DECouPLING ORBITAL AND ATTITUDE CONTROL OF COMMUNICATIONS TECHNOLOGY SATELLITE
DECOUPLING ORBITAL AND ATTITUDE CONTROL OF
COMMUNICATIONS TECHNOLOGY SATELLITE

by

Joe Yiu Yau

B.Sc. (Eng.) (University of Alberta, Edmonton)

A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree

Master of Engineering

McMaster University

April 1976
TITLE: Decoupling Orbital and Attitude Control of Communications Technology Satellite

AUTHOR: Joe Yiu Yau, B.Sc. (Eng.) (University of Alberta, Edmonton, 1974)

SUPERVISOR: Professor N.K. Sinha

NUMBER OF PAGES: (x), 87

SCOPE AND CONTENTS:

Based on Altman's model, a state model and a measurement model are obtained. Both models are linearized and the linear state variable feedback method is then applied to decouple the linearized system.

Computer programmes for decoupling are written and the results are then discussed.

Suggestions for future work are proposed.
ACKNOWLEDGEMENTS

With deep gratitude I wish to thank Professor N.K. Sinha for his guidance and encouragement during the preparation of this thesis.

I am also sincerely thankful to Mr. S. Altman and Dr. R. Mamen for their helpful suggestions and discussions.

I would like to acknowledge with thanks the useful discussions I had with S. Azim, S. Law and J. Hickin.

Financial support provided by the Communications Research Centre through Research Contract OSU5-0182 and by the Department of Electrical Engineering at McMaster University is gratefully acknowledged.

I wish to thank Theresa MacFarlane for typing this thesis.

(iii)
# TABLE OF CONTENTS

## CHAPTER 1
Introduction 1

## CHAPTER 2
The Unified State Model 3

### 2.1 Introduction 3

### 2.2 The Dynamical Variables 3

#### 2.2.1 The orbital trajectory state variables 4

#### 2.2.2 The attitude state variables 8

#### 2.2.3 The coordinate variables 10

### 2.3 The Dynamical Model 10

### 2.4 Nonlinear Model Equations 13

### 2.5 Linearization of the State Model 16

### 2.6 The Observables 38

### 2.7 The Linearized Measurement Model 39

### 2.8 Conclusions 46

## CHAPTER 3
The Decoupling Problem in Linear Multivariable Systems 47

### 3.1 Introduction 47

### 3.2 The Need for Decoupling 48

### 3.3 How the Problem can be Solved by State Variable Feedback 48

### 3.4 Necessary and Sufficient Conditions for Decoupling and Pole Placement 52

#### 3.4.1 Falb and Wolovich Method 52

#### 3.4.2 Sinha and Rozsa Method 53
<table>
<thead>
<tr>
<th>CHAPTER 4 Application to Communications Technology Satellite</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 Introduction</td>
<td>56</td>
</tr>
<tr>
<td>4.2 The Controllable Canonical Form</td>
<td>56</td>
</tr>
<tr>
<td>4.3 Decoupling Algorithms</td>
<td>58</td>
</tr>
<tr>
<td>4.4 Application and Results</td>
<td>58</td>
</tr>
<tr>
<td>4.4.1 Data for CTS</td>
<td>58</td>
</tr>
<tr>
<td>4.4.2 Results</td>
<td>62</td>
</tr>
<tr>
<td>4.5 Conclusions</td>
<td>67</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 5 Conclusions</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>68</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>APPENDIX A Definition of the Euler Parameters</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>71</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>APPENDIX B Computer Programme for Decoupling (Falb and Wolovich Method)</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>75</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>REFERENCES</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>84</td>
</tr>
</tbody>
</table>
# LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A, \gamma$</td>
<td>azimuth and elevation respectively, of a point referred to radar range coordinates about a ground-based site</td>
</tr>
<tr>
<td>$a, e$</td>
<td>semi-major axis and eccentricity respectively, of the orbital conic in position space</td>
</tr>
<tr>
<td>$C, R$</td>
<td>parametric variables of orbital velocity state</td>
</tr>
<tr>
<td>$[E]$</td>
<td>transformation matrix of body attitude rotation with elements $e_{ij}$</td>
</tr>
<tr>
<td>$e_1, e_2, e_3, e_4$</td>
<td>Euler parameters defining rotation of a coordinate set</td>
</tr>
<tr>
<td>$e_{a1}, e_{a2}, e_{a3}, e_{a4}$</td>
<td>Euler parameters defining attitude rotation of the orbital body about its center-of-mass</td>
</tr>
<tr>
<td>$e_{01}, e_{02}, e_{03}, e_{04}$</td>
<td>Euler parameters defining rotation of the orbital trajectory frame about the planetocentric origin 0</td>
</tr>
<tr>
<td>$[H]$</td>
<td>angular momentum matrix of the body attitude dynamics</td>
</tr>
<tr>
<td>$h$</td>
<td>scalar range from site to a given observed point inertia matrix of the orbital body</td>
</tr>
<tr>
<td>$L_a, L_0$</td>
<td>latitude and longitude respectively of topocentric coordinates</td>
</tr>
<tr>
<td>$m$</td>
<td>mass of orbital body</td>
</tr>
<tr>
<td>$q$</td>
<td>a unitary quaternion</td>
</tr>
<tr>
<td>$R_e$</td>
<td>spherical Earth's radius</td>
</tr>
<tr>
<td>$\mathbf{r}, \mathbf{v}, \mathbf{a}$</td>
<td>position, velocity and acceleration vectors of orbital motion, respectively</td>
</tr>
<tr>
<td>$[T]$</td>
<td>Torque vector of attitude perturbations</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>( w_1, w_3 )</td>
<td>angular velocity of the rotating polar coordinates of the orbital body motion about the planetocentric origin 0</td>
</tr>
<tr>
<td>( \delta )</td>
<td>angle of rotation of the axis X about the line-of-nodes LN, to axis X' in the instantaneous orbital plane</td>
</tr>
<tr>
<td>( \theta )</td>
<td>flight path angle</td>
</tr>
<tr>
<td>([a])</td>
<td>perturbing accelerations</td>
</tr>
<tr>
<td>( f_1, f_2, f_3 )</td>
<td>immediate set of unit vectors defining coordinates in instantaneous orbital plane, referred to axis X'</td>
</tr>
<tr>
<td>( g_1, g_2, g_3 )</td>
<td>planetocentric inertial set of unit vectors fixed in inertial space about the planetocentric origin 0</td>
</tr>
<tr>
<td>( \iota )</td>
<td>angle of incidence between the instantaneous orbital plane and the equatorial plane XY</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>angle of rotation from the axis X' to the orbital position vector ( \mathbf{r} ), in the instantaneous orbital plane</td>
</tr>
<tr>
<td>( \mu )</td>
<td>planetary gravitational constant</td>
</tr>
<tr>
<td>( \omega_1 )</td>
<td>angle from the u line-of-nodes LN to the orbital position vector ( \mathbf{r} )</td>
</tr>
<tr>
<td>( \Omega )</td>
<td>angle from the inertial axis X to the line-of-nodes LN</td>
</tr>
<tr>
<td>( w_e )</td>
<td>planetary rate of rotation about its spin axis Z</td>
</tr>
<tr>
<td>( X, Y, Z )</td>
<td>Cartesian components of position, referred to the planetocentric origin</td>
</tr>
<tr>
<td>( x, y, z )</td>
<td>Cartesian components of position, referred to the ground-based site as origin</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS - continued

SUBSCRIPTS

B orbital body axes

$e_1, e_2, e_3$ rotating polar components of orbital motion

$f_1, f_2$ intermediate coordinate components of orbital motion

$I$ inertial axes

$T_1, T_2, T_3, T_4$ Euler parameter components of the topocentric coordinates
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>DESCRIPTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>State maps of an orbit</td>
<td>5</td>
</tr>
<tr>
<td>2.2</td>
<td>Velocity space mapping of an orbit</td>
<td>7</td>
</tr>
<tr>
<td>2.3</td>
<td>State model geometry of an orbital trajectory</td>
<td>9</td>
</tr>
<tr>
<td>2.4</td>
<td>Azimuth-elevation coordinate system</td>
<td>40</td>
</tr>
<tr>
<td>2.5</td>
<td>Coordinate set for a ground-based site</td>
<td>43</td>
</tr>
<tr>
<td>3.1</td>
<td>Multivariable feedback systems</td>
<td>50</td>
</tr>
<tr>
<td>A.1</td>
<td>Geometry of the Euler parameters</td>
<td>72</td>
</tr>
<tr>
<td>A.2</td>
<td>Euler angles</td>
<td>72</td>
</tr>
<tr>
<td>TABLE</td>
<td>PAGE</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>61</td>
<td></td>
</tr>
</tbody>
</table>

Thruster moment arms and force components
CHAPTER 1

Introduction

Recently, man-made satellites for communication purposes have developed and progressed rapidly. Controlling the orbit and the attitude of the satellites, raises a very important and interesting question. In other words, the necessary corrective action of maintaining the satellites in a desired orbit as well as of monitoring it to spin at a given rotation rate, is highly recommended. In this thesis, Altman's unified state model [4] is used because of its invaluable analytical properties. Moreover, the interdependence between the trajectory dynamics of point-mass motion and the attitude dynamics of spacecraft body orientation about that point-mass also makes the use of unified dynamical model important.

The interdependence (or dynamic coupling) between the two dynamics, is introduced by the hydrazine thrusters. Apart from the thrusters, there exist other possible sources of coupling for satellites; these include the earth and sun sensors and the effect of spacecraft attitude on resultant solar radiation forces.

The object of this work is the decoupling of the
control of the orbit and the attitude of the satellites. Once the system is decoupled, the well-known single variable synthesis method can be applied. Therefore, the performance of the decoupled system can then be optimally controlled and designed with less effort than would be necessary without decoupling.

A detailed study of the unified state model is described in Chapter 2. After the necessary coordinate transformations to planetocenter, the nonlinear state model and measurement model are derived. These are then linearized to obtain the approximate linear model, from which, the desired matrices \( A, B, \) and \( C \) are obtained.

Chapter 3 presents the decoupling problem and how this can be solved by using the linear state variable feedback method. Two decoupling algorithms are discussed here. Furthermore, the comparison of the possibility of pole placements and decoupling, to be done simultaneously, based on the two different approaches, is also discussed.

Chapter 4 discusses application to the communication satellites. Computer programmes were written for decoupling and the results of using two different algorithms are compared and discussed.

The concluding chapter presents a general summary of the previous chapters and the suggestions for future work.
CHAPTER 2

The Unified State Model

2.1 Introduction

Samuel P. Altman has developed a unified state model of the orbital trajectory and attitude dynamics of an orbital spacecraft [4]. Usually, the synthesis and analysis of the orbital trajectories of spacecraft are considered separately from the attitude dynamics, largely due to the lack of dynamical models which are not only valid in the analytical sense, but are also realizable or efficient for computation. However, the interdependence between the trajectory dynamics of point-mass motion and the attitude dynamics of spacecraft body orientation about the point-mass makes the use of unified dynamical model essential for many advanced missions. This interdependence (or dynamic coupling) is due to: natural physical forces (such as gravity gradient and atmosphere drag) which affect both classes of motion, artificial (man-induced) forces and mass relocation; and energy interchange between the two classes of motion, due to structural flexure.

2.2 The Dynamical Variables

To facilitate easy and efficient machine computation,
the state and coordinate variables for a unified state model should define the orbital and attitude dynamics in a common form which is nevertheless simple and well-behaved in analytic operations. The state variables are momenta, a direct form of the attitude momentum, and a parametric form of the orbital momentum. The coordinate variables are the Euler parameters, a four-dimensional (quaternion) representation of the rotation transformation for a coordinate frame triad.

2.2.1 The Orbital Trajectory State Variables

An unperturbed orbital trajectory is represented by cyclic figures in position, velocity and acceleration vector spaces, as shown in Figure 2.1. However, as the orbital energy level changes (as shown in each row of Figure 2.1, from left to right), only the velocity space map (i.e. velocity hodograph) remains invariant in geometric figure. In other words, in the presence of perturbing forces, the differential formulation of the orbital trajectory dynamics will not encounter singularities in the state variables; that is, those velocity parameter variables are "regularized".

The velocity state parameters \((C,R)\) are functions of the radial momentum \((p_r = mv_1)\) and angular momentum \((p_\lambda = mrve_2)\), as defined by
Fig. 2.1

STATE MAPS OF AN ORBIT

CIRCULAR ORBIT
(e = 0, R = 0)

ELLIPTIC ORBIT
(e < 1, R < C)

PARABOLIC ORBIT
(e = 1, R = C)

HYPERBOLIC ORBIT
(e > 1, R > C)

IN POSITION VECTOR SPACE

IN VELOCITY VECTOR SPACE

IN ACCELERATION VECTOR SPACE
\[ C = \frac{\mu m}{P_t} \quad (2.1) \]
\[ R = \left[ \left( \frac{P}{m} \right)^2 + \left( \frac{P}{m} \frac{m}{r} - \frac{\mu m}{P_t} \right)^2 \right]^{1/2} \quad (2.2) \]

or
\[ R = \left[ 2E + C^2 \right]^{1/2} \quad (2.3) \]

where \( E \) = orbital energy per unit mass. As shown in Figure 2.2 it follows that the velocity space map corresponds to the position space map point-by-point.

The complete three-dimensional vector equations of position, velocity and acceleration are defined as functions of \((C,R)\) as follows,

\[ \vec{r} = r e^{i\phi} = \left[ \frac{\mu}{C} \left( C + R \cos \phi \right) \right] e^{i\phi} \quad (2.4) \]
\[ \vec{v} = \vec{R} + \vec{C} \times \left( \vec{r}/r \right) \quad (2.5) \]
\[ \vec{a} = \vec{R} + \vec{C} \times \left( \vec{r}/r \right) + \vec{a}_b \quad (2.6) \]

where \( \left| \vec{a}_b \right| = -C \lambda \), \( \vec{C}, \vec{R}, \vec{r} \) and \( \lambda \) are defined as shown in Figure 2.3.

Three coordinate sets of orthonormal unit vectors are used, as shown schematically in Figure 2.3. The planetocentric inertial set \((g_1, g_2, g_3)\) defines the coordinate axes \((X,Y,Z)\) fixed in inertial space about the origin \(O\), with \(X, Y\) in the equatorial plane and \(Z\) along the planetary spin axis. The intermediate set \((f_1, f_2, f_3)\) defines coordinate axes in the instantaneous orbital plane, with \(f_1\) directed along the intermediate axis \(X'\), \(f_3\) directed along the orbital angular momentum vector, and \(f_2\) completing the right-handed set. The axis \(X'\) is defined by rotation...
\[ \vec{r} = \text{POSITION VECTOR} \]
\[ \vec{v} = \text{VELOCITY VECTOR} \]
\[ \phi = \text{TRUE ANOMALY} \]
\[ \theta = \text{FLIGHT PATH ANGLE} \]

**Fig. 2.2**

- VELOCITY SPACE MAPPING OF AN ORBIT
of the planetocentric inertial axis X into the instantaneous orbital plane, about the line-of-nodes LN. The rotating polar set \((e_1, e_2, e_3)\) defines the coordinate axes rotating with the orbital body, with \(e_1\) directed along the position vector \(r\), \(e_2\) normal to \(e_1\) (in the direction of flight) but in instantaneous orbital plane, and \(e_3\) completing the right-hand set. Consequently, \(e_1\) defines the direction of \(r\), and the common direction of \(e_3\) and \(e_3\) defines the direction of \(C\).

