$+ (n-3)W_2'\dot{W}_2 + \phi'\dot{\phi} + \kappa^2 f e^{-2B_2}A_\theta'\dot{A}_\theta = 0, \qquad (\rho t) \quad (3.18)$$

while the dilaton and Maxwell equations become

$$\phi'' - (B_1' - B_2' - W_1' - (n-3)W_2')\,\phi' + e^{2(B_1-W_1)}\left[\ddot{\phi} + \left(\dot{B}_1 + \dot{B}_2 - \dot{W}_1 + (n-3)\dot{W}_2\right)\dot{\phi}\right]$$
$$- \kappa^2\left[e^{2B_1}\frac{\partial V}{\partial\phi} + \frac{1}{2}\frac{\partial f}{\partial\phi}e^{-2B_2}(A_\theta')^2 - \frac{1}{2}\frac{\partial f}{\partial\phi}e^{2(B_1-B_2-W_1)}(\dot{A}_\theta)^2\right] = 0, \qquad (3.19)$$

and

$$\partial_\rho\left(e^{-B_1-B_2+W_1+(n-3)W_2}f\partial_\rho A_\theta\right) - \partial_t\left(e^{B_1-B_2-W_1+(n-3)W_2}f\partial_t A_\theta\right) = 0, \qquad (3.20)$$

where prime denotes differentiation with respect to ρ, dot denotes differentiation with respect to t, and $\breve{R}$ is the Ricci scalar constructed from $\breve{g}_{ij}$.

As before, singularity positions in the extra dimensional metric, if they exist, are interpreted as the positions of branes which source the curvature of the bulk. The back-reaction of such branes can potentially have a strong effect on the low energy on-brane effective action, and must be studied thoroughly.

3.3 Boundary Conditions for Branes

In this section we deal with branes in the same way we dealt with them in the case of maximally symmetric branes. The same problem of divergences at the brane positions still exists for codimension-2 branes, so instead of worrying about this, we replace the codimension-2 branes with small cylindrical codimension-1 branes a small distance Δ, $\rho_b = \rho_s + \Delta$, from the singularity positions. This is accompanied by an interior geometry for $\rho < \rho_b$, which is capped off with a smooth solution to the bulk field equations of motion given by

$$ds^2_{\text{flat}} = e^{2B_1^{\text{flat}}}d\rho^2 + e^{2B_2^{\text{flat}}}d\theta^2 - e^{2W_1^{\text{flat}}}dt^2 + e^{2W_2^{\text{flat}}}\breve{g}_{ij}dx^i dx^j$$
$$= e^{2B_1^{\text{flat}}}d\rho^2 + e^{2B_3^{\text{flat}}}\rho^2 d\theta^2 - e^{2W_1^{\text{flat}}}dt^2 + e^{2W_2^{\text{flat}}}\breve{g}_{ij}dx^i dx^j, \qquad (3.21)$$

where the 'flat' variables do not depend on ρ. We can now relate the bulk and brane quantities at the brane positions, ρ_b, by using the junction conditions for codimension-1 branes.

3.3.1 Codimension-1 Branes

The junction conditions for codimension-1 branes serve as a stepping stone to generalizing to codimension-2 branes. We now compute the codimension-1 junction conditions.

Junction Conditions

The junction conditions for a codimension-1 brane are (see Appendix C)

$$\begin{aligned}\frac{1}{2\kappa^2}\left[\sqrt{-\bar{g}}\,(K^{mn}-K\bar{g}^{mn})\right]_{\rho_b}+\frac{\delta S_b^{(1)}}{\delta \bar{g}_{mn}}&=0.\\ \frac{1}{\kappa^2}\left[\sqrt{-\bar{g}}\partial_\rho\phi\right]_{\rho_b}+\frac{\delta S_b^{(1)}}{\delta\phi}&=0,\\ \left[\sqrt{-\bar{g}}fe^{B_1}F^{\rho M}\right]_{\rho_b}+\frac{\delta S_b^{(1)}}{\delta A_M}&=0,\end{aligned}\tag{3.22}$$

where $\bar{g}$ is the induced metric on the codimension-1 brane, K is the extrinsic curvature, and the notation $[F]_{\rho_b}$ for a bulk quantity denotes

$$[F]_{\rho_b}=\lim_{\epsilon\to 0}\left[F(\rho_b+\epsilon)-F(\rho_b-\epsilon)\right].\tag{3.23}$$

The components of the extrinsic curvature is given by $K_{mn}=\frac{1}{2}e^{-2B_1}\partial_\rho\bar{g}_{mn}$, and in particular for our ansatz, (3.11), this reduces to

$$\begin{aligned}K_{ij}&=e^{-2(B_1-W_2)}W_2'\,\check{g}_{ij}, & K_{\theta\theta}&=e^{-2(B_1-B_2)}B_2',\\ K_{tt}&=-e^{-2(B_1-W_1)}W_1', & K&=e^{-2B_1}\left[B_2'+W_1'+(n-3)W_2'\right],\end{aligned}\tag{3.24}$$

where $K=\bar{g}^{mn}K_{mn}$. Combining the expressions for the componenets of K_{mn}, (3.24), with the junction conditions, (3.22), the definition of $[F]_{\rho_b}$, and the flat geometry, (3.21) results in

$$\begin{aligned}\frac{1}{\kappa^2}e^{B_2+W_1+(n-3)W_2}\sqrt{\check{g}}\phi'+\frac{\delta S_b^{(1)}}{\delta\phi}&=0,\\ e^{-B_1-B_2+W_1+(n-3)W_2}\sqrt{\check{g}}fA_\theta'+\frac{\delta S_b^{(1)}}{\delta A_\theta}&=0,\\ \frac{1}{2\kappa^2}e^{-2B_1+W_1+(n-3)W_2}g^{tt}\sqrt{\check{g}}\left[e^{B_2}(B_2'+(n-3)W_2')-e^{B_3}\right]+\frac{\delta S_b^{(1)}}{\delta g_{tt}}&=0,\\ \frac{1}{2\kappa^2}e^{-2B_1+B_2+W_1+(n-3)W_2}g^{\theta\theta}\sqrt{\check{g}}\left[W_1'+(n-3)W_2'\right]-\frac{\delta S_b^{(1)}}{\delta g_{\theta\theta}}&=0,\\ \frac{1}{2\kappa^2}e^{-2B_1+W_1+(n-3)W_2}\check{g}^{ij}\sqrt{\check{g}}\left[e^{B_2}(B_2'+W_1'+(n-4)W_2')-e^{B_3}\right]-\frac{\delta S_b^{(1)}}{\delta\check{g}_{ij}}&=0,\end{aligned}\tag{3.25}$$

where it is now understood that all the fields are evaluated at the brane positions, ρ_b. We now rewrite the junction conditions in terms of the brane quantities $\mathcal{S}_\phi^{(1)}$, $\mathcal{S}_{A_\theta}^{(1)}$, $\mathcal{S}_t^{(2)}$, $\mathcal{S}_\theta^{(1)}$, and $\mathcal{S}_g^{(1)}$, as follows

$$\begin{aligned}
e^{B_2}\phi' &= e^{-(W_1+(n-3)W_2)}\mathcal{S}_\phi^{(1)}, \\
\kappa A_\theta' &= e^{B_1+B_2-W_1-(n-3)W_2}\frac{\mathcal{S}_{A_\theta}^{(1)}}{f}, \\
e^{B_2}B_2' - e^{B_3} &= -e^{2B_1-W_1-(n-3)W_2}\left[(n-3)\mathcal{S}_\theta^{(1)} + \frac{n-3}{n-2}\mathcal{S}_g^{(1)} + \frac{1}{n-2}\mathcal{S}_t^{(1)}\right], \\
e^{B_2}W_1' &= e^{2B_1-W_1-(n-3)W_2}\left[\mathcal{S}_\theta^{(1)} - \frac{n-3}{n-2}\left(\mathcal{S}_g^{(1)} - \mathcal{S}_t^{(1)}\right)\right], \\
e^{B_2}W_2' &= e^{2B_1-W_1-(n-3)W_2}\left[\mathcal{S}_\theta^{(1)} + \frac{1}{n-2}\left(\mathcal{S}_g^{(1)} - \mathcal{S}_t^{(1)}\right)\right],
\end{aligned} \tag{3.26}$$

where we have made the following definitions

$$\begin{aligned}
\mathcal{S}_\phi^{(1)} &= -\kappa^2\frac{1}{\sqrt{\breve{g}}}\frac{\delta S_b^{(1)}}{\delta\phi}, \\
\mathcal{S}_{A_\theta}^{(1)} &= -\kappa\frac{1}{\sqrt{\breve{g}}}\frac{\delta S_b^{(1)}}{\delta A_\theta}, \\
\mathcal{S}_t^{(1)} &= 2\kappa^2\frac{g_{tt}}{\sqrt{\breve{g}}}\frac{\delta S_b^{(1)}}{\delta g_{tt}}, \\
\mathcal{S}_\theta^{(1)} &= 2\kappa^2\frac{g_{\theta\theta}}{n-2}\frac{1}{\sqrt{\breve{g}}}\frac{\delta S_b^{(1)}}{\delta g_{\theta\theta}}, \\
\mathcal{S}_g^{(1)} &= -2\kappa^2\frac{\breve{g}_{ij}}{n-3}\frac{1}{\sqrt{\breve{g}}}\frac{\delta S_b^{(1)}}{\delta \breve{g}_{ij}}.
\end{aligned} \tag{3.27}$$

When considering a pure tension brane, $S_b^{(1)} = -\int_{\rho_b} d^{n-1}x\sqrt{\breve{g}}\,T_b$, the physical interpretations of $\mathcal{S}_\phi^{(1)}$, $\mathcal{S}_{A_\theta}^{(1)}$, and $\mathcal{S}_\theta^{(1)}$ are the same as in the case of maximally symmetric branes. They represent the derivative of the brane tension, $\partial_\phi T_b$, a brane contribution to the scalar potential within the low-energy on brane effective theory, and the microscopic axial currents within the brane, respectively. For an FRW-like brane, $\mathcal{S}_g^{(1)}$ and $\mathcal{S}_t^{(1)}$ represent the pressure and energy densities of the brane respectively.

