ELECTRON SCATTERING FROM NUCLEI

EIKONAL DISTORTED WAVE APPROXIMATION FOR HIGH ENERGY ELECTRON SCATTERING FROM SPHERICAL NUCLEI

By

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ABSTRACT

A detailed review of an eikonal approximation for high energy electrons in the Coulomb potential of a spherical nucleus is given. The matrix element for elastic scattering making use of this approximate distorted wave is then discussed. The matrix element is written in the form of a simple differential operator acting on the first Born amplitude. A possible method of evaluation for arbitrary nuclear charge densities is discussed and applied to the calculation of scattering cross sections for Ca, Ni, Sn, and Pb with realistic Fermi shapes.

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1. Introduction

The simplicity and physical appeal of the first Born approximation in scattering problems have made it a very useful and popular method. Unfortunately, its limitations in the case of electron scattering from heavy nuclei are well known. For example, in order to calculate accurate elastic cross sections for spherical nuclei, it is necessary to perform an "exact" phase shift analysis. For high electron energies (i.e. much greater than the electron rest energy) one of the earliest and best known analyses of this type was made by Yennie, Ravenhall, and Wilson (1954).

However, in attempting to treat more complicated problems, such as inelastic processes, scattering from deformed potentials, or magnetic interactions, the "exact" methods become extremely cumbersome. Here the simplicity of the Born approximation is of great importance. Therefore, it is interesting to attempt an improvement of the first Born approximation in such a manner that its accuracy is increased without destroying its basic simplicity.

One of the most successful attempts of this sort was made by Yennie, Boos, and Ravenhall (1965), who treated high energy elastic and inelastic scattering using an analytic distorted wave approximation. Their method gives

results which display excellent agreement with phase shift calculations. Specific application to Fermi charge densities was made by Petkov, Luk'yanov, and Pol' (1967) and utilized for scattering analysis by Shevchenko et.al. (1967). More general charge distributions were treated by Luk'yanov, Petkov, and Pol' (1969).

In this work, the analytic distorted wave approximation of Yennie is applied to the calculation of elastic electron scattering cross sections for spherical nuclear charge distributions of arbitrary shape. The method employed was inspired by the work of Knoll (1974). The main feature of the present approach is the transformation of the distorted wave matrix element into a simple differential operator acting on the first Born amplitude, and evaluated with an effective wave number k which is slightly larger than the free wave number k_0. This simple redefinition of the effective wave number already effects a tremendous improvement in the first Born approximation, a fact which had been recognized empirically long before its theoretical justification.

A detailed description of the eikonal-type distorted waves introduced by Yennie is given in sections 2 to 6. Then, the differential operator which yields the distorted wave scattering amplitude from the Born amplitude is discussed in sections 7 and 8. Finally, the accuracy of the approximation is compared with "exact" phase shift calculations in section 9.

2. High Energy Dirac Equation

The motion of an electron in the static Coulomb potential $V(\underset{\sim}{r})$ is given by the Dirac equation,

$$(\underset{\sim}{\alpha}\cdot\underset{\sim}{p}c + \beta mc^2 + V)\psi = E\,\psi \tag{2.1}$$

where $\psi(\underset{\sim}{r})$ is a four-component wave function, and the matrices $\underset{\sim}{\alpha}$ and β are chosen to be

$$\underset{\sim}{\alpha} = \begin{bmatrix} \underset{\sim}{\sigma} & 0 \\ 0 & -\underset{\sim}{\sigma} \end{bmatrix} \qquad \beta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{2.2}$$

This particular representation is very convenient in the case of electron energies much larger than the rest energy mc^2. If the mass term in (2.1) is negligibly small, then the Dirac equation simplifies to the following pair of uncoupled two-component equations;

$$(\underset{\sim}{\sigma}\cdot\underset{\sim}{p}c + V - E)\chi^{(1)} = 0 \tag{2.3a}$$

$$(-\underset{\sim}{\sigma}\cdot\underset{\sim}{p}c + V - E)\chi^{(2)} = 0. \tag{2.3b}$$

Furthermore, if the potential $V(\underset{\sim}{r})$ is spherically symmetric, then solutions of (2.3b) can be obtained from solutions of (2.3a) by a simple coordinate inversion. This implies that both equations will yield identical scattering solutions for any given initial momentum. Therefore, all further discussion is restricted to the following two-component

Dirac equation for high energy electrons;

$$(\underset{\sim}{\sigma}\cdot pc + V - E)\phi = 0. \tag{2.4}$$

Free plane wave solutions of (2.4) have the form

$$\phi = ve^{ik_0\cdot r} \tag{2.5}$$

where the free spinor $v(k)$ is given by

$$\underset{\sim}{\sigma}\cdot(\hbar k_0 c)v = Ev. \tag{2.6}$$

Using $E = pc = \hbar k_0 c$ for high energy electrons, (2.6) can be rewritten as

$$\underset{\sim}{\sigma}\cdot\hat{k}v = v. \tag{2.7}$$

Given the spherical angles (θ,ϕ) of the unit vector $\hat{k}$, (2.7) has the normalized solution

$$v = \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \; e^{i\phi} \end{bmatrix}. \tag{2.8}$$

3. The Eikonal Approximation

In order to account for distortions of the plane wave solution due to the potential $V(r)$, we assume an approximate solution of the form

$$\phi = u(\underset{\sim}{r})e^{iS(\underset{\sim}{r})} \tag{3.1}$$

where the spinor function $u(\underset{\sim}{r})$ plays the role of a slowly varying amplitude function.

Substitution of (3.1) into (2.4) yields the following expression for $u(\underset{\sim}{r})$

$$(\underset{\sim}{\sigma}\cdot\underset{\sim}{\nabla}S + V - E)u = i\underset{\sim}{\sigma}\cdot\underset{\sim}{\nabla}u \tag{3.2}$$

(where natural units are employed such that $\hbar c = 1$).

Under the assumption that $u(\underset{\sim}{r})$ is a slowly varying function, an approximation for (3.2) can be made by setting $\underset{\sim}{\nabla}u = 0$. This gives a simple homogeneous equation for $u(\underset{\sim}{r})$;

$$(\underset{\sim}{\sigma}\cdot\underset{\sim}{\nabla}S + V - E)u = 0. \tag{3.3}$$

Multiplication of (3.3) by $(\underset{\sim}{\sigma}\cdot\underset{\sim}{\nabla}S - V + E)$ then yields the following condition for the existence of a non-trivial $u(\underset{\sim}{r})$;

$$(\underset{\sim}{\nabla}S)^2 = (E - V)^2 . \tag{3.4}$$

Having found a phase function $S(\underset{\sim}{r})$ satisfying this condition, it is obvious that the solution of (3.3) can be

written in the form

$$u = f(\underset{\sim}{r})(\underset{\sim}{\sigma}\cdot\underset{\sim}{\nabla}S + E - V)v \tag{3.5}$$

where $f(\underset{\sim}{r})$ and the spinor v are arbitrary.

For convenience, the arbitrary spinor v is chosen to be the free spinor for the incident direction $\hat{\underset{\sim}{k}}$, as discussed in the previous section. Furthermore, $f(\underset{\sim}{r})$ is given the form

$$f(\underset{\sim}{r}) = N(\underset{\sim}{r})[2(E-V)]^{-1} . \tag{3.6}$$

Therefore, defining the unit vector $\hat{\underset{\sim}{n}}$ in the direction of $\underset{\sim}{\nabla}S$, the spinor function $u(\underset{\sim}{r})$ becomes

$$u = \frac{N}{2}(\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{n}} + 1)v \tag{3.7}$$

where

$$\hat{\underset{\sim}{n}} = \frac{\underset{\sim}{\nabla}S}{(E-V)} . \tag{3.8}$$

Notice that the form of $f(\underset{\sim}{r})$ has been chosen in such a manner that the normalization function $N(\underset{\sim}{r})$ simply equals unity in the limiting case of a free particle.