The orbital state variables of the unified state model are the three parameters \((C, R_{f1}, R_{f2})\) which define the velocity state. The velocity state variable \(R\) has been defined in two components \((R_{f1}, R_{f2})\): the component \(R_{f1}\) lies along the rotated axis \(X'\) and the component \(R_{f2}\) is normal to \(R_{f1}\), in the instantaneous plane of motion. The component pair \((R_{f1}, R_{f2})\) defines both \(R\) and the direction angle \(\chi\) of perigee apsis from the rotated axis \(X'\).

2.2.2 The Attitude State Variables

The attitude dynamics are effectively presented in terms of the attitude momentum state \(H\) in general form. The differential variation of the attitude momentum due to perturbing moments (on board or external) will define the consequent rotation rate of the body axis in inertial space. And so, the angular momentum and the kinetic energy of the attitude state would be the parametric state variables.
Fig. 2.3
STATE MODEL GEOMETRY OF AN ORBITAL TRAJECTORY
2.2.3 The Coordinate Variables

The use of Euler parameters as coordinate variables is based upon successful demonstration of typical exercises of the unified state model.

Hamilton's work on quaternions, in which he developed the orbital velocity hodograph, illustrates the operational value of the quaternion algebra for orbital mechanics. The Euler parameters define a unitary quaternion which can be employed to describe the rotation of a coordinate triad (see Appendix A). Consequently, the coordinate rotation of orbital motion (for the orbital trajectory) and of body motion about its point-mass (for the attitude dynamics) can be expressed by one matrix form of rotation transformation. Computation will define the Euler parameters of the orbital trajectory frame \( (e_{01}, e_{02}, e_{03}, e_{04}) \) and of the attitude frame \( (e_{a1}, e_{a2}, e_{a3}, e_{a4}) \) by means of the related constraint equation.

2.3 The Dynamical Model

The constraint equations for the attitude dynamics are defined by the angular momentum

\[
\frac{d}{dt} [H]_I = [T]_I
\]

\[
[H]_B = [E][H]_I
\]  \hspace{1cm} (2.7)

where \([E]\) = rotation transformation for body attitude in terms of \( (e_{a1}, e_{a2}, e_{a3}, e_{a4}) \) [Appendix A].
The body orientation is given by

\[
\begin{bmatrix}
e_{a1} \\
e_{a2} \\
e_{a3} \\
e_{a4}
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
0 & W_z & -W_y & W_x \\
-W_z & 0 & W_x & W_y \\
W_y & -W_x & 0 & W_z \\
-W_x & -W_y & -W_z & 0
\end{bmatrix}
\begin{bmatrix}
e_{a1} \\
e_{a2} \\
e_{a3} \\
e_{a4}
\end{bmatrix}
\]

(2.8)

where

\[
\begin{bmatrix}
W_x \\
W_y \\
W_z
\end{bmatrix} = \begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{bmatrix}^{-1}
\begin{bmatrix}
H_x \\
H_y \\
H_z
\end{bmatrix}
\]

(2.9)

and \([J]\) = Inertia matrix for the vehicle.

While the constraint equations for the orbital dynamics are defined by the orbital state,

\[
\begin{bmatrix}
dC \\
dR_{f1} \\
dR_{f2}
\end{bmatrix} = \begin{bmatrix}
0 & -p \\
\cos \lambda & -(1+p)\sin \lambda \\
\sin \lambda & (1+p)\cos \lambda
\end{bmatrix}
\begin{bmatrix}
ea_{a1} \\
ea_{a2} \\
ea_{a3} \\
ea_{a4}
\end{bmatrix}
\]

(2.10)

and the orbital coordinates

\[
\begin{bmatrix}
e_{o1} \\
e_{o2} \\
e_{o3} \\
e_{o4}
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
0 & W_3 & 0 & W_1 \\
-W_3 & 0 & W_1 & 0 \\
0 & -W_1 & 0 & W_3 \\
-W_1 & 0 & -W_3 & 0
\end{bmatrix}
\begin{bmatrix}
e_{o1} \\
e_{o2} \\
e_{o3} \\
e_{o4}
\end{bmatrix}
\]

(2.11)
where \( p = \frac{C}{\nu e_2} \)

and

\[
\begin{bmatrix}
  v_{e1} \\
  v_{e2}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  C
\end{bmatrix} + \begin{bmatrix}
  \cos \lambda & \sin \lambda \\
  -\sin \lambda & \cos \lambda
\end{bmatrix} \begin{bmatrix}
  R_{f1} \\
  R_{f2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \sin \lambda \\
  \cos \lambda
\end{bmatrix} = \frac{1}{(e_{03}^2 + e_{04}^2)} \begin{bmatrix}
  2 e_{03} e_{04} \\
  e_{04}^2 - e_{03}^2
\end{bmatrix}
\]

\( W_1 = \frac{a_{e3}}{\nu e_2} \)

\( W_3 = c \nu^2 e_2 / \mu \)

\[
[a] = \begin{bmatrix}
  a_{e1} \\
  a_{e2} \\
  a_{e3}
\end{bmatrix} = \Lambda a_i
\]

and \((e_{01}, e_{02}, e_{03}, e_{04})\) are defined in Appendix A.

Here, the source of coupling is due to the thruster forces and the resulting accelerations \((F/m)\) contribute to \(a_{e1}, a_{e2},\) and \(a_{e3}\), the perturbing accelerations. It should be remembered that these perturbing acceleration components do not include the central-force field acceleration due to the simple spherical potential function, \(V = m/r\).

Besides, the thruster torque components defined by Eq. (2.7) can be obtained from the following relation,
\[ \text{Torque} = \text{Force} \times \text{Effective Moment Arms} \quad (2.16) \]

Thus, we can already see the possible correlations between the attitude and the orbital dynamics.

2.4 Nonlinear Model Equations

The nonlinear system is described by the n-vector differential equation:

\[
\frac{dx_t}{dt} = f(x_t, t) + g(u_t, t) \quad (2.17)
\]

From the defined differential equations in Section 2.3, fourteen state variables are used in the model. From the attitude dynamic model, \( H_x, H_y, \) and \( H_z \) define the angular momentum state while \( e_{a1}, e_{a2}, e_{a3} \) and \( e_{a4} \) define the body orientation. From the orbital dynamic model, \( C, R_{f1} \) and \( R_{f2} \) define the velocity state while \( e_{01}, e_{02}, e_{03} \) and \( e_{04} \) define the coordinate variables. Also, \( T_x, T_y, \) and \( T_z \) define the forcing function.

The correlation between \([a]\) and \([T]\) is given by

\[
\begin{bmatrix}
    e_{a1} \\
    -e_{a2} \\
    e_{a3}
\end{bmatrix} = [E]^{-1} \begin{bmatrix}
    F_z/m \\
    F_x/m \\
    F_y/m
\end{bmatrix}_B
\]

\[
= [E]^{-1} \begin{bmatrix}
    T_z/mM_z \\
    T_x/mM_x \\
    T_y/mM_y
\end{bmatrix}_B \quad (2.18)
\]
and \[
\begin{bmatrix}
T_x \\
T_y \\
T_z
\end{bmatrix}_R = [E]^{-1}
\begin{bmatrix}
T_x \\
T_y \\
T_z
\end{bmatrix}_B
\] (2.19)

where \([E]\) is the transformation matrix given in the Appendix A,

\([M]\) is the effective moment arms,

\([T]\) is the thruster torque components,

subscripts \(R\) and \(B\) refer to reference state and body axes respectively.

Also, \(C = u/H\) (2.20)

From Section 2.3, those differential equations are rewritten as follows,

\[
\frac{dC}{dt} = -p^a e_2
\]

\[
\frac{dR_f1}{dt} = a_1 e_1 \cos \lambda - a_2 (1 + p) \sin \lambda
\]

\[
\frac{dR_f2}{dt} = a_1 e_1 \sin \lambda + (1 + p) a_2 \cos \lambda
\]

\[
\frac{de_{01}}{dt} = \frac{1}{2} (W_3 e_{02} + W_1 e_{04})
\]

\[
\frac{de_{02}}{dt} = \frac{1}{2} (-W_3 e_{01} + W_1 e_{03})
\]

\[
\frac{de_{03}}{dt} = \frac{1}{2} (-W_1 e_{02} + W_3 e_{04})
\]

\[
\frac{de_{04}}{dt} = -\frac{1}{2} (W_1 e_{01} + W_3 e_{03})
\]
\[
\begin{align*}
\frac{dH_x}{dt} &= T_{x_R} \\
\frac{dH_y}{dt} &= T_{y_R} \\
\frac{dH_z}{dt} &= T_{z_R} \\
\frac{de_{a1}}{dt} &= \frac{1}{2}(W_z e_{a2} - W_y e_{a3} + W_x e_{a4}) \\
\frac{de_{a2}}{dt} &= \frac{1}{2}(-W_z e_{a1} + W_x e_{a3} + W_y e_{a4}) \\
\frac{de_{a3}}{dt} &= \frac{1}{2}(W_y e_{a1} - W_x e_{a2} + W_z e_{a4}) \\
\frac{de_{a4}}{dt} &= -\frac{1}{4}(W_x e_{a1} + W_y e_{a2} + W_z e_{a3}) \\
\end{align*}
\]

and so, the nonlinear state model is defined by

\[
\begin{align*}
\dot{C} &= f_1(C, R_{f1}, R_{f2}, e_{01}, e_{02}, e_{03}, e_{04}, H_x, H_y, H_z, e_{a1}, e_{a2}, e_{a3}, e_{a4}) + g_1(T_{x_B}, T_{y_B}, T_{z_B}) \\
\dot{R}_{f1} &= f_2(C, e_{a4}) + g_2(T_{x_B}, T_{y_B}, T_{z_B}) \\
\dot{R}_{f2} &= f_3(C, e_{a4}) + g_3(T_{x_B}, T_{y_B}, T_{z_B}) \\
\dot{e}_{01} &= f_4(C, e_{a4}) + g_4(T_{x_B}, T_{y_B}, T_{z_B}) \\
\dot{e}_{02} &= f_5(C, e_{a4}) + g_5(T_{x_B}, T_{y_B}, T_{z_B}) \\
\dot{e}_{03} &= f_6(C, e_{a4}) + g_6(T_{x_B}, T_{y_B}, T_{z_B}) \\
\dot{e}_{04} &= f_7(C, e_{a4}) + g_7(T_{x_B}, T_{y_B}, T_{z_B}) \\
\end{align*}
\]
The linearization with respect to the states and thruster torque components gives the matrices $A$ and $B$ of the linearized continuous state model

$$\dot{x} = Ax + Bu$$

(2.23)

To linearize these differential equations (2.21), expansion in a 'Taylor' series about reference position $(X_R)$ and $(U_R)$ is used. For example [5]:

$$\dot{C} = f_1(C, \ldots, e_{a4}) + \frac{\partial f_1}{\partial C} (C-C_R) + \ldots + \frac{\partial f_1}{\partial e_{a4}} (e_{a4}-e_{a4_R}) \ldots \frac{\partial g_1}{\partial T_{x_B}} (T_{x_B}-T_{x_{R}}) + \ldots + \frac{\partial g_1}{\partial T_{z_B}} (T_{z_B}-T_{z_{R}}) + \text{higher order terms}$$

(2.24)
The partial derivatives in this expansion are evaluated at the reference position and without loss of accuracy, those higher order terms can be neglected, since the difference \((X-X_R)\) or \((U-U_R)\) is small. Finally, the equations are linearized.

The partial derivatives \(\partial f_i / \partial x\) where \(i=1, 14\) are as follows:

\[
\frac{\partial f_1}{\partial C} = -a e_2 \frac{\partial P}{\partial C}
\]

\[
\frac{\partial f_1}{\partial R_{f1}} = -a e_2 \frac{\partial P}{\partial R_{f1}}
\]

\[
\frac{\partial f_1}{\partial R_{f2}} = -a e_2 \frac{\partial P}{\partial R_{f2}}
\]

\[
\frac{\partial f_1}{\partial e_{01}} = -a e_2 \frac{\partial P}{\partial e_{01}}
\]

\[
\frac{\partial f_1}{\partial e_{02}} = -a e_2 \frac{\partial P}{\partial e_{02}}
\]

\[
\frac{\partial f_1}{\partial e_{03}} = -a e_2 \frac{\partial P}{\partial e_{03}} - a e_2 \frac{\partial P}{\partial e_{03}}
\]

\[
\frac{\partial f_1}{\partial e_{04}} = -a e_2 \frac{\partial P}{\partial e_{04}} - a e_2 \frac{\partial P}{\partial e_{04}}
\]

\[
\frac{\partial f_1}{\partial H_x} = -a e_2 \frac{\partial P}{\partial H_x}
\]
\[ \frac{\partial f_1}{\partial H_y} = -a e_2 \frac{\partial p}{\partial H_y} \]
\[ \frac{\partial f_1}{\partial H_z} = -a e_2 \frac{\partial p}{\partial H_z} \]
\[ \frac{\partial f_1}{\partial e_{a1}} = -p \frac{a e_2}{\partial e_{a1}} \]
\[ \frac{\partial f_1}{\partial e_{a2}} = -p \frac{a e_2}{\partial e_{a2}} \]
\[ \frac{\partial f_1}{\partial e_{a3}} = -p \frac{a e_2}{\partial e_{a3}} \]
\[ \frac{\partial f_1}{\partial e_{a4}} = -p \frac{a e_2}{\partial e_{a4}} \]
\[ \frac{\partial f_2}{\partial C} = -a e_2 \sin \lambda \frac{\partial p}{\partial C} \]
\[ \frac{\partial f_2}{\partial R_{f1}} = -a e_2 \sin \lambda \frac{\partial p}{\partial R_{f1}} \]
\[ \frac{\partial f_2}{\partial R_{f2}} = -a e_2 \sin \lambda \frac{\partial p}{\partial R_{f2}} \]
\[ \frac{\partial f_2}{\partial e_{01}} = \cos \lambda \frac{a e_1}{\partial e_{01}} - (1+P) \sin \lambda \frac{a e_2}{\partial e_{01}} \]
\[ \frac{\partial f_2}{\partial e_{02}} = \cos \lambda \frac{a e_1}{\partial e_{02}} - (1+P) \sin \lambda \frac{a e_2}{\partial e_{02}} \]
\[
\frac{\partial f_2}{\partial e_{03}} = \cos \lambda \frac{\partial a_{e1}}{\partial e_{03}} + a_{e1} \frac{\partial \cos \lambda}{\partial e_{03}} - (1+P) \sin \lambda \frac{\partial a_{e2}}{\partial e_{03}} - a_{e2}(1+P)
\]

\[
\frac{\partial \sin \lambda}{\partial e_{03}} - a_{e2} \sin \lambda \frac{\partial P}{\partial e_{03}}
\]

\[
\frac{\partial f_2}{\partial e_{04}} = \cos \lambda \frac{\partial a_{e1}}{\partial e_{04}} + a_{e1} \frac{\partial \cos \lambda}{\partial e_{04}} - (1+P) \sin \lambda \frac{\partial a_{e2}}{\partial e_{04}} - a_{e2}(1+P)
\]