Just as before, the brane constraint plays an important role in generalizing from codimension-1 branes to codimension-2 branes. Although the idea is essentially the same, when we explicitly include time dependence, there turns out to be two constraint equations to consider, rather than just one.

The Brane Constraints

Due to our more general ansatz (compared to the previous chapter), two components of the Bianchi identity are nontrivial rather than one. This implies that two of the field equations are not independent of the others. Related to this is the existence of two constraint equations which do not involve second derivatives with respect to ρ [16]. Both of these constraints are preserved when evolved using the field equations in the ρ direction by the Bianchi identities.

The first constraint equation is obtained by taking a linear combination of the field equations such that all second derivatives with respect to ρ of the bulk fields are eliminated. The relevant combination of the field equations that gives rise to this constraint is $(n-3)(ij)+(tt)+(\theta\theta)-(\rho\rho)$ and results in

$$\begin{aligned}&(n-4)(n-3)(W_2')^2+2(n-3)B_2'W_2'+2(n-3)W_1'W_2'+2B_2'W_1'-(\phi')^2\\&-e^{2(B_1-W_1)}\left[2(n-3)\ddot{W}_2+2\ddot{B}_2+(n-3)(n-2)(\dot{W}_2)^2+2(\dot{B}_2)^2-2\dot{B}_2\dot{W}_1-2(n-3)\dot{B}_2\dot{W}_2\right]\\&+e^{2(B_1-W_2)}\check{R}+2\kappa^2e^{2B_1}V-\kappa^2fe^{-2B_2}(A_\theta')^2-\kappa^2fe^{2(B_1-B_2-W_1)}(\dot{A}_\theta)^2=0.\end{aligned}\tag{3.28}$$

The second constraint equation is the (ρt) Einstein equation,

$$\begin{aligned}&\dot{B}_2'+(n-3)\dot{W}_2'+B_2'\dot{B}_2-B_2'\dot{B}_1-W_1'\dot{B}_2-(n-3)W_1'\dot{W}_2-(n-3)W_2'\dot{B}_1\\&+(n-3)W_2'\dot{W}_2+\phi'\dot{\phi}+\kappa^2fe^{-2B_2}A_\theta'\dot{A}_\theta=0.\end{aligned}\tag{3.29}$$

In order to turn these into constraints on the brane properties we multiply (3.28) and (3.29) through by $e^{-4B_1+2B_2+2W_1+2(n-3)W_2}$ and $e^{-2B_1+B_2+W_1+(n-3)W_2}$ respectively, take the limit $\rho\to\rho_b$ and trade B_2', W_1', W_2', ϕ', A_θ' for $\mathcal{S}_\phi^{(1)}$, $\mathcal{S}_{A_\theta}^{(1)}$, $\mathcal{S}_t^{(1)}$, $\mathcal{S}_\theta^{(1)}$, $\mathcal{S}_g^{(1)}$ using the junction conditions, (3.27), giving

$$\begin{aligned}&(n-3)(n-2)\left(\mathcal{S}_\theta^{(1)}\right)^2-2\left[(n-2)e^{2A_1}-(n-3)\mathcal{S}_g^{(1)}-\mathcal{S}_t^{(1)}\right]\mathcal{S}_\theta^{(1)}+\frac{n-3}{n-2}\left(\mathcal{S}_g^{(1)}-\mathcal{S}_t^{(1)}\right)^2\\&+e^{2A_2}\left[2(n-3)\ddot{W}_2+2\ddot{B}_2+(n-3)(n-2)(\dot{W}_2)^2+2(\dot{B}_2)^2-2\dot{B}_2\dot{W}_1-2(n-3)\dot{B}_2\dot{W}_2\right]\\&-e^{2A_3}\left[e^{-2W_2}\check{R}+V\right]+\frac{1}{f}e^{-2(B_1-B_2)}\left(\mathcal{S}_{A_\theta}^{(1)}\right)^2+\kappa^2fe^{-2(B_1-(n-3)W_2)}(\dot{A}_\theta)^2=0,\end{aligned}\tag{3.30}$$

and

$$\begin{aligned}&(n-2)\mathcal{S}_\theta^{(1)}+(n-3)\dot{W}_2\mathcal{S}_g^{(1)}+(\dot{W}_1-\dot{B}_1)\mathcal{S}_t^{(1)}+e^{-2B_1}\dot{\phi}\mathcal{S}_\phi^{(1)}+e^{2A_1}(\dot{B}_3-\dot{B}_1)\\&-\dot{\mathcal{S}}_t^{(1)}+\kappa\dot{A}_\theta e^{-B_1}\mathcal{S}_{A_\theta}^{(1)}=0,\end{aligned}\tag{3.31}$$

where

$$\begin{aligned}2A_1&=-2B_1+B_3+W_1+(n-3)W_2,\\2A_2&=-2(B_1-B_2-(n-3)W_2),\\2A_3&=2(B_1+B_2+W_1+(n-3)W_2).\end{aligned}\tag{3.32}$$

These constraints are now used to generalize the codimension-1 brane formalism to codimension-2 branes.

3.3.2 Codimension-2 Branes

The proper way to generalize to codimension-2 branes from the above codimension-1 formalism is to keep the form of the junction conditions and the definitions of the brane properties that still make sense (ie, that are defined in terms of fields that live on the brane), which are $\mathcal{S}_\phi^{(2)}$, $\mathcal{S}_g^{(2)}$, and $\mathcal{S}_t^{(2)}$. $\mathcal{S}_{A_\theta}^{(2)}$ and $\mathcal{S}_\theta^{(2)}$ are generalized the same way as was done in chapter 2. $\mathcal{S}_{A_\theta}^{(2)}$ is defined to be a tunable parameter describing any microscopic magnetic fluxes which enclose the brane, based on its physical interpretation discussed at the end of the codimension-1 section. To generalize $\mathcal{S}_\theta^{(2)}$ we let the brane constraint, (3.30), be its defining equation. In summary, for a codimension-2 brane action, $S_b^{(2)}$, the junction conditions are

$$\begin{aligned}
e^{B_2}\phi' &= e^{-(W_1+(n-3)W_2)}\mathcal{S}_\phi^{(2)},\\
\kappa A_\theta' &= e^{B_1+B_2-W_1-(n-3)W_2}\frac{\mathcal{S}_{A_\theta}^{(2)}}{f},\\
\kappa A_t' &= e^{B_1-B_2+W_1-(n-3)W_2}\frac{\mathcal{S}_{A_t}^{(2)}}{f},\\
e^{B_2}B_2' - e^{B_3} &= -e^{2B_1-W_1-(n-3)W_2}\left[(n-3)\mathcal{S}_\theta^{(2)} + \frac{n-3}{n-2}\mathcal{S}_g^{(2)} + \frac{1}{n-2}\mathcal{S}_t^{(2)}\right],\\
e^{B_2}W_1' &= e^{2B_1-W_1-(n-3)W_2}\left[\mathcal{S}_\theta^{(2)} - \frac{n-3}{n-2}\left(\mathcal{S}_g^{(2)} - \mathcal{S}_t^{(2)}\right)\right],\\
e^{B_2}W_2' &= e^{2B_1-W_1-(n-3)W_2}\left[\mathcal{S}_\theta^{(2)} + \frac{1}{n-2}\left(\mathcal{S}_g^{(2)} - \mathcal{S}_t^{(2)}\right)\right]
\end{aligned} \tag{3.33}$$

where $\mathcal{S}_{A_\theta}^{(2)}$ is a paramater describing the microscopic magnetic flux which enclosing the brane and

$$\mathcal{S}_\phi^{(2)} = -\frac{\kappa^2}{2\pi}\frac{1}{\sqrt{\breve{g}}}\frac{\delta S_b^{(1)}}{\delta\phi}, \quad \mathcal{S}_t^{(2)} = \frac{\kappa^2}{\pi}\frac{g_{tt}}{\sqrt{\breve{g}}}\frac{\delta S_b^{(1)}}{\delta g_{tt}}, \quad \mathcal{S}_g^{(2)} = -\frac{\kappa^2}{\pi}\frac{\breve{g}_{ij}}{n-3}\frac{1}{\sqrt{\breve{g}}}\frac{\delta S_b^{(1)}}{\delta \breve{g}_{ij}}. \tag{3.34}$$