The remaining arbitrariness of $N(\underset{\sim}{r})$ is removed by the current conservation condition,

$$\underset{\sim}{\nabla}\cdot(u^{\dagger}\underset{\sim}{\sigma}u) = 0. \tag{3.9}$$

Substituting (3.7) into (3.9) it can be shown that the normalization function $N(\underset{\sim}{r})$ satisfies the following

equation,

$$\underset{\sim}{\nabla}\cdot[\hat{\underset{\sim}{n}}\,\frac{N^2}{2}\,(\hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{n}}+1)]=0\ . \qquad (3.10)$$

4. An Approximation for S(r)

The phase function $S(\underset{\sim}{r})$ satisfies the equation

$$(\underset{\sim}{\nabla} S)^2 = (E - V)^2 \quad . \tag{4.1}$$

The parameter characterizing the strength of the potential energy V(r) in this equation is $\gamma = Z\alpha$, where α is the fine-structure constant $e^2/\hbar c$. We write $S(\underset{\sim}{r})$ as a power series in γ of the form

$$S = S(0) + \underset{\sim}{k}_0 \cdot \underset{\sim}{r} + S_1(\underset{\sim}{r}) + S_2(\underset{\sim}{r}) + \ldots \tag{4.2}$$

where $S_i(\underset{\sim}{r})$ is proportional to γ^i and vanishes at the origin.

Substituting (4.2) into (4.1) gives the following expressions for the lowest order terms;

$$\begin{aligned} \hat{\underset{\sim}{k}} \cdot \underset{\sim}{\nabla} S_1 &= -V \\ 2\underset{\sim}{k}_0 \cdot \underset{\sim}{\nabla} S_2 &= V^2 - (\underset{\sim}{\nabla} S_1)^2 \\ \underset{\sim}{k}_0 \cdot \underset{\sim}{\nabla} S_3 &= -\underset{\sim}{\nabla} S_1 \cdot \underset{\sim}{\nabla} S_2 \end{aligned} \tag{4.3}$$

etc. .

In this work only the first order correction will be considered, and terms of second or higher order in γ are neglected. We proceed, then, with the calculation of $S_1(\underset{\sim}{r})$. In order to simplify subsequent manipulations, the z-axis is oriented along $\hat{\underset{\sim}{k}}$, and the perpendicular distance to the

z-axis is denoted p.

$S_1(z,p)$ is determined by the equation

$$\frac{\partial S_1}{\partial z} = -V \qquad , \qquad (4.4)$$

which is obviously satisfied by

$$S_1 = -\int_{z_0}^{z} V(\sqrt{\lambda^2+p^2})\,d\lambda + A(p) \qquad (4.5)$$

where the lower limit z_0 and the arbitrary function $A(p)$ must be chosen to satisfy appropriate boundary conditions. For an incoming electron, $S(\underset{\sim}{r})$ must approach a free particle solution at $z = -\infty$. Therefore, recalling that $S_i(\underset{\sim}{r})$ vanishes at the origin,

and
$$S_1(0,0) = 0 \qquad (4.6)$$
$$S_1(-\infty,p) = \text{Constant.}$$

The boundary conditions (4.6) are satisfied by

$$S_1 = -\int_{-\infty}^{z} V(\sqrt{\lambda^2+p^2})\,d\lambda + \int_{-\infty}^{0} V(\lambda)\,d\lambda \qquad (4.7)$$

which can be rewritten as

$$S_1 = S_1^{(1)} + S_1^{(2)} \qquad (4.8)$$

$$S_1^{(1)} = -\int_{0}^{z} V(\sqrt{\lambda^2+p^2})\,d\lambda \qquad (4.9)$$

$$S_1^{(2)} = \int_0^\infty [V(\lambda) - V(\sqrt{\lambda^2+p^2})]d\lambda. \qquad (4.10)$$

In order to evaluate the integrals (4.9) and (4.10) we adopt the procedure of expanding $S(\underset{\sim}{r})$ in powers of $\underset{\sim}{r}$. This, of course, restricts the validity of the resulting wave function to the region near the origin. It will be shown, however, that this is not a serious shortcoming.

For a typical nuclear charge distribution roughly uniform at the centre, the potential energy can be expanded in the form

$$V(r) = V(0) + \frac{1}{2} ak^3r^2 + \ldots \qquad (4.11)$$

where the constant k is introduced for dimensional convenience and will be defined shortly. For example, in the case of a uniform charge distribution of radius R,

$$V(0) = -\frac{3\gamma}{2R} \quad , \quad a = \frac{\gamma}{(kR)^3} . \qquad (4.12)$$

Using the form (4.11) for V(r), $S_1^{(1)}$ is easily integrated;

$$S_1^{(1)} = -V(0)z - \frac{1}{2} a(k^3p^2z + \frac{1}{3} k^3z^3). \qquad (4.13)$$

Expanding the integrand of $S_1^{(2)}$ in powers of p^2 gives

$$S_1^{(2)} = -\frac{1}{2} p^2 \int_0^\infty \frac{1}{\lambda} \frac{\partial V}{\partial \lambda} d\lambda$$

$$- \frac{1}{8} p^4 \int_0^\infty \left(\frac{1}{\lambda^2} \frac{\partial^2 V}{\partial \lambda^2} - \frac{1}{\lambda^3} \frac{\partial V}{\partial \lambda}\right) d\lambda \qquad (4.14)$$

$$+ \ldots$$

$$= -b(kp)^2 + c(kp)^4 + \ldots \; . \qquad (4.15)$$

Equation (4.15) defines the parameters b and c.

The potential energy $V(\lambda)$ is related to the normalized nuclear charge distribution $\rho(\lambda)$ by the expression

$$V(\lambda) = -4\pi\gamma \left[\frac{1}{\lambda} \int_0^\lambda x^2 \rho dx + \int_\lambda^\infty x\rho dx\right] \; . \qquad (4.16)$$

Using (4.16) in (4.14), it can be shown that

$$b = \frac{\pi\gamma}{k^2} \int_0^\infty \rho(\lambda) d\lambda \qquad (4.17)$$

$$c = -\frac{1}{8} \frac{\pi\gamma}{k^4} \int_0^\infty \frac{1}{\lambda} \frac{\partial \rho}{\partial \lambda} d\lambda \; . \qquad (4.18)$$

For the uniform charge distribution, for example,

$$b = \frac{3}{4} \frac{\gamma}{(kR)^2} \quad , \quad c = \frac{3}{32} \frac{\gamma}{(kR)^4} \; . \qquad (4.19)$$

Therefore, to first order in γ and fourth order in $\underset{\sim}{r}$,

$$S = S(0) + k_0 z - V(0) z - \frac{1}{2} a(k^3 p^2 z + \frac{1}{3} k^3 z^3)$$

$$- b(kp)^2 + c(kp)^4 \ . \qquad (4.20)$$

Up to this point, the constant k has been arbitrary. Now define the effective wave number $k = k_0 - V(0)$, and use

$$kz = \underset{\sim}{k}\cdot\underset{\sim}{r} \quad , \quad k^2 p^2 = (\underset{\sim}{k}\times\underset{\sim}{r})^2 \qquad (4.21)$$

to obtain the following expression for $S(\underset{\sim}{r})$;

$$S = S(0) + \underset{\sim}{k}\cdot\underset{\sim}{r} - \frac{1}{6} a(\underset{\sim}{k}\cdot\underset{\sim}{r})[3k^2 r^2 - 2(\underset{\sim}{k}\cdot\underset{\sim}{r})^2]$$

$$- b(\underset{\sim}{k}\times\underset{\sim}{r})^2 + c(\underset{\sim}{k}\times\underset{\sim}{r})^4 \ . \qquad (4.22)$$

The solution (4.22) corresponds to the boundary conditions (4.6) for an incoming electron. It is therefore denoted $S^{(+)}$. However, it is also desirable to obtain the solution $S^{(-)}$ for an outgoing electron which approaches the free particle solution at $z = +\infty$. It is easily seen that $S^{(-)}$ is obtained from $S^{(+)}$ simply by reversing the signs of b and c.

5. Calculation of u(r)

Having obtained $S^{(+)}(\underset{\sim}{r})$ to first order in γ, we proceed to calculate the spinor function $u^{(+)}(\underset{\sim}{r})$ to the same order (i.e. first order in a,b,c).

