THE MARCEL RIESZ KERNEL AND A GREEN KERNEL FOR A HALF-SPACE

THE MARCEL RIESZ KERNEL AND A GREEN KERNEL FOR A HALF-SPACE

By

DAVID M. WALMSLEY, B.Sc.

A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Science

McMaster University

April 1977

MASTER OF SCIENCE (1977)
(Mathematics)

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: The Marcel Riesz Kernel and a Green Kernel for a Half-Space

AUTHOR: David Micheal Walmsley, B.Sc. (McMaster University)

SUPERVISOR: Mr. H.L. Jackson

NUMBER OF PAGES: (viii), 185

SCOPE AND CONTENTS: The principal object of this thesis is to study some aspects of the Marcel Riesz kernel and the notion of superharmonicity of fractional order. We define a Green kernel for a half-space and show that this kernel satisfies the principle of domination. This leads us to a new definition of superharmonicity of fractional order on a half-space.

ACKNOWLEDGEMENT

I would like to express my gratitude to Professor H.L. Jackson for his invaluable advice and encouragement during the preparation of this thesis. It is only through his suggestions and constructive criticism that this work has been completed.

My thanks are also due to McMaster University and the National Research Council for their generous financial support.

I gratefully acknowledge the prompt and efficient typing of C. Merritt.

Finally, I am indebted to my family and friends for their help throughout my stay at McMaster.

To the memory of my mother

TABLE OF CONTENTS

INTRODUCTION

In 1938, Marcel Riesz [35] extended the definition of the Riemann-Liouville integral from the real line to higher dimensional Euclidean space. The extended integral has since been called the M. Riesz kernel. We study the properties of the M. Riesz kernel and two associated Green kernels in this thesis.

We introduce the basic notation and results required throughout this thesis in chapter one.

In chapter two, we define the M. Riesz kernel and the M. Riesz potential. The concepts of capacity and energy relative to the M. Riesz kernel are defined. The principles of domination and balayage for the M. Riesz kernel are studied, which leads to the Otto Frostman [21] definition of a superharmonic function of fractional order. We prove the representation theorems for superharmonic functions of fractional order. We conclude chapter two by giving the O. Frostman definition of a superharmonic function of fractional order on a region. For this definition, we must define a suitable Green kernel of fractional order for the region.

Chapter three is devoted to the potential theory introduced by G.A. Hunt [24] in 1956. We show that the M. Riesz kernel is the potential kernel for the symmetric stable semigroup for certain orders.

We study another Green kernel for a half-space, defined by M. Essén and H. L. Jackson [17] in chapter four. In particular, we show that this Green kernel also satisfies the principles of domination and balayage. This allows us to define a convex cone of functions which strictly contains the superharmonic functions of fractional order on a half-space defined by O. Frostman.

CHAPTER 1

THE PRELIMINARIES

In this chapter we define our notation, and introduce the basic concepts to be used in the remaining chapters.

§1. Some Essential Concepts from Topology and Analysis

We denote the real line $(-\infty, +\infty)$ by $\mathbb{R}$ and p-dimensional Euclidean space by $\mathbb{R}^p$ $(p \geq 2)$. The Euclidean norm and inner product in $\mathbb{R}^p$ $(p \geq 1)$ are defined by

$$|x| = \left(\sum_{i=1}^{p} x_i^2\right)^{\frac{1}{2}}$$

and

$$x \cdot y = \sum_{i=1}^{p} x_i y_i$$

respectively, for $x = (x_1, \ldots, x_p)$, $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$. The extended real line $[-\infty, +\infty]$ is denoted by $\overline{\mathbb{R}}$. We denote by $\mathbb{N}$ the set of all integers $n \geq 1$ and by $\mathbb{C}$ the complex numbers.

In general, we follow the terminology of Bourbaki [3].

Definition 1.1.

A nonempty collection $\mathcal{F}$ of subsets of a set X is called a filter on X iff it satisfies the following three properties:

(F_I) $B \supset A \in \mathcal{F}$ implies that $B \in \mathcal{F}$;

(F_{II}) $A, B \in \mathcal{F}$ imply that $A \cap B \in \mathcal{F}$;

(F_{III}) $\emptyset \notin \mathcal{F}$, where $\emptyset$ is the empty subset of X.

Examples 1.1.

Let X be a set endowed with a topology τ. Then the set of all neighbourhoods of a point $x \in X$ with respect to τ, which we denote by $N(x)$, is a filter on X called the neighbourhood filter of x with respect to τ.

Let X be an infinite set. Then $\mathcal{F} = \{A \subset X \mid X \setminus A \text{ is finite}\}$ is a filter on X. The filter of complements of finite subsets of $\mathbb{N}$ is called the Fréchet filter.

Definition 1.2.

A nonempty collection $\mathcal{B}$ of subsets of a set X is called a filterbase on X iff it satisfies the following two properties:

(B_I) $A, B \in \mathcal{B}$ imply that there exists $C \in \mathcal{B}$ such that $A \cap B \supset C$;

(B_{II}) $\emptyset \notin \mathcal{B}$, where $\emptyset$ is the empty subset of X.

Examples 1.2.

Every filter on a set X is a filterbase on X.

Let X be a set endowed with a topology τ. We denote by $N'(x)$ the set of all deleted neighbourhoods of a point $x \in X$ with respect to τ. If x is not an isolated point of X, then $N'(x)$ is a filterbase on X and a filter on $X \setminus \{x\}$. In the remainder of this chapter, we consider only those topological spaces which contain no isolated points.

Let A be a subset of X and let a be an element in the closure $\overline{A}$ of A with respect to τ. Then the set $N'_A(a) = \{V \cap A \mid V \in N'(a)\}$

is a filterbase on X called the <u>trace of N'(a) on A</u>.

We mention the following proposition which is an immediate consequence of Definitions (1.1) and (1.2).

<u>Proposition 1.1.</u>

If $\mathcal{B}$ is a filterbase on a set X, then

$$\mathcal{F}(\mathcal{B}) = \{A \subset X \mid \text{there exists } B \in \mathcal{B} \text{ such that } A \supset B\}$$

is a filter on X called the <u>filter generated by $\mathcal{B}$</u>.

<u>Definition 1.3.</u>

A partial ordering $\prec$ defined on a nonempty set D is said to satisfy the <u>directive property</u> iff for every $k, \ell \in D$, there exists $m \in D$ such that $k \prec m$ and $\ell \prec m$. The pair $(D, \prec)$ is called a <u>directed set</u> iff $\prec$ is a partial ordering on the nonempty set D which satisfies the <u>directive property</u>.

<u>Examples 1.3.</u>

If $\mathcal{B}$ is a filterbase on a set X, then $(\mathcal{B}, \supset)$ is a directed set.

Let $(D, \prec)$ be a directed set. For each $k \in D$, the set

$$S(k) = \{\ell \in D \mid k \prec \ell\}$$

is called the <u>section of D relative to k</u>. The set of all sections of D is a filterbase on D called the <u>section filterbase on D</u>.

<u>Definition 1.4.</u>

Let $\Phi(X)$ denote the set of all filters on a set X. For $\mathcal{F}, \mathcal{F}' \in \Phi(X)$, $\mathcal{F}'$ is defined to be <u>finer than $\mathcal{F}$</u> iff $\mathcal{F}' \supset \mathcal{F}$. The relation $\supset$ is a partial ordering on $\Phi(X)$.

Remark 1.1.

Let $\mathcal{B}$ be a filterbase on a set X. Then any filter on X which contains $\mathcal{B}$ is finer than the filter $\mathcal{F}(\mathcal{B})$ generated by $\mathcal{B}$.

Definition 1.5.

Let X be a topological space, $\mathcal{F}$ a filter on X, and $\mathcal{B}$ a filterbase on X. A point $x \in X$ is defined to be a limit point of $\mathcal{F}$ iff $\mathcal{F} \supset N(x)$. A point $x \in X$ is defined to be a limit point of $\mathcal{B}$ iff x is a limit point of the filter $\mathcal{F}(\mathcal{B})$ generated by $\mathcal{B}$.

Remark 1.2.

Let $\mathcal{B}$ be a filterbase on topological space X. Then $x \in X$ is a limit point of $\mathcal{B}$ iff for every $V \in N(x)$, there exists $B \in \mathcal{B}$ such that $V \supset B$.

Definition 1.6.

Let X be a topological space, $\mathcal{F}$ a filter on X, and $\mathcal{B}$ a filterbase on X. A point $x \in X$ is defined to be a cluster point of $\mathcal{F}$ iff $V \cap A \neq \emptyset$ for every $A \in \mathcal{F}$ and $V \in N(x)$. A point $x \in X$ is defined to be a cluster point of $\mathcal{B}$ iff $V \cap B \neq \emptyset$ for every $B \in \mathcal{B}$ and $V \in N(x)$.

Remark 1.3.

Let $\mathcal{B}$ be a filterbase on a topological space X. It follows that $x \in X$ is a cluster point of $\mathcal{B}$ iff $x \in \cap \{\overline{B} | B \in \mathcal{B}\}$. The same is true for every filter on X.

Proposition 1.2.

Let f be a mapping from a set X into a set Y and $\mathcal{B}$ a filterbase on X. Then $f(\mathcal{B}) = \{f(B) | B \in \mathcal{B}\}$ is a filterbase on Y.

It is easy to show that $f(\mathcal{B})$ satisfies the conditions of Definition 1.2.

Definition 1.7.

Let f be a mapping from a set X into a topological space Y and $\mathcal{B}$ a filterbase on X. A point $y \in Y$ is defined to be a limit point (respectively cluster point) of f along $\mathcal{B}$, written $\lim_{\mathcal{B}} f = y$ (respectively $y \in Cl_{\mathcal{B}} f$), iff y is a limit point (respectively cluster point) of the filterbase $f(\mathcal{B})$.

Remark 1.4.

It follows (c.f. Remark 1.2) that $\lim_{\mathcal{B}} f = y$ iff for every $V \in N(y)$, there exists $B \in \mathcal{B}$ such that $V \supset f(B)$. By Remark 1.3, $Cl_{\mathcal{B}} f = \cap \{\overline{f(B)} | B \in \mathcal{B}\}$.

Remark 1.5.

If f is a mapping from a set X into a Hausdorff topological space Y and if $\mathcal{B}$ is a filterbase on X, then $\lim_{\mathcal{B}} f$ is unique whenever it exists.

Examples 1.4.

A sequence $Z = \{z_n\}_{n \in \mathbb{N}}$ of points in a topological space X is a mapping $Z: \mathbb{N} \to X$ defined by

$$Z(n) = z_n \qquad \text{for } n \in \mathbb{N}.$$

A point $z_0 \in X$ is defined to be a limit point of the sequence $Z = \{z_n\}_{n \in \mathbb{N}}$ as n tends to infinity, written $\lim_{n \to \infty} z_n = z_0$, iff z_0 is a limit point of the mapping Z along the Fréchet filter. It follows (c.f. Remark 1.4) that $\lim_{n\to\infty} z_n = z_0$ iff for every $V \in N(z_0)$, there exists $n_0 \in \mathbb{N}$ such that $z_n \in V$ for all integers $n \geq n_0$.

Let $(D, \prec)$ be a directed set and f a mapping from D into a topological space Y. A point $y \in Y$ is defined to be a <u>limit point of f along $(D, \prec)$</u>, written $\lim_D f = y$, iff y is a limit point of f along the section filterbase on D. By Examples 1.3 and Remark 1.4, $\lim_D f = y$ iff for every $V \in N(y)$, there exists $k \in D$ such that $f(\ell) \in V$ for every $\ell \in S(k)$, where $S(k)$ is the section of D relative to k.

<u>Definition 1.8.</u>

Let f be a mapping from a topological space X into a topological space Y. A point $y \in Y$ is defined to be a <u>limit point of f at $a \in X$</u>, written $\lim_{x\to a} f(x) = y$, iff y is a limit point of f along the filterbase $N'(a)$ of deleted neighbourhoods of a.

<u>Remark 1.6.</u>

It follows (c.f. Remark 1.4) that $\lim_{x\to a} f(x) = y$ iff for every $V \in N(y)$, there exists $W \in N'(a)$ such that $V \supset f(W)$.

<u>Definition 1.9.</u>

Let f be a mapping from a topological space X into a topological space Y, $A \subset X$, and $a \in \overline{A}$. A point $y \in Y$ is defined to be a <u>limit point of f at a relative to A</u>, written $\lim_{\substack{x\to a \\ x\in A}} f(x) = y$, iff y is a limit point of f along the trace $N'_A(a)$ of $N'(a)$ on A.

<u>Theorem 1.1.</u> (Theorem of Monotone Convergence).

Let $(D, \prec)$ be a directed set. Every monotonic extended real valued function f defined on D has a limit along $(D, \prec)$.

It follows (c.f. Examples 1.4) that

$$\lim_D f = \inf_{x \in D} f(x) \qquad \text{if } f \text{ is nonincreasing}$$

and

$$\lim_D f = \sup f(x) \qquad \text{if } f \text{ is nondecreasing.}$$

<u>Example 1.5.</u>

Let f be an extended real valued function defined on a set X and $\mathcal{B}$ a filterbase on X. Then $(\mathcal{B}, \supset)$ is a directed set and the function $F: \mathcal{B} \to \overline{\mathbb{R}}$ defined by

$$F(B) = \inf_{x \in B} f(x) \qquad \text{for } B \in \mathcal{B} \tag{1.1}$$

is nondecreasing (i.e. $F(B) \le F(C)$ if $B \supset C$). By Theorem 1.1, the function F has a limit along $(\mathcal{B}, \supset)$.

<u>Definition 1.10.</u>

Let f be an extended real valued function defined on a set X, $\mathcal{B}$ a filterbase on X, and F the extended real function defined on $\mathcal{B}$ which is given by (1.1). The limit of F along the directed set $(\mathcal{B}, \supset)$ is called the <u>lower limit of f along the filterbase $\mathcal{B}$</u>, and is denoted by $\liminf_{\mathcal{B}} f$.

<u>Remark 1.7.</u>

By Theorem 1.1,

$$\liminf_{\mathcal{B}} f = \sup_{B \in \mathcal{B}} (\inf_{x \in B} f(x)). \tag{1.2}$$

The upper limit of f along $\mathcal{B}$ is defined similarly, and is denoted by $\limsup_{\mathcal{B}} f$. It follows that

$$\limsup_{\mathcal{B}} f = \inf_{B \in \mathcal{B}} (\sup_{x \in B} f(x)) \tag{1.3}$$

Moreover, we have the result that

$$\inf_{x\in X} f(x) \le \liminf_{\mathcal{B}} f \le \limsup_{\mathcal{B}} f \le \sup_{x\in X} f(X) \qquad (1.4)$$

and

$$\limsup_{\mathcal{B}} f = -\liminf_{\mathcal{B}}(-f). \qquad (1.5)$$

Hence we need only consider the lower limit and observe that analogous results hold for the upper limit.

By Examples 1.4, $\liminf_{\mathcal{B}} f = y$ iff for every $V \in N(y)$, there exists $C \in \mathcal{B}$ such that $F(B) = \inf_{x\in B} f(x) \in \dot{V}$ for every $B \in \mathcal{B}$ with $C \supset B$.

<u>Proposition 1.3.</u>

Let f be an extended real valued function defined on a set X and $\mathcal{B}$ a filterbase on X. Then $\liminf_{\mathcal{B}} f \le y$ iff for every $\delta > 0$ and each $B \in \mathcal{B}$, $f(B) \cap [-\infty, y + \delta) \ne \emptyset$.

<u>Proof.</u>

The condition is necessary by (1.2). Suppose that $f(B) \cap [-\infty, y + \delta) \ne \emptyset$ for every $\delta > 0$ and each $B \in \mathcal{B}$, and let $a = \liminf_{\mathcal{B}} f$. For every $\delta > 0$, there exists $C \in \mathcal{B}$ such that $\inf_{x\in B} f(x) \in (a-\delta, +\infty]$ for every $B \in \mathcal{B}$ with $C \supset B$, and therefore, $a - \delta \le y + \delta$ and $\liminf_{\mathcal{B}} f = a \le y$.

The next proposition follows immediately from (1.2).

<u>Proposition 1.4.</u>

Let f be an extended real valued function defined on a set X and $\mathcal{B}$ a filterbase on X. Then $\liminf_{\mathcal{B}} f \ge y$ iff for every $\delta > 0$, there exists $B \in \mathcal{B}$ such that $f(B) \subset (y-\delta, +\infty]$.

The proof of the following theorem will be omitted, but can be found in Bourbaki ([3], Part 1, p. 354).

Theorem 1.2.

Let f be an extended real valued function defined on a set X and $\mathcal{B}$ a filterbase on X. Then $\liminf_{\mathcal{B}} f = \inf \mathrm{Cl}_{\mathcal{B}} f$.

For completeness we mention the following results, but shall omit the proofs.

Proposition 1.5.

Let f and g be extended real valued functions defined on a set X and $\mathcal{B}$ a filterbase on X. Then the following properties hold:

(a) $f \leq g$ implies that $\liminf_{\mathcal{B}} f \leq \liminf_{\mathcal{B}} g$;

(b) $\liminf_{\mathcal{B}} (f+g) \geq \liminf_{\mathcal{B}} f + \liminf_{\mathcal{B}} g$, whenever both sides of this inequality are defined;

(c) $\liminf_{\mathcal{B}} (\lambda f) = \lambda \liminf_{\mathcal{B}} f$ for every real number $\lambda \geq 0$.

Definition 1.11.

Let f be an extended real valued function defined on a topological space X. The lower limit (respectively upper limit) of f at $a \in X$, written $\liminf_{x \to a} f(x)$ (respectively $\limsup_{x \to a} f(x)$), is defined to be the lower limit (respectively upper limit) of f along the filterbase N'(a) of deleted neighbourhoods of a.

Definition 1.12.

An extended real valued function f defined on a topological space X is called lower semicontinuous (respectively upper semicontinuous) at $a \in X$ iff $\liminf_{x \to a} f(x) \geq f(a)$ (respectively $\limsup_{x \to a} f(x) \leq f(a)$).

A function is called <u>lower semicontinuous</u> (respectively <u>upper semicontinuous</u>) <u>on X</u> iff it is lower semicontinuous (respectively upper semicontinuous) at every point of X.

We are concerned primarily with lower semicontinuous functions in this thesis.

An extended real valued function defined on a topological space X is continuous at $a \in X$ iff it is both lower and upper semicontinuous at a.

By (1.5), f is lower semicontinuous at a point iff -f is upper semicontinuous at the same point, and hence we need only consider the properties of lower semicontinuous functions and observe that analogous properties hold for upper semicontinuous functions.

The following theorem is taken from Bourbaki ([3], Part 1, p. 361).

<u>Theorem 1.3.</u>

An extended real valued function f defined on a topological space X is lower semicontinuous on X iff for each $t \in \mathbb{R}$,

$$f^{-1}(t, +\infty] = \{x \in X \mid f(x) > t\}$$

is an open subset of X.

<u>Corollary 1.3.1.</u>

A subset U of a topological space X is open in X iff its characteristic function χ_u is lower semicontinuous on X.

<u>Proof.</u>

This result is an immediate consequence of the fact that $\chi_u{}^{-1}(t, +\infty]$ is empty if $t \geq 1$, is equal to U if $0 \leq t < 1$, and is equal to X if $t < 0$.

Proposition 1.6.

Let f and g be lower semicontinuous extended real valued functions defined on a topological space X. Then the functions $f \wedge g = \inf\{f,g\}$, $f \vee g = \sup\{f,g\}$, λf for every real number $\lambda \geq 0$, and $f+g$ (whenever it is defined) are lower semicontinuous on X.

Proof.