We also implicitly define $\mathcal{S}_\theta^{(2)}$ by the equation

$$\begin{aligned}
&(n-3)(n-2)\left(\mathcal{S}_\theta^{(2)}\right)^2 - 2\left[(n-2)e^{A_1} - (n-3)\mathcal{S}_g^{(2)} - \mathcal{S}_t^{(2)}\right]\mathcal{S}_\theta^{(2)} + \frac{n-3}{n-2}\left(\mathcal{S}_g^{(2)} - \mathcal{S}_t^{(2)}\right)^2\\
&+ e^{A_2}\left[2(n-3)\ddot{W}_2 + 2\ddot{B}_2 + (n-3)(n-2)(\dot{W}_2)^2 + 2(\dot{B}_2)^2 - 2\dot{B}_2\dot{W}_1 - 2(n-3)\dot{B}_2\dot{W}_2\right]\\
&- e^{A_3}\left[e^{-2W_2}\breve{R} + V\right] + \frac{1}{f}e^{-2(B_1-B_2)}\left(\mathcal{S}_{A_\theta}^{(2)}\right)^2 - \frac{1}{f}e^{-2(B_1-W_1)}\left(\mathcal{S}_{A_t}^{(2)}\right)^2\\
&+ \kappa^2 f e^{-2(B_1-(n-3)W_2)}(\dot{A}_\theta)^2 = 0.
\end{aligned} \tag{3.35}$$

We now turn our attention to studying the low-energy on-brane effective theory and how the back-reaction of codimension-2 branes affects the theory.

3.4 The Classical Low-Energy On-Brane Effective Action

We obtain the bulk contribution, V_B, to the effective potential, $V_{\rm eff}$, by substituting the extra-dimensional equations of motion into the full action. To eliminate the heavy fields we make use of the (ab) Einstein equations,

$$\frac{1}{2\kappa^2}(\mathcal{R}_{ab}+\partial_a\phi\partial_b\phi)+\frac{f}{2}F_a{}^PF_{bP}+\frac{1}{n-2}\left[V-\frac{f}{4}F_{PQ}F^{PQ}\right]h_{ab}=0. \tag{3.36}$$

These comprise of two independent equations, which we take to be the sum and difference of the $(\rho\rho)$ and $(\theta\theta)$ components. The difference, (3.17)-(3.16), gives

$$\begin{aligned}&W_1''+(n-3)W_2''+(W_1')^2+(n-3)(W_2')^2-B_1'W_1'-(n-3)B_1'W_2'-B_2'W_1'-(n-3)B_2'W_2'\\&+(\phi')^2-e^{2(B_1-W_1)}\Big[\ddot{B}_1-\ddot{B}_2+(\dot{B}_1)^2-(\dot{B}_2)^2-\dot{B}_1\dot{W}_1+(n-3)\dot{B}_1\dot{W}_2+\dot{B}_2\dot{W}_1\\&\qquad-(n-3)\dot{B}_2\dot{W}_2\Big]=0,\end{aligned} \tag{3.37}$$

while the sum is equivalent to contracting (3.36) with the extra-dimensional metric, h^{ab}, to give

$$\frac{1}{2\kappa^2}h^{ab}\left(\mathcal{R}_{ab}+\partial_a\phi\partial_b\phi\right)=\frac{f}{2(n-2)}F^{MN}F_{MN}-\frac{f}{2}F^{aN}F_{aN}-\frac{2}{n-2}V. \tag{3.38}$$

We begin by explicitly separating the on-brane metric from the extra-dimensional metric in the bulk action, using (3.12),

$$\begin{aligned}S_B&=-\int d^nx\sqrt{-g}\left[\frac{1}{2\kappa^2}g^{MN}\left(\mathcal{R}_{MN}+\partial_M\phi\partial_N\phi\right)+\frac{1}{4}f(\phi)F_{MN}F^{MN}+V(\phi)\right]\\
&=-\int d^nx\sqrt{-g}\Big\{\frac{1}{2\kappa^2}e^{-2W_2}\check{g}^{ij}\check{R}_{ij}+\frac{1}{2\kappa^2}e^{-2B_1}\left[W_1''+(n-3)W_2''+(W_1')^2+(n-3)^2(W_2')^2\right.\\
&\quad\left.-B_1'W_1'-(n-3)B_1'W_2'+B_2'W_1'+(n-3)B_2'W_2'+2(n-3)W_1'W_2'\right]\\
&\quad-\frac{1}{2\kappa^2}e^{-2W_1}\Big[\ddot{B}_1+\ddot{B}_2+2(n-3)\ddot{W}_2+(\dot{B}_1)^2+(\dot{B}_2)^2+(n-3)(n-2)(\dot{W}_2)^2+(\dot{\phi})^2\\
&\quad-\dot{B}_1\dot{W}_1+(n-3)\dot{B}_1\dot{W}_2-\dot{B}_2\dot{W}_1+(n-3)\dot{B}_2\dot{W}_2-2(n-3)\dot{W}_1\dot{W}_2\Big]\\
&\quad+\frac{1}{2\kappa^2}h^{ab}(\mathcal{R}_{ab}+\partial_a\phi\partial_b\phi)+\frac{1}{4}fF^{MN}F_{MN}+V\Big\}\\
&=-\int d^nx\sqrt{-g}\Big\{\frac{1}{2\kappa^2}e^{-2W_2}\check{g}^{ij}\check{R}_{ij}+\frac{1}{2\kappa^2}e^{-2B_1}\left[W_1''+(n-3)W_2''+(W_1')^2+(n-3)^2(W_2')^2\right.\\
&\quad\left.-B_1'W_1'-(n-3)B_1'W_2'+B_2'W_1'+(n-3)B_2'W_2'+2(n-3)W_1'W_2'\right]\\
&\quad-\frac{1}{2\kappa^2}e^{-2W_1}\Big[\ddot{B}_1+\ddot{B}_2+2(n-3)\ddot{W}_2+(\dot{B}_1)^2+(\dot{B}_2)^2+(n-3)(n-2)(\dot{W}_2)^2+(\dot{\phi})^2\\
&\quad-\dot{B}_1\dot{W}_1+(n-3)\dot{B}_1\dot{W}_2-\dot{B}_2\dot{W}_1+(n-3)\dot{B}_2\dot{W}_2-2(n-3)\dot{W}_1\dot{W}_2\Big]\\
&\quad+\frac{f}{4}\frac{n}{n-2}F^{MN}F_{MN}-\frac{f}{2}F^{aN}F_{aN}+\frac{n-4}{n-2}V\Big\}\end{aligned}$$

$$= -\int d^n x\sqrt{-g}\left\{\frac{1}{2\kappa^2}e^{-2W_2}\check{g}^{ij}\check{R}_{ij} + \frac{1}{2\kappa^2}e^{-2B_1}\left[(n-4)(n-3)(W_2')^2 + 2B_2'W_1'\right.\right.$$
$$+2(n-3)B_2'W_2' + 2(n-3)W_1'W_2' - (\phi')^2\Big] - \frac{1}{2\kappa^2}e^{-2W_1}\Big[2\ddot{B}_2 + 2(n-3)\ddot{W}_2 + 2(\dot{B}_2)^2$$
$$+(n-3)(n-2)(\dot{W}_2)^2 + (\dot{\phi})^2 - 2\dot{B}_2\dot{W}_1 + 2(n-3)\dot{B}_2\dot{W}_2 - 2(n-3)\dot{W}_1\dot{W}_2\Big]$$
$$\left.+\frac{f}{4}\frac{n}{n-2}F^{MN}F_{MN} - \frac{f}{2}F^{aN}F_{aN} + \frac{n-4}{n-2}V\right\} \tag{3.39}$$

where we have used both (3.37) and (3.38) in the above. The effective bulk action describing the low energy physics in the bulk is obtained by integrating out the extra dimensions. Unlike in the previous chapter, we cannot do this explicitly because the metric functions, W_1, W_2, B_1, and B_2, depend on both ρ and t, so integrating with respect to ρ has implications for the time-dependence of the final expression.