From the definition (3.8),

$$\hat{\underset{\sim}{n}} = \hat{\underset{\sim}{k}} + \frac{1}{k}[a(\underset{\sim}{k}\cdot\underset{\sim}{r}) + 2b - 4c(\underset{\sim}{k}\times\underset{\sim}{r})^2] \times [(\underset{\sim}{k}\cdot\underset{\sim}{r})\underset{\sim}{k} - k^2\underset{\sim}{r}] . \tag{5.1}$$

Notice that $\hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{n}} = 1$ to first order in γ, and therefore the current conservation condition (3.10) becomes

$$\underset{\sim}{\nabla}\cdot(\hat{\underset{\sim}{n}}N^2) = 0 \tag{5.2}$$

or equivalently,

$$\hat{\underset{\sim}{n}}\cdot(\underset{\sim}{\nabla}N^2) = -N^2(\underset{\sim}{\nabla}\cdot\hat{\underset{\sim}{n}}) . \tag{5.3}$$

From (5.1) we obtain

$$\underset{\sim}{\nabla}\cdot\hat{\underset{\sim}{n}} = -[2ak(\underset{\sim}{k}\cdot\underset{\sim}{r}) + 4bk - 16ck(\underset{\sim}{k}\times\underset{\sim}{r})^2] . \tag{5.4}$$

Now a solution for (5.3) is attempted of the form

$$N^2 = N_0^2(1 + \delta N^2) \tag{5.5}$$

where N_0^2 is the value of N^2 at the origin, and δN^2 is a first order function which vanishes at the origin.

Substituting (5.5) into (5.3) and retaining only first order terms gives

$$\hat{k}\cdot\nabla\delta N^2 = -\nabla\cdot\hat{n} \tag{5.6}$$

which is simply

$$\frac{\partial\delta N^2}{\partial z} = 2ak^2z + 4bk - 16ck^3p^2 . \tag{5.7}$$

(5.7) is easily integrated;

$$\delta N^2 = ak^2z^2 + 4bkz - 16ck^3p^2z + A(p) \tag{5.8}$$

where A(p) is an arbitrary function which vanishes at p = 0.

The constant N_0^2 and the function A(p) are determined by the value of N^2 in the transverse plane through the origin: i.e.

$$N^2(z=0) = N_0^2(1 + A(p)) . \tag{5.9}$$

These quantities can be calculated using classical conservation of energy and angular momentum. It is shown in Appendix A, that

$$\begin{aligned} N_0^2 &= (k/k_0)^2 \\ A(p) &= -2ak^2p^2 . \end{aligned} \tag{5.10}$$

Therefore, reverting back to vector notation,

$$N = \left(\frac{k}{k_0}\right)\left[1 + \frac{1}{2}a(\underset{\sim}{k}\cdot\underset{\sim}{r})^2 - a(\underset{\sim}{k}\times\underset{\sim}{r})^2 + 2b(\underset{\sim}{k}\cdot\underset{\sim}{r}) - 8c(\underset{\sim}{k}\cdot\underset{\sim}{r})(\underset{\sim}{k}\times\underset{\sim}{r})^2\right] . \tag{5.11}$$

Using (5.1), the spinor $(\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{n}} + 1)v$ is simply

$$(\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{n}}+1)v = [2+a(\underset{\sim}{k}\cdot\underset{\sim}{r})^2+2b(\underset{\sim}{k}\cdot\underset{\sim}{r})-4c(\underset{\sim}{k}\cdot\underset{\sim}{r})(\underset{\sim}{k}\times\underset{\sim}{r})^2]v$$
$$-k[a(\underset{\sim}{k}\cdot\underset{\sim}{r})+2b-4c(\underset{\sim}{k}\times\underset{\sim}{r})^2]\underset{\sim}{\sigma}\cdot\underset{\sim}{r}v \quad . \tag{5.12}$$

Finally, combining (5.11), (5.12) and (3.7), the spinor function $u(\underset{\sim}{r})$ is given by,

$$u^{(+)} = \frac{N}{2}(\underset{\sim}{\sigma}\cdot\hat{\underset{\sim}{n}} + 1)v$$
$$= (\frac{k}{k_0})\{[1 - a(\underset{\sim}{k}\times\underset{\sim}{r})^2 + a(\underset{\sim}{k}\cdot\underset{\sim}{r})^2 + 3b(\underset{\sim}{k}\cdot\underset{\sim}{r}) \tag{5.13}$$
$$- 10c(\underset{\sim}{k}\cdot\underset{\sim}{r})(\underset{\sim}{k}\times\underset{\sim}{r})^2]$$
$$- k[\frac{1}{2}a(\underset{\sim}{k}\cdot\underset{\sim}{r}) + b - 2c(\underset{\sim}{k}\times\underset{\sim}{r})^2]\underset{\sim}{\sigma}\cdot\underset{\sim}{r}\}v \quad .$$

Of course, $u^{(-)}$ is obtained from $u^{(+)}$ by reversing the signs of b and c.

6. The Matrix Element for Elastic Scattering

In this section we employ the symbols χ, ϕ, and ψ to represent plane waves, eikonal functions, and exact scattering solutions respectively. Furthermore, the subscript 1 shall denote an incident state and the subscript 2, a final state.

The exact T-matrix element for elastic scattering from the potential V(r) is given by

$$\langle 2|T|1\rangle = \langle \chi_2|V|\psi_1^{(+)}\rangle. \qquad (6.1)$$

In previous sections an eikonal approximation to the solution of the Dirac equation has been developed of the form $\phi(\underset{\sim}{r}) = u(\underset{\sim}{r})\exp iS(\underset{\sim}{r})$. It is obvious that this eikonal form does not contain the outgoing spherical waves of the exact solution. In other words, ϕ does not include any large angle scattering components. However, although ϕ is a very poor representation of ψ in the asymptotic region, it is a reasonably accurate approximation near the origin. Indeed, we have resigned ourselves to this fact when ϕ was expanded in powers of $\underset{\sim}{r}$.

Unfortunately, the matrix element (6.1) contains a long range Coulomb potential which would therefore render the substitution of $\phi^{(+)}$ for $\psi^{(+)}$ a poor approximation. However, a short range potential may be constructed by subtracting from V(r) some auxiliary potential $\tilde{V}(r)$, which is created by an

auxiliary charge distribution $\tilde{\rho}(r)$ with the same normalization and roughly the same size as $\rho(r)$. Therefore, we rewrite the T-matrix element using the well known two-potential formula,

$$\langle 2|T|1\rangle = \langle \chi_2|\tilde{V}|\tilde{\psi}_1^{(+)}\rangle + \langle \tilde{\psi}_2^{(-)}|V-\tilde{V}|\psi_1^{(+)}\rangle. \qquad (6.2)$$

If $\tilde{V}(r)$ is chosen to give negligible scattering compared to $V(r)$ in the angular region of interest, then the first term in (6.2) may be ignored. Moreover, $\tilde{V}$ may also be ignored in the second term. Therefore, (6.2) is approximated by

$$\langle 2|T|1\rangle \simeq \langle \phi_2^{(-)}|V|\phi_1^{(+)}\rangle \qquad (6.3)$$

where the short range of $(V-\tilde{V})$ has justified the substitution of ϕ for ψ, and the eikonal parameters of V and $\tilde{V}$ are assumed to be roughly equal.

Provided that some $\tilde{V}(r)$ exists with all of the stated characteristics, the approximation (6.3) is expected to be fairly reasonable. Crudely speaking, we require the existence of some $\tilde{\rho}(r)$ which is the same size as $\rho(r)$ and generates the same eikonal parameters, and yet is much "smoother" than $\rho(r)$ in that its higher Fourier components are negligibly small.

Clearly, the above requirements for the validity of (6.3) cannot be met for a charge distribution with rapidly decreasing high Fourier components. Therefore, we might expect our calculations to break down for "smooth" charge distributions.