The results for $f + g$ and λf follow from Proposition 1.5. For each $t \in \mathbb{R}$,

$$(f \wedge g)^{-1}(t, +\infty] = f^{-1}(t, +\infty] \cap g^{-1}(t, +\infty]$$

and

$$(f \vee g)^{-1}(t, +\infty] = f^{-1}(t, +\infty] \cup g^{-1}(t, +\infty],$$

and hence the results for $f \wedge g$ and $f \vee g$ follow from Theorem 1.3.

Proposition 1.7.

If $\{f_\alpha\}_{\alpha \in J}$ is any family of lower semicontinuous extended real valued functions defined on a topological space X, then the function $\bigvee_{\alpha \in J} f_\alpha$ is lower semicontinuous on X.

Proof.

The proposition is an immediate consequence of Theorem 1.3 and the fact that

$$\left(\bigvee_{\alpha \in J} f_\alpha\right)^{-1}(t, +\infty] = \bigcup_{\alpha \in J} f_\alpha^{-1}(t, +\infty] \text{ for each } t \in \mathbb{R}.$$

The converse of Proposition 1.7 is true if X is uniformizable. In fact, every lower semicontinuous function defined on a uniformizable space is the upper envelope of a family of continuous functions (c.f. Bourbaki ([3], Part 2, p. 146)).

A particular case of Proposition 1.7 is given by the following result.

Proposition 1.8.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a nondecreasing sequence of lower semicontinuous extended real valued functions defined on a topological space X and $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in X$. Then f is lower semicontinuous on X.

The converse of Proposition 1.8 is true if X is metrizable. In particular, the following proposition holds. We refer to Bourbaki ([3], Part 2, p. 155) for the proof.

Proposition 1.9.

Let X be a metrizable space and f a lower semicontinuous function defined on X with $f(X) \subset [a,b] \subset \overline{\mathbb{R}}$. Then there exists a nondecreasing sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous functions on X such that $f_n(X) \subset [a,b]$ for every $n \in \mathbb{N}$ and $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$.

Theorem 1.4.

Let f be a lower semicontinuous extended real valued function defined on a nonempty compact topological space X. Then there exists a point $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$.

Proof.

Let $A_t = f^{-1}[-\infty, t]$ for every $t \in f(X)$. Then $\mathcal{A} = \{A_t | t \in f(X)\}$ is a filterbase of compact subsets of X, and hence there exists a point

$x_0 \in \bigcap_{t \in f(X)} A_t$ by the finite intersection property. Therefore, $f(x_0) = \inf_{x \in X} f(x)$.

§2. Some Results from Measure Theory

We assume that the reader is familiar with basic measure theory as presented in Halmos [22] and Royden [36]. In the first few paragraphs of this section we establish our notation.

A pair $(E,\mathcal{E})$ consisting of a set E and a σ-algebra $\mathcal{E}$ on E is called a measurable space. A function f from a measurable space $(E,\mathcal{E})$ into a measurable space $(G,\mathcal{G})$ will be called measurable relative to $\mathcal{E}$ and $\mathcal{G}$ iff for every $B \in \mathcal{G}$, $f^{-1}(B) \in \mathcal{E}$.

Let Ω be a set, $(E,\mathcal{E})$ a measurable space, and $\{f_i\}_{i \in I}$ a family of maps from Ω to E. We denote by $\sigma(f_i; i \in I)$ the σ-algebra on Ω generated by $\{f^{-1}(B) \mid B \in \mathcal{E} \text{ and } i \in I\}$.

By a signed measure on a measurable space $(E,\mathcal{E})$ we will mean an extended real valued, σ-additive set function defined on $\mathcal{E}$ which takes on only one of the infinite values: $+\infty$ or $-\infty$. A measure on $(E,\mathcal{E})$ is a nonnegative signed measure. A measure space is a triple $(E,\mathcal{E},\mu)$ where $(E,\mathcal{E})$ is a measurable space and μ is a measure on $(E,\mathcal{E})$. A probability space is a measure space $(\Omega,\mathcal{G},P)$ with $P(\Omega) = 1$.

Let $(\Omega,\mathcal{G},P)$ be a probability space and let $A, B \in \mathcal{G}$ with $P(B) > 0$. The conditional probability of A given B, denoted by $P(A|B)$, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Let $\mathcal{M}$ be a σ-subalgebra of $\mathcal{G}$. The conditional probability of A given $\mathcal{M}$, denoted by $P(A|\mathcal{M})$, is defined to be any extended real valued function f defined on Ω which satisfies:

(a) f is measurable relative to $\mathcal{M}$ and the Borel subsets of $\overline{R}$;

(b) $\int_M f\,dP = P(A \cap M)$ for every $M \in \mathcal{M}$

(c.f. Loève ([31]; p. 341)). When $\mathcal{M}$ is of the form $\sigma(g)$ we write $P(A|g)$ instead of $P(A|\mathcal{M})$. If $\mathcal{M} = \mathcal{G}$, then $P(A|\mathcal{G}) = \chi_A$. It is also true that $P(A|\mathcal{M})$ is the conditional expectation of χ_A given $\mathcal{M}$.

Definition 1.13.

Let $(\Omega, \mathcal{G}, P)$ be a probability space, $(E, \mathcal{E})$ a measurable space, and T an arbitrary index set. A stochastic process with values in $(E, \mathcal{E})$ and parameter set T over $(\Omega, \mathcal{G}, P)$ is defined to be a collection $X = \{X_t\}_{t \in T}$ of maps X_t from Ω to E each of which is measurable relative to $\mathcal{G}$ and $\mathcal{E}$.

Definition 1.14.

Let $(\Omega, \mathcal{G}, P)$ be a probability space, $(E, \mathcal{E})$ a measurable space and T a subset of $\overline{R}$. A Markov process is defined to be a stochastic process $X = \{X_t\}_{t \in T}$ with values in $(E, \mathcal{E})$ over $(\Omega, \mathcal{G}, P)$ such that for each $t \in T$,

$$P(A \cap B \mid X_t) = P(A \mid X_t) \cdot P(B \mid X_t) \tag{1.6}$$

for every $A \in \sigma(X_s;\ s \le t)$ and $B \in \sigma(X_s;\ s \ge t)$.

We refer the reader to Blumenthal and Getoor ([2], Chapter 1) for a detailed exposition of Markov processes.

Condition (1.6) means that the σ-algebras $\sigma(X_s; s \leq t)$ and $\sigma(X_s; s \geq t)$ are conditionally independent given $\sigma(X_t)$. Hence the intuitive meaning of this condition is understood to be the assertion that the past, $\sigma(X_s, s \leq t)$, and the future, $\sigma(X_s, s \geq t)$, events are conditionally independent given the present, $\sigma(X_t)$, ones.

Let $(E, \mathcal{E}, \mu)$ be a measure space. A property holds <u>μ-almost everywhere</u> iff the set of points where it fails to hold has μ-measure zero. In particular, by almost everywhere we shall mean almost everywhere with respect to Lebesgue measure.

Let $(E, \mathcal{E}, \mu)$ be a measure space. The <u>restriction of the measure μ to a set $A \in \mathcal{E}$</u> is the measure μ_A, defined by

$$\mu_A(B) = \mu(A \cap B) \qquad \text{for } B \in \mathcal{E} .$$

The following theorem enables us to write any signed measure as the difference between two measures. We refer to Halmos ([22], p. 121) or Royden ([36], p. 235) for the proof.

<u>Theorem 1.5.</u> (Hahn Decomposition Theorem).

Let ν be a signed measure on a measurable space $(E, \mathcal{E})$. Then there exists a decomposition of E into two disjoint, ν-measurable sets A and B such that

$$\nu(F) \geq 0 \qquad \text{for every } \nu\text{-measurable set } F \subset A$$

and

$$\nu(G) \leq 0 \qquad \text{for every } \nu\text{-measurable set } G \subset B.$$

The pair of sets $\{A,B\}$ is called a Hahn decomposition of E with respect to the signed measure ν. Although a Hahn decomposition is not unique, the measures ν^+ and ν^-, defined by

$$\nu^+(F) = \nu(F \cap A)$$

and

$$\nu^-(F) = -\nu(F \cap B)$$

for ν-measurable sets $F \subset E$, are uniquely determined by ν. The measures ν^+ and ν^- are called the positive and negative variation of ν respectively. It is clear that $\nu = \nu^+ - \nu^-$. The measure $|\nu| = \nu^+ + \nu^-$ is called the total variation of ν.

Let X be a topological space. We denote by $C(X)$ the set of all continuous real valued functions defined on X, by $C_c(X)$ the set of all functions in $C(X)$ with compact support, and by $C_c^+(X)$ the set of all nonnegative functions in $C_c(X)$.

Definition 1.15.

Let X be a nonempty locally compact Hausdorff space. A real valued function I defined on $C_c(X)$ is called a nonnegative linear functional iff for each $f, g \in C_c(X)$ and every $\alpha \in \mathbb{R}$, the following three conditions are satisfied:

(a) $I(f + g) = I(f) + I(g)$;

(b) $I(\alpha f) = \alpha I(f)$;

(c) $I(f) \geq 0$ if $f \in C_c^{+}(X)$.

A nonnegative linear functional on $C_c(X)$ is called a Radon measure on X.

If I is a Radon measure on X, then it follows (c.f. Royden ([36], p. 304)) that there exists a Baire measure μ on X such that

$$I(f) = \int_X f(x)\, d\mu(x) \qquad \text{for each } f \in C_c(X).$$

Hence we refer to a Radon measure on X simply as a mass distribution on X. We denote by $\mathcal{M}_X^+$ the set of all mass distributions on X. If $X = \mathbb{R}^p$, then we write $\mathcal{M}_X^+ = \mathcal{M}^+$.

Let μ be a mass distribution on $\mathbb{R}^p$ and $G(\mu) = \sup \{G \subset \mathbb{R}^p \mid G \text{ open and } \mu(G) = 0\}$. The support of μ, denoted by $S(\mu)$, is the closed set

$$S(\mu) = \mathbb{R}^p \setminus G(\mu).$$

Definition 1.16.

Let X be a nonempty σ-compact Hausdorff space. A real valued function I defined on $C_c(X)$ is called a bounded linear functional iff for each f, $g \in C_c(X)$ and every $\alpha \in \mathbb{R}$, the following three conditions are satisfied:

(a) $I(f + g) = I(f) + I(g)$;

(b) $I(\alpha f) = \alpha I(f)$;

(c) There exists a real number $M > 0$ such that $|I(f)| \leq M\| f \|$, where M depends on the support of f and $\| f \| = \sup_{x \in X} |f(x)|$.

A bounded linear functional on $C_c(X)$ is called a signed Radon measure on X.

If I is a signed Radon measure on X, then it follows (c.f. Landkov ([30], p. 4)) that there exists a unique signed Baire measure ν on X such that

$$I(f) = \int_X f(x)\, d\nu(x) \qquad \text{for each } f \in C_c(X).$$

Hence we refer to a signed Radon measure on X simply as a charge distribution on X. We denote by $\mathcal{M}_X$ the set of all charge distributions on X. If $X = \mathbb{R}^p$, then we write $\mathcal{M}_X = \mathcal{M}$.

The support of a charge distribution ν on X is the closed set $S(\nu) = S(\nu^+) \cup S(\nu^-)$.

For a fixed charge distribution ν on X, we define the function $\nu : C_c(x) \to \mathbb{R}$ by

$$\nu(f) = \int_X f(x)\, d\nu(x) \qquad \text{for } f \in C_c(X). \tag{1.7}$$

Then ν becomes a linear functional on $C_c(X)$. Furthermore, suppose that we define convergence in $C_c(X)$ as follows: the sequence $\{f_n\}_{n \in \mathbb{N}}$ in $C_c(X)$ converges to $f \in C_c'(X)$ iff the supports of the functions f_n are contained in a fixed compact set K and f_n converges uniformly to f on X. It follows that

$$|\nu(f_n) - \nu(f)| \le \int_K |f_n(x) - f(x)|\, d|\nu|(x)$$

$$\le |\nu|(K) \max_{x \in K} |f_n(x) - f(x)| ,$$

and hence the linear functional ν defined by (1.7) is continuous on $C_c(X)$. Therefore, there exists a one-to-one correspondence between $\mathcal{M}_X$ and the topological dual space of $C_c(X)$.

We endow $\mathcal{M}_X$, considered as the topological dual space of $C_c(X)$, with the weak * topology or vague topology. Then the sequence $\{\nu_n\}_{n \in \mathbb{N}}$ of charge distributions on X <u>converges</u> to the charge distribution ν on X <u>in the vague topology</u>, denoted by $\nu_n \dot{\to} \nu$, iff

$$\lim_{n\to\infty} \nu_n(f) = \nu(f) \qquad \text{for every } f \in C_c(X).$$

<u>Proposition 1.10.</u>

Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of mass distributions on X such that $\mu_n \dot{\to} \mu$. Then $\mu \in \mathcal{M}_X^+$.

<u>Proof.</u>

For each $f \in C_c^+(X)$, $\mu_n(f) \ge 0$ and it follows that $\mu(f) \ge 0$. Furthermore, μ is linear so that $\mu \in \mathcal{M}_X^+$.

Proposition 1.11.

Let f be a lower semicontinuous, nonnegative extended real valued function defined on $\mathbb{R}^p (p \geq 2)$ and $\mu_n \dot{\to} \mu$ ($\mu_n \in \mathcal{M}^+$). Then

$$\liminf_{n\to\infty} \mu_n(f) \geq \mu(f) .$$

We refer to Landkov ([30], p. 8) for the proof of Proposition 1.11.

Definition 1.12.

A set of charge distributions $\mathcal{H} \subset \mathcal{M}_X$ is called vaguely bounded iff for any $f \in C_c(X)$, there exists a constant $C(f) > 0$ which is an upper bound for $\{|\mu(f)| \,|\, \mu \in \mathcal{H}\}$.

Theorem 1.6.

Every vaguely bounded set of charge distributions on a σ-compact Hausdorff space X is relatively compact in the vague topology.

We refer to Bourbaki ([4], p. 62) for the proof of Theorem 1.6. In the case where X is compact, Theorem 1.6 follows directly from the principle of uniform boundedness together with the Banach-Alaoglu theorem.

Let G be a σ-compact Hausdorff abelian topological group, where addition is the group operation. We will define a convolution operation on $\mathcal{M}_G$ (c.f. Cartan [6] and Landkov [30]).

We will consider $\mathcal{M}_G^+$ first. Let μ and ρ be mass distributions on G and $f \in C_c^+(G)$. The functions

$$g(x) = \int_G f(x+y)\, d\mu(y) \qquad \text{for } x \in G \tag{1.8}$$

and

$$h(y) = \int_G f(x+y)\, d\rho(x) \qquad \text{for } y \in G$$

are continuous, nonnegative real valued functions defined on G.

Hence

$$\rho(g) = \int_G d\rho(x) \int_G f(x+y)\, d\mu(y)$$

$$= \int_G d\mu(y) \int_G f(x+y)\, d\rho(x) \tag{1.9}$$

$$= \mu(h)$$

$$\leq + \infty$$

by Fubini's theorem.

If $\rho(g) = \mu(h) < +\infty$ for every $f \in C_c^+(G)$, then formula (1.9) defines a positive linear functional on $C_c(G)$. Hence there exists a mass distribution λ on G such that

$$\rho(g) = \int_G f(x)\, d\lambda(x) \qquad \text{for } f \in C_c(G).$$

The mass distribution λ is called the convolution of the mass distributions μ and ρ on G and is denoted by $\mu * \rho$. It is clear that $\mu * \rho = \rho * \mu$.

Proposition 1.12.

If μ is a mass distribution on G with compact support and if ρ is any mass distribution on G, then the convolution mass distribution $\mu * \rho$ exists and is a continuous function of the mass distribution ρ in the vague topology.

Proof.

This result is an immediate consequence of the fact that the function g defined by (1.8) belongs to $C_c^+(G)$ for every $f \in C_c^+(G)$.

Proposition 1.13.

Let $\mu, \rho, \lambda \in \mathcal{M}_G^+$. Then

$$(\mu * \rho) * \lambda = \mu * (\rho * \lambda) \tag{1.10}$$

provided that one side of (1.10) is defined.

Proof.

For each $f \in C_c^+(G)$,

$$\begin{aligned}((\mu * \rho) * \lambda)(f) &= \int_G d\lambda(x) \int_G f(x+y)\, d(\mu * \rho)(y) \\ &= \int_G d\lambda(x) \int_G d\rho(y) \int_G f(x+y+z)\, d\mu(z) \\ &= \int_G d(\rho * \lambda)(y) \int_G f(y+z)\, d\mu(z) \\ &= (\mu * (\rho * \lambda))(f)\end{aligned}$$

by Fubini's theorem, and hence the proposition holds.

We can extend the convolution operation to $\mathcal{M}_G$ in the following way: for $\nu, \zeta \in \mathcal{M}_G$, we define

$$\nu * \zeta = \nu^+ * \zeta^+ - \nu^+ * \zeta^- - \nu^- * \zeta^+ + \nu^- * \zeta^- \qquad (1.11)$$

provided that all convolutions of mass distributions on the right hand side of (1.11) are defined

It is clear that Proposition 1.12 remains valid for charge distributions on G. Proposition 1.13 generalizes in the following way.

<u>Proposition 1.14.</u>

Let $\nu, \zeta, \eta \in \mathcal{M}_G$. Then

$$(\nu * \zeta) * \eta = \nu * (\zeta * \eta)$$

provided that each of the eight possible convolution mass distributions

$$\nu^{\pm} * \zeta^{\pm} * \eta^{\pm}$$

is defined.

We denote by m_p the Lebesgue measure on $\mathbb{R}^p (p \geq 2)$ and by dx the Lebesgue measure on $\mathbb{R}$. Let Φ be a locally Lebesgue integrable function defined on $\mathbb{R}^p (p \geq 1)$. Then there exists a charge distribution ν_Φ on $\mathbb{R}^p$ defined by

$$\nu_\Phi(f) = \int_{\mathbb{R}^p} f(x)\Phi(x)dm_p(x) \qquad \text{for } f \in C_c(\mathbb{R}^p). \qquad (1.12)$$

The function Φ is called the <u>density of the charge distribution ν_Φ with respect to Lebesgue measure</u>. A charge distribution on $\mathbb{R}^p$ having a density function with respect to Lebesgue measure is called <u>absolutely continuous with respect to Lebesgue measure</u>. If ζ is a charge distribution on $\mathbb{R}^p$ such that $\zeta * \nu_\Phi$ is defined, then

$$(\zeta * \nu_\Phi)(f) = \int_{\mathbb{R}^p} \Phi(x)dm_p(x)\int_{\mathbb{R}^p} f(x+y)d\zeta(y)$$

$$= \int_{\mathbb{R}^p} f(z) \left(\int_{\mathbb{R}^p} \Phi(z-y)d\zeta(y)\right)dm_p(z)$$

for each $f \in C_c(\mathbb{R}^p)$. It follows that the convolution charge distribution $\zeta * \nu_\Phi$ is absolutely continuous with respect to Lebesgue measure and its density function is given by

$$(\zeta * \Phi)(x) = \int_{\mathbb{R}^p} \Phi(x-y)d\zeta(y) \qquad \text{for } x \in \mathbb{R}^p.$$

<u>Examples 1.6.</u>

We consider certain examples of mass distributions on $\mathbb{R}^p (p \geq 1)$. We denote by σ_{p-1} the (p-1) - dimensional surface measure on $\mathbb{R}^p$. For any real number $r > 0$ and each $x \in \mathbb{R}^p$, we denote by $S(x;r)$ the sphere with radius r and center x and by $B(x;r)$ the open ball with radius r and center x. By ω_p we shall mean the surface measure of the unit sphere in $\mathbb{R}^p$.