\[
\frac{\partial \sin \lambda}{\partial e_{04}} - a_{e2} \sin \lambda \frac{\partial P}{\partial e_{04}}
\]

\[
\frac{\partial f_2}{\partial H_x} = -a_{e2} \sin \lambda \frac{\partial P}{\partial H_x}
\]

\[
\frac{\partial f_2}{\partial H_y} = -a_{e2} \sin \lambda \frac{\partial P}{\partial H_y}
\]

\[
\frac{\partial f_2}{\partial H_z} = -a_{e2} \sin \lambda \frac{\partial P}{\partial H_z}
\]

\[
\frac{\partial f_2}{\partial e_{a1}} = \cos \lambda \frac{\partial a_{e1}}{\partial e_{a1}} - (1+P) \sin \lambda \frac{\partial a_{e2}}{\partial e_{a1}}
\]

\[
\frac{\partial f_2}{\partial e_{a2}} = \cos \lambda \frac{\partial a_{e1}}{\partial e_{a2}} - (1+P) \sin \lambda \frac{\partial a_{e2}}{\partial e_{a2}}
\]

\[
\frac{\partial f_2}{\partial e_{a3}} = \cos \lambda \frac{\partial a_{e1}}{\partial e_{a3}} - (1+P) \sin \lambda \frac{\partial a_{e2}}{\partial e_{a3}}
\]

\[
\frac{\partial f_2}{\partial e_{a4}} = \cos \lambda \frac{\partial a_{e1}}{\partial e_{a4}} - (1+P) \sin \lambda \frac{\partial a_{e2}}{\partial e_{a4}}
\]

\[
\frac{\partial f_3}{\partial C} = a_{e2} \cos \lambda \frac{\partial P}{\partial C}
\]
\[
\frac{\partial f_3}{\partial R_{\text{f1}}} = a_2 e_2 \cos \lambda \frac{\partial p}{\partial R_{\text{f1}}}
\]

\[
\frac{\partial f_3}{\partial R_{\text{f2}}} = a_2 e_2 \cos \lambda \frac{\partial p}{\partial R_{\text{f2}}}
\]

\[
\frac{\partial f_3}{\partial e_{01}} = \sin \lambda \frac{\partial a_{e_1}}{\partial e_{01}} + (1+P) \cos \lambda \frac{\partial a_{e_2}}{\partial e_{01}}
\]

\[
\frac{\partial f_3}{\partial e_{02}} = \sin \lambda \frac{\partial a_{e_1}}{\partial e_{02}} + (1+P) \cos \lambda \frac{\partial a_{e_2}}{\partial e_{02}}
\]

\[
\frac{\partial f_3}{\partial e_{03}} = \sin \lambda \frac{\partial a_{e_1}}{\partial e_{03}} + a_{e_1} \frac{\partial \sin \lambda}{\partial e_{03}} + a_{e_2} \cos \lambda \frac{\partial p}{\partial e_{03}} + (1+P) \cos \lambda \frac{\partial a_{e_2}}{\partial e_{03}}
\]

\[
\frac{\partial a_{e_2}}{\partial e_{03}} + (1+P) a_{e_2} \frac{\partial \cos \lambda}{\partial e_{03}}
\]

\[
\frac{\partial f_3}{\partial e_{04}} = \sin \lambda \frac{\partial a_{e_1}}{\partial e_{04}} + a_{e_1} \frac{\partial \sin \lambda}{\partial e_{04}} + a_{e_2} \cos \lambda \frac{\partial p}{\partial e_{04}} + (1+P) \cos \lambda \frac{\partial a_{e_2}}{\partial e_{04}}
\]

\[
\frac{\partial a_{e_2}}{\partial e_{04}} + (1+P) a_{e_2} \frac{\partial \cos \lambda}{\partial e_{04}}
\]

\[
\frac{\partial f_3}{\partial H_x} = a_2 e_2 \cos \lambda \frac{\partial p}{\partial H_x}
\]

\[
\frac{\partial f_3}{\partial H_y} = a_2 e_2 \cos \lambda \frac{\partial p}{\partial H_y}
\]

\[
\frac{\partial f_3}{\partial H_z} = a_2 e_2 \cos \lambda \frac{\partial p}{\partial H_z}
\]

\[
\frac{\partial f_3}{\partial e_{al}} = \sin \lambda \frac{\partial a_{e_{al}}}{\partial e_{al}} + (1+P) \cos \lambda \frac{\partial a_{e_2}}{\partial e_{al}}
\]
\[ \frac{\partial f_3}{\partial e_{a2}} = \sin \lambda \frac{\partial a_{e1}}{\partial e_{a2}} + (1+P) \cos \lambda \frac{\partial a_{e2}}{\partial e_{a2}} \]

\[ \frac{\partial f_3}{\partial e_{a3}} = \sin \lambda \frac{\partial a_{e1}}{\partial e_{a3}} + (1+P) \cos \lambda \frac{\partial a_{e2}}{\partial e_{a3}} \]

\[ \frac{\partial f_3}{\partial e_{a4}} = \sin \lambda \frac{\partial a_{e1}}{\partial e_{a4}} + (1+P) \cos \lambda \frac{\partial a_{e2}}{\partial e_{a4}} \]

\[ \frac{\partial f_4}{\partial C} = \frac{\lambda}{d} (e_{02} \frac{\partial w_3}{\partial C} + e_{04} \frac{\partial w_1}{\partial C}) \]

\[ \frac{\partial f_4}{\partial R_{f1}} = \frac{\lambda}{d} (e_{02} \frac{\partial w_3}{\partial R_{f1}} + e_{04} \frac{\partial w_1}{\partial R_{f1}}) \]

\[ \frac{\partial f_4}{\partial R_{f2}} = \frac{\lambda}{d} (e_{02} \frac{\partial w_3}{\partial R_{f2}} + e_{04} \frac{\partial w_1}{\partial R_{f2}}) \]

\[ \frac{\partial f_4}{\partial e_{01}} = \frac{\lambda}{d} e_{04} \frac{\partial w_1}{\partial e_{01}} \]

\[ \frac{\partial f_4}{\partial e_{02}} = \frac{\lambda}{d} (e_{04} \frac{\partial w_1}{\partial e_{02}} + w_3) \]

\[ \frac{\partial f_4}{\partial e_{03}} = \frac{\lambda}{d} (e_{02} \frac{\partial w_3}{\partial e_{03}} + e_{04} \frac{\partial w_1}{\partial e_{03}}) \]

\[ \frac{\partial f_4}{\partial e_{04}} = \frac{\lambda}{d} (e_{02} \frac{\partial w_3}{\partial e_{04}} + \omega_1 + e_{04} \frac{\partial w_1}{\partial e_{04}}) \]

\[ \frac{\partial f_4}{\partial H_x} = \frac{\lambda}{d} (e_{02} \frac{\partial w_3}{\partial H_x} + e_{04} \frac{\partial w_1}{\partial H_x}) \]
\[
\frac{\partial f_5}{\partial e_{03}} = \frac{1}{2} (-e_{01} \frac{\partial w_3}{\partial e_{03}} + w_1 + e_{03} \frac{\partial w_1}{\partial e_{03}})
\]

\[
\frac{\partial f_5}{\partial e_{04}} = \frac{1}{2} (-e_{01} \frac{\partial w_3}{\partial e_{04}} + e_{03} \frac{\partial w_1}{\partial e_{04}})
\]

\[
\frac{\partial f_5}{\partial H_x} = \frac{1}{2} (-e_{01} \frac{\partial w_3}{\partial H_x} + e_{03} \frac{\partial w_1}{\partial H_x})
\]

\[
\frac{\partial f_5}{\partial H_y} = \frac{1}{2} (-e_{01} \frac{\partial w_3}{\partial H_y} + e_{03} \frac{\partial w_1}{\partial H_y})
\]

\[
\frac{\partial f_5}{\partial H_z} = \frac{1}{2} (-e_{01} \frac{\partial w_3}{\partial H_z} + e_{03} \frac{\partial w_1}{\partial H_z})
\]

\[
\frac{\partial f_5}{\partial e_{a1}} = \frac{1}{2} e_{03} \frac{\partial w_1}{\partial e_{a1}}
\]

\[
\frac{\partial f_5}{\partial e_{a2}} = \frac{1}{2} e_{03} \frac{\partial w_1}{\partial e_{a2}}
\]

\[
\frac{\partial f_5}{\partial e_{a3}} = \frac{1}{2} e_{03} \frac{\partial w_1}{\partial e_{a3}}
\]

\[
\frac{\partial f_5}{\partial e_{a4}} = \frac{1}{2} e_{03} \frac{\partial w_1}{\partial e_{a4}}
\]

\[
\frac{\partial f_6}{\partial C} = \frac{1}{2} (-e_{02} \frac{\partial w_1}{\partial C} + e_{04} \frac{\partial w_3}{\partial C})
\]

\[
\frac{\partial f_6}{\partial R_{f1}} = \frac{1}{2} (-e_{02} \frac{\partial w_1}{\partial R_{f1}} + e_{04} \frac{\partial w_3}{\partial R_{f1}})
\]

\[
\frac{\partial f_6}{\partial R_{f2}} = \frac{1}{2} (-e_{02} \frac{\partial w_1}{\partial R_{f2}} + e_{04} \frac{\partial w_3}{\partial R_{f2}})
\]
\[
\frac{\partial f_6}{\partial e_{01}} = -\frac{1}{2} e_{02} \frac{\partial w_1}{\partial e_{01}} \\
\frac{\partial f_6}{\partial e_{02}} = \frac{1}{2} (-w_1 - e_{02} \frac{\partial w_1}{\partial e_{02}}) \\
\frac{\partial f_6}{\partial e_{03}} = \frac{1}{2} (-e_{02} \frac{\partial w_1}{\partial e_{03}} + e_{04} \frac{\partial w_3}{\partial e_{03}}) \\
\frac{\partial f_6}{\partial e_{04}} = \frac{1}{2} (-e_{02} \frac{\partial w_1}{\partial e_{04}} + w_3 + e_{04} \frac{\partial w_3}{\partial e_{04}}) \\
\frac{\partial f_6}{\partial H_x} = \frac{1}{2} (-e_{02} \frac{\partial w_1}{\partial H_x} + e_{04} \frac{\partial w_3}{\partial H_x}) \\
\frac{\partial f_6}{\partial H_y} = \frac{1}{2} (-e_{02} \frac{\partial w_1}{\partial H_y} + e_{04} \frac{\partial w_3}{\partial H_y}) \\
\frac{\partial f_6}{\partial H_z} = \frac{1}{2} (-e_{02} \frac{\partial w_1}{\partial H_z} + e_{04} \frac{\partial w_3}{\partial H_z}) \\
\frac{\partial f_6}{\partial e_{a1}} = -\frac{1}{2} e_{02} \frac{\partial w_1}{\partial e_{a1}} \\
\frac{\partial f_6}{\partial e_{a2}} = -\frac{1}{2} e_{02} \frac{\partial w_1}{\partial e_{a2}} \\
\frac{\partial f_6}{\partial e_{a3}} = -\frac{1}{2} e_{02} \frac{\partial w_1}{\partial e_{a3}} \\
\frac{\partial f_6}{\partial e_{a4}} = -\frac{1}{2} e_{02} \frac{\partial w_1}{\partial e_{a4}}
\]
\[ \frac{\partial f_f}{\partial c} = -\frac{1}{2}(e_0 + 1 \frac{\partial w_1}{\partial c} + e_0 + 3 \frac{\partial w_3}{\partial c}) \]
\[ \frac{\partial f_f}{\partial R_{f1}} = -\frac{1}{2}(e_0 + 1 \frac{\partial w_1}{\partial R_{f1}} + e_0 + 3 \frac{\partial w_3}{\partial R_{f1}}) \]
\[ \frac{\partial f_f}{\partial R_{f2}} = -\frac{1}{2}(e_0 + 1 \frac{\partial w_1}{\partial R_{f2}} + e_0 + 3 \frac{\partial w_3}{\partial R_{f2}}) \]
\[ \frac{\partial f_f}{\partial e_{01}} = -\frac{1}{2}(w_1 + e_0 + 1 \frac{\partial w_1}{\partial e_{01}}) \]
\[ \frac{\partial f_f}{\partial e_{02}} = -\frac{1}{2} e_0 + 1 \frac{\partial w_1}{\partial e_{02}} \]
\[ \frac{\partial f_f}{\partial e_{03}} = -\frac{1}{2}(e_0 + 1 \frac{\partial w_1}{\partial e_{03}} + w_3 + e_0 + 3 \frac{\partial w_3}{\partial e_{03}}) \]
\[ \frac{\partial f_f}{\partial e_{04}} = -\frac{1}{2}(e_0 + 1 \frac{\partial w_1}{\partial e_{04}} + e_0 + 3 \frac{\partial w_3}{\partial e_{04}}) \]
\[ \frac{\partial f_f}{\partial H_x} = -\frac{1}{2}(e_0 + 1 \frac{\partial w_1}{\partial H_x} + e_0 + 3 \frac{\partial w_3}{\partial H_x}) \]
\[ \frac{\partial f_f}{\partial H_y} = -\frac{1}{2}(e_0 + 1 \frac{\partial w_1}{\partial H_y} + e_0 + 3 \frac{\partial w_3}{\partial H_y}) \]
\[ \frac{\partial f_f}{\partial H_z} = -\frac{1}{2}(e_0 + 1 \frac{\partial w_1}{\partial H_z} + e_0 + 3 \frac{\partial w_3}{\partial H_z}) \]
\[ \frac{\partial f_f}{\partial e_{a1}} = -\frac{1}{2} e_0 + 1 \frac{\partial w_1}{\partial e_{a1}} \]
\[ \frac{\partial f_f}{\partial e_{a2}} = -\frac{1}{2} e_0 + 1 \frac{\partial w_1}{\partial e_{a2}} \]
$$\frac{\delta f_7}{\delta e_{a3}} = -\frac{1}{4} e_{01} \frac{\delta w_1}{\delta e_{a3}}$$

$$\frac{\delta f_7}{\delta e_{a4}} = -\frac{1}{4} e_{01} \frac{\delta w_1}{\delta e_{a4}}$$

$$\frac{\delta f_8}{\delta C} = 0$$

$$\frac{\delta f_8}{\delta R_{f1}} = 0$$

$$\frac{\delta f_8}{\delta R_{f2}} = 0$$

$$\frac{\delta f_8}{\delta e_{01}} = \frac{\delta T x_R}{\delta e_{01}}$$

$$\frac{\delta f_8}{\delta e_{02}} = \frac{\delta T x_R}{\delta e_{02}}$$

$$\frac{\delta f_8}{\delta e_{03}} = \frac{\delta T x_R}{\delta e_{03}}$$

$$\frac{\delta f_8}{\delta e_{04}} = \frac{\delta T x_R}{\delta e_{04}}$$

$$\frac{\delta f_8}{\delta H_x} = 0$$

$$\frac{\delta f_8}{\delta H_y} = 0$$

$$\frac{\delta f_8}{\delta H_z} = 0$$
\[ \frac{\partial f_8}{\partial e_{a1}} = \frac{\partial T_x}{\partial e_{a1}} \]

\[ \frac{\partial f_8}{\partial e_{a2}} = \frac{\partial T_x}{\partial e_{a2}} \]

