TESTING AN ASSUMED DISTRIBUTION

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ABSTRACT

Testing for an assumed distribution has been a major area of statistical research, both in theory and in practice. A reason for this interest is that many statistical procedures are based on certain distributional assumptions. Two new statistics, B_N and B_L, are suggested in this thesis for testing for normal and logistic distributions. The formulation of these statistics is based on the best linear unbiased estimator of the population scale parameter σ, using order statistics. The distributions of B_N and B_L tend to normal very rapidly, effectively for sample size $n \geq 20$. In general, B_N and B_L have good power properties. They are particularly sensitive in testing against skew distributions or symmetric distributions with large kurtosis. The power of B_N is comparable with other available test-statistics.

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CHAPTER 1

GOODNESS-OF-FIT TESTS

1.1 INTRODUCTION

In statistical methodology and practice,we usually make certain assumptions about the population from which a sample is drawn. For instance, we may assume that the life span of electric bulbs is an exponential function of time;or that the individuals' income in a country is normally distributed. But how valid are our assumptions ? In practice, we might therefore like to test for a particular distribution. A procedure used to test the validity of an assumed distribution on the basis of a random sample is called a "goodness-of-fit" test.

Let, $x_1, x_2, \ldots, x_n$ (1)

be a random sample of size n drawn from a population with c.d.f. (cumulative distribution function) $F(x;\theta)$.We wish to test the null hypothesis

$$H_0 : F(x;\theta) = F_o(x;\theta_1,\theta_2, \ldots, \theta_r) \text{ for all } x, \quad (2)$$

against the general alternative

$$H_1 : F(x;\theta) = F_1(x;\theta_1,\theta_2, \ldots, \theta_r) \text{ for some } x \quad (3)$$

Depending on whether all parameters θ_i in (2) are known or not, the hypothesis is simple or composite.

1.2 SIMPLE HYPOTHESIS

A simple null hypothesis is one in which F_o is specified and $(\theta_1, \theta_2, \ldots, \theta_r)$ are known. Usually, the functional form of F_1 is not known.

1.2.2 The Likelihood Ratio And Pearson Tests For Testing Simple Hypothesis

Suppose the n observations given in (1) are grouped into k mutually exclusive classes.

Let the probability that an observation falls into the ith class $(i=1,2, \ldots ,k)$ be p_{oi} under H_o and p_{1i} under H_1, i.e. ,

$$H_o : p_i = p_{oi} ,$$

$$H_1 : p_i = p_{1i} , i = 1,2, \ldots ,k \quad (p_{1i}\text{'s not known})$$

The joint density function of the observed n_i in the k classes under H_o and H_1 are respectively

$$\frac{n!}{n_1!n_2! \ldots n_k!} (p_{o1})^{n_1} (p_{o2})^{n_2} \cdots (p_{ok})^{n_k} \tag{4}$$

and

$$\frac{n!}{n_1!n_2! \ldots n_k!} (p_{11})^{n_1} (p_{12})^{n_2} \ldots (p_{1k})^{n_k} ; \tag{5}$$

Their likelihood functions are respectively

$$(L \mid H_0) \quad \propto \quad \prod_{i=1}^{k} (p_{0i})^{n_i} \tag{6}$$

$$(L \mid H_1) \quad \propto \quad \prod_{i=1}^{k} (p_{1i})^{n_i} \tag{7}$$

The likelihood function (7) is maximized when p_{1i} is substituted by its ML(maximum Likelihood) estimator $\hat{p}_{1i} = \frac{n_i}{n}$. Therefore from (7) we have

$$\text{Max } (L \mid H_1) \quad \propto \prod_{i=1}^{k} \left(\frac{n_i}{n} \right)^{n_i} \tag{8}$$

The likelihood ratio statistic for testing H_0 against H_1 is

$$\lambda \quad = \frac{\text{Max } (L \mid H_0)}{\text{Max } (L \mid H_1)}$$

$$= \prod_{i=1}^{k} \left(\frac{n\, p_{0i}}{n_i} \right)^{n_i} \tag{9}$$

The exact distribution of statistic λ is difficult to obtain. However, the statistic

$$-2 \log \lambda = 2 \sum_{i=1}^{k} n_i \log\left(\frac{n_i}{np_{0i}}\right) \tag{10}$$

is asymptotically equivalent to the following χ^2 statistic and is , therefore, asymptotically distributed as Chi-square.

Karl Pearson (1900) proposed a statistic X^2 as a measure of the discrepancy between the frequency n_i and its expected value of np_{0i} in a multinormal situation. That is

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - np_{0i})^2}{n\, p_{0i}} \tag{11}$$

For large n and hence large $n = \sum_{i=1}^{k} n_i$, the distribution of χ^2 is approximately chi-square with (k-1) degrees of freedom.

By letting $\Delta_i = \dfrac{n_i - np_{0i}}{np_{0i}}$, we will have

$$-2 \log \lambda = 2 \sum_{i=1}^{k} n_i \log (1 + \Delta_i) \tag{12}$$

But $n_i = np_{0i} (1 + \Delta_i)$ so that

$$-2 \log \lambda = \sum_{i=1}^{k} n\, p_{0i} (1 + \Delta_i) \log (1 + \Delta_i) \tag{13}$$

$$\simeq \sum_{i=1}^{k} n\, p_{0i} (1 + \Delta_i)\, \Delta_i$$

$$\simeq \sum_{i=1}^{k} n\, p_{0i}\, \Delta_i + \sum_{i=1}^{k} n\, p_{0i}\, \Delta_i^2 . \tag{14}$$

But $\sum_{i=1}^{k} n\, p_{0i}\, \Delta_i = \sum_{i=1}^{k} (n_i - n\, p_{0i}) = 0$, thus (14) reduces to

$$-2 \log \lambda \simeq \sum_{i=1}^{k} n\, p_{0_i}\, \Delta_i^2 = \sum_{i=1}^{k} \frac{(n_i - np_{0i})^2}{n\, p_{0i}} = \chi^2 \tag{15}$$

Pearsons' χ^2 test is one of the oldest and best known goodness-of-fit test. It can be used for testing normal, binormal, Poisson or some other known type distribution. Remarkable as it may seem, this test nonetheless suffered from several disadvantages : (a) it does not work well for small samples, (b) the hypothesis must be completely specified , (c) the sample must be grouped into classes and, in practice, one does not know how many groups to make.