We denote by ε the Dirac measure whose support is $\{0\}$; that is, the unit mass distribution concentrated at the point 0. For each $f \in C_c(\mathbb{R}^p)$,

$$\varepsilon(f) = f(0), \tag{1.13}$$

and for each $\nu \in \mathcal{M}$,

$$\nu * \varepsilon = \varepsilon * \nu = \nu .$$

We denote by $\varepsilon^{(r)}$ the mass distribution obtained by uniformly distributing a unit mass over the surface of the sphere $S(0;r)$. It is possible to show that

$$r^{p-1}\,\omega_p = \sigma_{p-1}\,(S(0;r)),$$

and hence for $f \in C_c(\mathbb{R}^p)$,

$$\varepsilon^{(r)}(f) = \frac{1}{r^{p-1}\omega_p} \int_{S(0;r)} f(y)\, d\sigma_{p-1}(y). \tag{1.14}$$

It is clear that $\varepsilon^{(r)} \overset{\bullet}{\to} \varepsilon$ as r tends to zero.

We denote by $m^{(r)}$ the mass distribution obtained by uniformly distributing a unit mass over the open ball $B(0;r)$. We note that

$$m_p(B(0;r)) = \frac{r}{p}\,\sigma_{p-1}(S(0;r)) = \frac{r^p}{p}\,\omega_p ,$$

and hence for each $f \in C_c(\mathbb{R}^p)$,

$$m^{(r)}(f) = \frac{p}{r^p\omega_p} \int_{B(0;r)} f(y)\, dm_p(y). \tag{1.15}$$

It is also clear that $m^{(r)} \overset{\bullet}{\to} \varepsilon$ as r tends to zero.

For $x \in \mathbb{R}^p$, we define the <u>translation</u> τ_x by

$$\tau_x(y) = x + y \qquad \text{for } y \in \mathbb{R}^p.$$

For any function f on $\mathbb{R}^p$, $\tau_x f$ denotes the right translation of f through x defined by

$$\tau_x f(y) = f(y-x) \qquad \text{for } y \in \mathbb{R}^p.$$

For any mass distribution μ on $\mathbb{R}^p$, μ_x denotes the mass distribution on $\mathbb{R}^p$ defined by

$$\mu_x(f) = \mu \circ \tau_x^{-1}(f) = \mu \circ \tau_{-x}(f) = \int_{S(\mu)} f(y+x)\, d\mu(y)$$

for $f \in C_c(\mathbb{R}^p)$. Then $S(\mu_x) = \tau_x(S(\mu))$.

Therefore, from (1.13), (1.14) and (1.15) we obtain the following formulae

$$\varepsilon_x(f) = f(x),$$

$$\varepsilon_x^{(r)}(f) = \frac{1}{r^{p-1}\,\omega_p} \int_{S(x;r)} f(y)\, d\sigma_{p-1}(y),$$

and

$$m_x^{(r)}(f) = \frac{p}{r^p\,\omega_p} \int_{B(x;r)} f(y)\, dm_p(y)$$

for any $x \in \mathbb{R}^p$ and $f \in C_c(\mathbb{R}^p)$.

Let $\mathcal{D}$ be the set of all infinitely differentiable functions on $\mathbb{R}^p$ with compact support. For any vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ of nonnegative integers α_i, we define the differential operator

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\cdots\partial x_p^{\alpha_p}}$$

whose order is $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_p$. We define convergence in $\mathcal{D}$ as follows: the sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}$ converges to zero in $\mathcal{D}$ iff the supports of the functions f_n are contained in a fixed compact set and the sequence of derivatives $\{D^{\alpha} f_n\}_{n\in\mathbb{N}}$ of order $|\alpha|$ converges uniformly to zero on $\mathbb{R}^p$ for any given vector α.

Any continuous linear functional defined on $\mathcal{D}$ is called a Schwartz distribution on $\mathbb{R}^p$. We denote by $\mathcal{D}^*$ the set of all Schwartz distributions on $\mathbb{R}^p$.

We endow $\mathcal{D}^*$ with the weak * topology. For the sake of abbreviation we shall say that the sequence $\{T_n\}_{n\in\mathbb{N}}$ of Schwartz distributions converges weakly to the Schwartz distribution T iff

$$\lim_{n\to\infty} T_n(f) = T(f) \qquad \text{for every } f \in \mathcal{D}.$$

Under this definition of weak convergence, the subset $\mathcal{M}$ of charge distributions on $\mathbb{R}^p$ is dense in $\mathcal{D}^*$ (c.f. Landkov [30] , p. 22).

§3. Superharmonic Functions

We will begin this section by defining harmonic functions. We refer the reader to the book of Helms [23] for a detailed exposition of the topic.

We denote by Δ the Laplace operator. By a region $D \subset \mathbb{R}^p (p \geq 1)$ we shall mean an open and connected subset of $\mathbb{R}^p$.

Definition 1.18.

A real valued function h defined on the region $D \subset \mathbb{R}^p (p \geq 1)$ is called harmonic on D iff it satisfies the following two conditions:

(a) h has continuous second order partial derivatives on D;

(b) $\Delta h(x) = 0$ for every $x \in D$.

Theorem 1.7.

Let h be a harmonic function on the region $D \subset \mathbb{R}^p$. Then for each $x \in D$ and $r > 0$ such that $\overline{B(x;r)} \subset D$,

$$h(x) = \varepsilon_x^{(r)} (h). \tag{1.16}$$

The proof of Theorem 1.7 can be found in Helms ([23], p. 11). We summarize Theorem 1.7 by saying that harmonic functions are mean valued or satisfy the averaging principle. A partial converse to this result is given by the following theorem. We refer to Helms ([23], p. 25) for the proof.

Theorem 1.8.

Let h be a continuous real valued function defined on the region $D \subsetneq \mathbb{R}^p$. If for each $x \in D$ and $r > 0$ such that $\overline{B(x;r)} \subset D$,

$$h(x) = \varepsilon_x^{(r)}(h) ,$$

then h is harmonic on D.

We will follow the book of Landkov [30] throughout the remainder of this section.

Definition 1.19.

An extended real valued function f defined on $\mathbb{R}^p (p \geq 2)$ is called superharmonic on $\mathbb{R}^p$ iff it satisfies the following three conditions:

(a) $-\infty < f(x) \leq +\infty$ for each $x \in \mathbb{R}^p$ and $f \not\equiv +\infty$;

(b) f is lower semicontinuous on $\mathbb{R}^p$;

(c) For every $x \in \mathbb{R}^p$ and $r > 0$,

$$f(x) \geq \varepsilon_x^{(r)} (f). \tag{1.17}$$

Since

$$m_x^{(r)}(f) = \frac{1}{r} \int_0^r \varepsilon_x^{(\rho)}(f)\, d\rho,$$

it follows that inequality (1.17) is equivalent to the following inequality

$$f(x) \geq m_x^{(r)} (f) . \tag{1.18}$$

Therefore, every superharmonic function on $\mathbb{R}^p$ is locally Lebesgue integrable.

Proposition 1.15.

If f is superharmonic on $\mathbb{R}^p$, then

$$f(x) = \lim_{r\to o} \varepsilon_x^{(r)} (f) = \lim_{r\to o} m_x^{(r)} (f) \tag{1.19}$$

for every $x \in \mathbb{R}^p$.

Proof.

For every $x \in \mathbb{R}^p$, we obtain the following inequalities

$$f(x) \geq \limsup_{r\to o} \varepsilon_x^{(r)} (f)$$

by (1.17) and

$$f(x) \geq \limsup_{r\to o} m_x^{(r)} (f)$$

by (1.18). Since $\varepsilon_\cdot^{(r)} \to \varepsilon$ and $m_\cdot^{(r)} \to \varepsilon$ as r tends to zero and f is lower semicontinuous on $\mathbb{R}^p$, it follows (c.f. Proposition 1.11) that

$$f(x) \leq \liminf_{r\to o} \varepsilon_x^{(r)}(f)$$

and

$$f(x) \leq \liminf_{r\to o} m_x^{(r)}(f)$$

for every $x \in \mathbb{R}^p$. Then equation (1.19) follows from (1.4).

Proposition 1.16.

If f is superharmonic on $\mathbb{R}^p$, then

$$f(x) = \liminf_{y \to x} f(y) \qquad \text{for every } x \in \mathbb{R}^p. \qquad (1.20)$$

Proof.

Since f is lower semicontinuous on $\mathbb{R}^p$, we obtain the result that

$$f(x) \leq \liminf_{y \to x} f(y) \qquad \text{for every } x \in \mathbb{R}^p.$$

Let $x \in \mathbb{R}^p$ and suppose that

$$b = \liminf_{y \to x} f(y) - f(x) > 0.$$

By Proposition 1.4, there exists $r > 0$ such that

$$f(y) \geq f(x) + b/2 \qquad \text{for every } y \in \mathbb{R}^p \text{ with } 0<|y-x|<r,$$

and hence

$$m_x^{(r)}(f) \geq f(x) + b/2 > f(x)$$

which contradicts (1.18). Therefore, equation (1.20) holds.

Let f be a superharmonic function on $\mathbb{R}^p$ and D a bounded region in $\mathbb{R}^p$. We denote by ∂D the boundary of D. Since f is lower semicontinuous on $\mathbb{R}^p$, there exists a point $x_o \in \overline{D}$ such that $f(x_o) = \inf_{x \in \overline{D}} f(x)$ (c.f. Theorem 1.4). Furthermore, we obtain the following minimum principle for superharmonic functions in a bounded region.

Theorem 1.9.

Let f be a superharmonic function on $\mathbb{R}^p$ and D a bounded region in $\mathbb{R}^p$. If there exists a point $x_o \in D$ such that $f(x_o) = \inf_{x \in \overline{D}} f(x)$, then $f(x) = f(x_o)$ for every $x \in \overline{D}$.

Proof.

Let $m = \inf_{x \in \overline{D}} f(x) < +\infty$ and

$$A = \{x \in D \mid f(x) = m\} = \{x \in \mathbb{R}^p \mid f(x) \leq m\} \cap D.$$

Since f is lower semicontinuous on $\mathbb{R}^p$, the set A is closed relative to D (c.f. Theorem 1.3). The set A is nonempty because $x_o \in A$.

We will show that the set A is also open relative to D. Suppose that $a \in A$ and choose $r > 0$ such that $\overline{B(a;r)} \subset D$. By (1.18), we obtain the result that

$$\int_{B(a,r)} (f(x) - f(a))\, dm_p(x) \leq 0 ,$$

and hence

$$\int_{B(a,r)} (f(x) - f(a))\, dm_p(x) = 0$$

since $f(x) \geq f(a)$ for every $x \in D$. Let $g(x) = f(x) - f(a)$ for $x \in D$. Then g is a nonnegative lower semicontinuous function defined on D which satisfies the equation

$$\int_{B(a;r)} g(x)\, dm_p(x) = 0 . \tag{1.21}$$

We will now prove that $g(x) = 0$ for every $x \in B(a;\, r/2)$. Suppose that there exists a point $b \in B(a;r/2)$ such that $g(b) = 2\delta > 0$. Since g is lower semicontinuous on D, there exists $0 < \rho < r/2$ such that $g(x) \geq \delta$ for every $x \in B(b;\rho)$ (c.f. Proposition 1.4). It follows that

$$\int_{B(b;\rho)} g(x)\, dm_p(x) \geq \delta m_p\, (B(b;\rho)) > 0$$

which contradicts the facts that g is nonnegative and satisfies (1.21). Therefore, $B(a;r/2) \subset A$, and hence A is open relative to D.

We can conclude that $A = D$ since D is connected, and therefore, $f(x) = f(x_o)$ for every $x \in D$. The theorem follows immediately because f is lower semicontinuous on $\overline{D}$.

In a completely analogous way we prove the following proposition.

<u>Proposition 1.17.</u> (Principle of Harmonic Minorant).

Let f be a superharmonic function on $\mathbb{R}^p$ and h a function which is harmonic on the bounded region D and continuous on $\overline{D}$. If

$$f(x) \geq h(x) \qquad \text{for every } x \in \partial D,$$

then

$$f(x) \geq h(x) \qquad \text{for every } x \in D.$$

In particular, if there exists a point $x_o \in D$ such that $f(x_o) = h(x_o)$, then $f(x) = h(x)$ for every $x \in \overline{D}$.

If D is a bounded region in $\mathbb{R}^p$ and if f is a continuous real valued function defined on ∂D, then the <u>classical Dirichlet problem</u> consists of finding a function h harmonic on D such that

$$\lim_{\substack{y \to x \\ y \in D}} h(y) = f(x) \qquad \text{for every } x \in \partial D.$$

The classical Dirichlet problem is solvable for an open ball $B(x;r)$ given a continuous boundary function f. The solution is given by the Poisson integral, denoted by

$$H_f(y) = \frac{1}{r\omega_p} \int_{S(x;r)} f(z)\left(\frac{r^2 - |y-x|^2}{|z-x|^p}\right) d\sigma_{p-1}(z) \tag{1.22}$$

for every $y \in B(x;r)$ and Lebesgue integrable function f defined on $S(x;r)$ (c.f. Helms ([23], p. 25)).

Theorem 1.10.

An extended real valued function f defined on $\mathbb{R}^p$ is superharmonic on $\mathbb{R}^p$ iff it satisfies the following three conditions:

(a) $-\infty < f(x) \leq +\infty$ for each $x \in \mathbb{R}^p$ and $f \not\equiv +\infty$;

(b) f is lower semicontinuous on $\mathbb{R}^p$;

(c') for every $x \in \mathbb{R}^p$ and $0 < r < r_o(x)$,

$f(x) \geq \varepsilon_x^{(r)}(f)$,

where $r_o(x)$ is a constant which depends on x.

Proof.

The necessity of the conditions is clear.

Conversely, we first notice that the proof of Theorem 1.9 remains valid under condition (c'), and hence the principle of harmonic minorant is preserved. We will show that the principle of harmonic minorant implies condition (c) of Definition (1.19).

Let x be any point in $\mathbb{R}^p$ and $r > 0$. Since f is lower semi-continuous on $\mathbb{R}^p$, there exists a nondecreasing sequence $\{f_n\}_{n \in \mathbb{N}}$ of continuous real valued functions defined on $S(x;r)$ such that

$$f(y) = \lim_{n\to\infty} f_n(y) \qquad\qquad \text{for each } y \in S(x;r)$$

(c.f. Proposition 1.9). For every $n \in \mathbb{N}$,

$$f_n(y) \le f(y) \qquad\qquad \text{for each } y \in S(x;r),$$

and consequently,

$$H_{f_n}(y) \le f(y) \qquad\qquad \text{for each } y \in B(x;r)$$

by the principle of harmonic minorant. It follows (c.f. Theorem 1.8) that

$$\varepsilon_x^{(r)}(f_n) \le f(x),$$

and hence

$$\varepsilon_x^{(r)}(f) = \lim_{r\to\infty} \varepsilon_x^{(r)}(f_n) \le f(x)$$

by the Lebesgue monotone convergence theorem.

Theorem 1.10 allows us to define in a natural way a superharmonic function on the region $D \subset \mathbb{R}^p$.

<u>Definition 1.20.</u>

An extended real valued function f defined on the region $D \subset \mathbb{R}^p$ is called <u>superharmonic on D</u> iff it satisfies the following three conditions:

(a) $-\infty < f(x) \leq +\infty$ for each $x \in D$ and $f \not\equiv +\infty$;

(b) f is lower semicontinuous on D;

(c) For each $x \in D$ and $r > 0$ such that $\overline{B(x;r)} \subset D$,

$$f(x) \geq \varepsilon_x^{(r)}(f).$$

We mention the following more general form of the harmonic minorant principle. We refer to Landkov ([30], p. 55) for the proof.

Proposition 1.18.

Let f be a superharmonic function on the bounded region $D \subset \mathbb{R}^p$ and h a function which is harmonic on D and continuous on $\overline{D}$. If

$$\liminf_{\substack{y \to x \\ y \in D}} f(y) \geq h(x) \qquad \text{for every } x \in \partial D,$$

then

$$f(x) \geq h(x) \qquad \text{for every } x \in D.$$

In particular, if there exists a point $x_o \in D$ such that $f(x_o) = h(x_o)$, then $f(x) = h(x)$ for every $x \in \overline{D}$.

Proposition 1.18 is called the minimum principle for superharmonic functions if we take h to be a constant function.

We denote by $\mathcal{S}(D)$ the set of all superharmonic functions on the region D. It is possible to show that $\mathcal{S}(D)$ forms a convex cone. Furthermore, the infimum of any two superharmonic functions on D is superharmonic on D and the limit of a monotone increasing sequence of superharmonic functions on D is superharmonic on D or $\equiv +\infty$.

We refer the reader to Landkov ([30], p. 56) for the proofs of the next two results.

Theorem 1.11.

Let f be a nonnegative superharmonic function on $\mathbb{R}^p$ and $\mu \in \mathcal{M}^+$. If

$$f * \mu(x) = \int_{\mathbb{R}^p} f(x-y)\, d\mu(y) \qquad \text{for } x \in \mathbb{R}^p$$

is not identically infinite, then $f * \mu$ is superharmonic on $\mathbb{R}^p$.

Corollary 1.11.1.

Every superharmonic function f on $\mathbb{R}^p$ is the limit of a monotone nondecreasing sequence $\{f_n\}_{n \in \mathbb{N}}$ of continuous superharmonic functions on $\mathbb{R}^p$.

It is sufficient to set

$$f_n(x) = f * m^{(1/n)}(x) = m_x^{(1/n)}(f)$$

for each $n \in \mathbb{N}$ and $x \in \mathbb{R}^p$

Example 1.7.

Any real valued function f defined on the region $D \subset \mathbb{R}^p$ which has continuous second order partial derivatives and satifies the inequality

$$\Delta f(x) \leq 0 \qquad \text{for every } x \in D$$

is superharmonic on D (c.f. Landkov ([30], p. 57)).

An extended real valued function f defined on D is called subharmonic on D iff −f is superharmonic on D. Moreover, a real valued function h is harmonic on D iff h is both subharmonic and superharmonic on D (c.f. Theorem 1.8).

CHAPTER 2

THE M. RIESZ KERNEL

In 1938, Marcel Riesz published a paper [35] which extended the definition of the Riemann-Liouville integral from $\mathbb{R}$ to $\mathbb{R}^p$. This extension is called the M. Riesz potential and the associated kernel is called the M. Riesz kernel. We will study the properties of the M. Riesz kernel and the M. Riesz potential throughout this chapter. We will follow the books of Landkov [30] and the papers of Riesz [35] and Frostman ([19], [20] and [21]).

§1. The Definition and Basic Properties of the M. Riesz Kernel and the M. Riesz Potential

A non-elementary function which will be needed in our work is the gamma function. We give the following definition of this function.

Definition 2.1.

The gamma function is defined to be the improper integral

$$\Gamma(s) = \int_0^\infty x^{s-1}\, e^{-x}\, dx \qquad \text{for } s > 0.$$

It is easy to show that the gamma function satisfies the recurrence formula $\Gamma(s) = (s-1)\,\Gamma(s-1)$ for $s > 1$.

Definition 2.2.

The extended real valued function k_α defined on $\mathbb{R}^p (p \geq 2)$, which is given by

$$k_\alpha(x) = H(p,\alpha)\, |x|^{\alpha-p} \qquad \text{for } x \in \mathbb{R}^p,$$

where $0 < \alpha < p$ and

$$H(p,\alpha) = \frac{\Gamma(\frac{p-\alpha}{2})}{\pi^{p/2} 2^{\alpha} \Gamma(\alpha/2)} ,$$

is called the <u>M. Riesz kernel of order α</u>.

This is the definition given by Riesz ([35], p. 3). We remark that in the definition of the M. Riesz kernel of order α given by Landkov ([30], p. 43), the constant factor $H(p,\alpha)$ is replaced by $A(p,\alpha) = (2\pi)^{\alpha} H(p,\alpha)$.

