CLASSIFICATION OF CERTAIN 6-MANIFOLDS

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by

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ABSTRACT

Oriented, simply-connected, differentiable 6-manifolds, with vanishing second Stiefel-Whitney class and integral homology groups $H_2(M) = H_3(M) = Z/n$, where $n \not\equiv 0 \bmod 4$, are shown to be classified up to orientation preserving diffeomorphism by the following invariants: the cohomology ring of M with coefficients in the ring Z/n, the first Pontrjagin class of the tangent bundle of M, and the Pontrjagin cubing cohomology operation.

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To Stanley

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INTRODUCTION

Until 1960 the problem of classifying manifolds up to diffeomorphism was solved only in dimensions less than three. In that year, Smale [13], working from ideas of Morse, developed his handlebody theory which enabled him to settle the generalized Poincaré conjecture. Smale proved that every closed n-manifold of dimension greater than four, which has the homotopy type of the n-sphere, is homeomorphic to the n-sphere.

Using handlebody theory, Smale [14] also classified simply-connected 5-manifolds with vanishing second Stiefel-Whitney class, the latter restriction being removed in a paper by Barden [1]. In the same paper, Smale showed that any 2-connected 6-manifold is either the 6-sphere S^6 or a connected sum of copies of $S^3 \times S^3$.

Wall [18] classified simply-connected 6-manifolds with torsion free homology and vanishing second Stiefel-Whitney class. Again the restriction on the vanishing of $w_2(M)$ was removed in a later paper by Jupp [11].

In this thesis, we begin the study of simply-connected 6-manifolds with torsion in their homology. In [18], Wall showed that any simply-connected 6-manifold M can be split as a connected sum $M = M_o \# M_1$ where M_o is a connected sum of copies of $S^3 \times S^3$ and $H_3(M_1)$ is finite ($H_3(M_1)$ is isomorphic to the torsion subgroup of $H_2(M_1)$). The group $H_2(M_1)$ can be any finitely generated abelian group. Our main theorem (5.2) classifies simply-connected 6-manifolds M with $w_2(M) = 0$ and $H_2(M) = H_3(M)$, a cyclic torsion group Z/n where $n \not\equiv 0 \bmod 4$. Thus, our results are a step towards the solution of the classification problem for simply-connected 6-manifolds.

In chapter one, we review some topics in differential and algebraic topology including Smale's handlebody theory and the closely related concept of surgery. Chapter two is a résumé of Wall's classification [18] of simply-connected 6-manifolds with torsion free homology. Among the manifolds classified by Wall are those with homology the same as $S^2 \times S^4$. In this thesis, these are referred to as Wall manifolds.

In chapter three, we define a torsion manifold to be a simply-connected 6-manifold M with $w_2(M) = 0$ and $H_2(M) = H_3(M) = Z/n$ for some integer $n > 1$. We discuss some invariants of torsion manifolds and show that every torsion manifold can be obtained by surgery on a Wall manifold. Thus, for a fixed integer $n > 1$, the class of Wall manifolds is partitioned into equivalence classes where two Wall manifolds are equivalent if they reduce via surgery to diffeomorphic torsion manifolds.

Chapter four contains the computation of the invariants for a torsion manifold derived from a Wall manifold. We also discuss a geometric construction which generates new Wall manifolds from a given one.

In chapter five, we prove that the Wall manifolds generated by our construction are all equivalent (i.e. they reduce via surgery to diffeomorphic torsion manifolds). From this, we derive our main theorem (see §5.2) by showing that if two Wall manifolds reduce via surgery to torsion manifolds with the same invariants, then these Wall manifolds are related by the construction of chapter four. It then follows that the invariants of chapter three suffice to determine the diffeomorphism class of a torsion manifold.

CHAPTER 1

§1.1 DIFFERENTIAL TOPOLOGY

To establish notation, we begin by describing some fundamental notions in differential and algebraic topology.

A *differentiable n-manifold* (of class C^r) is a locally Euclidean, paracompact space together with a covering by open sets U_i, called charts, each homeomorphic to R^n by a map $h_i : U_i \longrightarrow R^n$, such that the overlap functions $h_j h_i^{-1} \;:\; h_i(U_i \cap U_j) \longrightarrow h_j(U_i \cap U_j)$ are r-times continuously differentiable. We assume that all manifolds are of class C^∞.

A *manifold with boundary* is defined similarly except that the homeomorphisms h_i are allowed to be onto the half space $R^n_+ = \{ (x_1,\ldots,x_n) \mid x_n \geq 0\}$. The *boundary* of a manifold M^n, denoted ∂M^n, is an n-1 dimensional manifold without boundary.

The closed unit disc in R^n is denoted by D^n and its boundary, $\partial D^n = S^{n-1}$ is the n-1 sphere.

A differentiable map f between manifolds M and N is called an *immersion* at $x \in M$ if the induced linear map $(Df)_x : T_x M \longrightarrow T_{f(x)} N$ between tangent spaces is one to one. The map f is called an *embedding* if it is an immersion at each point of its domain and if it maps its domain homeomorphically onto its image.

Two embeddings $f_o, f_1 : M \longrightarrow N$ are called *isotopic* if there is a differentiable map $F : M \times I \longrightarrow N$ such that for each $t \in I = [0,1]$ the map $F_t : M \longrightarrow N$ given by $F_t(x) = F(x,t)$ is an embedding, with $F_o = f_o$ and $F_1 = f_1$. The map F is called an *isotopy* between f_o and f_1.

If $f_o = id_M$ and f_1 is a diffeomorphism of the manifold M, then f_1 is called *diffeotopic to the identity* if id_M and f_1 are isotopic as embeddings where the intermediate maps F_t are also diffeomorphisms. The map F in this case is called a *diffeotopy* or *ambient isotopy*.

The connection between isotopy and diffeotopy is contained in the following.

THEOREM 1.1 *(Isotopy extension theorem) let* V *be a compact submanifold of* M *and* $F : V \times I \longrightarrow M$ *an isotopy of the inclusion of* V *in* M. *If either* $F(V \times I) \subset \partial M$ or $F(V \times I) \subset M - \partial M$, *then* F *extends to a diffeotopy of* M.

A proof can be found in Hirsch [10]. $\pi_i(M)$ and $H_i(M)$ denote respectively the i-dimensional homotopy and integral homology groups of M. A space M is called q-connected if its homotopy groups are trivial in dimensions less than or equal to q. The following theorem from Haefliger [5] establishes the relationship between isotopy, homotopy and connectivity.

THEOREM 1.2 *Let* V^n *and* M^m *be two differentiable manifolds which are respectively* (k-1)*-connected and* k*-connected. Then*

(a) any continuous map of V^n *in* M^m *is homotopic to an embedding if* $m \geq 2n - k+1$ and $2k \leq n$

(b) two differentiable embeddings of V^n in M^m *which are homotopic as continuous maps are differentiably isotopic if* $m \geq 2n - k+2$ *and* $2k < n+1$.

Isotopy of embeddings is an important concept in Smale's handlebody theory, which is described in the next section (cf. Smale [13]).

§1.2 SURGERY AND HANDLEBODY THEORY

If M^n is an n-manifold with boundary $\partial M^n = Q^{n-1}$ and $f : \partial D^k \times D^{n-k} \longrightarrow Q^{n-1}$ is a differentiable embedding then the space obtained from the disjoint union of M^n and $D^k \times D^{n-k}$ by identifying $(x,y) \in \partial D^k \times D^{n-k}$ with $f((x,y)) \in Q^{n-1}$ is a manifold with boundary except for a "corner" along $\partial D^k \times \partial D^{n-k}$. By a standard technique this "corner" can be straightened (cf. Conner and Floyd [3],§3) to give a new differentiable manifold which is denoted $\chi(M^n, Q^{n-1} ; f)$. The construction of the manifold $\chi(M^n, Q^{n-1} ; f)$ is called *attaching a k-handle to* M^n along the k-1 sphere $f(\partial D^k \times 0)$ in Q^{n-1}.

Given several embeddings $f_i : \partial D^k \times D^{n-k} \longrightarrow Q^{n-1}$, $i = 1,2,\ldots s$, whose images are disjoint we can attach several handles to M^n and we denote the result by $\chi(M^n, Q^{n-1} ; f_1,\ldots.f_s)$.

The boundary of the manifold $\chi(M^n, Q^{n-1} ; f)$ is an n-1 manifold which is related to the boundary Q^{n-1} of M^n in the following way. Let $Q_o = Q^{n-1} - f(\partial D^k \times 0)$ and consider the disjoint union $Q_o \cup D^k \times \partial D^{n-k}$. If we identify the points $f((x,ty))$ and (tx,y) for each $x \in \partial D^k$, $y \in \partial D^{n-k}$ and $0 < t \leqslant 1$ then the resulting differentiable manifold is said to be obtained from Q^{n-1} by *surgery* on the k-1 sphere $f(\partial D^k \times 0)$ and we denote it by $\chi(Q^{n-1} ; f)$. It turns out that $\chi(Q^{n-1};f)$ is diffeomorphic to $\partial\chi(M^n,Q^{n-1};f)$ (for details on surgery see Browder [2]).

Two closed n-manifolds M^n and N^n are called *cobordant* if there is an n+1 manifold W^{n+1}, called a *cobordism* between M^n and N^n, whose boundary is the disjoint union of M^n and N^n. If $N^n = \chi(M^n ; f)$ for some embedding $f : \partial D^{k+1} \times D^{n-k} \longrightarrow M^n$ then by attaching a k+1-handle to $M^n \times I$ along the k-sphere $f(\partial D^{k+1} \times 0) \subset M^n \times 1$ we obtain a manifold

$W^{n+1} = \chi(M^n \times I, M^n \times 1 ; f)$, whose boundary, $\partial W^{n+1} = M^n \cup N^n$, is the disjoint union of M^n and N^n. The manifold W^{n+1} is called the cobordism between M^n and N^n associated to the surgery $\chi(M^n ; f)$.

The manifold $W^{n+1} = \chi(M^n \times I, M^n \times 1 ; f)$ has the homotopy type of M^n with a k+1 cell attached and consequently the pair (W^{n+1}, M^n) is homotopy equivalent to the pair $(D^{k+1}, \partial D^{k+1})$. Now $N^n = \chi(M^n ; f)$ is constructed as a quotient space of $M_o \cup D^{k+1} \times \partial D^{n-k}$ where $M_o = M^n - f(\partial D^{k+1} \times 0)$ and thus has a naturally embedded $D^{k+1} \times \partial D^{n-k}$. Surgery on this embedding $g : D^{k+1} \times \partial D^{n-k} \longrightarrow N^n$ yields the original manifold M^n. Thus W^{n+1} also has a representation $W^{n+1} = \chi(N^n \times I, N^n \times 1 ; g)$ and as above (W^{n+1}, N^n) is homotopy equivalent to $(D^{n-k}, \partial D^{n-k})$.

If M^n is oriented then $\chi(M^n ; f)$, the result of doing surgery on the embedding $f : \partial D^{k+1} \times D^{n-k} \longrightarrow M^n$, is, for $k > 0$, oriented as follows. The orientation of M^n gives a local orientation around a point $x \in M_o = M^n - f(\partial D^{k+1} \times 0)$ and hence it specifies a local orientation around $x \in \chi(M^n ; f) = M_o \cup D^{k+1} \times \partial D^{n-k}$ which can be lifted to an orientation of $\chi(M^n ; f) = N^n$.