1.2.2 Tests Based On the Difference Between The Empirical And The Hypothesized Distribution Functions

Let $X_1 \leqslant X_2 \leqslant \ldots \leqslant X_n$ (16)

denote the order statistics of a random sample from (1). The empirical distribution function of this sample is defined as

$$S_n(x) = \begin{cases} 0 & \text{if } x < X_1 \\ i/n & \text{if } X_i \leqslant x < X_{i+1} \text{ for } i = 1,2, \ldots n-1 \\ 1 & \text{if } x \geqslant X_n \end{cases} \tag{17}$$

Let $f_0(x)$ be the p.d.f. and $F_0(x)$ be the c.d.f of the population specified by the null hypothesis. Let

$$u_i = \int_{-\infty}^{X_i} f_0(x) \, dx \tag{18}$$

If x_i are independent and $f_0(x)$ is completely known, the transformations u's will be independent and uniformly distributed within the interval [0,1].

Several goodness-of-fit tests are based on the maximum difference between $S_n(x)$ and $F_0(x)$. Depending on whether $S_n(x)$ is below or above $F_0(x)$, we can have

$$D_n^+ = \max_{1 \le i \le n} \left(\frac{i}{n} - u_i \right) \quad \text{if } S_n(x) \text{ is above } F_0(x)$$

$$D_n^- = \max_{1 \le i \le n} \left(u_i - \frac{i-1}{n} \right) \quad \text{if } S_n(x) \text{ is below } F_0(x).$$

Test statistics that are functions of such differences are :

Kolmogorov $$D_n = \max_n | S_n(x) - F_0(x) | \tag{19}$$

Cramer-von Mises $$W^2 = \sum_{i=1}^{k} (u_i - \frac{2i-1}{2n}) + \frac{1}{12n} \tag{20}$$

Kuiper $$V = D_n^+ + D_n^- \tag{21}$$

Watson $$u^2 = W^2 - n(\bar{u} - 0.5)^2, \quad \bar{u} = \frac{1}{n}\sum_{i=1}^{n} u_i \tag{22}$$

Anderson-Darling $$u = \frac{1}{n}[-\{ \sum_{i=1}^{n}(2i - 1)(\log u_i + \log(1-u_{n-i+1}))\}]-n \tag{23}$$

The Kolmogorov test and the Chi-square test are alike in two respects : (i) the null hypothesis must be completely specified in both cases, (ii) they are omnibus goodness-of-fit tests in the sense that they can be used for testing any distributions. However, the Kolmogorov test may be preferred over the Chi-square test if the sample size is small, because the Kolmogorov test is exact even for small samples, while the Chi-square test assumes that the number of observations is large enough so that the Chi-square distribution provides a good approximation for its distribution. The Kolmogorov test, in most situations, is also more powerful than the Chi-square test. Nevertheless, like the Chi-square test, the Kolmogorov test is only suitable for testing a simple hypothesis. In testing normality and exponentiality,the Kolmogorov test has been modified by Lilliefors [9] so that it may be used to test the composite hypothesis (see also Stephens [18]).

The Cramer-von Mises is another omnibus goodness-of-fit test. Like the Kolmogorov test,this test is a function of the difference between $S_n(x)$ and $F_o(x)$. Also like the Kolmogorov test, this test is suitable only for simple hypothesis. However, unlike the Kolmogorov test which only considers the greatest discrepancy between $S_n(x)$ and $F_o(x)$, the Cramer-von Mises takes into account all n differences between the two curves. Intuitively, it seems that the Cramer-von Mises test statistic given in (20) makes full use of the data and therefore should be much more effective than the Kolmogorov test statistic, but suprisingly, this is not the case.

1.3 COMPOSITE HYPOTHESIS

A composite hypothesis is the one in which one or more parameters of the hypothesized distribution are unknown.

1.3.1 The Lilliefors Test

This test is a modification of the Kolmogorov test to test the composite hypothesis of normality, that is, the parameters μ and σ are unknown.

Recall that Kolmogorov's test statistic D_n defined by (19) for testing a completely specified distribution $F_0(x)$, is a distribution-free statistic. But if the $F_0(x)$ is not completely known, the transformtion u's defined by (18) may not be independent and uniformly distributed. However, David and Johnson [2] showed that if $F_0(x)$ only depends on the location and scale parameters μ and σ (e.g. a normal distribution) and if these parameters are replaced by proper estimators $\hat{\mu}$ and $\hat{\sigma}$, then the random variable $u = F(x;\hat{\mu},\hat{\sigma})$ under H_0 will be distribution-free, i.e. does not depend on the parameters. Because of this, Lilliefors [9] suggested the use of

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{to replace } \mu \text{ and}$$

$$s = \left[\sum_{i=1}^{n} (x_i - \bar{x})^2 / (n-1) \right]^{\frac{1}{2}} \quad \text{to replace } \sigma.$$

In Lilliefors' test, the sample values x's have to be transformed by the equation

$$z_i = \frac{x_i - \bar{x}}{s}$$

and the test statistic is defined as

$$T_2 = \text{Max} \mid S_n(z) - F_0(z) \mid \tag{24}$$

Lilliefor has determined the percentage points of T_2 by Monte Carlo methods. He [10] took a similar approach with an exponential distribution; (see also Stephens [19]).

1.3.2 The Geary, $\sqrt{b_1}$, b_2 And The Studentized Range Tests

These four tests are for testing normality of a distribution. Define the rth sample moment and the sample standard deviations as

$$m_r = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^r}{n} ,$$

$$s = \left[\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}\right]^{1/2} \quad \text{where } \bar{x} = \sum_{i=1}^{n} x_i / n .$$

The test statistics are

The Studentized Range $$u = \frac{X_n - X_1}{s} \tag{25}$$

Skewness $$\sqrt{b_1} = m_3 / m_2^{3/2} \tag{26}$$

Kurtosis $$b_2 = m_4 / m_2^2 \tag{27}$$

Mean absolute-deviation $$a = \frac{\sum_{i=1}^{n} | X_i - \bar{X} |}{\{n\sum_{i=1}^{n} (X_i - \bar{X})^2\}^{1/2}} \tag{28}$$

The Studentized range statistic was suggested to pick out the large departure from normality. Departure of $\sqrt{b_1}$ from the normal value of zero is an indication of skewness of the sample population, while deviation in b_2 from the normal value of 3 is an indication of abnormal kurtosis. The statistic u is very powerful against distributions which are symmetric (zero skewness) and have small values of kurtosis (less than 3). $\sqrt{b_1}$ is powerful against skew distribution and b_2 is powerful against symmetric non-normal distributions. However, the sampling distributions of these statistics are not known, although some Monte Carlo percentage points are available [12].