\[ \frac{\partial f_8}{\partial e_{a3}} = \frac{\partial T_x}{\partial e_{a3}} \]

\[ \frac{\partial f_8}{\partial e_{a4}} = \frac{\partial T_x}{\partial e_{a4}} \]

\[ \frac{\partial f_9}{\partial C} = 0 \]

\[ \frac{\partial f_9}{\partial R_{f1}} = 0 \]

\[ \frac{\partial f_9}{\partial R_{f2}} = 0 \]

\[ \frac{\partial f_9}{\partial e_{01}} = \frac{\partial T_y}{\partial e_{01}} \]

\[ \frac{\partial f_9}{\partial e_{02}} = \frac{\partial T_y}{\partial e_{02}} \]

\[ \frac{\partial f_9}{\partial e_{03}} = \frac{\partial T_y}{\partial e_{03}} \]

\[ \frac{\partial f_9}{\partial e_{04}} = \frac{\partial T_y}{\partial e_{04}} \]

\[ \frac{\partial f_9}{\partial R_x} = 0 \]
\[ \frac{\partial f_9}{\partial H_y} = 0 \]

\[ \frac{\partial f_9}{\partial H_z} = 0 \]

\[ \frac{\partial f_9}{\partial e_{a1}} = \frac{\partial T_y}{\partial e_{a1}} \]

\[ \frac{\partial f_9}{\partial e_{a2}} = \frac{\partial T_y}{\partial e_{a2}} \]

\[ \frac{\partial f_9}{\partial e_{a3}} = \frac{\partial T_y}{\partial e_{a3}} \]

\[ \frac{\partial f_9}{\partial e_{a4}} = \frac{\partial T_y}{\partial e_{a4}} \]

\[ \frac{\partial f_{10}}{\partial C} = 0 \]

\[ \frac{\partial f_{10}}{\partial R_{f1}} = 0 \]

\[ \frac{\partial f_{10}}{\partial R_{f2}} = 0 \]

\[ \frac{\partial f_{10}}{\partial e_{01}} = \frac{\partial T_z}{\partial e_{01}} \]

\[ \frac{\partial f_{10}}{\partial e_{02}} = \frac{\partial T_z}{\partial e_{02}} \]

\[ \frac{\partial f_{10}}{\partial e_{03}} = \frac{\partial T_z}{\partial e_{03}} \]
\[ \frac{\partial f_{10}}{\partial e_{04}} = \frac{\partial T_z}{\partial e_{04}} \]
\[ \frac{\partial f_{10}}{\partial H_x} = 0 \]
\[ \frac{\partial f_{10}}{\partial H_y} = 0 \]
\[ \frac{\partial f_{10}}{\partial H_z} = 0 \]
\[ \frac{\partial f_{10}}{\partial e_{a1}} = \frac{\partial T_z}{\partial e_{a1}} \]
\[ \frac{\partial f_{10}}{\partial e_{a2}} = \frac{\partial T_z}{\partial e_{a2}} \]
\[ \frac{\partial f_{10}}{\partial e_{a3}} = \frac{\partial T_z}{\partial e_{a3}} \]
\[ \frac{\partial f_{10}}{\partial e_{a4}} = \frac{\partial T_z}{\partial e_{a4}} \]
\[ \frac{\partial f_{11}}{\partial C} = 0 \]
\[ \frac{\partial f_{11}}{\partial R_{f1}} = 0 \]
\[ \frac{\partial f_{11}}{\partial R_{f2}} = 0 \]
\[ \frac{\partial f_{11}}{\partial e_{01}} = 0 \]
\[
\begin{align*}
\frac{\partial f_{11}}{\partial e_{02}} &= 0 \\
\frac{\partial f_{11}}{\partial e_{03}} &= 0 \\
\frac{\partial f_{11}}{\partial e_{04}} &= 0 \\
\frac{\partial f_{11}}{\partial H_x} &= \frac{1}{4} \left( e_{a2} J_{31} - e_{a3} J_{21} + e_{a4} J_{11} \right) \\
\frac{\partial f_{11}}{\partial H_y} &= \frac{1}{4} \left( e_{a2} J_{32} - e_{a3} J_{22} + e_{a4} J_{12} \right) \\
\frac{\partial f_{11}}{\partial H_z} &= \frac{1}{4} \left( e_{a2} J_{33} - e_{a3} J_{23} + e_{a4} J_{13} \right) \\
\frac{\partial f_{11}}{\partial e_{a1}} &= 0 \\
\frac{\partial f_{11}}{\partial e_{a2}} &= \frac{1}{4} \left( J_{31} H_x + J_{32} H_y + J_{33} H_z \right) \\
\frac{\partial f_{11}}{\partial e_{a3}} &= -\frac{1}{4} \left( J_{21} H_x + J_{22} H_y + J_{23} H_z \right) \\
\frac{\partial f_{11}}{\partial e_{a4}} &= \frac{1}{4} \left( J_{11} H_x + J_{12} H_y + J_{13} H_z \right) \\
\frac{\partial f_{12}}{\partial c} &= 0 \\
\frac{\partial f_{12}}{\partial R_{f1}} &= 0
\end{align*}
\]
\[ \frac{\partial f_{13}}{\partial C} = \frac{\partial f_{13}}{\partial R_{f1}} = \frac{\partial f_{13}}{\partial R_{f2}} = 0 \]

\[ \frac{\partial f_{13}}{\partial e_{01}} = 0 \]

\[ \frac{\partial f_{13}}{\partial e_{02}} = 0 \]

\[ \frac{\partial f_{13}}{\partial e_{03}} = 0 \]

\[ \frac{\partial f_{13}}{\partial e_{04}} = 0 \]

\[ \frac{\partial f_{13}}{\partial H_x} = \frac{1}{2}(e_{a1} J_{21} - e_{a2} J_{11} + e_{a4} J_{31}) \]

\[ \frac{\partial f_{13}}{\partial H_y} = \frac{1}{2}(e_{a1} J_{22} - e_{a2} J_{12} + e_{a4} J_{32}) \]

\[ \frac{\partial f_{13}}{\partial H_z} = \frac{1}{2}(e_{a1} J_{23} - e_{a2} J_{13} + e_{a4} J_{33}) \]

\[ \frac{\partial f_{13}}{\partial e_{a1}} = \frac{1}{2}(J_{21} H_x + J_{22} H_y + J_{23} H_z) \]

\[ \frac{\partial f_{13}}{\partial e_{a2}} = -\frac{1}{2}(J_{11} H_x + J_{12} H_y + J_{13} H_z) \]

\[ \frac{\partial f_{13}}{\partial e_{a3}} = 0 \]

\[ \frac{\partial f_{13}}{\partial e_{a4}} = \frac{1}{2}(J_{31} H_x + J_{32} H_y + J_{33} H_z) \]
\[ \frac{\partial f_{14}}{\partial e_{01}} = 0 \]
\[ \frac{\partial f_{14}}{\partial e_{02}} = 0 \]
\[ \frac{\partial f_{14}}{\partial e_{03}} = 0 \]
\[ \frac{\partial f_{14}}{\partial e_{04}} = 0 \]
\[ \frac{\partial f_{14}}{\partial H_x} = -\frac{\gamma}{2} (e_{a1} J_{11} + e_{a2} J_{21} + e_{a3} J_{31}) \]
\[ \frac{\partial f_{14}}{\partial H_y} = -\frac{\gamma}{2} (e_{a1} J_{12} + e_{a2} J_{22} + e_{a3} J_{32}) \]
\[ \frac{\partial f_{14}}{\partial H_z} = -\frac{\gamma}{2} (e_{a1} J_{13} + e_{a2} J_{23} + e_{a3} J_{33}) \]
\[ \frac{\partial f_{14}}{\partial e_{a1}} = -\frac{\gamma}{2} (J_{11} H_x + J_{12} H_y + J_{13} H_z) \]
\[ \frac{\partial f_{14}}{\partial e_{a2}} = -\frac{\gamma}{2} (J_{21} H_x + J_{22} H_y + J_{23} H_z) \]
\[ \frac{\partial f_{14}}{\partial R_{f1}} = 0 \]
\[ \frac{\partial f_{14}}{\partial R_{f2}} = 0 \]
\[
\frac{\partial f_{14}}{\partial e_{a3}} = -\frac{1}{2}(J_{31} H_x + J_{32} H_y + J_{33} H_z)
\]

\[
\frac{\partial f_{14}}{\partial e_{a4}} = 0
\]

(2.25)

The partial derivatives for \( \partial g_i/\partial u_j \), where \( i = 1,14 \)
and \( j = 1,3 \) are obtained as follows,

\[
\frac{\partial g_1}{\partial T_x B} = -p \frac{\partial e_2}{\partial T_x B}
\]

\[
\frac{\partial g_1}{\partial T_y B} = -p \frac{\partial e_2}{\partial T_y B}
\]

\[
\frac{\partial g_1}{\partial T_z B} = -p \frac{\partial e_2}{\partial T_z B}
\]

\[
\frac{\partial g_2}{\partial T_x B} = \cos \lambda \frac{\partial e_1}{\partial T_x B} - (1+p) \sin \lambda \frac{\partial e_2}{\partial T_x B}
\]

\[
\frac{\partial g_2}{\partial T_y B} = \cos \lambda \frac{\partial e_1}{\partial T_y B} - (1+p) \sin \lambda \frac{\partial e_2}{\partial T_y B}
\]

\[
\frac{\partial g_2}{\partial T_z B} = \cos \lambda \frac{\partial e_1}{\partial T_z B} - (1+p) \sin \lambda \frac{\partial e_2}{\partial T_z B}
\]

\[
\frac{\partial g_3}{\partial T_x B} = \sin \lambda \frac{\partial e_1}{\partial T_x B} + (1+p) \cos \lambda \frac{\partial e_2}{\partial T_x B}
\]

\[
\frac{\partial g_3}{\partial T_y B} = \sin \lambda \frac{\partial e_1}{\partial T_y B} + (1+p) \cos \lambda \frac{\partial e_2}{\partial T_y B}
\]
\[ \frac{\partial g_3}{\partial T_B} = \sin \lambda \frac{\partial \alpha_1}{\partial T_B} + (1+\rho) \cos \lambda \frac{\partial \alpha_2}{\partial T_B} \]

\[ \frac{\partial g_4}{\partial T_B} = \frac{1}{2} e_0^2 \frac{\partial \omega_1}{\partial T_B} \]

\[ \frac{\partial g_4}{\partial T_B} = \frac{1}{2} e_0^2 \frac{\partial \omega_1}{\partial T_B} \]

\[ \frac{\partial g_5}{\partial T_B} = \frac{1}{2} e_0^2 \frac{\partial \omega_1}{\partial T_B} \]

\[ \frac{\partial g_5}{\partial T_B} = \frac{1}{2} e_0^2 \frac{\partial \omega_1}{\partial T_B} \]

\[ \frac{\partial g_6}{\partial T_B} = -\frac{1}{2} e_0^2 \frac{\partial \omega_1}{\partial T_B} \]

\[ \frac{\partial g_6}{\partial T_B} = -\frac{1}{2} e_0^2 \frac{\partial \omega_1}{\partial T_B} \]

\[ \frac{\partial g_6}{\partial T_B} = -\frac{1}{2} e_0^2 \frac{\partial \omega_1}{\partial T_B} \]

\[ \frac{\partial g_7}{\partial T_B} = -\frac{1}{2} e_0^1 \frac{\partial \omega_1}{\partial T_B} \]
\[ \frac{\partial g_7}{\partial T_y_B} = -\frac{1}{2} e_{01} \frac{\partial u_1}{\partial T_y_B} \]

\[ \frac{\partial g_7}{\partial T_z_B} = -\frac{1}{2} e_{01} \frac{\partial u_1}{\partial T_z_B} \]

\[ \frac{\partial g_8}{\partial T_x_B} = \frac{\partial T_x_R}{\partial T_x_B} \]

\[ \frac{\partial g_8}{\partial T_y_B} = \frac{\partial T_y_R}{\partial T_y_B} \]

\[ \frac{\partial g_8}{\partial T_z_B} = \frac{\partial T_z_R}{\partial T_z_B} \]

\[ \frac{\partial g_9}{\partial T_x_B} = \frac{\partial T_y_R}{\partial T_x_B} \]

\[ \frac{\partial g_9}{\partial T_y_B} = \frac{\partial T_y_R}{\partial T_y_B} \]

\[ \frac{\partial g_9}{\partial T_z_B} = \frac{\partial T_z_R}{\partial T_z_B} \]

\[ \frac{\partial g_{10}}{\partial T_x_B} = \frac{\partial T_z_R}{\partial T_x_B} \]

\[ \frac{\partial g_{10}}{\partial T_y_B} = \frac{\partial T_y_R}{\partial T_y_B} \]

\[ \frac{\partial g_{10}}{\partial T_z_B} = \frac{\partial T_z_R}{\partial T_z_B} \]
\frac{\partial g_{11}}{\partial T x_B} = 0

\frac{\partial g_{11}}{\partial T y_B} = 0

\frac{\partial g_{12}}{\partial T x_B} = 0

\frac{\partial g_{12}}{\partial T y_B} = 0

\frac{\partial g_{12}}{\partial T z_B} = 0

\frac{\partial g_{13}}{\partial T x_B} = 0

\frac{\partial g_{13}}{\partial T y_B} = 0

\frac{\partial g_{13}}{\partial T z_B} = 0

\frac{\partial g_{14}}{\partial T x_B} = 0

\frac{\partial g_{14}}{\partial T y_B} = 0
From the partial derivatives, the Jacobian matrices for the states and torque components are defined and the linearized perturbation equation is in the form:

$$\frac{d}{dt} [\hat{X}] = [F] [\hat{X}] + [G] [\hat{U}]$$

(2.27)

where

$$\hat{X} = X - X_R$$

$$\hat{U} = U - U_R$$

(2.28)

and $[F] = \text{Jacobian matrix evaluated at the reference state } X_R$.

$[G] = \text{Jacobian matrix evaluated at the reference state } U_R$.

In this case,

$$[F] = \left[\frac{\partial f_i}{\partial x_j}\right] i = 1,14 \text{ and } j = 1,14$$

$$[G] = \left[\frac{\partial g_i}{\partial u_j}\right] i = 1,14 \text{ and } j = 1,3$$

(2.29)

That is, $[F]$ is a 14 x 14 matrix and $[G]$ is a 14 x 3 matrix.

2.6 The Observables

Five observables are defined here. From the attitude dynamic, the observations are pitch angle $\phi$, and roll angle $\psi$. While from the orbital dynamic, the observations
are the slant range $h$, elevation angle $\gamma$ and azimuth angle $\alpha$.

However, from Figure 2.4, we see that

$$\alpha = \pi/2 + \Omega$$

$$\gamma = 1$$

(2.30)

Thus, the number of observables is reduced to three and so, decoupling theory can be applied since there is equal number of inputs and outputs.

2.7 The Linearized Measurement Model

To obtain the linearized model, two successive transformations are used. First, the mapping of the position observation in the orbit plane results in a two-dimensional problem. Later, the three-dimensional problem results by mapping back to the inertial (geocentric) system from the orbital plane system.

The nonlinear measurement model in three dimensions can be obtained very easily by converting rectilinear coordinates $(X,Y,Z)$ to polar coordinates $(\alpha,\gamma,h)$. They are given as follows:

$$X = h \cos \gamma \sin \alpha$$

$$Y = h \cos \gamma \cos \alpha$$

$$Z = h \sin \gamma$$

where $X$, $Y$, $Z$ are referred to radar site coordinates.