It is clear that k_α is nonnegative, and continuous on $\mathbb{R}^p \setminus \{0\}$. If we let

$$k_\alpha^{(1/n)}(x) = \begin{cases} H(p,\alpha)|x|^{\alpha-p} & \text{for } |x| > 1/n, \\ H(p,\alpha)\, n^{p-\alpha} & \text{for } |x| \le 1/n \end{cases} \tag{2.1}$$

for each $n \in \mathbb{N}$, then k_α is the limit of the monotone increasing sequence $\{k_\alpha^{(1/n)}\}_{n\in\mathbb{N}}$ of continuous truncated kernels, and hence k_α is lower semicontinuous on $\mathbb{R}^p$ (c.f. Proposition 1.8). It follows that k_α is a locally Lebesgue integrable function on $\mathbb{R}^p$, and therefore, it defines an absolutely continuous mass distribution with respect to Lebesgue measure (c.f. (1.12)), which we will also denote by k_α.

The M. Riesz kernel can be extended to negative values of α by the method of analytic continuation on the parameter α, considered as a complex number (c.f. Landkov ([30], pp. 45-48)). Then we regard k_α as a Schwartz distribution and the following theorem holds for real values of α.

Theorem 2.1.

The following rule of composition for M. Riesz kernels

$$k_\alpha * k_\beta = k_{\alpha+\beta} \tag{2.2}$$

holds under the conditions

$$\alpha + \beta < p \text{ and } \alpha, \beta \neq p + 2n \ (n = 0, 1, \dots).$$

In particular,

$$k_0 = \lim_{\alpha\to 0} k_\alpha = \varepsilon,$$

where the limit means the weak convergence of Schwartz distributions, and hence

$$k_\alpha * k_{-\alpha} = \varepsilon \text{ for any } \alpha \neq \pm (p+2n) \ (n = 0, 1, \dots). \tag{2.3}$$

Furthermore,

$$k_{-2} = - \Delta \varepsilon.$$

Since the factor $H(p,\alpha)$ has a simple pole at $\alpha = p + 2n$ ($n=0, 1, \dots$), it follows that k_α does not have a pointwise or vague limit as α tends to p. It becomes necessary to introduce the following kernel.

Definition 2.3.

The extended real valued function k_p defined on $\mathbb{R}^p (p \geq 2)$, which is given by

$$k_p(x) = H(p) \ln \frac{1}{|x|} \qquad \text{for } x \in \mathbb{R}^p,$$

where

$$H(p) = \frac{1}{(2\pi)^{p/2}\Gamma(p/2)},$$

is called the logarithmic kernel.

For any function $f \in C_c(\mathbb{R}^p)$ which satisfies the integral constraint

$$\int_{\mathbb{R}^p} f(x)\, dm_p(x) = 0 ,$$

it follows (c.f. Landkov ([30], p. 50)) that

$$\lim_{\alpha \to p-} \int_{\mathbb{R}^p} f(x)\, k_\alpha(x)\, dm_p(x) = \int_{\mathbb{R}^p} f(x)\, k_p(x)\, dm_p(x).$$

The M. Riesz kernel of order 2 on $\mathbb{R}^p$ $(p \geq 3)$ is called the <u>Newtonian kernel</u>. It satisfies the equation

$$-(\Delta\varepsilon) * k_2 = k_{-2} * k_2 = \varepsilon \qquad (2.5)$$

by (2.3) and (2.4). The logarithmic kernel on $\mathbb{R}^2$ satisfies the equation

$$-(\Delta\varepsilon) * k_2 = \varepsilon. \qquad (2.6)$$

These kernels are called <u>classical</u>.

The Laplacean is a local operator with support $\{0\}$ which acts as an inverse for the classical kernels in the sense of equations (2.5) and (2.6). It is replaced by the non-local operator $k_{-\alpha}$ with support $\mathbb{R}^p$ for the M. Riesz kernel of order α, where $0 < \alpha < 2$ (c.f. (2.3)).

<u>Theorem 2.2.</u>

For $2 < \alpha < p$, the M. Riesz kernel of order α is superharmonic on $\mathbb{R}^p$. It is not harmonic on $\mathbb{R}^p \setminus \{0\}$.

The classical kernels are superharmonic on $\mathbb{R}^p$ and harmonic on $\mathbb{R}^p \setminus \{0\}$.

For $0 < \alpha < 2$, the M. Riesz kernel of order α is subharmonic on $\mathbb{R}^p \setminus \{0\}$

Proof.

The results for the classical kernels follow directly from (2.5) and (2.6).

The other results follow from the fact that

$$\Delta k_\alpha(x) = H(p,\alpha)\,\frac{(p-\alpha)(2-\alpha)}{|x|^{p-\alpha+2}} \qquad \text{for } x \in \mathbb{R}^p\setminus\{0\}.$$

Definition 2.4.

Let ν be a charge distribution on $\mathbb{R}^p (p \geq 2)$ and $0 < \alpha < p$. The extended real valued function $k_\alpha * \nu$ defined on $\mathbb{R}^p$, which is given by

$$k_\alpha * \nu(x) = \int_{S(\nu)} k_\alpha(x-y)d\nu(y) = H(p,\alpha)\int_{S(\nu)} \frac{d\nu(y)}{|x-y|^{p-\alpha}} \quad \text{for } x \in \mathbb{R}^p,$$

is called the M. Riesz potential of order α of the charge distribution ν.

Definition 2.5.

Let ν be a charge distribution on $\mathbb{R}^2$. The extended real valued function $k_2 * \nu$ defined on $\mathbb{R}^2$, which is given by

$$k_2 * \nu(x) = \int_{S(\nu)} k_2(x-y)d\nu(y) = \frac{1}{2\pi}\int_{S(\nu)} \ln\frac{1}{|x-y|}\,d\nu(y) \quad \text{for } x \in \mathbb{R}^2,$$

is called the logarithmic potential of the charge distribution ν.

In the cases when we wish to speak of a potential of any of the above two types, then we will simply write U^ν.

Proposition 2.1.

Let μ be a mass distribution on $\mathbb{R}^p (p \geq 2)$ with compact support. Then the following properties hold:

(a) U^μ is lower semicontinuous on $\mathbb{R}^p$ and for any point $x \in \mathbb{R}^p$,

$$U^\mu(x) = \liminf_{y \to x} U^\mu(y) \ ;$$

(b) For any point $x \in \mathbb{R}^p$,

$$-\infty < U^\mu(x) \leq +\infty,$$

and in all cases except the logarithmic one

$$0 \leq k_\alpha * \mu(x) \leq +\infty.$$

Moreover,

$$0 < k_\alpha * \mu(x) \leq +\infty \ , \text{ if } \mu \not\equiv 0 \ ;$$

(c) The set $\{x \in \mathbb{R}^p \,|\, U^\mu(x) = +\infty\}$ has Lebesgue measure zero.

Proof.

To prove property (a), we let $k_\alpha^{(1/n)}$ be the continuous truncated kernel defined by (2.1) in all cases except the logarithmic one and we set

$$k_2^{(1/n)}(x) = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x|} & \text{for } |x| > 1/n, \\ \frac{1}{2\pi} \ln n & \text{for } |x| \leq 1/n \end{cases}$$

in the logarithmic case. Then for each $x \in \mathbb{R}^p$,

$$U^{\mu}(x) = \int_{S(\mu)} \lim_{n\to\infty} k_{\alpha}^{(1/n)}(x-y)\, d\mu(y)$$

$$= \lim_{n\to\infty} \int_{S(\mu)} k_{\alpha}^{(1/n)}(x-y)\, d\mu(y)$$

by the Lebesgue monotone convergence theorem. The result follows directly from Proposition 1.8.

Property (b) is a trivial result.

To prove property (c), we will show that U^{μ} is locally Lebesgue integrable on $\mathbb{R}^p$. For any $r > 0$,

$$\int_{B(0;r)} U^{\mu}(x)\, dm_p(x) = \int_{B(0;r)} dm_p(x) \int_{S(\mu)} k_{\alpha}(x-y)\, d\mu(y)$$

$$= \int_{S(\mu)} d\mu(y) \int_{B(0;r)} k_{\alpha}(x-y)\, dm_p(x) < +\infty$$

by Fubini's theorem.

In the remainder of this section, we will also consider M. Riesz potentials of order α due to mass distributions or charge distributions ν without compact support. However, we shall assume that both the potentials $k_{\alpha} * \nu^{+}$ and $k_{\alpha} * \nu^{-}$ are finite almost everywhere on $\mathbb{R}^p$ in all cases. Then

$$k_{\alpha} * \nu(x) \doteq k_{\alpha} * \nu^{+}(x) - k_{\alpha} * \nu^{-}(x)$$

is defined almost everywhere on $\mathbb{R}^p$.

We refer to Landkov ([30], p. 62) for the following theorem.

Theorem 2.3. (The Principle of Descent).

If $\mu_n \dot{\to} \mu (\mu_n \in \mathcal{M}^+)$, then

$$k_\alpha * \mu(x) \leq \liminf_{n\to\infty} k_\alpha * \mu_n(x) \qquad \text{for any } x \in \mathbb{R}^p;$$

that is, if k_α is given, then the mapping from μ to $k_\alpha * \mu$ is lower semi-continuous on $\mathcal{M}^+$.

Theorem 2.4.

Let $\{k_\alpha * \mu_n\}_{n\in\mathbb{N}}$ be a nondecreasing sequence of M. Riesz potentials due to mass distributions on $\mathbb{R}^p$. Then either

$$\lim_{n\to\infty} k_\alpha * \mu_n(x) = +\infty \qquad \text{for every } x \in \mathbb{R}^p$$

or

$$\lim_{n\to\infty} k_\alpha * \mu_n(x) = k_\alpha * \mu(x) + A$$

almost everywhere on $\mathbb{R}^p$, where $\mu \in \mathcal{M}^+$ and $A \geq 0$.

For the proof of Theorem 2.4, we refer the reader to Landkov ([30], pp. 63-65).

Theorem 2.5.

Let μ be a mass distribution on $\mathbb{R}^p$. Then the following three properties hold:

(a) For $2 < \alpha < p$, $k_\alpha * \mu$ is superharmonic on $\mathbb{R}^p$;

(b) For $\alpha = 2$, $k_2 * \mu$ is superharmonic on $\mathbb{R}^p$ and harmonic on $\mathbb{R}^p \setminus S(\mu)$;

(c) For $\alpha < 2$, $k_\alpha * \mu$ is subharmonic on $\mathbb{R}^p \setminus S(\mu)$.

These results follow immediately from Theorems 1.11 and 2.2.

Theorem 2.6.

Let μ be a mass distribution on $\mathbb{R}^p (p \geq 2)$ and $0 < \alpha \leq 2$. If

$$k_\alpha * \mu(x) \leq M \qquad \mu\text{ - almost everywhere on } \mathbb{R}^p,$$

then

$$k_\alpha * \mu(x) \leq M \qquad \text{for every } x \in \mathbb{R}^p.$$

We refer to Landkov ([30], pp. 67-72) for the proof of Theorem 2.6.

Theorem 2.7.

Let ν be a charge distribution on $\mathbb{R}^p$. Then

$$k_\alpha * \nu * m^{(r)}(x) = \frac{p}{r^p \omega_p} \int_{B(x;r)} dm_p(z) \int_{S(\nu)} \frac{d\nu(y)}{|z-y|^{p-\alpha}}$$

is a continuous function on $\mathbb{R}^p$. Moreover,

$$k_\alpha * \nu(x) = \lim_{r \to 0} k_\alpha * \nu * m^{(r)}(x) \qquad \text{for every } x \in \mathbb{R}^p.$$

The proof of Theorem 2.7 can be found in Landkov ([30], pp. 72-74).

We complete this section by stating the uniqueness theorem for M. Riesz potentials. We refer the reader to Landkov ([30], pp. 74-76) for the proof.

<u>Theorem 2.8.</u>

If ν is a charge distribution on $\mathbb{R}^p$ such that $k_\alpha * \nu(x) = 0$ almost everywhere on $\mathbb{R}^p$, then $\nu \equiv 0$, and consequently, $k_\alpha * \nu(x) = 0$ for every $x \in \mathbb{R}^p$.

§2. The Definition and Basic Properties of Energy

For $\nu,\rho \in \mathcal{M}$ and $0 < \alpha < p$, we consider the integral

$$I_\alpha(\nu,\rho) = \iint_{R^p} k_\alpha(x-y)\ d\nu(x)\ \ d\rho(y).$$

Then

$$I_\alpha(\nu,\rho) = I_\alpha\ (\nu^+,\rho^+) + I_\alpha\ (\nu^-,\rho^-) - (I_\alpha(\nu^-,\ \rho^+) + I_\alpha(\nu^+,\rho^-))$$

will be well-defined under the conditions

$$I_\alpha(\nu^-,\rho^+) = \iint_{\mathbb{R}^p}\ k_\alpha(x-y)\ d\nu^-(x)\ d\rho^+\ (y) = \int_{\mathbb{R}^p} k_\alpha * \rho^+(x)d\nu^-(x) < +\infty$$

and

$$I_\alpha(\nu^+,\rho^-) = \iint_{\mathbb{R}^p}\ k_\alpha(x-y)\ d\nu^+(x)\ d\rho^-\ (y) = \int_{\mathbb{R}^p} k_\alpha * \rho^-(x)d\nu^+(x) < +\infty$$

by Fubini's theorem. Then

$$I_\alpha(\nu,\rho) = \int_{\mathbb{R}^p} k_\alpha * \nu(x)d\rho(x) = \int_{\mathbb{R}^p} k_\alpha * \rho(x)d\nu(x) = I_\alpha(\rho,\nu)\ .$$

<u>Definition 2.6.</u>

Let $0 < \alpha < p$ and $\nu,\rho \in \mathcal{M}$ such that

$$\int_{\mathbb{R}^p} k_\alpha * \rho^+(x)\, d\nu^-(x) < +\infty \text{ and } \int_{\mathbb{R}^p} k_\alpha * \rho^-(x)\, d\nu^+(x) < +\infty .$$

We define the mutual α-energy of the charge distributions ν, ρ to be the integral

$$I_\alpha(\nu,\rho) = \iint_{\mathbb{R}^p} k_\alpha(x-y)\, d\nu(x)\, d\rho(y) .$$

Definition 2.7.

Let $0 < \alpha < p$ and $\nu \in \mathcal{M}$ such that

$$\int_{\mathbb{R}^p} k_\alpha * \nu^+(x)\, d\nu^-(x) < +\infty \text{ and } \int_{\mathbb{R}^p} k_\alpha * \nu^-(x)\, d\nu^+(x) < +\infty . \quad (2.7)$$

We define the α-energy of the charge distribution ν to be the integral

$$I_\alpha(\nu) = \iint_{\mathbb{R}^p} k_\alpha(x-y) d\nu(x)\, d\nu(y) .$$

Theorem 2.9.

If $\mu_n \dot{\to} \mu$ and $\lambda_n \dot{\to} \lambda$ $(\mu_n, \lambda_n \in \mathcal{M}^+)$, then

$$I_\alpha(\mu,\lambda) \le \liminf_{n\to\infty} I_\alpha(\mu_n,\lambda_n) .$$

Moreover,

$$I_\alpha(\mu) \le \liminf_{n\to\infty} I_\alpha(\mu_n) .$$

The mass distributions $\mu_n \times \lambda_n$ on $\mathbb{R}^p \times \mathbb{R}^p$ converge vaguely to $\mu \times \lambda$, and hence Theorem 2.9 is an immediate consequence of Propositions 1.11 and 2.1.

For the remainder of this section, we shall consider only those charge distributions which satisfy condition (2.7).

Theorem 2.10.

If $\nu \in \mathcal{M}$, then

$$I_\alpha(\nu) \geq 0 .$$

Moreover,

$$I_\alpha(\nu) = 0 \qquad \text{iff } \nu \equiv 0 .$$

Proof.

By Theorem 2.1,

$$k_\alpha = k_{\alpha/2} * k_{\alpha/2} .$$

Suppose that $I_\alpha(\nu) < +\infty$. Then by condition (2.7),

$$I_\alpha(|\nu|) = \iint_{\mathbb{R}^p} k_\alpha(x-y)\, d|\nu|(x)\, d|\nu|(y) < +\infty$$

and therefore, from Fubini's theorem we obtain the result that

$$I_\alpha(\nu) = \iint_{\mathbb{R}^p} k_{\alpha/2} * k_{\alpha/2}\,(x-y)\, d\nu(x)\, d\nu(y)$$

$$= \iint_{\mathbb{R}^p} d\nu(x)\, d\nu(y) \int_{\mathbb{R}^p} k_{\alpha/2}(x-y-z)\, k_{\alpha/2}(z)\, dm_p(z)$$

$$= \iint_{\mathbb{R}^p} d\nu(x)\, d\nu(y) \int_{\mathbb{R}^p} k_{\alpha/2}(x-u) \cdot k_{\alpha/2}(u-y)\, dm_p(u)$$

$$= \int_{\mathbb{R}^p} dm_p(u) \int_{\mathbb{R}^p} k_{\alpha/2}(x-u)\, d\nu(x) \int_{\mathbb{R}^p} k_{\alpha/2}(u-y)\, d\nu(y)$$

$$= \int_{\mathbb{R}^p} [k_{\alpha/2} * \nu(u)]^2 \, dm_p(u) .$$

Hence $I_\alpha(\nu) \geq 0$. In particular, if $I_\alpha(\nu) = 0$, then $k_{\alpha/2} * \nu(x) = 0$ almost everywhere on $\mathbb{R}^p$, and consequently, $\nu \equiv 0$ by Theorem 2.8.

We denote by $\mathcal{E}_\alpha$ the set of all charge distributions on $\mathbb{R}^p$ having finite α-energy and by $\mathcal{E}_\alpha^+$ the set of all mass distributions on $\mathbb{R}^p$ having finite α-energy.

By Theorem 2.10, we can introduce an inner product in $\mathcal{E}_\alpha$ by setting

$$(\nu,\lambda)_\alpha = I_\alpha(\nu,\lambda)$$

for ν, $\lambda \in \mathcal{E}_\alpha$. We define the norm of the charge distribution $\nu \in \mathcal{E}_\alpha$ to be

$$\| \nu \|_\alpha = \sqrt{(\nu,\nu)_\alpha} .$$

We remark that $\mathcal{E}_\alpha$ endowed with the above inner product is a pre-Hilbert space and that $\mathcal{E}_\alpha^+$ is a complete convex cone in $\mathcal{E}_\alpha$ (c.f. Landkov ([30], p. 90)).

§3. Capacity and the Equilibrium Mass Distribution

In this section, we will define α-capacity as a nonnegative set function defined on the Borel subsets of $\mathbb{R}^p (p \geq 2)$, where we assume that $0 < \alpha < p$. We refer the reader to Landkov ([30], pp. 167-177) for logarithmic capacity ($\alpha = p = 2$).

Let K be a compact subset of $\mathbb{R}^p$. We denote by $\mathcal{M}_K^+$ the set of all mass distributions with support contained in K and by $\overset{\circ}{\mathcal{M}}{}_K^+$ the set of all mass distributions in $\mathcal{M}_K^+$ with unit total mass. It follows (c.f. Theorem 1.6) that $\overset{\circ}{\mathcal{M}}{}_K^+$ is vaguely compact.