The other terms contributing to the full low-energy on-brane effective action are the boundary contributions due to the Gibbons-Hawking action, S_{GH}, as well as the brane action, $S_b^{(2)}$, itself. With two branes located at positions ρ_b, $b = \{0, 1\}$, the Gibbons-Hawking action is,

$$\begin{aligned} S_{GH} &= \sum_{b=0}^{1}\int_{\rho_b} d\theta d^{n-2}x\frac{1}{\kappa^2}\sqrt{-g}K \\ &= \frac{2\pi}{\kappa^2}\sum_{b=0}^{1}(-)^b\int_{\rho_b} d^{n-2}x\sqrt{\check{g}}e^{-2B_1+B_2+W_1+(n-3)W_2}\left[B_2' + W_1' + (n-3)W_2'\right] \\ &= -\frac{2\pi}{\kappa^2}\sum_{b=0}^{1}\int_{\rho_b} d^{n-2}x\sqrt{\check{g}}\left[\mathcal{S}_\theta^{(2)} - \frac{n-3}{n-2}\mathcal{S}_g^{(2)} - \frac{1}{n-2}\mathcal{S}_t^{(2)}\right]. \end{aligned} \tag{3.40}$$

Thus the effective potential of the low-energy on-brane theory is

$$V_{\text{eff}} = V_B - \frac{2\pi}{\kappa^2}\sum_{b=0}^{1}\left[\mathcal{S}_\theta^{(2)} - \frac{n-3}{n-2}\mathcal{S}_g^{(2)} - \frac{1}{n-2}\mathcal{S}_t^{(2)} - \frac{\kappa^2}{2\pi}\mathcal{L}_b^{(2)}\right], \tag{3.41}$$

where V_B is obtained by integrating (3.39) over the extra dimensions when a solutions to the field equations are given.

This completes the description of the low-energy on-brane theory for FRW-like branes. This is the end of calculations we consider, but we now discuss several ways to test this theory out.

3.5 Future Work

We now discuss three different ways which could potentially be used to test the above formulation and generate new results. The first idea is rather straight forward. If time dependent solutions to the field equations, (3.14)-(3.20), are found (given choices for f and V in (2.2)) then an analysis which is similar to that done for the brane-axion couplings in $6D$ in chapter 2 can be done. This will involve studying the bulk solutions and calculating the Ricci scalar in the higher dimensional picture. Then using the junction conditions, (3.27), we can study thc brane properties and relate the bulk and brane fields to each other at the brane positions, which is then used to calculate the effective potential, (3.41). The lower dimensional field equations are then solved and used to calculate the Ricci scalar in the lower dimensional picture and compared to that of the higher dimensional calculation.

However, expecting to find explicit solutions to the time dependent field equations may be a little unrealistic. If none are found, studying the near-brane regime will serve as a good approximation and will greatly simplify the equations of motion. We already considered this approach in the second example in chapter 2, although in the static case. The process here is similar to when explicit solutions are known. The bulk equations should first be solved and analyzed, followed by a study of the brane properties using the junction conditions. This will ultimately lead to considering the lower dimensional theory and comparing it to results in the higher dimensional picture.

Another way to test this formalism is to make use of known scaling solutions to the $6D$ gauged chiral supergravity lagrangian, (2.78). It can be shown that if we consider ansätze of the form [16]

$$\begin{aligned} ds^2 &= (H_0 t)^{2+c}\left[e^{2B_1(\rho)}d\rho^2 + e^{B_2(\rho)}\right] + (H_0 t)^c\left[-e^{W_1(\rho)}dt^2 + e^{W_2(\rho)}\delta_{ij}dx^i dx^j\right], \\ e^\phi &= e^{\Phi(\rho)}(H_0 t)^{-(2+c)}, \qquad\qquad A_\theta = A_\theta(\rho), \end{aligned} \tag{3.42}$$

then the time dependence of the field equations completely drops out and we are left with a set of differential equations in ρ only. The problem then becomes a static problem and can be approached in the same way as was done in chapter 2. However the scaling symmetry possessed by (2.78) offers us another avenue for testing as well. We can perform the scaling symmetry transformation

$$e^\phi \to e^{\phi+\varphi(t)}, \qquad g_{MN} \to e^{-\varphi(t)}g_{MN}, \tag{3.43}$$

and only follow the time dependence generated by φ, which represents a classically massless KK zero mode coming from a combination of the metric and ϕ fields. When this transformation is applied to the bulk contribution to the potential, V_B, we can ignore all contributions due to the radial profiles (these will just integrate out to a constant with the choices we have made) since we will only be interested in tracking the time dependence caused by φ. The resulting effective theory will be of the form

$$S_{\text{eff}} = -\int d^4x\sqrt{\check{g}}\left[X(\varphi)\check{g}^{ij}\check{R}_{ij} + Y(\varphi)\partial_t\varphi\partial_t\varphi + V_{\text{eff}}(\varphi)\right], \tag{3.44}$$

and the time dependence in its field equations due to φ can be solved for and compared to the time depenedence in (3.42).

Thus we have described at least three of the different approaches one could take to test the time dependent formalism created in this chapter, and potentially study new and interesting physics.

Chapter 4

Conclusion

If extra dimensions exist, they will have many cosmological applications and predictions. For this reason, the goal of this thesis has been to derive the junction conditions and low-energy on-brane effective theory for FRW-like branes.

We begin by considering maximally symmetric branes. The ideas developed in this scenario are identical to that for FRW-like branes, but the mathematics will be less cumbersome and so this serves as a good check when we finally investigate FRW-like branes. It turns out that certain solutions to the field equations have singularities in the bulk. These singularities physically represent the position of a codimension-2 brane which sources the curvature in the bulk. However, working with codimension-2 branes is difficult because of the possibility of the bulk fields diverging at the brane position. This makes relating bulk fields at the brane to brane properties nearly impossible.

To overcome this problem we introduce a renormalization scheme which involves replacing the codimension-2 branes with small cylindrical codimension-1 branes located a small distance, Δ, from the singularity positions. With this setup we then compute the codimension-1 junction conditions without worrying about divergences of bulk fields at the brane position. This codimension-1 picture is used as a stepping stone and guides us as we generalize the junction conditions for codimension-1 branes to codimension-2 branes.

Once the codimension-2 brane formalism is set, we make use of it when deriving the low-energy on-brane effective theory from which we study the back-reaction of the codimension-2 branes. This is then applied to two examples, and in both examples we show that the high and low energy theories are in agreement with one another.

Using this line of thought as a guide, we turn to solutions which are axially symmetric in the bulk directions and maximally symmetric in the spatial directions of the (FRW-like) brane. The same process leads us to the codimension-2 junction conditions and the low-energy on-brane effective action. We conclude by discussing three possible routes that can be taken to verify the time dependent theory and which can be used to study new physics.

Appendix A

Derivation of Field Equations

Here we derive the equations of motion of the following n dimensional scalar-tensor bulk action

$$S_B = -\int_{\mathcal{M}} d^n x\sqrt{-g}\left[\frac{1}{2\kappa^2} g^{MN}\left(\mathcal{R}_{MN} + \partial_M\phi\partial_N\phi\right) + \frac{1}{4}f(\phi)F_{MN}F^{MN} + V(\phi)\right]. \tag{A.1}$$

$\mathcal{R}_{MN}$ is the n-dimension Ricci tensor constructed from the Einstein-frame metric, g_{MN}, ϕ is a real scalar field, and $F = dA$ is the Maxwell field.

A.1 The Einstein Equations

The Einstein equations are obtained from

$$\frac{\partial\mathcal{L}_B}{\partial g_{MN}} - \partial_P\left(\frac{\partial\mathcal{L}_B}{\partial\left(\partial_P g_{MN}\right)}\right) = 0, \tag{A.2}$$

but before computing the Einstein equations it is useful to have a couple identities ready to use :

$$\begin{aligned}
\frac{\partial\sqrt{-g}}{\partial g_{PQ}} &= \frac{1}{2}\sqrt{-g}g^{PQ}, \\
\frac{\partial g^{MN}}{\partial g_{PQ}} &= -\frac{1}{2}(g^{MP}g^{NQ} + g^{MQ}g^{NP}).
\end{aligned} \tag{A.3}$$

It is also be necessary to write $\mathcal{L}_B$ in such a way as to make all the factors of g explicit, doing this results in

$$\mathcal{L}_B = -\sqrt{-g}\left[\frac{1}{2\kappa^2}g^{MN}\left(\mathcal{R}_{MN} + \partial_M\phi\partial_N\phi\right) + \frac{1}{4}fg^{MR}g^{NS}F_{MN}F_{RS} + V\right]. \tag{A.4}$$

Computing $\partial_{\partial_P g_{MN}}\mathcal{L}_B$ is very cumbersome since the Ricci tensor is a function of the Christoffel symbols which are a function of derivatives of g_{MN}, so to avoid this we use the Palatini approach, where we treat Γ^{Q}_{MN} and g_{MN} as independent variables and vary the action with respect to each individually. Varying the action with respect to the Chrsitoffel symbols simply yields the well known formula,

$$\Gamma^{Q}_{MN} = \frac{1}{2} g^{QP} (\partial_M g_{PN} + \partial_N g_{PM} - \partial_P g_{MN}), \tag{A.5}$$

and so with this approach the Einstein equations become

$$\frac{\partial \mathcal{L}_B}{\partial g_{PQ}} = 0. \tag{A.6}$$

Using (A.3) we get

$$\begin{aligned}\frac{\partial \mathcal{L}_B}{\partial g_{PQ}} = \sqrt{-g} \Big[&\frac{1}{2\kappa^2} \mathcal{R}^{PQ} + \frac{1}{2\kappa^2} \partial^P \phi \partial^Q \phi - \frac{1}{4\kappa^2} g^{PQ} \mathcal{R} - \frac{1}{4\kappa^2} g^{PQ} \partial^N \phi \partial_N \phi - \frac{1}{2} g^{PQ} V \\ &- \frac{1}{8} f g^{PQ} F^{MN} F_{MN} + \frac{1}{2} f F^{PN} F^{Q}_{\ \ N} \Big]. \end{aligned} \tag{A.7}$$

