AUTOMORPHISMS OF RIEMANN SURFACES

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By

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A Thesis Submitted to the
School of Graduate Studies
in Partial Fulfilment of the
Requirements for the Degree of
Master of Science

McMaster University

Master of Science (2009)
McMaster University
Hamilton, Ontario
Department of Mathematics and Statistics

TITLE: Automorphisms of Riemann Surfaces
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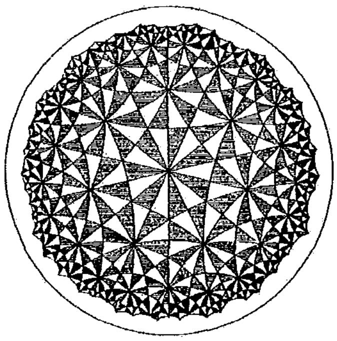
NUMBER OF PAGES: vi, 67

Abstract

This paper consists of mainly two parts. First it is a survey of some results on automorphisms of Riemann surfaces and Fuchsian groups. A theorem of Hurwitz states that the maximal automorphism group of a compact Riemann surface of genus g has order at most $84(g-1)$. It is well-known that the Klein quartic is the unique genus 3 curve that attains the Hurwitz bound. We will show in the second part of the paper that, in fact, the Klein curve is the unique non-singular curve in $\mathbb{C}P^2$ that attains the Hurwitz bound. The last section concerns automorphisms of surfaces with cusps or punctured surfaces.



Acknowledgements

I would like to first thank and express my sincere appreciation to my thesis advisor, Dr. Ian Hambleton, for suggesting the topic of Hurwitz's theorem as well as his continued patience and insightful discussions throughout this project. I am also especially thankful to Dr. McKenzie Wang and Dr. Maung Min-Oo for helpful discussions on invariant surfaces and related topics. Lastly, I'm grateful to my family for their support and encouragement.

Contents

1 Introduction

Automorphisms of a Riemann surface X are conformal homeomorphisms and we denote the full group of automorphisms by $Aut(X)$. We say that a group G acts on X if there exists an injection $G \hookrightarrow Aut(X)$. It was shown by Schwarz that for genus $g \geq 2$, the full automorphism group of a compact Riemann surface is finite. Later, in 1893, Hurwitz proved that the order of the automorphism group is bounded by $84(g-1)$ and cited Klein's quartic, $x^3y + y^3z + z^3x = 0$, to give an example of a curve of genus 3 that attains the upper bound with 168 automorphisms [Hur93].

Hurwitz went further to show that any finite group G acts as a group of automorphisms on some compact Riemann surface and attains the upper bound $84(g-1)$ if and only if G has generators x and y such that $x^2 = y^3 = (xy)^7 = 1$. The abstract group defined by this presentation is called the triangle group $\Delta(2,3,7)$ and Hurwitz's result shows that those finite groups that attain the upper bound, now called Hurwitz groups, are exactly non-trivial homomorphic images of $\Delta(2,3,7)$. It follows in particular, that a Hurwitz group of smallest order must be a simple group, and as there is no simple group of order 84, the automorphism group of the Klein quartic, $PSL(2,7)$, is simple.

For non-singular curves in $\mathbb{C}P^2$, automorphisms can be realized as the restriction of projective linear maps [Cha78], so there exists an injective homomorphism $Aut(X) \hookrightarrow PGL(3,\mathbb{C})$. It is well-known that $PSL(2,7)$ is the unique simple group of order 168, and by checking the invariant polynomials for this group we see that the Klein quartic is the unique genus 3 curve that attains the Hurwitz bound. By using Blichfeldt's classification [MBD16] for finite subgroups of $PGL(3,\mathbb{C})$, we show in Theorem 7.4.1, that there are no other Hurwitz groups in the list, so that the Klein quartic turns out to be the unique non-singular curve in $\mathbb{C}P^2$ that attain the Hurwitz bound.

For non-compact surfaces obtained from compact ones by removing a finite number of points, say s, the automorphisms are again finite and the upper bound for the order of the automorphism group is given by $12(g-1)+6s$, see Theorem 8.1.4 or [Oik56]. In this case those groups that can be realized as maximal symmetry groups are homomorphic images of the modular group $PSL(2,\mathbb{Z})$, see Theorem 8.2.2. Interestingly, the Klein quartic with 24 punctures attains simultaneously both the $12(g-1)+6s$ bound and the Hurwitz upper bound, see Proposition 8.3.3. and Theorem 8.3.4.

2 Preliminaries

In this section we give an overview of some background material and results that will be used in later sections.

2.1 Riemann Mapping Theorem

Since we are interested in automorphisms of Riemann surfaces, we begin by first considering simply connected surfaces and their automorphism groups. This classical result states that up to biholomorphic equivalence, there are only three simply connected Riemann surfaces:

Theorem 2.1.1 (Riemann Mapping Theorem). *Every simply connected Riemann surface is biholomorphic to one of the following:*
(1) Riemann sphere Σ
(2) $\mathbb{C}$
(3) $\mathbb{H}$*, upper-half plane.*

The automorphisms of each simply connected Riemann surface is computed in the following

Theorem 2.1.2. *Automorphisms of the Riemann sphere are given by Möbius transformations*

$$\frac{\alpha z+\beta}{\gamma z+\delta}$$

where $\alpha\delta-\beta\gamma\neq 0$ *with* α,β,γ *and* δ *are in* $\mathbb{C}\cup\{\infty\}$. *Automorphisms of the complex plane are given by*

$$Aut(\mathbb{C})=\{f|f(z)=az+b, \text{with } a\neq 0\}$$

and the Möbius transformations with α,β,γ *and* δ *in* $\mathbb{R}\cup\{\infty\}$ *and* $\alpha\delta-\beta\gamma=1$ *give the automorphisms of the upper-half plane* $\mathbb{H}$ *which is denoted by* $PSL(2,\mathbb{R})$.

The Cayley transform $\varphi(z)=\frac{z-i}{z+i}$ is a biholomorphic mapping from the upper-half plane to the unit disk $\{|z|<1\}$. As a result, the automorphisms of the open unit disk are determined by applying the Cayley transform to $PSL(2,\mathbb{R})$ and using the following result which can be found in [Car73, pg. 180]

Lemma 2.1.3. *Let D be an open set of the Riemann sphere and let G be a subgroup of $Aut(D)$. If G acts transitively on D and for at least one point of D has stabilizer contained in G then $G = Aut(D)$.*

Proof. Let T be any automorphism of D. Suppose $z_0 \in D$ is the point which has stabilizer contained in G. By transitivity, there exists $S \in G$ such that $S(z_0) = T(z_0)$ so we have $S^{-1}T$ fixes z_0 and so is an element of G. It follows that T is in G and so we have $Aut(D) = G$. □

If we take G to be $\varphi Aut(\mathbb{H})\varphi^{-1}$, then certainly G is a subgroup of $Aut(\mathbb{D})$ and acts transitively on $\mathbb{D}$. The stabilizer of the origin in the disk is given by Schwarz' lemma so that any automorphism of $\mathbb{D}$ that fixes 0 is given by $T(z) = e^{i\theta}z$ for some angle θ and it can be checked that this stabilizer is contained in G by conjugating the following automorphisms of $\mathbb{H}$

$$\frac{z + tan(\theta/2)}{1 - ztan(\theta/2)}$$

by the Cayley transformation. It follows from the lemma that $Aut(\mathbb{D}) = G$. We record this in the following

Theorem 2.1.4. *Automorphisms of the unit disk $\mathbb{D}$ are given by transformations given by*

$$\frac{az + b}{\overline{b}z + \overline{a}}$$

where a and b are complex numbers such that $|a|^2 - |b|^2 = 1$. By rearranging we can also express the automorphisms as transformations given by:

$$\omega\frac{z - \alpha}{1 - \overline{\alpha}z}$$

where $|\omega| = 1$ and $|\alpha| < 1$.

We note here that every surface can be given a complex structure and so realized as a Riemann surface.

Theorem 2.1.5. *Every connected surface (2-manifold) admits a complex structure.*

In particular, every simply connected surface is determined by the Riemann mapping theorem. It's important to note, however, that though the unit disk is homeomorphic to the plane, they are not biholomorphic; they are not the same as Riemann surfaces. It's also useful to know that we can extend topological group actions on surfaces to conformal automorphisms.

Theorem 2.1.6. *If G acts on a compact orientable surface as a group of orientation-preserving homeomorphisms, then there exists a complex structure on the surface that extends the group action as a group of automorphisms.*

2.2 Hyperbolic Geometry

One of Riemann's fundamental insights into the foundations of geometry was that a metric is not inherent to the space on which it acts; that is, the same space can be given a different metric and thus giving a different geometry on the space. Non-Euclidean or Hyperbolic geometry replaces the Euclidean metric with a metric that yields a geometry that violates Euclid's fifth axiom; that given a line L and a point not on L, there exists a infinitely many lines containing that point and parallel to L. We will mostly mention only the results which will be used in later sections.

2.2.1 Hyperbolic Metric

Two models of hyperbolic geometry are the unit disk $\mathbb{D}$ and the upper-half plane $\mathbb{H}$. We will use both depending on which is most convenient in the given context. In the upper-half plane, the hyperbolic metric is given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Suppose $z(t) = x(t) + iy(t)$ is a path $\gamma : [0,1] \to \mathbb{H}$ then the length of γ is given by

$$\ell(\gamma) = \int_0^1 \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{y(t)} dt = \int_0^1 \frac{|\frac{dz}{dt}|}{y(t)} dt.$$

The hyperbolic distance function $\rho(z,w)$ is given by

$$\rho(z,w) = \inf\{\ell(\gamma) | \gamma \text{ is a path from } z \text{ to } w\}$$

and it can be checked that it defines a metric on the upper-half plane. Homeomorphisms T that preserve distance, i.e. $\rho(Tz, Tw) = \rho(z, w)$, are called **isometries**. We know that transformation of $PSL(2, \mathbb{R})$ are conformal homeomorphisms of the upper-half plane, but the following theorem shows that they also isometries.

Theorem 2.2.1. *For any $T \in PSL(2, \mathbb{R})$ and any path γ we have $\ell(T(\gamma) = \ell(\gamma)$.*

As a result, transformations of $PSL(2, \mathbb{R})$ preserve the hyperbolic distance function. If we denote the isometry group by $Isom(\mathbb{H})$ then we have $PSL(2, \mathbb{R}) \subset Isom(\mathbb{H})$. For any $z \in \mathbb{H}$, the tangent space $T_z\mathbb{H}$ has inner product

$$< x, y >= \frac{< x, y >_e}{Im(z)^2},$$

where x,y are tangent vectors and $< \cdot, \cdot >_e$ is the Euclidean inner-product. It is easily checked using $\cos(\theta) = \frac{<x,y>}{|x||y|}$ that the angle defined in hyperbolic geometry is the same as the Euclidean angle. Now since isometries preserve the inner-product, it follows that they are conformal if and only if they preserve the orientation. Thus we have the following

Theorem 2.2.2 (Poincaré, 1882). *The orientation-preserving isometries of $\mathbb{H}$ are given by*

$$\frac{\alpha z + \beta}{\gamma z + \delta}$$

where α, β, γ and δ are real numbers such that $\alpha\delta - \beta\gamma = 1$. Thus we have $Isom^+(\mathbb{H}) = PSL(2, \mathbb{R})$.

Theorem 2.2.3. *Geodesics in $\mathbb{H}$ are either semi-circles with center on the real axis or Euclidean lines perpendicular to the real axis.*

Theorem 2.2.4. *The action of $PSL(2, \mathbb{R})$ is*
(1) transitive on $\mathbb{H}$,
(2) transitive on the set of all geodesics in $\mathbb{H}$,
(3) doubly transitive on $\mathbb{R} \cup \{\infty\}$.

2.2.2 Gauss-Bonnet Theorem

For a subset F of the hyperbolic plane, area is defined by

$$\mu(F) = \int\int_F \frac{dxdy}{y^2}.$$

Theorem 2.2.5. *Hyperbolic area is invariant under the action of $PSL(2,\mathbb{R})$.*

Theorem 2.2.6 (Gauss-Bonnet Theorem). *Let Δ be a hyperbolic triangle with angles α, β and γ. Then $\mu(\Delta) = \pi - \alpha - \beta - \gamma$.*

Theorem 2.2.7. *There exists a hyperbolic triangle with angles α, β, γ if and only if $\alpha + \beta + \gamma < \pi$.*

Proof. Suppose that α, β, and γ are non-zero and that $\alpha + \beta + \gamma < \pi$. Consider the disk model $\mathbb{D}$ and two rays from the origin L_1 and L_2 that make an angle of α. Consider the triangle with one vertex at the origin, $P \in L_1$ and $\infty \in L_2$. As P varies from $\infty \in L_1$ to the origin we have that the angle at P, $\theta(P)$ satisfies

$$0 < \theta(P) < \pi - \alpha$$

since $\beta < \beta + \gamma < \pi - \alpha$, at some point on L_1 we must have $\theta(P) = \beta$. Now for each point Q on the line segment from the origin to P there exists a geodesic connecting Q to some point $R(Q)$ on L_2 which makes an angle β at Q and some angle $\gamma(Q)$ at $R(Q)$. As Q approaches the origin the hyperbolic area of this triangle tends to zero and thus by the Gauss-Bonnet theorem the angle $\gamma(Q)$ approaches $\pi - \alpha - \beta$. As Q approaches P, $\gamma(Q)$ tends to zero. Thus $\gamma(Q)$ satisfies

$$0 < \gamma(Q) < \pi - \alpha - \beta$$

now since $\gamma < \pi - \alpha - \beta$ we can choose Q on L_1 so that $\gamma(Q) = \gamma$. The triangle with vertices at the origin, Q and $R(Q)$ now have angles α, β and γ and this completes the proof. □

Theorem 2.2.8. *Let $\mathscr{P}$ be a n-sided convex hyperbolic polygon with angles $\{\alpha_i\}_1^n$. Then its hyperbolic area is given by $\mu(\mathscr{P}) = (n-2)\pi - \sum \alpha_i$.*

Proof. Since $\mathscr{P}$ is convex, there is a point which connects to the vertices by hyperbolic geodesics, each contained in $\mathscr{P}$. So now we get a triangulation and we can sum the areas of the triangles. □

2.2.3 Classification of Isometries

The isometries of $PSL(2,\mathbb{R})$ are classified by the number of fixed points. Let $T = \frac{az+b}{cz+d}$, where $ad - bc = 1$. The fixed points of T are $Tz = z$, which gives $z = [(a-d) \pm \sqrt{\Delta}]/2c$, where $\Delta = tr(T)^2 - 4$ and $tr(T) = a + d$.

Hyperbolic: $\Delta > 0$

In this case T is called **hyperbolic** and there are two fixed points α and $\beta \in \mathbb{R}$. Consider the mapping $S(z) = \frac{z-\alpha}{z-\beta}$ which maps α to 0 and β to ∞. Now the mapping $R := STS^{-1}$ is in the same conjugacy class as T but fixes 0 and ∞. Thus R has the form $R(z) = \lambda z$, with $\lambda > 0$. For $\lambda > 1$ the fixed point 0 is repelling, whereas infinity is attracting and vice-versa for $0 < \lambda < 1$. The diagram below illustrates the action of T on the upper-half plane.

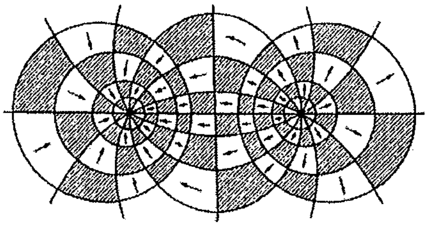


Figure 1: [DFN85] Action of a hyperbolic element on the upper-half plane.

The family of Euclidean circles through α and β are called **hypercycles**[1] and are T-invariant. Let $w = T(z)$, then the family of Euclidean concentric circles in the w-plane, $|w| = |\frac{z-\alpha}{z-\beta}| = k$ pulls back under S to a family of concentric circles in the z-plane known as the **circles of Apollonius**[2], each orthogonal to every hypercycle.

Parabolic: $\Delta = 0$

In this case T is called **parabolic** and there is one fixed point $\alpha \in \mathbb{R}$. Consider the mapping $S(z) = \frac{1}{z-\alpha}$ which maps α to ∞. Now the mapping $R := STS^{-1}$ is conjugate to T and fixes ∞. Thus R has the form $R(z) = z+b$ and its action on $\mathbb{H}$ in the w-plane is given by translation. Each Euclidean circle tangent to the real-axis at α is T-invariant and is called a **horocycle**.

[1] Also called Steiner circles of the first kind.

[2] Also called Steiner circles of the second kind.

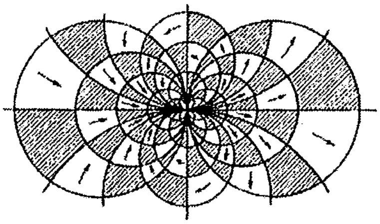


Figure 2: [DFN85] Action of a parabolic element on the upper-half plane.

Elliptic: $\Delta < 0$

In this case T is called **elliptic** and there are two fixed points ζ and $\overline{\zeta}$ where we choose $\zeta \in \mathbb{H}$. Consider the map $S(z)$ which maps ζ to i and $\overline{\zeta}$ to $-i$. Thus by them map $R := STS^{-1}$, every elliptic element is conjugate to a map R that fixes i and $-i$. Now apply the Cayley transform on the upper-half plane ($\varphi(z) = \frac{z-i}{z+i}$) and we get that $\varphi R \varphi^{-1}$ is a biholomorphic map of the unit disk that fixes 0. Thus the action of an elliptic element in the disk model is just rotation, $\varphi R \varphi^{-1}(z) = e^{i\theta} z$. Each Euclidean circle that has ζ and $\overline{\zeta}$ as inverse points is T-invariant and is called an **oricycle**.

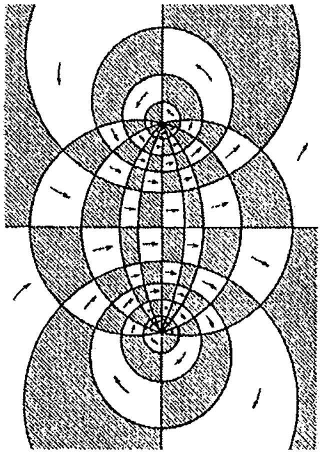


Figure 3: [DFN85] Action of an elliptic element on the upper-half plane.