1.3.3 Shapiro And Wilks' W for Testing Normality

Shapiro and Wilk had been doing some work on the plotting of observed order statistics against the corresponding expected normal order statistics. Partly initiated by an attempt to summarize those probability plots, they suggested a test based on the use of the order statistics $X_1 \leq X_2 \leq \ldots \leq X_n$. The test statistics [15] proposed was

$$W = \frac{\left(\sum_{i=1}^{n} a_i X_i\right)^2}{\sum_{i=1}^{n} (X_i - \bar{X})^2} \qquad (29)$$

which is proportional to the square of an linear combination of the sample order statistics divided by the usual estimate of variance.

The linear function $\sum_{i=1}^{n} a_i X_i$ they choose is the BLUE of σ, i.e.

$$\sigma^* = \frac{\underset{\sim}{\alpha}\, \underset{\sim}{V}^{-1} \underset{\sim}{X}'}{\underset{\sim}{\alpha}\, \underset{\sim}{V}^{-1}\, \underset{\sim}{\alpha}'} \tag{30}$$

where $\underset{\sim}{\alpha}$ is the vector of mean of the ordered statistics and $\underset{\sim}{V}$ their variance-covariance matrix.

The distribution of W is, however, not known. The value of a_i are available for sample size n ⩽ 50 (see Shapiro & Wilk [15]). The statistic W is very powerful against most non-normal distribution [16].

1.3.4 Tikus' T Statistic

If the ordered sample from (16) is trimmed of r_1 smallest and r_2 largest observations, the resulting censored sample is

$$X_a \leqslant X_{a+1} \leqslant \dots \leqslant X_b \qquad (a=r_1+1, b=n-r_2) \tag{31}$$

Let σ_c be the maximum likelihood estimator (or modified maximum likelihood estimator [20,21,22] of the population standard deviation σ calculated from (31) and $\hat{\sigma}$ be the maximum likelihood estimator of σ calculated from (16). Tiku [25] defined his statistic T as

$$T = h\ (\sigma_c / \hat{\sigma}) \tag{32}$$

where $h = E(\hat{\sigma})/E(\sigma_c)$ is a constant.

Based on (32), Tiku proposed T_N, T_U, T_E and T_{LN} to test 4 different distributions :

(i) Assumed distribution : normal

$$H_0 : \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty , \quad \sigma > 0 \qquad (33)$$

$$\text{Test Statistic : } T_N = (1 - \tfrac{1}{n})\, \sigma_c \,/\, (1 - \tfrac{1}{nA})\, \hat{\sigma} \qquad (34)$$

where $A = 1 - q_1 - q_2$,

$$n\hat{\sigma} = \sum_{i=1}^{n} (X_i - \bar{X})^2$$

σ_c = Tiku's [20] modified MLE of σ based on a censored sample

$$= \{B + \sqrt{(B^2 + 4AC)}\} / 2A$$

$$B = q_2\alpha_2 X_b - q_1\alpha_1 X_a - (q_2\alpha_2 - q_1\alpha_1)K, \quad (a=r_1+1, b=n-r_2)$$

$$C = (\tfrac{1}{n}) \sum_{i=a}^{b} X_i^2 + q_2\beta_2 X_b - q_1\beta_1 X_a - (1-q_1-q_2+q_2\beta_2-q_1\beta_1)K ,$$

$$K = (\tfrac{1}{n}\sum_{i=a}^{b} X_i + q_2\beta_2 X_b - q_1\beta_1 X_a)/(1-q_1-q_2+q_2\beta_2-q_1\beta_1)$$

The values of K, α_1, β_1, α_2, β_2 are chosen to give good fit, i.e.

$$f(z) / P(z) \simeq \alpha_1 + \beta_1 z , \quad f(z) / Q(z) \simeq \alpha_2 + \beta_2 z ,$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \exp\{-\tfrac{1}{2}t^2\} , \quad P(t_1) = \int_{-\infty}^{t_1} f(t)\, dt = q_1 ,$$

$$P(t_2) = 1 - q_2 .$$

Testing normality against positively skew distributions, one chooses $q_1=0$, $q_2=[\frac{1}{2}+0.3n]/n$, and against symmetric distributions $q_1=q_2=[\frac{1}{2}+0.3n]/n$.

Asymptotically, $\alpha_1 = 0$, $\beta_1 = -1$, $\alpha_2 = 0.7902$ and $\beta_2 = 0.5773$ if the H_1 distribution is positively skew, or $\alpha_1 = \alpha_2 = 0.7733$, $\beta_2 = -\beta_1 = 0.7355$ if the H_1 distribution is symmetric. However, one can also use these values for small n. The asymptotic null distribution of T_N is normal with

$$E(T_N) = 1 \quad , \text{ and}$$

$$nV(T_N) \simeq \{\frac{(1-1/n)}{(1-1/nA)}\}^2 . \{\frac{1}{2(1-q_1-q_2)-(q_2\alpha_2 t_2 - q_1\alpha_1 t_1)} - 0.5 - 0.0532(q_1+q_2) + 1.1284(q_2\alpha_2 - q_1\alpha_1)\}$$

Statistic T_N, in general, has good power properties. Against skew distributions, or symmetric distributions with kurtosis larger than 3, T_N is usually more powerful than Shapiro and Wilks' W. However, in the use of T_N, one assumes a priori knowledge of whether the H_1 distribution is skew or symmetric.

(ii) Assumed distribution : uniform

$$H_0 : f_0 = \frac{1}{\beta} \quad , \quad \alpha \leq x \leq \alpha + \beta$$

Test Statistic :

$$T_U = \frac{n-1}{n-r-1} \cdot \frac{X_{n-r} - X_1}{X_n - X_1} \quad \text{if } H_1 \text{ distribution is skew} \quad (35)$$

$$= \frac{n-1}{n-2r-1} \cdot \frac{X_{n-r} - X_{r+1}}{X_n - X_1} \quad \text{if } H_1 \text{ distribution is symmetric} \quad (36)$$

$$= \frac{n+1}{n-r} \cdot X_{n-r} . \quad \text{if } \alpha = 0 \text{ , } \beta = 1 \quad (37)$$

This test statistic seems to be very powerful too. For instance, in testing against non-uniform distributions with positive skewness, the power of T_U is greater than that of the Kolmogorov test.