Setting this to zero gives the Einstein equation,

$$\begin{aligned}&\frac{1}{2\kappa^2} \mathcal{R}^{PQ} + \frac{1}{2\kappa^2} \partial^P \phi \partial^Q \phi - \frac{1}{4\kappa^2} g^{PQ} \mathcal{R} - \frac{1}{4\kappa^2} g^{PQ} \partial^N \phi \partial_N \phi - \frac{1}{2} g^{PQ} V \\ &\quad - \frac{1}{8} f g^{PQ} F^{MN} F_{MN} + \frac{1}{2} f F^{PN} F^{Q}_{\ \ N} = 0, \end{aligned} \tag{A.8}$$

but further manipulations make it easier to work with. We first contract (A.8) with g_{PQ} and solve for $\mathcal{R}$,

$$\frac{(n-2)}{4\kappa^2} \mathcal{R} = -\frac{(n-2)}{4\kappa^2} \partial^N \phi \partial_N \phi - \frac{n}{2} V - \frac{(n-4)}{8} f F^{MN} F_{MN}. \tag{A.9}$$

Putting this back into (A.8) and lowering indices yields

$$\frac{1}{2\kappa^2} (\mathcal{R}_{MN} + \partial_M \phi \partial_N \phi) + \frac{f}{2} F_M^{\ \ P} F_{NP} + \frac{1}{n-2} \left[V - \frac{f}{4} F_{PQ} F^{PQ} \right] g_{MN} = 0, \tag{A.10}$$

which are the Einstein equations in the desired form.

A.2 The Dilaton Equation

The dilaton equation is obtained from

$$\frac{\partial \mathcal{L}_B}{\partial \phi} - \partial_P \left(\frac{\partial \mathcal{L}_B}{\partial (\partial_P \phi)} \right) = 0. \tag{A.11}$$

We begin by computing $\partial_\phi \mathcal{L}_B$ and $\partial_{\partial_P \phi} \mathcal{L}_B$:

$$\frac{\partial \mathcal{L}_B}{\partial \phi} = -\sqrt{-g}\left[\frac{\partial V}{\partial \phi} + \frac{1}{4}\frac{\partial f}{\partial \phi} F_{MN}F^{MN}\right], \tag{A.12}$$

$$\frac{\partial \mathcal{L}_B}{\partial(\partial_P \phi)} = -\sqrt{-g}\frac{1}{\kappa^2}\partial^P \phi. \tag{A.13}$$

Now notice that,

$$\begin{aligned}\Box\phi &= \nabla_P \nabla^P \phi \\ &= \nabla_P \partial^P \phi \\ &= \frac{1}{\sqrt{-g}}\nabla_P\left(\sqrt{-g}\partial^P \phi\right) \\ &= \frac{1}{\sqrt{-g}}\partial_P\left(\sqrt{-g}\partial^P \phi\right),\end{aligned} \tag{A.14}$$

where we have used the facts that $\nabla_M(\sqrt{-g}) = 0$ and $\nabla_M\left[\sqrt{-g}\partial^M \phi\right] = \partial_M\left[\sqrt{-g}\partial^M \phi\right]$. So

$$\begin{aligned}\partial_P\left(\frac{\partial \mathcal{L}_B}{\partial(\partial_P \phi)}\right) &= -\partial_P\left(\sqrt{-g}\frac{1}{\kappa^2}\partial^P \phi\right) \\ &= -\sqrt{-g}\frac{1}{\kappa^2}\Box\phi.\end{aligned} \tag{A.15}$$

Combining these results gives the dilaton equation,

$$\Box\phi - \kappa^2\left[\frac{\partial V}{\partial \phi} + \frac{1}{4}\frac{\partial f}{\partial \phi} F_{MN}F^{MN}\right] = 0. \tag{A.16}$$

A.3 The Maxwell Equations

The Maxwell equations are obtained from

$$\frac{\partial \mathcal{L}_B}{\partial A_M} - \partial_P\left(\frac{\partial \mathcal{L}_B}{\partial\left(\partial_P A_M\right)}\right) = 0, \tag{A.17}$$

but before computing anything it is necessary to make all factors of A_M,

$$F_{MN} = \partial_M A_N - \partial_N A_M, \tag{A.18}$$

in the Lagrangian explicit. Notice that,

$$\begin{aligned}F^{MN}F_{MN} &= g^{MR}g^{NS}F_{MN}F_{RS} \\ &= g^{MR}g^{NS}(\partial_M A_N - \partial_N A_M)(\partial_R A_S - \partial_S A_R),\end{aligned} \tag{A.19}$$

so the Lagrangian can be written as

$$\mathcal{L}_B = -\sqrt{-g}\left[\frac{1}{2\kappa^2}g^{MN}\left(\mathcal{R}_{MN} + \partial_M\phi\partial_N\phi\right)\right.$$
$$\left. + \frac{1}{4}f(\phi)g^{MR}g^{NS}(\partial_M A_N - \partial_N A_M)(\partial_R A_S - \partial_S A_R) + V(\phi)\right]. \tag{A.20}$$

It can now be computed that

$$\frac{\partial \mathcal{L}_B}{\partial A_C} = 0, \tag{A.21}$$
$$\frac{\partial \mathcal{L}_B}{\partial(\partial_D A_C)} = \sqrt{-g}fF^{CD}, \tag{A.22}$$

using

$$\frac{\partial(\partial_M A_N)}{\partial(\partial_D A_C)} = \delta^D_M \delta^C_N. \tag{A.23}$$

Combining these results yields

$$\partial_M(\sqrt{-g}fF^{MN}) = 0, \tag{A.24}$$

but further manipulations can still be made. Notice that

$$\begin{aligned}
\partial_M\left(\sqrt{-g}fF^{MN}\right) &= \sqrt{-g}\partial_M\left(fF^{MN}\right) + \frac{1}{2\sqrt{-g}}\partial_M g f F^{MN} \\
&= \sqrt{-g}\partial_M\left(fF^{MN}\right) + \sqrt{-g}\Gamma^P_{PM}fF^{MN} \\
&= \sqrt{-g}\nabla_M\left(fF^{MN}\right)
\end{aligned} \tag{A.25}$$

where we've used the facts that $\partial_M g = 2g\Gamma^P_{PM}$ and $\Gamma^N_{MD}F^{MD} = 0$ since F_{MN} is antisymmetric and Γ^P_{MN} is symmetric. Thus the Maxwell field equations are

$$\nabla_M\left(fF^{MN}\right) = 0. \tag{A.26}$$

Appendix B

Dimensional Reduction of the Ricci Tensor

Consider an n dimensional metric g_{MN} given by

$$g_{MN} = \begin{pmatrix} e^{2C(x^\mu)}\tilde{h}_{ab}(x^c) & 0 \\ 0 & e^{2W(x^a)}\hat{g}_{\mu\nu}(x^\lambda) \end{pmatrix}.$$

$\tilde{h}$, and $\hat{g}$ have dimensions d and n_1 respectively, such that

$$d + n_1 = n. \tag{B.1}$$

Our convention for indicies is as follows

$$\begin{aligned} &\{a, b, ..., h\} = \{1, 2, ..., d\} && \text{associated with } \tilde{h}_{ab} \\ \text{and} \quad &\{\alpha, \beta, ...\} = \{d+1, d+2, ..., d+n_1\} && \text{associated with } \hat{g}_{\mu\nu} \end{aligned} \tag{B.2}$$

We also have assumed that $\tilde{h}_{ab} = \tilde{h}_{ab}(x^c)$ and $\hat{g}_{\mu\nu} = \hat{g}_{\mu\nu}(x^\lambda)$ only depend on the coordinates x^c and x^λ respectively. The warp factors $C = C(x^\mu)$ and $W = W(x^a)$ are depend only on the coordinates x^μ and x^a respectively. Any quantity with a $\hat{}$ on it is constructed solely from $\hat{g}$, and similiarly for $\tilde{}$. The Christoffel symbols can be calculated and the non-zero ones are

$$\begin{aligned} &\Gamma^c_{ab} = \tilde{\Gamma}^c_{ab}, \quad \Gamma^c_{\mu\nu} = -e^{2(W-C)}\tilde{h}^{cd}\hat{g}_{\mu\nu}\tilde{\nabla}_d W, \quad \Gamma^c_{\mu b} = \delta^c_b \hat{\nabla}_\mu C, \\ &\Gamma^\kappa_{\mu\nu} = \hat{\Gamma}^\kappa_{\mu\nu}, \quad \Gamma^\kappa_{ab} = -e^{-2(W-C)}\tilde{h}_{ab}\hat{g}^{\kappa\sigma}\hat{\nabla}_\sigma C, \quad \Gamma^\kappa_{a\nu} = \delta^\kappa_\nu \tilde{\nabla}_a W. \end{aligned} \tag{B.3}$$