7. Calculation of the Matrix Element

In terms of the spinor and phase functions derived in previous sections, the matrix element (6.3) becomes

$$\int u_2^{(-)\dagger} u_1^{(+)} e^{i[S_1^{(+)} - S_2^{(-)}]} V(r) d^3\underset{\sim}{r} . \qquad (7.1)$$

Using (4.22) and (5.13) and retaining first order terms in γ, the matrix element (7.1) is simply

$$\left(\frac{k}{k_0}\right)^2 \int m(\underset{\sim}{r}) e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}} V(r) d^3\underset{\sim}{r} \qquad (7.2)$$

where $\underset{\sim}{q} = \underset{\sim}{k}_1 - \underset{\sim}{k}_2$ and,

$$\begin{aligned} m(\underset{\sim}{r}) = v_2^\dagger v_1 \{ & 1 + 3b[(\underset{\sim}{k}_1\cdot\underset{\sim}{r}) - (\underset{\sim}{k}_2\cdot\underset{\sim}{r})] \\ & + (2a + ib)[(\underset{\sim}{k}_1\cdot\underset{\sim}{r})^2 + (\underset{\sim}{k}_2\cdot\underset{\sim}{r})^2] \\ & + (10c + i\tfrac{a}{3})[(\underset{\sim}{k}_1\cdot\underset{\sim}{r})^3 - (\underset{\sim}{k}_2\cdot\underset{\sim}{r})^3] \\ & + ic[(\underset{\sim}{k}_1\cdot\underset{\sim}{r})^4 + (\underset{\sim}{k}_2\cdot\underset{\sim}{r})^4] \\ & - 2(a+ib)k^2r^2 + 2ick^4r^4 \\ & - (10c + i\tfrac{a}{2})[(\underset{\sim}{k}_1\cdot\underset{\sim}{r}) - (\underset{\sim}{k}_2\cdot\underset{\sim}{r})]k^2r^2 \\ & - 2ic[(\underset{\sim}{k}_1\cdot\underset{\sim}{r})^2 + (\underset{\sim}{k}_2\cdot\underset{\sim}{r})^2]k^2r^2\} \\ & - kv_2^\dagger \underset{\sim}{\sigma}\cdot\underset{\sim}{r}\, v_1 \{\tfrac{1}{2} a[(\underset{\sim}{k}_1\cdot\underset{\sim}{r}) + (\underset{\sim}{k}_2\cdot\underset{\sim}{r})] \\ & + 2c[(\underset{\sim}{k}_1\cdot\underset{\sim}{r})^2 - (\underset{\sim}{k}_2\cdot\underset{\sim}{r})^2]\} \end{aligned} \qquad (7.3)$$

However, it is well known from Fourier transform theory that

$$\int m(\underset{\sim}{r}) e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}} V(r) d^3\underset{\sim}{r} = m(-i\underset{\sim}{\nabla})\tilde{V}(\underset{\sim}{q}) \qquad (7.4)$$

where $\tilde{V}(\underset{\sim}{q})$ is the Fourier transform of $V(r)$ and the gradient operator acts with respect to the variable $\underset{\sim}{q}$.

For a spherically symmetric potential, the Fourier transform $\tilde{V}(\underset{\sim}{q})$ is also spherically symmetric, and the matrix element for scattering simply becomes

$$\left(\frac{k}{k_0}\right)^2 m(-i\underset{\sim}{\nabla})\tilde{V}(q) \; . \qquad (7.5)$$

Therefore, the problem is transformed into an evaluation of the differential operator $m(-i\underset{\sim}{\nabla})$. This is a relatively straightforward but laborious procedure, and the details are omitted here. The final result, however, is very simple:

$$\langle 2|T|1\rangle = \left(\frac{k}{k_0}\right)^2 \cos\frac{\theta}{2} \; O(D)\tilde{V}(q) \qquad (7.6)$$

where the real and imaginary parts of the operator $O(D)$ are given by (D denotes differentiation),

$$\begin{aligned} \mathrm{Re}O(D) = 1 &+ \left(a\frac{k^2}{q} + \frac{3}{4}aq\right)D \\ &+ \left(2ak^2 - \frac{3}{4}aq^2\right)D^2 \\ &+ \left(\frac{1}{2}ak^2q - \frac{1}{12}aq^3\right)D^3 \end{aligned} \qquad (7.7)$$

$$\mathrm{Im}\mathcal{O}(d) = \left(2b\frac{k^2}{q} - \frac{5}{2}bq + 2c\frac{k^4}{q^3} - 3c\frac{k^2}{q} + \frac{45}{8}cq\right)D$$
$$+ \left(2bk^2 - \frac{1}{2}bq^2 + 3ck^2 - 2c\frac{k^4}{q^2} - \frac{45}{8}cq^2\right)D^2$$
$$+ \left(4c\frac{k^4}{q} - 8ck^2q + \frac{7}{4}cq^3\right)D^3$$
$$+ \left(2ck^4 - ck^2q^2 + \frac{1}{8}cq^4\right)D^4 \quad . \qquad (7.8)$$

In (7.6) the angle θ is the scattering angle (i.e. the angle between $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$), and the effective momentum transfer q is given by the familiar expression,

$$q = 2k\sin\frac{\theta}{2} \quad . \qquad (7.9)$$

It is desirable, of course, to work directly with the charge distribution ρ rather than the potential energy. Therefore, we recall the following well known relation between the Fourier transforms of ρ and V;

$$\tilde{V}(q) = -4\pi\gamma\frac{\tilde{\rho}(q)}{q^2} \quad . \qquad (7.10)$$

The scattering cross section is given by

$$\frac{d\sigma}{d\omega} = \left(\frac{k_0}{2\pi}\right)^2 |\langle 2|T|1\rangle|^2 \quad . \qquad (7.11)$$

Therefore, combining (7.6), (7.10) and (7.11) we obtain the following expression for the electron scattering cross section;

$$\frac{d\sigma}{d\omega} = (2k_0\gamma)^2 \left(\frac{k}{k_0}\right)^4 \cos^2\frac{\theta}{2}\left|\mathcal{O}(D)\frac{\tilde{\rho}(q)}{q^2}\right|^2 \quad . \qquad (7.12)$$

8. Computational Procedure

The implementation of the method described in the preceding section for an arbitrary charge distribution would seem to be a hopeless prospect, since the operator $O(D)$ involves up to fourth order derivatives. This poses a serious problem, due to the difficulty of numerical differentiation.

It is possible, however, to circumvent this difficulty by expanding the charge distribution in some suitable set of basis functions which can be handled by algebraic means. For example, the charge density $\rho(r)$ may be approximated by a finite Fourier Bessel series over the interval $0 \leq r \leq R_0$, i.e.

$$r\rho(r) = \sum_{\ell=1}^{L} a_\ell \sin\left(\frac{\ell\pi r}{R_0}\right) \tag{8.1}$$

where the Fourier coefficients a_ℓ are given by the familiar formula

$$a_\ell = \frac{2}{R_0}\int_0^{R_0} r\rho(r)\sin\left(\frac{\ell\pi r}{R_0}\right)dr \ . \tag{8.2}$$

Using (8.1), we easily obtain

$$\frac{\tilde{\rho}(q)}{q^2} = \frac{4\pi}{q^3}\int r\rho(r)\sin(qr)\,dr$$

$$= \frac{4\pi^2}{R_0}\sum_\ell \ell a_\ell (-)^\ell F_\ell(q) \tag{8.3}$$

where

$$F_\ell(q) = \frac{\sin(qR_0)}{q^3[q^2 - (\frac{\ell\pi}{R_0})^2]} \quad . \tag{8.4}$$

Successive derivatives of (8.3) are obtained by differentiating $F_\ell(q)$ analytically. The results of this exercise are given in Appendix B.

In order to apply (7.12), however, it is also necessary to determine the parameters $k = k_0 - V(0)$, a, b, and c. First of all, the total charge must be normalized using

$$\rho(\text{total}) = 4R_0^2 \sum_\ell (-1)^{\ell+1} \frac{a_\ell}{\ell}$$

$$= 1 \tag{8.5}$$

Then the potential energy V(r) is calculated for small r using

$$V(r) = -4\pi\gamma[\frac{1}{r}\int_0^r x^2\rho dx + \int_r^\infty x\rho dx] \quad . \tag{8.6}$$

Comparing the results with equation (4.11), we extract the parameters V(0) and a,

$$V(0) = -\frac{\gamma}{R_0} - 4R_0\gamma \sum_\ell \frac{a_\ell}{\ell} \tag{8.7}$$

$$a = \frac{4}{3}\frac{\pi^2\gamma}{k^3R_0} \sum_\ell \ell a_\ell \quad . \tag{8.8}$$

The values of b and c are determined using (4.17) and (4.18);

$$b = \frac{\pi\gamma}{k^2} \sum_{\ell} a_{\ell} \, Si(\ell\pi) \tag{8.9}$$

$$c = \frac{1}{16} \frac{\pi\gamma}{k^4} \left(\frac{\pi}{R_0}\right)^2 \sum_{\ell} \ell^2 a_{\ell} \left[\frac{(-)^{\ell}}{\ell\pi} + Si(\ell\pi)\right] \tag{8.10}$$

where $Si(x) = \int_0^x \frac{\sin u}{u} \, du.$

Unfortunately, equations (8.7)-(8.10) may yield unreasonable results if the finite Fourier series is a poor representation of $\rho(r)$. This is especially true in the case of the parameter c, which depends very sensitively on the shape of the charge distribution. For example, charge densities with "sharp" features are very poorly approximated by finite Fourier expansions, and experience has indicated that the eikonal parameters calculated using (8.7)-(8.10) are unreliable in such extreme cases.