These sets of equations must be linearized and by expanding in a 'Taylor' series, we obtain
Fig. 2.4. AZIMUTH–ELEVATION COORDINATE SYSTEM
\[
\begin{align*}
\frac{\partial F_1}{\partial y} &= -h \sin \gamma \sin A \\
\frac{\partial F_1}{\partial h} &= \cos \gamma \sin A \\
\frac{\partial F_1}{\partial A} &= h \cos \gamma \cos A \\
\frac{\partial F_2}{\partial y} &= -h \sin \gamma \cos A \\
\frac{\partial F_2}{\partial h} &= \cos \gamma \cos A \\
\frac{\partial F_2}{\partial A} &= -h \cos \gamma \sin A \\
\frac{\partial F_3}{\partial y} &= h \cos \gamma \\
\frac{\partial F_3}{\partial h} &= \sin \gamma \\
\frac{\partial F_3}{\partial A} &= 0 \\
\end{align*}
\]

And so, the linearized measurement model in the non-inertial site coordinate set is given by

\[
\begin{bmatrix}
\sim X \\
\sim Y \\
\sim Z \\
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial h} & \frac{\partial F_1}{\partial A} \\
\frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial h} & \frac{\partial F_2}{\partial A} \\
\frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial h} & \frac{\partial F_3}{\partial A} \\
\end{bmatrix}
\begin{bmatrix}
\sim \gamma \\
\sim \delta h \\
\sim \delta A \\
\end{bmatrix}
\]

(2.32)

\[
\begin{bmatrix}
\sim \gamma \\
\sim \delta h \\
\sim \delta A \\
\end{bmatrix}
OBS
= \begin{bmatrix}
\frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial h} & \frac{\partial F_1}{\partial A} \\
\frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial h} & \frac{\partial F_2}{\partial A} \\
\frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial h} & \frac{\partial F_3}{\partial A} \\
\end{bmatrix}
\begin{bmatrix}
\sim \gamma \\
\sim \delta h \\
\sim \delta A \\
\end{bmatrix}
R
\]

(2.33)
where the subscript R means the reference point, and the Jacobian matrix is evaluated at the reference position.

Usually, observations are made from a coordinate system that is rotating and so, it is necessary to transform the system back to the inertial axes.

Observations from a ground-based site are referred to the geocenter and the defined inertial axes by means of the site topocentric coordinates shown in Figure 2.5.

\[
\begin{bmatrix}
X \\
Y \\
Z + \text{Re}
\end{bmatrix}
= \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}_{\text{OBS}} = [E]_{\text{TOPO}} \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}_{\text{IN}}
\]

or \([X]_{\text{OBS}} = [E]_{\text{TOPO}} [\tilde{x}]_{\text{IN}}\)

where \(\text{Re} = \text{radius of Earth}\).

The transformation from inertial to topocentric axes in terms of Euler angles is in the order:

\[
\begin{align*}
\Omega &= \pi/2 + \text{Lo} \\
\iota &= \pi/2 - \text{La} \\
\psi &= 0
\end{align*}
\]

The rotation matrix \([E]_{\text{TOPO}}\) is defined in terms of the longitude (Lo) and latitude (La) as follows:

\[
e_{T1} = \frac{1}{2} \{\cos \left(\frac{\text{La} - \text{Lo}}{2}\right) - \sin \left(\frac{\text{La} + \text{Lo}}{2}\right)\}
\]
Fig. 2.5

COORDINATE SET FOR A GROUND-BASED SITE
\[ e_{T2} = \frac{1}{2} \cos \left( \frac{L_a + L_o}{2} \right) - \sin \left( \frac{L_a - L_o}{2} \right) \]

\[ e_{T3} = \frac{1}{2} \cos \left( \frac{L_a - L_o}{2} \right) + \sin \left( \frac{L_a + L_o}{2} \right) \]

\[ e_{T4} = \frac{1}{2} \cos \left( \frac{L_a + L_o}{2} \right) + \sin \left( \frac{L_a - L_o}{2} \right) \]

and \[ [E]_{TOPO} = [\varepsilon] \]

where \( \varepsilon \) is given in terms of the Euler parameters in Appendix A.

Substituting in Eqn. (2.36), we get

\[
[E]_{TOPO} = \begin{bmatrix}
-sin L_o & cos L_o & 0 \\
-cos L_o sin L_a & -sin L_o sin L_a & cos L_a \\
cos L_o cos L_a & sin L_o cos L_a & sin L_a
\end{bmatrix}
\]

(2.37)

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}_{IN} = [E]_0^T \begin{bmatrix}
r \\
0 \\
0
\end{bmatrix}_{IN}
\]

(2.38)

where \( r = \mu/C_v e_{a2} \)

and \( E_0 \) is defined in Appendix A using the Euler parameters \( (e_{01}, e_{02}, e_{03}, e_{04}) \).

From Eqn. (2.34) we can derive the general form

\[ \overline{X}_{OBS} = f(C, \ldots, \varepsilon) \]

(2.39)

By linearizing about a reference position, we obtain

\[ \overline{X} - \overline{X}_R = (Jacobian \ matrix)(State \ perturbation) \]

(2.40)

Using Eqns. (2.33), (2.34) and (2.40), we finally obtain the linearized measurement model referred to the geocenter.
LEAF 45 OMITTED IN PAGE NUMBERING.
Therefore, in the linear continuous state system representation, Eq. (2.46) can be written as follows:

\[ y = Cx \]  

\[(2.42)\]

where \( C = [C] \) = output matrix.

2.8 Conclusions

From the linearized model, the matrices \( A, B \) and \( C \) of the system

\[ \dot{x} = Ax + Bu \]

\[ y = Cx \]  

\[(2.43)\]

can thus be determined.

In doing so, decoupling theory can then be applied to the linearized system.
Chapter 3

The Decoupling Problem in Linear Multivariable Systems

3.1 Introduction

The design of multivariable control systems has received considerable attention in recent years. In particular, the design procedure for the compensation of linear time-invariant multivariable systems, with several inputs and outputs, is greatly complicated by coupling or interaction. By interaction, we mean that one input affects more than one output and an initial condition on one output variable affects more than one output. Therefore, it is highly desirable to somehow transform the systems so that it is noninteracting, that is, each input affects one and only one output.

In this chapter, a general review of the necessary and sufficient conditions for decoupling and pole-placement, presented by Sinha and Rozsa [1], and a different approach by Falb and Wolovich [2], are described. The method of Falb and Wolovich is very good in determining the feasibility of decoupling. However, not necessarily all the poles can be placed. Furthermore, placing them at desired locations can only be achieved by trial-and-error. For the method of Sinha and Rozsa, the synthesis procedure leads to a very simple and efficient algorithm which enables
one to tell at once whether the system can be decoupled and all of its poles can be assigned simultaneously. This is often more useful.

3.2 The Need for Decoupling

Needless to say, there is a great advantage for such a design. Once noninteraction or decoupling is obtained, the system is thus reduced to a number of single variable systems to which the well-known linear design technique can be applied. In other words, a decoupled system is said to exist if the resulting input-output transfer-function matrix is diagonal, that is,

\[
D(s) = \begin{bmatrix}
D_1(s) & 0 & 0 \\
0 & D_2(s) & 0 \\
0 & 0 & D_m(s)
\end{bmatrix}
\]

(3.1)

where \( D(s) \) is the diagonal matrix of proper rational functions of \( s \) and there is an equal number of inputs and outputs which will be denoted by the integer, \( m \).

3.3 How the Problem Can be Solved by State Variable Feedback

Now consider the time-invariant linear system

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

(3.2)

where \( x \) is an \( n \)-vector called the state,
u is an m-vector called the control (or input),
y is an m-vector called the output,
and A, B and C are nxn, nmx, and mxn matrices respectively, and m ≤ n.

If F is an mxn matrix and G a nonsingular mxm constant matrix, then the substitution of the state feedback law

\[ u = Fx + Gv \]  

(3.3)

where v represents the new m-vector control (Fig. 3.1), into (3.2) shall be called linear state variable feedback.

If the system can be decoupled then the resulting closed-loop system transfer function matrix

\[ D(s) = C(sI - A - BF)^{-1}BG \]  

(3.4)

is a diagonal matrix of proper rational functions of s.

Before answering the question of how to obtain the necessary and sufficient conditions on F and G for decoupling, one should examine how the state variable feedback method can solve the decoupling problem and why is such a method desirable?

To answer the question how, following Falb and Wolovich [2], F and G, with G non-singular, decouple the system, if

\[ y_i^{(n)} = \sum_{k=0}^{n-1} P_k \{F\} y_i^{(k)} \equiv \text{tr}(L_1^i\{F,G\} \Omega) \]

\[ \equiv \text{tr}(L_1^i\{F,G\} \Omega^i) \]

(3.5)

where \( i = 1, 2, \ldots, m \)
Fig. 3.1 MULTIVARIABLE FEEDBACK SYSTEM
and if

$$\text{tr}(L_i^{(F,G) \Omega}) \neq 0, \ i = 1, 2, \ldots, m$$  \hspace{1cm} (3.6)

To answer the question why, it is understood that a method is said to be desirable if it can be used to implement certain control objectives. In other words, the design procedures of state variable feedback method have many advantages: they are comparatively simple; they require little algebraic manipulation of transfer-function matrices; they permit much of the tedious work to be done on a digital computer; they yield compensators of low order and permit rigorous stability analyses to be made. Besides, it has been shown that it is possible by state variable feedback to obtain decoupling without an increase in system order. Above all, the controllability property is preserved so that the eigenvalues or pole placements, of \((A + BF)\) can be arbitrarily assigned. Nonetheless, one point worth mentioning is that in practice, actual measurement of the state is almost never possible, and so an observer [3] might be required to estimate the system state. It is well known, however, that for the system (3.2), any linear input-output structure realized via state feedback control may also be realized via "observed" state feedback control. Thus, assuming that \(y\) and \(u\) may be measured and that the system is completely controllable and observable, the problem of decoupling with state feedback is the same as the problem of decoupling with an observer plus output feedback.
### 3.4 Necessary and Sufficient Conditions for Decoupling and Pole Placement

#### 3.4.1 Falb and Wolovich Method

Falb and Wolovich method [2] is good for decoupling only, while pole placement can, sometimes, be achieved by trial-and-error. In other words, pole placement and decoupling cannot be done simultaneously.

First of all, a necessary and sufficient condition for decoupling is given as follows:

Let $B^*$ be the $m \times m$ matrix given by

$$B^* = \begin{bmatrix}
c_1 A^{d_1} B \\
c_2 A^{d_2} B \\
\vdots \\
c_m A^{d_m} B
\end{bmatrix}$$

(3.7)

where $d_i = n - 1$ if $c_i A^j B = 0$ for all $j$

Then there is a pair of matrices $F$ and $G$ which decouple the system (3.2) if and only if

$$\det B^* \neq 0$$

(3.8)

Thus, once the necessary and sufficient condition is satisfied, $F$ and $G$ matrices which decouple the system can then be determined from

$$F = B^*^{-1} \left[ \delta \sum_{K=0}^{\delta} M_K C A^K - A^* \right]$$
\[ G = B^{*-1} \]

where \( \delta = \text{max. } d_i \),

\[ M_k = \text{diag } [M_k^1, M_k^2, \ldots, M_k^m] \]

\[ A^* = A^{**}A \]

\[ A^{**} = \begin{bmatrix} c_1 A^{d_1} \\ c_2 A^{d_2} \\ \vdots \\ c_m A^{d_m} \end{bmatrix} \quad (3.10) \]

3.4.2 Sinha and Rozsa Method

Sinha and Rozsa method is very useful since it does decoupling and pole placement simultaneously. Necessary and sufficient conditions are established for decoupling and pole placement in a linear multivariable system, which is represented in the controllable canonical form, by using linear state variable feedback and a constant non-singular transformation of the input. Since these conditions are based only on the matrix \( C \) of the triple \( A, B, C \) in the canonical representation, they lead to a very simple and efficient algorithm [1]. This is a straightforward method for being able to tell when a multivariable system may be decoupled as well as the poles of the closed-loop system may be placed arbitrarily. Such an approach leads to determine the "static" precompensator instead of "dynamic" precompensator and the feedback matrix for decoupling and desired pole placement.
Assume the system equations are already in the controllable canonical form. Then the necessary and sufficient conditions for decoupling and pole placement are described as follows,

(i) the matrix \([c_{nj} - \alpha_i](ij)\) is non-singular, where \(i, j = 1, 2, \ldots, m\) and \(\alpha_i\) is the Degree of simplicity of the \(i\)th row of \(C\),

(ii) for \(n_j < n_i\), \(c_K(ij) = 0\), for \(K = 1, 2, \ldots, n_j - \alpha_i\), and

(iii) the fundamental polynomials, \(L_{ij}(s)\), constructed from the elements of \(C\) for which \(n_j \geq n_i\) have \(L_{ii}(s)\) as a factor for each given \(i\) and all \(j\).

Thus, we can see that had all the conditions been satisfied,

(i) the matrix \([c_{nj} - \alpha_i](ij)\) should give the inverse of the matrix \(G\).

(ii) with \(L_{ij}(s)\), have \(\frac{L_{ii}(s)}{c_{nj} - \alpha_i}(ii)\) as a factor for each given \(i\) and all \(j\), then

\[
R_i(s) = \frac{L_{ii}(s)}{c_{nj} - \alpha_i}(ii) s^{\beta_i} \tag{3.11}
\]

and immediately, \(F\)-matrix can be found from

\[
F = - GP - B^T A \tag{3.12}
\]

where
and the coefficients $q_i(i)$ can be determined from the pre-assigned poles.

Therefore, as a comparison, if only to decide the feasibility of decoupling a system, Falb and Wolovich method is easier to apply. However, the main drawback is the poles cannot be pre-assigned arbitrarily and above all, in most cases, not all the poles can be specified.

The method of Sinha and Rozsa allows decoupling and pole placement to be done simultaneously if the three necessary and sufficient conditions are satisfied. It follows that the systems for which decoupling and pole placement can be done simultaneously are a subset of the systems which can be decoupled by state feedback.
CHAPTER 4

Application to the Communication Technology Satellite

4.1 Introduction

In order to decouple the system, one has to transform the A, B and C matrices into controllable canonical form first. Therefore, three computer programmes were written; one for controllable canonical form transformation and the other two for decoupling, using two different methods as described in Chapter 3.