(iii) Assumed distribution : exponential

$$H_0 : \frac{1}{\sigma} \exp\{-(x - \theta)/\sigma\} \quad , \quad \theta \leq x \leq \infty \quad \theta \text{ and } \sigma \text{ are unknown}$$

$$\text{Test Statistic : } T_E = (1 - \tfrac{1}{n})\sigma_c / (1 - \frac{1}{n-r})\hat{\sigma} \quad , \qquad (38)$$

where

$$q = r / n$$

$$r = [\tfrac{1}{2} + 0.5\, n]$$

$$\sigma_c = \{ (\tfrac{1}{n}) \sum_{i=1}^{n-r} X_i + q\, X_{n-r} - X_1\} / (1-q)$$

$$\hat{\sigma} = (\tfrac{1}{n}) \sum_{i=1}^{n} X_i - X_1$$

The null distribution of $\{(n-r-1)/(n-1)\}\, T_E$ is Beta $\beta(n-r-1,r)$. The power of T_E and Shapiro & Wilks' W_E [17] , on the whole, are of the same magnitudes.

(iv) Assumed distribution : log normal (τ known)

$$H_0 : \quad f_0 = \frac{1}{\sqrt{(2\pi)}\sigma(x-\tau)} \exp[-\tfrac{1}{2}\{\log(x-\tau) + \gamma\}^2 / \sigma^2] \; , \; \tau < x < \infty$$

$$\text{Test Statistic : } T_{LN} = (1 - \tfrac{1}{n})\sigma_c / (1 - \frac{1}{nA})\hat{\sigma} \; , \qquad A = 1 - q_1 - q_2$$

(39)

where σ_c and $\hat{\sigma}$ are given by (34) with X_i replaced by $-\log(X_i-\tau)$.

The null distribution of T_{LN} is exactly the same as T_N. Monte Carlo values of the power of T_{LN} indicate that T_{LN} is particularly sensitive to positively skew distributions such as the $\chi^2_{(1)}$ and the exponential distributions.

Tiku [25] also extended his T into the generalized T^* statistic for testing k independent random samples with excellent power properties. No other goodness-of-fit test admits such a simple and straight forward generalization.

CHAPTER 2

NEW STATISTICS B_N AND B_L FOR TESTING DEPARTURES FROM NORMAL AND LOGISTIC DISTRIBUTIONS

2.1 INTRODUCTION

The Statistic T suggested by Tiku not only works fine in case of a single sample for testing a distribution which has the functional form $\frac{1}{\sigma} f_0 (\frac{x-\mu}{\sigma})$, but it can also be generalized to test the goodness-of-fit of k independent random samples. The only problem with $T \propto \sigma_c / \hat{\sigma}$ is that, for some distributions (e.g. logistic distribution), the estimators σ_c and $\hat{\sigma}$ are not available as the corresponding maximum likelihood equations have no explicit solutions. Therefore it seems desirable to use some other appropriate estimators, for instance, the best linear unbiased estimators (BLUE) σ_c^* and σ^*. In this chapter, two statistics-B_N and B_L- are suggested for testing normal and logistic distributions respectively. These statistics are defined as $B = \sigma_c^* / \sigma^*$; where σ_c^* and σ^* are the BLUE of σ calculated from censored and complete samples,respectively. The properties and power of B_N and B_L are discussed in the following sections.

2.2 Statistic B_N For Testing Normality

2.2.1 Definition OF B_N

Let $$X_1 \leq X_2 \leq \ldots \leq X_n \tag{40}$$
be a random, ordered sample of size n drawn from a normal distribution with unknown parameters μ and σ and let

$$X_{r_1+1} \leq X_{r_1+2} \leq \ldots \leq X_{n-r_2} \tag{41}$$

be the censored sample. By using Lloyd's [11] generalized least square method, one can show that the best linear unbiased estimator of σ based on (41) is

$$\sigma_c^* = \sum_{i=r_1+1}^{n-r_2} a_i X_i = \underset{\sim}{1}' \underset{\sim}{\Gamma} \underset{\sim}{X} \tag{42}$$

where $\underset{\sim}{a} = \underset{\sim}{1}' \underset{\sim}{\Gamma} = (a_{r_1+1}\ a_{r_1+2}\ \cdots\ a_{n-r_2})$ is a vector of $(n-r_1-r_2)$ BLUE coefficient for σ,

$\underset{\sim}{X}' = (X_{r_1+1}\ X_{r_1+2}\ \cdots\ X_{n-r_2})$ is a vector of $(n-r_1-r_2)$ ordered observations,

$\underset{\sim}{1}' = (1\ 1\ 1\ \ldots\ 1)$ is a vector of $(n-r_1-r_2)$ unities,

$\underset{\sim}{\Gamma} = \underset{\sim}{V}_c^{-1} (\underset{\sim}{1}\,\underset{\sim}{\alpha}' - \underset{\sim}{\alpha}\,\underset{\sim}{1}')\, \underset{\sim}{V}_c^{-1} / \underset{\sim}{\Delta}$,

$\underset{\sim}{\alpha}' = (\alpha_{r_1+1}\ \alpha_{r_1+2}\ \cdots\ \alpha_{n-r_2})$; α_i is the expected value of the ith order statistic from a sample of size n drawn from a normal distribution $N(0,1)$.

$\underset{\sim}{V}_c$ = the variance-covariance matrix of $(n-r_1-r_2)$ apropriate normalized order statistics,

$$\underset{\sim}{\Delta} = (\underset{\sim}{1}' \underset{\sim}{V}_c^{-1} \underset{\sim}{1})(\underset{\sim}{\alpha}' \underset{\sim}{V}_c^{-1} \underset{\sim}{\alpha}) - (\underset{\sim}{1}' \underset{\sim}{V}_c^{-1} \underset{\sim}{\alpha})^2$$

Based on the complete, ordered sample, the BLUE for σ is given by

$$\sigma^* = \sum_{i=1}^{n} b_i X_i - \underset{\sim}{1}' \underset{\sim}{\Gamma} \underset{\sim}{X} \qquad (43)$$

where $\underset{\sim}{1}'$, $\underset{\sim}{\Gamma}$ and $\underset{\sim}{X}$ are the same as in (42) except that r_1 and r_2 are equal to zero in (43).

Now define

$$B_N = \sigma_c^* / \sigma^* \qquad (44)$$

as the statistic for testing normality. B_N is similar to Tiku's T_N in that both statistics are defined as the ratio of two estimators, one based on a censored sample, the other on the complete sample, of the population parameter σ.