On the other hand if we orient $M^n \times I$ so that the induced orientation on $M^n \times 0$ is the given one on M^n then $\chi(M^n \times I, M^n \times 1 ; f)$ has a natural orientation and this induces an orientation on the boundary component N^n. The orientation on N^n induced in this way from W^{n+1} is the opposite orientation to the one obtained in the previous paragraph by surgery on M^n.

The following theorem is proven in Smale [13]

THEOREM 1.3 *Let* $f_i : \partial D_i^k \times D_i^{n-k} \longrightarrow Q^{n-1}$ *and* $f_i' : \partial D_i^k \times D^{n-k} \longrightarrow Q^{n-1}$, $i = 1,2,\ldots,s$, *be two sets of embeddings each with disjoint images where*

$Q^{n-1} = \partial M^n$. *Then* $\chi(M,Q ; f_1 \ldots f_s)$ *and* $\chi(M,Q ; f_1', \ldots f_s')$ *are diffeomorphic if*

(a) there is a diffeomorphism $h : M \longrightarrow M$ *such that* $f_i' = hf_i$; *or*

(b) these exist diffeomorphisms $h_i : D^k \times D^{n-k} \longrightarrow D^k \times D^{n-k}$ *such that* $f_i' = f_i h_i, i = 1,2,\ldots s$; *or*

(c) the f_i' *are permutations of the* f_i.

An immediate corollary of this theorem and the isotopy extension theorem is

COROLLARY 1.4 *If* $f_1, f_2 : \partial D^k \times D^{n-k} \longrightarrow Q^{n-1} = \partial M^n$ *are two isotopic embeddings then* $\chi(M,Q ; f_1)$ *is diffeomorphic to* $\chi(M,Q ; f_2)$.

In particular, the result of surgery on an embedded $\partial D^k \times D^{n-k}$ in a manifold depends only on the isotopy class of the embedding.

We quote the following important theorem of Smale [13] (compare Wallace [19]).

THEOREM 1.5 *(Handle addition theorem) Let* $f_1, \ldots f_i \ldots f_s$ *be disjoint embeddings* $f_i : \partial D_i^{k+1} \times D^{n-k-1} \longrightarrow Q = \partial M^n$. *Denote by* $[f_i]$ *the class of the restrictions* $f_i | \partial D^{k+1} \times 0$ in $\pi_k(Q)$. *Suppose* $\bar{f}_j' : \partial D^{k+1} \times 0 \longrightarrow Q^{n-1}$, $j = 1,2,\ldots s$, *are disjoint embeddings such that* $[\bar{f}_j'] = \sum_{i=1}^{s} a_{ij}[f_i]$ *where* $\det(a_{ij}) = +1$. *If* $n \geqslant 2k + 2$ *and* $k > 1$ *then the* $\bar{f}_j'$ *can be extended to embeddings* $f_j' : \partial D^{k+1} \times D^{n-k-1} \longrightarrow Q^{n-1}$ *such that* $\chi(M,Q ; f_1 \ldots f_s)$ *is diffeomorphic to* $\chi(M,Q ; f_1' \ldots f_s')$.

In the special case when M is the disc D^n and the $f_i : \partial D^k \times D^{n-k} \longrightarrow S^{n-1} = \partial D^n$, $i = 1,2,\ldots s$, are disjoint embeddings, the manifold $\chi(D^n, S^{n-1} ; f_1, \ldots f_s)$ is called, following Smale, *a handlebody*.

We conclude this section with the definition of an important operation on manifolds. If M_1 and M_2 are connected oriented n-manifolds and $f_i : D^n \longrightarrow M_i$ $i = 1,2$ are embeddings, the first preserving orientation and the second reversing it, then the *connected sum* of M_1 and M_2, written $M_1 \# M_2$, is the oriented manifold obtained from $(M_1 - f_1(0)) \cup (M_2 - f_2(0))$ by identifying $f_1(tx)$ and $f_2((1-t)x)$ for $x \in S^{n-1}$ and $0 < t \leq 1$.

§1.3 CHARACTERISTIC CLASSES

In this final section, we discuss characteristic classes of differentiable manifolds, and in particular a geometric condition, $w_2(M) = 0$, which allows us to do surgery.

The Stiefel-Whitney characteristic classes of a manifold M^n are mod 2 cohomology classes defined as follows. The classifying space for n-plane bundles is the infinite Grassmanian $G_n(R^\infty)$ which is the union of the Grassmanians $G_n(R^{n+k})$ for $k > 0$. $G_n(R^{n+k})$ is the set of n-dimensional subspaces of R^{n+k} topologized in the usual way. The mod 2 cohomology ring of $G_n(R^\infty)$, $H^*(G_n(R^\infty) ; Z/2) = Z/2[w_1 \ldots w_n]$ is a polynomial algebra over $Z/2$ with generators w_i (cf. Milnor [12]).

If M^n is a differentiable manifold then by a well known result (cf. Hirsch [10]) M^n can be embedded in R^{n+k} for sufficiently large k. Given an embedding $i : M^n \longrightarrow R^{n+k}$ we may define the Gauss map $\tau : M^n \longrightarrow G_n(R^{n+k}) \subset G_n(R^\infty)$ which takes a point $x \in M^n$ into the subspace of R^{n+k} parallel to the plane in R^{n+k} tangent to $i(M^n)$ at $i(x)$. It can be shown that the homotopy class of the map $\tau : M^n \longrightarrow G_n(R^\infty)$ is independent of the particular embedding chosen.

The i^{th} Stiefel-Whitney class of M^n is then defined to be $w_i(M^n) = \tau^*(w_i)$ where $\tau^* : H^*(G_n(R^\infty) ; Z/2) \longrightarrow H^*(M^n, Z/2)$ is induced by τ on cohomology.

For a manifold M^n, the universal coefficient theorem gives the exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_1(M\ ;\ Z),\ Z/2) \longrightarrow H^2(M\ ;\ Z/2) \longrightarrow \mathrm{Hom}(H_2(M\ ;\ Z),\ Z/2) \longrightarrow 0$$

and hence if M^n is simply connected we have

$$H^2(M^n\ ;\ Z/2) \cong \mathrm{Hom}(H_2(M^n\ ;\ Z),\ Z/2).$$

Thus the second Stiefel-Whitney class $w_2(M^n) \in H^2(M^n\ ;\ Z/2)$ can be considered as a homomorphism from the integral homology group $H_2(M^n\ ;\ Z)$ to $Z/2$. As M^n is simply-connected the Hurewicz homomorphism $h : \pi_2(M^n) \longrightarrow H_2(M^n\ ;\ Z)$ is an isomorphism, whence every 2-dimensional homology class in M^n is represented by an map $f : S^2 \longrightarrow M^n$. If $n \geqslant 5$ theorem 1.2 implies that f may be taken to be an embedding. The normal bundle of this embedding will then be a stable bundle over the 2-sphere. Now there are only two such bundles over S^2 corresponding to the stable group $\pi_1(SO) = Z/2$. It turns out that the homomorphism $w_2(M^n)$ for $n \geqslant 5$ evaluates 0 on an embedding $f : S^2 \longrightarrow M^n$ if the normal bundle is trivial and 1 if the normal bundle is non-trivial.

Thus the condition that the second Stiefel-Whitney class of a simply-connected 6-manifold M^6 vanish, is equivalent to the condition that every embedded 2-sphere in M^6 have a trivial normal bundle.

CHAPTER 2

§2.1 TORSION FREE MANIFOLDS

In this chapter we review Wall's classification of simply-connected 6-manifolds with vanishing second Stiefel-Whitney class and torsion free homology. This classification was begun in Smale [14] and completed in Wall [18]. Wall's results depend essentially upon the earlier study by Haefliger [6,7,8] of the isotopy classes of embeddings of the 3-sphere in the 6-sphere.

In [14] Smale showed that every 2-connected 6-manifold was either S^6 or a connected sum of copies of $S^3 \times S^3$. Wall [18] begins with the observation that any simply-connected, 6-manifold M can be written as a connected sum $M = M_1 \# M_2$ where $H_3(M_1)$ is finite and M_2 is a connected sum of copies of $S^3 \times S^3$. If the homology of M is torsion free, then this splitting is unique up to diffeomorphism.

It suffices then to consider simply-connected 6-manifolds M^6 with torsion free homology, vanishing second Stiefel-Whitney class, $w_2(M)$, and $H_3(M) = 0$. Wall shows that such manifolds are obtained by surgery on a disjoint set of embeddings $g_i : S^3 \times D^3 \longrightarrow S^6$ (Wall [18], theorem 2), or in other words these manifolds are the boundaries of the handlebodies formed by attaching 4-handles to the 7-disc along 3-spheres embedded in the 6-sphere.

§2.2 KNOTS AND LINKS

A single embedding $\bar{g} : S^3 \longrightarrow S^6$ is called a knot in S^6 while an embedding $g : S^3 \times D^3 \longrightarrow S^6$ is called a framed knot. The isotopy classes of embeddings of S^3 in S^6 form a group, which Haefliger [8] denotes C_3^3.

and the isotopy classes of embeddings of $S^3 \times D^3$ in S^6 form a group, FC_3^3. Restriction of an embedding $g : S^3 \times D^3 \longrightarrow S^6$ to $\bar{g} = g|S^3 \times 0 \longrightarrow S^6$ defines a homomorphism from FC_3^3 to C_3^3. Moreover, there is a map τ from $\pi_3(SO_3)$ to FC_3^3 given by taking an element $[\tau] \varepsilon \pi_3(SO_3)$ to the embedding given by the composition of the diffeomorphism $(x,y) \longrightarrow (x, \tau(x)y)$ of $S^3 \times D^3$ with the natural inclusion $S^3 \times D^3 \subset S^6$. This gives a short exact sequence

$$0 \longrightarrow \pi_3(SO_3) \longrightarrow FC_3^3 \longrightarrow C_3^3 \longrightarrow 0.$$

In [8] Haefliger calculates these groups and shows that $FC_3^3 = Z \oplus Z$ and $C_3^3 = Z$. Thus the isotopy class of a framed knot $S^3 \times D^3$ in S^6 is determined by a pair of integers $(a,b) \varepsilon FC_3^3$.

A collection of disjoint embeddings $g_i : S^3 \times D^3 \longrightarrow S^6$ is called a framed link. These were studied by Haefliger [7] where he determines the invariants of isotopy classes of framed links. If $g_i : S^3 \times D^3 \longrightarrow S^6$ are disjoint embeddings then linking coefficients $c_j^i \varepsilon \pi_3(S^2) = Z$ are defined by taking the homotopy class of the composite of the embedding $\bar{g}_i$ of $S^3 \times 0$ into $S^6 - g_j(S^3 \times 0)$ and a homotopy equivalence of $S^6 - g_j(S^3 \times 0)$ with S^2, chosen by taking an embedded copy of S^2 whose linking number with $g_j(S^3 \times 0)$ is + 1. For example choose S^2 to be the boundary of one fibre in the tubular neighborhood $g_j(S^3 \times D^3)$ of $g_j(S^3 \times 0)$.