4.2 The Controllable Canonical Form

The algorithm [6] for transforming any given set of dynamical equations to the controllable canonical form has been described by Luenberger. From the columns of the controllability matrix of the system, a set of n independent vectors are selected in the following sequence,

\[ U = [b_1 \ b_2 \ \ldots \ b_m, \ Ab_1 \ Ab_2 \ \ldots \ Ab_m, \ A^2 b_1 \ \ldots ] \] (4.1)

where \( b_i \), \( i = 1, 2, \ldots, m \), is the column vectors of \( B \), dropping any vector which is a linear combination of those occurring before it. The resulting matrix \( U \) is, therefore, non-singular, and can be inverted. From the rows of \( U^{-1} \), a selection is then made of the row corresponding to the last vector in each group and is designated as \( e^1, e^2 \) etc. With
n such row vectors, a square matrix $R$, is outlined as follows,

$$
R = \begin{bmatrix}
  e^1 \\
  e^1 A \\
  \vdots \\
  e^1 A^m \\
  e^2 \\
  \vdots \\
  e^2 A \\
  \vdots 
\end{bmatrix}
$$

(4.2)

Then, the transformation matrix $P = R^{-1}$ gives the controllable canonical form representation as follows:

$$
\bar{A} = P^{-1} AP \\
\bar{B} = P^{-1} B \\
\bar{C} = CP
$$

(4.3)

Usually, $\bar{B}$ has the form

$$
\bar{B} = \begin{bmatrix}
  0 & 0 & \cdots & \cdots \\
  0 & 0 & \cdots & \cdots \\
  1 & X & X & X \\
  \vdots & \vdots & \vdots & \vdots \\
  X & 1 & X & X \\
  \vdots & \vdots & \vdots & \vdots 
\end{bmatrix}
$$

(4.4)
and so a transformation of the input alone is then required to give $\hat{B} = BQ$ in the desired form, that is,

$$
\begin{bmatrix}
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\begin{bmatrix}
Q
\end{bmatrix}
\begin{bmatrix}
\hat{B}
\end{bmatrix}
$$

(4.5)

The transformation matrix, $Q$, is any given non-singular matrix.

4.3 Decoupling Algorithms

Programmes for the two different approaches were written. However, only the programme of Halb and Wolovich method is included, since it is proved to be the more useful one in this case. Thorough descriptions of the two algorithms can be found in the references, [1] and [2].

4.4 Application and Results

4.4.1 Data for CTS

In view of the linearized model derived in Chapter 2, and by evaluating the Jacobian matrices at the reference position, the matrices $A$, $B$, and $C$ of the linear, time-invariant, multivariable system, given by
\dot{x} = Ax + Bu
\dot{y} = Cx \quad (4.6)
can thus be obtained.

The following initial values obtained from Mamen are used:

\( \mu = 0.3986 \times 10^6 \text{ km}^2/\text{sec}^2 \)

\( \text{Re} = 0.637814 \times 10^4 \text{ km/sec} \)

\( v_{e1} = 0.0 \text{ km/sec} \)

\( v_{e2} = 3.074661027 \text{ km/sec} \)

\( \sigma = 3.074656 \text{ km/sec} \)

\( R_{f1} = 0.0 \text{ km/sec} \)

\( R_{f2} = 0.0 \text{ km/sec} \)

\( \psi = 0.8^\circ \)

\( \Omega = 240^\circ \)

\( u_1 = -\Omega \)

\( w_e = 0.72921159 \times 10^{-4} \text{ radians/sec} \)

\( Hx_B = 0.0 \text{ ft-lb-sec} \)

\( Hy_B = 15.0 \text{ ft-lb-sec} \)

\( Hz_B = 0.0 \text{ ft-lb-sec} \)

\( \gamma = 24.6^\circ \)

\( h = 39180 \text{ km} \)

\( A = 229.8^\circ \)

\( J_{11} = 822.4 \text{ slug ft}^2 \)
\begin{align*}
J_{12} &= 1.83 \text{ slug ft}^2 \\
J_{13} &= 0.0 \quad \text{slug ft}^2 \\
J_{21} &= 1.83 \quad \text{slug ft}^2 \\
J_{22} &= 65.3 \quad \text{slug ft}^2 \\
J_{23} &= 0.0 \quad \text{slug ft}^2 \\
J_{31} &= 0.0 \quad \text{slug ft}^2 \\
J_{32} &= 0.0 \quad \text{slug ft}^2 \\
J_{33} &= 846.5 \quad \text{slug ft}^2
\end{align*}

The two high-level thrusters (Ra, Ax) produce nominal five pounds force in radial and axial directions respectively. The sixteen low level thrusters (comprising Yaw, East, West, pitch, roll and offset thrusters) are rated at one quarter pound force level. The mass, \( m \), varies from 765.781 lb. at beginning of life (BOL) to 731.465 lb. at end of life (EOL).
### TABLE 1  Thruster Moment Arms and Force Components

<table>
<thead>
<tr>
<th>Thruster</th>
<th>Effective Moments Arms (in inches)</th>
<th>Force Components</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>About Roll</td>
<td>About Pitch</td>
</tr>
<tr>
<td>$Y_1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Y_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Y_3$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Y_4$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$E$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$W$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0</td>
<td>-37.50</td>
</tr>
<tr>
<td>$P_2$</td>
<td>0</td>
<td>+37.50</td>
</tr>
<tr>
<td>$P_3$</td>
<td>0</td>
<td>-36.00</td>
</tr>
<tr>
<td>$P_4$</td>
<td>0</td>
<td>+36.00</td>
</tr>
<tr>
<td>$R_1$</td>
<td>-24.868</td>
<td>0</td>
</tr>
<tr>
<td>$R_2$</td>
<td>+24.868</td>
<td>0</td>
</tr>
<tr>
<td>$O_1$</td>
<td>-24.505</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>tan9.9</td>
<td>0</td>
</tr>
<tr>
<td>$O_2$</td>
<td>+24.505</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>tan9.9</td>
<td>0</td>
</tr>
<tr>
<td>$O_3$</td>
<td>-23.190</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>tan9.9</td>
<td>0</td>
</tr>
<tr>
<td>$O_4$</td>
<td>+23.190</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>tan9.9</td>
<td>0</td>
</tr>
<tr>
<td>$R_x$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$A_x$</td>
<td>+3.234</td>
<td>+36.002</td>
</tr>
</tbody>
</table>
Finally, the reference orbit plane axes consist of $X_R$, $Y_R$, and $Z_R$, with $Z_R$ towards the centre, $Y_R$ along the southerly orbit normal and $X_R$ completing the right hand set. The body axes $X_B$, $Y_B$ and $Z_B$ lie along the spacecraft roll, pitch and yaw axes, and nominally are aligned with the reference orbit plane axes. The relation between these orbit-plane axes and the rotating polar set ($e_1$, $e_2$ and $e_3$) is, $e_1$ is antiparallel to $Z_R$, $e_2$ is parallel to $X_R$ and $e_3$ is antiparallel to $Y_R$.

4.4.2 Results

It is found that Falb and Wolovich method can decouple the system while Sinha and Rozsa method cannot. Thus, it shows that simultaneously decoupling and pole placement is impossible in this case.

Once the system is proved to be decoupled, then it is desired to stabilize the decoupled system. Therefore, by using the synthesis procedure, as described in Chapter 3, a number of different matrices $M_k$ are tried, until a satisfactory result is achieved.

The result is as follows,

With $M = \begin{bmatrix} -0.74 & 0 & 0 \\ 0 & -0.74 & 0 \\ 0 & 0 & -0.74 \end{bmatrix}$,

The eigenvalues are
<table>
<thead>
<tr>
<th>REAL PART</th>
<th>IMAGINARY PART</th>
<th>REAL PART</th>
<th>IMAGINARY PART</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.312</td>
<td>491.364</td>
<td>-2.077</td>
<td>513.432</td>
</tr>
<tr>
<td>7.312</td>
<td>-491.364</td>
<td>-2.077</td>
<td>-513.432</td>
</tr>
<tr>
<td>-2.096</td>
<td>490.251</td>
<td>-7.788</td>
<td>467.295</td>
</tr>
<tr>
<td>-2.096</td>
<td>-490.251</td>
<td>-7.788</td>
<td>-467.295</td>
</tr>
<tr>
<td>-5.220</td>
<td>0.108</td>
<td>-12.347</td>
<td>16.136</td>
</tr>
<tr>
<td>-5.220</td>
<td>-0.108</td>
<td>-12.347</td>
<td>-16.136</td>
</tr>
<tr>
<td>0.008</td>
<td>0</td>
<td>-4.654</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-0.193</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-0.368</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-0.474</td>
<td>0.149</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-0.474</td>
<td>-0.149</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-0.180</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-0.740</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The feedback matrix and feed forward matrix are respectively,

\[
F = \begin{bmatrix}
8.525 \times 10^6 & -1.731 \times 10^{12} & -2.082 \times 10^{12} & 1.974 \times 10^8 \\
5.801 \times 10^4 & -1.177 \times 10^{10} & -1.417 \times 10^{10} & 1.311 \times 10^6 \\
-6.194 \times 10^4 & 1.257 \times 10^{10} & 1.513 \times 10^{10} & -1.405 \times 10^6 
\end{bmatrix}
\]
\[-1.448 \times 10^3 \quad -2.1 \times 10 \quad -3.308 \times 10^2 \quad -9.708 \times 10^7 \]
\[-0.088 \quad -0.138 \quad -2.250 \quad -6.555 \times 10^4 \]
\[-5.046 \quad 0.147 \quad 2.401 \quad 6.971 \times 10^4 \]
\[-1.188 \times 10^7 \quad 3.379 \times 10^5 \quad -4.525 \times 10^2 \quad 1.822 \times 10^7 \]
\[-8.014 \times 10^4 \quad 2.298 \times 10^3 \quad -3.078 \quad 1.244 \times 10^5 \]
\[8.519 \times 10^4 \quad -2.454 \times 10^3 \quad 3.286 \quad -1.330 \times 10^5 \]
\[2.172 \times 10^7 \quad 3.206 \times 10^5 \]
\[1.485 \times 10^5 \quad 2.181 \times 10^3 \]
\[-1.589 \times 10^5 \quad -2.329 \times 10^3 \]

and

\[
\begin{bmatrix}
-6.94820.387 & 913869.888 & 5430483.572 \\
-12565.462 & 2061.232 & 10839.434 \\
-2138.362 & -15.194 & 2154.363
\end{bmatrix}
\]
THE TRANSFER FUNCTION MATRIX ARE AS FOLLOWS,
OPEN-LOOP SYSTEM, $D(S) = \frac{R(S)}{Q(S)}$, WHERE

\[
R(S) = \begin{bmatrix}
I & 2855365213E-02 & 8004235884E+00 & 7521127628E-02 & I \\
I & 2412970592E-02 & 6764179704E+00 & 6440192499E-02 & I \\
I & 2851254778E+00 & 7992629667E+00 & 7619348407E-02 & I \\
I & -138653214E+05 & 1676108898E+05 & 3825038557E+05 & I \\
I & -1173429085E+05 & 1416391044E+05 & 3232404219E+05 & I \\
I & -1386495223E+05 & 1673660840E+05 & 3829443399E+05 & I \\
I & 1433535620E+04 & 3856900570E+04 & 3823700239E+04 & I \\
I & 1212775171E+04 & 3259370239E+04 & 3243874449E+04 & I \\
I & 1431290810E+04 & 3851308018E+04 & 3833260836E+04 & I \\
I & -6789252581E+10 & 8094056612E+10 & 1839114281E+11 & I \\
I & -5737504052E+10 & 6839862512E+10 & 1554194185E+11 & I \\
I & -6779297897E+10 & 8082238726E+10 & 1836452571E+11 & I \\
I & -1780629091E+11 & 4260956450E+11 & 4813702040E+11 & I \\
I & -1504735881E+11 & 3600742471E+11 & 4067883979E+11 & I \\
I & -1778017693E+11 & 4254797155E+11 & 4800662659E+11 & I \\
I & -8304848282E+15 & 9771720308E+15 & 2211071320E+16 & I \\
I & -7014622672E+15 & 8257568156E+15 & 1868497657E+16 & I \\
I & -8283314169E+15 & 9757453505E+15 & 2207837049E+16 & I \\
I & -4242608396E+16 & -1062177274E+16 & 4133990245E+17 & I \\
I & -3560020969E+16 & -8978243336E+15 & 9582949164E+16 & I \\
I & -4206431137E+16 & -1060589108E+16 & 1132331579E+17 & I \\
I & -9261924912E+14 & -3170153732E+17 & -8209946131E+15 & I \\
I & -7827126722E+14 & -2679054043E+17 & -5938114519E+15 & I \\
I & -9248344471E+14 & -3165505401E+17 & -8197980850E+15 & I \\
I & -7725734112E+12 & -2650127435E+15 & -6859637592E+13 & I \\
I & -16528911775E+12 & -2239586299E+15 & -5969850570E+13 & I \\
I & -7714406294E+12 & -2646241699E+15 & -5849579683E+13 & I \\
I & -8038661798E+10 & -2232393994E+13 & -6433592108E+11 & I \\
I & -6792953211E+10 & -1878959320E+13 & -5436906130E+11 & I \\
I & -8269490490E+10 & -2220134016E+13 & -5424164014E+11 & I \\
I & -4132647420E+13 & -9535547920E+13 & -8085670632E+13 & I \\
I & -3492266281E+13 & -8058158667E+13 & -7441313432E+13 & I \\
I & -4126614594E+13 & -9521597718E+13 & -8792797372E+13 & I \\
I & -3394561824E+16 & -4692987515E+16 & -1932513132E+16 & I \\
I & -2868600924E+16 & -3968004556E+16 & -1633550247E+16 & I \\
I & -3339601022E+16 & -4688631533E+16 & -1929692520E+16 & I \\
I & -1066474563E+19 & -2358048863E+19 & -2079477124E+19 & I \\
I & -9029808946E+19 & -1992704610E+19 & -1757231197E+19 & I \\
I & -1066914571E+19 & -2354599149E+19 & -2076437112E+19 & I \\
I & 7952158725E+21 & -1102571804E+22 & -5068786107E+21 & I \\
I & 6720032332E+21 & -9317465201E+21 & -5284356445E+21 & I \\
I & 7940535569E+21 & -1100958566E+22 & -5061386517E+21 & I \\
\end{bmatrix}

Q(S) = \begin{bmatrix}
-1000E+015 & +0.8313E-02 & +4817E+06S & +2527E+10S \\
+0.6060E+12S & +1583E+13S & +5038E+11S & S \\
+0.0050E+09S & +0.0050E+09S & +0.0050E+09S & S \\
\end{bmatrix}
CLOSED-LOOP SYSTEM, $D(S) = R(S)/Q(S)$, WHERE

$$R(S) = \begin{align*}
&I \quad 9.999847412E+00 \quad 0. \\
&I \quad 0. \\
&I \quad 0. \\
&+ I \quad -6.927664614E+06 \\
&I \quad 0. \\
&+ I \quad -3.846674800E+08 \\
&I \quad 0. \\
&+ I \quad -7.551372194E+13 \\
&I \quad 0. \\
&+ I \quad -8.260684106E+15 \\
&I \quad 0. \\
&+ I \quad -1.879389579E+19 \\
&I \quad 0. \\
&+ I \quad 1.016260868E+20 \\
&I \quad 0. \\
&+ I \quad -1.651478653E+22 \\
&I \quad 0. \\
&+ I \quad -9.145238996E+24 \\
&I \quad 0. \\
&+ I \quad 3.131801522E+27 \\
&I \quad 0. \\
&+ I \quad 2.613977371E+30 \\
&I \quad 0. \\
&+ I \quad -5.75888405E+32 \\
&I \quad 0. \\
&+ I \quad -7.422358327E+35 \\
&I \quad 0. \\
&+ I \quad 9.861468180E+37 \\
&I \quad 0.
\end{align*}$$

$$Q(S) = \begin{align*}
&1.000E+01S^{14} + 5.151E+02S^{13} + 4.833E+06S^{12} + 2.035E+08S^{11} + 5.803E+11S^{10} + 1.834E+13S^9 + 3.465E+14S^8 + 1.886E+15S^7 + 3.429E+15S^6 + 2.990E+15S^5 + 1.413E+15S^4 + 3.674E+14S^3 + 4.888E+13S^2 + 2.583E+12S + 0.
\end{align*}$$
4.5 Conclusions

Sinha and Rozsa method is unsuccessful in this case because it requires both decoupling and pole placement to be done simultaneously. Since there are no zero elements in the C-matrix, in other words, for \( n_i < n_j \), \( c_{K}(ij) \neq 0 \), for \( K = 1, 2, \ldots, n_j - n_i \), the second condition is not satisfied and so the fundamental polynomials cannot be constructed. However, due to the fact that the first condition is satisfied it is certain that the system can be decoupled and the feed forward matrix \( G \) is thus obtained.