From this we can compute the (relevant) components of the Ricci tensor, which are

$$\mathcal{R}_{\mu\nu} = \hat{R}_{\mu\nu} + n_1 e^{2(W-C)}\hat{g}_{\mu\nu}\tilde{h}^{ab}\tilde{\nabla}_a W\tilde{\nabla}_b W + e^{2(W-C)}\hat{g}_{\mu\nu}\tilde{\Box}W + d\hat{\nabla}_\mu C\hat{\nabla}_\nu C + d\hat{\nabla}_\mu\hat{\nabla}_\nu C, \tag{B.4}$$

$$\mathcal{R}_{ab} = \tilde{R}_{ab} + de^{-2(W-C)}\hat{g}^{\mu\nu}\tilde{h}_{ab}\hat{\nabla}_\mu C\hat{\nabla}_\nu C + e^{-2(W-C)}\tilde{h}_{ab}\hat{\Box}C + n_1\tilde{\nabla}_a W\tilde{\nabla}_b W + n_1\tilde{\nabla}_a\tilde{\nabla}_b W. \tag{B.5}$$

In calculating the components of the Ricci tensor the following convention was employed

$$\mathcal{R}_{AB} = \Gamma^C_{AC,B} - \Gamma^C_{AB,C} + \Gamma^C_{EB}\Gamma^E_{AC} - \Gamma^C_{EC}\Gamma^E_{AB} \tag{B.6}$$

where the indicies $\{A, B, C, ...\}$ run over all coordinates. If we now also assume that $C = 0$ so that our metric is

$$g_{MN} = \begin{pmatrix} \tilde{h}_{ab}(x^c) & 0 \\ 0 & e^{2W(x^a)}\hat{g}_{\mu\nu}(x^\lambda) \end{pmatrix}.$$

our equations become

$$\mathcal{R}_{\mu\nu} = \hat{R}_{\mu\nu} + n_1 e^{2W}\hat{g}_{\mu\nu}\tilde{h}^{ab}\tilde{\nabla}_a W\tilde{\nabla}_b W + e^{2W}\hat{g}_{\mu\nu}\tilde{\Box}W, \tag{B.7}$$

$$\mathcal{R}_{ab} = \tilde{R}_{ab} + n_1\tilde{\nabla}_a W\tilde{\nabla}_b W + n_1\tilde{\nabla}_a\tilde{\nabla}_b W. \tag{B.8}$$

Appendix C

Boundary Considerations

C.1 Gauss-Codazzi Equations

Any n dimensional space, M, can be thought of as being a series of $n-1$ dimensional hypersurfaces, Σ, whos curvature is characterized by two quantities, the intrinsic and extrinsic curvatures. The Gauss-Codazzi equations relate the curvature of the full n dimensional space to the intrinsic and extrinsic curvatures of the hypersurfaces. To simplify the derivation of these equations, we choose coordinates such that these hypersurfaces are surfaces of constant coordinate, ρ, and for which the metric is

$$\begin{aligned} ds^2 &= g_{MN} dx^M dx^N \\ &= e^{2C} d\rho^2 + \bar{g}_{mn} dx^m dx^n, \end{aligned} \tag{C.1}$$

where C can depend on all coordinates. Our convention here is that capital letters run over all indices and small letters run over all indices except ρ. In these coordinates $\bar{g}_{mn} = \bar{g}_{mn}(\rho, x)$ defines the intrinsic geometry on the surfaces of constant ρ, Σ_ρ. Although $\bar{g}_{mn}$ is simply g_{MN} restricted to all coordinates except ρ, we introduce the bar notation because we deal with quantities constructed solely from $\bar{g}_{mn}$, and this distinction allows for more clarity in future calculations. Any barred variable indicates it has been constructed from $\bar{g}_{mn}$. The Christoffel symbols of g_{MN} are

$$\begin{aligned} &\Gamma^r_{mn} = \bar{\Gamma}^r_{mn}, \quad &&\Gamma^\rho_{mn} = -\frac{1}{2} e^{-2C} \partial_\rho \bar{g}_{mn}, \quad &&\Gamma^m_{\rho n} = \frac{1}{2} \bar{g}^{mr} \partial_\rho \bar{g}_{nr}, \\ &\Gamma^\rho_{\rho n} = \partial_n C, \quad &&\Gamma^r_{\rho\rho} = -e^{2C} \bar{g}^{rm} \partial_m C, \quad &&\Gamma^\rho_{\rho\rho} = \partial_\rho C. \end{aligned} \tag{C.2}$$

The Riemann (intrinsic) curvature on the surfaces Σ_ρ is defined through the Christoffel symbols, $\bar{\Gamma}^m_{nr} = \frac{1}{2}\bar{g}^{ms}(\partial_n \bar{g}_{rs} + \partial_r \bar{g}_{ns} - \partial_s \bar{g}_{nr})$, in the usual way,

$$\bar{R}^m{}_{nrs} = \partial_s \bar{\Gamma}^m_{nr} + \bar{\Gamma}^m_{sq}\bar{\Gamma}^q_{nr} - (r \leftrightarrow s). \tag{C.3}$$

If we consider a particular surface, $\Sigma_{\rho_b} = \{\rho = \rho_b\}$, one can think of dividing our space, M, into two parts, $M_\pm$, lying on either side of Σ_{ρ_b}, with M_+ denoting the space $\rho > \rho_b$, and M_- denoting the space $\rho < \rho_b$. The extrinsic curvature of the surfaces Σ_ρ, K_{mn}, is defined in terms of the outward pointing unit normal,

$$N^M = \pm e^{-C}\delta^M_\rho, \tag{C.4}$$

by

$$K_{MN} = P_M{}^P P_N{}^R \nabla_P N_R \tag{C.5}$$

where $P_M{}^N = \delta^N_M - N_M N^N$ is the projector onto the surfaces Σ. The $\pm$ must be chosen appropriately depending on which side of the surface Σ_ρ we are considering, with the understanding that $N_M dx^M = \mp e^C d\rho^2$ in the space $M_\pm$. Without loss of generality, we'll do calculations as though we are in M_-, and any time we need to consider being in M_+ we can simply perform the transformation $K_{MN} \to -K_{MN}$. A quick calculation yields,

$$K_{MN} = -e^C \Gamma^\rho_{MN} + \delta^\rho_N e^C \Gamma^\rho_{\rho M} + \delta^\rho_M e^C \Gamma^\rho_{\rho N}, \tag{C.6}$$

and in particular

$$\begin{aligned} K_{mn} &= -e^C \Gamma^\rho_{mn} = \frac{1}{2}e^{-C}\partial_\rho \bar{g}_{mn} \\ K_{\rho\rho} &= e^C \partial_\rho C, \qquad K_{\rho m} = 0. \end{aligned} \tag{C.7}$$

It's useful to express the Christoffel symbols, (C.2), in terms of K_{MN} when possible, and doing so yields

$$\Gamma^m_{\rho n} = e^C K^m{}_n, \quad \Gamma^\rho_{mn} = -e^{-C} K_{mn}, \quad \Gamma^\rho_{\rho\rho} = e^{-C} K_{\rho\rho}. \tag{C.8}$$

All nonzero components of the Riemann tensor, calculated in the usual way, are related to the following components by the symmetries of the Riemann tensor,

$$\begin{aligned}
\mathcal{R}^{m}{}_{nrs} &= \bar{R}^{m}{}_{nrs} + K^{m}{}_{r}K_{ns} - K^{m}{}_{s}K_{nr}, \\
\mathcal{R}^{\rho}{}_{mnr} &= \bar{\nabla}_{n}(e^{-C}K_{mr}) - \bar{\nabla}_{r}(e^{C}K_{mn}), \\
\mathcal{R}^{\rho}{}_{m\rho n} &= e^{-C}\partial_{\rho}K_{mn} - K_{mq}K^{q}\ \ s + \bar{\nabla}_{m}\bar{\nabla}_{n}C + \bar{\nabla}_{m}C\bar{\nabla}_{n}C \\
&= e^{-C}\nabla_{\rho}K_{mn} + K_{mq}K^{q}{}_{n} + \bar{\nabla}_{m}\bar{\nabla}_{n}C + \bar{\nabla}_{m}C\bar{\nabla}_{n}C
\end{aligned} \tag{C.9}$$

The last line makes use of

$$\begin{aligned}
\nabla_{\rho}K_{mn} &= \partial_{\rho}K_{mn} - \Gamma^{s}_{\rho m}K_{sn} - \Gamma^{s}_{\rho n}K_{sm} \\
&= \partial_{\rho}K_{mn} - 2e^{C}K_{ms}K^{s}{}_{n}.
\end{aligned} \tag{C.10}$$