2.2.2 Proportions of Censoring

The proportions $q_1=r_1/n$ and $q_2=r_2/n$ of the censoring are chosen in such a fashion that the power of the test is maximised. Tiku [24] found out that the power of T_N is generally an increasing function of (q_1+q_2) and this power for $q_1 + q_2 > 0.6$ is not substantially higher than that for $q_1 + q_2 = 0.6$. He also found out that for $(q_1 + q_2) > 0.6$, the null distribution of T_N tends to normality more slowly than for

$(q_1 + q_2) = 0.6$. The choice of q_1 and q_2 also depends on whether the alternative distribution is skew or symmetric. He therefore suggested the following :

(i) Choose $r_1=nq_1=0$, $r_2=nq_2=$ integer part of [½ + 0.6n] if the H_1 distribution is positively skew. In case the H_1 distribution is negatively skew, the transformation $y = -x$ changes it into a positively skew distribution.

(ii) Choose $r_1 = r_2 =$ integer part of [½ + 0.3n] if the distribution is symmetric.

Since B_N is similar to Tiku's T_N, the same choices of r_1 and r_2 happen to be optimal for B_N.

2.2.3 Mean And Variance Of B_N

Since both σ_c^* and σ^* are unbiased, hence for large n, $E(B_N) = E(\sigma_c^*) / E(\sigma^*) = 1$ (see Kendall & Stuart [8,p.232]). For small samples, the following Monte Carlo values indicate that $E(B_N) \approx 1$.

TABLE 1

Empirical Values* Of The Mean Of B_N

n	$q_1 = 0.0, q_2 = 0.6$	$q_1 = 0.3, q_2 = 0.3$
10	0.9988	0.9802
20	0.9951	1.0083
30	0.9911	1.0065
50	0.9890	1.0084

* *values were determined from 20000 random samples for n=10; 10000 for n=20,30; 6000 for n=50*

An approximation to the variance of B_N can be obtained by Kendall & Stuart's method [8,p.232] . Through a Taylor expansion, they worked out the approximate variance of the ratio x_1/x_2 as

$$V\left(\frac{x_1}{x_2}\right) \simeq \left\{\frac{E(x_1)}{E(x_2)}\right\}^2 \left\{ \frac{V(x_1)}{E^2(x_1)} + \frac{V(x_2)}{E^2(x_2)} - \frac{2Cov(x_1,x_2)}{E(x_1)E(x_2)} \right\} \tag{45}$$

Simply replacing x_1 and x_2 in (45) by σ_c^* and σ^*, respectively, and using the fact that the expected values of σ_c^* and σ^* are equal to σ, it follows that

$$V(B_N) = V\left(\frac{\sigma_c^*}{\sigma^*}\right) \doteq \frac{1}{\sigma^2}\{V(\sigma_c^*) + V(\sigma^*) - 2\,Cov(\sigma_c^*,\sigma^*)\}$$

$$= \frac{1}{\sigma^2}\{V\left(\sum_{i=r_1+1}^{n-r_2} a_i X_i\right) + V\left(\sum_{i=1}^{n} b_i X_i\right)$$

$$+ 2\,Cov\left(\sum_{i=r_1+1}^{n-r_2} a_i X_i , \sum_{i=1}^{n} b_i X_i\right)\}$$

$$= V\left(\sum_{i=r_1+1}^{n-r_2} a_i Z_i\right) + V\left(\sum_{i=1}^{n} b_i Z_i\right) + 2Cov\left(\sum_{i=r_1+1}^{n-r_2} a_i Z_i , \sum_{i=1}^{n} b_i Z_i\right)\} \tag{46}$$

$$= \underset{\sim}{a}\, \underset{\sim}{V}_c\, \underset{\sim}{a}' + \underset{\sim}{b}\, \underset{\sim}{V}\, \underset{\sim}{b}' - 2\underset{\sim}{a}\, \underset{\sim}{V}\, \underset{\sim}{b}' \tag{47}$$

where Z_i is the standardized value of X_i and matrices $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{V}_c$, $\underset{\sim}{V}$ have been defined in (42) and (43).

It is clear from (47) that $V(B_N)$ depends on the known values of $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{V}_c$ and $\underset{\sim}{V}$. These values are given in Sarhan and Greenberg [14, p.222] for $n \leqslant 20$. For $n > 20$, we suggest the following :

(i) Replace the BLUE coefficients a_i and b_i by Gupta's simplified BLUE coefficients. That is, choose

$$a_i = \frac{\alpha_i - \bar{\alpha}_c}{\sum_{i=r_1+1}^{n-r_2} (\alpha_i - \bar{\alpha}_c)^2} \quad \text{for the censored sample,} \tag{48}$$

and

$$b_i = \frac{\alpha_i - \bar{\alpha}}{\sum_{i=1}^{n} (\alpha_i - \bar{\alpha})^2} \quad \text{for the complete sample;} \tag{49}$$

$$\bar{\alpha}_c = \sum_{i=r_1+1}^{n-r_2} \alpha_i \,/\, (n-r_1-r_2), \quad \bar{\alpha} = \sum_{i=1}^{n} \alpha_i \,/n .$$

The values of α_i are available for sample sizes up to 400 (see [7])

(ii) For n > 20, close approximations to $V(X_i)$ and $Cov(X_i ,X_j)$ had been suggested by David and Johnson [3] :

$$V(X_i) = \frac{c_i d_i}{n+2} + \frac{c_i d_i}{(n+2)^2}\{(2d_i - c_i)\, X_i + c_i d_i (1+\tfrac{5}{2} X_i^2)\} + \dots, \tag{50}$$

$$Cov\,(X_i, X_j) = \frac{c_i d_j}{n+2} + \frac{c_i d_j}{(n+2)^2}\{(d_j - c_i) X_i + \tfrac{1}{2} c_i d_i (1+2X_i^2)$$

$$+ \tfrac{1}{2} c_i d_j (1+2X_j^2) + \tfrac{1}{2} c_i d_j X_i X_j \} + \dots . \tag{51}$$

with $i < j$,

$$P_i = 1 - Q_i = \frac{i}{n+1} ,$$

$$X_i = F^{-1}(P_i) = F^{-1}\left(\frac{i}{n+1}\right) ,$$

$$Z_i = \frac{1}{\sqrt{2\pi}} \exp\left(-X_i^2/2\right),$$

$$c_i = P_i / Z_i ,$$

$$d_i = Q_i / Z_i .$$

It is, of course, time-consuming to evaluate the expressions (50) and (51). However, we found out that the simplified expressions

$$V(X_i) \simeq \frac{c_i d_i}{n+2} \qquad (52)$$

$$\mathrm{Cov}(X_i, X_j) \simeq \frac{c_i d_j}{n+2} \qquad (53)$$

suffice to give adequate approximations for our purposes, especially for $n \geq 30$. The comparison between the values of $V(B_N)$ obtained empirically and those approximated by equation (47) for sample sizes of 10, 20, 30, 50 is presented in Table 2. In spite of their grossly simplified nature, equations (47), (48), (49), (52) and (53) provide accurate approximations to $V(B_N)$.