Theorem 2.2.9. *Two non-identity elements of $PSL(2, \mathbb{R})$ commute if and only if they have the same fixed-point set.*

Proof. Suppose T_1 and T_2 have the same fixed point set. We have the following cases:
Case(1): T_1 and T_2 are elliptic fixing $\xi \in \mathbb{D}$. Choose an isometry R with $R(\xi) = 0$, then for $i := 1, 2$: $\widetilde{T_i} := RT_iR^{-1}$ fixes the origin and so is a rotation. Thus $\widetilde{T_1}$ and $\widetilde{T_2}$ commute which implies that T_1 and T_2 commute.
Case(2): T_1 and T_2 are parabolic fixing $\alpha \in \mathbb{R}$, as above we can consider conjugation by an appropriate isometry so that we may assume $\alpha = \infty$. Then T_1 and T_2 are translations and so they commute.
Case(3): T_1 and T_2 are hyperbolic fixing α and β in $\mathbb{R}$. As the isometries act doubly transitive on $\mathbb{R} \cup \{\infty\}$ we may conjugate and assume $\alpha = 0$ and $\beta = \infty$. Then the transformations have the form: $T_i(z) = \lambda z$ and so they commute.

Conversely, suppose T_1 and T_2 commute. Again we have three cases:
Case(1): Suppose T_1 is elliptic fixing $\xi \in \mathbb{D}$. Then T_1 fixes $T_2(\xi)$ and so T_2 is also elliptic with the same fixed-point set as T_1.
Case(2): Suppose T_1 is parabolic fixing $\alpha \in \mathbb{R}$, then T_1 fixes $T_2(\alpha)$. Since α is the only fixed point for T_1 it follows that $T_2(\alpha) = \alpha$. Now we check that T_2 cannot fix any other point: Suppose $T_2(\beta) = \beta$ and $\beta \neq \alpha$. Then T_2 must also fix $T_1(\beta)$ but as it cannot have more than two fixed points it follows that $T_1(\beta)$ is either α or β. Since T_1 is parabolic it has only one fixed-point and so it cannot have another, so we get $T_1(\beta) = \alpha$. But now using that T_1 is a 1-1 mapping it follows that $\alpha = \beta$ and we have a contradiction. So T_2 has no other fixed points and the result follows.
Case(3): Suppose T_1 is hyperbolic fixing α and β in $\mathbb{R}$. Then T_1 fixes $T_2(\alpha)$ and $T_2(\beta)$. So we either have $T_2(\alpha) = \alpha$, $T_2(\beta) = \beta$ or $T_2(\alpha) = \beta$, $T_2(\beta) = \alpha$. In the first case there is nothing more to prove so we check that the second case leads to a contradiction. T_2 must have at least one fixed point, call it γ. Then T_2^2 must be the identity map as it fixes three points: α, β, and γ. But every element of finite order is elliptic and so γ is the unique fixed point of T_2. But now T_2 fixes $T_1(\gamma)$ which implies that $T_1(\gamma) = \gamma$. This contradicts that T_1 cannot have more than two fixed points and the proof is finished. □

2.3 Properly Discontinuous Group Actions

Let X be a Hausdorff, locally compact topological space and G a subgroup of homeomorphisms of X. Recall that G acts **discontinuously** on X if and only if for every point $x \in X$ there exists a neighbourhood V such that $\forall g \neq Id\ g(V) \cap V = \emptyset$. Such an action is fixed-point free[3] with discrete orbits. If we want to allow fixed-points in the group action, we will need to refine the definition as follows:

Definition 2.3.1. G acts **properly discontinuously** if $\forall x \in X$ the G-orbit Gx is locally finite (i.e. $\forall K$ compact $\subset X$ $\{g \in G | g(x) \in K\} = Gx \cap K$ is finite).

Recall that a space is discrete if and only if no subset has a limit point. So for example $\{\frac{1}{n}\} \subset [0,1]$ is not discrete. Note that if a group acts properly discontinuously then the orbits are closed sets. Notice, also, that if G acts properly discontinuously on a compact Hausdorff space, then the group G is necessarily finite, since for any point $x \in X$ we have a bijection $G \leftrightarrow Gx$ with the orbits of x. But as the orbits are closed and discrete in a compact space they are finite.[4] We have the following proposition which characterizes properly discontinuous actions.

Proposition 2.3.2. *G acts properly discontinuously if and only if each G-orbit is discrete and each point has finite stabilizer (i.e. $\forall x \in X$ $Stab_G(x)$ is finite).*

Proof. If G acts properly discontinuously and a G-orbit Gx had a limit point p in X, then by local compactness there would exist a compact set containing p that doesn't satisfy the locally finite condition. Similarly the stabilizer of x must be finite, otherwise the singleton $\{x\}$ would be a compact set which doesn't satisfy the locally finite condition. The converse follows since the orbits would be closed and discrete in a compact set K, so must be finite. □

There are several equivalent definitions throughout literature, some of which we record here for convenience.

[3]Conversely, a free action by a finite group on a Hausdorff space acts discontinuously.

[4]Recall, a discrete space is compact if and only if it is finite. In particular, if X is compact with $A \subset X$ closed and discrete, then A is finite.

Theorem 2.3.3. *The following are equivalent:*
(1) G acts properly discontinuously on X.
(2) $\forall x \in X$, $\exists$ neighbourhood of x, V_x such that $\forall g \in G$ $g(V_x) \cap V_x \neq \emptyset$ then $g(x) = x$ and $Stab_G(x)$ is finite.
(3) $\forall x \in X$, $\exists$ neighbourhood of x, V_x such that $\{g \in G | g(V_x) \cap V_x \neq \emptyset\}$ is finite.

Proof. (2) $\Leftrightarrow$ (1): This follows since the first part of condition (2) says that G-orbits are discrete unless they are elements of the stabilizer.
(1) $\Rightarrow$ (3): Since the orbits are discrete, we can choose a sufficiently small neighbourhood V around each point x such that $\{g \in G | g(V) \cap V \neq \emptyset\} = Stab_G(x)$ and since the stabilizer of x is finite, condition (3) follows.
(3) $\Rightarrow$ (1): Suppose that the G-orbit Gx is not discrete, so it has a limit point, say $p \in X$. Then we can find a neighbourhood V_p about the point p such that $\{g \in G | g(V_p) \cap V_p \neq \emptyset\}$ is infinite which gives a contradiction. Similarly, the stabilizer of any point x is finite. □

There is another common definition of properly discontinuous action that is stronger than the one given here:

Proposition 2.3.4. *If $\forall K$ compact $\subset X$ $\{g \in G | g(K) \cap K \neq \emptyset\}$ is finite, then G acts properly discontinuously.*

Proof. Let $K = \{x\}$ for any point $x \in X$, then the condition says that the stabilizer of x is finite. If the G-orbit of x had a limit point p, then by local compactness we would be able to choose a compact neighbourhood K around p such that $\{g \in G | g(K) \cap K \neq \emptyset\}$ is an infinite set which contradicts the condition. So the G-orbits are discrete with finite stabilizers and so G acts properly discontinuously. □

The converse in general is false as the following gives a counter-example.

Example 2.3.1. Consider $X = \mathbb{R}^2/\{0\}$ with subspace topology and G is the infinite cyclic group generated by $T(x, y) = (2x, \frac{1}{2}y)$. The only fixed point is the origin which is not in the space X, so G acts properly discontinuously. Consider the unit cube K in X, K=$\{(x, y) \in X||x| \leq 1, |y| \leq 1\}$. K is compact in the subspace topology since it is the intersection of the closed unit cube in $\mathbb{R}^2$ centered at the origin. By considering the image of K by elements of G, $T^n(x, y) = (2^n x, \frac{1}{2^n}y)$, we see that the set $\{g \in G \mid g(K) \cap K \neq \emptyset\}$ is infinite.

If, however, X is metrizable and the group G acts as isometries then the definitions become equivalent.

Theorem 2.3.5. *Let (X, d) be a locally compact metric space with G acting as isometries. Then G acts properly discontinuously if and only if for any compact set K in X, $\{g \in G | g(K) \cap K \neq \emptyset\}$ is finite.*

First we need a lemma:

Lemma 2.3.6. *Let $\{g_n\}$ be distinct sequence of isometries and with $y_n = g_n(x_n)$. Then $d(g_m \circ g_n^{-1}(y_k), y_k) \leq 2d(y_k, y_n) + d(x_n, x_m) + d(y_m, y_n)$.*

Proof. Applying the triangle inequality twice and using that the group acts as isometries we get:

$$\begin{aligned} d(g_m \circ g_n^{-1}(y_k), y_k) &\leq d(g_m \circ g_n^{-1}(y_k), g_m \circ g_n^{-1}(y_n)) + d(g_m \circ g_n^{-1}(y_n), y_k) \\ &\leq d(g_m \circ g_n^{-1}(y_k), g_m \circ g_n^{-1}(y_n)) + d(g_m \circ g_n^{-1}(y_n), y_n) + d(y_n, y_k) \\ &= d(y_k, y_n) + d(g_m \circ g_n^{-1}(y_n), y_n) + d(y_n, y_k) \\ &= 2d(y_n, y_k) + d(g_m(x_n), y_n) \end{aligned}$$

and $d(g_m(x_n), y_n) \leq d(g_m(x_n), g_m(x_m)) + d(g_m(x_m), y_n)$, but again since the group acts as isometries, this equals $d(x_n, x_m) + d(y_m, y_n)$ and this completes the proof. □

Proof. (of theorem 1.3.5)Let G act properly discontinuously so that the orbits are discrete and stabilizers are finite. Suppose for some compact K we have $\{g \in G | gK \cap K \neq \emptyset\}$ is infinite so that there exist a sequence of distinct elements of G, $\{g_n\}$ and elements $\{x_n\}$, $\{y_n\}$ of K with $y_n = g_n(x_n)$. Since K is compact we can take subsequences and get $x_n \to x$ and $y_n \to y$ for some x and y in K. By lemma 1.3.6 we have

$$d(g_m \circ g_n^{-1}(y_k), y_k) \leq 2d(y_k, y_n) + d(x_n, x_m) + d(y_m, y_n)$$

then let $k \to \infty$ and we get

$$d(g_m \circ g_n^{-1}(y), y) \leq 2d(y, y_n) + d(x_n, x_m) + d(y_m, y_n).$$

Now for any ε we can choose m and n sufficiently large so that $d(y, y_n)$, $d(x_n, x_m)$ and $d(y_m, y_n)$ are less than ε. Then

$$d(g_m \circ g_n^{-1}(y), y) \leq 2\varepsilon + \varepsilon + \varepsilon = 4\varepsilon.$$

So infinitely many elements $g_m \circ g_n^{-1}(y)$ in the orbit of y get infinitely close to y. Since the orbits are discrete these must be infinitely many distinct elements in the stabilizer and this gives a contradiction. □

2.4 Uniformization Theorem

The universal cover for a torus is constructed by tessellating the Euclidean plane by parallelograms with sides identified. For a compact surface of genus $g > 1$ the corresponding polygon model cannot tessellate the plane by congruent polygons, however such a tessellation can be constructed in the hyperbolic plane using non-euclidean geometry.

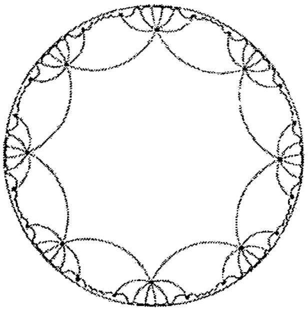


Figure 4: [Hen79] Universal cover of a compact genus two surface.

It follows from the uniformization theorem that almost all Riemann surfaces have the hyperbolic plane as their universal cover.

Theorem 2.4.1 (Uniformization Theorem). *Every connected Riemann surface X is conformally equivalent to $\widetilde{X}/\Lambda$, where $\widetilde{X}$ is the universal cover of X and Λ is a subgroup of $Aut(\widetilde{X})$ acting discontinuously on $\widetilde{X}$. Moreover, if X is not the Riemann sphere, the plane, the punctured plane or a torus then $\widetilde{X}$ is the upper-half plane $\mathbb{H}$ with Λ a subgroup of $PSL(2, \mathbb{R})$.*

Proof. See [JS87] □

Remark 2.4.2. We will be mostly working with Riemann surfaces with universal covering $\mathbb{H}$ so henceforth Λ will always be a subgroup of $PSL(2, \mathbb{R})$ acting discontinuously on $\mathbb{H}$.

Theorem 2.4.3. *The Riemann surface $\mathbb{H}/\Lambda_1$ is conformally equivalent to $\mathbb{H}/\Lambda_2$ if and only if Λ_1 and Λ_2 are conjugate subgroups in $PSL(2, \mathbb{R})$.*

Proof. Suppose $\varphi : \mathbb{H}/\Lambda_1 \to \mathbb{H}/\Lambda_2$ is a biholomorphic map. Then we can lift φ to an automorphism T of the universal cover and so is an element of $PSL(2,\mathbb{R})$ such that the following diagram commutes $\varphi \circ \pi_1 = \pi_2 \circ T$:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{T} & \mathbb{H} \\ {\scriptstyle \pi_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \pi_2} \\ \mathbb{H}/\Lambda_1 & \xrightarrow[\varphi]{} & \mathbb{H}/\Lambda_2 \end{array}$$

equivalently, $\varphi([z]_{\Lambda_1}) = [T(z)]_{\Lambda_2}$. We want to show that $T\Lambda_1T^{-1} = \Lambda_2$ but first we claim that $T\Lambda_1T^{-1} \subseteq \Lambda_2$. For any $R_1 \in \Lambda_1$ we have $\varphi \circ \pi_1 = \varphi \circ \pi_1 \circ R_1$, which together with the commutativity of the diagram implies $\varphi \circ \pi_1 = \varphi \circ \pi_1 \circ R_1 = \pi_2 \circ T \circ R_1$. This gives that for all $z \in \mathbb{H}$ $[T(z)]_{\Lambda_2} = [T(R_1(z))]_{\Lambda_2}$. This implies that there exists $R_2 \in \Lambda_2$ such that $R_2(T(z)) = T(R_1(z))$ i.e. $R_2 = TR_1T^{-1}$ which proves the claim. Similarly, since T and φ are automorphisms, their arrows in the diagram above can be reversed to give another commutative diagram proving $T^{-1}\Lambda_2T \subseteq \Lambda_1$. These results prove $T\Lambda_1T^{-1} = \Lambda_2$.

Conversely, let Λ_1 and Λ_2 be conjugate groups. Then there exists T in $PSL(2,\mathbb{R})$ such that $T\Lambda_1T^{-1} = \Lambda_2$. Define a map:

$$\begin{aligned} \varphi : \mathbb{H}/\Lambda_1 &\longrightarrow \mathbb{H}/\Lambda_2, \\ [z]_{\Lambda_1} &\mapsto [T(z)]_{\Lambda_2} \end{aligned}$$

We show that φ is well-defined, bijective and biholomorphic.
Well-defined: Suppose $[z]_{\Lambda_1} = [z']_{\Lambda_1}$, then $z' = R(z)$ for some $R \in \Lambda_1$. Now

$$\varphi([z']_{\Lambda_1}) = [T(z')]_{\Lambda_2} = [TR(z)]_{\Lambda_2} = [TRT^{-1}(T(z))]_{\Lambda_2} = [T(z)]_{\Lambda_2} = \varphi([z]_{\Lambda_1})$$

The second last equality follows from $TRT^{-1} \in \Lambda_2$.
One-to-one: Suppose $\varphi([z_1]_{\Lambda_1}) = \varphi([z_2]_{\Lambda_1})$ which gives $[T(z_1)]_{\Lambda_2} = [T(z_2)]_{\Lambda_2}$ and so $\exists S \in \Lambda_2$ such that $S(T(z_1)) = T(z_2)$ so that $T^{-1}ST(z_1) = z_2$ with $T^{-1}ST \in \Lambda_1$. So $[z_1]_{\Lambda_1} = [z_2]_{\Lambda_1}$ and φ is one-to-one.
Onto: Let $[z_2]_{\Lambda_2} \in S_2$. Then there exists $z_1 \in \mathbb{H}$ such that $T(z_1) = z_2$ since T is a bijection of the upper-half plane. Now $\varphi([z_1]_{\Lambda_1}) = [T(z_1)]_{\Lambda_2} = [z_2]_{\Lambda_2}$ and so φ is onto.
Biholomorphic: In local coordinates, φ is given by T which is biholomorphic on $\mathbb{H}$. □

Corollary 2.4.4. *If X is uniformized by Λ, i.e. $X = \mathbb{H}/\Lambda$, then*
(1) $\varphi \in Aut(\mathbb{H}/\Lambda)$ if and only if $\varphi([z]_\Lambda) = [T(z)]_\Lambda$ for some unique T in $N(\Lambda)$, where $N(\Lambda)$ is the normalizer in $PSL(2,\mathbb{R})$.
(2) $Aut(\mathbb{H}/\Lambda) \cong N(\Lambda)/\Lambda$.

Proof. (1) This follows from the proof of the previous theorem.
(2) By (1) be can consider the homomorphism $\psi : N(\Lambda) \to Aut(\mathbb{H}/\Lambda)$ which has kernel Λ. Now by first isomorphism theorem, we have $Aut(\mathbb{H}/\Lambda) \cong N(\Lambda)/\Lambda$. □

2.5 Riemann Surfaces

This section covers the main properties of holomorphic maps that will be used later; in particular the Riemann-Hurwitz formula between holomorphic maps of compact Riemann surfaces will be used to prove the Hurwitz bound for the maximal order of the automorphism group. For more information on Riemann surfaces and for proofs see [For81], [FK80], or [Mir95].

2.5.1 Holomorphic maps and Branched Coverings

The following useful theorem can be proved by applying the identity theorem of complex analysis in local coordinates.

Theorem 2.5.1 (Identity Theorem). *Suppose X and Y are Riemann surfaces and $f, g : X \to Y$ such that $f = g$ on a subset having a limit point in X. Then $f \equiv g$.*

Theorem 2.5.2 (Riemann's Removable Singularities Theorem). *Let f be a bounded holomorphic function on the punctured disk $\mathbb{D}^*$, then there exists a unique extension $\tilde{f}$ holomorphic on $\mathbb{D}$.*

This theorem can be applied to local coordinates and so still holds for open subsets of Riemann surfaces.

Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces and let $p \in X$. Then it is a theorem of complex analysis that there are charts centered at p and $f(p)$ such that locally the map f is given by $z \mapsto z^n$ for some unique integer n called the **multiplicity** of f at p, denoted by $mult_p(f)$. The points in X with non-trivial multiplicity ($n \geq 2$) are called **ramification** points and they form a discrete subset. The images under f of the ramification points are called **branch points**.

Theorem 2.5.3. *Suppose X and Y are Riemann surfaces and $f : X \to Y$ is a non-constant holomorphic map.*
(1) f is an open and discrete map.
(2) If f is one-to-one then f is biholomorphic onto $f(X)$.
(3) If X is compact, then Y is compact with f surjective.

Note that if f is unbranched (no ramification or branch points) then it is a local homeomorphism.