However, for typical nuclear shapes, it is sufficiently accurate to replace $\rho(r)$ with an equivalent uniform sphere for the purpose of calculating the distorted wave. The equivalent uniform radius is given by

$$R = \sqrt{\frac{5}{3} \langle r^2 \rangle} \tag{8.11}$$

where the mean square radius is simply

$$\langle r^2 \rangle = R_0^2 + \frac{24 R_0^4}{\pi^2} \sum_{\ell} (-)^{\ell} \frac{a_{\ell}}{\ell^3} . \tag{8.12}$$

The eikonal parameters are then given by the simple formulae (4.12) and (4.19).

The above procedure is adequate for charge distributions of roughly uniform shape. However, a slightly more sophisticated approach involves the calculation of V(0), a, b, and c using a Fermi distribution of the form

$$\rho_F = \frac{\rho_0}{1 + \exp[(r-d)/t]} . \tag{8.13}$$

If the ratio t/d is small, then d is the radius at which ρ_F falls to half of its maximum value ρ_0, and the "skin thickness" T = 4.4t is the distance over which ρ_F decreases from 90% to 10% of its maximum value.

For $\varepsilon = \frac{t}{d} << 1$,

$$\rho_0 = \frac{3}{4\pi d^3} (1 + \pi^2\varepsilon^2)^{-1} \tag{8.14}$$

$$V(0) = -\frac{3}{2}\frac{\gamma}{d} (1 + \frac{\pi^2}{3}\varepsilon^2)(1 + \pi^2\varepsilon^2)^{-1} \tag{8.15}$$

$$a = \frac{\gamma}{(kd)^3} (1 + \pi^2\varepsilon^2)^{-1} \tag{8.16}$$

$$b = \frac{3}{4}\frac{\gamma}{(kd)^2} (1 + \pi^2\varepsilon^2)^{-1} \tag{8.17}$$

$$c \approx \frac{3}{32}\frac{\gamma}{(kd)^4} (1 + 3.2899\ \varepsilon^2 + 45.458\ \varepsilon^4 + 1419.2\ \varepsilon^6) \times (1 + \pi^2\varepsilon^2)^{-1}. \tag{8.18}$$

9. Example Calculations

The eikonal approximation to first order in γ, as outlined in the previous sections, has been tested against an "exact" phase-shift calculation for the nuclei Ca($\gamma = 0.146$), Ni($\gamma = 0.204$), Sn($\gamma = 0.365$) and Pb($\gamma = 0.598$).

A charge distribution of Fermi form (8.13) is assumed in each case, and the eikonal parameters are calculated using equations (8.15)-(8.18). The resulting cross sections are plotted on the following pages, where the curve type denotes,

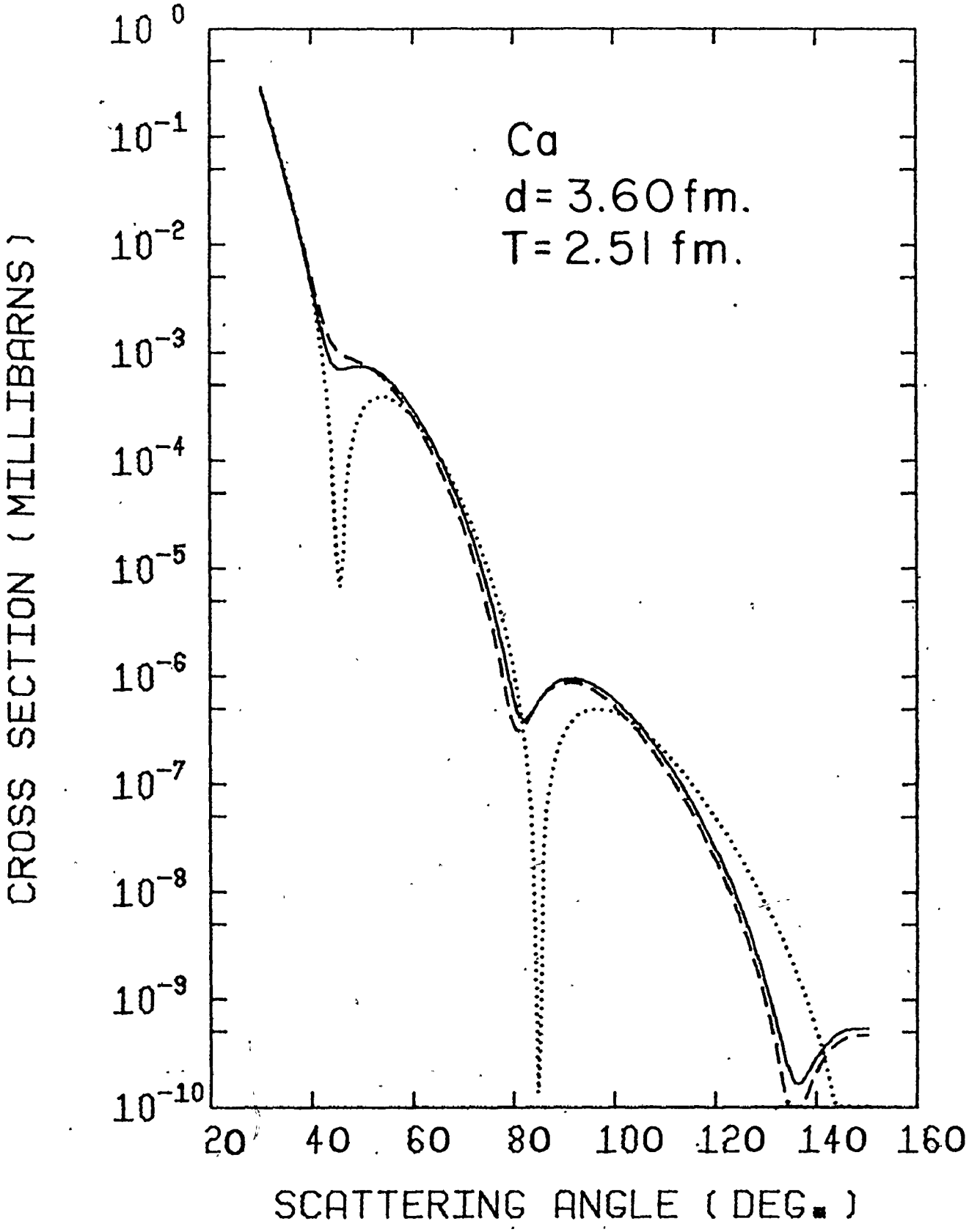
Dotted : Born Approximation

Dashed : Eikonal Approximation

Solid : Phase-shift Calculation.

In all cases, the electron energy is E = 300 MeV.

The accuracy of the method is seen to be reasonably good for the medium-Z nuclei Ca and Ni. In the case of Sn and Pb, however, agreement with the phase shift calculation is poorer, due to the much larger values of γ. Notice, however, that agreement is noticeably better for the "sharper" charge densities of Figures 5 and 6. This improvement seems reasonable in light of the discussion following equation (6.3).