Linking coefficients c_{jk}^i are defined by considering the map $\bar{g}_i$ of $S^3 \times 0$ into $S^6 - (g_j(S^3 \times 0) \cup g_k(S^3 \times 0))$. This last space has the homotopy type of $S^2 \vee S^2$ so $\bar{g}_i$ determines a class in

$$\pi_3(S^2 \vee S^2) \cong \pi_3(S^2) \oplus \pi_3(S^2) \oplus \pi_3(S^3) \qquad \text{(cf. Hilton [9])}$$

and c_{jk}^i is defined to be the projection of $[g_i] \varepsilon \pi_3(S^2 \vee S^2)$ on the factor $\pi_3(S^3) = Z$, which is injected into $\pi_3(S^2 \vee S^2)$ by the Whitehead product of the inclusions of S^2 in $S^2 \vee S^2$.

Using these invariants of framed links Wall shows that the isotopy class of a framed link $g_i : S^3 \times D^3 \longrightarrow S^6$ is determined by the knot class $(a_i, b_i) \in FC_3^3$ of each of its components and by the linking coefficients c_j^i and c_{jk}^i. These linking coefficients are subject to the relations $c_j^i \equiv c_i^j \bmod 2$ and the c_{jk}^i are symmetric in i,j and k.

§2.3 CLASSIFICATION OF TORSION FREE MANIFOLDS

As mentioned above, all simply-connected 6-manifolds M with $w_2(M) = 0$ and $H_3(M) = 0$ can be constructed by attaching 4-handles to the 7-disc along a framed link of 3-spheres in the 6-sphere and taking the boundary of the resulting handlebody. Theorem 1.3 implies that the diffeomorphism class of the result only depends on the isotopy class of the framed link.

Given a disjoint set of embeddings $g_i : S^3 \times D^3 \longrightarrow S^6$, $i = 1,2,\ldots k$, let W be the handlebody formed by attaching $D_i^4 \times D_i^3$ to D^7 along the g_i and let $M^6 = \partial W$. The handles $D_i^4 \times D_i^3$ determine homology classes in $H_4(W,D^7) \simeq H_4(W) \simeq H_4(M) \simeq H^2(M)$ where the first two isomorphisms come from the exact sequences of the pairs (W,D^7), (W,M) and the last isomorphism is Poincaré duality for $M^6 = \partial W$.

The oriented, simply-connected 6-manifold $M^6 = \partial W$ has torsion free homology and $w_2(M) = 0$. Moreover $H_3(M) = 0$ and $H^2(M) = \bigoplus_{i=1}^{k} e_i Z$ where the classes $e_i \in H^2(M)$ correspond via the above isomorphisms to the handles $D_i^4 \times D_i^3$. If the knot type of g_i is $(a_i, b_i) \in FC_3^3 = Z \oplus Z$ and the linking coefficients are c_j^i and c_{jk}^i then Wall shows that the cup product structure of M^6 is determined by

$$< e_i \cup e_j \cup e_k, [M] > = c_{jk}^i \qquad i,j,k \text{ distinct}$$

$$< e_i \cup e_i \cup e_i, [M] > = 6a_i + b_i$$

$$< e_i \cup e_i \cup e_j, [M] > = c_i^j$$

and the first Pontrjagin class of M, $p_1(M) \in H^4(M,Z)$ is determined by $< p_1(M) \cup e_i, [M] > = 4b_i$ (cf. Wall [18], theorem 4).

This gives Wall's classification theorem for simply-connected 6-manifolds:

THEOREM 2.1 *(Wall [18]) Diffeomorphism classes of oriented simply-connected 6-manifolds with vanishing second Stiefel-Whitney class and torsion free homology, correspond to systems of invariants:*

two free abelian groups H,G

a symmetric trilinear map $\mu : H \times H \times H \longrightarrow Z$

a homomorphism $p_1 : H \longrightarrow Z$

subject to : for $x,y \in H$

$$\mu(x,x,y) \equiv \mu(x,y,y) \pmod 2$$

and for $x \in H$

$$p_1(x) \equiv 4\mu(x,x,x) \pmod{24} .$$

For a given manifold M^6 this system of invariants is given by $H = H^2(M ; Z)$, $G = H^3(M ; Z)$ and for $x,y,z \in H$

$$\mu(x,y,z) = < x \cup y \cup z, [M] >$$

and $p_1(x) = < x \cup p_1(M), [M] >$.

§2.4 WALL MANIFOLDS

In chapters 3 and 4 we will need some particular cases of Wall's results. Let M^6 be a simply-connected 6-manifold with vanishing second Stiefel-Whitney class. We call M^6 a *Wall manifold* if it has the same homology groups as $S^2 \times S^4$. M^6 is called a *double Wall manifold* if it has the same homology groups as $S^2 \times S^4 \# S^2 \times S^4$.

By the remarks above any Wall manifold M^6 is obtained by surgery on a framed knot $g : S^3 \times D^3 \longrightarrow S^6$ with knot invariants $(a,b) \varepsilon FC_3^3$.

If W^7 is the handlebody obtained by attaching the handle $D^4 \times D^3$ to D^7 along the framed knot $g : S^3 \times D^3 \longrightarrow S^6$ and $M^6 = \partial W^7$, then the core of the handle $D^4 \times 0$ is embedded in W and we may form the cone on $\partial D^4 \times 0$ to the center of D^7. This gives a map $f : S^4 \longrightarrow W^7$: which determines a homology class in $H_4(W^7)$. The exact sequence of the pair (W^7, M^6) is

$$H_5(W^7,M^6) \longrightarrow H_4(M^6) \longrightarrow H_4(W^7) \longrightarrow H^4(W^7,M^6)$$

and $\quad H_i^*(W^7,M^6) \simeq H^{7-i}(W^7)$ by Lefschetz duality.

But W^7 is homotopy equivalent to S^4 and therefore $H^2(W) = H^3(W) = 0$ and from the exact sequence we obtain $H_4(M^6) \simeq H_4(W^7)$. Thus the class given by $f : S^4 \longrightarrow W^7$ gives a class V^4 in $H_4(M^6)$. Let $e \varepsilon H^2(M^6)$ be the Poincaré dual of V^4 and let $\hat{e} \varepsilon H^4(M^6)$ be the cohomology dual of V^4, that is $\hat{e}\ (V^4) = +1$. Then the cup product structure is given by $e \cup \hat{e} = \alpha$ where $< \alpha, [M]> = 1$ and $e \cup e = (6a + b)\hat{e}$. The first Pontrjagin class $p_1(M^6)$ is given by $p_1(M^6) = 4b\hat{e}$.

Double Wall manifolds are obtained by surgery on a framed link $g_i : S^3 \times D^3 \longrightarrow S^6$, $i = 1,2$. In terms of the knot invariants $(a_i, b_i) \varepsilon FC_3^3$ and the linking coefficients c_j^i (the coefficients c_{jk}^i are undefined here) the cup product structure is as follows. We form the cones on $\partial D_i^4 \times 0$ to the center of D^7 to give homology classes in $H_4(W^7)$ where $W^7 = \chi(D^7, S^6 ; g_1, g_2)$. As in the previous paragraph $H_4(W) \simeq H_4(M^6)$ where $M^6 = \partial W^7$ and we obtain classes V_1^4 and V_2^4 in $H_4(M^6)$. Let $e_1, e_2 \varepsilon H^2(M^6)$ be the Poincaré duals of V_1^4 and V_2^4 respectively and take $\hat{e}_1, \hat{e}_2 \varepsilon H^4(M^6)$ to be the duals in cohomology (ie. $\hat{e}_i(V_j^4) = \delta_{ij}$). Then cup products in M^6

are determined by

$$e_i \cup \hat{e}_j = \delta_{ij}\,\alpha \quad \text{where } \langle \alpha, [M] \rangle = 1$$

$$e_1 \cup e_1 = (6a_1 + b_1)\hat{e}_1 + c_1^2\,\hat{e}_2$$

$$e_1 \cup e_2 = c_1^2\,\hat{e}_1 + c_2^1\,\hat{e}_2$$

$$e_2 \cup e_2 = c_2^1\,\hat{e}_1 + (6a_2 + b_2)\hat{e}_2$$

and the first Pontrjagin class is given by

$$p_1(M) = 4b_1\hat{e}_1 + 4b_2\hat{e}_2 \ .$$

CHAPTER 3

§3.1 TORSION MANIFOLDS

The manifolds of main interest in this thesis are oriented, simply-connected 6-manifolds M with $w_2(M) = 0$ and integral homology groups $H_2(M) = Z/n = H_3(M)$ for some integer $n > 1$. Manifolds satisfying these conditions will be referred to as *torsion manifolds.*

THEOREM 3.1 *Every torsion manifold* M *is obtained by surgery on a 2-sphere in some Wall manifold* N.

Proof: Let $\bar{f}:S^3 \longrightarrow M$ be a differentiable embedding representing a generator of $H_3(M;Z) = Z/n$. This is possible since the Hurewicz homomorphism $h : \pi_3(M) \longrightarrow H_3(M;Z)$ is onto and by theorem 1.2 every homotopy class in $\pi_3(M)$ is representable by a differentiable embedding. The normal bundle to this embedding is trivial since it is classified by an element in $\pi_2 SO(3) = 0$. Extend the embedding of S^3 to an embedding $f : S^3 \times D^3 \longrightarrow M$ and let $N = \chi(M;f)$ be the result of surgery on the embedded $S^3 \times D^3$.

Let W be the cobordism between M and N described in chapter one. Then

$$H_q(W,M) = \begin{cases} Z & q = 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_q(W,N) = \begin{cases} Z & q = 3 \\ 0 & \text{otherwise} \end{cases} .$$

From the exact sequence for the pair (W,M)

$$\begin{array}{ccccccc} H_4(M) & \longrightarrow & H_4(W) & \longrightarrow & H_4(W,M) & \longrightarrow & H_3(M) \\ \| & & & & \| & & \| \\ 0 & & & & Z & & Z/n \end{array}$$

$$H_3(W,M) \longrightarrow H_2(M) \longrightarrow H_2(W) \longrightarrow H_2(W,M)$$

we conclude that $H_4(W) = Z$ and $H_2(W) = Z/n$.

Using this in the exact sequence of (W,N)

$$\begin{array}{ccccccccccc} H_5(W,N) & \longrightarrow & H_4(N) & \longrightarrow & H_4(W) & \longrightarrow & H_4(W,N) & \longrightarrow & H_3(N) & \longrightarrow & H_3(W) \\ \| & & & & \| & & \| & & & & \| \\ 0 & & & & Z & & 0 & & & & 0 \end{array}$$

$$\begin{array}{cccccccc} \longrightarrow & H_3(W,N) & \longrightarrow & H_2(N) & \longrightarrow & H_2(W) & \longrightarrow & H_2(W,N) \\ & \| & & & & \| & & \| \\ & Z & & & & Z/n & & 0 \end{array}$$

yields $H_4(N) = Z$ and $0 = H_3(N) \approx H^3(N)$, by Poincaré duality for N, which implies that $H_2(N) = Z$ since torsion $H^3(N) =$ torsion $H_2(N)$. Thus N has the homology of $S^2 \times S^4$ and is simply-connected. Reversing the procedure exhibits M as the result of surgery on an embedded 2-sphere $g : S^2 \longrightarrow N$ in N whose homotopy class is n-times a generator of $\pi_2(N) = H_2(N)$.