Falb and Wolovich method is more successful because it does not require that many constraints and above all, once the necessary and sufficient condition for decoupling is satisfied, that is, \( B^{-1} \) is nonsingular, then the matrices \( F \) and \( G \) can easily be found without much problem. Finally, it is also shown that the system after decoupled can be stabilized if \( M_K \) is suitably chosen. Thus, a necessary closed-loop poles configuration can be specified by varying \( M_K \).
CHAPTER 5

Conclusions

In this thesis, a unified state model of attitude and orbital trajectory dynamics is obtained, based on the work derived by Altman [4]. The reason for using such a model is due to its many analytical properties, discussed in Chapter 2. In particular, this type of model is essential for advanced missions, especially, when we speak of interdependence (or coupling) between the attitude and the orbital trajectory dynamics.

The interdependence (or dynamic coupling) to be considered here, is introduced by the hydrazine thrusters, which may produce moments about the centre of mass as well as translational accelerations. However, these thruster forces are referred to the body-fixed coordinates and so it is necessary to transform them back to the orbit-plane axes, in order to solve for orbital motion resulting from the thrusting. Similarly, thruster torque components may be rotated into the planetocentric inertial set for solution of the attitude dynamics. Therefore, by expanding those non-linear differential equations into 'Taylor' series, about the reference position, a linearized model is obtained. From that linearized state model, fourteen state parameters are realized along with the input forcing function of the
thruster torque components. Thus, the state matrix $A$ and the input matrix $B$ of the linear, time-invariant, multi-variable systems are obtained.

The observables to be measured are slant range $h$, azimuth $A$ and elevation $\gamma$. The relation between the other two attitude observables, pitch and roll angles, and the foregoing mentioned ones, is discussed in Chapter 2. Since these observations are obtained from the radar site coordinate system; a coordinate transformation back to the planetocentric inertial system is required. However, to obtain the linearized measurement model, a successive two-step procedure is needed. First, the system is expressed in a two-dimensional non-inertial set and then it is converted back to the inertial set by the coordinate transformation, described in Chapter 2. Finally, the $C$-matrix is obtained from the linearized measurement model after the successive transformations.

Since there exists a coupling between the two dynamics, it is desirable to decouple the system, so that, we may have one input controlling one and only one output. In other words, we may have a number of single variable sub-systems after decoupling, so that the well-known single variable synthesis method can be used to simulate the system. In this thesis, the linear state variable feedback method is used because it has a great deal of advantages over other methods, as described in Chapter 3.
To apply the decoupling theory, two algorithms are used, as described in Chapter 3. It is found out that Falb and Wolovich method is more successful because it requires less constraints and the matrices F and G, to decouple the system, are obtained quite easily once the necessary and sufficient condition for decoupling is satisfied. In fact, using Sinha and Rozsa method, feed forward matrix G, can also be obtained but due to the fact that the second necessary condition is not satisfied, the feedback matrix F is undetermined.

Both methods can be used to determine the feasibility of decoupling a system, as described in Chapter 4. However, unlike Falb and Wolovich method, Sinha and Rozsa method requires both decoupling and pole placement to be done simultaneously. Therefore, it is obvious that more constraints are imposed upon the algorithm and so in this case, only G-matrix is obtained.

It should be pointed out here, although the system is proved to be decoupled, it is necessary to specify the closed-loop poles configurations for optimal control purposes. Further work in this area is required to obtain the said closed-loop poles and also, the question of how to control the attitude and orbit after decoupling, remains to be studied.
APPENDIX A

Definition of the Euler Parameters

The rotation of a triad set can be generated by a scalar rotation about a directed line from the origin of the inertial space. This rotation is defined by the unitary quaternion

\[ q = e_4 + (g_1 e_1 + g_2 e_2 + g_3 e_3) \]  

(A.1)

where the set of four Euler parameters

\[
\begin{bmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  e_4
\end{bmatrix} = \begin{bmatrix}
  \cos \alpha \sin \frac{u}{2} \\
  \cos \beta \sin \frac{u}{2} \\
  \cos \gamma \sin \frac{u}{2} \\
  \cos \frac{u}{2}
\end{bmatrix}
\]  

(A.2)

consists of real scalars such that

\[ e_1^2 + e_2^2 + e_3^2 + e_4^2 = 1 \]  

(A.3)

The spherical angles \((\alpha, \beta, \gamma)\) in Eq. (A.2) are shown in Figure A.1. As an alternative, the Euler parameters may be defined in terms of the Euler angles \((\Omega, \iota, \upsilon)\) shown in Figure A.2, in accordance with
Fig. A.1
GEOMETRY FOR THE EULER PARAMETERS

Fig. A.2
EULER ANGLES
The variation of the Euler parameters with angular velocity of rotation (w) of the triad set is defined as follows:

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
0 & w_z & -w_y & w_x \\
-w_z & 0 & w_x & w_y \\
w_y & -w_x & 0 & w_z \\
-w_x & -w_y & -w_z & 0
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4
\end{bmatrix}
\]  \hspace{1cm} (A.5)

In matrix notation, a coordinate triad \([X]_{\text{Body}}\) is rotated from an inertial reference triad \([X]_{\text{Inertial}}\) by a transformation \([E]\) such that

\[
[X]_{\text{Body}} = [E] [X]_{\text{Inertial}}
\]  \hspace{1cm} (A.6)

where

\[
[E] = \begin{bmatrix}
e_{11} & e_{12} & e_{13} \\
e_{21} & e_{22} & e_{23} \\
e_{31} & e_{32} & e_{33}
\end{bmatrix}
\]  \hspace{1cm} (A.7)
and
\[
\begin{align*}
\varepsilon_{11} &= 1 - 2(e_2^2 + e_3^2) \\
\varepsilon_{12} &= 2(e_1 e_2 + e_3 e_4) \\
\varepsilon_{13} &= 2(e_1 e_3 - e_2 e_4) \\
\varepsilon_{21} &= 2(e_1 e_2 - e_3 e_4) \\
\varepsilon_{22} &= 1 - 2(e_1^2 + e_3^2) \\
\varepsilon_{23} &= 2(e_2 e_3 + e_1 e_4) \\
\varepsilon_{31} &= 2(e_1 e_3 + e_2 e_4) \\
\varepsilon_{32} &= 2(e_2 e_3 - e_1 e_4) \\
\varepsilon_{33} &= 1 - 2(e_1^2 + e_2^2)
\end{align*}
\]

(A.8)

Note that the transformation matrix in Equation (A.5) is skew-symmetric.
APPENDIX B

RB IS THE NUMBER OF ROWS IN MATRIX B
CB IS THE NUMBER OF COLUMNS IN MATRIX B
N IS THE ARRAY OF MATRIX A
NI IS THE NUMBER OF INPUTS
NO IS THE NUMBER OF OUTPUTS
D IS THE LIMIT OF DELTA.
S=D
A IS THE STATE MATRIX OF DIMENSION(N,N)
B IS THE INPUT MATRIX OF DIMENSION(RB,CB)
C IS THE OUTPUT MATRIX OF DIMENSION(NO,N)
G1 IS THE FEEDFORWARD MATRIX OF DIMENSION(NO,NO)
H IS THE FEEDBACK MATRIX OF DIMENSION(NO,N)
W IS THE POLEPLACEMENT MATRIX OF DIMENSION(NO,NO)

INPUT DATA - A, B, C, RB, CB, N, NI, NO, D, W, S

OUTPUT PRINT-OUT A, B, C, (IN CONTROLLABLE CANONICAL FORM), G1, H, AND
TRANSFER FUNCTION MATRIX WHERE THE REAL AND IMAGINARY PART OF THE
POLES ARE STORED IN E(N) AND E1(N) RESPECTIVELY. COEFFICIENTS OF THE
NUMERATOR ARE STORED IN T(NO, NO, N) AND COEFFICIENTS OF THE CHARACTERISTIC
POLYNOMIAL ARE STORED IN G(N1)

CY IS THE NUMBER OF COLUMNS IN Y
(NMAX-1) IS THE MAXIMUM POWER OF A THAT CAN GO

NMAX=N-NI+1
CY=NI*NMAX
NI=N+1

SUBROUTINE RD(RB, CB, N, NI, NO, D, CY, NMAX, PB, A, B, C, U, P, PINV, R, T, NI,
*Y, IC, G, Q, B1, E1, E, F1, L1, L2, Z, BCAP, LS, NS, H, G1, R, S)
INTEGER CY, RB, CB, PB(NMAX)
DIMENSION A(N,N), B(RB,CB), C(NO,N), U(N,N), P(N,N), PINV(N,N),
*Y(N,CY), IC(CY), G(CY), Q(N), B1(RB), E1(N), E1(N), F1(N1),
*L1(N), L2(N), Z(N,CY), BCAP(RB,CB), LB(NI), NB(NI),
*G1(NO, NO), H(NO, NO), T(NO, NO, N)

JY=1
EX=(2.0)**(-48.0)
HPW=NMAX-1

TO OBTAIN THE EIGENVALUES OF THE OPEN-LOOP SYSTEM

DO 112 I=1, N
DO 112 I=1, N
PINV(I,J)=A(I,J)
CONTINUE
112 CONTINUE
CALL REIGEN(N, N, PINV, E, E1, Q, L1, S, EX)
PRINT 812

812 FORMAT(14, 12X, "OPEN-LOOP SYSTEM", 12X, "REAL PART OF THE POLE", 40X,
*IMAGINARY PART OF THE POLE")
PRINT 813, (E(I), E(I), I(I), I=1, N)
813 FORMAT(14, 12X, E20.10, 44X, E20.10, 7X, "/")
TO OBTAIN THE TRANSFER FUNCTION OF THE OPEN-LOOP SYSTEM

CALL TRAFUN(A, B, C, E, E1, T, G, B1, F1, U, R, N, NO, N1, Y, Z, JY)

CALL CALHY(A, B, Y, N, RB, CB, CY, MPH, B1, F1)

DO 400 J = 1, CY
   DO 401 I = 1, N
      Z(I, J) = Y(I, J)
   CONTINUE
   400 CONTINUE

TRANSFORM Y INTO HERMITE NORMAL FORM

CALL RANK(Y, G, Q, IC, CY, N, RB, D, NY, NRUN)
IF(NY .LT. RB) GO TO 34

CALCULATE MATRIX U

DO 26 KA = 1, N
   DO 27 KB = 1, N
      U(KB, KA) = Z(KB, IC(KA))
   CONTINUE
   26 CONTINUE

INVERSION OF MATRIX U

CALL MINV(U, N, X, L1, L2).

DETERMINE THE LAST ROWS OF B*S AND NUMBER OF TIMES OF REPETITION

NUMBER = 0
DO 9 K = 1, NI
   KSUM = 0
   DO 2 I = 1, NMAX
      PB(I) = (I-1) * NI + K
   DO 29 J = 1, N
      IF(IC(J) .EQ. PB(I)) GO TO 30
   GO TO 29
   30 KSUM = KSUM + 1
   NUMBER = NUMBER + 1
   LB(K) = J
   NB(K) = KSUM
   IF(NUMBER .EQ. N) GO TO 9
   GO TO 2
   CONTINUE
   CONTINUE
   CONTINUE

CALCULATE MATRIX PINV

CALL CALMP(U, PINV, P, N, A, NI, LB, NB, E1, E)

CALCULATE INVERSION OF MATRIX PINV

CALL MINV(P, N, X, L1, L2)
CALCULATE MATRIX A IN CONTROLLABLE CANONICAL FORM
CALL MPRD(PINV, A, R, N, N, 0, 0, N)
CALL MPRD(R, P, A, N, N, 0, 0, N)

CALCULATE MATRIX B IN CONTROLLABLE CANONICAL FORM
CALL MPRD(PINV, B, BCAP, N, N, 0, 0, NI)
DO 24 J = 1, NI
DO 25 I = 1, RB
B(I, J) = 0.0
CONTINUE
24 CONTINUE
CONTINUE
DO 40 J = 1, NI
DO 41 I = 1, RB
IF (ABS(BCAP(I, J) - 1.0) .GT. 0) GO TO 41
B(I, J) = 1.0
CONTINUE
GO TO 40
CONTINUE
CONTINUE
CONTINUE
25 CONTINUE
CONTINUE

CALCULATE MATRIX C IN CONTROLLABLE CANONICAL FORM
CALL MPRD(C, H, NO, N, 0, 0, N)
DO 92 J = 1, N
DO 93 I = 1, NO
C(I, J) = H(I, J)
CONTINUE
CONTINUE

TO OBTAIN MATRICES F AND G
CALL FINDW(A, B, C, U, B1, E1, Q, G1, N, NO, D, IDET, L1, L2, IC, H, P, R, W, MAXDEL)
IF (IDET .LT. -6) GO TO 33

TO OBTAIN MATRIX A OF THE CLOSED-LOOP SYSTEM
CALL MPRD(B, H, R, N, NO, 0, 0, N)
CALL MAOD(A, R, P, N, N, 0, 0)

TO OBTAIN MATRIX B OF THE CLOSED-LOOP SYSTEM
CALL MPRD(B, G1, BCAP, N, NO, 0, 0, NO)

TO DETERMINE WHETHER A FEEDBACK MATRIX F IS REALLY OBTAINED
CALL FINDQ(P, C, B, IC, Y, N, NO, Q, E, E1, LB, NY, NRUN, )
IF (NY .GT. 1) GO TO 2000