Ca
d = 3.60 fm.
T = 2.51 fm.
CROSS SECTION (MILLIBARNS)
SCATTERING ANGLE (DEG.)
10 0
10-1
10-2
10-3
10-4
10-5
10-6
10-7
10-8
10-9
10-10
20
40
60
80
100
120
140
160

fi l

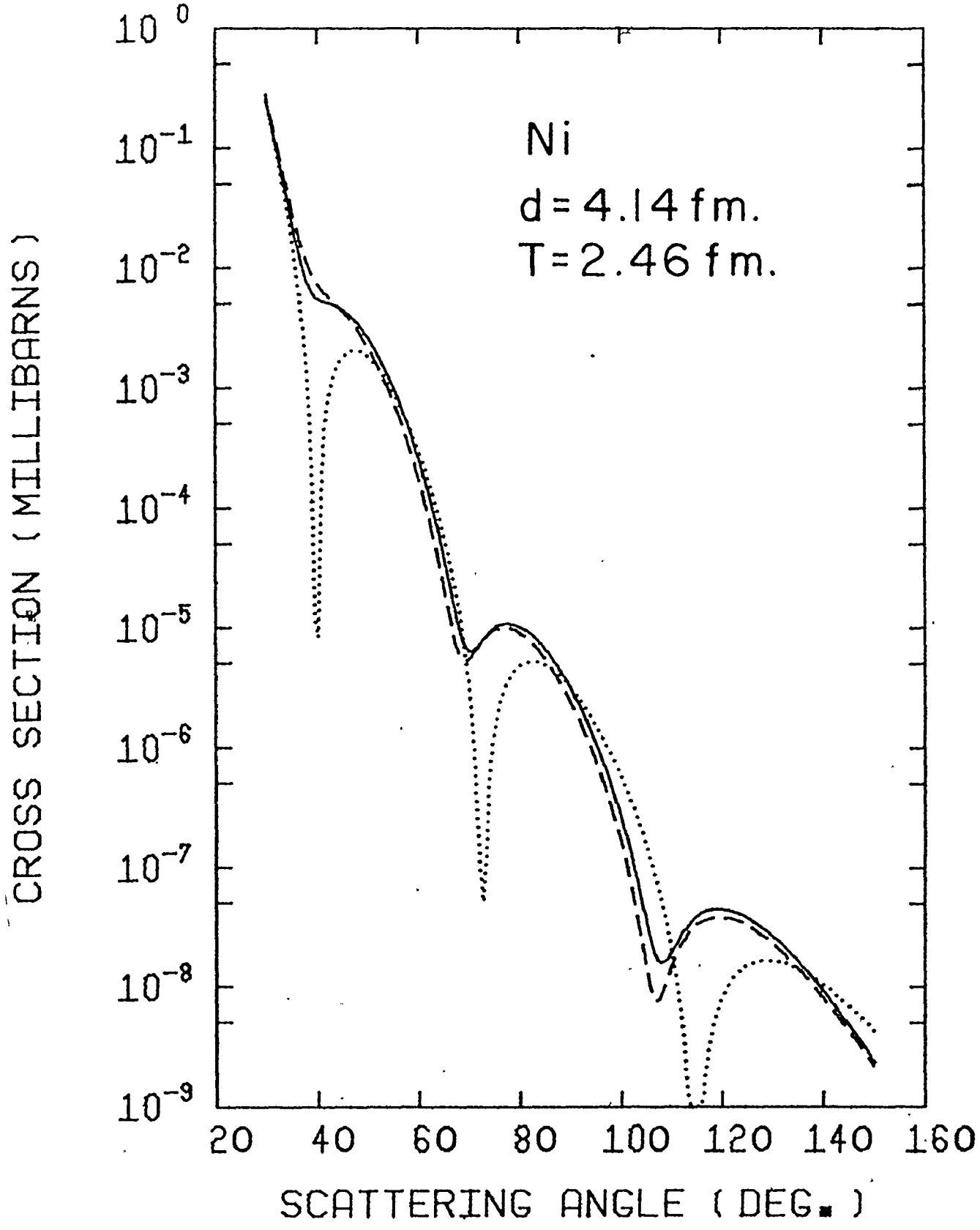

Ni
d = 4.14 fm.
T = 2.46 fm.
CROSS SECTION (MILLIBARNS)
SCATTERING ANGLE (DEG.)
10 0
10-1
10-2
10-3
10-4
10-5
10-6
10-7
10-8
10-9
20
40
60
80
100
120
140
160

fig 2

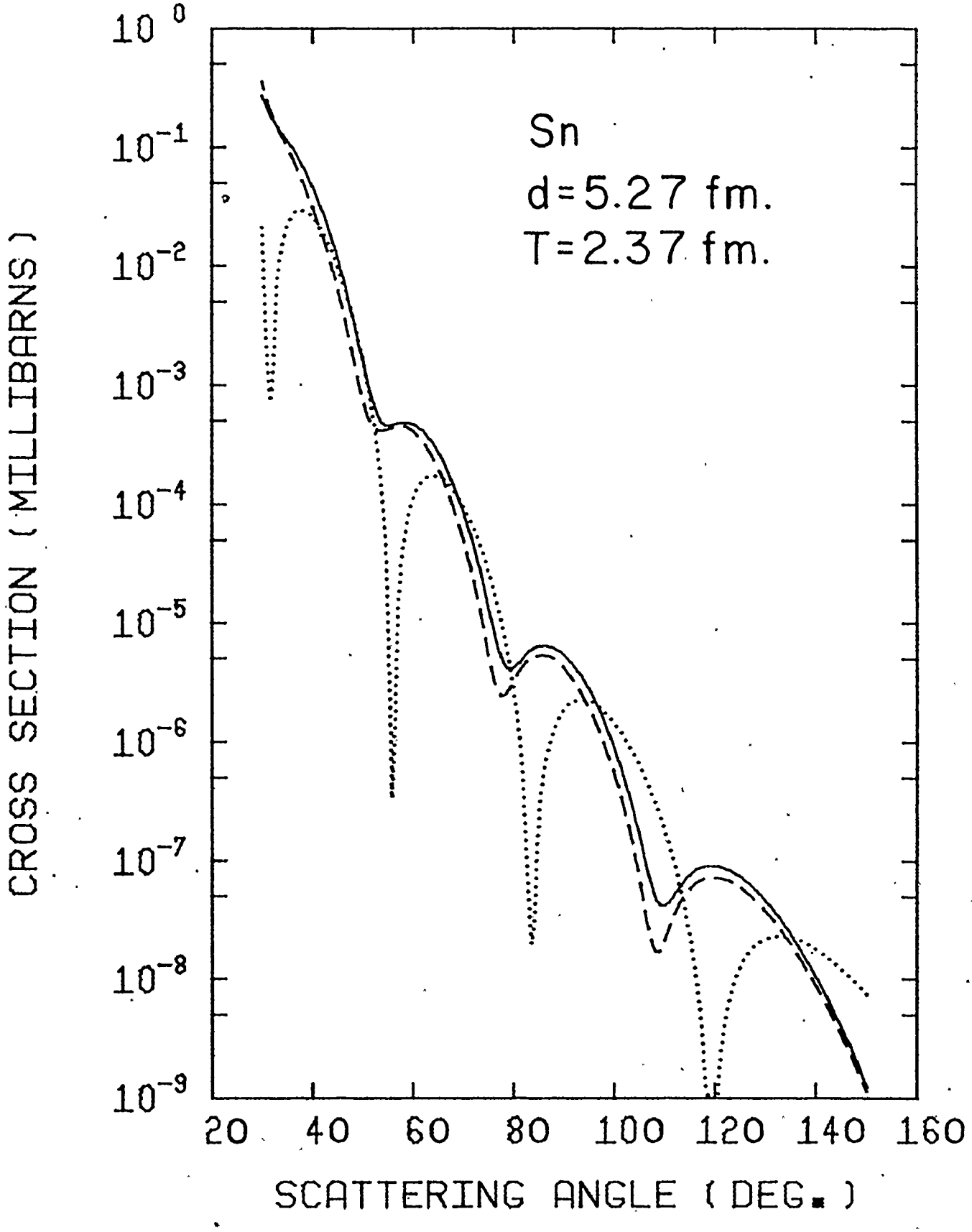
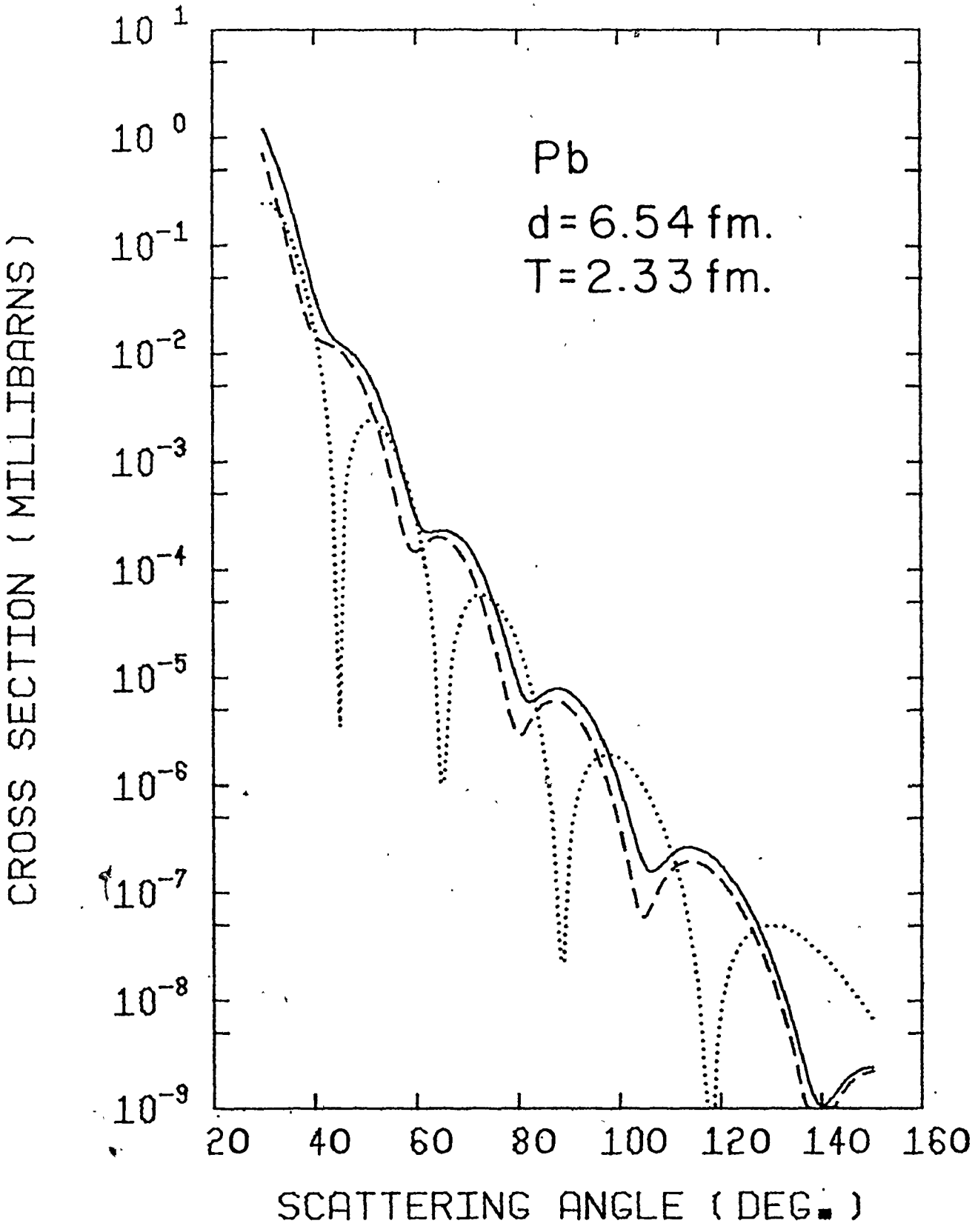