TABLE 2

Comparison Of The Exact And Approximate Values of $V(B_N)$

Sample Size n	$q_1 = 0.3$, Approx.*	$q_2 = 0.3$ Exact**	$q_1 = 0.0$, Approx.*	$q_2 = 0.6$ Exact**
10	0.178	0.174	0.167	0.165
20	0.077	0.077	0.068	0.067
30	0.051	0.056	0.047	0.053
50	0.032	0.033	0.029	0.031

* *The values of $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{V}_c$ and $\underset{\sim}{V}$ were taken from [14] for n=10,20. For n=30,50,these values were approximated by equations(48), (49), (52),and (53).*

** *The number of random samples were 20000 for n=10;10000 for n=20, 30;6000 for n=50.*

2.2.4 Null Distribution of B_N

SINCE σ^* is an estimator based on a complete sample while σ_c^* is based on a censored sample, therefore σ^* converges to σ faster than σ_c^*. Thus for sufficiently large n, the distribution of B_N will effectively be the same as of σ_c^* / σ. Ali and Chan [1] proved that the distribution of Gupta's linear estimate of σ tends to be normal for $n \to \infty$. Therefore the asymptotic distribution of B_N based on Gupta's estimator will be normal. In fact, the Monte Carlo values in Table 3 indicate that the distribution of B_n tends to normality very rapidly.

TABLE 3

Values of β_1 and β_2 For The Null Distribution of B_N

Sample size n	$q_1 = 0.3$, β_1	$q_2 = 0.3$ β_2	$q_1 = 0.0$, β_1	$q_2 = 0.6$ β_2
10	0.132	2.744	0.089	2.723
20	0.035	2.832	0.025	2.890
30	0.057	2.950	0.060	2.989
50	0.031	2.973	0.026	3.034

To verify this further, the empirical percentage points of B_N were compared with those obtained from normal approximation.The lower percentage points of B_N were determined,based on 20000 random samples from N(0,1) for n=10;10000 for n=20,30;6000 for n=50,using the CDC 6400 Computer of McMaster University. The empirical and the approximated percentage points of 1%, 2.5%, 5% and 10% are given in Table 4. On the whole, the approximations are good except those for very small n or extreme tails. Therefore, the distribution of B_N can successfully be approximated by a normal distribution with mean 1 and variance given by (47) for samples of size $n \geq 20$. However, it is difficult to obtain the distribution of a linear function $\sum_{i=1}^{n} l_i X_i$ of ordered observations in general, even asymptotically. Therefore , unlike Tiku's statistic T, the asymptotic normality of the statistic B_N does not follow, in general.

TABLE 4

Lower Percentage Points Of B_N

	P	$q_1 = 0.3$, $q_2 = 0.3$ Approx.	Emp.	$q_1 = 0.0$, $q_2 = 0.6$ Approx.	Emp.
n = 10	0.01	0.017	0.192	0.049	0.206
	0.025	0.172	0.266	0.200	0.290
	0.050	0.305	0.348	0.327	0.370
	0.100	0.458	0.454	0.476	0.481
n = 20	0.010	0.355	0.424	0.394	0.434
	0.025	0.457	0.498	0.490	0.510
	0.050	0.544	0.572	0.572	0.578
	0.100	0.649	0.654	0.666	0.668
n = 30	0.010	0.474	0.510	0.500	0.500
	0.025	0.557	0.570	0.576	0.572
	0.05	0.628	0.632	0.644	0.626
	0.100	0.710	0.710	0.722	0.696
n = 50	0.010	0.587	0.610	0.606	0.588
	0.025	0.652	0.662	0.668	0.658
	0.050	0.708	0.720	0.722	0.708
	0.100	0.772	0.776	0.783	0.770

2.2.5 Empirical Power Study

A useful statistic, not only must be relatively easy to compute, but also must be highly sensitive to departures from the assumed distribution, that is, must have high power.

To evaluate the power of B_N , an empirical sampling investigation was conducted. The alternatives that were chosen for this study cover a wide varieties of populations - from the very symmetric and short-tailed Tukey distribution to the very skew and long-tailed Weibull distribution.

The null distribution of B_N used for the study was determined as described above. For all other non-normal alternatives, 1000 random samples of sizes of 10, 20, 30, and 50 were generated for each of them. The empirical power for 10% significance level for these selected alternatives are given in Table 5.

The results of Table 5 indicate that B_N is particularly sensitive to the non-normal distributions with large skewness. For instance, the power of B_N against a Weibull distribution with parameter k = 0.5 is very close to 1 even for a sample size as small as 10. B_N is also very powerful against other populations such as Chi-square, log normal, Johnson's $S\beta$ (γ = 0.5333, δ = 0.5 and γ = 1, δ = 1), Tukey (a = 10,λ=3.1) and Weibull (k = 2.0) which all are skew. B_N also has the ability to detect any significant difference between the normal and the non-normal distributions due to kurtosis as it is reflected in the cases of Johnson's unbounded (γ = 0, δ = 0.9, and γ = 0, δ = 1) and logistic distributions which are symmetric (zero skewness) but have kurtosis larger than 3.0. Statistic B_N , however, is insensitive to symmetric non-normal populations

TABLE 5

Empirical Power For 10% Tests For Some Selected Alternative Distributions

H_1 distribution	$\sqrt{\beta_1}$	β_2	n=10	n=20	n=30	n=50
Johnson's S_β						
$\gamma = 0, \delta = .7071$	0.00	1.87	.049	.026	.014	.004
$\gamma = 1, \delta = 2$	0.28	2.77	.145	.217	.275	.368
$\gamma = .5333, \delta = .5$	0.65	2.13	.627	.882	.995	1.000
$\gamma = 1, \delta = 1$	0.73	2.91	.353	.649	.856	.977
Tukey						
$a = 1, \lambda = 1.5$	0.00	1.75	.044	.028	.012	.001
$a = 1, \lambda = 0.1$	0.00	3.21	.092	.106	.125	.001
$a = 10, \lambda = 3.1$	0.97	2.80	.721	.952	.999	.999
Logistic	0.00	4.20	.130	.182	.190	.250
Johnson's unbounded						
$\gamma = 0, \delta = 3$	0.00	3.52	.100	.122	.127	.145
$\gamma = 0, \delta = 2$	0.00	4.51	.116	.154	.169	.221
$\gamma = 0, \delta = 1$	0.00	36.19	.221	.397	.535	.717
$\gamma = 0, \delta = .9$	0.00	83.08	.255	.479	.652	.824
Weibull						
$k = 2.0$	0.63	3.25	.214	.342	.544	.770
$k = 0.5$	6.62	87.72	.957	1.000	1.000	1.000
Chi-square						
$\upsilon = 2$	2.00	9.00	.664	.927	.998	1.000
$\upsilon = 1$	2.83	15.00	.878	.992	1.000	1.000
Log Normal	6.18	113.94	.758	.981	.999	1.000

with tails shorter than normal (i.e. $\beta_2 < 3$).