Following an argument in Landkov ([30], pp. 130-133), we will now show the existence of a unique mass distribution $\lambda_K \in \overset{\circ}{\mathcal{M}}{}_K^+$ such that

$$\| \lambda_K \|_\alpha \leq \| \mu \|_\alpha \qquad \text{for every } \mu \in \overset{\circ}{\mathcal{M}}{}_K^+ .$$

We define

$$W_\alpha(K) = \inf \{ \| \mu \|_\alpha^2 \ \ | \mu \in \overset{\circ}{\mathcal{M}}{}_K^+ \} .$$

If we let d be the diameter of K, then

$$k_\alpha(x-y) \geq \frac{H(p,\alpha)}{d^{p-\alpha}} \qquad \text{for any } x,y \in K,$$

and hence we obtain the result that

$$0 < W_\alpha(K) \leq +\infty .$$

We assume that $W_\alpha(K)$ is finite. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $\overset{\circ}{\mathcal{M}}{}_K^+$ such that

$$\lim_{n\to\infty} \| \mu_n \|_\alpha^2 = W_\alpha(K) .$$

Since $\overset{\circ}{\mathcal{M}}{}_K^+$ is vaguely compact, there exists a subsequence $\{\mu_{n(i)}\}_{i\in\mathbb{N}}$ of $\{\mu_n\}_{n\in\mathbb{N}}$ such that $\mu_{n(i)} \dot{\to} \lambda_K$, where $\lambda_K \in \overset{\circ}{\mathcal{M}}{}_K^+$. Then by Theorem 2.9,

$$\| \lambda_K \|_\alpha^2 \le \lim_{i\to\infty} \| \mu_{n(i)} \|_\alpha^2 = W_\alpha(K) ,$$

and hence

$$\| \lambda_K \|_\alpha^2 = W_\alpha(K), \tag{2.8}$$

or equivalently,

$$\| \lambda_K \|_\alpha \le \| \mu \|_\alpha \qquad \text{for every } \mu \in \overset{\circ}{\mathcal{M}}{}_K^+. \tag{2.9}$$

To prove that the mass distribution λ_K is unique, we suppose that λ' is another mass distribution in $\overset{\circ}{\mathcal{M}}{}_K^+$ which satisfies (2.8). Then

$$\sqrt{W_\alpha(K)} \le \| \tfrac{1}{2} (\lambda_K + \lambda') \|_\alpha$$

$$\le \tfrac{1}{2}(\| \lambda_K \|_\alpha + \| \lambda' \|_\alpha) = \sqrt{W_\alpha(K)} ,$$

and hence

$$W_\alpha(K) = \| \tfrac{1}{2}(\lambda_K + \lambda') \|_\alpha^2 .$$

Therefore, we obtain (c.f. Cartan([6], p. 84)) the identity

$$\| \lambda_K - \lambda' \|_\alpha^2 = 2\| \lambda_K \|_\alpha^2 + 2\| \lambda' \|_\alpha^2 - 4 \|\tfrac{1}{2}(\lambda_K + \lambda')\|_\alpha^2 = 0$$

which implies that $\lambda_K \equiv \lambda'$ (c.f. Theorem 2.10).

We call λ_K the minimizing mass distribution for the compact set K.

Definition 2.8.

The α-capacity of a compact set $K \subset \mathbb{R}^p$ is defined to be the nonnegative real number

$$C_\alpha(K) = \frac{1}{W_\alpha(K)},$$

if $W_\alpha(K) < +\infty$. If $W_\alpha(K) = +\infty$, then $C_\alpha(K)$ is defined to be zero.

Theorem 2.11.

For any compact set $K \subset \mathbb{R}^p$,

$$C_\alpha(K) = 0 \quad \text{iff} \quad \mu(K) = 0 \qquad \text{for every } \mu \in \mathcal{E}_\alpha^+ .$$

Proof.

If $\mu(K) = 0$ for every $\mu \in \mathcal{E}_\alpha^+$, then $\overset{\circ}{\mathcal{M}}{}_K^+ \cap \mathcal{E}_\alpha^+$ is empty, and hence $C_\alpha(K) = 0$. Conversely, assume that $C_\alpha(K) = 0$. For any $\mu \in \mathcal{E}_\alpha^+$,

$$\| \mu_K \|_\alpha^2 \le \| \mu \|_\alpha^2 < +\infty ,$$

where μ_K is the restriction of the mass distribution μ to K, and consequently, $\mu_K \in \mathcal{E}_\alpha^+$. Since $W_\alpha(K) = +\infty$, $\mu_K \equiv 0$ and it follows that $\mu(K) = 0$.

Proposition 2.2.

If K is a compact subset of $\mathbb{R}^p$ with α-capacity zero, then for every $\mu \in \mathcal{M}_K^+ \setminus \{0\}$ there exists a point $x \in S(\mu)$ such that

$k_\alpha * \mu(x) = +\infty$.

Proof.

Otherwise, we would obtain the following result

$$\| \mu \|_\alpha^2 = \int_{S(\mu)} k_\alpha * \mu(x)\, d\mu(x) < +\infty ,$$

and hence $C_\alpha(K) > 0$.

Definition 2.9.

A subset E of $\mathbb{R}^p$ is defined to have inner α-capacity zero iff for any compact set $K \subset E$, $C_\alpha(K) = 0$.

Definition 2.10.

A property is defined to hold α-nearly everywhere iff the set of points where it fails to hold has inner α-capacity zero.

Proposition 2.3.

Let λ_K be the minimizing mass distribution for the compact set $K \subset \mathbb{R}^p$. Then the following properties hold:

(a) $k_\alpha * \lambda_K(x) \geq W_\alpha(K) = \| \lambda_K \|_\alpha^2$ α-nearly everywhere on K;

(b) For every point $x \in S(\lambda_K) \subset K$,

$$k_\alpha * \lambda_K(x) \leq W_\alpha(K) = \| \lambda_K \|_\alpha^2 ;$$

(c) $k_\alpha * \lambda_K(x) = W_\alpha(K)$ α-nearly everywhere on $S(\lambda_K)$

and

$k_\alpha * \lambda_K(x) \leq W_\alpha(K)$ for every $x \in S(\lambda_K)$.

We refer the reader to Landkov ([30], pp. 136 and 137) for the straightforward proof of Proposition 2.3.

Theorem 2.12.

Let λ_K be the minimizing mass distribution for the compact set $K \subset \mathbb{R}^p$. Then for $0 < \alpha \leq 2$ ($0 < \alpha < 2$ if $p = 2$),

$$k_\alpha * \lambda_K(x) = W_\alpha(K) \qquad \alpha\text{-nearly everywhere on } K$$

and

$$k_\alpha * \lambda_K(x) \leq W_\alpha(K) \qquad \text{for every } x \in \mathbb{R}^p.$$

This result is an immediate consequence of Proposition 2.3 and Theorem 2.6.

Definition 2.11.

Let λ_K be the minimizing mass distribution for the compact set $K \subseteq \mathbb{R}^p$. The mass distribution

$$\gamma_K = \frac{\lambda K}{W_\alpha(K)} = C_\alpha(K)\ \lambda_K$$

is called the equilibrium mass distribution for the compact set K.

It is clear that

$$\| \gamma_K \|^2 = \gamma_K(K) = C_\alpha(K). \tag{2.10}$$

From Proposition 2.3 we easily obtain the following proposition.

Proposition 2.4.

Let γ_K be the equilibrium mass distribution for the compact set $K \subset \mathbb{R}^p$. Then the following properties hold:

(a) $k_\alpha * \gamma_K(x) \geq 1$ α-nearly everywhere on K;

(b) For every point $x \in S(\gamma_K) \subset K$,

$$k_\alpha * \gamma_K(x) \leq 1;$$

(c) $k_\alpha * \gamma_K(x) = 1$ α-nearly everywhere on $S(\gamma_K)$

and

$$k_\alpha * \gamma_K(x) \leq 1 \qquad \text{for every } x \in S(\gamma_K).$$

Theorem 2.13.

Let γ_K be the equilibrium mass distribution for the compact set $K \subset \mathbb{R}^p$. Then for $0 < \alpha \leq 2$ $(0 < \alpha < 2$ if $p = 2)$,

$$k_\alpha * \gamma_K(x) = 1 \qquad \alpha\text{-nearly everywhere on } K$$

and

$$k_\alpha * \gamma_K(x) \leq 1 \qquad \text{for every } x \in \mathbb{R}^p.$$

The properties of γ_K are easily obtained from Theorem 2.12.

Theorem 2.14.

If E is a subset of $\mathbb{R}^p$ with inner α-capacity zero and if $\mu \not\equiv 0$ is a mass distribution with $S(\mu) \subset E$, then there is a point $x \in E$ such that

$$k_\alpha * \mu(x) = +\infty .$$

The proof of Theorem 2.14 can be found in Landkov ([30], p. 140).

Corollary 2.14.1.

If $\mu \in \mathcal{M}^+$ and if

$$k_\alpha * \mu(x) \leq +\infty \qquad \text{for every } x \in \mathbb{R}^p,$$

then

$$\mu(K) = 0$$

for every compact set $K \subset \mathbb{R}^p$ with α-capacity zero.

Proposition 2.5.

The α-capacity of a compact set satisfies the following properties:

(a) C_α is a monotone nondecreasing set function, that is,

$$C_\alpha(K_1) \leq C_\alpha(K_2) \qquad \text{if } K_1 \subset K_2;$$

(b) C_α is countably subadditive, that is, if

$$K = \bigcup_{i=1}^{\infty} K_i \text{ is compact, then } C_\alpha(K) \leq \sum_{i=1}^{\infty} C_\alpha(K_i) \ ;$$

(c) C_α is right continuous, that is, for any compact set K and any $\delta > 0$ there exists an open set $U \supset K$ such that

$$C_\alpha(K') - C_\alpha(K) < \delta$$

for any compact set K' with $K \subset K' \subset U$.

Proof.

To prove property (a), we first notice that $\mathcal{M}^+_{K_1} \subset \mathcal{M}^+_{K_2}$.

Then $W_\alpha(K_1) \geq W_\alpha(K_2)$, and hence $C_\alpha(K_1) \leq C_\alpha(K_2)$.

To prove property (b), we let γ be the equilibrium mass distribution for K and γ_i the restriction of γ to K_i. It follows (c.f. (2.9) and (2.10)) that

$$\gamma_i(K_i) \leq \| \gamma_i \|^2_\alpha = \int_{S(\gamma_i)} k_\alpha * \gamma_i(x)\, d\gamma_i(x)$$

$$\leq \gamma_i(K_i) \sup \{ k_\alpha * \gamma_i(x) \mid x \in S(\gamma_i) \}$$

$$\leq C_\alpha(K_i) \sup \{ k_\alpha * \gamma_i(x) \mid x \in S(\gamma_i) \},$$

and hence

$$C_\alpha(K) = \gamma(K) \leq \sum_{i=1}^\infty \gamma_i(K_i)$$

$$\leq \sum_{i=1}^\infty C_\alpha(K_i) \sup \{ k_\alpha * \gamma_i(x) \mid x \in S(\gamma_i) \}$$

$$\leq \sum_{i=1}^\infty C_\alpha(K_i) \sup \{ k_\alpha * \gamma(x) \mid x \in S(\gamma) \}$$

$$= \sum_{i=1}^\infty C_\alpha(K_i)$$

by Proposition 2.4(c).

We refer to Landkov ([30], p. 141) for a detailed proof of property (c).

Definition 2.12.

The inner α-capacity of a subset E of $\mathbb{R}^p$ is defined to be

$$\underline{C}_\alpha(E) = \sup \{ C_\alpha(K) \mid K \subset E \text{ and } K \text{ is compact} \} .$$

Definition 2.13.

The outer α-capacity of a subset E of $\mathbb{R}^p$ is defined to be

$$\overline{C}_\alpha(E) = \inf \{ \underline{C}_\alpha(U) \mid U \supset E \text{ and } U \text{ is open} \} .$$

Proposition 2.6.

The inner and outer α-capacity of a set satisfy the following properties:

(a) Both $\underline{C}_\alpha$ and $\overline{C}_\alpha$ are monotone nondecreasing set functions which are finite on bounded sets;

(b) For any open subset U of $\mathbb{R}^p$,

$$\underline{C}_\alpha(U) = \overline{C}_\alpha(U) ;$$

(c) For any subset E of $\mathbb{R}^p$,

$$\underline{C}_\alpha(E) \le \overline{C}_\alpha(E) ;$$

(d) $\overline{C}_\alpha$ is countably subadditive.

Proof.

Properties (a) and (b) follow easily from Proposition 2.5. We refer to Landkov ([30], p. 144) for the proof of property (d). We will prove property (c). For any $\delta > 0$, there exists a compact set K and an open set U such that $K \subset E \subset U$ and

$$\underline{C}_\alpha(E) - \delta < C_\alpha(K), \qquad \underline{C}_\alpha(U) < \overline{C}_\alpha(E) + \delta .$$

Since $C_\alpha(K) \leq \underline{C}_\alpha(U)$,

$$\underline{C}_\alpha(E) - \delta < \overline{C}_\alpha(E) + \delta$$

which implies that

$$\underline{C}_\alpha(E) \leq \overline{C}_\alpha(E) .$$

Definition 2.14.

A subset E of $\mathbb{R}^p$ is defined to be α-capacitable iff

$$\underline{C}_\alpha(E) = \overline{C}_\alpha(E) .$$

Then we write $C_\alpha(E)$ for the common value of $\underline{C}_\alpha(E)$ and $\overline{C}_\alpha(E)$.

Theorem 2.15.

Every compact set and every open set is α-capacitable.

We have already shown that every open set is α-capacitable in Proposition 2.6(b). The result for compact sets is an immediate consequence of Proposition 2.5(c).

Definition 2.15.

A property is defined to hold α-quasi-everywhere iff the set of points where it fails to hold has outer α-capacity zero. If α = 2, we simply write quasi-everywhere.

We refer to Landkov ([30], p. 145) for the next theorem.

Theorem 2.16.

Let E be a subset of $\mathbb{R}^p$ with inner α-capacity $\underline{C}_\alpha(E) < +\infty$. Then there exists a unique mass distribution γ_E with $S(\gamma_E) \subset \overline{E}$ which satisfies the following conditions:

(a) $\| \gamma_E \|_\alpha^2 = \gamma_E(\overline{E}) = \underline{C}_\alpha(E)$;

(b) $k_\alpha * \gamma_E(x) \geq 1$ α-nearly everywhere on E;

(c) $k_\alpha * \gamma_E(x) \leq 1$ for every $x \in S(\gamma_E)$.

The mass distribution γ_E is called the inner equilibrium mass distribution for E.

Remark 2.1.

It follows (c.f. (b) and (c)) that

$k_\alpha * \gamma_E(x) = 1$ α-nearly everywhere on $S(\gamma_E)$

and

$k_\alpha * \gamma_E(x) \leq 1$ for every $x \in S(\gamma_E)$.

In particular, for $0 < \alpha \leq 2$ $(0 < \alpha < 2$ if $p = 2)$

$k_\alpha * \gamma_E(x) = 1$ α-nearly everywhere on E

and

$k_\alpha * \gamma_E(x) \leq 1$ for every $x \in \mathbb{R}^p$

by Theorem 2.6. Then it follows (c.f. Theorems 2.13 and 2.16) that the inner equilibrium mass distribution and the equilibrium mass distribution for a compact subset K of $\mathbb{R}^p$ with α-capacity $C_\alpha(K) > 0$ are identical.

We take the following theorem from Landkov ([30], p. 149).

Theorem 2.17.

Let E be a subset of $\mathbb{R}^p$ with outer α-capacity $\overline{C}_\alpha(E) < +\infty$. Then there exists a unique mass distribution $\gamma_E{}^*$ with $S(\gamma_E^*) \subset \overline{E}$ which satisfies the following conditions:

(a) $\| \gamma_E^* \|_\alpha^2 = \gamma_E^*(\overline{E}) = \overline{C}_\alpha(E)$;

(b) $k_\alpha * \gamma_E^*(x) \geq 1$ α-quasi-everywhere on E;

(c) $k_\alpha * \gamma_E^*(x) \leq 1$ for every $x \in S(\gamma_E^*)$.

The mass distribution γ_E^* is called the outer equilibrium mass distribution for E.

Remark 2.2.

As in Remark 2.1, it follows (c.f. (b) and (c)) that

$$k_\alpha * \gamma_E^*(x) = 1 \qquad \alpha\text{-quasi-everywhere on } S(\gamma_E^*)$$

and

$$k_\alpha * \gamma_E^*(x) \leq 1 \qquad \text{for every } x \in S(\gamma_E^*) .$$

In particular, for $0 < \alpha \leq 2$ ($0 < \alpha < 2$ if $p = 2$)

$$k_\alpha * \gamma_E^*(x) = 1 \qquad \alpha\text{-quasi-everywhere on } E$$

and

$$k_\alpha * \gamma_E^*(x) \leq 1 \qquad \text{for every } x \in \mathbb{R}^p$$

by Theorem 2.6.

The next theorem characterizes the α-capacitability of a set E in terms of the inner and outer equilibrium mass distributions for E. We refer to Landkov ([30], p. 154) for the proof.

Theorem 2.18.

A subset E of $\mathbb{R}^p$ with outer α-capacity $\overline{C}_\alpha(E) < +\infty$ is α-capacitable iff $\gamma_E = \gamma_E^*$.

Remark 2.3.

For $0 < \alpha \leq 2$ ($0 < \alpha < 2$ if $p = 2$), it follows (c.f. Remark 2.1 and Theorems 2.15 and 2.17) that the outer equilibrium mass distribution and the equilibrium mass distribution for a compact subset K of $\mathbb{R}^p$ with α-capacity $C_\alpha(K) > 0$ are identical. Hence we obtain the following result from Theorems 2.13 and 2.17.

Theorem 2.19.

Let K be a compact subset of $\mathbb{R}^p$ with α-capacity $C_\alpha(K) > 0$, where $0 > \alpha \leq 2$ ($0 < \alpha < 2$ if $p = 2$). Then the equilibrium mass distribution γ_K for K is the unique mass distribution which satisfies the following conditions:

(a) $\|\gamma_K\|_\alpha^2 = \gamma_K(K) = C_\alpha(K)$;

(b) $k_\alpha * \gamma_K(x) = 1$ α-quasi-everywhere on K;

(c) $\quad k_\alpha * \gamma_K(x) \leq 1 \qquad$ for every $x \in \mathbb{R}^p$.

Theorem 2.20.

Every Borel subset of $\mathbb{R}^p$ is α-capacitable.

We refer to Landkov ([30], pp. 156-158) for the proof of Theorem 2.20.

§4. The Balayage Principle for the M. Riesz Kernel

Consider a bounded region D in $\mathbb{R}^p (p \geq 2)$ with compact boundary ∂D and let ν be a charge distribution with $S(\nu) \subset \bar{D}$ such that the potential $k_2 * \nu$ is well-defined. The classical problem of balayaging ν out (or sweeping out) of D consists of defining a charge distribution ν' with $S(\nu') \subset \partial D$ such that

$$k_2 * \nu'(x) = k_2 * \nu(x) \qquad \text{quasi-everywhere on } \mathbb{R}^p \setminus D \quad (2.11)$$

If we can replace quasi-everywhere by everywhere in (2.11), then the strict balayage problem can be solved.

The above problem generalizes as follows. Let K be a compact subset of $\mathbb{R}^p (p \geq 2)$ and ν a charge distribution with $S(\nu) \subset \mathbb{R}^p \setminus K$ such that $k_2 * \nu$ is well-defined. The classical problem of balayaging ν onto K consists of defining a charge distribution ν' with $S(\nu') \subset \partial K$ such that

$$k_2 * \nu'(x) = k_2 * \nu(x) \qquad \text{quasi-everywhere on } K.$$

This problem can also be generalized to the case when K is an arbitrary closed subset of $\mathbb{R}^p$.

We will demonstrate that the M. Riesz kernel of order α on $\mathbb{R}^p (p \geq 2)$ satisfies the principle of balayage onto a closed set for $0 < \alpha \leq 2$ ($0 < \alpha < 2$ if $p = 2$); that is, for any closed set F in $\mathbb{R}^p$ and charge distribution ν on $\mathbb{R}^p$ such that the potential $k_\alpha * \nu$ is well-defined, there exists a charge distribution ν' with $S(\nu') \subset F$ such that

$$k_\alpha * \nu'(x) = k_\alpha * \nu(x) \qquad \alpha\text{-quasi-everywhere on } F.$$

Moreover, if ν is a mass distribution, then

$$k_\alpha * \nu'(x) \leq k_\alpha * \nu(x) \qquad \text{for every } x \in \mathbb{R}^p.$$

The proof, which is based on the Kelvin transformation and Theorem 2.19, is due to Riesz [35].