Theorem 2.5.4. *Let $f : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then the number of ramification points are finite and $f : X \setminus f^{-1}(B) \to Y \setminus B$ where B is the set of branch points, is an n-sheeted covering map for some non-negative integer n. Moreover, for any point $q \in Y$ we have*

$$\sum_{p \in f^{-1}(q)} mult_p(f) = n.$$

It will be useful to know that every lift of a holomorphic map to a covering is also holomorphic.

Theorem 2.5.5. *Suppose $p : X \to Y$ is an unbranched holomorphic map and $f : Z \to Y$ is any holomorphic map. Then every lift $\widetilde{f} : Z \to X$ is also holomorphic.*

Theorem 2.5.6. *Suppose $p : X \to Y$ is a local homeomorphism between a Riemann surface X and a Hausdorff space Y. Then there exists a unique complex structure on Y such that p is locally biholomorphic map. Conversely, if $p : X \to Y$ is a covering map and Y is a Riemann surface, then there exists a unique Riemann surface structure on X such that p is a holomorphic covering.*

The following theorem describes finite-sheeted coverings of the punctured disk $\mathbb{D}^* = \{0 < |z| < 1\}$.

Theorem 2.5.7. *Suppose X is a Riemann surfaces and $f : X \to \mathbb{D}^*$ is a finite-sheeted holomorphic covering (unbranched). Then there exists a biholomorphic map $\varphi : X \to \mathbb{D}^*$ such that $f = p_k \circ \varphi$ where $p_k(z) = z^k$ for some $k \geq 1$.*

Example 2.5.1. Let $\mathbb{D}^*$ be the punctured unit disk. Then the fundamental group of $\mathbb{D}^*$ is infinite cyclic and every finite-sheeted n-covering of $\mathbb{D}^*$, $f : \mathbb{D}^* \to \mathbb{D}^*$ is defined by $z \mapsto z^n$. This unramified covering can be extended to a branched covering by defining $f(0) = 0$.

Theorem 2.5.8 (Extending Coverings). *Let X be a connected Riemann surface with B a discrete set in X. Then any finite-sheeted covering of $X^* = X \setminus B$, $p : Y^* \to X^*$, can be extended to a finite-sheeted branched covering $p : Y \to X$ with branch points B and a biholomorphic map $\varphi : Y \setminus p^{-1}(B) \to Y^*$ that preserves the fibers.*

Let X^* denote $X \setminus B$ where X is compact and B a finite subset. If p is a finite-sheeted covering $p : Y^* \to X^*$ then any covering transformation of Y^* can be extended to a covering transformation of Y.

Theorem 2.5.9 (Riemann-Hurwitz Formula). *Let $\varphi : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces with genus g and $\tilde{g}$ respectively. Then*

$$2g - 2 = deg(\varphi)(2\tilde{g} - 2) + \sum_{p \in X} (mult_p(\varphi) - 1).$$

Remark 2.5.10. Note that the ramification points are a discrete subset of a compact space X, and so are finite. So there are only a finite set of points with non-trivial multiplicity and this makes the summation in the Riemann-Hurwitz formula a finite sum.

Proof. Let the branch points be the vertices of a triangulation of Y which has V vertices, E edges and F faces. When we lift this triangulation to X we get deg(φ) faces and deg(φ) edges. We need to count the number of vertices we get on X. Let $q \in Y$ be a vertex, then the number of vertices on X is

$$\sum_{\text{vertices q in Y}} |\varphi^{-1}(q)|.$$

But we can write $|\varphi^{-1}(q)| = \sum_{p \in \varphi^{-1}(q)} 1 = \deg(\varphi) + \sum_{p \in \varphi^{-1}(q)} (1 - mult_p(\varphi))$. After substituting we get that the total number of vertices on X is

$$\deg(\varphi)V - \sum_{\text{vertices p in X}} (mult_p(\varphi) - 1).$$

Now calculating the Euler characteristic on X, we get the Riemann-Hurwitz formula. □

2.5.2 Riemann-Roch Theorem

Let X be compact Riemann surface and $M(X)$ its field of meromorphic functions. A **divisor** D is finite 0-chain $\sum n_i p_i$ on X with $n_i \in \mathbb{Z}$ and $p_i \in X$. The **degree** of D is defined to be $\sum n_i$ and **effective** if each $n_i \geq 0$. For an effective divisor D we write $D \geq 0$ and similarly $D \geq \widetilde{D}$ if $D - \widetilde{D} \geq 0$ for any two divisors D and $\widetilde{D}$. If $f \in M(X)$ then (f) is the associated divisor

$$-\sum n_i p_i + \sum n_j q_j,$$

where p_i and q_j are the finite number of poles and zeros respectively with n_i is the order of the pole or zero at the corresponding point. Similarly, a meromorphic 1-form ω is locally $\omega = f(z)dz$ for some meromorphic function f, and the associated divisor (ω) is defined to be

$$\sum_{p \in X} ord_p(w),$$

where the $ord_p(w)$ is the order of the local meromorphic function f at the point p. A **canonical divisor** K is the divisor of any meromorphic differential.

For any divisor on X define $L(D) = \{f \in M(X) | (f) + D \geq 0\}$. Then $L(D)$ is a finite dimensional vector space over the complex numbers define its dimension to be $\ell(L(D))$.

Theorem 2.5.11 (Riemann-Roch Theorem). *Let X be a compact Riemann surface of genus g. If D is any divisor on X and K is a canonical divisor then*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

As an application of the Riemann-Roch theorem, it can be proved that the space or moduli of complex structures on a surface of genus g has dimension $3g - 3$. This is Riemann's parameter count. We will show how to count parameters for hyperelliptic surfaces. Recall that a compact Riemann surface X of genus $g \geq 2$ is called **hyperelliptic** if X is a two sheeted branched covering of the Riemann sphere, i.e. there exists a 2 to 1 map $f : X \to \mathbb{P}$. By the Riemann-Hurwitz formula there must exist $2g + 2$ branch points. Conversely, given any $2g + 2$ points on $\mathbb{P}$, it's possible to construct a unique hyperelliptic Riemann surface branched over these $2g + 2$ points[GH78]. But there always exists an automorphism of the Riemann sphere that maps any 3 branch points to 0,1 and ∞. Therefore, the number of parameters needed to describe a hyperelliptic surfaces of genus g is $(2g+2)-3=2g-1$. In particular, we have the following result

Theorem 2.5.12. *The generic Riemann surface of genus g is not hyperelliptic.*

Definition 2.5.13. Let X be compact Riemann surface of genus $g \geq 2$. Then a point $p \in X$ is a **Weierstrass point** if and only if there exists a meromorphic function f on X which has a pole of order $\leq g$ at the point p and holomorphic everywhere else on X.

The following results about hyperelliptic Riemann surfaces will be useful later.

Proposition 2.5.14. *Let X be a compact Riemann surface. Then*
(1) X has at least $2g+2$ Weierstrass points and exactly $2g+2$ if and only if X is hyperelliptic.
(2) X is hyperelliptic if and only if there exists an automorphism J such that $J^2=1$ and fixes $2g+2$ points, and in this case the fixed points of J are exactly the Weierstrass points.
(3) If X is hyperelliptic, then any non-identity automorphism distinct from J has at most four 4 fixed points.

Proof. See [FK80]. □

2.5.3 Complex Algebraic Curves

A complex plane curve is defined to be $C=\{[x,y,z]\in\mathbb{C}P^2|P(x,y,z)=0\}$ for some irreducible homogeneous polynomial P. If the polynomial is non-singular then the curve C defines a compact Riemann surface in the complex projective plane.

Theorem 2.5.15 (Hilbert's Nullstellensatz, [Kir92]). *Complex polynomials $P(x,y)$ and $Q(x,y)$ define the same zero-set $\{P=0\}=\{Q=0\}$ if and only if P and Q have the same irreducible factors with possibly different multiplicities. Moreover, if they have no repeated factors then $P=\lambda Q$ for some non-zero constant $\lambda\in\mathbb{C}$.*

Theorem 2.5.16. *The holomorphic automorphisms of $\mathbb{C}P^2$ are given by $PGL(3,\mathbb{C})$.*

For a proof of the following theorem see [Kir92] or [Mir95].

Theorem 2.5.17 (Bezout's Theorem). *Let C and D be non-singular complex projective curves in $\mathbb{C}P^2$ with degree n and m respectively. If they have no common component then the number of intersections counting multiplicities is nm.*

In particular, any two curves in $\mathbb{C}P^2$ intersect in at least one point. As a corollary we have

Corollary 2.5.18. *Every non-singular homogeneous polynomial is irreducible.*

Proof. Suppose $f = P(x,y,z)Q(x,y,z)$ is non-singular and consider the two curves given by $\{P = 0\}$ and $\{Q = 0\}$ with an intersection point p. Then $\nabla f = (\nabla P)Q + P(\nabla Q)$ vanishes at p and this contradicts that f is non-singular and so f must be irreducible. □

Theorem 2.5.19 (Plücker's Formula). *Let C be a nonsingular plane curve given by $\{[x,y,z] \in \mathbb{C}P^2 | P(x,y,z) = 0\}$ with $P(x,y,z)$ a homogeneous polynomial of degree d. Then the genus is given by*

$$g = \frac{(d-1)(d-2)}{2}.$$

In particular, not all Riemann surfaces can be realized as non-singular curves in $\mathbb{C}P^2$.

Example 2.5.2 (Fermat Curve). Consider the complex projective curve defined by the non-singular homogeneous polynomial $z_1^d + z_2^d + z_3^d$ for $d \geq 1$. Then it follows from the degree-genus formula that the defining compact Riemann surface has genus $\frac{(d-1)(d-2)}{2}$.

It is useful to know when an abstract compact Riemann surface can be realized as a smooth plane curve.

Theorem 2.5.20. *Every compact Riemann surface of genus $g = 3$ can be realized as a non-singular plane curve if it is not hyperelliptic.*

Proof. See [FK80] □

It turns out that every compact Riemann surface can be embedded in $\mathbb{C}P^3$.

3 Hurwitz's Theorem

We know that the automorphism group of the Riemann sphere is the infinite group of Möbius transformations $\frac{\alpha z+\beta}{\gamma z+\delta}$ where $\alpha\delta - \beta\gamma \neq 0$. For Riemann surfaces of genus $g = 1$, the automorphism groups are infinite. In 1878 Schwarz proved that for genus $g \geq 2$, surprisingly, the automorphism groups are finite. It was Hurwitz, which in the same 1893 paper where he gives his well-known Riemann-Hurwitz formula, gives the upper-bound $84(g-1)$ for the number of automorphisms. That is, regardless of the complex structure, the topology of the surface (determined by the genus) gives the restriction on the maximal order of the automorphism group.

3.1 Schwarz's Theorem

Lemma 3.1.1. *Let X be a compact Riemann surface of genus $g \geq 2$. Any non-identity automorphism of X has at most $2g+2$ fixed points. Moreover, Weierstrass points get mapped to Weierstrass points.*

Proof. See [FK80] □

Theorem 3.1.2 (H.A. Schwarz, 1878). *The automorphism group of a compact Riemann surface X of genus $g \geq 2$ is finite.*

Proof. We have an action of $Aut(X)$ on Weierstrass points $W(X)$, so consider the homomorphism $\varphi : Aut(X) \to Perm(W(X))$. Suppose X is hyperelliptic, then the hyperelliptic involution J fixes all the Weierstrass points so we have $\langle J \rangle \leq ker(\varphi)$. Now if $T \in Aut(X)$ with $T \neq J$ then T has at most 4 fixed points so it cannot fix all the $2g+2$ Weierstrass points on X. This shows that $ker(\varphi) = \langle J \rangle$ and it follows from the first isomorphism theorem that $Aut(X)$ is a finite group. Now suppose X is non-hyperelliptic, then there are more than $2g+2$ Weierstrass points and so if an automorphism fixes these points it would have more than $2g+2$ fixed points. By the lemma above, this can only be the identity map. Thus the action is faithful and so φ is injective. Again, it follows that the automorphism group is finite. □

3.2 Finite Group Actions on Riemann Surfaces

Given a finite group G, we want to find a compact surface that G acts on. We can do this just by using covering space theory. Suppose G has g

generators. Then consider the fundamental group of a surface of genus g, $\pi_1(X_g) = \langle a_1, b_1, ..., a_g, b_g | \prod_i [a_i, b_i] = 1 \rangle$ and a surjective map $\varphi : \pi_1(X_g) \to G$ such that $\varphi(a_i) = x_i$ and $\varphi(b_i) = 1$. Now the kernel of φ corresponds to a normal covering space $P : X \to X_g$. Since X_g is compact, X is also a compact surface. So we can write $X = X_{\widetilde{g}}$ for some $\widetilde{g}$. Now the deck transformation group $G(X_{\widetilde{g}})$ is isomorphic to $\pi_1(X_g)/Ker(\varphi) \cong G$. Since the deck transformation group acts freely on the covering space $X_{\widetilde{g}}$, so does G. In particular, if P is a k-sheeted covering map then $\chi(X_{\widetilde{g}}) = k\chi(X_g)$. So $\widetilde{g} = 1 - k + kg$. We get the following

Theorem 3.2.1 (Hurwitz). *Every finite group G acts freely on some compact orientable surface. Moreover, if G has n generators, then the genus on which it acts is given by $g = 1 + |G|(n-1)$.*

Proof. All that remains to show is the formula for the genus. But this follows by noting that the number of sheets k in the covering is given by the index of the kernel in the fundamental group. But in this case, this is the order of G. □

If X_g above is a Riemann surface then we can lift the complex structure to the cover X and the the transformation group acts as a group of automorphisms.

Lemma 3.2.2. *Let X be a Riemann Surface and a group G acting discontinuously on X then there exists a complex structure on X/G that makes the projection map $X \to X/G$ a holomorphic unramified covering map.*

We want to extend this result so as to allow group actions with fixed points.

Theorem 3.2.3. *Suppose we have a finite group action on a Riemann surface X, then the following hold:*
(1) The stabilizer of every point is a finite cyclic.
(2) The points with non-trivial stabilizers are a discrete subset of X.

Proof. (1) Let G be the stabilizer subgroup of a point $p \in X$ and D be a small disk neighbourhood D about p and consider

$$D^G \subset \bigcap_{g \in G} g(D),$$

where D^G is a G-invariant disk about p biholomorphic to the unit disk in the plane. By the uniformization theorem we can lift G to a finite group of automorphisms of the unit disk with the origin fixed. It follows that G is finite cyclic.
(2) Changing notation, suppose G is the finite group acting on X and $\{p_n\} \subset X$ is an infinite set of points with non-trivial stabilizers and a limit point p. The stabilizer of each p_n is finite cyclic and so is generated by some $g_n \in G$. This induces an infinite tuple $(g_1, g_2, ...)$ but as G is finite this tuple must have repetitions. So by considering a subsequence of $\{p_n\}$ we can assume each stabilizer is generated by some fixed g in G. By continuity, $g(p) = g(\lim_n p_n) = \lim_n g(p_n) = \lim_n p_n = p$ and so g also fixes p. Let I be the identity map, we then have that g and I agree on an infinite set with a limit point and so by the identity theorem of complex analysis they are equal: $g = I$. But this contradicts that g was chosen as a generator of each stabilizer which was assumed to be non-trivial, and so this proves that the points $\{p_n\}$ are discrete on X. □

The above proof tell us that locally around a fixed point, the action is given by rotation by some n-th root of unity, where n is the order of the stabilizer of the fixed point.

Theorem 3.2.4. *Let G be a finite group acting on a Riemann surface X. Then the quotient space X/G is a Riemann surface and the canonical projection map $\pi : X \to X/G$ is holomorphic branched covering map of degree $|G|$ and $\forall p \in X$ $mult_p(\pi) = |Stab_G(p)|$.*

Proof. Let $X^* = X - \{\text{fixed points}\}$ then G acts discontinuously on X^* and so gives a holomorphic unramified $|G|$-sheeted covering $X^* \to X^*/G$. Now we can extend the map to a holomorphic branched covering $X \to X/G$. □

3.3 $\mathbb{Z}/p$ - Actions on Surfaces

In this section be prove a result which can be found in a paper by Czes Kosniowski(see [Kos78]) about $\mathbb{Z}/p$-actions on surfaces with p odd prime. First we need a preliminary theorem:

Theorem 3.3.1. *For p odd , $\mathbb{Z}_p$ cannot act on a closed oriented surface S with exactly one fixed point.*

Proof. This action gives a branched covering map $\pi : S \to S/\mathbb{Z}_p$ with exactly one ramification point (fixed point) of order p and one branch point. If we remove small disks around these points then we have a normal covering map $P : \widetilde{X} \to X$ where $\widetilde{X} = S - \{\text{open disk}\}$ and $X = S/\mathbb{Z}_p - \{\text{open disk}\}$ with $\mathbb{Z}_p$ as the deck transformation group. As π is locally of the form $z \mapsto z^p$ this gives a connected p-sheeted covering on the boundary components: $(\partial\widetilde{X}, \widetilde{x}_0) = S^1 \to (\partial X, x_0) = S^1$. Now consider two possible cases:
Case(1): Suppose $S/\mathbb{Z}_p$ is orientable. The covering P gives a homomorphism φ and the composition of homomorphisms $\pi_1(\partial X, x_0) \overset{i_*}{\hookrightarrow} \pi_1(X, x_0) \overset{\varphi}{\to} \mathbb{Z}_p$ is trivial. This can be seen by looking at the polygon model for the orientable surface: the generator for ∂X is homotopic to a product of commutators in X and φ maps this into an abelian group, so the composition is trivial. This trivial homomorphism corresponds to a trivial covering of ∂X. Choose a component of $P^{-1}(\partial X)$ that contains $\widetilde{x}_0$. Now we have two distinct lifts of ∂X which agree on x_0, but as ∂X is connected and satisfies $i_*(\pi_1(\partial X, x_0)) \subset P_*(\pi_1(\partial\widetilde{X}, \widetilde{x}_0))$ the two lifts must be unique, which is a contradiction.
Case(2): Suppose $S/\mathbb{Z}_p$ is non-orientable with $P : \widetilde{X} \to X$ a normal p-sheeted covering. X has an orientable double cover S_2 and we have $\widetilde{X} \overset{k-sheets}{\to} S_2 \overset{2-sheets}{\to} X$ and this gives $\chi(\widetilde{X}) = 2k\chi(X) = p\chi(X)$ so p is even, contradiction. □

Corollary 3.3.2. *A finite group action on an orientable surface cannot have exactly one fixed point.*

Proof. Let G be the finite group acting on an orientable surface S with genus g. If an element of G fixes a point then it generates a finite cyclic subgroup which again acts with exactly one fixed point. It's enough to assume G is finite cyclic and to show that $|G|$ must be odd, then the previous theorem gives the result. Since orientation does not affect the counting of the vertices in a triangulation, we can still apply the Riemann-Hurwitz formula. Suppose S/G has a triangulation with V vertices (including the branch point), E edges and F faces. Then the number of edges $\widetilde{E}$ and faces $\widetilde{F}$ on S would be $|G|E$ and $|G|F$. Since there is only one fixed point, hence one ramification point, the number of vertices $\widetilde{V}$ on S is $|G|V - (|G| - 1)$. Then we have:

$$2g - 2 = -\chi(S) = -\widetilde{V} + \widetilde{E} - \widetilde{F} = |G|(\chi(S/G) + 1) - 1,$$

now if $|G|$ is even, the right-hand side is odd but the left-hand side is always even. This proves that $|G|$ cannot be even. □

Theorem 3.3.3. *Let n, a be non-negative integers and $g \geq 2$. (1) Any orientable closed surface S_g admits a finite cyclic group action with exactly 2-fixed points.*
(2) The surface S_{an} admits a $\mathbb{Z}_n$-action with exactly 2-fixed points.
(3) The surface S_{1+an} admits a free action by $\mathbb{Z}_n$.