To check that $w_2(N) = 0$ let $i : S^2 \longrightarrow N$ be any embedded 2-sphere. By general position $i(S^2)$ may be assumed disjoint from $g(S^2 \times D^4)$ and hence it may be considered as an embedding into M. Since $w_2(M) = 0$ the normal bundle of i is trivial in M and therefore in N, implying that $w_2(N) = 0$. □

If N is a Wall manifold then N is determined by a framed knot $(a,b) \in FC_3^3$. Given an integer $n > 1$ we may perform surgery on a 2-sphere representing n-times a generator in $H_2(N) = \pi_2(N)$. By theorem 1.2 the diffeomorphism class of the result depends only on the homotopy class of the embedded sphere. We denote the torsion manifold so constructed by M(a,b,n). Theorem 3.1 then says that any torsion manifold can be written as $M(a,b,n)$ for some $a,b \in Z \quad n > 1$.

§3.2 COHOMOLOGY RINGS OF TORSION MANIFOLDS

If M is an oriented torsion manifold then its integral cohomology groups are

$$H^q(M;Z) = \begin{cases} Z & q = 0,6 \\ Z/n & q = 3,4 \\ 0 & \text{otherwise} . \end{cases}$$

The cup product structure in the ring $H^*(M,Z)$ is trivial, and two such rings are isomorphic if and only if there torsion subgroups are isomorphic.

By the universal coefficient theorem the cohomology groups of M with coefficients in the group Z/n where $H^3(M;Z) = Z/n$ are

$$H^q(M;Z/n) = \begin{cases} Z/n & q = 0,2,4,6 \\ Z/n \oplus Z/n & q = 3 . \end{cases}$$

The ring structure in $H^*(M;Z/n)$ can be specified by considering the commutative subring

$$H^{2*}(M;Z/n) = \bigoplus_{i=0}^{3} H^{2i}(M;Z/n)$$

together with the skew-symmetric, non-singular bilinear form on $H^3(M;Z/n)$ given by cup product.

By Poincaré duality a basis for $H^{2*}(M;Z/n)$ can be chosen so that

$$H^{2*}(M;Z/n) = 1\, Z/n \oplus eZ/n \oplus \hat{e}Z/n \oplus \alpha Z/n$$

where e and $\hat{e}$ are dual generators for $H^2(M;Z/n)$ and $H^4(M;Z/n)$ respectively, and α is the orientation class of M determined by the Z-orientation of M. With respect to this basis the product structure in this ring is determined by

$$1 \cup x = x \qquad \text{for all } x$$

$$e \cup \hat{e} = \alpha$$

$$e \cup e = k\hat{e} \qquad \text{for some } k \in Z/n .$$

The first Pontrjagin class of M, $p_1(M) \in H^4(M;Z) = Z/n$ can be considered as a class $\bar{p}_1(M) \in H^4(M;Z/n)$ since by the universal coefficient theorem $0 \longrightarrow H^4(M;Z) \otimes Z/n \longrightarrow H^4(M;Z/n) \longrightarrow \mathrm{Tor}(H^5(M;Z),\ Z/n) \longrightarrow 0$ is exact and $H^5(M;Z)$ is zero. With the above basis we can write $\bar{p}_1(M) = \ell\hat{e}$ for some $\ell \in Z/n$.

For a given torsion manifold M with $H_2(M;Z) = Z/n$ let R(M) denote the even dimensional cohomology ring with Z/n coefficients, together with a distinguished element $\bar{p}_1(M) \in H^4(M;Z/n)$. The augmented ring R(M) is determined by two integers, k and ℓ mod n. Since our manifolds are oriented the rings that arise have a distinguished generator in dimension 6 given by the orientation class. Moreover, by Poincaré duality, the product structure in these rings is such that the pairing

$$H^2(M;Z/n) \times H^4(M;Z/n) \longrightarrow H^6(M;Z/n)$$

is non-singular. Isomorphisms of such rings are those graded ring isomorphisms which preserve the distinguished element $\bar{p}_1(M)$ and the orientation class. In the next theorem we determine the isomorphism classes of such rings.

THEOREM 3.2 *For each* $n > 1$ *the isomorphism classes of rings* R(M) *as above are in one-one correspondence with the orbits in* $Z/n \oplus Z/n$ *under the action of the group of units* $U(Z/n)$ *given by* $m\cdot(k,\ell) = (m^3k,\ m\ell)$ *for* $m \in U(Z/n)$ *and* $(k,\ell) \in Z/n \oplus Z/n$.

Proof: Let $R = 1Z/n \oplus eZ/n \oplus \hat{e}Z/n \oplus \alpha Z/n$ and $R' = 1'Z/n \oplus e'Z/n \oplus \hat{e}'Z/n \oplus \alpha'Z/n$. Suppose $e \cup e = k\hat{e}$, $\bar{p} = \ell\hat{e}$ in R and $e' \cup e' = k'\hat{e}'$, $\bar{p}' = \ell'\hat{e}'$ in R'. If $f : R \longrightarrow R'$ is an isomorphism then $f(1) = 1'$, $f(\alpha) = \alpha'$, $f(\bar{p}) = \bar{p}'$ and since f preserves grading $f(e) = se'$ and $f(\hat{e}) = t\hat{e}'$ where $s, t \in U(Z/n)$. Now

$$\alpha' = f(\alpha) = f(e \cup \hat{e}) = f(e) \cup f(\hat{e}) = se' \cup t\hat{e}' = st\,\alpha'$$

and therefore $t = s^{-1}$. From $f(\bar{p}) = \bar{p}'$ we have

$$\ell'\hat{e}' = \bar{p}' = f(\ell\hat{e}) = \ell f(\hat{e}) = \ell s^{-1}\hat{e}' \text{ which implies } \ell' = \ell s^{-1} \text{ or}$$

equivalently $\ell = s\ell'$. Finally $ks^{-1}\hat{e}' = k\,f(\hat{e}) = f(k\hat{e}) = f(e \cup e)$

$$= f(e) \cup f(e) = se' \cup se' = s^2k'\hat{e}'$$

which gives $\quad ks^{-1} = s^2k' \quad$ or $\quad k = s^3k'$.

Thus if R and R' are isomorphic then

$$k = s^3k' \text{ and } \ell = s\ell' \text{ for some } s \in U(Z/n).$$

Conversely if R and R' are defined as above with $k = s^3k'$ and $\ell = s\ell'$ then f as above will be an isomorphism of R with R'. □

The augmented ring R(M) is by definition an invariant of the oriented diffeomorphism type of the torsion manifold M since the cohomology ring and the Pontrjagin classes are diffeomorphism invariants.

We consider now the skew-symmetric bilinear form on $H^3(M;Z/n)$ given by

$$b(x,y) = \langle x \cup y, [M]\rangle \in Z/n .$$

The coefficient sequence

$$0 \longrightarrow Z/n \longrightarrow Z/n^2 \longrightarrow Z/n \longrightarrow 0$$

induces the Bockstein homomorphism

$$\beta_n : H^q(M;Z/n) \longrightarrow H^{q+1}(M;Z/n)$$

which satisfies (c.f. Spanier [15], p.281) $\beta_n \cdot \beta_n = 0$

and $\quad \beta_n(u \cup v) = \beta_n(u) \cup v + (-1)^{\deg u} u \cup \beta_n(v)$.

From the long exact sequence induced by the coefficient sequence we conclude

that $\quad \beta_n : H^2(M;Z/n) \longrightarrow H^3(M;Z/n)$ is injective

and $\quad \beta_n : H^3(M;Z/n) \longrightarrow H^4(M,Z/n)$ is surjective.

Together with $\beta_n \cdot \beta_n = 0$ this implies that the sequence

$$0 \longrightarrow H^2(M;Z/n) \xrightarrow{\beta_n} H^3(M;Z/n) \xrightarrow{\beta_n} H^4(M;Z/n) \longrightarrow 0 \quad \text{is exact.}$$

Let $e \in H^2(M;Z/n)$ be a generator and set $x = \beta_n(e)$. Let $y \in H^3(M;Z/n)$ be such that $\beta_n(y) = \hat{e} \in H^4(M;Z/n)$ where $\hat{e}$ is the dual generator to e. Using the product formula for β_n we have

$$0 = \beta_n(e \cup x) = \beta_n(e) \cup x + (-1)^2\, e \cup \beta_n(x)$$

since $e \cup x \in H^5(M;Z/n) = 0$. Now $\beta_n(x) = 0$

so $\quad 0 = \beta_n(e) \cup x = x \cup x.$

Applying the product formula again gives

$$0 = \beta_n(y \cup e) = \beta_n(y) \cup e + (-1)^3\, y \cup \beta_n(e)$$

since $y \cup e \in H^5(M;Z/n) = 0$. This implies

$$\alpha = \hat{e} \cup e = \beta_n(y) \cup e = y \cup \beta_n(e) = y \cup x .$$

Thus the matrix of the form b with respect to the basis $\{x,y\}$ of $H^3(M;Z/n)$

is $\begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix}$ where $a = -a$ in Z/n.

If n is odd then $2a = 0$ implies $a = 0$. If $n \equiv 2 \bmod 4$ we have the following commutative diagram, where Sq^i is the Steenrod squaring operation $Sq^i : H^n(M;Z/2) \longrightarrow H^{n+i}(M;Z/2)$ (cf. Spanier [15], p.270)

$$\begin{array}{ccc} H^3(M;Z/n) & \xrightarrow{\text{squaring}} & H^6(M;Z/n) \\ \big\downarrow r & & \big\downarrow r \\ H^3(M;Z/2) & \xrightarrow{Sq^3} & H^6(M;Z/2) \end{array}$$

and where r is reduction of coefficients.

Since $\frac{n}{2}$ is odd, $r(\frac{n}{2}) = \bar{1} \in Z/2$. But $r(y \cup y) = Sq^3(r(y)) = 0$, since $Sq^3 = Sq^1\, Sq^2$ by the Adem relations for the Steenrod squares.

(i.e. Sq^3 factors through the group $H^5(M;Z/2) = 0$)

Therefore $y \cup y = 0$ if $n \equiv 2 \mod 4$.

In the case $n \equiv 0 \mod 4$ a has two possible values, 0 or $\frac{n}{2}$.

§3.3 PONTRJAGIN POWERS

In this section we consider a final invariant of torsion manifolds, the Pontrjagin cubing operation. These cohomology operations are a special instance of the Pontrjagin p^{th} power operations defined by E. Thomas [16].