We will begin by defining the Kelvin transformation. Let x_o be an arbitrary point in $\mathbb{R}^p$. The point transformation which maps each $x \in \mathbb{R}^p \setminus \{x_o\}$ to the point x^* on the ray through x which issues from x_o such that

$$|x - x_o| \cdot |x^* - x_o| = 1$$

is called <u>inversion relative to the sphere $S(x_o;1)$</u>. Then we obtain the result that

$$x^* = x_o + \frac{(x - x_o)}{|x - x_o|^2}$$

and

$$|x^* - y^*| = \frac{|x - y|}{|x-x_o| \cdot |y-x_o|} \tag{2.12}$$

for $x, y \in \mathbb{R}^p \setminus \{x_o\}$. The inversion transformation is a one-to-one bicontinuous mapping of $\mathbb{R}^p \setminus \{x_o\}$ onto itself. Therefore, every closed set in $\mathbb{R}^p \setminus \{x_o\}$ is transformed under inversion to a closed and bounded set in $\mathbb{R}^p \setminus \{x_o\}$. In particular, every compact set in $\mathbb{R}^p \setminus \{x_o\}$ is mapped by inversion onto a compact set in $\mathbb{R}^p \setminus \{x_o\}$. If we denote the one-point compactification of $\mathbb{R}^p$ by $\overline{\mathbb{R}^p} = \mathbb{R}^p \cup \{\infty\}$, then the inversion transformation extends to a one-to-one, bicontinuous mapping of $\overline{\mathbb{R}^p}$ onto itself under which x_o is mapped to ∞ and ∞ is mapped to x_o.

Theorem 2.21.

Let K be a compact subset of $\mathbb{R}^p \setminus \{x_o\}$ and let K^* denote the compact set obtained from K by inversion relative to the sphere $S(x_o;1)$. Then for $0 < \alpha < p$, K^* has α-capacity zero iff K has α-capacity zero.

Proof.

We can establish a one-to-one correspondence between the elements of $\mathcal{M}_K^+$ and $\mathcal{M}_{K^*}^+$ by setting

$$\mu'(B^*) = \mu(B) \qquad \text{for any Borel subsets } B^* \text{ of } K^*$$

for each $\mu \in \mathcal{M}_K^+$, and

$$\mu(B) = \mu'(B^*) \qquad \text{for any Borel subsets } B \text{ of } K$$

for each $\mu' \in \mathcal{M}_{K^*}^+$.

There exists two positive constants m_1 and m_2 such that

$$m_1|x-y| < |x^*-y^*| < m_2|x-y|$$

for any $x, y \in K$ (c.f. (2.12)). Hence we obtain the inequalities

$$m_1^{\alpha-p} \| \mu \|_\alpha^2 = m_1^{\alpha-p} H(p,\alpha) \iint_{S(\mu)} \frac{1}{|x-y|^{p-\alpha}} \, d\mu(x)\, d\mu(y)$$

$$> H(p,\alpha) \iint_{S(\mu)} \frac{1}{|x^*-y^*|^{p-\alpha}} \, d\mu(x)\, d\mu(y)$$

$$= H(p,\alpha) \iint_{S(\mu')} \frac{1}{|x^*-y^*|^{p-\alpha}} \, d\mu'(x^*)\, d\mu'(y^*)$$

$$= \| \mu' \|_\alpha^2$$

and

$$m_2^{\alpha-p} \| \mu \|_\alpha^2 < \| \mu' \|_\alpha^2$$

for $\mu \in \mathcal{M}_K^+$. Therefore,

$$m_1^{\alpha-p} W_\alpha(K) > W_\alpha(K^*) > m_2^{\alpha-p} W_\alpha(K)$$

which implies that

$$m_1^{p-\alpha} C_\alpha(K) < C_\alpha(K^*) < m_2^{p-\alpha} C_\alpha(K). \tag{2.13}$$

Hence K^* has α-capacity zero iff K has α-capacity zero.

Let E be any subset of $\mathbb{R}^p \setminus \{x_o\}$ and let E^* denote the set obtained from E by inversion relative to the sphere $S(x_o;1)$. From (2.13), we obtain the result that for $0 < \alpha < p$, E^* has inner α-capacity zero iff E has inner α-capacity zero. The same is true if we consider outer α-capacity.

We take the following definition from Landkov ([30], p. 261) and Riesz ([35], p. 13).

Definition 2.16.

Let ν be a charge distribution on $\mathbb{R}^p$ for which $\nu(\{x_o\}) = 0$. The Kelvin transform of ν is defined to be the charge distribution ν^* on $\mathbb{R}^p$ which is given by

$$d\nu^*(x^*) = |x-x_o|^{\alpha-p}\, d\nu(x) \qquad (2.14)$$

for $x \in \mathbb{R}^p$ and $0 < \alpha < p$.

From (2.12) and (2.14), we obtain the result that

$$\begin{aligned} d(\nu^*)^*(x) &= |x^*-x_o|^{\alpha-p}\, d\nu^*(x^*) \\ &= (|x^*-x_o|\cdot|x-x_o|)^{\alpha-p}\, d\nu(x) \\ &= d\nu(x) , \end{aligned}$$

and hence $(\nu^*)^* = \nu$. Moreover,

$$k_\alpha * \nu(x_o) = \int_{S(\nu)} \frac{d\nu(x)}{|x-x_o|^{p-\alpha}} = \nu^*(\mathbb{R}^p),$$

$$k_\alpha * \nu^*(x_o) = \nu(\mathbb{R}^p) \qquad (2.15)$$

and for each $x \in \mathbb{R}^p \setminus \{x_o\}$,

$$\begin{aligned} k_\alpha * \nu^*(x^*) &= \int_{S(\nu^*)} \frac{d\nu^*(y^*)}{|x^*-y^*|^{p-\alpha}} = \int_{S(\nu)} \frac{|x-x_o|^{p-\alpha}}{|x-y|^{p-\alpha}}\, d\nu(y) \\ &= |x-x_o|^{p-\alpha}\, k_\alpha * \nu(x). \end{aligned} \qquad (2.16)$$

If the charge distribution ν on $\mathbb{R}^p$ has an atomic constituent at x_o, then we define ν^* to be the Kelvin transform of the restriction of ν to $\mathbb{R}^p \setminus \{x_o\}$. Hence (2.16) becomes

$$k_\alpha * \nu^*(x^*) + \nu(\{x_o\}) = |x-x_o|^{p-\alpha} k_\alpha * \nu(x)$$

for each $x \in \mathbb{R}^p \setminus \{x_o\}$.

We are now ready to prove the balayage principle for the M. Riesz kernel.

We shall assume that $p \geq 2$ and $0 < \alpha \leq 2$ ($0 < \alpha < 2$ if $p = 2$) for the remainder of this section. Also, we will omit the normalization factor $H(p,\alpha)$ from the M. Riesz potential of order α due to a charge distribution ν and will simply write

$$k_\alpha * \nu(x) = \int_{S(\nu)} \frac{d\nu(y)}{|x-y|^{p-\alpha}} \qquad \text{for } x \in \mathbb{R}^p.$$

Let F be a closed subset of $\mathbb{R}^p$ with positive α-capacity (if F has α-capacity zero, then the balayage principle for the M. Riesz kernel is immediate). Take any point $x_o \notin F$ and let F^* denote the set obtained from F by inversion relative to the sphere $S(x_o;1)$. Then F^* is a compact set with positive α-capacity. If we let γ^* be the equilibrium mass distribution for F^* and ε'_{x_o} the Kelvin transform of γ^*, then it follows (c.f. (2.15) and (2.16)) that

$$\varepsilon'_{x_o}(\mathbb{R}^p) = k_\alpha * \gamma^*(x_o), \tag{2.17}$$

and

$$k_\alpha * \varepsilon'_{x_o}(x) = \frac{1}{|x-x_o|^{p-\alpha}}\ k_\alpha * \gamma^*(x^*)$$

for $x \in \mathbb{R}^p \setminus \{x_o\}$. By Theorem 2.19,

$$k_\alpha * \gamma^*(x^*) = 1 \qquad \alpha\text{-quasi-everywhere on } F^*$$

and

$$k_\alpha * \gamma^*(x) \leq 1 \qquad \text{for every } x \in \mathbb{R}^p.$$

Therefore,

$$k_\alpha * \varepsilon'_{x_o}(x) = \frac{1}{|x-x_o|^{p-\alpha}} \qquad \alpha\text{-quasi-everywhere on } F, \quad (2.18)$$

and

$$k_\alpha * \varepsilon'_{x_o}(x) \leq \frac{1}{|x-x_o|^{p-\alpha}} \qquad \text{for every } x \in \mathbb{R}^p, \quad (2.19)$$

since sets having outer α-capacity zero are mapped to sets having outer α-capacity zero by inversion relative to the sphere $S(x_o;1)$. Since $S(\varepsilon'_{x_o}) \subset F$, the mass distribution ε'_{x_o} solves the problem of balayaging the Dirac measure ε_{x_o} onto F.

The mass distribution ε'_{x_o} will be called the <u>Green measure for F relative to the pole x_o</u>.

Remark 2.4.

Since $k_\alpha * \varepsilon'_{x_o}$ is bounded on F (c.f. (2.19)), it follows that ε'_{x_o} is zero on all subsets of F with outer α-capacity zero (c.f. Theorem 2.14).

For $x_o, y_o \notin F$, we obtain the result that

$$\begin{aligned}
k_\alpha * \varepsilon'_{x_o}(y_o) &= \int_F \frac{d\varepsilon'_{x_o}(x)}{|x-y_o|^{p-\alpha}} \\
&= \int_F k_\alpha * \varepsilon'_{y_o}(x)\, d\varepsilon'_{x_o}(x) \\
&= \int_F d\varepsilon'_{x_o}(x) \int_F \frac{d\varepsilon'_{y_o}(y)}{|x-y|^{p-\alpha}} \\
&= \iint_F \frac{1}{|x-y|^{p-\alpha}}\, d\varepsilon'_{x_o}(x)\, d\varepsilon'_{y_o}(y)
\end{aligned}$$

(c.f. (2.18) and Remark 2.4). Therefore,

$$k_\alpha * \varepsilon'_{x_o}(y_o) = k_\alpha * \varepsilon'_{y_o}(x_o), \tag{2.20}$$

or in the language of mutual α-energy

$$(\varepsilon'_{x_o}, \varepsilon_{y_o})_\alpha = (\varepsilon_{x_o}, \varepsilon'_{y_o})_\alpha .$$

Definition 2.17.

The Green function of order α for the open set $\mathbb{R}^p \setminus F$ is defined to be the extended real valued function G_α defined on $\mathbb{R}^p \times (\mathbb{R}^p \setminus F)$, which is given by

$$G_\alpha(x,y) = k_\alpha * \varepsilon_y(x) - k_\alpha * \varepsilon'_y(x) \quad \text{for } x \in \mathbb{R}^p,\ y \in \mathbb{R}^p \setminus F.$$

From (2.20) we obtain the result that

$$G_\alpha(x,y) = G_\alpha(y,x) \qquad \text{for } x,y \in \mathbb{R}^p \setminus F. \tag{2.21}$$

For any $y \in \mathbb{R}^p \setminus F$, it follows (c.f. (2.18) and (2.19)) that

$$G_\alpha(x,y) = 0 \qquad \alpha\text{-quasi-everywhere on } \Gamma, \tag{2.22}$$

and

$$G_\alpha(x,y) \geq 0 \qquad \text{for every } x \in \mathbb{R}^p. \tag{2.23}$$

In particular, we will show that $G_\alpha(x,y) = 0$ for every interior point of F. Let x_o be any interior point of F and choose $r > 0$ such that $B(x_o;r) \subset F$. By Theorem 2.11 and (2.22),

$$G_\alpha * m_{x_o}^{(r)}(y) = \int_{B(x_o;r)} G_\alpha(x,y)\, dm^{(r)}(x) = 0,$$

and hence

$$G_\alpha(x_o,y) = \lim_{r\to 0} G_\alpha * m_{x_o}^{(r)}(y) = 0$$

by Theorem 2.7.

<u>Definition 2.18.</u>

The points $x \in \partial F$ where $G_\alpha(x,y) > 0$ for some $y \in \mathbb{R}^p \setminus F$ are called the <u>irregular points of F</u>.

We denote by F'' the set of all irregular points of F. It follows from (2.22) that F'' has outer α-capacity zero. We will show that the set of irregular points of F does not depend on the position of the pole in the connected components of $\mathbb{R}^p \setminus F$ later in this section.

Definition 2.19.

We set $F' = F \setminus F''$ and define the points of F' to be the regular points of F.

We will define the Green measure for F relative to any regular point $x \in F$ to be the Dirac measure ε_x. If the Green measure is defined in this way, equation (2.20) remains valid for $x_o, y_o \notin F''$.

Theorem 2.22.

Let ν be a charge distribution with $S(\nu) \subset F$. Then for any $y \in \mathbb{R}^p \setminus F''$,

$$k_\alpha * \nu(y) = \int_{F'} k_\alpha * \nu(x)\, d\varepsilon'_y(x) + \int_{F''} G_\alpha(x,y)\,d\nu(x).$$

Proof.

For every $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^p \setminus F''$, we obtain from Definition 2.17 the result that

$$\frac{1}{|x-y|^{p-\alpha}} = k_\alpha * \varepsilon_y(x) = k_\alpha * \varepsilon'_y(x) + G_\alpha(x,y),$$

and hence by Fubini's theorem,

$$k_\alpha * \nu(y) = \int_F \frac{d\nu(x)}{|x-y|^{p-\alpha}}$$

$$= \int_F k_\alpha * \varepsilon'_y(x)\, d\nu(x) + \int_F G_\alpha(x,y)\, d\nu(x)$$

$$= \int_{F'} k_\alpha * \nu(x)\, d\varepsilon'_y(x) + \int_{F''} G_\alpha(x,y)\, d\nu(x)$$

because $\varepsilon'_y(F'') = 0$ and $G_\alpha(x,y) = 0$ if $x \in F'$.

Corollary 2.22.1.

Let ν be a charge distribution with $S(\nu) \subset F'$. Then for any $y \in \mathbb{R}^p \setminus F''$,

$$k_\alpha * \nu(y) = \int_{F'} k_\alpha * \nu(x)\, d\varepsilon'_y(x). \tag{2.24}$$

Moreover, (2.24) holds α-quasi-everywhere on $\mathbb{R}^p$.

Definition 2.20.

Let ν be a charge distribution on $\mathbb{R}^p$. We define a charge distribution ν' with $S(\nu') \subset F$ by

$$\nu'(f) = \int_{S(\nu)} \varepsilon'_x(f)\, d\nu(x) \qquad \text{for } f \in C_c(\mathbb{R}^p) \tag{2.25}$$

For any $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^p \setminus F''$, we obtain from Definition 2.17 the result that

$$\frac{1}{|x-y|^{p-\alpha}} = k_\alpha * \varepsilon_y(x) = k_\alpha * \varepsilon'_y(x) + G_\alpha(x,y),$$

and hence

$$\begin{aligned}
k_\alpha * \nu(y) &= \int_{S(\nu)} \frac{d\nu(x)}{|x-y|^{p-\alpha}} \\
&= \int_{S(\nu)} k_\alpha * \varepsilon'_y(x)\, d\nu(x) + \int_{S(\nu)} G_\alpha(x,y)\, d\nu(x) \\
&= \int_{S(\nu)} \frac{d\nu'(x)}{|x-y|^{p-\alpha}} + \int_{S(\nu)} G_\alpha(x,y)\, d\nu(x) \\
&= k_\alpha * \nu'(y) + \int_{S(\nu)} G_\alpha(x,y)\, d\nu(x) .
\end{aligned}$$

It follows (c.f. (2.22)) that

$$k_\alpha * \nu'(y) = k_\alpha * \nu(y) \tag{2.26}$$

α-quasi-everywhere on F. In particular, equation (2.26) holds for every regular point $y \in F$. Therefore, ν' solves the problem of sweeping the charge distribution ν onto F. Moreover, if ν is a mass distribution on $\mathbb{R}^p$, then we obtain (c.f. (2.23)) the result that

$$k_\alpha * \nu'(y) \le k_\alpha * \nu(y) \qquad \text{for every } y \in \mathbb{R}^p.$$

Corollary 2.22.2.

Let ν_1 and ν_2 be two charge distributions with $S(\nu_1)$, $S(\nu_2) \subset F'$. If

$$k_\alpha * \nu_1(x) = k_\alpha * \nu_2(x) \qquad \alpha\text{-quasi-everywhere on } F,$$

then $\nu_1 \equiv \nu_2$. Moreover, the Green measure for F relative to any pole $x \notin F''$ is unique.

Proof.

If we set $\nu = \nu_1 - \nu_2$, then $S(\nu) \subset F'$, and hence

$$k_\alpha * \nu(y) = \int_{F'} k_\alpha * \nu(x)\, d\varepsilon'_y(x) \quad \alpha\text{-quasi-everywhere on } \mathbb{R}^p$$

by Corollary 2.22.1. Since $k_\alpha * \nu(x) = 0$ α-quasi-everywhere on F, it follows that $k_\alpha * \nu(y) = 0$ α-quasi-everywhere on $\mathbb{R}^p$ (c.f. Remark 2.4). By Theorems 2.8 and 2.11, $\nu \equiv 0$, and consequently, $\nu_1 \equiv \nu_2$.

Let F_1 and F_2 be closed sets in $\mathbb{R}^p$ such that $F_1 \subset F_2$. Take any point $x_o \notin F_2$, and let ε'_{x_o} and ε''_{x_o} be the Green measures for F_1 and F_2 relative to x_o respectively. If we let $(\varepsilon''_{x_o})'$ denote the mass distribution obtained by sweeping the Green measure ε''_{x_o} onto F_1, then we obtain the result that

$$k_\alpha * (\varepsilon''_{x_o})'(x) = k_\alpha * \varepsilon''_{x_o}(x) = \frac{1}{|x-x_o|^{p-\alpha}} = k_\alpha * \varepsilon'_{x_o}(x)$$

α-quasi-everywhere on F_1 (c.f. (2.18) and (2.26)). By Corollary 2.22.2, $(\varepsilon''_{x_o})' = \varepsilon'_{x_o}$, and therefore,

$$\varepsilon'_{x_o}(f) = \int_{F_2} \varepsilon'_x(f)\, d\varepsilon''_{x_o}(x) \qquad \text{for } f \in C_c(\mathbb{R}^p) \tag{2.27}$$

(c.f. (2.25)). Similarly for any charge distribution ν on $\mathbb{R}^p$,

$$\nu'(f) = \int_{F_2} \varepsilon'_x(f)\, d\nu''(x) \qquad \text{for } f \in C_c(\mathbb{R}^p), \tag{2.28}$$

where ν' and ν'' are the charge distributions obtained by sweeping ν onto

F_1 and F_2 respectively. If we denote by ν''_{F_1} the restriction of the charge distribution ν'' to F_1, then we obtain from (2.28) the result that

$$\nu'(f) = \nu''_{F_1}(f) + \int_{F_2 \setminus F_1} \varepsilon'_x(f)\, d\nu''(x) \quad \text{for } f \in C_c(\mathbb{R}^p). \tag{2.29}$$

Example 2.1.