Proof. (1) Consider an inscribed regular g-polygon on the equator of a sphere perpendicular to an axis of rotation. Attaching g handles to the vertices gives S_g with $\mathbb{Z}_g$ acting as rotational symmetry fixing two points corresponding to the axis.
(2) This follows from (1) by attaching handles to handles.
(3) Consider $\mathbb{Z}_n$ acting freely on the torus by rotational symmetry. Then by attaching handles as before gives a free $\mathbb{Z}_n$ action on S_{1+n}. Now by attaching handles to handles again we get the same free action on S_{1+an}. □

Theorem 3.3.4. *Let S_g be a compact surface of genus g and p a prime number. Then $\mathbb{Z}_p$ acts on S_g if and only if $g = 1 + ap$ or $g = ap + b(\frac{p-1}{2})$ for some non-negative integers a and b.*

Suppose $\mathbb{Z}_p$ acts on S_g, then the quotient-space is a compact surface $S_{\tilde{g}}$, where g and $\tilde{g}$ are related by the Riemann-Hurwitz formula:

$$2g - 2 = |G|\,(2\tilde{g} - 2) + \sum_{x \in S_g} (mult_x(\pi) - 1).$$

Since the stabilizer is a subgroup of $\mathbb{Z}_p$ and p is prime, it follows that points with non-trivial stabilizer are fixed by every element of the group and since our surface is compact, these points form a finite subset. Suppose there are n such points, and using $\sum_{x \in S_g} (mult_x(\pi) - 1) = \sum_{i=1}^{n} p - 1 = n(p-1)$, then the Riemann-Hurwitz formula now becomes:

$$g = p\tilde{g} + \frac{(n-2)(p-1)}{2}.$$

By the remarks at the beginning of this section, we cannot have $n = 1$. So it remains to check (1) $n = 0$,(2) $n \geq 2$.
Case(1): If there are no points fixed by the group action, then we have a p-sheeted unbranched covering map $\pi : S_g \rightarrow S_{\tilde{g}}$ and we can also say that

$g \geq 1$ since finite group actions on the sphere have fixed points. Now using the Euler characteristic equation $\chi(S_g) = p\chi(S_{\tilde{g}})$ we can deduce that $\tilde{g} \geq 1$. This allows us to write $\tilde{g} = a + 1$ for some non-negative integer a. Now, substituting in the Riemann-Hurwitz relation gives $g = 1 + ap$.
Case(2): The Riemann-Hurwitz formula gives: $g = p\tilde{g} + \frac{(n-2)(p-1)}{2} = ap + b(\frac{p-1}{2})$, where $a = \tilde{g}$ and $b = n - 2$ are non-negative integers. This finishes the proof. For the converse see [Kos78].

Corollary 3.3.5. *For p odd prime, $\mathbb{Z}_p$ acts freely on S_g if and only if $g = 1 + ap$ for some non-negative integer a.*

Corollary 3.3.6 ($\mathbb{Z}_p \times \mathbb{Z}_q$-Action)**.** *For p and q odd primes, if $\mathbb{Z}_p \times \mathbb{Z}_q$ acts freely on S_g then $g = 1 + n$ where n is an integer which has both p and q as divisors.*

3.4 Hurwitz's Theorem

We will now use the Riemann-Hurwitz formula to prove Hurwitz's upper-bound for automorphisms of compact Riemann surfaces of genus $g \geq 2$.

Theorem 3.4.1 (Hurwitz, 1893)**.** *Let G be a finite group acting on a compact Riemann surface X of genus $g \geq 2$, then $|G| \leq 84(g-1)$.*

Proof. Let G act on X and let $\widetilde{X} = X/G$ be a compact Riemann surface of genus $\widetilde{g}$. Then we get a branched covering map $\pi : X \to \widetilde{X}$. Suppose there are n branch points $[P_1], ..., [P_n]$ then we have the following:

$$\sum_{x \in X_g} (mult_x(\pi) - 1) = \sum_{i=1}^{n} \sum_{p \in [P_i]} (mult_p(\pi) - 1).$$

The points $p \in [P_i]$ belong to the same orbit and thus have conjugate stabilizers. So for these points, the order of the stabilizer is the same and since there are $\frac{|G|}{|Stab_G(P_i)|}$ points in the orbit, the last sum above becomes:

$$\sum_{i=1}^{n} \frac{|G|}{Stab_G(P_i)} [Stab_G(P_i) - 1] = |G| \sum_{i=1}^{n} (1 - \frac{1}{m_i}),$$

where $m_i = Stab_G(P_i)$. So we get from the Riemann-Hurwitz formula:

$$2g-2=|G|\,[2\widetilde{g}-2+\sum_{i=1}^{n}(1-\frac{1}{m_i})].$$

Now to maximize $|G|$ we must minimize the expression on the right hand side determined by the branching data $(\widetilde{g}; m_1, ..., m_n)$. It can be checked that $(0; 2, 3, 7)$ is the minimum attained and this gives $|G| \leq 42(2g-2) = 84(g-1)$. □

As a corollary of the proof, we can see that if G acts freely on a compact Riemann surface X, then $|G| = \frac{2g-2}{2\widetilde{g}-2} \leq g-1$. A finite group G is called a Hurwitz group if it acts as a group of automorphisms on some compact Riemann surface X with genus $g \geq 2$ with order $84(g-1)$. Hurwitz went further and gave a criterion for when a finite group can be realized as a Hurwitz group.

Theorem 3.4.2 (Hurwitz's Criterion). *A non-trivial finite group G is a Hurwitz group if and only if G is generated by x and y which satisfy the relations: $x^2 = y^3 = (xy)^7$ with possibly some other relations.*

Proof. Suppose G is generated by x and y with the above relations. The fundamental group of the sphere with 3 punctures is given by

$$\pi_1(\mathbb{P}-\{0,1,\infty\}) = \langle \gamma_1, \gamma_2, \gamma_3 | \gamma_1\gamma_2\gamma_3 = 1\rangle.$$

Let $\mathbb{P}^* = \mathbb{P} - \{0, 1, \infty\}$ and consider the homomorphism

$$\pi_1(\mathbb{P}-\{0,1,\infty\}) \to G \to 1,$$

sending γ_1 to x, γ_2 to y and γ_3 to $(xy)^{-1}$. Then the kernel corresponds to a normal covering $X^* \to \mathbb{P}^*$. We can fill in the punctures and we get a branched covering $X \to \mathbb{P}$, with branching order 2, 3 and 7. We can lift the complex structure to make X into a compact Riemann surface. The deck transformations also extend and so G is isomorphic to $G(X/\mathbb{P})$ and acts on X. The Riemann-Hurwitz formula applied to this branching gives $|G| = 84(g-1)$. □

We end this section by noting that if we define an abstract group by

$$\Delta(2,3,7) = \langle x, y | x^2 = y^3 = (xy)^7 = 1\rangle,$$

then Hurwitz's result states that Hurwitz groups are exactly nontrivial homomorphic images of $\Delta(2,3,7)$. This leads to Fuchsian groups as well as a different perspective on Hurwitz's theorem and automorphisms of surfaces.

4 Fuchsian Groups

Fuchsian groups are related to lattices in the hyperbolic plane and give an interpretation of automorphisms groups of compact Riemann surfaces as tilings on the surface. By lifting the tiles to the universal cover, we get a tessellation of the hyperbolic half-plane(or disk). An important theorem of Siegel shows that these tiles cannot be arbitrary small, more precisely, their hyperbolic area is bounded from below by $\frac{\pi}{21}$. It is this result that forces the automorphism groups to be bounded by $84(g-1)$, the Hurwitz bound.

4.1 Discrete Subgroups of $PSL(2,\mathbb{R})$

If a subgroup of a topological group has discrete subspace topology then it's called a **discrete subgroup**.

Example 4.1.1. Discrete subgroups of $\mathbb{R}$ are isomorphic to infinite cyclic group $\mathbb{Z}$.

Example 4.1.2. Discrete subgroups of S^1 are isomorphic to finite cyclic groups $\mathbb{Z}_n$.

Definition 4.1.1. A **Fuchsian group** is a discrete subgroup of $PSL(2,\mathbb{R})$.

Proposition 4.1.2. *Let G be a topological group. Then G is a discrete group if and only if the identity element e is isolated (i.e. for any sequence of elements of G, $\{g_n\}_{n=1}^{\infty}$, such that $\lim g_n = e$ implies $g_n = e$ for large n).*

Proof. We show that if the identity element e is isolated then so is any other point of G. If a point g is not isolated then there exists a sequence $\{g_n\}$ that has g as a limit point. Now consider the sequence $\{g^{-1}\cdot g_n\}$ which has e as a limit point. This gives a contradiction. □

Proposition 4.1.3. *For any Fuchsian group Γ, there exists at least one point in $\mathbb{H}$ that is not fixed by any non-identity element of Γ.*

Proof. Let $w \in \mathbb{H}$. Since the Γ-orbits are discrete there exists an ϵ-ball $B_\epsilon(w)$ such that if $\forall T \in \Gamma\ T(B_\epsilon) \cap B_\epsilon \neq \emptyset \Rightarrow T \in Stab_\Gamma(w)$. Now choose $z \in B_\epsilon(w)$ distinct from w. If any $T \in \Gamma$ fixes z then it also fixes w and any element of $PSL(2,\mathbb{R})$ which fixes two distinct points in $\mathbb{H}$ must be the identity map. So z is not fixed by any non-identity element of Γ. □

Lemma 4.1.4. *Given a compact subset K of $\mathbb{H}$, then for any point $z \in \mathbb{H}$, we have $E := \{T \in PSL(2,\mathbb{R}) \mid T(z) \in K\}$ is a compact subset of $PSL(2,\mathbb{R})$.*

Proof. Consider the continuous map $\varphi : SL(2,\mathbb{R}) \xrightarrow{\pi} PSL(2,\mathbb{R}) \xrightarrow{e_z} \mathbb{H}$, where π is the projection mapping and e_z is the evaluation map at the point z $\in \mathbb{H}$. E is closed since it's the inverse image of a closed set. We want to show that $\widetilde{E} = \pi^{-1}(E)$ is a bounded subset of $\mathbb{R}^4$. Since K is compact, there exists constants C_1 and C_2 such that (1) $|\varphi(\widetilde{E})| < C_1$ and (2) $Im(\varphi(\widetilde{E})) \geq C_2$. Let $(a,b,c,d) \in \mathbb{R}^4$ such that $ad - bc = 1$. Then (1) is equivalent to $| \frac{az+b}{cz+d} |< C_1$ and (2) is equivalent to $Im(\frac{az+b}{cz+d}) = \frac{Im(z)}{|cz+d|^2} \geq C_2$. These conditions give that (a,b,c,d) are bounded. □

Theorem 4.1.5. *Let Γ be a subgroup of $PSL(2,\mathbb{R})$. Then the following are equivalent:*
(1) Γ is a discrete subgroup(i.e. Fuchsian).
(2) Γ acts properly discontinuously on $\mathbb{H}$.
(3) Γ-orbits are discrete in $\mathbb{H}$.

Proof. (1) $\Rightarrow$ (2): We show that for any point $z \in \mathbb{H}$, the Γ-orbit of z is locally finite. Let K be any compact subset of $\mathbb{H}$ and consider the set $\{T \in \Gamma \mid T(z) \in K\}$ which by the lemma above is a discrete subgroup of compact set and thus is finite. So Γ acts properly discontinuously on $\mathbb{H}$.
(2) $\Rightarrow$ (3): This is trivial since if Γ acts properly discontinuously its Γ-orbits are already discrete. (3) $\Rightarrow$ (1): Suppose Γ acts properly discontinuously but is not a discrete subgroup. Choose a point z that is not fixed by any element of Γ except the identity. Then there exists a sequence $\{T_n\}_{n=1}^{\infty}$ such that $T_n \to I$ as $n \to \infty$. Now the Γ-orbit of z has a limit point at z which contradicts that the orbits are discrete. □

Corollary 4.1.6. *Any Fuchsian group is at most countable.*

Proof. Let Γ be a Fuchsian group, then for any $z \in \mathbb{H}$ we have a bijection between the group and each orbit: $\Gamma \longleftrightarrow \Gamma z$ and each Γ-orbit is a discrete subset of the upper-half plane and thus at most countable. □

4.2 Algebraic Properties of Fuchsian Groups

Theorem 4.2.1. *If all non-identity elements of a Fuchsian group Γ have the same fixed-point set then Γ is cyclic.*

Proof. Suppose Γ consists only of hyperbolic elements, then by conjugation we can suppose the fixed point set is 0 and ∞. It follows that Γ is a discrete subgroup of $\{T|T(z) = \lambda z, \lambda > 0\} \cong \mathbb{R}^*$, the multiplicative group of positive real numbers. The only discrete subgroups of $\mathbb{R}^* \cong \mathbb{R}$ are the infinite cyclic groups.
Similarly if Γ contains only parabolic elements fixing ∞ then Γ must be a discrete subgroup of $\{T|T(z) = z + \beta, \beta \in \mathbb{R}\} \cong \mathbb{R}$ and Γ is again infinite cyclic. And lastly, suppose Γ consists only of elliptic elements, if we are using the disk model we can suppose the fixed point set is the origin. Then Γ is a discrete subgroup of $\{T|T(z) = e^{i\theta}\theta \in \mathbb{R}\} \cong S^1$ the circle group. Thus again, Γ is cyclic, this time finite. □

Corollary 4.2.2. *The only elements of finite order in a Fuchsian group are the elliptic elements.*

Another way to see this is to consider a cyclic group generated by any elliptic element, say T. Then by conjugation we can suppose $T(z) = e^{i\theta}z$. If θ is a irrational number then all the powers, T^n are different, otherwise for $m \neq n$ we would have $(m-n)\theta \in \mathbb{Z}$, a contradiction. So the set $\{e^{i\theta n}|n \in \mathbb{Z}\}$ has a limit point on the unit circle and this contradicts discreteness of the Fuchsian group.

Proposition 4.2.3. *Let Γ be a Fuchsian group and $z \in \mathbb{H}$. Then the stabilizer of z, $Stab_\Gamma(z)$ is a finite cyclic group.*

Proof. Since each element of the stabilizer of z have the same fixed points, this subgroup is abelian and hence cyclic by the above theorem. Since it is generated by an elliptic element it has finite order and so the $Stab_\Gamma(z)$ is finite cyclic. □

Corollary 4.2.4. *A Fuchsian group is abelian if and only if it is cyclic.*

Proof. The elements of an abelian Fuchsian group must have the same fixed-points so by the theorem must be a cyclic group. □

Theorem 4.2.5. *The normalizer of an non-cyclic Fuchsian group is Fuchsian.*

Proof. Let Λ be a non-cyclic Fuchsian group and $\mathrm{N}(\Lambda)$ the normalizer of Λ in $PSL(2,\mathbb{R})$. If $N(\Lambda)$ is not a Fuchsian group then its not discrete, so there exists a sequence of distinct elements $\{T_n\}_{n=1}^{\infty}$ in the normalizer such that

$T_n \to I$ as $n \to \infty$. Recall that two elements of $PSL(2,\mathbb{R})$ commute if and only if they have the same fixed-point set.
Claim: For large n, T_n commutes with every element of Λ and hence has the same fixed-point sets as every element of Λ.
Proof of Claim: $\forall S \in \Lambda \{T_n S T_n^{-1}\}_{n=1}^{\infty} \subset \Lambda$ such that $\{T_n S T_n^{-1}\} \to S$ as $n \to \infty$. But as Λ is discrete, its points are isolated and so for large n, $T_n S T_n^{-1} = S$ and the claim is proved.
Now since Λ is non-cyclic, it is non-abelian and so there exists at least two elements S_1 and S_2 which have different fixed-point sets but both have the same fixed-points sets as T_n for large n. This gives a contradiction, so $N(\Lambda)$ is discrete and hence Fuchsian. □

4.3 Fundamental Regions and their Tessellations

Let Γ be a Fuchsian group. Define an equivalence relation on the upper-half plane $\mathbb{H}$ as follows: $z \sim z'$ if $z' = T(z)$ for some $T \in \Gamma$. We get a disjoint union of orbits $\Gamma z = \{T(z) \mid T \in \Gamma\}$:

$$\mathbb{H} = \coprod_{z \in \mathbb{H}} \Gamma z.$$

We want to find a set $\mathfrak{F} \subset \mathbb{H}$ that contains exactly one representative point from each orbit, so we can use it as a tile to tessellate the upper-half plane: $\mathbb{H} = \coprod_{T \in \Gamma} T(\mathfrak{F})$. Every element in $\mathbb{H}$ is equivalent to an element in this set which is called a **fundamental set**. Though such a set exists by the axiom of choice, it isn't unique[5][Leh66]. Furthermore we cannot say a priori that such a set has any topological properties such as being convex or locally finite, but we can say that it cannot be an open set since points on the boundary would then not be equivalent to any representative. So instead we work with the concept of a fundamental region which we now define:

Definition 4.3.1. Let $\mathscr{F}$ be closed subset of $\mathbb{H}$(or the disk model $\mathbb{D}$). Then $\mathscr{F}$ is called a **fundamental region** if
(1) $\bigcup_{T \in \Gamma} T(\mathscr{F}) = \mathbb{H}$ and
(2) $int(\mathscr{F}) \cap T(int(\mathscr{F})) = \emptyset$ for all $T \neq I$ in Γ.

From the definition, we can conclude that if we have a fundamental region $\mathscr{F}$, then there exists a fundamental set $\mathfrak{F}$ such that $int(\mathscr{F}) \subset \mathfrak{F} \subset \mathscr{F}$. For

[5]For any subset A of $\mathfrak{F}$, $\mathfrak{F}/A \cup T(A)$ is still a fundamental set for any $T \neq I$ in Γ, although not connected.

each T in Γ we call $T(\mathscr{F})$ a **hyperbolic tile** and $\bigcup_{T\in\Gamma} T(\mathscr{F}) = \mathbb{H}$ the corresponding **tessellation** of the upper-half plane. Clearly, two tiles can only intersect along their boundaries and any succession of tiles connected along their boundaries is called a **chain**. Before we prove the existence of fundamental regions, we first show that its hyperbolic area does not depend on the choice of such a region but only on the Fuchsian group.