The power of B_N is also compared with other well-known tests for normality (Table 6) . These are

(1) Tiku's T_N ,

(2) D' Agnostinos' D ,

(3) Shapiro and Wilk's W ,

(4) $\sqrt{b_1}$,

(5) b_2 ,

(6) Studentized range u.

Values in Table 6 indicate that B_N and Tiku's T_N behave in a similar manner, probably because of the similarity in their formulation -both statistics are defined as the ratios of two estimators of the population σ , one estimator is based on a censored sample, the other on a complete sample. Also similar to T_N is the use of B_N in which one assumes a priori knowledge about the skewness of the alternative distribution.

2.2.6 A Numerical Example

The following is a purely hypothetical example, suggested by the **author** to illustrate the operational procedure of using B_N as a test statistic for testing whether or not a random sample comes from a normal distribution with unknown parameters μ and σ .

Suppose from a certain public examination written by a large number of students, a random sample of 20 marks was chosen. These marks,

TABLE 6

Comparison Of Power For 10% Tests Of B_N With
Some Other Tests Of Normality
n = 50

H_1 distribution	B_N	(1) T_N	(2) D	(3) W	(4) $\sqrt{b_1}$	(5) b_2	(6) u
Johnson's Sβ							
γ = 0 , δ = .7071	.004	.01	.71	.85	.01	.89	.94
γ = 1 , δ = 2	.368	.32	.07	.19	.17	.11	.13
γ =.5333, δ=.5	1.000	.99	.23	1.00	.67	.51	.99
γ = 1 , δ = 1	.977	.90	.17	.08	.09	.08	.10
Tukey							
a = 1 , λ = 1.5	.001	.01	.70	.97	.01	.98	.99
a = 1 , λ = 0.1	.123	.10	.12	.08	.14	.13	.12
a =10 , λ = 3.1	.999	1.00	—	—	—	—	—
Logistic	.250	.21	.30	.20	.29	.34	.29
Johnson's unbounded							
γ = 0 , δ = 3	.145	.15	.16	—	.18	.14	—
γ = 0 , δ = 2	.221	.27	.30	—	.34	.26	—
γ = 0 , δ = 1	.717	.82	.88	—	.66	.80	—
γ = 0 , δ = .9	.824	.89	.91	—	.74	.89	—
Weibull							
k = 2.0	.770	.67	.14	.24	.17	.10	.14
k = 0.5	1.000	1.00	1.00	1.00	1.00	.98	.44
Chi-square							
υ = 2	1.000	1.00	.90	1.00	1.00	.81	.21
υ = 1	1.000	1.00	1.00	1.00	1.00	.96	.29
Log-normal	1.000	1.00	.99	1.00	1.00	.94	.43

— : *values are not available.*

arranged in an ascending order, were

68, 70, 70, 70, 70, 70, 73, 73, 73, 75,

79, 82, 86, 89, 91, 92, 94, 96, 96, 97.

The null and alternative hypotheses are respectively

H_0 : these 20 observations come from a normal distribution.

H_1 : negation of the null hypothesis.

A rough plot of the data suggests a population distribution with marked positive skewness. However, it is desirable to establish a statistical conclusion on an objective basis.

Inasmuch as the positive skewness suggested by the rough plot, we should censor the sample with $q_1 = 0.0$, $q_2 = 0.6$. The resulting truncated sample is left with

68, 70, 70, 70, 70, 70, 73, 73, —, —,

—, —, —, —, —, —, —, —, —, — .

The BLUE coefficients a_i and b_i for determining σ_c^* and σ^* are [11,p.222] :

σ_c^* : -.3363, -.2088, -.1550, -.1151, -.0820, -.0531, -.0268, .9771.

σ^* : -.1128, -.0765, -.0611, -.0497, -.0402, -.0318, -.0241, -.0169, -.0101, -.0033, .0033 , .0101, .0169, .0241, .0318, .0402, .0497, .0611, .0765 , .1128.

Based on equations (42) and (43), we obtain

$$\sigma_c^* = (-.3363)(68) + (-.2088)(70) + \quad .... \quad + (.9771)(77)$$

$$= 3.5235$$

$$\sigma^* = (-.1128)(68) + (-.0765)(70) + ... + (.0101)(77) + ... +(.1128)(91)$$

$\sigma^{*} = 10.3032$

Therefore, $B_N = \sigma_c^{*} / \sigma^{*} = 3.5235 / 10.3032 = 0.3420$.

Small values of B_N indicate non-normality, i.e. lead to rejection of H_o. Referring to Table 4 for $n = 20$, $q_1 = 0.0$, $q_2 = 0.6$, one finds that the calculated value (0.3420) of B_N is smaller than the tabulated 1% point, which is 0.434. Therefore the test- statistic B_N reflects non-normality of the given sample, and so do Tiku statistic T and Shapiro & Wilk statistic W.

2.3 Statistic B_L For Testing A Logistic Distribution

The logistic distribution is described by the density function

$$f(x;\mu,\sigma) = \frac{\pi}{\sigma\sqrt{3}} \exp\{-\pi\,(x-\mu)/\,\sigma\sqrt{3}\}\,/[\,1+\exp\{-\,\pi(x-\mu)/\sigma\sqrt{3}\}]^2 \qquad (54)$$

where $-\infty < x < \infty \quad -\infty < \mu < \infty \quad \sigma\sqrt{3}/\pi > 0.$

This distribution is symmetric with mean μ and variance σ . Suppose we want to test the null hypothesis

H_o : The ordered random sample $X_1 \leq X_2 \leq \ldots \leq X_n$ comes from a logistic distribution with parameters μ and σ that may not be known.