If p is a prime, the Pontrjagin p^{th} power operations are functions

$$\mathbf{P}_p : H^k(M;Z/pm) \longrightarrow H^{kp}(M;Z/p^2m)$$

defined for any cell complex M and any positive integer m. The functions $\mathbf{P}_p$, being cohomology operations, are natural with respect to maps of spaces. If $r : Z/p^2m \longrightarrow Z/pm$ is the reduction homomorphism and r_* the induced map on cohomology

$$r_* : H^{kp}(M;Z/p^2m) \longrightarrow H^{kp}(M;Z/pm)$$

then the Pontrjagin p^{th} power satisfies

$$r_*(\mathbf{P}_p(u)) = u^p \quad \text{(p-fold cup product) for any } u \in H^k(M;Z/pm).$$

In [17] Thomas lists additional properties of these operations and computes the operations for the infinite complex projective space CP^∞.

Let M be a torsion manifold with $H_2(M;Z) = Z/n$ and assume that $n = 3m$. We consider the operation with $k = 2$ and $p = 3$

$$\mathbf{P}_3 : H^2(M;Z/3m) \longrightarrow H^6(M;Z/9m) .$$

In the next chapter we calculate these operations for torsion manifolds using Thomas' results for the space CP^∞, which we now summarize.

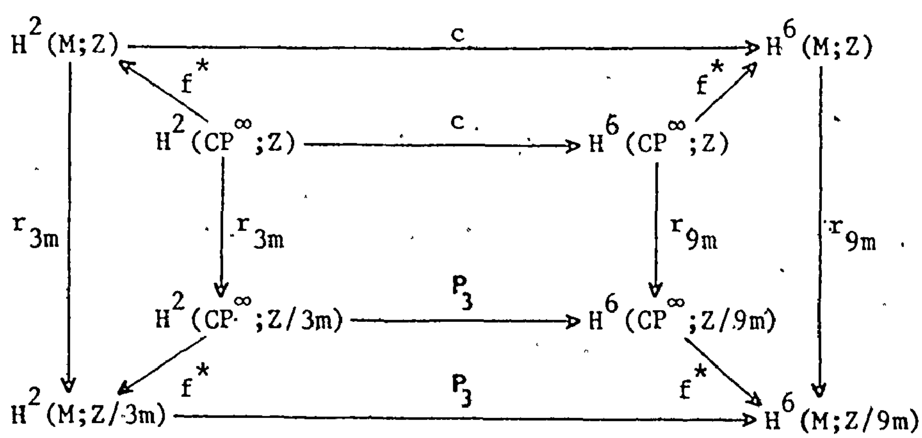
Let u be a generator for the cohomology group $H^2(CP^\infty;Z) = Z$.

Then u^3 generates the group $H^6(CP^\infty;Z) = Z$. For any integer $k > 1$ let $r_k : H^n(CP^\infty;Z) \longrightarrow H^n(CP^\infty;Z/k)$ be the coefficient homomorphism from Z to Z/k, then $r_{9m}(u^3)$ is the image of u^3 in $H^6(CP^\infty;Z/9m)$ and according to Thomas ([17], appendix, theorem 3)

$$P_3(r_{3m}(u)) = r_{9m}(u^3) \qquad (1) .$$

The space CP^∞ is the Eilenberg-MacLane space K(Z,2) (cf. Spanier [15], p.425) and thus for any cell complex M, the second integral cohomology group of M, $H^2(M;Z)$, is in one to one correspondence with the homotopy classes of maps from M into $K(Z;2) = CP^\infty$. This correspondence associates a homotopy class $[f] \in [M, K(Z;2)]$ with the image $f^*(u)$ of the fundamental class of $u \in H^2(K(Z,2);Z)$ under the map $f^* : H^2(K(Z,2);Z) \longrightarrow H^2(M,Z)$ induced by f.

If M is any Wall manifold then $H^2(M;Z) = Z$ and we can choose a map $f : M \longrightarrow CP^\infty$ such that $f^* : H^2(CP^\infty;Z) \longrightarrow H^2(M;Z)$ maps the class u to a generator of $H^2(M;Z)$. Then the map $f^* : H^2(CP^\infty;Z) \longrightarrow H^2(M;Z)$ is an isomorphism and we consider the following diagram



where $c(x) = x \cup x \cup x$ and r_k is reduction of coefficients mod k.

The inner square commutes by equation (1) and the trapezoids commute by naturality. A simple diagram chase, using the fact that f^* is an isomorphism in the upper left corner, yields the commutativity of the outer square. Thus the Pontrjagin cube for any Wall manifold M is determined by

$$P_3(r_{3m}(e)) = r_{9m}(e^3)$$

where $e \in H^2(M;Z)$ is an integral cohomology class.

CHAPTER 4

§4.1 SURGERY ON WALL MANIFOLDS

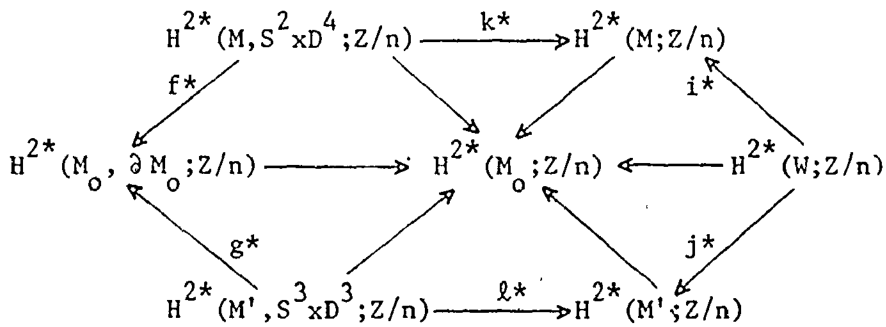
By theorem 3.1 each torsion manifold is of the form M(a,b,n), where M(a,b,n) is the manifold obtained by doing surgery on a 2-sphere representing n-times a generator of the second homotopy group of the Wall manifold determined by the framed knot $(a,b) \in FC_3^3$. In this section we calculate the invariants of chapter 3 for the torsion manifolds M(a,b,n).

Let M be the Wall manifold determined by the framed knot $(a,b) \in FC_3^3$, and let $\bar{f} : S^2 \longrightarrow M$ be an embedding representing n-times a generator of $\pi_2(M)$. By theorem 1.2 such an embedding exists and its isotopy class is determined by the integer n. By assumption $w_2(M) = 0$ so $f(S^2)$ has a trivial normal bundle and since $\pi_2\ SO(4) = 0$ any two extensions of $\bar{f}$ to $S^2 \times D^4$ are isotopic. Thus we can do surgery on $\bar{f}$ to obtain a manifold $M' = \chi(M,\bar{f})$ and the diffeomorphism class of M' is determined by the knot invariants (a,b) and by the integer n. In our previous notation M' = M(a,b,n).

Let $W = M \times I \cup_f D^3 \times D^4$ where $f : S^2 \times D^4 \longrightarrow M$ is an extension of $\bar{f}$ as above. Then W is a cobordism between the Wall manifold M and the torsion manifold M' and $\partial W = M \cup M'$. Setting $M_0 = M - f(S^2 \times D^4)$ and $M' = M_0 \cup S^3 \times D^3$ gives an alternative description of M' = M(a,b,n). We remark that these two descriptions of M' give opposite orientations (cf. section 1.2) and the orientation we want for M' is the one given by surgery.

From the exact sequences of the pairs, $(M, S^2 \times D^4)$ and

$(M', S^3 \times D^3)$ with Z/n coefficients (see figure 1) we obtain the following diagram :



where i,j,k and ℓ are inclusion maps f and g are excisions, and all maps to $H^{2*}(M_o;Z/n)$ are induced by inclusions.

The map $h = \ell^* \, g^{*-1} \, f^* \, k^{*-1}$ is an isomorphism $h : H^{2*}(M;Z/n) \longrightarrow H^{2*}(M';Z/n)$ of the even dimensional cohomology rings of M and M'.

From chapter 2, §4 the Wall manifold M has a basis $\underline{e} \in H^2(M;Z)$ and $\underline{\hat{e}} \in H^4(M;Z)$ with cup product and Pontrjagin class given by

$$<\underline{e} \cup \underline{\hat{e}}, [M]> = 1$$
$$\underline{e} \cup \underline{e} = (6a + b)\underline{\hat{e}}$$
$$p_1(M) = 4b\underline{\hat{e}} \ .$$

Since the Wall manifold M has torsion free homology the coefficient map $r_n : Z \longrightarrow Z/n$ induces a ring surjection $r_{n*}H^*(M;Z) \longrightarrow H^*(M;Z/n)$ and we take the images of $\underline{e}$ and $\underline{\hat{e}}$, $e = r_{n*}(\underline{e})$ and $\hat{e} = r_{n*}(\underline{\hat{e}})$. The cohomology ring $H^*(M;Z/n)$ of M with Z/n coefficients is then determined by the generators $e \in H^2(M;Z/n)$ and $\hat{e} \in H^4(M;Z/n)$ subject to

$$<e \cup \hat{e}, [M]> = 1 \in Z/n \qquad (1)$$
$$e \cup e = r_n(6a + b)\,\hat{e} \ .$$

FIGURE 1
(Z/n coefficients)

$$\begin{array}{ccccccccc}
H^2(M,S^2\times D^4) \to & H^2(M) \to & H^2(S^2\times D^4) \to & H^3(M,S^2\times D^4) \to & H^3(M) \to & H^3(S^2\times D^4) \to & H^4(M,S^2\times D^4) \to & H^4(M) \to & H^4(S^2\times D^4) \\
Z/n \;\cong & Z/n & Z/n \;\cong & Z/n & 0 & 0 & Z/n \;\cong & Z/n & 0 \\
H^2(M',S^3\times D^3) \to & H^2(M') \to & H^2(S^3\times D^3) \to & H^3(M',S^3\times D^3) \to & H^3(M') \to & H^3(S^3\times D^3) \to & H^4(M',S^3\times D^3) \to & H^4(M') \to & H^4(S^3\times D^3) \\
Z/n \;\cong & Z/n & 0 & Z/n & Z/n \oplus Z/n & Z/n & Z/n \;\cong & Z/n & 0
\end{array}$$

$$\begin{array}{ccccccccc}
H^2(M_0,\partial M_0) \to & H^2(M_0) \to & H^2(\partial M_0) \to & H^3(M_0,\partial M_0) \to & H^3(M_0) \to & H^3(\partial M_0) \to & H^4(M_0,\partial M_0) \to & H^4(M_0) \to & H^4(\partial M_0) \\
Z/n \;\cong & Z/n & Z/n \;\cong & Z/n & Z/n \;\cong & Z/n & Z/n \;\cong & Z/n & 0
\end{array}$$

$$\begin{array}{ccccccccc}
H^2(M,M_0) \to & H^2(M) \to & H^2(M_0) \to & H^3(M,M_0) \to & H^3(M) \to & H^3(M_0) \to & H^4(M,M_0) \to & H^4(M) \to & H^4(M_0) \\
0 & Z/n \;\cong & Z/n & 0 & 0 & Z/n \;\cong & Z/n & Z/n \;\cong & Z/n \\
H^2(M',M_0) \to & H^2(M') \to & H^2(M_0) \to & H^3(M',M_0) \to & H^3(M') & H^3(M_0) \to & H^4(M',M_0) \to & H^4(M') \to & H^4(M_0) \\
0 & Z/n \;\cong & Z/n & Z/n & Z/n \oplus Z/n & Z/n & 0 & Z/n \;\cong & Z/n
\end{array}$$

$$\begin{array}{ccccccccc}
H^2(W,M) \to & H^2(W) \to & H^2(M) \to & H^3(W,M) \to & H^3(W) \to & H^3(M) \to & H^4(W,M) \to & H^4(W) \to & H^4(M) \\
0 & Z/n \;\cong & Z/n & Z/n \;\cong & Z/n & 0 & 0 & Z/n \;\cong & Z/n \\
H^2(W,M') \to & H^2(W) \to & H^2(M') \to & H^3(W,M') \to & H^3(W) \to & H^3(M') \to & H^4(W,M') \to & H^4(W) \to & H^4(M') \\
0 & Z/n \;\cong & Z/n & 0 & Z/n & Z/n \oplus Z/n & Z/n & Z/n \;\cong & Z/n
\end{array}$$

The images $h(e)$ and $h(\hat{e})$ (which we denote by the same symbols e and $\hat{e}$) under the isomorphism h of the rings $H^{2*}(M;Z/n)$ and $H^{2*}(M';Z/n)$ give a basis of $H^{2*}(M';Z/n)$ which is subject to the same equations (1).