Theorem 4.3.2. *If $\mathscr{F}_1$ and $\mathscr{F}_2$ are both fundamental regions for a Fuchsian group Γ, then they have the same hyperbolic area: $\mu(\mathscr{F}_1) = \mu(\mathscr{F}_2)$.*

Proof. $\mu(\mathscr{F}_1) = \sum_{T\in\Gamma} \mu(\mathscr{F}_1 \cap T(\mathscr{F}_2)) = \sum_{T\in\Gamma} \mu(T^{-1}(\mathscr{F}_1) \cap \mathscr{F}_2) = \mu(\mathscr{F}_2)$ □

We now prove the existence of a convex, locally finite fundamental region by a construction due to Dirichlet in the Euclidean plane and Poincaré in the hyperbolic plane. Suppose $p \in \mathbb{H}$ is not fixed by any element of Γ and for each $T \in \Gamma$ define the half-plane $H_T(p) = \{z \in \mathbb{H} | \rho(z,p) \leq \rho(z, T(p))\}$ with boundary given by the geodesic $L_T(p) = \{z \in \mathbb{H} | \rho(z,p) = \rho(z, T(p))\}$ which is the perpendicular bisector of the geodesic joining p and $T(p)$. We can now define the Dirichlet region to be

$$D_p(\Gamma) = \bigcap_{T\in\Gamma, T\neq I} H_T(p).$$

Definition 4.3.3. Let Γ be a Fuchsian group. Consider a point $p \in \mathbb{H}$ not fixed by any non-identity element of Γ. Then $D_p(\Gamma) = \{z \in \mathbb{H} \mid \rho(p,z) \leq \rho(z, T(p))\ \forall T \in \Gamma\}$ is called the **Dirichlet region** for Γ centered at p.

Remark 4.3.4. By the invariance of the metric under isometries, we have $\rho(z, T(p)) = \rho(T^{-1}(z), p)$ for each $T \in \Gamma$. So the Dirichlet region can also be described as the following set: $D_p(\Gamma) = \{z \in \mathbb{H} \mid \rho(z,p) \leq \rho(T(z),p)\ \forall T \in \Gamma\}$.

Theorem 4.3.5. *Let Γ be a Fuchsian group. Then the Dirichlet region is a convex, connected fundamental region.*

Proof. It is non-empty since it contains, first of all p, but also since the orbits are discrete it actually contains some small neighbourhood of p. Since each half-plane is closed and convex, we see that $D_p(\Gamma)$ is closed and convex and hence connected. We now show that $D_p(\Gamma)$ contains at least one point from

each orbit and if it contains two points from the same Γ-orbit, these points lie on the boundary. This then implies that there exists a fundamental set $\mathfrak{F}$, with $int(D_p(\Gamma)) \subset \mathfrak{F} \subset D_p(\Gamma)$ and so the Dirichlet region is a fundamental region for Γ.
For any Γ-orbit, since it's discrete, has a point closest to p and so is a point in $D_p(\Gamma)$. Thus the Dirichlet region contains at least one point from each orbit. Now suppose that more than one point is closest to p in the same Γ-orbit, say z and $z' = T(z)$ for some $T \in \Gamma$. Then $\rho(z,p) = \rho(T(z),p) = \rho(z, T^{-1}(p))$, so z lies on the boundary of a half-plane, $L_{T^{-1}}(p)$. Since z is in $D_p(\Gamma)$, the only way this can happen is if $z \in \partial D_p(\Gamma)$. Similarly for z'. In particular, the interior contains at most one point from each orbit and the only congruent points lie on the boundary of the Dirichlet region. This completes the proof. □

Definition 4.3.6. Let $\mathscr{F}$ be a fundamental region for a Fuchsian group Γ. $\mathscr{F}$ is called **locally finite** if every compact set K intersects only finitely many tessellations of $\mathscr{F}$: $\{T \in \Gamma \mid T(\mathscr{F}) \cap K\}$ is finite.

Theorem 4.3.7. *The Dirichlet region of a Fuchsian group is locally finite.*

Proof. Let $\mathscr{F}$ be the Dirichlet region $D_p(\Gamma)$. We can always choose a large enough disk centered at p so that contains any compact set. If we can show that there are always finitely many intersection of $T(\mathscr{F})$ with this disk then it proves the result. So let K be a compact disk centered at p of radius r. Now if $T(\mathscr{F}) \cap K \neq \emptyset$ then for some $z \in \mathscr{F}$, we have $\rho(p, T(z)) \leq r$. By the triangle inequality and invariance under isometries, we have

$$\rho(p, T(p)) \leq \rho(p, T(z)) + \rho(T(z), T(p)) \leq r + r = 2r.$$

Since the Γ orbits are locally finite, the above can only hold for finitely many T and this proves the result. □

Theorem 4.3.8. *Let Λ be a subgroup of finite index in a Fuchsian group Γ, with coset decomposition $\{\Lambda, \Lambda T_1, ..., \Lambda T_n\}$. If the fundamental region of Γ is $\mathscr{F}_\Gamma$ then we have the following:*
(1) $\mathscr{F}_\Lambda = \mathscr{F}_\Gamma \cup T_1(\mathscr{F}_\Gamma) ... \cup T_n(\mathscr{F}_\Gamma)$ is a fundamental region for Λ.
(2) $\mu(\mathscr{F}_\Lambda) = [\Gamma : \Lambda]\mu(\mathscr{F}_\Gamma)$.

Proof. (1) We first show $\mathcal{U} = \bigcup_{S \in \Lambda}(\mathscr{F}_\Lambda)$: let $z \in \mathcal{U}$, then $z = T(z')$ for some $T \in \Gamma$ and $z' \in \mathscr{F}_\Gamma$. We have $T \in \Lambda T_i$ for some i, so write $T = ST_i$ for some

$S \in \Lambda$, it then follows that $z = S(w)$ where $w = T_i(z') \in \mathscr{F}_\Lambda$. Thus the Λ-tessellation covers the upper-half plane.
We now show $int(\mathscr{F}_\Lambda) \cap S(int(\mathscr{F}_\Lambda)) = \emptyset$ for all $S \neq I$: Suppose $z \in int(\mathscr{F}_\Lambda)$ and $z = S(z')$ for some non-identity element S in Λ and some $z' \in int(\mathscr{F}_\Lambda)$. Then $z \in T_i(\mathscr{F}_\Gamma)$ and $z' \in T_j(\mathscr{F}_\Gamma)$ for some i and j, which implies that the tiles $T_i(\mathscr{F}_\Gamma)$ and $ST_j(\mathscr{F}_\Gamma)$ intersect with interior points which is impossible unless $T_i = ST_j$. But now this implies that T_i and T_j belong to the same coset and so $i = j$ and we that that z and z' are Γ-equivalent in $T_i(\mathscr{F}_\Gamma)$ which is a contradiction and the theorem is proved. □

Corollary 4.3.9. *If Λ is a subgroup of a Fuchsian group Γ such that $\mathscr{F}_\Lambda$ has finite area, then Λ is of finite index in Γ and $\mathscr{F}_\Gamma$ has finite area.*

4.4 Structure of a Fundamental Region

Now that we have proven the existence of a convex and locally finite fundamental region, we use these properties to develop its polygonal structure. For more information see [Bea83]. Throughout this section we let Γ be a Fuchsian group and $\mathscr{P}$ be its convex and locally finite Fundamental region.

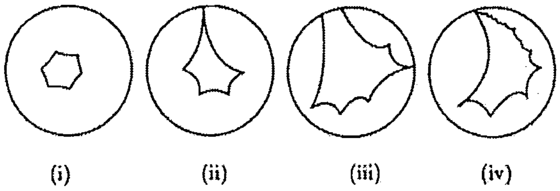


Figure 5: [Kat92, pg.68] By construction, the Dirichlet region is a countable intersection of half-planes, so we expect that its fundamental region is bounded by at most countably many geodesic segments and possibly segments on $\partial\mathbb{D}$. (i) is an example of a compact fundamental region. (ii) has a vertex on the boundary of the unit disk and is called an **ideal vertex.** (iii) has a segment on the boundary and is called a **free side** of the fundamental region. (iv) has countably many sides with an ideal vertex.

Proposition 4.4.1. *The only points of $\mathscr{P}$ that are equivalent(i.e. in the same Γ-orbit) are the points on the boundary, $\partial\mathscr{P}$, and every point z on the boundary, there exists some non-identity element $T \in \Gamma$ such that $T(z) \in \partial\mathscr{P}$.*

Proof. If z lies on the boundary, by local finiteness, it intersects only finitely many tiles: $z \in \mathscr{P} \cap T_1(\mathscr{P}) ... \cap T_k(\mathscr{P})$ for $k \geq 1$. So there exists at least one other element z' with $z = T(z')$ and this must also lie on the boundary. □

- *Sides of* $\mathscr{P}$: For any non-identity $T \in \Gamma$, $\mathscr{P} \cap T(\mathscr{P})$ is either empty or at most a geodesic segment called a **side** of $\mathscr{P}$. To see this note that $\mathscr{P} \cap T(\mathscr{P})$ is convex and so if it contains more than one point it must contain a geodesic segment. Now since the intersection occurs along the boundary of $\mathscr{P}$, it must be a set of measure zero and thus cannot contain any three non-collinear points. It follows that a non-empty intersection is either a single point or a geodesic segment.

- *Vertices of* $\mathscr{P}$: For any distinct non-identity elements $T_1, T_2 \in \Gamma$, $\mathscr{P} \cap T_1(\mathscr{P}) \cap T_2(\mathscr{P})$ is either empty or a single point called a **vertex** of $\mathscr{P}$. This can be seen by considering $\mathscr{P} \cap T_1(\mathscr{P}) \cap \mathscr{P} \cap T_2(\mathscr{P})$. So a nonempty intersection must consist of a single point. Moreover, a vertex cannot by an interior point of a side by convexity.

Theorem 4.4.2. *Let $\mathscr{P}$ be a convex, locally finite fundamental region, then we have the following:*
(1) $\mathscr{P}$ has at most countably many sides and vertices,
(2) Any compact set K intersects finitely many sides and vertices,
(3) $\partial\mathscr{P}$ is a union of sides,
(4) Intersection of two sides is either empty or is a vertex that is the common end-point of each side.
(5) Conversely, every vertex is the intersection of exactly two sides and is the common end-point of each side.
(6) Vertices are isolated, in particular a compact $\mathscr{P}$ can only have a finite number of vertices and sides.

Proof. (1) follows since Γ is countable. By local finiteness only finitely many tiles intersect K (i.e. $\{T \in \Gamma \mid T(\mathscr{P}) \cap K \neq \emptyset\}$ is finite), and since each side is of the form $\mathscr{P} \cap T(\mathscr{P})$ only finitely many sides can intersect K. Similarly for vertices and this proves (2). For (3), let $z \in \partial\mathscr{P}$, by local finiteness z lies at the intersection of finitely many tiles and we cannot have $\mathscr{P} \cap T(\mathscr{P}) = \{z\}$ for all $T \in \Gamma$ so z lies on some side. To see (4) let $s_1 = \mathscr{P} \cap T_1(\mathscr{P})$ and $s_2 = \mathscr{P} \cap T_2(\mathscr{P})$ be two sides with non-empty intersection, then $\mathscr{P} \cap T_1(\mathscr{P}) \cap T_2(\mathscr{P}) \neq \emptyset$ and so must be a vertex and since it cannot by the interior of either side, it must be the common endpoint of each.

(5) A vertex must be contained in a finite number of sides by (2) and cannot be an interior point to any side.
(6) A small compact disk around any point intersects only finitely many vertices by (2). □

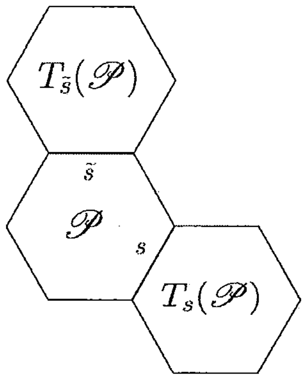


Figure 6: Side paired polygon

Definition 4.4.3. $\mathscr{P}$ is called a **side-paired polygon** if for each side s there exists a unique side $\tilde{s}$ such that $\tilde{s} = T(s)$ for some $T \in \Gamma$. We include the possibility that $\tilde{s} = s$, in which case the corresponding transformation T is elliptic of order 2 and we consider the midpoint of side s to be a vertex which is the unique fixed point of T.

In particular, a side-paired polygon has an even number of sides.

Proposition 4.4.4. *Let* $\Gamma^* = \{T \in \Gamma \mid \mathscr{P} \cap T(\mathscr{P}) \text{is a side}\}$ *and let* $\mathcal{S}$ *be the collection of sides of* $\mathscr{P}$. *Then we have a bijection:* $\Gamma^* \leftrightarrow \mathcal{S}$.

Theorem 4.4.5 (Side Condition). $\mathscr{P}$ *is a side-paired polygon.*

Proof. If s is a side then it corresponds to a unique $T_s \in \Gamma^*$ and $s = \mathscr{P} \cap T_s(\mathscr{P})$. Then $T_s^{-1}(s) = \mathscr{P} \cap T_s^{-1}(\mathscr{P})$ and is also a side of $\mathscr{P}$, call it $\tilde{s}$. So we have $T_{\tilde{s}} = T_s^{-1}$ and an involution $\Phi : \mathcal{S} \to \mathcal{S}$ defined by $\Phi(s) = \tilde{s}$: $\Phi^2(s) = \Phi(\tilde{s}) = T_{\tilde{s}}^{-1}(\tilde{s}) = T_s(\tilde{s}) = s$. □

Lemma 4.4.6. *Any two tiles of a tessellation are connected by a finite chain.*

Proof. Choose two interior points, one in each tile and consider the closed geodesic segment connecting them. This geodesic is a compact subset of $\mathcal{U}$ and so by local finiteness it intersects only finitely many tiles. □

Now suppose two tiles have non-empty intersection, let $e = T_1(\mathscr{P}) \cap T_2(\mathscr{P})$ be the edge they intersect, then $s := T_1^{-1}(e) = \mathscr{P} \cap T_1^{-1}T_2(\mathscr{P})$ is in $\mathcal{S}$, and so $T_1^{-1}T_2 = T_s \in \Gamma^*$. With this observation we can prove the following algebraic result:

Theorem 4.4.7. Γ^* *generates* Γ.

Proof. Let $T \in \Gamma$ be a non-identity element, then T corresponds to a unique tile $T(\mathscr{P})$ and there exists a finite chain of tiles from $\mathscr{P}$ to $T(\mathscr{P})$ say given by $\{T_i(\mathscr{P})\}_{i=1}^{n}$. If the edges connecting the tiles are $\{e_i\}$ then $T_{i+1} = T_iT_{s_i}$ for some s_i in $\mathcal{S}$. □

We will be working with Fuchsian groups which have fundamental regions of finite area. For the following theorem see [Kat92].

Theorem 4.4.8. *Let* Γ *be a Fuchsian group with fundamental region* $\mathscr{P}$. *Then the following are equivalent.*
(1) $\mu(\mathscr{P}) < \infty$
(2) $\mathscr{P}$ *has a finite number of sides and no free sides.*

Note that if $\mathscr{P}$ has finite area then it follows that Γ is finitely generated.

Now let $\mathcal{V}$ be the collection of vertices of $\mathscr{P}$ and consider an equivalence relation defined by $v_1 \sim v_2$ if and only if $v_2 = T(v_1)$ for some $T \in \Gamma$. The equivalence classes are called **cycles**.

Theorem 4.4.9 (Cycle Condition). *Let* $\mathscr{P}$ *be a fundamental region for a Fuchsian group and* $[v_1] = \{v_1, ..., v_n\}$ *be a cycle of vertices with internal angles* $\theta_1, ..., \theta_n$. *The stabilizer of the vertices from the cycle are conjugate and hence are of the same order, say* m. *Then we have*

$$\theta_1 + \ldots + \theta_n = \frac{2\pi}{m}.$$

Proof. There exists elements $\{T_i\}_{i=1}^{n}$ in Γ such that $T_i(v_i) = v_1$ and so $T_i(\mathscr{P})$ are tiles with v_1 as a vertex and angle θ_i. Since the stabilizer of any vertex

is a finite cycle group, let $Stab_{\Gamma}(v_1) = \{I, S, S^2, ..., S^{m-1}\}$.
Claim: The set of all tiles with v_1 as a vertex is a union of cosets:

$$\bigcup_{i=1}^{n} Stab_{\Gamma}(v_1)T_i.$$

The result will follow since the elements of the stabilizer are hyperbolic rotations, each tile from the coset $Stab_{\Gamma}(v_1)T_j$ has vertex v_1 and angle θ_j and since there are m such tiles in each coset we have $m(\theta_1 + \ldots + \theta_n) = 2\pi$.
Proof of Claim: A tile T has v_1 as a vertex (i.e. $v_1 \in T(\mathscr{P})$) if and only if $v_1 = T(v_i)$ for some i if and only if $TT_i^{-1} \in Stab_{\Gamma}(v_1)$ if and only if $T \in Stab_{\Gamma}(v_1)T_i$. □

4.5 Quotient Riemann Surface

We have seen that every connected Riemann surface with universal cover $\mathbb{H}$ can be uniformized by a group Λ acting discontinuously on $\mathbb{H}$. So in particular Λ is a Fuchsian group; conversely if Λ is a torsion-free Fuchsian group then it acts discontinuously on the upper-half plane so that $\mathbb{H}/\Lambda$ can be given a Riemann surface structure. If a Fuchsian group Γ has elliptic elements, however, $\mathbb{H}/\Gamma$ can still be given the structure of a Riemann surface.

Theorem 4.5.1. *Let Γ be a Fuchsian group, then the quotient space $\mathbb{H}/\Gamma$ is a connected Riemann Surface such that the canonical projection map $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ is holomorphic.*

Proof. See [JS87]. □

Theorem 4.5.2. *Let Γ be a Fuchsian group with locally finite fundamental region $\mathscr{P}$. Then we have the following:*
(1) $\mathbb{H}/\Gamma$ is homeomorphic to $\mathscr{P}/\Gamma$.
(2) $\mathbb{H}/\Gamma$ is compact if and only if $\mathscr{P}$ is compact.

Proof. See [JS87]. □

Definition 4.5.3. A Fuchsian group is called **cocompact** if it has compact fundamental region.