Sn
d=5.27 fm.
T=2.37 fm.
CROSS SECTION (MILLIBARNS)
SCATTERING ANGLE (DEG.)
10 0
10-1
10-2
10-3
10-4
10-5
10-6
10-7
10-8
10-9
20
40
60
80
100
120
140
160

fig.3


Pb
d=6.54 fm.
T=2.33 fm.
CROSS SECTION (MILLIBARNS)
10 1
10 0
10-1
10-2
10-3
10-4
10-5
10-6
10-7
10-8
10-9
20
40
60
80
100
120
140
160
SCATTERING ANGLE (DEG.)

fig

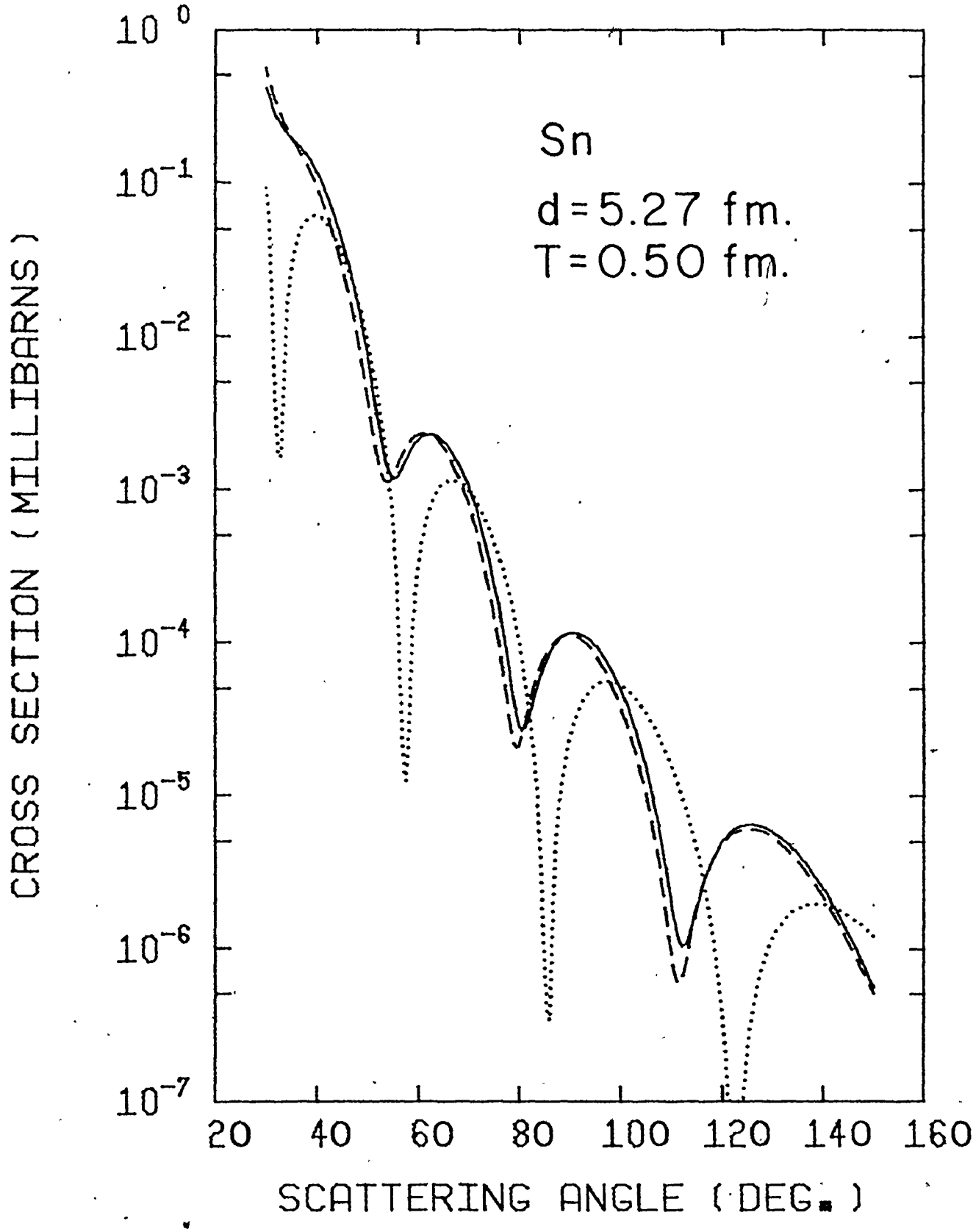

Sn
d=5.27 fm.
T=0.50 fm.
CROSS SECTION (MILLIBARNS)
SCATTERING ANGLE (DEG.)
10 0
10-1
10-2
10-3
10-4
10-5
10-6
10-7
20
40
60
80
100
120
140
160