The test statistic suggested for testing H_o is

$$B_l = \frac{\hat{\sigma}_c}{\hat{\sigma}} \cdot \frac{\sum_{i=r_1+1}^{n-r_2} a_i X_i}{\sum_{i=1}^{n} b_i X_i} , \tag{55}$$

where $\hat{\sigma}_c$ and $\hat{\sigma}$ are the best linear unbiased estimators of σ calculated from the censored sample and the complete sample respectively. The matrix forms of $\hat{\sigma}_c$ and $\hat{\sigma}$ are similar to those for σ_c^* and σ^* given in (42) and (43) except that in (55) the random variable X has a logistic distribution. Gupta, Qureishi and Shah [6] give the BLUE coefficients a_i and b_i for sample sizes of n = 2,5(5)25. They also give the variance and covariance of the order statistics from a logistic distribution with mean equals to zero and variance equal to one for sample sizes n=5(5)25.

2.3.1 Distribution of B_L

The exact distribution of B_L is difficult to derive. However, Monte Carlo simulations indicate that as sample size increases, the distribution of B_L tends to normality. This is indicated by the values of β_1 and β_2 of B_L given in Table 7.

TABLE 7

Empirical* Mean, β_1 And β_2 OF B_L

Sample Size N	$q_1 = 0.3$ $q_2 = 0.3$			$q_1 = 0.0$ $q_2 = 0.6$		
	$E(B_L)$	β_1	β_2	$E(B_L)$	β_1	β_2
10	1.005	.063	2.610	1.000	.167	2.820
20	1.003	.009	2.787	1.000	.043	2.837

* *20000 random samples for n=10, 10000 for n = 20*

The asymptotic mean of B_L is very close to 1.0. The variance of B_L, like that for B_N, can be successfully approximated by

$$V(B_L) = V\left(\frac{\hat{\sigma}_c}{\hat{\sigma}}\right) \approx \left\{\frac{E(\hat{\sigma}_c)}{E(\hat{\sigma}_c)}\right\} \left\{\frac{V(\hat{\sigma}_c)}{E^2(\hat{\sigma})} + \frac{V(\hat{\sigma})}{E^2(\hat{\sigma})} - \frac{2\mathrm{Cov}(\hat{\sigma}_c,\hat{\sigma})}{E(\hat{\sigma}_c)\,E(\hat{\sigma})}\right\} \qquad (56)$$

For instance the values of $V(B_L)$ obtained by Monte Carlo simulation for n=10 and n=20 are 0.167 and 0.070 for $q_1 = q_2 = 0.3$; 0.1842 and 0.0780 for $q_1 = 0.0$, $q_2 = 0.6$. Those approximated by equation (56) are respectively 0.172, 0.070 and 0.186 and 0.078.

The fact that the empirical distribution of B_L is close to be normal and equation (56) provides good approximations to the variance of B_L makes it possible to determine the percentage points using normal approximation. The lower percentage points obtained from a normal approximation and those obtained through Monte Carlo simulations are compared in Table 8.

TABLE 8

Lower Percentage Points Of B_L

	P	$q_1 = 0.3$, $q_2 = 0.3$ Approx.	Emp.	$q_1 = 0.0$, $q_2 = 0.6$ Approx.	Emp.
n = 10	0.010	0.035	0.208	0.002	0.206
	0.025	0.187	0.284	0.155	0.278
	0.050	0.318	0.367	0.291	0.356
	0.100	0.469	0.481	0.447	0.462
n = 20	0.010	0.385	0.421	0.351	0.413
	0.025	0.482	0.500	0.453	0.488
	0.050	0.565	0.574	0.541	0.561
	0.100	0.661	0.663	0.642	0.642

2.3.2 Empirical Power Study of B_L

The Monte Carlo percentage points for B_L were computed from 20000 random samples for n=10, 10000 for n=20. For non-logistic distributions, 1000 random samples were generated for each of them. The proportions q_1 and q_2 of censoring were chosen exactly as for testing normality. Values of the power of B_L against various H_1 distributions are given in Table 9.

The power-properties of B_L are similar to those of B_N. The B_L statistic is very sensitive against those distributions with large kurtosis and skewness. However, B_L is insensitive against those symmetric distributions with small kurtosis.

SUMMARY

The distributions of B_N and B_L are approximatedly normal with means equal to one and variances given by equations (46) and (56). The power of B_N and B_L are good in general. They are particularly effective in detecting the differences between the symmetric H_0 distributions and the skewed H_1 distributions. However, like many other test statistics, they do not work so well against certain type of H_1 distributions.

In this thesis, only several sample sizes were chosen to work with. A more complete table of percentage points for other sample sizes can be completed by the methods suggested in (48), (49), (52) and (53) since the expected values α_i for normal order statistics have been computed by Harter [7]. However, for B_L, a more complete table of per-

TABLE 9

Empirical Power For 10% Tests For Some Selected Alternative Distributions

H_1 distribution	$\sqrt{\beta_1}$	β_2	n=10	n=20
Johnson's S_β				
$\gamma = 0,\ \delta = .7071$	0.00	1.87	0.045	0.220
$\gamma = 1,\ \delta = 2$	0.28	2.77	0.157	0.234
$\gamma = .5333,\ \delta = .5$	0.65	2.13	0.618	0.849
$\gamma = 1,\ \delta = 1$	0.73	2.91	0.362	0.610
Tukey				
$a = 1,\ \lambda = 1.5$	0.00	1.75	0.041	0.023
$a = 1,\ \lambda = 0.1$	0.00	3.21	0.077	0.069
$a = 10,\ \lambda = 3.1$	0.97	2.80	0.709	0.921
Normal	0.00	3.00	0.116	0.107
Johnson's unbounded				
$\gamma = 0,\ \delta = 3$	0.00	3.52	0.075	0.067
$\gamma = 0,\ \delta = 2$	0.00	4.51	0.085	0.097
$\gamma = 0,\ \delta = 1$	0.00	36.19	0.172	0.260
$\gamma = 0,\ \delta = 9$	0.00	83.08	0.201	0.325
Weibull				
$k = 2$	0.63	3.25	0.216	0.315
$k = .5$	6.62	87.72	0.945	0.999
Chi-square				
$\nu = 2$	2.00	9.00	0.622	0.876
$\nu = 1$	2.83	15.00	0.854	0.981
Log - normal	6.18	113.94	0.729	0.956

centage points cannot be obtained till the BLUE coefficients a_i, b_i and the values of $V(X_i)$, $Cov(X_i, X_j)$ are also available.

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