The components of the Ricci tensor, $\mathcal{R}_{MN} = \mathcal{R}^{P}{}_{MPN}$, are

$$\begin{aligned}
\mathcal{R}_{mn} &= \bar{R}_{mn} + e^{-C}\nabla_{\rho}K_{mn} + KK_{mn} + \bar{\nabla}_{m}\bar{\nabla}_{n}C + \bar{\nabla}_{m}C\bar{\nabla}_{n}C, \\
\mathcal{R}_{\rho m} &= e^{2C}\bar{\nabla}_{m}(e^{-C}K) - e^{2C}\bar{\nabla}^{n}(e^{-C}K_{mn}), \\
\mathcal{R}_{\rho\rho} &= e^{C}\partial_{\rho}K + e^{2C}K_{mn}K^{mn} + e^{2C}\bar{\Box}C + e^{2C}\bar{g}^{mn}\bar{\nabla}_{m}C\bar{\nabla}_{n}C,
\end{aligned} \tag{C.11}$$

where $K = \bar{g}^{mn}K_{mn} = K^{m}{}_{m}$. The Ricci scalar, $\mathcal{R} = g^{MN}\mathcal{R}_{MN}$, is

$$\mathcal{R} = \bar{R} + 2e^{-C}\partial_{\rho}K + K_{mn}K^{mn} + K^{2} + 2\bar{g}^{mn}\bar{\nabla}_{m}C\bar{\nabla}_{n}C + 2\bar{\Box}C. \tag{C.12}$$

Notice that

$$\partial_{\rho}\left(\sqrt{-g}\right) = \frac{1}{2}\sqrt{-g}\,\bar{g}^{mn}\partial_{\rho}\bar{g}_{mn} = \sqrt{-g}\,e^{C}K \tag{C.13}$$

which also implies

$$\partial_{\rho}\left(\sqrt{-g}F\right) = \sqrt{-g}\left(\partial_{\rho}F + e^{C}KF\right) \tag{C.14}$$

when F is any scalar, in particular when applied to $F = K$ we get

$$\sqrt{-g}\partial_{\rho}K = \partial_{\rho}\left(\sqrt{-g}K\right) - \sqrt{-g}e^{C}K^{2}. \tag{C.15}$$

Applying this to $\sqrt{-g}\mathcal{R}$ yields

$$\begin{aligned}\sqrt{-g}\mathcal{R} &= \sqrt{-g}\left(\bar{R} + 2e^{-C}\partial_\rho K + K_{mn}K^{mn} + K^2 + 2\bar{g}^{mn}\bar{\nabla}_m C\bar{\nabla}_n C + 2\bar{\Box}C\right) \\ &= \partial_\rho\left(2\sqrt{-\bar{g}}K\right) + \sqrt{-g}\left(\bar{R} + K_{mn}K^{mn} - K^2 + 2\bar{g}^{mn}\bar{\nabla}_m C\bar{\nabla}_n C + 2\bar{\Box}C\right). \qquad (C.16)\end{aligned}$$

C.2 The Gibbons-Hawking Action

One of the main considerations in this text is to determine how discontinuites of the bulk fields are governed by brane properties. To study this we'll consider a system composed of a bulk action, S_B, given by (2.2), and a codimension-1 brane action, $S_b^{(1)}$, which is a boundary term. But notice that the Einstein-Hilbert action (which is part of the bulk action),

$$S_{EH} = -\frac{1}{2\kappa^2}\int d^n x\,\sqrt{-g}\mathcal{R}, \qquad (C.17)$$

can be written as the sum of a bulk and boundary term using (C.16),

$$S_{EH} = -\frac{1}{2\kappa^2}\int d^n x\sqrt{-g}\left(\bar{R} + K_{mn}K^{mn} - K^2 + 2\bar{g}^{mn}\bar{\nabla}_m C\bar{\nabla}_n C + 2\bar{\Box}C\right) - \frac{1}{\kappa^2}\int d^{n-1}x\sqrt{-\bar{g}}K. \qquad (C.18)$$

When varying the action, $S_B + S_b^{(1)}$, the boundary contribution due S_B leads to terms like $\partial_\rho\delta\bar{g}_{mn}$, while varying $S_b^{(1)}$ leads to terms like $\delta\bar{g}_{mn}$ on the boundary. Since $\partial_\rho\delta\bar{g}_{mn}$ and $\delta\bar{g}_{mn}$ can be varied independently on the boundary, the result is an over-constrained problem. Thus the variation of the action $S_B + S_b^{(1)}$ is not a well posed problem. In order to compensate for this, the boundary term arising from the bulk action must be explicitly removed by the addition of an appropriate boundary term to the system's action, this is the Gibbons-Hawking action,

$$S_{GH} = +\frac{1}{\kappa^2}\int d^{n-1}x\,\sqrt{-\bar{g}}K. \qquad (C.19)$$

So the full action is obtained by supplementing the bulk and brane actions with the Gibbons-Hawking action,

$$S = S_B + S_{GH} + S_b^{(1)}. \qquad (C.20)$$

C.3 Derivation of Junction Conditions

The junction conditions describe how the presence of a brane action affects the bulk fields at the boundary. They are the result of minimizing the boundary term of the variation of the full action.

We'll use the notation $[F]_{\rho_b}$ for a bulk quantity which denotes

$$[F]_{\rho_b} = \lim_{\epsilon \to 0} \left[F(\rho_b + \epsilon) - F(\rho_b - \epsilon) \right]. \tag{C.21}$$

C.3.1 The Metric Junction Condition

Before varying the action with respect to $\bar{g}_{mn}$ it is be useful to note that

$$\begin{aligned} \delta(K_{mn}) &= \frac{1}{2} e^{-C} \partial_\rho \delta \bar{g}_{mn}, \\ \delta K = \delta(\bar{g}^{mn} K_{mn}) &= -\bar{g}^{mr} \bar{g}^{ns} K_{mn} \delta \bar{g}_{rs} + \frac{1}{2} e^{-C} \bar{g}^{mn} \partial_\rho \delta \bar{g}_{mn}. \end{aligned} \tag{C.22}$$

Since we only keep track of the boundary terms in the variation of the action $S_{B'} = S_B + S_{GH}$ with respect to $\bar{g}_{mn}$, we only explicitly show the variation of terms that lead to terms that go like $\partial_\rho \delta \bar{g}_{mn}$. Thus

$$\begin{aligned} \delta S &= \delta S_B + \delta S_{GH} + \delta S_b^{(1)} \\ &= -\frac{1}{2\kappa^2} \int d^n x \sqrt{-g} \left[\delta(K_{mn} K^{mn}) - \delta(K^2) \right] + \delta S_b^{(1)} + \ldots \\ &= -\frac{1}{2\kappa^2} \int d^n x \sqrt{-\bar{g}} \left[K^{mn} \partial_\rho \delta \bar{g}_{mn} - K \bar{g}^{mn} \partial_\rho \delta \bar{g}_{mn} \right] + \delta S_b^{(1)} + \ldots \\ &= -\frac{1}{2\kappa^2} \int d^n x \, \partial_\rho \left(\sqrt{-\bar{g}} \left[K^{mn} \delta \bar{g}_{mn} - K \bar{g}^{mn} \delta \bar{g}_{mn} \right] \right) + \delta S_b^{(1)} + \ldots \\ &= \frac{1}{2\kappa^2} \int_{\rho_b + \epsilon} d^{n-1} x \, \sqrt{-\bar{g}} \left[K^{mn} \delta \bar{g}_{mn} - K \bar{g}^{mn} \delta \bar{g}_{mn} \right] \\ &\quad - \frac{1}{2\kappa^2} \int_{\rho_b - \epsilon} d^{n-1} x \, \sqrt{-\bar{g}} \left[K^{mn} \delta \bar{g}_{mn} - K \bar{g}^{mn} \delta \bar{g}_{mn} \right] + \delta S_b^{(1)} + \ldots . \end{aligned} \tag{C.23}$$

In going from the second to third line we have used,

$$\begin{aligned} \partial_\rho \left(\sqrt{-\bar{g}} K^{mn} \delta \bar{g}_{mn} \right) &= \sqrt{-\bar{g}} K^{mn} \partial_\rho \delta \bar{g}_{mn} + \partial_\rho \left(\sqrt{-\bar{g}} K^{mn} \right) \delta \bar{g}_{mn}, \\ \partial_\rho \left(\sqrt{-\bar{g}} K \bar{g}^{mn} \delta \bar{g}_{mn} \right) &= \sqrt{-\bar{g}} K \bar{g}^{mn} \partial_\rho \delta \bar{g}_{mn} + \partial_\rho \left(\sqrt{-\bar{g}} K \bar{g}^{mn} \right) \delta \bar{g}_{mn}. \end{aligned} \tag{C.24}$$

The elipses denote all other terms in the variation that do not lead to boundary terms, in fact these terms lead to the field equations, which are derived in Appendix A. In going from the third to fourth line we have assumed that the brane action is defined at $\rho = \rho_b$, then integrated, and used the appropriate normal vectors at the positions $\rho_b + \epsilon$ and $\rho_b - \epsilon$. Now taking the limit as $\epsilon \to 0$ and setting the variation equal to 0 gives us our metric junction condition, which is

$$\frac{1}{2\kappa^2} \left[\sqrt{-\bar{g}} \left(K^{mn} - K \bar{g}^{mn} \right) \right]_{\rho_b} + \frac{\delta S_b^{(1)}}{\delta \bar{g}_{mn}} = 0. \tag{C.25}$$