We refer to Riesz ([35], pp. 16-18) for the following results. Let $F = \overline{B(0;r)}$ and $x_o \notin F$. Then the Green measure for F relative to the pole x_o has density

$$\varepsilon'_{x_o}(x) = \begin{cases} A(p,\alpha)(|x_o|^2-r^2)^{\alpha/2}(r^2-|x|^2)^{-\alpha/2}|x-x_o|^{-p}, & |x|<r, \\ 0 \quad ; & |x| \geq r, \end{cases} \tag{2.30}$$

where

$$A(p,\alpha) = \pi^{-(p/2+1)}\,\Gamma(p/2)\ \sin\frac{\pi\alpha}{2}\,.$$

Similarly, if $F = \mathbb{R}^p \setminus B(0;r)$ and $x_o \notin F$, then the Green measure for F relative to the pole x_o has density

$$\varepsilon'_{x_o}(x) = \begin{cases} A(p,\alpha)(r^2-|x_o|^2)^{\alpha/2}(|x|^2-r^2)^{-\alpha/2}|x-x_o|^{-p}, & |x|>r, \\ 0 \quad , & |x| \leq r. \end{cases} \tag{2.31}$$

Theorem 2.23. (The Generalized Harnack Inequality).

Let F be the interior or exterior of the sphere $S(x_o;r)$ and ν a charge distribution on $\mathbb{R}^p$ such that

$$k_\alpha * \nu(x) \geq 0 \qquad \text{for every } x \in F.$$

For $y, y_o \in \mathbb{R}^p \setminus F$,

$$k_\alpha * \nu(y) \leq \left|\frac{r^2-|y-x_o|^2}{r^2-|y_o-x_o|^2}\right|^{\alpha/2} \left|\frac{r-|y-x_o|}{r+|y_o-x_o|}\right|^{-p} k_\alpha * \nu(y_o) .$$

Proof.

For $y, y_o \in \mathbb{R}^p \setminus F$, we obtain (c.f. Corollary 2.22.1 and Example 2.1) the result that

$$\frac{k_\alpha*\nu(y)}{k_\alpha*\nu(y_o)} = \left|\frac{r^2-|y-x_o|^2}{r^2-|y_o-x_o|^2}\right|^{\alpha/2} \frac{\int_F k_\alpha*\nu(x)\left|r^2-|x-x_o|^2\right|^{-\alpha/2}|x-y|^{-p}dm_p(x)}{\int_F k_\alpha*\nu(x)\left|r^2-|x-x_o|^2\right|^{-\alpha/2}|x-y_o|^{-p}\,dm_p(x)}$$

$$\leq \left|\frac{r^2-|y-x_o|^2}{r^2-|y_o-x_o|^2}\right|^{\alpha/2} \left|\frac{r-|y-x_o|}{r+|y_o-x_o|}\right|^{-p}$$

since $\dfrac{|x-y|}{|x-y_o|} \geq \dfrac{\left|r-|y-x_o|\right|}{\left|r+|y_o-x_o|\right|}$ for $x \in F$.

Corollary 2.23.1.

Let F be the interior or exterior of a sphere. If the sequence of nonnegative potentials $\{k_\alpha * \nu_n\}_{n\in\mathbb{N}}^{\infty}$, where $S(\nu_n) \subset F$, converges to zero at a point $y_o \notin F$, then it converges uniformly to zero on any compact subset of $\mathbb{R}^p \setminus F$.

Corollary 2.23.2.

Let F be a closed subset of $\mathbb{R}^p$. If the sequence of nonnegative potentials $\{k_\alpha * \nu_n\}_{n\in\mathbb{N}}$, where $S(\nu_n) \subset F$, converges to zero at a point $y_o \notin F$, and if D is the connected component of $\mathbb{R}^p \setminus F$ which contains y_o, then the sequence converges uniformly to zero on any compact subset of D.

Theorem 2.24.

Let F be a closed subset of $\mathbb{R}^p$. If

$$G_\alpha(x_o,y_o) = 0 \text{ for } x_o \in \partial F \text{ and } y_o \in \mathbb{R}^p\setminus F, \text{ then } G_\alpha(x_o,y) = 0$$

for any point y in the connected component D of $\mathbb{R}^p\setminus F$ which contains y_o.

Proof.

Let $\{x_n\}_{n\in\mathbb{N}}$ be any sequence of points in $\mathbb{R}^p\setminus F$ which converges to x_o. Then

$$0 \le G_\alpha(x_n,y_o) = k_\alpha * (\varepsilon_{y_o} - \varepsilon'_{y_o})\,(x_n)$$

(c.f. (2.23) and (2.21)), and

$$\lim_{n\to\infty} G_\alpha(x_n,y_o) = 0$$

since $G_\alpha(x_o,y_o) = 0$ and G_α is upper semicontinuous in a neighbourhood of F. Hence we obtain from Corollary 2.23.2 the result that

$$\lim_{n\to\infty} G_\alpha(x_n,y) = 0$$

for any point $y \in D$.

We will now show that $G_\alpha(x_o,y) = 0$. By Theorem 2.7,

$$G_\alpha(x_o,y) = \lim_{r\to 0} \int_{B(x_o;r)} G_\alpha(x,y)\, dm^{(r)}(x) .$$

If $G_\alpha(x_o,y) > 0$, then there exists a sequence $\{x_n\}_{n\in \mathbb{N}}$ of points in D which converges to x_o and

$$\limsup_{n\to\infty} G_\alpha(x_n,y) > 0,$$

which leads to a contradiction.

We obtain from Theorem 2.24 the result that x is a regular point of F iff $G_\alpha(x,y) = 0$ for every $y \in \mathbb{R}^p \setminus F$, and hence the set of irregular points of F does not depend on the position of the pole in the connected components of $\mathbb{R}^p \setminus F$.

§5. The Domination Principle for the M. Riesz Kernel

In this section, we will show that the M. Riesz kernel of order α on $\mathbb{R}^p (p \geq 2)$ satisfies the domination principle iff $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$; that is, for any mass distribution $\mu \in \mathcal{E}_\alpha^+$ and mass distribution λ on $\mathbb{R}^p$, if

$$k_\alpha * \mu(x) \leq k_\alpha * \lambda(x) \qquad \mu\text{-almost everywhere on } \mathbb{R}^p,$$

then

$$k_\alpha * \mu(x) \leq k_\alpha * \lambda(x) \qquad \text{for every } x \in \mathbb{R}^p.$$

First we shall assume that $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$. Suppose that F is a closed subset of $\mathbb{R}^p$ and $x \notin F$. We will denote by $\varepsilon_x^{(F)}$ the Green measure for F relative to the pole x. If $F = \mathbb{R}^p \setminus \overline{B(x_o;r)}$, then we write $\varepsilon_{x_o}^{(F)} = \varepsilon^{(r;x_o)}$. It is possible to show that $\varepsilon^{(r;x_o)} \to \varepsilon_{x_o}$ as r tends to zero (c.f. (2.31)).

We refer to Frostman ([21], p. 3) for the following definition.

Definition 2.21.

An extended real valued function f defined on $\mathbb{R}^p$ is called superharmonic of order α (or simply, α-superharmonic) on $\mathbb{R}^p$ iff it satisfies the following three conditions:

(a) $f \not\equiv +\infty$;

(b) f is lower semicontinuous on $\mathbb{R}^p$;

(c) For every closed subset F of $\mathbb{R}^p$ and $x \notin F$,

$$f(x) \geq \int_F f(y)d\varepsilon_x^{(F)}(y) \; . \tag{2.32}$$

We set

$$f^{(F)}(x) = \int_F f(y)\, d\varepsilon_x^{(F)}(y) \qquad \text{for } x \in \mathbb{R}^p \setminus F$$

and call the function $f^{(F)}$ the <u>M. Riesz extension of f to $\mathbb{R}^p \setminus F$</u> (c.f. Riesz ([35], p. 27)).

Consider an α-superharmonic function f on $\mathbb{R}^p$. Then there exists a point $x \in \mathbb{R}^p$ such that $f(x) < +\infty$. By Corollary 2.21.2, $\varepsilon_y^{(\{x\})} \equiv 0$ for every $y \in \mathbb{R}^p \setminus \{x\}$, and hence the M. Riesz extension of f to $\mathbb{R}^p \setminus \{x\}$, which is given by

$$f^{(\{x\})}(y) = f(x)\; \varepsilon_y^{(\{x\})}(\{x\}) \qquad \text{for } y \in \mathbb{R}^p \setminus \{x\} \; ,$$

is identically equal to zero. It follows (c.f. (2.32)) that $f \geq 0$. If $\alpha = 2$ and $p \geq 3$, then f is necessarily a nonnegative superharmonic function on $\mathbb{R}^p$ defined in the classical sense of F. Riesz.

<u>Proposition 2.7.</u>

If f is α-superharmonic on $\mathbb{R}^p$, then

$$f(x) = \lim_{r \to 0} \varepsilon^{(r;x)}(f) \tag{2.33}$$

for every $x \in \mathbb{R}^p$.

This is proved in exactly the same way as the analogous property for ordinary superharmonic functions (see Proposition 1.15).

If μ is a mass distribution on $\mathbb{R}^p$, then $k_\alpha * \mu$ is either identically equal to infinity or is an α-superharmonic function on $\mathbb{R}^p$. This follows from the facts that $k_\alpha * \mu$ is lower semicontinuous on $\mathbb{R}^p$ (c.f. Proposition 2.1(a)) and for every closed subset F of $\mathbb{R}^p$ and $x \notin F$,

$$(k_\alpha * \mu)^{(F)}(x) = \int_F d\varepsilon_x^{(F)}(y) \int_{S(\mu)} \frac{d\mu(z)}{|y-z|^{p-\alpha}}$$

$$= \int_{S(\mu)} d\mu(z) \int_F \frac{d\varepsilon_x^{(F)}(y)}{|y-z|^{p-\alpha}}$$

$$\leq \int_{S(\mu)} \frac{d\mu(z)}{|x-z|^{p-\alpha}} = k_\alpha * \mu(x)$$

by Fubini's theorem and (2.19). Moreover, if $S(\mu) \subset F'$, then

$$(k_\alpha * \mu)^{(F)}(x) = k_\alpha * \mu(x) \tag{2.34}$$

by (2.18). In particular, if $k_\alpha * \mu$ is bounded, then μ is zero on all sets of outer α-capacity zero (c.f. Proposition 2.2), and hence equation (2.34) holds.

Proposition 2.8.

If f and g are two α-superharmonic functions on $\mathbb{R}^p$, then $f \wedge g$ is α-superharmonic on $\mathbb{R}^p$.

Proof.

It is clear that $f \wedge g \not\equiv +\infty$, and furthermore, $f \wedge g$ is lower semicontinuous on $\mathbb{R}^p$ (c.f. Proposition 1.6). For every closed subset F of $\mathbb{R}^p$ and $x \notin F$,

$$(f \wedge g)^{(F)}(x) = \int_F (f \wedge g)(y)\, d\varepsilon_x^{(F)}(y)$$

$$\leq (f^{(F)} \wedge g^{(F)})\,(x)$$

$$\leq (f \wedge g)(x).$$

This completes the proof.

The following results are an immediate consequence of Proposition 2.8.

Proposition 2.9.

Let μ and λ be two mass distributions on $\mathbb{R}^p$ and f an α-superharmonic function on $\mathbb{R}^p$. Then $f \wedge (k_\alpha * \mu)$ is an α-superharmonic function on $\mathbb{R}^p$. Moreover, $(k_\alpha * \mu) \wedge (k_\alpha * \lambda)$ is either identically equal to infinity or is also an α-superharmonic function on $\mathbb{R}^p$.

Proposition 2.10.

If $\{f_n\}_{n \in \mathbb{N}}$ is a nondecreasing sequence of α-superharmonic functions on $\mathbb{R}^p$, then

$$f(x) = \lim_{n\to\infty} f_n(x)$$

is either identically equal to infinity or is also an α-superharmonic function on $\mathbb{R}^p$.

Proof.

By Proposition 1.8, f is lower semicontinuous on $\mathbb{R}^p$. Furthermore, for every closed subset F of $\mathbb{R}^p$ and $x \notin F$,

$$f^{(F)}(x) = \int_F (\lim_{n\to\infty} f_n(y))\, d\varepsilon_x^{(F)}(y)$$

$$= \lim_{n\to\infty} \int_F f_n(y)\, d\varepsilon_x^{(F)}(y)$$

$$\le \lim_{n\to\infty} f_n(x)$$

$$= f(x)$$

by the Lebesgue monotone convergence theorem.

Theorem 2.25.

Let $\{k_\alpha * \mu_n\}_{n\in\mathbb{N}}$ be a nondecreasing sequence of M. Riesz potentials of mass distributions on $\mathbb{R}^p$. Then either

$$\lim_{n\to\infty} k_\alpha * \mu_n(x) = +\infty \qquad \text{for every } x \in \mathbb{R}^p,$$

or

$$\lim_{n\to\infty} k_\alpha * \mu_n(x) = k_\alpha * \mu(x) + A \qquad \text{for every } x \in \mathbb{R}^p, \tag{2.35}$$

where $\mu \in \mathcal{M}^+$ and $A \ge 0$.

Proof.

It follows from Theorem 2.4 that if $\lim_{n\to\infty} k_\alpha * \mu_n \not\equiv +\infty$, then equation (2.35) holds almost everywhere on $\mathbb{R}^p$. Furthermore, $\lim_{n\to\infty} k_\alpha * \mu_n$ is an α-superharmonic function on $\mathbb{R}^p$ by Proposition 2.10. Hence by Proposition 2.7, equation (2.35) holds everywhere on $\mathbb{R}^p$.

We will now establish a representation theorem for α-superharmonic functions on $\mathbb{R}^p$, which is due to Frostman ([20] and [21]).

Theorem 2.26.

Let K be a compact subset of $\mathbb{R}^p$ with α-capacity $C_\alpha(K) > 0$ and f a continuous real valued function defined on K. Then there exists a mass distribution $\mu^{(K)}$ which is a minimum for the Gauss-Frostman functional

$$J_\alpha(\mu) = \tfrac{1}{2}\| \mu \|_\alpha^2 - \int_K f(x)\, d\mu(x),$$

where $\mu \in \mathcal{E}_\alpha^+$ and $S(\mu) \subset K$. Moreover,

$$k_\alpha * \mu^{(K)}(x) \geq f(x) \tag{2.36}$$

α-quasi-everywhere on K. In particular, equation (2.36) holds for any regular point $x \in K$.

Proof.

If $f(x) \leq 0$ for every $x \in K$, then we need only set $\mu^{(K)} \equiv 0$. Therefore, the theorem is only interesting if $\sup_{x \in K} f(x) > 0$.

Since $J_\alpha(\mu) = 0$ if $\mu \equiv 0$, the Gauss-Frostman functional has infimum $\ell \leq 0$. Furthermore,

$$\tfrac{1}{2}\| \mu \|_\alpha^2 \geq \tfrac{1}{2}\mu(K)^2 \left(\sup_{x,y \in K} |x-y| \right)^{\alpha-p} > 0$$

and

$$\int_K f(x)\, d\mu(x) \leq \mu(K) \sup_{x \in K} f(x) < +\infty,$$

and consequently,

$$J_\alpha(\mu) \geq \zeta m^2 - \beta m,$$

where $\zeta = \frac{1}{2}\left(\sup_{x,y\in K}|x-y|\right)^{\alpha-p} > 0$, $\beta = \sup_{x\in K} f(x) < +\infty$, and $m = \mu(K)$. It follows that

$$J_\alpha(\mu) \geq -\frac{\beta^2}{4\zeta} > -\infty ,$$

and hence $-\infty < \ell \leq 0$.

Suppose that $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of mass distributions with finite α-energy such that $S(\mu_n) \subset K$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} J_\alpha(\mu_n) = \ell.$$

Then there exists $n_o \in \mathbb{N}$ such that

$$1 > J_\alpha(\mu_n) \geq \zeta m_n^2 - \beta m_n \qquad \text{for every } n \geq n_o,$$

where $m_n = \mu_n(K)$. Therefore,

$$m_n < \frac{\beta + \sqrt{\beta^2 + 4\zeta}}{2\zeta} < +\infty \qquad \text{for every } n \geq n_o,$$

and hence the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is vaguely bounded on K. By Theorem 1.6, there exists a subsequence $\{\mu_{n(i)}\}_{i\in\mathbb{N}}$ of $\{\mu_n\}_{n\in\mathbb{N}}$ such that $\mu_{n(i)} \dot{\to} \mu^{(K)}$, where $\mu^{(K)}$ is a mass distribution with $S(\mu^{(K)}) \subset K$. It follows that

$$\lim_{i\to\infty} J_\alpha(\mu_{n(i)}) = \ell$$

and

$$\lim_{i\to\infty} \int_K f(x)\, d\mu_{n(i)}(x) = \int_K f(x)\, d\mu^{(K)}(x) ,$$

and therefore,

$$\| \mu^{(K)} \|_\alpha^2 \le \lim_{i\to\infty} \| \mu_{n(i)} \|_\alpha^2 < +\infty$$

(c.f. Theorem 2.9). Hence we obtain the result that

$$\ell = \lim_{i\to\infty} J_\alpha(\mu_{n(i)}) \ge J_\alpha(\mu^{(K)}) \ge \ell$$

by the minimality of ℓ. Therefore, $\mu^{(K)}$ is a minimum for the Gauss-Frostman functional.

To prove that $k_\alpha * \mu^{(K)}(x) \ge f(x)$ α-quasi-everywhere on K, we take any mass distribution $\nu \in \mathcal{E}_\alpha^+$ with $S(\nu) \subset K$. Let B be any Borel subset of K and ν_B the restriction of ν to B. Then for each $\delta > 0$, we obtain the result that $\delta\nu_B \in \mathcal{E}_\alpha^+$ with $S(\delta\nu_B) = S(\nu_B) \subset B$, and hence

$$\begin{aligned} 0 &\le J_\alpha(\mu^{(K)} + \delta\nu_B) - J_\alpha(\mu^{(K)}) \\ &= \delta\left((\mu^{(K)},\nu_B)_\alpha - \int_B f(x)\, d\nu(x)\right) + \frac{\delta^2}{2} \| \nu_B \|_\alpha^2 \\ &= \delta \int_B (k_\alpha * \mu^{(K)}(x) - f(x))\, d\nu(x) + \frac{\delta^2}{2} \| \nu_B \|_\alpha^2 \end{aligned}$$

which implies that

$$0 \le \int_B (k_\alpha * \mu^{(K)}(x) - f(x))\, d\nu(x) + \frac{\delta}{2} \| \nu_B \|_\alpha^2 . \qquad (2.37)$$

Taking the limit of (2.37) as δ tends to zero, we obtain the result that

$$\int_B (k_\alpha * \mu^{(K)}(x) - f(x))\, d\nu(x) \ge 0.$$

Since B is an arbitrary Borel subset of K, it follows that

$$k_\alpha * \mu^{(K)}(x) \geq f(x) \qquad \nu\text{-almost everywhere on } K.$$

Since $\nu \in \mathcal{E}_\alpha^+$ with $S(\nu) \subset K$ is arbitrary, we obtain (c.f. Theorem 2.11) the result that

$$k_\alpha * \mu^{(K)}(x) \geq f(x) \qquad \alpha\text{-quasi-everywhere on } K. \tag{2.38}$$

We will now show that equation (2.36) holds for any regular point $x \in K$. Let $A_r = K \setminus B(x;r)$ for $r > 0$. Since $\mu^{(K)}$ has finite α-energy, it follows that $k_\alpha * \mu^{(K)} \not\equiv +\infty$, and hence $k_\alpha * \mu^{(K)}$ is an α-superharmonic function on $\mathbb{R}^p$. Therefore,

$$\begin{aligned}(k_\alpha * \mu^{(K)} - f)^{(A_r)}(x) &= (k_\alpha * \mu^{(K)})^{(A_r)}(x) - f^{(A_r)}(x) \\ &\leq k_\alpha * \mu^{(K)}(x) - f^{(A_r)}(x).\end{aligned}$$

By Remark 2.4 and (2.38), it follows that

$$k_\alpha * \mu^{(K)}(x) \geq f^{(A_r)}(x). \tag{2.39}$$

It follows from the regularity of x that $\varepsilon_x^{(A_r)} \dot{\to} \varepsilon_x$ as r tends to zero, and consequently,

$$k_\alpha * \mu^{(K)}(x) \geq f(x)$$

by taking the limit of (2.39) as r tends to zero.