Theorem 4.5.4. *A cocompact Fuchsian group does not contain any parabolic elements.*

Corollary 4.5.5. *If Λ is a non-cyclic, torsion-free Fuchsian group, then the group of automorphisms of $\mathbb{H}/\Lambda$ is isomorphic to a quotient of Fuchsian groups Γ/Λ with $\Lambda \triangleleft \Gamma$.*

Proof. Since Λ is torsion-free, $Aut(\mathbb{H}/\Lambda) = N(\Lambda)/\Lambda$ and as Λ is also non-cyclic, its normalizer is a Fuchsian group and this completes the proof. □

4.6 Fricke-Klein Signatures

Note that if p is fixed by an elliptic element S in a Fuchsian group, then it must lie on the boundary of a tile, say $T(\mathscr{P})$, so $T^{-1}(p)$ is an elliptic fixed point on $\partial\mathscr{P}$ fixed by $T^{-1}ST$. Similarly for parabolic fixed points. With this we have the following:

Theorem 4.6.1. *Let Γ be a Fuchsian group with fundamental region $\mathscr{P}$ with v_i elliptic fixed points on $\partial\mathscr{P}$ which has stabilizer generated by T_i. Then any elliptic element of Γ is conjugate to some power of T_i. Similarly for parabolic fixed points.*

Proof. Let $T^{-1}(p) = v_i$ for some i, then since $T^{-1}ST$ fixes v_i we must have $T^{-1}ST \in \langle T_i \rangle$. □

We now restrict to the case when the quotient space is compact, so we have that the fundamental region $\mathscr{F}$ is compact and thus is a polygon with a finite number of sides and vertices. In particular there are no parabolic elements only a finite number of elliptic cycles, let m_i denote the order of the stabilizer for the elliptic cycle $[v_i]$, then we define the **signature** of Γ to be $(g; m_1, \ldots, m_r)$ where g is the genus of the quotient space.

Theorem 4.6.2. *Let Γ be a Fuchsian group with signature $(g; m_1, ..., m_r)$. If the fundamental region for Γ is μ-measurable, then its area is given by:*

$$\mu(F) = 2\pi\{(2-2g) + \sum_{i=1}^{r}(1 - \frac{1}{m_i})\}.$$

Proof. Let F the Dirichlet region for Γ. There are r elliptic cycles $[v_1], \ldots, [v_r]$ and say s classes of non-elliptic vertices. The sum of the angles for elliptic cycle $[v_i]$ is $\frac{2\pi}{m_i}$. So the sum of all the angles at elliptic vertices is $\sum_{i=1}^{r} \frac{2\pi}{m_i}$.

The sum of all the angles at non-elliptic vertices is $2\pi s$. Thus the total sum of all the internal angles for F is

$$2\pi\{(\sum_{i=1}^{r}\frac{1}{m_i}) + s\}.$$

Now we find the Euler-characteristic of the closed oriented surface $\mathbb{H}/F = X_g$ of genus g. The triangulation of F gives a triangulation on the quotient surface. Since the number of edges in F come in pairs, lets say there are 2n-edges. When glued together we get n-edges, $(r+s)$-vertices and 1-face on X_g. So we get $\chi(X_g) = V - E + F = (r+s) - n + 1 = 2 - 2g$. Now we can also use the Gauss-Bonnet theorem for polygons to calculate the area of F as:

$$\mu(F) = (2n-2)\pi - 2\pi\{(\sum_{i=1}^{r}\frac{2\pi}{m_i}) + s\}.$$

Solving for n in the Euler characteristic and substituting we get the resulting formula. □

Theorem 4.6.3 (Siegel's Theorem, 1945). *Let $\mathbb{H}/\Gamma$ be compact Riemann surface with Γ a Fuchsian group. Then the hyperbolic area of $\mathscr{F}$ is bounded from below. In particular, $\mu(F) \geq \frac{\pi}{21}$ and equality implies Γ has signature* $(0; 2, 3, 7)$.

Proof. We need to show $\mu(\mathscr{F}) = 2\pi[(2g-2) + \sum_i^r\{1 - \frac{1}{m_i}\}] \geq \frac{\pi}{21}$. If $g > 1$ then the result is trivial. If $g = 1$ then there must be periods ($r \geq 1$) since $\mu(\mathscr{F}) > 0$, and since $1 - \frac{1}{m_i} \geq \frac{1}{2}$ we get $\mu(\mathscr{F}) = 2\pi(\sum 1 - \frac{1}{m_i}) \geq 2\pi\frac{r}{2} \geq \pi$. If $g = 0$ then we get $\mu(\mathscr{F}) \geq \pi(r-4)$ and now we check possible cases for r. If $r \geq 5$ then $\mu(\mathscr{F}) \geq \pi$. If $r = 4$ then signature $(0; 2, 2, 2, 2)$ gives zero area so a minimum is attained for signature $(0; 2, 2, 2, 3)$ which gives area $\mu(\mathscr{F}) \geq \frac{\pi}{3}$. For $r \leq 2$ we get negative area which is not possible, so we have reduced to the case when Γ has signature $(0; m_1, m_2, m_3)$. We have

$$\mu(\mathscr{F}) = 2\pi\{1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\},$$

and by permuting if necessary, we can assume $m_1 \leq m_2 \leq m_3$. Consider the following cases:

Case(1) $m_1 \geq 4$: Then for signature $(0; 4, 4, 4)$ we get minimum area $\frac{\pi}{2}$.
Case(2) $m_1 = 3$: Then signature $(0; 3, 3, 3)$ gives zero area and signature

$(0;3,3,4)$ gives minimum area $\frac{\pi}{6}$.
Case(3) $m_1 = 2$: Since the area must be positive we have $m_2 > 2$, and if $m_2 \geq 4$ gives minimum $\mu(\mathscr{F}) \geq 2\pi(1-\frac{1}{2}-\frac{1}{4}-\frac{1}{m_3}) \geq 2\pi\frac{1}{20} = \frac{\pi}{10}$. So consider when $m_2 = 3$ this gives $\mu(\mathscr{F}) = 2\pi\{1-\frac{1}{2}-\frac{1}{3}-\frac{1}{m_3}\}$ and as the area again must be positive this forces $m_3 > 6$ and we get minimum when $m_3 = 7$ and so $\mu(\mathscr{F}) \geq 2\pi\{1-\frac{1}{2}-\frac{1}{3}-\frac{1}{7}\} = 2\pi(\frac{1}{42}) = \frac{\pi}{21}$ for signature $(0;2,3,7)$. □

4.7 Theorems of Poincaré

We have seen that the Dirichlet region for a Fuchsian group satisfies side and cycle conditions. Poincaré's theorem proves the converse, if a compact polygon satisfies these conditions then the side transformations generate a Fuchsian group whose fundamental region is the given polygon. The theorem is also true in the non-compact case, however Poincaré's original proof contained a gap, for a complete proof see [Mas71].

Theorem 4.7.1 (Compact Polygon Theorem, Poincaré, 1882). *Let $\mathscr{P}$ be a compact polygon with satisfies the side and angle conditions. Let Γ be the group generated by the side-pairing transformations for $\mathbb{H}$. Then Γ is a Fuchsian group and $\mathscr{P}$ is its fundamental domain.*

Proof. See [Sti92], [KL07] or [Ive92]. □

Theorem 4.7.2 (Poincaré 1882). *There exists a cocompact Fuchsian group with signature $(g;m_1,...,m_r)$ if and only if $2g-2+\sum(1-\frac{1}{m_i}) > 0$ and $g \geq 0, m_i \geq 2$.*

Proof. Consider the Poincaré unit disk model $\mathbb{D}$ and a regular $(4g+r)$-sided hyperbolic polygon inscribed in a circle of radius t for $0 < t < 1$ and origin as its center. For each of the r sides construct an external isosceles hyperbolic triangle with angle $\frac{2\pi}{m_i}$. The union of the regular polygon with the external isosceles triangles gives a star-like hyperbolic $(4g+2r)$-sided polygon $N(t)$. As $t \to 0$ we get $\mu(N(t)) \to 0$ and we also have

$$\lim_{t\to 1}\mu(N(t)) = (4g+2r-2)\pi - \sum(\text{internal angles})$$

which equals

$$(4g+2r-2)\pi - \sum\frac{2\pi}{m_i} = 2\pi[(2g-1) - \sum(1-\frac{1}{m_i})]$$

and by continuity we can choose t^* such that $\mu(N(t^*)) = 2\pi[(2g-2) - \sum(1-\frac{1}{m_i})]$. By considering the side transformations as indicated by the figure, we get $(4g+r)$ congruent vertices and r elliptic cycles where the angle conditions of Poincaré's compact polygon theorem are satisfied. If Γ is the group generated by the side transformations then we get a Fuchsian group with fundamental domain $N(t^*)$ and signature $(g; m_1, ..., m_r)$. □

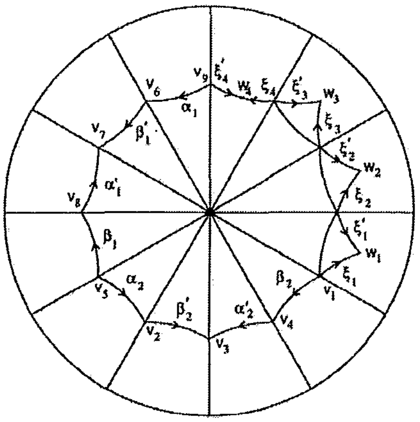


Figure 7: [Kat92] Proof of Poincaré's theorem.

4.8 Hurwitz's Theorem Using Fuchsian Groups

We can now give a proof of Hurwitz's theorem using Fuchsian groups. Note that this approach proves at once both the finiteness of the automorphism group and gives the upper-bound.

Theorem 4.8.1 (Hurwitz, 1893). *Let X be a compact Riemann surface of genus $g \geq 2$. Then $|Aut(X)| \leq 84(g-1)$.*

Proof. Let $X = \mathbb{H}/\Lambda$, where Λ is a torsion-free Fuchsian group. If F_Λ is the fundamental domain for Λ then its hyperbolic area is given by $\mu(F_\Lambda) = 2\pi(2g-2)$. Now the automorphism group, $Aut(X)$, is isomorphic to the quotient of Fuchsian groups: Γ/Λ with $\Lambda \triangleleft \Gamma$ for some Γ. Now the quotient $\mathbb{H}/\Gamma$ is compact - to see this, consider the surjective continuous map $\mathbb{H}/\Lambda \longrightarrow \mathbb{H}/\Gamma$, so the fundamental domain for Γ, F_Γ, is compact and its area is finite and bounded from below by $\frac{\pi}{21}$. Thus we have that $|Aut(X)| = [\Gamma : \Lambda] = \frac{\mu(F_\Lambda)}{\mu(F_\Gamma)} \leq \frac{2\pi(2g-2)21}{\pi} = 84(g-1)$. □

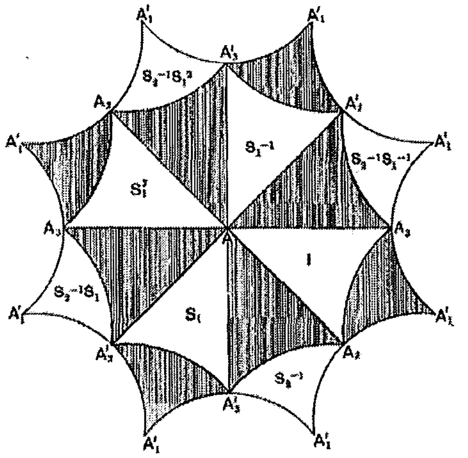


Figure 8: [Bur55] Tiling inside an octagon model for a genus two surface.

We can now give an interpretation of the automorphism group as tiling on the surface. Suppose G acts on a compact Riemann surface X. Then G is isomorphic to a quotient of Fuchsian groups Γ/Λ with Λ torsion-free. The fundamental region of Λ, F_Λ, is a compact hyperbolic polygon such that when the sides are identified we get the surface X. Moreover, F_Λ is tessellated by finer tiles which correspond to the fundamental regions for Γ and the number of these finer tiles in F_Λ is exactly the order of G. So the surface X is tessellated by $|G|$ fine tiles and by identifying sides of these tiles we get the quotient Riemann surface X/G. Since fundamental regions have the same area we have:

$$\mu(F_\Lambda) = |G|\mu(F_\Gamma).$$

4.9 Group Actions Using Fuchsian Groups

Suppose we are given a Fuchsian group Γ and a finite group G with a surjective map φ that has torsion-free kernel K. Then we get a short exact sequence of groups: $1 \to K \to \Gamma \xrightarrow{\varphi} G \to 1$.

Theorem 4.9.1. *The above short exact sequence gives a well-defined group action, Ψ: $G \times \mathbb{H}/K \to \mathbb{H}/K$, given by $\Psi(\mathbf{g}, [p]) = [g(p)]$.*

Proof. First note that by the first isomorphism theorem, G is isomorphic to Γ/K. So let $\mathbf{g} = gK$ for some $g \in \Gamma$. Now suppose that $[p] = [q]$, then there exists $T \in K$ such that $T(p) = q$ and we need to show that $[\mathrm{g}(p)] = [\mathrm{g}(q)]$.

Figure 9: [Bur55] Tiling on a genus two surface.

Consider the map $f := gTg^{-1}$, which is an element of K by normality of the kernel. Then $f(g(p)) = g(q)$ and so the action does not depend on the representative of the orbit.

Next we need to show that $\Psi(\mathbf{g}, [p]) = \Psi(\mathbf{h}, [p])$ for $gK = hK \in \Gamma/K$. Then consider the map $f := gh^{-1} \in K$. We have that $f(h(p)) = g(p)$, so $[g(p)] = [h(p)]$. □

Corollary 4.9.2. *A finite group G acts on a compact Riemann surface X of genus $g \geq 2$ if and only if G is isomorphic to a quotient of Fuchsian groups Γ/Λ where Λ is torsion-free with $X = \mathbb{H}/\Lambda$ and $X/G = \mathbb{H}/\Gamma$.*

Theorem 4.9.3 (Chih-Han Sah, 1969). *Let Γ be a Fuchsian group with signature $(g; m_1, ..., m_r)$ and a normal subgroup $H \trianglelefteq \Gamma$ which has finite index d. If t_i is the order of c_iH in the quotient group, then the signature of H is given by $(\widetilde{g}; f(i,j))$ where $f(i,j) = \frac{m_i}{t_i}$, $1 \leq j \leq \frac{d}{t_i}$,$1 \leq i \leq r$ with $f(i,j) = 1$ is deleted and*

$$\widetilde{g} - 1 = d(g-1) + \frac{d}{2}\sum_{i=1}^{r}(1 - \frac{1}{t_i}).$$

Proof. See [Bre00]. □

Corollary 4.9.4 (W.J. Harvey, 1966). *Let Γ be a Fuchsian group and consider a homomorphism $\varphi : \Gamma \to G$ onto a finite group G. Then the kernel of φ is torsion-free if and only if φ preserves the orders of the elliptic generators.*

Theorem 4.9.5. *Every finite group G acts as a group of automorphisms of some compact Riemann surface X of genus $g \geq 2$.*

5 Hurwitz Groups

In this section we prove Hurwitz's criterion for a finite group G to be a Hurwitz group, this time using the Fuchsian groups approach. We will also see that $PSL(2, \mathbb{F}_7)$ (sometimes denoted $PSL(2, 7)$ is a Hurwitz group of smallest order.

5.1 Hurwitz's Criterion

Definition 5.1.1. A group G that acts as a group of automorphisms on some compact Riemann surface of genus $g \geq 2$ and has order $84(g-1)$ is called a **Hurwitz Group**.

Theorem 5.1.2. *A non-trivial finite group G is a Hurwitz group if and only if it is the homomorphic image of the triangle group $\Delta(2,3,7) = \langle X, Y | X^2 = Y^3 = (XY)^7 = 1\rangle$.*

Proof. Suppose G is a Hurwitz group that acts on a compact Riemann Surface $\mathbb{H}/\Lambda$ where Λ has signature $(g; -)$ for some $g \geq 2$. Now the group of automorphisms of $\mathbb{H}/\Lambda$ is isomorphic to Γ/Λ for some Fuchsian group Γ with $\Lambda \trianglelefteq \Gamma$. So we have the following short exact sequence: $1 \to \Lambda \to \Gamma \xrightarrow{\pi} G \to 1$, where π is the canonical projection homomorphism and $G \cong \Gamma/\Lambda$. Since $84(g-1) = |G| = |\Gamma/\Lambda| = \frac{\mu(F_\Lambda)}{F_\Gamma}) = \frac{2\pi(2g-2)}{\mu(F_\Gamma)}$ we see that Γ must be the triangle group $\Delta(2,3,7)$.
Conversely, suppose we have G as the homomorphic image of the triangle group $\Gamma = \Delta(2,3,7)$, then we have the following: $1 \to \Lambda \to \Gamma \xrightarrow{\varphi} G \to 1$, where φ is the homomorphism and $\Lambda = Ker\varphi$.
Claim(1): Λ is torsion-free and has signature $(g; -)$,
Claim(2): G is a Hurwitz group.
Proof of Claim(1): If Λ contains an elliptic element, then it must be conjugate to some power of X, Y or XY. Since these elements have prime order, Λ contains X, Y or XY, but then the homomorphic image G is reduced to the trivial group. Thus there are no elliptic elements in the kernel Λ and it has signature $(g; -)$ where $g \geq 2$.
Proof of Claim(2): The short exact sequence gives that G acts on a

compact Riemann surface of genus $g \geq 2$ and $|G| = |\Gamma/\Lambda| = \frac{2\pi(2g-2)}{\frac{\pi}{21}} = 84(g-1)$ and so G is a Hurwitz group. □

Theorem 5.1.3. *(1) The quotient group of a Hurwitz group is also a Hurwitz group.*
(2) There is no Hurwitz group of order 84.
(3) $PSL(2, \mathbb{Z}_7)$ is a Hurwitz group of order 168.

Proof. (1) follows from the above theorem since if H is any normal subgroup of a Hurwitz group G, consider the projection homomorphism $\pi : G \to G/H$. If G satisfies $x^2 = y^3 = (xy)^7 = 1$, then so does G/H: $\pi(x)^2 = \pi(y)^3 = (\pi(x)\pi(y))^7 = 1$.
(2) If H is a Hurwitz group of order 84 then it is a Hurwitz group of smallest order. So H must be simple, otherwise a factor group would also be a Hurwitz group with order smaller than 84. But by using Sylow's theorem it can be shown that there is no simple group of order 84, contradiction.
(3) $PSL(2, \mathbb{Z}_7)$ has generating elements x and y such that $x^2 = y^3 = (xy)^7 = 1$. See [JS87, pg.265]. □

The above theorem tells us that $PSL(2, \mathbb{Z}_7)$ is a Hurwitz group of smallest order and hence is a simple group. So the Hurwitz bound is attained for genus 3 but not genus 2. In the next section we will see that the Hurwitz bound is attained and not attained infinitely often.
It will be useful later to eliminate given groups as being Hurwitz, so we now look at some necessary conditions for Hurwitz groups. The simplest condition is to note that by Hurwitz's theorem such a group must have order divisible by 84. There is a less obvious condition, first recall that the triangle group $\Delta(2,3,7)$ has presentation $\langle x, y | x^2 = y^3 = (xy)^7 = 1 \rangle$, so if x and y commute, then $x^7y^7 = 1$ which gives $xy = 1$. So the generators are inverses of each other so should have the same order but since they have order 2 and 3 respectively this implies $x = y = 1$. Thus $\Delta/[\Delta, \Delta] = \{1\}$ and so is a perfect group. Since homomorphic images of perfect groups are perfect we get that a Hurwitz group must be perfect. We summarize this in the following:

Theorem 5.1.4. *Suppose G is a Hurwitz group. Then we have the following:*
(1) $|G|$ is divisible by 84.
(2) G is a perfect group.
(3) If G is not simple it has a maximal normal subgroup K such that G/K is a simple non-abelian Hurwitz group.