fig 5

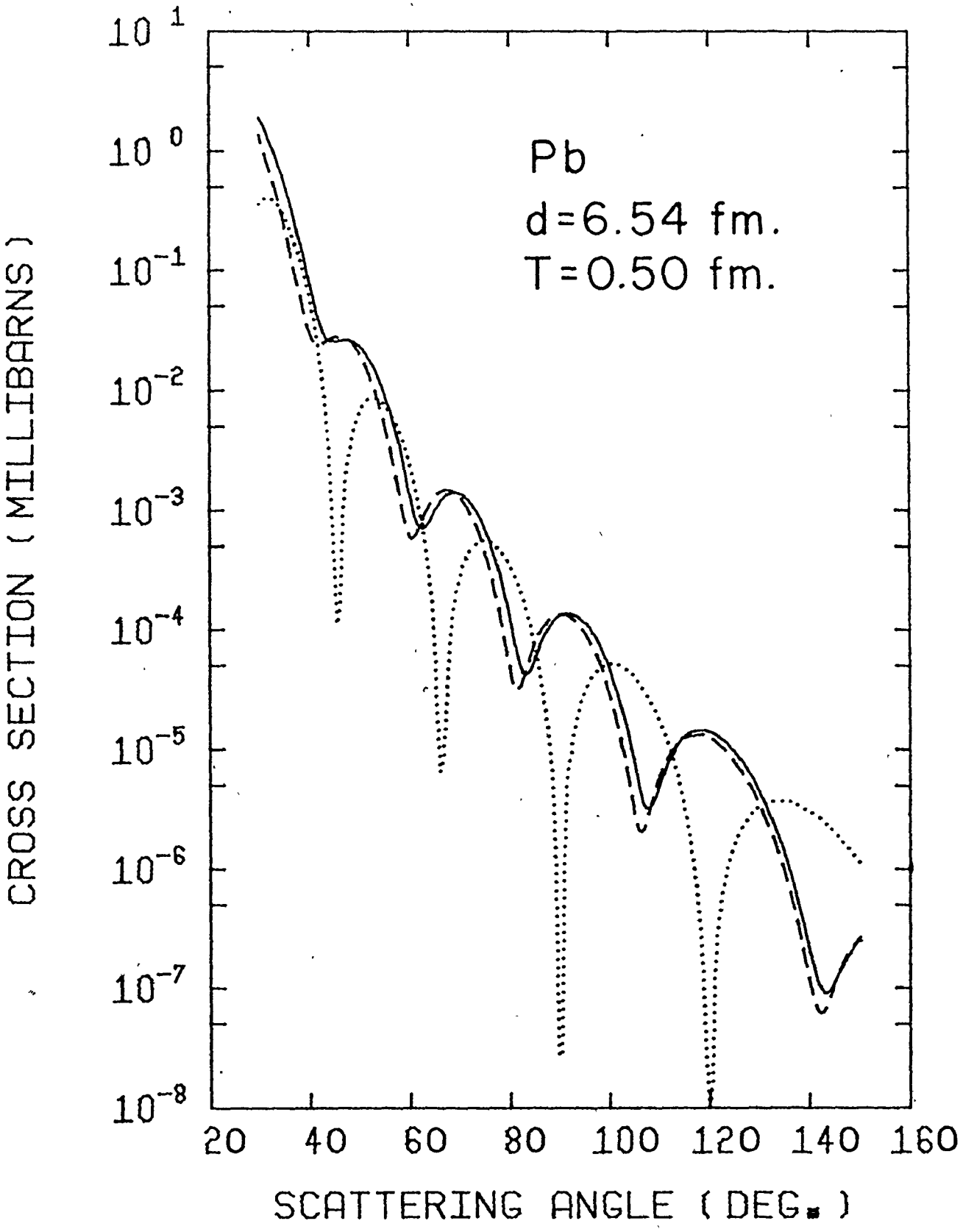

Pb
d=6.54 fm.
T=0.50 fm.
CROSS SECTION (MILLIBARNS)
SCATTERING ANGLE (DEG.)
10 1
10 0
10-1
10-2
10-3
10-4
10-5
10-6
10-7
10-8
20
40
60
80
100
120
140
160

fig 6

Appendix A

The value of the normalization function $N(\underset{\sim}{r})$ in the transverse plane through the origin can be calculated by appealing to the semiclassical aspects of the eikonal approximation. In order to exploit the connection with classical mechanics, we notice that equation (3.4) is just the Hamilton-Jacobi equation for particles of zero rest mass. Therefore, the vector $\hat{\underset{\sim}{n}}$ defined by (3.8) gives the direction of the classical trajectory at any point.

Now consider a small tube of trajectories extending from $-\infty$ to the transverse plane through the origin. The transverse cross sectional areas at $z = -\infty$ and $z = 0$ are denoted A_0 and A respectively. Integrating (5.2) over the volume of the tube gives

$$A_0 = \hat{\underset{\sim}{k}} \cdot \hat{\underset{\sim}{n}} \; N^2 A. \tag{A.1}$$

Therefore, to first order in γ, $(\hat{\underset{\sim}{k}} \cdot \hat{\underset{\sim}{n}} = 1)$

$$N^2 = \frac{A_0}{A} . \tag{A.2}$$

The area ratio (A.2) can be calculated using classical conservation of energy and angular momentum along any given trajectory.

$$k' + V = k_0 \quad \text{(Energy Conservation)} \tag{A.3}$$

$$k' r(\hat{\underset{\sim}{k}} \cdot \hat{\underset{\sim}{n}}) = k_0 h \quad \text{(Angular Momentum Conservation)} \tag{A.4}$$

where h is the classical impact parameter.

Substituting (A.3) into (A.4),

$$\begin{aligned} k_0 h &= r(k_0 - V) \\ &= r(k_0 - V(0) - \tfrac{1}{2} ak^3 r^2) \\ &= r(k - \tfrac{1}{2} ak^3 r^2) \ . \end{aligned} \tag{A.5}$$

Therefore,

$$k_0 h = kr(1 - \tfrac{1}{2} ak^2 r^2) . \tag{A.6}$$

Differentiating (A.6) yields

$$k_0 dh = kdr(1 - \tfrac{3}{2} ak^2 r^2) \ . \tag{A.7}$$

Therefore, from (A.6) and (A.7) we obtain

$$hdh = (\frac{k}{k_0})^2 rdr(1 - 2ak^2 r^2) . \tag{A.8}$$

This result gives the desired area ratio for use in (A.2),

$$N^2 = \frac{hdh}{rdr} = (\frac{k}{k_0})^2 (1 - 2ak^2 r^2) . \tag{A.9}$$

Comparison with (5.9) finally gives

$$\begin{aligned} N_0^2 &= (\frac{k}{k_0})^2 \\ A(p) &= -2ak^2 p^2 \ . \end{aligned} \tag{A.10}$$

Appendix B

For notational convenience, we employ the symbols

$$s = \sin(qR_0)$$

$$c = \cos(qR_0)$$

$$B = [q^2 - (\tfrac{\ell\pi}{R_0})^2]$$

$$F_\ell(q) = s/q^3B \tag{B.1}$$

$$DF_\ell(q) = -3s/q^4B - 2s/q^2B^2 + R_0c/q^3B \tag{B.2}$$

$$D^2F_\ell(q) = 12s/q^5B + 10s/q^3B^2 - 6R_0c/q^4B + 8s/qB^3 - 4R_0c/q^2B^2 - R_0^2s/q^3B \tag{B.3}$$

$$D^3F_\ell(q) = -60s/q^6B - 54s/q^4B^2 + 36R_0c/q^5B - 48s/q^2B^3 + 30R_0c/q^3B^2 + 9R_0^2s/q^4B - 48s/B^4 + 24R_0c/qB^3 + 6R_0^2s/q^2B^2 - R_0^3c/q^3B \tag{B.4}$$

$$D^4F_\ell(q) = 360s/q^7B + 336s/q^5B^2 - 240R_0c/q^6B + 312s/q^3B^3 - 216R_0c/q^4B^2 - 72R_0^2s/q^5B + 288s/qB^4 - 192R_0c/q^2B^3 - 60R_0^2s/q^3B^2 + 12R_0^3c/q^4B + 384qs/B^5 - 192R_0c/B^4 - 48R_0^2s/qB^3 + 8R_0^3c/q^2B^2 + R_0^4s/q^3B \tag{B.5}$$

A problem arises in the application of these expressions, owing to the presence of singularities in the derivatives at the points $q_\ell = \ell\pi/R_0$. Experience has indicated, however, that these singularities are extremely narrow (i.e. <0.001 fm^{-1}) and offending points are simply ignored. In other words, the calculated scattering amplitude is interpolated in the negligibly small regions $(q_\ell - 0.001)$ to $(q_\ell + 0.001)$ fm^{-1}.

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