TO OBTAIN THE EIGENVALUES OF THE CLOSED-LOOP SYSTEM
DO 815 I = 1, N
DO 816 J = 1, N
PINV(I,J) = P(I,J)
CONTINUE
CONTINUE
CONTINUE
JY = 0
CALL REIGEN(N,N,P,E,E1,Q,L1,D,EX)
PRINT 810
FORMAT(*I12,11X,*CLOSED-LOOP SYSTEM,11X,*REAL PART OF THE POLE*,
*40X,*IMAGINARY PART OF THE POLE*)
PRINT 811,(E(I),E1(I),L1(I),I=1,N)
FORMAT(*C8,40X,E20.10,E20.10,E20.10,7X,I2//)
CONTINUE
TO OBTAIN THE TRANSFER FUNCTION OF THE CLOSED-LOOP SYSTEM
CALL TRAFUN(PINV,BCAP,C,E,T,G,B1,F1,U,R,N,NO,N1,Y,Z,JY)
GO TO 2000
PRINT 1000
1000 PRINT *,18X,*SORRY, THE SYSTEM CANNOT BE DECOUPLED BY THIS ALGORITHM, SINCE IT DOES NOT SATISFY
* THE NECESSARY CONDITION *
GO TO 2000
34 PRINT 6000, NY
4000 PRINT *,54X,*THE RANK OF THE MAT
2000 RETURN
END
SUBROUTINE PRINT(T,G,N,NO,N1,JY) DIMENSION T(NO,NO,N);G(N1)
IIFI(JY,EQ.0)GO TO 1
PRINT 22
FORMAT(*1X,46X,*THE TRANSFER FUNCTION MATRIX ARE AS FOLLOWS,*)
PRINT 2
FORMAT(*0A,50X,*OPEN-LOOP SYSTEM, D(S)=R(S)/Q(S), WHERE*)
PRINT 3, ((1,J), J=1,NO)
3 FORMAT(*0A,39X,*3E18.10,2X,*12X,*13#)
PRINT 4, ((2,J), J=1,NO)
4 FORMAT(*0A,39X,*3E18.10,2X,*12X,*13#)
PRINT 5, ((3,J), J=1,NO)
5 FORMAT(*0A,39X,*3E18.10,2X,*12X,*13#)
DO 6 I=2, N
L=N-I
PRINT 7, ((1,J), J=1,NO),L
7 FORMAT(*0A,39X,*3E18.10,2X,*12X,*13#)
PRINT 8, ((2,J), J=1,NO)
8 FORMAT(*0A,39X,*3E18.10,2X,*12X,*13#)
PRINT 9, ((3,J), J=1,NO)
9 FORMAT(*0A,39X,*3E18.10,2X,*12X,*13#)
CONTINUE
GO TO 10
10 DO 12 K=1, N
12 IF(J=I)GO TO 14
14 T(I,J,K) = 0.0
CONTINUE
CONTINUE
CONTINUE
CONTINUE
CONTINUE
CONTINUE
CONTINUE
CONTINUE
CONTINUE
CONTINUE
CONTINUE
FORMAT(*1X,50X,*CLOSED-LOOP SYSTEM, D(S)=R(S)/Q(S), WHERE*)
GO TO 15
15 PRINT 16
16 FORMAT(*1X,50X,*14#,12X,*13#,12X,*12#,12X,*11#,12X,*10#)
PRINT 17, (G(I), I=1,5)
17 FORMAT(*1X,32X,*Q(S)=*,2X,E10.4,*,S#,2X,*,E10.4,*,S#,2X,*,
*E10.4,*,S#,2X,*,E10.4,*,S#,2X,*,E10.4,*,S#)
PRINT 18
18 FORMAT(*1X,51X,*9#,13X,*8#,13X,*7#,13X,*6#,13X,*5#)
PRINT 19, (G(I), I=6,10)
19 FORMAT(*1X,38X,*E10.4,*,S#,2X,*,E10.4,*,S#,2X,*,E10.4,*,S#,
*E10.4,*,S#,2X,*,E10.4,*,S#)
PRINT 20
20 FORMAT(*1X,51X,*9#,13X,*8#,13X,*7#,13X,*6#,13X,*5#)
PRINT 21, (G(I), I=11,15)
21 FORMAT(*1X,38X,*E10.4,*,S#,2X,*,E10.4,*,S#,2X,*,E10.4,*,S#,
*E10.4,*,S#,2X,*,E10.4)
RETURN
END
SUBROUTINE TRAFUN(A,B,C,E1,E1,T,G,B1,F1,U,R,N,NO,N1,Y,Z,JY)
DIMENSION A(N,N),B(N,NO),C(NO,N),E1(N),E1(N),T(NO,NO,N),G(N1),B1(N)
,F1(N1),U(NO,NO),R(NO,N),Y(N,N),Z(N,N)
C
TO OBTAIN THE CHARACTERISTIC POLYNOMIAL
C
D=0.01
DO 1 I=1,N
IF(ABS(E(I)) .GT. 0) GO TO 1
E(I)=0.0
C
1 CONTINUE
IF(ABS(E1(I)) .GT. D) GO TO 4
G(1)=1.0
G(2)=-E(I)
IDIHX=2
J=2
GO TO 3

2 G(1)=1.0
G(2)=(-2.0)*E(I)
G(3)=E(I)*E(I)+E1(I)*E1(I)
IDIHX=3
J=3

3 IDIHY=1
DO 5 I=J,N
IF(ABS(E1(I)) .LT. D) GO TO 6
IDIHY=IDIHY+1
IF(IDIHY .LT. 3) GO TO 5
B1(I)=1.0
B1(2)=(-2.0)*E(I)
B1(3)=E(I)*E(I)+E1(I)*E1(I)
IDIHZ=IDIHX+IDIHY-1
GO TO 7

6 IDIHY=IDIHY+1
B1(1)=1.0
B1(2)=E(I)
IDIHZ=IDIHX+IDIHY-1
CALL PMYP(F1,IDIHZ,G,IDIHX,B1,IDIHY)
DO 8 K=1,IDIHZ
G(K)=F1(K)
8 CONTINUE
IDIHX=IDIHZ
IDIHY=1
CONTINUE
C
TO OBTAIN THE NUMERATOR OF THE TRANSFER FUNCTION MATRIX
C
CALL CNUM(A,B,C,G,U,T,R,Y,Z,N,NO,N1,JY)
RETURN
END

SUBROUTINE PANK(Y,G,Q,IC,CY,N,RB,D,NY,NRUN)
INTEGER CY,RB
DIMENSION Y(N,CY),G(CY),Q(N),IC(CY)
NRUN=0
NY=0
DO 3 IA=1,CY
IB=1
IF(NY.EQ.RB) GO TO 23
IF(NRUN.EQ.NY) GO TO 10
NRUN=NY
NRUN=NRUN+1
PIVOT=ABS(Y(IB,IA))
DO 4 JB=2,N
IF(PIVOT.GE.(ABS(Y(JB,IA)))) GO TO 4
PIVOT=ABS(Y(JB,IA))
IB=JB
4 CONTINUE
IF(PIVOT.LT.D) GO TO 3
IC(NRUN)=IA
CALL YAU(Y,CY,IA,IB,N,G,Q,D)
NY=NY+1
3 CONTINUE
RETURN
END
SUBROUTINE CNUM(A,B,C,G,U,T,P,Y,Z,N,NO,N1,JY)
DIMENSION A(N,N),B(N,NO),C(NO,N),G(N1),U(NO,NO),T(NO,NO,N),R(NO,N),
Y(N,N),Z(N,N)
DO 9 I=1,N
 IF(I-Z)1,2,3
1 CALL MPRO(C,B,U,NO,N,0,0,NO)
16 DO 10 J=1,NO
10 DO 11 K=1,NO
11 CALL MPRO(C,U,J,K)=U(J,K)
11 CONTINUE
10 CONTINUE
GO TO 9
2 DO 13 J=1,N
13 DO 14 K=1,N
14 Y(J,K)=A(J,K)
14 CONTINUE
13 CONTINUE
DO 15 J=1,N
15 Y(J,J)=Y(J,J)+G(I)
15 CONTINUE
15 CONTINUE
GO TO 16
DO 17 J=1,N
17 Z(J,J)=Z(J,J)+G(I)
17 CONTINUE
17 CONTINUE
DO 18 J=1,N
18 DO 19 K=1,N
19 Y(J,K)=Z(J,K)
19 CONTINUE
19 CONTINUE
18 CONTINUE
17 CONTINUE
TO PRINT OUT THE TRANSFER FUNCTION MATRIX
CALL PRINT(T,G,N,NO,N1,JY)
RETURN
END

SUBROUTINE FINDQ(P,C,B,G,Y,N,NO,Q,E,EB,LB,NR1:NUN,D)
DIMENSION P(N,N),C(NO,N),B(NO,N),G(NO),Y(N,NO),Q(N),E(NO),
E1(N1),LB(NO)
DO 708 L=1,NO
708 CONTINUE
DO 700 I=1,N
700 CONTINUE
DO 701 J=1,N
701 CONTINUE
DO 709 M=1,N
709 CONTINUE
Q(M)=C(LM)
709 CONTINUE
J=N-1C(L)
709 CONTINUE
K=IC(L)
DO 702 I=1,J
702 CONTINUE
K=K+1
710 IF(K-Z)1,711,710,711
711 CALL MPRO(Q,P,E1,1,N,0,0,NO)
712 DO 713 JJ=1,NO
713 CONTINUE
711 CONTINUE
GO TO 702
711 CALL MPRO(Q,P,E1,1,N,0,0,NO)
712 DO 713 JJ=1,NO
713 CONTINUE
711 CONTINUE
GO TO 702
712 CONTINUE
CALL RANK(Y,G,Q,LB,NO,N,NO,D,NY,NR1:NUN)
708 CONTINUE
714 RETURN
END
SUBROUTINE FANDW(A,B,C,U,B1,E1,Q,G1,N,NO,D,IDET,L1,L2,IC,H,P,R, *HAXDEL)
DIMENSION A(N,N),B(N,NO),C(NO,N),U(NO,NO),B1(N),E1(NO),Q(N), *G1(NO,NO),L1(NO),L2(NO),IC(NO),H(NO,N),P(NO,N),R(NO,N),W(NO,NO)
L0=0
JK=0
DO 100 I=1,NO
DO 101 J=1,N
B1(J)=C(I,J)
101 CONTINUE
CALL MPRO(B1,B,E1,1,N,0,0,NO)
DO 102 K=1,NO
IF(ABS(E1(K)).LT.0)GO TO 102
DO 103 J=1,NO
102 CONTINUE
DO 107 JJ=1,LD
JK=JJ
CALL MPRO(B1,A,Q,1,N,0,0,N)
DO 105 H=1,N
B1(H)=Q(M)
105 CONTINUE
CALL MPRO(Q,B,E1,1,N,0,0,NO)
DO 106 II=1,NO
IF(ABS(E1(II)).LT.0)GO TO 106
DO 103 J=1,NO
106 CONTINUE
IC(I)=JK
DO 104 L=1,NO
U(I,L)=E1(I)
104 CONTINUE
CALL MD(U,NO,NO,NO,DIDET,G1)
IF(DIDET.LT.(-1))GO TO 106
CALL HINV(U,NO,X,L1,L2)
DO 120 I=1,NO
DO 121 J=1,NO
G1(I,J)=U(I,J)
121 CONTINUE
120 CONTINUE
C TO DETERMINE MAXIMUM D#I
MAXDEL=IC(I)
DO 206 I=2,NO
IF(MAXDEL.GE.IC(I))GO TO 206
MAXDEL=IC(I)
206 CONTINUE
K=MAXDEL+1
C TO OBTAIN H#K*C#A#K
DO 209 I=1,K
CALL MPRO(H,C,R,NO,NO,0,0,N)
J=I-1
IF(J.EQ.0)GO TO 210
DO 214 L=1,J
CALL MPRO(R,A,P,NO,N,0,0,N)
DO 215 H=1,NO
DO 216 HM=1,N
R(N,HM)=P(H,HM)
216 CONTINUE
215 CONTINUE
214 CONTINUE
CALL MADD(H,P,R,NO,N,0,0)
210 DO 211 II=1,NO
DO 212 JJ=1,N
H(II,JJ)=R(II,JJ)
212 CONTINUE
211 CONTINUE
209 CONTINUE
CALL CALMF(A,C,U,B1,N,NO,IC,H,P,R,Q)
108 RETURN
SUBROUTINE YAU(Y,CY,IA,IB,N,G,Q,D)
TO LOCATE THE INDEPENDENT COLUMN VECTORS IN MATRIX Y

INTEGER CY
DIMENSION Y(N,CY),G(CY),Q(N)
DO 13 JA=1,CY
  G(JA)=Y(IB,JA)
CONTINUE
Y(IB,IA)=Y(IB,IA)-1.0
DO 14 JB=1,N
  Q(JB)=Y(JB,IA)
CONTINUE
Y(IB,IA)=Y(IB,IA)+1.0
YSUM=Y(IB,IA)
DO 11 JC=1,CY
  ESP=ABS(G(JC)/YSUM)
  IF(ESP.LT.0.0)GO TO 11
  DO 12 JD=1,N
    Y(JD,JC)=Y(JD,JC)-((1.0/YSUM)*Q(JD)*G(JC))
CONTINUE
11 CONTINUE
DO 2 I=1,CY
  Y(IB,I)=0.0
CONTINUE
RETURN
END

SUBROUTINE CALMV(A,B,Y,N,RB,CB,CY,MPW,F1)
NUMBER OF PARTITIONS OF MATRIX B DEPENDS ON NUMBER OF INPUTS

INTEGER CY,RB,CB
DIMENSION A(N,N),B(RB,CB),Y(N,CY),B1(RB),F1(RB)
DO 101 J=1,CY
  DO 102 I=1,RB
    Y(I,J)=B(I,J)
    CONTINUE
101 CONTINUE
CALCULATE A*B1,(A**2)*B1,...,A*B2,(A**2)*B2,...,A*B3,(A**2)*B3,
...,A*B4,(A**2)*B4,...,AND SO ON

DO 100 K=1,CY
  DO 105 KK=1,PB
    B1(KK)=B(KK,K)
    CONTINUE
105 CONTINUE
DO 103 J=1,MPW
  CALL HPRD(A,B1,F1,N,N,0,0,1)
  DO 104 I=1,RB
    IF(CB+K)
5 I=J*CB+K
    Y(I,I)=F1(I)
    B1(I)=F1(I)
    CONTINUE
103 CONTINUE
100 CONTINUE
RETURN
END
SUBROUTINE CALMF(A,C,U,B1,N,N0,IC,H,P,P,Q)
DIMENSION A(N,N),C(N0,N),U(N0,N0),B1(N),IC(N0),H(N0,N),P(N0,N)
       R(N0,N),Q(N)
DO 200 I=1,N0
DO 203 J=1,N
B1(J)=C(I,J)
203 CONTINUE
IF(II(J).EQ.0) GO TO 201
K=IC(J)
DO 204 L=1,K
CALL MRPO(B1,A,Q,1,N0,0,0)
DO 205 M=1,N
B1(M)=Q(M)
205 CONTINUE
204 CONTINUE
201 DO 202 J=1,N
R(I,J)=B1(J)
202 CONTINUE
200 CONTINUE
CALL MRPO(P,A,P,N0,N0,0,0)
CALL MSUB(H,P,R,N0,N0,0)
CALL MRPO(U,P,H,N0,N0,0,0)
RETURN
END

SUBROUTINE CALMP(U,PINV,P,N,A,NI,LB,NB,E1,E)
DIMENSION U(N,N),PINV(N,N),P(N,N),A(N,N),LB(NI),NB(NI),E1(N),E(N)
L=0
H=0
DO 215 K=1,NI
DO 200 J=1,N
E1(J)=U(LB(K),J)
200 CONTINUE
H=H+1
L=NB(K)*L
IF(L.EQ.1) GO TO 212
DO 201 I=M,L
CALL MRPO(E1,A,E,1,N0,0,0)
DO 202 JJ=1,N
PINV(J,J)=E1(JJ)
E1(JJ)=E(JJ)
202 CONTINUE
201 CONTINUE
212 H=L
215 CONTINUE
DO 209 J=1,N
DO 210 I=1,N
P(I,J)=PINV(I,J)
210 CONTINUE
209 CONTINUE
RETURN
END
REFERENCES