Since perfect groups are not solvable groups, we also get that Hurwitz groups are not solvable. Recall the following sequence of inclusions from group theory:

$$\textit{cyclic} \subset \textit{abelian} \subset \textit{nilpotent} \subset \textit{solvable}.$$

It follows that neither of such groups can be Hurwitz. In particular, since p-groups and dihedral groups are solvable they also cannot be Hurwitz groups.

Corollary 5.1.5. *(1) p-groups and dihedral groups are not Hurwitz.*
(2) Any group of odd order cannot be Hurwitz.

5.2 Theorems of Macbeath

Recall that a subgroup K of G is characteristic if it is invariant under any group automorphism of G.

Theorem 5.2.1 (Macbeath, 1961). *The Hurwitz bound is attained for infinitely many X_g.*

Proof. Let Γ be the triangle group with signature $\Delta(2,3,7)$ and choose a torsion-free normal subgroup Λ. Consider the following sequence of subgroups $\{\Lambda^m[\Lambda,\Lambda]\}_{m\in\mathbb{Z}^+}$. Let $K_m = \Lambda^m[\Lambda,\Lambda]$. Each element of the sequence is a characteristic subgroup of Λ (hence normal in Γ) and the factor group Λ/K_m is isomorphic to the finite abelian group $\mathbb{Z}_m^{2g}$. Let the signature of K_m be $(\widetilde{g};-)$ and now since Γ/K_m is a Hurwitz group (canonical projection gives nontrivial homomorphism from triangle group Δ) we have $84(\widetilde{g}-1) = |\Gamma/K_m|$. So we get an exact sequence: $1 \to K_m \to \Gamma \to \Gamma/K_m \to 1$ with the Hurwitz group Γ/K_m acting on the compact Riemann surface $\mathbb{H}/K_m$ of genus $\widetilde{g_m} = (g-1)\cdot m^{2g}+1$. This last formula comes from finding the index : $|\Gamma/K_m| = |\Gamma/\Lambda|\cdot|\Lambda/K_m| = 84(g-1)\cdot m^{2g}$. □

Theorem 5.2.2 (Macbeath). *The Hurwitz bound is not attained for infinitely many X_g.*

Proof. Consider genus g so that $g-1$ is a prime number p greater than 84. This gives us an infinite sequence of Riemann surfaces $\{X_g\}$, for which we show that each such surface cannot attain the Hurwitz bound. Let $G = Aut(X_g)$ and suppose on the contrary that $|G| = 84p$. Then it is enough to show that G is a simple group, since this contradicts Sylow's theorem that G should contain a nontrivial normal subgroup.

If G has a nontrivial normal subgroup H then we have the surjective canonical homomorphism given by $\pi : G \to G/H$. But a nontrivial homomorphism of a Hurwitz group is also a Hurwitz group and so for some integer k, $84k = |G/H| = \frac{|G|}{|H|} = \frac{84p}{|H|}$. This gives $p = |H| \cdot k$. Now since p is prime and H is a nontrivial subgroup, $k = 1$ and this contradicts that there is no Hurwitz group of order 84. So G is a simple group.

Sylow's theorems tell us that G must have a p-subgroup of order p and if n_p is the number of such subgroups, then $n_p \equiv 1 mod p$ and n_p divides 84. These two conditions imply that $n_p = 1$ and so this subgroup must be normal. □

6 Automorphisms of Compact Riemann Surfaces

This section contains a survey of some results on automorphisms of compact Riemann surfaces.

6.1 Automorphisms of Hyperelliptic Surfaces

Recall that a subgroup H in G is central if H lies in the center of the group (i.e. $H \leq Z(G)$), where the center contains elements of G that commute with every other element in the group. A group G is called an **extension** of Q by H if we have an exact sequence of groups

$$1 \to H \to G \to Q \to 1$$

An extension is called **central** if H is central in G. With this terminology we can completely describe the automorphism groups of hyperelliptic Riemann surfaces.

Theorem 6.1.1. *Let X be a hyperelliptic compact Riemann surface of genus $g \geq 2$ with J the unique involution that fixes $2g+2$ points. Then the group generated by J is normal and central in $Aut(X)$ and the quotient group $Aut(X)/\langle J \rangle$ is a finite Möbius group.*

Proof. For any automorphism of X, T, we have that TJT^{-1} is also an involution that fixes $2g+2$ points. Since J is the unique involution that satisfies this property we have $TJT^{-1} = J$. Thus the group $\mathbb{Z}/2$ generated by J is both normal and central in $Aut(X)$. Since $X/\langle J\rangle$ is the Riemann sphere with $2g+2$ branch points we have that the quotient group $Aut(X)/\langle J\rangle$ must consist of automorphisms of the Riemann sphere that act as permutations on these points. Such a group must be a finite Möbius group. □

Corollary 6.1.2. *Let X be a compact hyperelliptic Riemann surface of genus $g \geq 2$. Then the full automorphism group of X is a central extension of Q by $\mathbb{Z}/2$ where Q is one of the finite Möbius groups:*

$$1 \to \mathbb{Z}/2 \to Aut(X) \to Q \to 1$$

These groups have all been determined[Wea02].

Corollary 6.1.3. *Automorphisms of hyperelliptic curves X do not attain the Hurwitz bound.*

Proof. Suppose $Aut(X)$ is Hurwitz, then it must be perfect and since homomorphic images of perfect group are perfect, one of the finite Möbius groups is perfect. The only such group is the icosahedral group of order 60. So $|Aut(X)| = 2|Q| = 2(60) = 120$ and this is not divisible by 84 and so $Aut(X)$ cannot be Hurwitz. □

6.2 Lifting Automorphisms and Accola's Bound

Let X be a compact Riemann surface. If φ is an automorphism of X, we can define an induced action on the fundamental group $\pi(X,x)$ by defining a map $\varphi_* : \pi(X,x) \to \pi(X,x)$ which sends γ to $\alpha^{-1}\varphi(\gamma)\alpha$, where α is a path joining x and $\varphi(x)$.

Theorem 6.2.1 (Lifting Automorphisms, A.M. Macbeath 1961, [Mac61]). *Let X be compact Riemann surface with an automorphism φ and D a normal subgroup of $\pi(X,x)$. If D is invariant under φ_* then φ lifts to an automorphism $\widetilde{\varphi}$ of the Galois covering corresponding to D such that it preserves the fibers.*

For a fixed genus g let $\mu(g)$ denote the largest automorphism group that acts on a compact Riemann surface of genus g. Hurwitz's theorem shows that $\mu(g) \leq 84(g-1)$. The following theorem shows that $\mu(g) \geq 8g+8$.

Theorem 6.2.2 (Accola 1968 [Acc68], Maclachlan 1969, [Mac69]). *For every $g \geq 2$, there exists a group of order $8g+8$ acting on a hyperelliptic Riemann surface of genus g.*

6.3 Abelian Groups of Automorphisms

In this section we examine the results of Maclachlan and Harvey on abelian and cyclic groups of automorphisms. See Breuer's book, [Bre00], for more information. Let G be a finite Abelian group of order n which has prime decomposition $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then by the Kronecker decomposition theorem we have $G \cong A_1 \times \cdots \times A_k$ where A_i is a p-group of order $p_i^{\alpha_i}$.

Theorem 6.3.1 (C. Maclachlan, 1964). *Let G be a finite Abelian group and Γ a Fuchsian group with signature $(g; m_1, ..., m_r)$ with $M = lcm(m_1, ..., m_r)$. Then there is a surface kernel epimorphism $\varphi : \Gamma \to G$ if and only if the following are satisfied:*
(1) There exists at least one epimorphism,
(2) $lcm(m_1, ..., \widehat{m_i}, ..., m_r) = M$, for all i
(3) M divides $exp(G)$
(4) $r \neq 1$ and if $g = 0$, $r \geq 3$,
(5) If M is even and exactly one of the Abelian invariants of G is divisible by the maximum power of 2 dividing M, then the number of m_i divisible by the maximum power of 2 dividing M is even.

As a corollary we get Harvey's result for cyclic groups of automorphisms:

Theorem 6.3.2 (W.J. Harvey, 1964). *Let Γ be a Fuchsian group with signature $(g; m_1, ..., m_r)$, and let $M = lcm(m_1, ..., m_r)$. Then there is a surface kernel epimorphism epimorphism $\varphi : \Gamma \to \mathbb{Z}_n$ if and only if the following are satisfied:*
(1) $lcm(m_1, ..., \widehat{m_i}, ..., m_r) = M$, for all i
(2) M divides n, and for $g = 0$, $M = n$
(3) $r \neq 1$ and if $g = 0$, $r \geq 3$,
(4) If M is even, then the number of periods m_i divisible by the maximum power of 2 dividing M is even.

Corollary 6.3.3. *Let X be compact Riemann surface of genus $g \geq 2$. Then*
(1) The maximum order of an automorphism is $4g + 2$.
(2) If X admits an abelian group of automorphisms G, then $|G| \leq 4g + 4$.

In particular, the upper-bound for a cyclic group action is $4g + 2$. Similarly, the following are other Hurwitz-type bounds for various classes of groups.

Theorem 6.3.4 ([Wea02]). *For each of the following classes we have the given upper-bound for the group of automorphisms:*
(1) Abelian: $4g+4$
(2) Solvable: $48(g-1)$
(3) Cyclic: $4g+2$
(4) Nilpotent: $16(g-1)$

$$(5)\ p\text{-}group = \begin{cases} 16(g-1), & \text{if } p=2 \\ 9(g-1), & \text{if } p=3 \\ \frac{2p}{p-3}(g-1), & \text{if } p>3 \end{cases}$$

For a finite group G, let $\mu(G)$ denote the minimum genus greater then 1 for which G acts as a group of automorphisms. The following is the result of W. Harvey [Har66] on the minimum genus for cyclic groups.

Theorem 6.3.5 (W.J. Harvey, 1966). *For the cyclic group of order* n, $\mathbb{Z}_n$, *let* $p_1^{r_1}\cdots p_t^{r_t}$ *be the prime factorization of* n. *Then we have the following:*

$$\mu(\mathbb{Z}_n) = \begin{cases} max\{2, (\frac{p_1-1}{2})(\frac{n}{p_1})\}, & \text{if } r_1>1 \text{ or } n \text{ is prime;} \\ max\{2, (\frac{p_1-1}{2})(\frac{n}{p_1}-1)\}, & \text{if } r_1=1. \end{cases}$$

6.4 Kulkarni's Theorem

Let G be a finite group and G_p a Sylow p-subgroup of order p^{n_p}. If the exponent of G_p is $exp(G_p)=p^{e_p}$ then n_p-e_p is called the **cyclic p-deficiency** of G. Define f_p as follows:

$$f_p = \begin{cases} n_p-e_p, & \text{if } p \text{ is odd prime or } p=2 \text{ and } n_2=e_2 \\ n_2-e_2-1, & \text{if } p=2 \text{ and } n_2>e_2 \end{cases}$$

Now we define an integer that depends on G given by:

$$n_0(G) = \eta\prod_p p^{f_p}$$

where the product is over the prime divisors of $|G|$ and η depends on G_2. A Sylow 2-group is of **Type II** if it is non-cyclic and there exists a homomorphism $\varphi: G\to\mathbb{Z}_2$ with $\varphi^{-1}(0)$ containing elements of order strictly less than 2^{e_2} and $\varphi^{-1}(1)$ containing elements of order 2^{e_2}. Otherwise it is defined to be of **Type I**. Now define $\eta=1$ or 2 depending on if G_2 is of Type I or II respectively.

Theorem 6.4.1 (Kulkarni, 1987, [Kul87]). *Let G be a finite group. Then there exists an integer $n_0(G) \geq 1$ such that*
(1) If G acts on S_g, then $g \equiv 1(mod\, n_0(G))$.
(2) Conversely, if $g \equiv 1(mod\, n_0(G))$ is large enough, then G acts on S_g.

Example 6.4.1. Let $G = \mathbb{Z}_p$ for p odd prime. Then $G_p = G$ and $exp(G) = p$ and this gives $n_p = e_p = 1$ so that $n_0(G) = 1$. So if $\mathbb{Z}_p$ acts on S_g, then $g = 1 + a$ for some nonnegative integer a. Conversely, we have already seen that every genus of the form $g = 1 + n$ has a free action of $\mathbb{Z}_n$.

Example 6.4.2. Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$, for p odd prime. Then $G_p = G$ and $exp(G) = p$. In this case $n_0(G) = p$ so that if G acts on S_g then $g = 1 + ap$ for some nonnegative integer a. Similarly, if G is an elementary abelian group $E_{p^n} = \overbrace{\mathbb{Z}_p \times \ldots \times \mathbb{Z}_p}^{n}$, then $exp(E_{p^n}) = p$ so that $f_p = n_p - e_p = n - 1$ and this gives $n_0(G) = p^{n-1}$. So if E_{p^n} acts on S_g then $g = 1 + ap^{n-1}$ for some nonnegative integer a.

7 Plane Algebraic Curves

In this section, it will be proved that the Klein quartic is the unique non-singular plane curve that attains the Hurwitz bound.

7.1 Finite Collineation Groups

We state the classification theorem of finite collineation groups in three variables;that is, finite subgroups of $PGL(3, \mathbb{C})$. This can be found in Miller and Blichfeldt's book [MBD16, pg. 235]. See also [HL88].
(A) Abelian groups of rank ≤ 2
(B) Finite subgroups of $U(2)$
(C) A group generated by the permutation $T : (x_1x_2x_3)$ and an abelian group of transformations H of the form:

$$H : \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix} T : \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(D) The group generated by H, T from (C) and a transformation R of the form:

$$R: \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}$$

(E) The group of order 36 generated by S_1, T, and V:

$$S_1: \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix} V: \rho \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

where $\rho = \frac{1}{\omega - \omega^2}$ and $\omega^3 = 1$.

(F) The group of order 72 generated by S_1, T, V and UVU^{-1} with:

$$U: \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon\omega \end{pmatrix}$$

where $\epsilon^3 = \omega^2$.

(G) The Hessian group of order 216 generated by S_1, T, V, and U.

(H) Alternating group A_5 of order 60, generated by X, Y, and Z:

$$X: \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^4 & 0 \\ 0 & 0 & \zeta \end{pmatrix} Y: \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} Z: \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix}$$

where $\zeta^5 = 1$, $s = \zeta^2 + \zeta^3$ and $t = \zeta + \zeta^4$ with $\sqrt{5} = t - s$.

(I) Alternating group A_6 of order 360, generated by X, Y, Z and W:

$$W: \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \lambda_1 & \lambda_1 \\ 2\lambda_2 & s & t \\ 2\lambda_2 & t & s \end{pmatrix}$$

(J) The group $PSL(2, \mathbb{F}_7)$ of order 168, generated by S, T, and R with relations $R^2 = T^3 = S^7 = I$, $T^{-1}ST = S^4$, $R^{-1}TR = T^2$, $(RS)^4 = I$ where

$$S = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix} T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$R = h \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

where $\beta^7 = 1$, $a = \beta^4 - \beta^3$, $b = \beta^2 - \beta^5$, $c = \beta - \beta^6$, and $h = \frac{1}{7}(\beta + \beta^2 + \beta^4 - \beta^6 - \beta^5 - \beta^3) = \frac{1}{\sqrt{-7}}$

7.2 Chang's Theorem and Moduli of Plane Curves

Theorem 7.2.1 (H.C. Chang, 1978, [Cha78]). *Any isomorphism of non-singular plane algebraic curves C_1 and C_2 is induced by a projective linear transformation in $PGL(3, \mathbb{C})$. In particular, for a smooth plane curve C this gives an imbedding $\imath : Aut(C) \hookrightarrow PGL(3, \mathbb{C})$.*

Let $\mathbb{C}_d[x, y, z]$ be the space of homogeneous polynomials of degree d. Each polynomial of fixed degree d in x, y and z will have $\frac{(d+1)(d+2)}{2}$ coefficients. So we can identify $\mathbb{C}_d[x, y, z]$ with the complex space $\mathbb{C}^N$ where $N = \frac{(d+1)(d+2)}{2}$. We want to consider the non-singular polynomials in this space:

Lemma 7.2.2. *The subset of non-singular polynomials $\mathbb{C}_d^{nonsing}[x, y, z]$ in $\mathbb{C}_d[x, y, z]$ is open and dense.*

Theorem 7.2.3. *The number of parameters need to describe the space of non-singular complex plane curves of genus g is*

$$\frac{(g+1)(g+2)}{2} - 9.$$

Proof. By Hilbert's Nullstellensatz theorem, the zero sets of two polynomials are equal: $\{P = 0\} = \{Q = 0\}$ if and only if P and Q have the same irreducible factors. But all polynomials in $\mathbb{C}^N$ are non-singular and hence irreducible; so this occurs if and only if $P = \lambda Q$ for some constant λ. Thus identifying polynomials in $\mathbb{C}^N$ which give the same locus in $\mathbb{C}P^2$ gives the projective space $\mathbb{C}P^{N-1}$. Now we have to identify isomorphic curves in $\mathbb{C}P^2$, this is where we use Chang's theorem since an isomorphism of curves is given by the restriction of a $PGL(3, \mathbb{C})$ map. This gives an action of $PGL(3, \mathbb{C})$ on $\mathbb{C}P^{N-1}$. The action is proper since for any compact set K in $\mathbb{C}P^{N-1}$ the set $\{g \in PGL(3, \mathbb{C}) \mid gK \cap K \neq \emptyset\}$ is a closed subset of $\mathbb{C}P^8$ so is compact. Note also, that for a given $P \in \mathbb{C}P^{N-1}$ the stabilizer $\{g \in PGL(3, \mathbb{C}) \mid gP = P\} = Aut(P)$ is finite by Hurwitz's theorem. So we have

$$\begin{aligned} dim(\mathbb{C}P^{N-1}/PGL(3,\mathbb{C})) &= dim(\mathbb{C}P^{N-1}) - dim(PGL(3,\mathbb{C}) \\ &= N-1-8 = \frac{(d+1)(d+2)}{2} - 9. \end{aligned}$$

□

7.3 Invariant Surfaces in $\mathbb{C}P^2$

Given a finite group G in $PGL(3,\mathbb{C})$, we want to find a G-invariant non-singular polynomial $f(x,y,z)$ so that its zero-set is an invariant compact smooth algebraic curve in $\mathbb{C}P^2$. For more information see [MBD16, pg. 253] and [Ben93].
First we consider the simplest case of a cyclic group:

Example 7.3.1. Let G be a finite cyclic group of order d generated by the following matrix:

$$A_{mn} = \begin{pmatrix} 1 & & \\ & \zeta^m & \\ & & \zeta^n \end{pmatrix}$$

where m and n are non-negative integers and $\zeta = e^{\frac{2\pi i}{d}}$. Then $f(x,y,z) = x^d + y^d + z^d$ is non-singular and hence irreducible and so $X_G = \{f = 0\}$ is a G-invariant compact Riemann surface in $\mathbb{C}P^2$.