The inclusions $M \xrightarrow{i} W \xleftarrow{j} M'$ give the following relations among the tangent bundles of M, M' and W.

The restrictions of the tangent bundle τ_W of W to the manifolds M and M' are just the pull backs of the tangent bundle of W along the respective inclusions

$$\tau_W|M = i^*(\tau_W) \text{ and } \tau_W|M' = j^*(\tau_W).$$

Taking Pontrjagin classes of these bundles gives

$$p_1(\tau_W|M) = i^*(p_1\tau_W) = i^*(p_1(W))$$

and $$p_1(\tau_W|M') = j^*(p_1\tau_W) = j^*(p_1(W)).$$

But M and M' are boundary components of W, hence $\tau_W|M = \tau_M \oplus \varepsilon^1$ and $\tau_W|M' = \tau_{M'} \oplus \varepsilon^1$ where ε^1 is the trivial line bundle.

Finally, $p_1(\tau_M \oplus \varepsilon^1) = p_1(\tau_M) = p_1(M)$. (Milnor [12], lemma 15.2) which implies $p_1(M) = i^*(p_1(W))$ and similarly $p_1(M') = j^*(p_1(W))$.

We have the following commutative diagram:

$$\begin{array}{ccccc}
Z & & Z & & Z/n \\
\shortparallel & & \shortparallel & & \shortparallel \\
H^4(M;Z) & \xleftarrow[\cong]{i^*} & H^4(W;Z) & \xrightarrow{j^*} & H^4(M';Z) \\
\downarrow r_n & & \downarrow r_n & & \downarrow r_n \\
H^4(M;Z/n) & \xleftarrow[\cong]{i^*} & H^4(W;Z/n) & \xrightarrow[\cong]{j^*} & H^4(M';Z/n)
\end{array}$$

Now $p_1(M) = 4b\hat{e}$ and thus the reduced Pontrjagin class of M, $\bar{p}_1(M) \in H^4(M;Z/n)$

is $r_n(4b)\hat{e}$. Therefore the reduced Pontrjagin class of M' is $\bar{p}_1(M') = r_n(4b)\hat{e} \in H^4(M';Z/n)$.

When n is divisible by 3, say n = 3m, we have the Pontrjagin cubing operation, which we now calculate. At the end of chapter 3 we calculated this operation for a Wall manifold M determined by a framed knot $(a,b) \in FC_3^3$.

With M_o and $M' = M(a,b,n)$ as above we have the following diagram in which all vertical maps are isomorphisms.

$$\begin{array}{ccc}
H^2(M;Z/3m) & \xrightarrow{P_3} & H^6(M;Z/9m) \\
\uparrow & & \uparrow \\
H^2(M,S^2 \times D^4;Z/3m) & \xrightarrow{P_3} & H^6(M,S^2 \times D^4;Z/9m) \\
\downarrow & & \downarrow \\
H^2(M_o, \partial M_o;Z/3m) & \xrightarrow{P_3} & H^6(M_o, \partial M_o;Z/9m) \\
\uparrow & & \uparrow \\
H^2(M',S^3 \times D^3;Z/3m) & \xrightarrow{P_3} & H^6(M',S^3 \times D^3;Z/9m) \\
\downarrow & & \downarrow \\
H^2(M';Z/3m) & \xrightarrow{P_3} & H^6(M';Z/9m)
\end{array}$$

Thus with $e \in H^2(M';Z/3m)$ the image of $e \in H^2(M;Z/3m)$ under the isomorphism h

$$P_3(e) = r_{9m}(6a + b)\ \alpha_{9m}$$

where α_{9m} is the orientation class in

$$H^6(M';Z/9m) \quad (\quad \alpha_{9m} = r_{9m}(\alpha)) .$$

We summarize these results in the following

THEOREM 4.1 *A basis for the augmented ring* R(M(a,b,n)) *of the torsion manifold* M(a,b,n) *may be chosen with*

$$R(M(a,b,n)) = 1Z/n \oplus eZ/n \oplus \hat{e}Z/n \oplus \alpha Z/n$$

where $1 \cup x = x \qquad e \cup \hat{e} = \alpha$

$$e \cup e = r_n(6a + b)\hat{e}$$

$$\bar{p}_1(M(a,b,n)) = r_n(4b)\hat{e}$$

and if $n \equiv 0 \bmod 3$

$$P_3(e) = r_{3n}(6a + b)\ \alpha_{3n}\ .$$

§4.2 SURGERY ON DOUBLE WALL MANIFOLDS

Let N be the double Wall manifold determined by the framed knots $(a_1 b_1)$, $(a_2,b_2) \in FC_3^3$ with linking coefficients $c_2^1 = x$, $c_1^2 = x+2y$. From chapter 2, §4, generators $e_1, e_2 \in H^2(N;Z)$ $\hat{e}_1, \hat{e}_2 \in H^4(N;Z)$ can be chosen so that the cup product structure of N is given by

$$e_i \cup \hat{e}_j = \delta_{ij}\ \alpha \qquad \text{where } \langle \alpha, [M] \rangle = 1$$

$$e_1 \cup e_1 = (6a_1 + b_1)\hat{e}_1 + c_1^2\ \hat{e}_2$$

$$e_1 \cup e_2 = c_1^2\ \hat{e}_1 + c_2^1\ \hat{e}_2$$

$$e_2 \cup e_2 = c_2^1\ \hat{e}_1 + (6a_2 + b_2)\hat{e}_2$$

and the first Pontrjagin class is given by $p_1(M) = 4b_1\hat{e}_1 + 4b_2\hat{e}_2$.

The homology group $H_2(N;Z) = Z \oplus Z$ is generated by two classes S_1^2 and S_2^2 which are Poincaré duals of the classes e_1 and e_2 in $H^4(N;Z)$. The classes S_1^2 and S_2^2 are represented in N by the 2-spheres

$$0 \times \partial D_1^3 \subset D_1^4 \times D_1^3 \qquad \text{and} \qquad 0 \times \partial D_2^3 \subset D_2^4 \times D_2^3$$

where the $D_i^4 \times D_i^3$ are the handles attached to D^7 along the knots (a_i,b_i) to form the handlebody whose boundary is N. Surgery on these two classes in N would yield the 6-sphere S^6.

Now let m and n be relatively prime integers. The class $mS_1^2 + nS_2^2$ in $H_2(N;Z)$ can be represented by an embedded 2-sphere by theorem 1.2 and since $w_2(N) = 0$ it will have trivial normal bundle. Again by theorem 1.2 and the fact that $\pi_2(SO4) = 0$ the isotopy class of the resulting embedding $f : S^2 \times D^2 \longrightarrow N$ is completely determined by the class $mS_1^2 + nS_2^2$. Let $N' = \chi(N,f)$ be the result of surgery on the embedded $S^2 \times D^4$. Since $(m,n) = 1$ the manifold N' will have the same homology as $S^2 \times S^4$. Moreover N' is simply-connected and by the argument used in the proof of theorem 3.1, $w_2(N') = 0$. Thus N' is a Wall manifold and is therefore determined by a framed knot $(a,b) \in FC_3^3$. The remainder of this section is devoted to the proof of the following theorem.

THEOREM 4.2 *Let* N *and* N' *be as above. Then* N' *is determined by the framed knot* $(a,b) \in FC_3^3$ *where*

$$a = a_2 m^3 + b_2\left(\frac{m^3-m}{6}\right) + \frac{nm(n-m)}{2}\, x + n^2 my - a_1 n^3 - b_1\left(\frac{n^3-n}{6}\right)$$

and $b = b_2 m - b_1 n$.

Proof: Let $W = N \times I \cup_f D^3 \times D^4$ where $f : S^2 \times D^4 \longrightarrow N \times \{1\}$ is an embedding representing the class $mS_1^2 + nS_2^2$. Let i and j be the inclusions $i : N \longrightarrow \partial W \subset W$, $j : N' \longrightarrow \partial W \subset W$ of N and N' into W. From the exact sequence of the pair (W,N)

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_3(W,N) & \longrightarrow & H_2(N) & \xrightarrow{i*} & H_2(W) & \longrightarrow & H_2(W,N) \\ & \| & & \| & & \| & & \| \\ & Z & & Z\oplus Z & & Z & & 0 \end{array}$$

we have $i_*(mS_1^2 + nS_2^2) = 0$. Since $(m,n) = 1$ then there exist integers k,ℓ with $km + \ell n = 1$ and then $\{mS_1^2 + nS_2^2, -\ell S_1^2 + kS_2^2\}$ form a basis for $H_2(N;Z)$. Let $V_1^4, V_2^4 \in H_4(N;Z)$ be classes dual to S_1^2 and S_2^2 respectively. That is $V_i^4 \cdot S_j^2 = \delta_{ij}$ where $\cdot$ is the intersection number (cf. Dold [4]). Then the classes $kV_1^4 + \ell V_2^4$ and $-nV_1^4 + mV_2^4$ in $H_4(N;Z)$ form a dual base to $mS_1^2 + nS_2^2$ and $-\ell S_1^2 + kS_2^2$.