C.3.2 The Scalar Junction Condition

The only term in bulk action that leads to surface terms upon variation with respect to ϕ comes from the bulk action, (2.1), and is

$$S_\phi = -\frac{1}{2\kappa^2}\int d^n x\,\sqrt{-g}g^{MN}\partial_M\phi\partial_N\phi. \tag{C.26}$$

Varying results in

$$\begin{aligned}
\delta S &= \delta S_\phi + \delta S_b^{(1)} + \dots \\
&= -\frac{1}{2\kappa^2}\int d^n x\,\sqrt{-g}g^{MN}\delta(\partial_M\phi\partial_N\phi) + \delta S_b^{(1)} + \dots \\
&= -\frac{1}{\kappa^2}\int d^n x\,\sqrt{-g}g^{MN}\partial_M\phi\partial_N\delta\phi + \delta S_b^{(1)} + \dots \\
&= -\frac{1}{\kappa^2}\int_{\rho_b+\epsilon} d^{n-1}x\,\sqrt{-g}N^M\partial_M\phi\delta\phi - \frac{1}{\kappa^2}\int_{\rho_b-\epsilon} d^{n-1}x\,\sqrt{-g}N^M\partial_M\phi\delta\phi + \delta S_b^{(1)} + \dots \\
&= \frac{1}{\kappa^2}\int_{\rho_b+\epsilon} d^{n-1}x\,\sqrt{-\bar{g}}\partial_\rho\phi\delta\phi - \frac{1}{\kappa^2}\int_{\rho_b-\epsilon} d^{n-1}x\,\sqrt{-\bar{g}}\partial_\rho\phi\delta\phi + \delta S_b^{(1)} + \dots,
\end{aligned} \tag{C.27}$$

where we have used the divergence theorem to go from the second to third line. In the same way as before, this leads to the junction condition

$$\frac{1}{\kappa^2}\left[\sqrt{-\bar{g}}\partial_\rho\phi\right]_{\rho_b} + \frac{\delta S_b^{(1)}}{\delta\phi} = 0. \tag{C.28}$$

C.3.3 The Maxwell Junction Condition

The relevant term in the action is

$$S_A = -\frac{1}{4}\int d^n x\,\sqrt{-g}fF^{MN}F_{MN}. \tag{C.29}$$

Varying gives

$$\begin{aligned}
\delta S &= \delta S_A + \delta S_b^{(1)} + \dots \\
&= -\frac{1}{2}\int d^n x\,\sqrt{-g}fg^{MR}g^{NS}\left(\partial_M A_N\partial_R\delta A_S - \partial_M A_N\partial_S\delta A_R - \partial_N A_M\partial_R\delta A_S + \partial_N A_M\partial_S\delta A_R\right) \\
&\qquad + \delta S_b^{(1)}\dots \\
&= -\int d^n x\,\sqrt{-g}fF^{MN}\partial_M\delta A_N + \delta S_b^{(1)} + \dots \\
&= -\int d^n x\,\partial_M\left(\sqrt{-g}fF^{MN}\delta A_N\right) + \delta S_b^{(1)} + \dots,
\end{aligned} \tag{C.30}$$

where we've used

$$\partial_M \left(\sqrt{-g} f F^{MN} \delta A_N\right) = \sqrt{-g} f F^{MN} \partial_M \delta A_N + \partial_M \left(\sqrt{-g} f F^{MN}\right) \delta A_N, \tag{C.31}$$

and ignored irrelevant terms. In particular if we take $M = \rho$, then

$$\delta S = \int_{\rho_b+\epsilon} d^{n-1}x \sqrt{-\bar{g}} e^C f F^{\rho N} \delta A_N - \int_{\rho_b-\epsilon} d^{n-1}x \sqrt{-\bar{g}} e^C f F^{\rho N} \delta A_N + \delta S_b^{(1)} + \ldots, \tag{C.32}$$

which gives us our Maxwell junction condition

$$\left[\sqrt{-\bar{g}} e^C f F^{\rho M}\right]_{\rho_b} + \frac{\delta S_b^{(1)}}{\delta A_M} = 0. \tag{C.33}$$

Appendix D

Solutions to Field Equations

D.1 Maximally Symmetric Branes

Consider the following bulk action

$$S_B = -\int_{\mathcal{M}} d^6x\sqrt{-g}\left[\frac{1}{2\kappa^2}g^{MN}\left(\mathcal{R}_{MN} + \partial_M\phi\partial_N\phi\right) + \frac{1}{4}F_{MN}F^{MN} + \Lambda\right], \tag{D.1}$$

which is (2.2) with $n = 6$, $f = 1$, $V = \Lambda$. We now find solutions to the field equations of this action for maximally symmetric branes. Recall our ansatz,

$$\begin{aligned} ds^2 &= d\rho^2 + e^{2B(\rho)}d\theta^2 + e^{2W(\rho)}\hat{g}_{\mu\nu}(x^\sigma)dx^\mu dx^\nu, \\ A &= A_\theta(\rho)d\theta, \\ \phi &= \phi(\rho), \end{aligned} \tag{D.2}$$

where $\hat{g}_{\mu\nu}$ is a four dimensional maximally symmetric Minkowski-signature metric, which yield the field equations

$$\frac{1}{4}e^{-2W}\hat{R} + W'' + 4(W')^2 + W'B' - \frac{1}{4}\kappa^2 e^{-2B}(A_\theta')^2 + \frac{1}{2}\kappa^2\Lambda = 0, \qquad (\mu\nu) \tag{D.3}$$

$$B'' + (B')^2 + 4W'B' + \frac{3}{4}\kappa^2 e^{-2B}(A_\theta')^2 + \frac{1}{2}\kappa^2\Lambda = 0, \qquad (\theta\theta) \tag{D.4}$$

$$4W'' + 4(W')^2 + B'' + (B')^2 + (\phi')^2 + \frac{3}{4}\kappa^2 e^{-2B}(A_\theta')^2 + \frac{1}{2}\kappa^2\Lambda = 0, \qquad (\rho\rho) \tag{D.5}$$

$$\phi'' + [B' + 4W']\phi' = 0, \qquad \text{(Dilaton)} \tag{D.6}$$

$$\left(e^{-B+4W}A_\theta'\right)' = 0. \qquad \text{(Maxwell)} \tag{D.7}$$

To simplify our search for solutions, we'll look for solutions with $W = 0$ and $\phi = \phi_0$, a constant. Our field equations then become

$$\hat{R} + 2\kappa^2\Lambda - \kappa^2 e^{-2B}(A'_\theta)^2 = 0, \tag{D.8}$$

$$B'' + (B')^2 + \frac{1}{2}\kappa^2\Lambda + \frac{3}{4}\kappa^2 e^{-2B}(A'_\theta)^2 = 0, \tag{D.9}$$

$$\left(e^{-B}A'_\theta\right)' = 0. \tag{D.10}$$

Equation (D.10) gives

$$e^{-B}A'_\theta = \mathcal{B}_0 \tag{D.11}$$

where $\mathcal{B}_0$ is a constant, and when used in (D.8) we find that

$$\hat{R} = 2\kappa^2\left(\frac{\mathcal{B}_0^2}{2} - \Lambda\right). \tag{D.12}$$

Multiplying (D.9) by e^B yields

$$\left(e^B\right)'' + \frac{1}{L^2}e^B = 0 \tag{D.13}$$

where L is defined by $\frac{1}{L^2} = \frac{\kappa^2}{2}\left(\frac{3\mathcal{B}_0^2}{2} + \Lambda\right)$. The solution to (D.13) is

$$e^B = \alpha L \sin\left(\frac{\rho}{L}\right), \tag{D.14}$$

and so

$$A'_\theta = \alpha\mathcal{B}_0 L \sin\left(\frac{\rho}{L}\right). \tag{D.15}$$

In summary, a solution to (D.1) subject to the ansatz (D.2) is

$$\begin{aligned} ds^2 &= d\rho^2 + \alpha^2 L^2 \sin^2\left(\frac{\rho}{L}\right) d\theta^2 + \hat{g}_{\mu\nu}dx^\mu dx^\nu, \\ F_{\rho\theta} &= \alpha\mathcal{B}_0 L \sin\left(\frac{\rho}{L}\right), \\ \phi &= \phi_0. \end{aligned} \tag{D.16}$$

With this we can calculate the christoffel symbols of the extra dimensional metric which are

$$\begin{aligned} \Gamma^\rho_{\theta\theta} &= -\alpha^2 L \sin\left(\frac{\rho}{L}\right)\cos\left(\frac{\rho}{L}\right), \\ \Gamma^\theta_{\rho\theta} &= \frac{1}{L}\cot\left(\frac{\rho}{L}\right), \end{aligned} \tag{D.17}$$

and the components of the Ricci tensor, and the Ricci scalar are

$$\begin{aligned}
\mathcal{R}_{\rho\rho} &= -\frac{1}{L^2} \\
\mathcal{R}_{\theta\theta} &= -\alpha^2 \sin^2\left(\frac{\rho}{L}\right) \\
\mathcal{R}_{(2)} &= -\frac{2}{L^2}
\end{aligned} \tag{D.18}$$

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