We now list examples of invariants for some finite collineation groups.

(C) Suppose we have $G = \mathbb{Z}/3 \times \mathbb{Z}/3$ given by

$$X = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix} Y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where $\omega^3 = 1$ and $X^3 = Y^3 = I$. Then the Fermat curve of degree three: $x^3 + y^3 + z^3$ is a non-singular invariant polynomial for the group. We can generate a family of Fermat curves that are invariant by considering $\{x^{3d} + y^{3d} + z^{3d} = 0\}$ for any $d \geq 1$. These are still non-singular and invariant to arbitrary high degree with genus $\frac{(3d-1)(3d-2)}{2}$. In particular, by Kulkarni's theorem the period for G is $n_0(G) = 3$ so that if G acts on a non-singular curve in $\mathbb{C}P^2$ then the genus must be $g = 1+3a$ for some non-negative integer a. Solving for the genus in terms of the degree we get:

$$a = \frac{(18d-9)^2 - 81}{216},$$

$a = 0, 1, 3, 9, 18, 30, 45, \ldots$. Similarly for $G = \mathbb{Z}/p \times \mathbb{Z}/p$ for p odd prime, represented as a group of the abelian type (A) has the family of Fermat curves invariant $\{x^{dp} + y^{dp} + z^{dp} = 0\}$ for any $d \geq 1$. Again by Kulkarni's theorem the period for G is $n_0(G) = p$ and there exists a non-negative integer a such that

$$\frac{(pd-1)(pd-2)}{2} = 1 + ap$$

for any $d \geq 1$.

(D) Let G be the group generated by H, R and T, where

$$H = \langle \begin{pmatrix} 1 & & \\ & 1 & \\ & & \omega \end{pmatrix} \rangle R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and $\omega^d = 1$. This group generates the full automorphism group for the Fermat curve $\{x^d + y^d + z^d = 0\}$ for any $d \geq 3$ by the following theorem.

Theorem 7.3.1. *[FG67] Let G be the set of linear transformations leaving the polynomial $z_1^d + z_2^d + z_3^d$ invariant. Then G is the group of linear transformations given by $\acute{z}_i = \epsilon_i z_{\sigma(i)}$ where $\epsilon_i^d = 1$ and σ is a permutation in S_3.*

(G) Let $\phi = xyz$, $\psi = x^3 + y^3 + z^3$, and $\chi = x^3y^3 + y^3z^3 + x^3z^3$, then the generators are given by

$$\begin{aligned} \sigma_1 &= \psi^2 - 12\chi, \\ \sigma_2 &= \psi(\psi^3 + 216\phi^3), \\ \sigma_3 &= \phi(27\phi^3 - \psi^3), \\ \sigma_4 &= (x^3 - y^3)(y^3 - z^3)(x^3 - x^3) \end{aligned}$$

which satisfy the following relation

$$(432\sigma_4^2 - \sigma_1^3 + 3\sigma_1\sigma_2)^2 = 4(1728\sigma_3^2 + \sigma_2)$$

If $\mathscr{I}$ is the ideal generated by this relation, then the invariant polynomials are $\mathbb{C}^G[x,y,z] = \mathbb{C}[\sigma_1,\sigma_2,\sigma_3,\sigma_4]/\mathscr{I}$.

(H) There are four generators for $\mathbb{C}^H[x,y,z]$. Let A be the non-singular polynomial x^2+yz and $B = x(x^5+y^5+z^5+5xy^2z^2-5x^3yz)$ then A and B are two generators.

(I) Again, there are four generators for the invariant polynomials. $F = A+\lambda B$ where $\lambda = \frac{-9\pm3\sqrt{-15}}{20}$ is a generator of lowest degree. The other invariants are the determinant of the Hessian of F, the bordered Hessian and the Jacobian of the previous invariants.

(J) $PSL(2,\mathbb{F}_7)$ is isomorphic to $PSL(3,\mathbb{F}_2)$ and has as generators for the invariant polynomials $\mathbb{C}^J[x,y,z]$ the following polynomials of degree 4, the Hessian of degree 6, bordered Hessian of degree 14 and the Jacobian of the previous polynomials of degree 21:

$$f = xy^3+yz^3+zx^3,$$
$$H(f) = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix},$$
$$X = \begin{vmatrix} f_{11} & f_{12} & f_{13} & f_1 \\ f_{21} & f_{22} & f_{23} & f_2 \\ f_{31} & f_{32} & f_{33} & f_3 \\ f_1 & f_2 & f_3 & 0 \end{vmatrix},$$

and finally, the Jacobian of $F = (f, H, X)$:

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \\ \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \end{vmatrix}.$$

Notice that the invariant of lowest degree f is the **Klein quartic** and so gives a Hurwitz surface in $\mathbb{C}P^2$ of genus 3 with 168 automorphisms. In the next section we will see that this curve is the unique non-singular plane curve which attains the Hurwitz bound.

7.4 Hurwitz Curves in $\mathbb{C}P^2$

If C is a Hurwitz surface in $\mathbb{C}P^2$, then its automorphism group is a Hurwitz group in $PGL(3, \mathbb{C})$. So finding Hurwitz curves in the complex projective plane is now reduced to finding Hurwitz groups among Blichfeldt's classification of finite collineation groups. We know that $PSL(2,7)$ is a Hurwitz group for the Klein quartic, the next theorem shows that it's the only Hurwitz group.

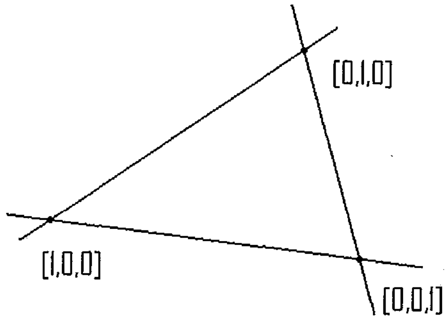


Figure 10: Fundamental triangle in $\mathbb{C}P^2$.

Theorem 7.4.1. *The only Hurwitz group in $PGL(3, \mathbb{C})$ is $PSL(2,7)$.*

Proof. Since the order of a Hurwitz group must be divisible by 84 we have only to check subgroups of type (A),(B),(C) and (D). Subgroups of type (A) are not possible as abelian groups never reach the Hurwitz bound. Suppose G is of type (B), that is, a finite subgroup of $U(2)$. If G is Hurwitz then it must be perfect and so is actually a finite subgroup of $SU(2)$. But the only perfect finite subgroup of $SU(2)$ is the binary icosahedral group of order 120, and this is not divisible by 84. It only remains to check subgroups G from type (C) and (D) and we will show that no such group can be perfect and hence cannot be a Hurwitz group. The fundamental triangle $\{x_1 = 0\}$, $\{x_2 = 0\}$, $\{x_3 = 0\}$ is invariant under these transformations and those of from H fix the points $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$ and $p_3 = [0, 0, 1]$. T cyclically permutes these points while transformations of the form R fix p_1 and interchange p_2 and p_3. Thus G acts on these points and so gives a homomorphism φ to the symmetry group of the fundamental triangle: $\varphi : G \to S_3$. This action is not faithful as transformations of the form H are in the kernel. We have

$$Im(\varphi) = \begin{cases} \mathbb{Z}_3, & \text{if } G \text{ is type (C)} \\ S_3, & \text{if } G \text{ is type (D)} \end{cases}$$

In the first case we have $1 \to H \to G \to \mathbb{Z}_3 \to 1$ and so G has a non-trivial abelian quotient and thus cannot be perfect. Similarly, in the second case we have $G/Ker(\varphi) \cong S_3$ and as $\mathbb{Z}_3$ is a normal subgroup of S_3 there is a subgroup K of G containing $Ker(\varphi)$ such that $G/K \cong \mathbb{Z}_3$ and again we have that G cannot be perfect. □

Corollary 7.4.2. *The Klein quartic is the unique non-singular complex plane curve that attains the Hurwitz bound.*

Proof. If $C = \{f = 0\}$ is a non-singular plane curve that attain the Hurwitz bound, f must be an invariant quartic polynomial of $PSL(2, 7)$ and so is a constant multiple of the Klein quartic. □

Theorem 7.4.3. *Let C be a non-singular plane curve, C^* its dual and $\widetilde{C}$ the desingularization of the dual, all of which have the same genus g. Then we have the following:*

$$Aut(C) \subset Aut(C^*) \subset Aut(\widetilde{C}).$$

Proof. $Aut(C) = Lin(C) = Lin(C^*) \subset Aut(C^*)$. And now since each automorphism of the dual curve lifts to an automorphism of the desingularization[BD04] we have $Aut(C^*) \subset Aut(\widetilde{C})$. □

Example 7.4.1. Let C be the Klein quartic of genus 3 and degree $d = 4$, then $Aut(C) = Aut(C^*) = Aut(\widetilde{C})$. It is well-known that this curve has 24 flexes f and 28 bitangents b. By the classical Plücker's formula C^* has degree $d^* = d(d-1) = 12$ and the genus is given by $g = \frac{(d^*-1)(d^*-2)}{2} - b - f = \frac{10*11}{2} - 52 = 3$. By the above theorem we have $Aut(C) \subset Aut(\widetilde{C})$. But $\widetilde{C}$ also has genus 3 and cannot be hyperelliptic since such curves never reach the Hurwitz bound, so the desingularization of C is embeddable in the complex plane $\mathbb{C}P^2$ and therefore must be the Klein quartic, $\widetilde{C} = C$.

Since every non-singular plane curve has genus of type $g = \frac{(d-1)(d-2)}{2}$, it's natural to ask the following question: does there exist a Hurwitz surface of genus $g = \frac{(d-1)(d-2)}{2}$ with $d \geq 5$ that is not embeddable in the complex plane $\mathbb{C}P^2$ as a non-singular curve?

8 Automorphisms of Non-Compact Surfaces

This section contains theorems about automorphisms of surfaces with cusps. There is a similar bound for these surfaces in terms of their topology, determined by the genus and and the number of cusps. Interestingly, even though a torus has infinitely many automorphisms, the punctured torus does not. We will also give a criterion for when a finite group can be realized as a group of automorphisms that attains the upper-bound.

8.1 Bound for Cusped Surfaces

Let X be a non-compact Riemann surface that is uniformized by a torsion-free Fuchsian group Λ with parabolic elements. So we have $X \cong \Gamma/\Lambda$ where the fundamental region for Λ is non-compact but has finite hyperbolic area. The topology of X is determined by two parameters; the genus g and the number of cusps s that go off to infinity and $\pi_1(X) \cong \Lambda$. Since Λ has fundamental region with finite area it has finite number of sides and is finitely generated. The signature is given by $(g; -; s)$ where $s > 2$ is the number of maximal conjugacy classes of parabolic infinite cyclic subgroups and we denote this by $\Lambda_{g,s}$

Theorem 8.1.1. *The fundamental region for a Fuchsian group with signature $(g; m_1, ..., m_r; s)$ has hyperbolic area*

$$2\pi\{(2g-2)+s+\sum_{i=1}^{r}(1-\frac{1}{m_i})\}.$$

Remark 8.1.2. Note that for a torsion-free Fuchsian group $(g; -; s)$, if $g = 0$ then the fundamental region has positive area if $s \geq 3$. Thus, there is no Fuchsian group with signature $(0; -; s)$ with $s < 3$. Note that the punctured sphere is conformal to the complex plane and the twice punctured sphere is conformal to the the punctured plane. So by the uniformization theorem, the three-punctured sphere has hyperbolic structure. Similarly, for a Fuchsian group of signature $(1; -; s)$, we must have $s > 0$(i.e. the torus has no hyperbolic structure but the punctured torus does).

Theorem 8.1.3. *Let $\mathscr{P}$ be the fundamental region for a Fuchsian group with signature $(g; m_1, ..., m_r; s)$. Then*

$$\mu(\mathscr{P}) \geq \frac{\pi}{3},$$

and this minimum is attained by the modular group $PSL(2, \mathbb{Z})$ with signature $(0; 2, 3; 1)$.

The following bound was found by Oikawa using covering space theory, here we give a Fuchsian groups proof of the bound.

Theorem 8.1.4 (Kotaro Oikawa, 1956,[Oik56]). *Let X be a Riemann surface of genus g with $s > 0$ cusps that go off to infinity. The automorphisms of X are finite and we have the following bound:*

$$|Aut(X)| \leq 12(g-1) + 6s.$$

Proof. Let X be uniformized by $\Lambda = \Lambda_{g,s}$. Since Λ is torsion-free and non-cyclic, $Aut(X) \cong N(\Lambda)/\Lambda$ where $N(\Lambda)$ is the normalizer in $PSL(2, \mathbb{R})$ and is itself a Fuchsian group, call it Γ. Let F_Λ and F_Γ be the fundamental regions. Then we have

$$|Aut(X)| = [\Gamma : \Lambda] = \frac{\mu(F_\Lambda)}{\mu(F_\Gamma)} = \frac{2\pi(2g-2+s)}{\mu(F_\Gamma)}.$$

For a general Fuchsian group with parabolic elements, the lower bound for the fundamental region is $\frac{\pi}{3}$(in the compact case it was $\frac{\pi}{21}$), so we have:

$$|Aut(X)| \leq \frac{2\pi(2g-2+s)3}{\pi} = 12(g-1) + 6s.$$

□

8.2 Criterion for Maximal Symmetry

The modular group $PSL(2, \mathbb{Z})$ has signature $(0; 2, 3, 1)$ and has presentation $\langle X, Y | X^2 = Y^3 = 1 \rangle$, so it is a free product $\mathbb{Z}/2 * \mathbb{Z}/3$.

Lemma 8.2.1. *Let Γ be the modular group $PSL(2,\mathbb{Z})$, with Λ a normal subgroup of finite index greater than* 3. *Then Λ has signature $(g; -; s)$ for some $s > 0$.*

Proof. We show that Λ cannot contain any elliptic elements. Every elliptic element in $PSL(2,\mathbb{Z})$ is conjugate to a power of X or Y, but as these have prime order it must be conjugate to either X or Y. Suppose Λ contains X, then Γ/Λ is generated by an element of order less than or equal to 3 and this contradicts that the condition on the index. Similarly, Λ cannot contain Y. □

Theorem 8.2.2. *Let G be a finite group with $|G| > 3$. Then G can be realized as a maximal symmetry group of a Riemann surface with cusps if and only if G is a homomorphic image of the modular group $PSL(2,\mathbb{Z})$.*

Proof. Suppose G is a homomorphic image of the modular group Γ given by π with kernel Λ.

$$1 \to \Lambda \to PSL(2,\mathbb{Z}) \to G \to 1$$

By the lemma we have $\Lambda = \Lambda_{g,s}$ for some g and s. Then we have

$$|G| = |\Gamma/\Lambda_{g,s}| = \frac{\mu(F_{\Lambda_{g,s}})}{\mu(F_\Gamma)} = \frac{2\pi(2g-2+s)}{\frac{\pi}{3}} = 12(g-1)+6s.$$

Conversely, if G acts as maximal symmetry group for a Riemann surface with cusps uniformized by $\Lambda_{g,s}$. Then $G \cong \Gamma/\Lambda_{g,s}$ for some Γ. Then

$$12(g-1)+6s = |G| = |\Gamma/\Lambda_{g,s}| = \frac{\mu(\Lambda_{g,s})}{\mu(\Gamma)} = \frac{2\pi(2g-2+s)}{\mu(\Gamma)}$$

and we see that Γ must be the modular group $PSL(2,\mathbb{Z})$. This completes the proof. □

8.3 Punctured Surfaces in $\mathbb{C}P^2$

The following can be found as an exercise in [For81] pg.59.

Theorem 8.3.1. *Let X and Y be compact Riemann surfaces and X_r and Y_s be r and s-punctures on X and Y respectively. Then every isomorphism $f : X_r \to Y_s$ extends to an isomorphism $\tilde{f} : X \to Y$.*

Proof. Since in particular, every isomorphism is a homeomorphism, we have $r = s$. Let U_p be a small neighbourhood of a puncture p in Y_s, then $f^{-1}(U_p)$ is a small neighbourhood of a puncture p' in X_s (since f is an isomorphism, locally this is a unbranched covering of a punctured disk and so $f^{-1}(U_p)$ is itself isomorphic to a punctured disk). Now we can extend f by defining $f(p') = p$ and carry this out for each puncture. The extended map $\widetilde{f}$ is an isomorphism of X and Y. □

Corollary 8.3.2. *$Aut(X_s)$ is a subgroup of $Aut(X)$.*

Proposition 8.3.3. *For $s > 0$, if the automorphism group of the s-punctured Klein quartic attains its upper-bound, then $s = 24$.*

Proof. Let G be the automorphism group of the s-punctured Klein quartic. Then $|G| = 24 + 6s$ and as every automorphism extends to an automorphism of the compact Klein quartic it follows that G is a subgroup of order greater than 24. But the largest proper subgroup of $PSL(2,7)$ has order 24, so $|G| = 168$ and this gives $s = 24$. □

We now attempt to generalize the above proposition. We call X a **simple Hurwitz surface** if X is a Hurwitz surface and $Aut(X)$ is a simple group. Then we have the following theorem

Theorem 8.3.4. *Let X_s be a s-punctured simple Hurwitz surface for $s > 0$. Then $Aut(X_s)$ is either trivial or $s = 12(g-1)$ and in this case $Aut(X_s)$ attains the upper-bound $12(g-1) + 6s$.*

Proof. It's enough to show that $Aut(X_s)$ is a normal subgroup of $Aut(X)$. Let φ be any automorphism of the punctured surface, then for any $T \in Aut(X)$, $T^{-1}\varphi T$ is an automorphism of X but permutes the s-punctures and so can be realized as an automorphism of X_s. Since $Aut(X)$ is a simple group, we have that either $Aut(X_s)$ is trivial or $Aut(X_s) = Aut(X)$ which forces $s = 12(g-1)$. □

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