For the pair (W,N') we have

$$0 \longrightarrow H_3(W,N') \longrightarrow H_2(N') \xrightarrow{j^*} H_2(W) \longrightarrow H_2(W,N')$$
$$\begin{array}{cccc} \| & \| & \| & \| \\ 0 & Z & Z & 0 \end{array}$$

and hence j* is an isomorphism. If we set $i_*(-\ell S_1^2 + kS_2^2) = \bar{S}^2 \in H_2(W)$ and $S^2 = j_*^{-1}(\bar{S}^2)$, then S^2 is a generator for the group $H_2(N') = Z$. Again from the exact sequences of the pairs (W,N) and (W,N')

$$0 \longrightarrow H_4(N) \xrightarrow{i^*} H_4(W) \longrightarrow H_4(W,N)$$
$$\begin{array}{ccc} \| & \| & \| \\ Z \oplus Z & Z \oplus Z & 0 \end{array}$$

$$0 \longrightarrow H_4(N') \xrightarrow{j^*} H_4(W) \xrightarrow{h^*} H_4(W,N') \longrightarrow H_3(W)$$
$$\begin{array}{cccc} \| & \| & \| & \| \\ Z & Z \oplus Z & Z & 0 \end{array}$$

we obtain $H_4(N') = Z$. If $V^4 \in H_4(N')$ is the generator dual to $S^2 \in H_2(N')$ (that is $V^4 \cdot S^2 = +1$ in N' where N' is oriented by way of the cobordism W) then $j_*(V^4) = n\bar{V}_1^4 - m\bar{V}_2^4$ where $\bar{V}_i^4 = i_*(V_i^4)$, $i = 1,2$. To see this notice that with this choice of sign for the image of V^4 under j_* we obtain $i_*^{-1} j_*(V^4) = nV_1^4 - mV_2^4 \in H_4(N)$. On the other hand $j_*^{-1} i_*(-\ell S_1^2 + kS_2^2) = S^2 \in H_2(N')$ Since N and N' have opposite local orientations the two classes $-\ell S_1^2 + kS_2^2$ and $nV_1^4 - mV_2^4$ must have intersection number -1 in N. Since $(-\ell S_1^2 + kS_2^2) \cdot (nV_1^4 - mV_2^4) = -\ell n - km = -1$, $j_*(V^4) = n\bar{V}_1^4 - m\bar{V}_2^4$.

Thus we have the following maps

$$H_2(N) \xrightarrow{i^*} H_2(W) \xleftarrow{j^*} H_2(N')$$

$$H_4(N) \xrightarrow{i^*} H_4(W) \xleftarrow{j^*} H_4(N')$$

with $\quad i_*(-\ell S_1^2 + kS_2^2) = \bar{S}^2 , \quad j_*(S^2) = \bar{S}^2,$

$$i_*(mS_1^2 + nS_2^2) = 0 \ ,$$

$$i_*(V_1^4) = \bar{V}_1^4 \quad , \qquad i_*(V_2^4) = \bar{V}_2^4 \quad , \qquad j_*(V^4) = n\bar{V}_1^4 - m\bar{V}_2^4 \ ,$$

which implies $i_*(S_1^2) = -n\bar{S}^2$ and $i_*(S_2^2) = m\bar{S}^2$.

Since the homology groups of N, W and N' are torsion free, the universal coefficient theorem implies that we have the following maps for the dual groups

$$H^i(K) = \mathrm{Hom}(H_i(K),Z) \text{ for } K = N,\ N',\ W$$

$$H^2(N) \xleftarrow{i^*} H^2(W) \xrightarrow{j^*} H^2(N')$$

$$H^4(N) \xleftarrow{i^*} H^4(W) \xrightarrow{j^*} H^4(N') \ .$$

Let $e_1, e_2 \in H^2(N)$ be a dual base in cohomology to S_1^2 and S_2^2, $\bar{e} \in H^2(W)$ dual to $\bar{S}^2$, $e \in H^2(N)$ dual to S^2, $\hat{e}_1, \hat{e}_2 \in H^4(N)$ dual to V_1^4, V_2^4, $\tilde{e}_1, \tilde{e}_2 \in H^4(W)$ dual to $\bar{V}_1^4, \bar{V}_2^4$ and $\hat{e} \in H^4(N')$ dual to V^4.

Then the maps i^* and j^* are determined by applying the functor $\mathrm{Hom}(-,Z)$. They are given by:

$$i^*(\bar{e}) = -ne_1 + me_2 \ ,$$

$$i^*(\tilde{e}_1) = \hat{e}_1 \ , \quad i^*(\tilde{e}_2) = \hat{e}_2 \ ,$$

$$j^*(\bar{e}) = e \quad , \quad j^*(\tilde{e}_1) = n\hat{e} \quad , \quad j^*(\tilde{e}_2) = -m\hat{e} \ .$$

The maps i^* and j^* in cohomology are multiplicative, thus

$$i^*(\bar{e}^2) = (i^*(\bar{e}))^2 = (-ne_1 + me_2)^2 = n^2 e^2_{\ 1} - 2nme_1e_2 + m^2 e_2^2$$

$$= n^2((6a_1 + b_1)\hat{e}_1 + c_1^2\hat{e}_2) - 2nm(c_1^2\hat{e}_1 + c_2^1\hat{e}_2)$$

$$+ m^2(c_2^1\hat{e}_1 + (6a_2 + b_2)\hat{e}_2) \ .$$

Applying i^{*-1} to both sides of this equation gives

$$\bar{e}^2 = n^2(6a_1 + b_1)\tilde{e}_1 + n^2c_1^2\tilde{e}_2 - 2nmc_1^2\tilde{e}_1 - 2nmc_2^1\tilde{e}_2$$
$$+ m^2c_2^1\tilde{e}_1 + m^2(6a_2 + b_2)\tilde{e}_2 \ .$$

Therefore $e^2 = (j^*\bar{e})^2 = j^*(\bar{e}^2)$

$$= n^2(6a_1 + b_1)n\hat{e} + n^2c_1^2(-m\hat{e}) - 2nmc_1^2 n\hat{e}$$
$$- 2nmc_2^1(-m\hat{e}) + m^2c_2^1\, n\hat{e} + m^2(6a_2 + b_2)(-m\hat{e})$$
$$= \{n^3(6a_1 + b_1) - 3n^2mc_1^2 + 3nm^2c_2^1 - m^3(6a_2 + b_2)\}\hat{e} \ .$$

The Pontrjagin class of W satisfies $i^*(p_1(W)) = p_1(N)$ giving $p_1(W) = i^{*-1}p_1(N)$

$$= i^{*-1}(4b_1\hat{e}_1 + 4b_2\hat{e}_2) = 4b_1\tilde{e}_1 + 4b_2\tilde{e}_2 \ .$$

Thus $\quad p_1(N') = j^*(p_1(W)) = j^*(4b_1\tilde{e}_1 + 4b_2\tilde{e}_2) = (4nb_1 - 4mb_2)\hat{e}$.

The classes e and $\hat{e}$ in $H^2(N')$ and $H^4(N')$ have cup product $e \cup \hat{e} = \alpha$ where α is the orientation class of N' determined by the orientation on the cobordism W between N and N', chosen so that the induced orientation on N is the given one. Thus the invariants of N', with the orientation induced from N by surgery, are given in the basis $e \in H^2(N')$ and $\hat{e} \in H^4(N')$ by

$$\langle e \cup \hat{e}, [N'] \rangle = 1 \ ,$$
$$e \cup e = \{m^3(6a_2 + b_2) - 3nm^2c_2^1 + 3n^2mc_1^2 - n^3(6a_1 + b_1)\}\hat{e} \ ,$$
$$p_1(N') = (4mb_2 - 4nb_1)\hat{e} \ .$$

Now N' is a Wall manifold and is therefore determined by a framed knot $(a,b) \in FC_3^3$. From chapter 2, §4, the cup product and Pontrjagin class of N' are given by

$$e \cup e = (6a + b)\,\hat{e} \qquad \text{and} \qquad p_1(N') = 4b\hat{e} \ .$$

Setting $c_2^1 = x$ and $c_1^2 = x + 2y$ and solving for a and b we obtain:

$$a = a_2 m^3 + b_2\left(\frac{m^3-m}{6}\right) + nm\frac{(n-m)}{2}x + n^2my - a_1n^3 - \left(\frac{n^3-n}{6}\right)b_1 ,$$

$$b = mb_2 - nb_1 .$$

CHAPTER 5

§5.1 A DIFFEOMORPHISM CRITERION

In section 4.2 surgery was done on the double Wall manifold N determined by the framed knots (a_1,b_1), (a_2,b_2) and linking coefficients $c_2^1 = x$ and $c_1^2 = x + 2y$. If, in the same manifold N, we do surgery on the disjoint 2-spheres S_1^2 and nS_2^2 then the resulting manifold will be a torsion manifold. In fact surgery on the 2-sphere S_1^2 just cancels the surgery on the S^3 in S^6 corresponding to the framed knot (a_1,b_1), and the result is the Wall manifold determined by the framed knot (a_2,b_2). Following this by surgery on the class nS_2^2 in this Wall manifold yields the torsion manifold $M(a_2,b_2,n)$.

Now let m be relatively prime to n and consider surgery on the following classes in N;

$$\bar{S}_1^2 = S_1^2 ,$$

$$\bar{S}_2^2 = mS_1^2 + nS_2^2 .$$

By the handle addition theorem (see §1.2), the result of this surgery is diffeomorphic to the torsion manifold $M(a_2,b_2,n)$ obtained by surgery on the classes S_1^2 and nS_2^2. Thus we can write

$$\chi(N;\ \bar{S}_1^2,\ \bar{S}_2^2) \cong \chi(N;\ S_1^2,\ nS_2^2) \cong M(a_2,b_2,n).$$

Performing the surgery $\chi(N;\ \bar{S}_1^2,\ \bar{S}_2^2)$, first on $\bar{S}_2^2$, and then on $\bar{S}_1^2$, we see that the intermediate manifold $M = \chi(N;\ \bar{S}_2^2)$ is a Wall manifold which, by theorem 4.2, is determined by the framed knot (a,b) where

$$a = a_2m^3 + b_2\left(\frac{m^3-m}{6}\right) + nm\left(\frac{n-m}{2}\right) x + n^2my - a_1n^3 - b_1\left(\frac{n^3-n}{6}\right) \qquad (*)$$

and $$b = b_2m - b_1n.$$

The image of the class $\bar{S}_1^2$ in $H_2(M) = Z$ is, by the previous section, n-times a generator of this group, and surgery on this class yields the torsion manifold $M(a,b,n)$.

This proves the following

THEOREM 5.1 *If the framed knots* (a,b) *and* (a_2,b_2) *satisfy equations* (*) *for integers* n,m,a_1,b_1,x *and* y *with* $(m,n) = 1$, *then the torsion manifolds* $M(a,b,n)$ *and* $M(a_2,b_2,n)$ *are diffeomorphic.*

In the next section we apply theorem 5.1 to classify certain 6-dimensional torsion manifolds.

§5.2 CLASSIFICATION OF TORSION MANIFOLDS

Before proving our classification theorem we prove a number of lemmas which follow directly from theorem 5.1.

LEMMA 1. *Let* k *be any integer, then*

(a) *if* n *is odd* $M(a,b,n)$ *is diffeomorphic to* $M(a + kn,b,n)$,

(b) *if* n *is even* $M(a,b,n)$ *is diffeomorphic to* $M(a + k\frac{n}{2},b,n)$.

Proof: Writing equations (*) with $m = 1, a_1 = 0$ and $b_1 = 0$ we have,

$$a = a_2 + n\left(\frac{n-1}{2}\right) x + n^2y ,$$

$$b = b_2 .$$

If n is odd then

$$a = a_2 + n[(\frac{n-1}{2})\ x + ny]$$

and since g.c.d. $\{(\frac{n-1}{2}), n\} = 1$ we can choose x and y so that $(\frac{n-1}{2})\ x + ny = -k$. Then with $a_2 = a + kn$ part (a) follows.

If n is even then

$$a = a_2 + \frac{n}{2}\ [(n-1)\ x + 2ny]$$

and again g.c.d. $\{(n-1), 2n\} = 1$ so we can choose x and y with

$$(n-1)\ x + 2ny = -k\ .$$

Setting $a_2 = a + k\frac{n}{2}$ gives part (b). □

LEMMA 2. *If* $n \not\equiv 0$ *mod* 3 *then for any integer* ℓ, $M(a,b,n)$ *is diffeomorphic to* $M(a,b + \ell n,n)$.

Proof: In equations (*) take $b_1 = \ell$, $a_1 = 0$, $m = 1$ and $y = 0 = x$ to obtain

$$a = a_2 - \ell(\frac{n^3-n}{6}) = \begin{cases} a_2 - \ell(\frac{n^2-1}{6})\ n & \text{if } n \text{ is odd} \\ a_2 - \ell(\frac{n^2-1}{3})\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases},$$

$$b = b_2 - \ell n\ .$$

Thus $M(a,b,n) \cong M(a + \ell(\frac{n^2-1}{6})\ n,\ b + \ell n, n)$ if n is odd

and $M(a,b,n) \cong M(a + \ell(\frac{n^2-1}{3})\ \frac{n}{2},\ b + \ell n, n)$ if n is even.

In either case by lemma 1

$$M(a,b,n) \cong M(a,\ b + \ell n, n).$$ □

LEMMA 3. *If* m *is relatively prime to* n *and*

$$a = m^3 a' + \frac{m^3-m}{6} b',$$

$$b = mb',$$

then $M(a,b,n) \cong M(a',b',n)$.

Proof: Write equations (*) with $a_2 = a'$, $b_2 = b'$, $a_1 = b_1 = x = y = 0$. □

LEMMA 4. *Suppose* n *is odd and* $n \equiv 0$ *mod* 3.

If $a = a' + \ell\frac{n}{3}$ *and* $2\ell + \ell' \equiv 0$ *mod* 3,

$$b = b' + \ell' n,$$

then $M(a,b,n) \cong M(a',b',n)$.

Proof: Since $2\ell + \ell' \equiv 0 \bmod 3$ then $\ell \equiv \ell' \bmod 3$. Let $\ell = \ell' + 3k$. Then the equations above become

$$a = a' + kn + \ell'\frac{n}{3},$$

$$b = b' + \ell' n.$$

In equations (*) let $m = 1$, $b_1 = -\ell'$, $a_1 = 0$ and choose x and y so that $(\frac{n-1}{2})x + ny = k - \ell'(\frac{n^2-3}{6})$ yielding (with $a_2 = a'$, $b_2 = b'$)

$$a = a' + n(k - \ell'(\frac{n^2-3}{6})) + \ell'(\frac{n^3-n}{6}),$$

$$b = b' + \ell' n$$

and thus by theorem 5.1, $M(a,b,n) \cong M(a',b',n)$. □

LEMMA 5. *Let* $n \equiv 2$ *mod* 4 *and* $n \equiv 0$ *mod* 3.

If $a = b' + (3k - \ell)\frac{n}{6}$

and $b = b' + \ell n$

then $M(a,b,n)$ *is diffeomorphic to* $M(a',b',n)$.

Proof: We consider two cases depending on the parity of $3k - \ell$.

Case 1: $3k - \ell = 2r$ or $\ell = 3k - 2r$.

In equations (*) put $m = 1$, $b_1 = 2r - 3k$, $a_1 = 0$ and choose x and y so that

$$(n-1)\, x + 2ny = Z$$

where $$Z = \frac{-3kn^2 + 2rn^2 + 3k}{3}.$$

Then equations (*) with $a' = a_2$, $b' = b_2$,

become

$$a = a' + \frac{n}{2}\left(\frac{-3kn^2 + 2rn^2 + 3k}{3}\right) - (2r - 3k)\left(\frac{n^3-n}{6}\right),$$

$$b = b' + \ell n$$

which are the equations of the lemma.

Therefore $M(a,b,n) \cong M(a',b',n)$.

Case 2: $3k - \ell = 2r + 1$ or $\ell = 3k - 2r - 1$.

In equations (*) put $m = 1$, $b_1 = -3k + 2r + 1$,

$a_1 = 0$ and choose x and y so that

$$(n-1)\, x + 2ny = Z$$

where $$Z = \frac{3k - 3kn^2 + 2rn^2 + n^2}{3}.$$

With $a_2 = a'$ and $b_2 = b'$ equations (*) become

$$a = a' + \frac{n}{2}\left(\frac{3k - 3kn^2 + 2rn^2 + n^2}{3}\right)$$

$$+ (3k - 2r - 1)\left(\frac{n^3-n}{6}\right),$$

$$b = b' + \ell n$$

which reduce to the equations of the lemma.

Thus $M(a,b,n) \cong M(a',b',n)$. □

To a given oriented torsion manifold M, we have associated the augmented ring R(M) described in section 3.2, and if n is divisible by 3 where $H_2(M;Z) = Z/n$, we have the Pontrjagin cubing operation $P_3 : H^2(M;Z/n) \longrightarrow H^6(M;Z/3n)$, which is determined by its value on a generator of $H^2(M;Z/n)$. Our main theorem shows that these invariants are sufficient to determine the oriented diffeomorphism class of M if $n \not\equiv 0 \bmod 4$.

THEOREM 5.2 *If* $n \not\equiv 0$ *mod* 4 *then two oriented torsion manifolds* M *and* M' *are diffeomorphic by an orientation-preserving diffeomorphism if and only if, (a) there is an isomorphism between their augmented rings*

$$f : R(M) \longrightarrow R(M') \qquad \textit{and}$$

(b) if $n \equiv 0$ *mod* 3 *the Pontrjagin cubing operations of* M *and* M' *satisfy*

$$< P_3(e), [M] > = < P_3(f(e)), [M'] > .$$

where e *is a generator of* $H^2(M;Z/n)$.

Proof: Since the invariants described are invariants of oriented diffeomorphism type, these conditions are necessary.

Suppose now that $f : R(M) \longrightarrow R(M')$ is an isomorphism in the sense of section 3.2. We may assume by theorem 3.1, that $M = M(a,b,n)$ and $M' = M(a',b',n)$. Theorem 4.1 then implies that R(M) is determined by the reduced integers $r_n(6a + b)$ and $r_n(4b)$ while R(M') is determined by $r_n(6a' + b')$ and $r_n(4b')$. From theorem 3.2 the isomorphism f implies the existence of an integer m, relatively prime to n, such that

$$6a + b \equiv m^3(6a' + b') \quad \text{mod } n \qquad (1)$$

$$\text{and} \quad 4b \equiv m4b' \quad \text{mod } n \qquad (2).$$

Moreover if 3 divides n then the Pontrjagin cubing operation yields the congruence $6a + b \equiv m^3(6a' + b') \bmod 3n$. (3)

We complete our proof by showing that these modular relations imply integral relations which have the form of equations (*). It then follows from theorem 5.1 that $M(a,b,n)$ is diffeomorphic to $M(a',b',n)$

We consider separately four cases.

Case 1: n odd and $n \not\equiv 0 \bmod 3$.

Then we have from (1) and (2)

$$6a + b + kn = m^3(6a' + b'),$$

$$4b + \ell n = m4b'.$$

Since n is odd, $4|\ell$, say $\ell = 4\ell'$;

thus $$b + \ell' n = mb'.$$

Substituting gives

$$6a + (k - \ell')n = 6m^3a' + b'(m^3 - m).$$

Now $6 \mid m^3 - m$ so $6 \mid (k - \ell')$ yielding

$$a + \left(\frac{k-\ell'}{6}\right)n = m^3a' + b'\left(\frac{m^3-m}{6}\right),$$

$$b + \ell' n = mb'.$$

By lemma 3, $M(a + (\frac{k-\ell'}{6})n,\ b + \ell'n, n) \cong M(a',b',n)$ and by lemmas 1 and 2

$$M(a,b,n) \cong M(a + (\tfrac{k-\ell'}{6})n,\ b + \ell'n, n).$$

Therefore $M(a,b,n) \cong M(a',b',n)$.

Case 2: $n \equiv 2 \mod 4$ and $n \not\equiv 0 \mod 3$,

$$6a + b + kn = m^3(6a' + b'),$$

$$4b + \ell n = 4b'.$$

Since n is even the first equation yields $b \equiv b' \mod 2$, which implies

$$4(b' + 2h) + \ell n = m4b',$$

$$8h + \ell n = 4b'(m-1),$$

$$4h + \ell\frac{n}{2} = 2b'(m-1),$$

from which it follows that $\ell \equiv 0 \mod 4$,

say $\ell = 4\ell'$.

Therefore $b + \ell' n = b'$ and substituting in (1) gives

$$6a + (k - \ell')n = m^3 6a' + b'(m^3 - m)$$

whence $k - \ell'$ is divisible by 3

yielding $$a + \left(\frac{k-\ell'}{3}\right)\frac{n}{2} = m^3 a' + b'\left(\frac{m^3-m}{6}\right).$$

$$b + \ell' n = b'.$$

Thus $$M\left(a + \left(\frac{k-\ell'}{3}\right)\frac{n}{2},\ b + \ell' n, n\right) \cong M(a', b', n)$$

by lemma 3, and by lemmas 1 and 2

$$M(a,b,n) \cong M\left(a + \left(\frac{k-\ell'}{3}\right)\frac{n}{2},\ b + \ell' n, n\right).$$

Therefore $M(a,b,n) \cong M(a',b',n)$.

Case 3: n odd, $n \equiv 0 \mod 3$.

As in the first case equation (2) becomes

$$b + \ell' n = b'm.$$

From the congruence (3) (since $n \equiv 0 \mod 3$) we have

$$6a + b + k3n = m^3(6a' + b').$$

Substituting for b yields

$$6a + n(3k - \ell') = m^3\, 6a' + b'(m^3 - m)$$

which implies that $3k - \ell' = 2\ell$.

We have

$$a + \frac{\ell n}{3} = m^3 a' + b'(\frac{m^3-m}{6}),$$

$$b + \ell' n = mb'$$

and $$2\ell + \ell' \equiv 0 \mod 3 .$$

By lemma 4, $M(a,b,n) \cong M(a + \frac{\ell n}{3}, b + \ell' n, n)$

and lemma 3 implies

$$M(a + \frac{\ell n}{3}, b + \ell' n, n) \cong M(a',b',n) ;$$

thus $$M(a,b,n) \cong M(a',b',n).$$

Case 4: $n \equiv 2 \mod 4$ and $n \equiv 0 \mod 3$.

Equations (3) and (2) become

$$6a + b + k3n = m^3(6a' + b'),$$

$$4b + \ell n = m4b' .$$

As in case 2 we obtain

$$b + \ell' n = mb'.$$

Substituting for b in the first equation, we obtain

$$6a + (3k - \ell')n = m^3 6a' + b'(m^3 - m),$$

$$a + (3k - \ell')\frac{n}{6} = m^3 a' + b'(\frac{m^3-m}{6}),$$

$$b + \ell' n = mb' .$$

By lemma 3

$$M(a + (3k - \ell')\frac{n}{6}; b + \ell' n, n) \cong M(a',b',n)$$

and by lemma 5

$$M(a + (3k - \ell')\frac{n}{6}, b + \ell'n, n) \cong M(a,b,n),$$

yielding $M(a,b,n) \cong M(a',b',n)$. □

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