Theorem 2.27.

Let K be a compact subset of $\mathbb{R}^p$ with α-capacity $C_\alpha(K) > 0$ and f a continuous α-superharmonic function on $\mathbb{R}^p$. Then there exists a unique mass distribution $\mu^{(K)} \in \mathcal{E}_\alpha^+$ with $S(\mu^{(K)}) \subset K$ such that

$$k_\alpha * \mu^{(K)}(x) = f(x) \qquad \text{for every regular point } x \in K$$

and

$$k_\alpha * \mu^{(K)}(x) \leq f(x) \qquad \text{for every } x \in \mathbb{R}^p.$$

Proof.

We will show that the mass distribution $\mu^{(K)}$ defined in Theorem 2.26 satisfies the conditions of Theorem 2.27.

If $f \equiv 0$ on K, then we set $\mu^{(K)} \equiv 0$. Otherwise, we let $S = S(\mu^{(K)})$. First we will show that

$$k_\alpha * \mu^{(K)}(x) \leq f(x) \qquad \text{for every } x \in S. \tag{2.40}$$

We define

$$\Phi(x) = k_\alpha * \mu^{(K)}(x) - f(x) \qquad \text{for } x \in \mathbb{R}^p.$$

Suppose that $\Phi(x_o) > 0$ for some $x_o \in S$. Since Φ is lower semicontinuous on $\mathbb{R}^p$, there exists a neighbourhood V of x_o such that $\Phi(x) > 0$ for every $x \in V$ (c.f. Proposition 1.4). It follows that

$$k_\alpha * \mu^{(K)}(x) > f(x) \geq 0 \qquad \text{for every } x \in V,$$

since f is α-superharmonic on $\mathbb{R}^p$. If $\mu_V^{(K)}$ is the restriction of $\mu^{(K)}$ to V, then

$$0 < \iint_V k_\alpha * \mu^{(K)}(x)\, d\mu^{(K)}(x) = \| \mu_V^{(K)} \|_\alpha^2 \leq \| \mu^{(K)} \|_\alpha^2 < +\infty$$

and

$$\int_V \Phi(x)\, d\mu^{(K)}(x) > 0 \, .$$

Therefore,

$$\begin{aligned} J_\alpha(\mu^{(K)}-\delta\mu_V^{(K)})-J_\alpha(\mu^{(K)}) &= \tfrac{1}{2}\| \mu^{(K)}-\delta\mu_V^{(K)} \|_\alpha^2 - \int_K f(x)d(\mu^{(K)}-\delta\mu_V^{(K)})(x) \\ &\quad - \tfrac{1}{2}\| \mu^{(K)} \|_\alpha^2 + \int_K f(x)d\mu^{(K)}(x) \\ &= \frac{\delta^2}{2} \| \mu_V^{(K)} \|_\alpha^2 - \delta\left((\mu^{(K)},\mu_V^{(K)})_\alpha - \int_V f(x)d\mu^{(K)}(x)\right) \\ &= \frac{\delta^2}{2} \| \mu_V^{(K)} \|_\alpha^2 - \delta \int_V \Phi(x)d\mu^{(K)}(x) \\ &< 0 \end{aligned}$$

for any $\delta > 0$ such that

$$\delta < \inf\left\{ \frac{2\left(\int_V \Phi(x)\ d\mu^{(K)}(x)\right)}{| \mu_V^{(K)} |_\alpha^2} , 1 \right\}.$$

Since $\mu^{(K)} - \delta\mu_V^{(K)} \in \mathcal{E}_\alpha^+$ has support contained in K for any $0 < \delta < 1$, this contradicts the results of Theorem 2.26. Therefore, $\Phi(x) \leq 0$ for every $x \in S$, or equivalently, inequality (2.40) holds.

We will extend inequality (2.40) to $\mathbb{R}^p \setminus S$. Since f is continuous on $\mathbb{R}^p$ and $S \subset K$, then f is bounded on S. It follows from (2.40) that $k_\alpha * \mu^{(K)}$ is bounded on S, and consequently by (2.34),

$$(k_\alpha * \mu^{(K)})^{(S)}(x) = k_\alpha * \mu^{(K)}(x) \qquad \text{for every } x \in \mathbb{R}^p \setminus S.$$

Since f is α-superharmonic on $\mathbb{R}^p$, we obtain the result that

$$0 \geq \Phi^{(S)}(x) = (k_\alpha * \mu^{(K)})^{(S)}(x) - f^{(S)}(x) \geq k_\alpha * \mu^{(K)}(x) - f(x)$$

for every $x \in \mathbb{R}^p \setminus S$, and therefore,

$$k_\alpha * \mu^{(K)}(x) \leq f(x) \qquad \text{for every } x \in \mathbb{R}^p$$

and

$$k_\alpha * \mu^{(K)}(x) = f(x) \qquad \text{for any regular point } x \in K$$

(c.f. Theorem 2.26).

To prove that the mass distribution $\mu^{(K)}$ is unique, we assume that there exists another mass distribution $\mu' \in \mathcal{E}_\alpha^+$ with $S(\mu') \subset K$ such that

$$k_\alpha * \mu'(x) = f(x) \qquad \text{for every regular point } x \in K.$$

Set $\nu = \mu^{(K)} - \mu'$. Then

$$\begin{aligned} \|\nu\|_\alpha^2 &= \int_K (k_\alpha * \mu^{(K)}(x) - k_\alpha * \mu'(x)) d(\mu^{(K)} - \mu')(x) \\ &= \int_K (k_\alpha * \mu^{(K)}(x) - k_\alpha * \mu'(x)) d\mu^{(K)}(x) - \int_K (k_\alpha * \mu^{(K)}(x) - k_\alpha * \mu'(x)) d\mu'(x) \\ &= 0 \end{aligned}$$

by Theorem 2.11. Therefore, $\nu \equiv 0$ (c.f. Theorem 2.10), and consequently, $\mu^{(K)} \equiv \mu'$.

Theorem 2.28.

Let f be a continuous α-superharmonic function on $\mathbb{R}^p$. Then there exists a mass distribution μ on $\mathbb{R}^p$ and a constant $A \geq 0$ such that

$$f(x) = k_\alpha * \mu(x) + A \qquad \text{for every } x \in \mathbb{R}^p. \quad (2.41)$$

The pair (μ, A) in the representation (2.41) of f is uniquely determined.

Proof.

Let $K_n = \overline{B(0;n)}$ for each $n \in \mathbb{N}$. Then $\{K_n\}_{n \in \mathbb{N}}$ forms a nondecreasing sequence of regular compact sets such that $\mathbb{R}^p = \bigcup_{n \in \mathbb{N}} K_n$. For each $n \in \mathbb{N}$, there exists a unique mass distribution $\mu_n \in \mathcal{E}_\alpha^+$ with $S(\mu_n) \subset K_n$ such that

$$k_\alpha * \mu_n(x) = f(x) \qquad \text{for every } x \in K_n \quad (2.42)$$

by Theorem 2.27.

If we let μ'_{n+1} be the mass distribution obtained by sweeping μ_{n+1} onto K_n, then we obtain the result that

$$k_\alpha * \mu'_{n+1}(x) = k_\alpha * \mu_{n+1}(x) = f(x)$$

for every point $x \in K_n \subset K_{n+1}$ (c.f. (2.27)). It follows from (2.42) that

$$k_\alpha * \mu'_{n+1}(x) = k_\alpha * \mu_n(x) \qquad \text{for every } x \in K_n,$$

and consequently, $\mu'_{n+1} \equiv \mu_n$ by Corollary 2.22.2. Therefore,

$$k_\alpha * \mu_n(x) = k_\alpha * \mu'_{n+1}(x) = (k_\alpha * \mu_{n+1})^{(K_n)}(x) \leq k_\alpha * \mu_{n+1}(x)$$

for every $x \in K_n$. Hence the sequence $\{k_\alpha * \mu_n\}_{n \in \mathbb{N}}$ of M. Riesz potentials is nondecreasing, and by Theorem 2.25,

$$f(x) = \lim_{n\to\infty} k_\alpha * \mu_n(x) = k_\alpha * \mu(x) + A \quad \text{for every } x \in \mathbb{R}^p,$$

where $\mu \in \mathcal{M}^+$ and $A \geq 0$.

By (2.31),

$$\lim_{r\to\infty} \varepsilon^{(r;x)}(k_\alpha * \mu) = 0,$$

and hence

$$A = \lim_{r\to\infty} \varepsilon^{(r;x)}(f). \tag{2.43}$$

Since $\mu_n \overset{\cdot}{\to} \mu$, μ is unique. This proves the uniqueness of the representation (2.41).

Theorem 2.29.

Let f be an α-superharmonic function on $\mathbb{R}^p$. Then there exists a mass distribution μ on $\mathbb{R}^p$ and a constant $A \geq 0$ such that

$$f(x) = k_\alpha * \mu(x) + A \qquad \text{for every } x \in \mathbb{R}^p. \tag{2.44}$$

The pair (μ, A) in the representation (2.44) of f is uniquely determined.

Proof.

We obtain from Corollary 1.11.1 the result that f is the limit of a nondecreasing sequence $\{f_i\}_{i \in \mathbb{N}}$ of continuous α-superharmonic functions on $\mathbb{R}^p$. Let $K_n = \overline{B(0;n)}$ for each $n \in \mathbb{N}$. For each $i \in \mathbb{N}$ and $n \in \mathbb{N}$, there exists a unique mass distribution $\mu_i^{(n)} \in \mathcal{E}_\alpha^+$ with $S(\mu_i^{(n)}) \subset K_n$ such that

$$k_\alpha * \mu_i^{(n)}(x) = f_i(x) \qquad \text{for every } x \in K_n$$

by Theorem 2.27. The sequence $\{k_\alpha * \mu_i^{(n)}\}_{n \in \mathbb{N}}$ of M. Riesz potentials is nondecreasing, and by Theorem 2.25,

$$f(x) = \lim_{i\to\infty} f_i(x) = \lim_{i\to\infty} k_\alpha * \mu_i^{(n)}(x) = k_\alpha * \mu_n(x) + A_n$$

for every $x \in K_n$, where $\mu_n \in \mathcal{M}^+$ with $S(\mu_n) \subset K_n$ and $A_n \geq 0$. If we let f_{K_n} be the restriction of f to K_n, then

$$\lim_{r\to\infty} \varepsilon^{(r;x)}(f_{K_n}) = 0$$

and

$$\lim_{r\to\infty} \varepsilon^{(r;x)}(k_\alpha * \mu_n) = 0$$

for every $x \in K_n$, and consequently, $A_n = 0$. Therefore,

$$f(x) = k_\alpha * \mu_n(x) \qquad \text{for every } x \in K_n.$$

We need only repeat the proof of Theorem 2.28 to obtain the representation (2.44).

Corollary 2.29.1.

Let μ be a mass distribution on $\mathbb{R}^p$ and f an α-superharmonic function on $\mathbb{R}^p$. Then there exists a unique mass distribution λ on $\mathbb{R}^p$ such that

$$((k_\alpha * \mu)\wedge f)(x) = k_\alpha * \lambda(x) \qquad \text{for every } x \in \mathbb{R}^p$$

Proof.

By Proposition 2.9 and Theorem 2.29, there exists a mass distribution λ on $\mathbb{R}^p$ and a constant $A \geq 0$ such that

$$((k_\alpha * \mu)\wedge f)(x) = k_\alpha * \lambda(x) + A \quad \text{for every } x \in \mathbb{R}^p.$$

Moreover, the pair (μ, A) is unique. By (2.44),

$$0 \le A = \lim_{r\to\infty} \varepsilon^{(r;x)}((k_\alpha * \mu)\wedge f) \le \lim_{r\to\infty} \varepsilon^{(r;x)}(k_\alpha * \mu) = 0,$$

and therefore, $A = 0$. This completes the proof.

Corollary 2.29.2.

Let μ and ρ be mass distributions on $\mathbb{R}^p$. If $k_\alpha * \mu \not\equiv +\infty$ or $k_\alpha * \rho \not\equiv +\infty$, then there exists a unique mass distribution λ on $\mathbb{R}^p$ such that

$$((k_\alpha * \mu) \wedge (k_\alpha * \rho))(x) = k_\alpha * \lambda(x) \quad \text{for every } x \in \mathbb{R}^p.$$

Theorem 2.30. (The Principle of Domination).

Let f be an α-superharmonic function on $\mathbb{R}^p$ and $\mu \in \mathcal{E}_\alpha^+$. If

$$k_\alpha * \mu(x) \le f(x) \qquad \mu\text{-almost everywhere on } \mathbb{R}^p,$$

then

$$k_\alpha * \mu(x) \le f(x) \qquad \text{for every } x \in \mathbb{R}^p.$$

Proof.

Set $g = (k_\alpha * \mu) \wedge f$. By Corollary 2.29.1, there exists a unique mass distribution λ on $\mathbb{R}^p$ such that

$$g(x) = k_\alpha * \lambda(x) \qquad \text{for every } x \in \mathbb{R}^p.$$

It is sufficient to show that $\mu \equiv \lambda$.

Since

$$k_\alpha * \lambda(x) \le k_\alpha * \mu(x) \qquad \text{for every } x \in \mathbb{R}^p,$$

we obtain the following inequality

$$\begin{aligned} \| \lambda \|_\alpha^2 &= \int_{\mathbb{R}^p} k_\alpha * \lambda(x)\, d\lambda(x) \\ &\le \int_{\mathbb{R}^p} k_\alpha * \mu(x)\, d\lambda(x) \\ &= \int_{\mathbb{R}^p} k_\alpha * \lambda(x)\, d\mu(x) \\ &\le \int_{\mathbb{R}^p} k_\alpha * \mu(x)\, d\mu(x) \\ &= \| \mu \|_\alpha^2 < +\infty \end{aligned}$$

by Fubini's theorem. Set

$$E = \{x \in \mathbb{R}^p \mid k_\alpha * \mu(x) > f(x)\} = \{x \in \mathbb{R}^p \mid k_\alpha * \mu(x) > k_\alpha * \lambda(x)\}.$$

Then $\mu(E) = 0$ and

$$k_\alpha * \mu(x) = k_\alpha * \lambda(x) \qquad \text{for every } x \in \mathbb{R}^p \setminus E.$$

Therefore,

$$\| \mu - \lambda \|_\alpha^2 = \int_E (k_\alpha * \mu(x) - k_\alpha * \lambda(x))\, d(\mu - \lambda)(x)$$

$$= -\int_E (k_\alpha * \mu(x) - k_\alpha * \lambda(x))\, d\lambda(x)$$

$$\leq 0,$$

and consequently, $\| \mu-\lambda \|_\alpha^2 = 0$ by Theorem 2.10. Hence $\mu \equiv \lambda$ by Theorem 2.10.

Since every M. Riesz potential due to a mass distribution having finite α-energy is α-superharmonic on $\mathbb{R}^p$, we obtain the following corollary to Theorem 2.30.

Corollary 2.30.1. (The Principle of Domination).

Let λ be a mass distribution on $\mathbb{R}^p$ and $\mu \in \mathcal{E}_\alpha^+$. If

$$k_\alpha * \mu(x) \leq k_\alpha * \lambda(x) \qquad \mu\text{-almost everywhere on } \mathbb{R}^p,$$

then

$$k_\alpha * \mu(x) \leq k_\alpha * \lambda(x) \qquad \text{for every } x \in \mathbb{R}^p.$$

Cartan and Deny ([8], p. 88) proved that the balayage principle and the principle of domination are equivalent for the M. Riesz kernel. Furthermore, both principles are equivalent to the principle that for any $\mu, \lambda \in \mathcal{E}_\alpha^+$;

$$\inf \{k_\alpha * \mu,\ k_\alpha * \lambda\}$$

is the potential of a mass distribution having finite α-energy.

We are now ready to prove that the M. Riesz kernel of order α on $\mathbb{R}^p$ does not satisfy the principle of domination for $2 < \alpha \leq p$.

We denote the gradient operator by

$$\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_p} \right) .$$

The fundamental harmonic function for $\mathbb{R}^p (p \geq 2)$ with pole y is given by

$$h_y(x) = \begin{cases} +\infty & \text{for } x = y, \\ \log \dfrac{1}{|x-y|} & \text{for } x \in \mathbb{R}^2 \setminus \{y\} \end{cases}$$

for $p = 2$, and

$$h_y(x) = \begin{cases} +\infty & \text{for } x = y, \\ \dfrac{1}{|x-y|^{p-2}} & \text{for } x \in \mathbb{R}^p \setminus \{y\} \end{cases}$$

for $p \geq 3$.

We refer to Helms ([23], p. 9) for the following result.

Lemma 2.1. (Green's Third Identity).

Let $B = B(y;r)$ be an open ball in $\mathbb{R}^p (p \geq 2)$ with boundary ∂B, $n(x)$ the outer unit normal vector to the surface ∂B at the point $x \in \partial B$, and υ a twice continuously differentiable function on $\overline{B} = B \cup \partial B$. Then

$$\upsilon(y) = -\frac{1}{C_p \omega_p} \left(\int_B h_y(x) \Delta \upsilon(x) dm_p(x) + \int_{\partial B} \left(\upsilon(x) \frac{\partial h_y}{\partial n}(x) - h_y(x) \frac{\partial \upsilon}{\partial n}(x) \right) d\sigma_{p-1}(x) \right) ,$$

where

$$C_p = \begin{cases} 1 & \text{for } p = 2, \\ p-2 & \text{for } p \geq 3, \end{cases}$$

$$\frac{\partial \upsilon}{\partial n}(x)=\vec{\nabla}\upsilon(x)\cdot n(x), \text{ and } \frac{\partial h_y}{\partial n}(x)=\vec{\nabla}h_y(x)\cdot n(x) .$$

Lemma 2.2.

Let f be a three times continuously differentiable function on $\mathbb{R}^p (p \geq 2)$ with compact support S(f). Then the function

$$\Phi(x) = -\Delta f(x) \qquad \text{for } x \in \mathbb{R}^p$$

has compact support $S(\Phi) \subset S(f)$ and satisfies

$$k_2 * \Phi(x) = f(x) \qquad \text{for every } x \in \mathbb{R}^p.$$

Proof.

For any $x \in \mathbb{R}^p$, there exists $r > 0$ such that $S(f) \subset B(x;r)$, because S(f) is a compact subset of $\mathbb{R}^p$. Since $S(\Delta f) \subset S(\vec{\nabla} f) \subset S(f) \subset B(x;r)$, we obtain from Lemma 2.1 the result that

$$\begin{aligned} f(x) &= -\frac{1}{C_p \omega_p} \int_{B(x;r)} h_x(y)\Delta f(y) dm_p(y) \\ &= -\frac{1}{C_p \omega_p} \int_{\mathbb{R}^p} h_x(y)\Delta f(y) dm_p(y). \end{aligned}$$

Then for $p = 2$,

$$f(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^p}\left(\log\frac{1}{|y-x|}\right)\Delta f(y)\,dm_p(y)$$

$$= \int_{R^p}\left(\frac{1}{2\pi}\log\frac{1}{|y-x|}\right)(-\Delta f(y))\,dm_p(y)$$

$$= k_2 * \Phi(x),$$

and for $p \geq 3$,

$$f(x) = -\frac{1}{(p-2)\omega_p}\int_{\mathbb{R}^p}\frac{1}{|y-x|^{p-2}}\Delta f(y)\,dm_p(y)$$

$$= \int_{\mathbb{R}^p}\frac{1}{(p-2)\omega_p}\cdot\frac{1}{|y-x|^{p-2}}(-\Delta f(y))\,dm_p(y)$$

$$= k_2 * \Phi(x).$$

<u>Theorem 2.31.</u>

Let ν be a charge distribution on $\mathbb{R}^p (p \geq 3)$ such that $k_2 * \nu$ is well-defined, K a compact subset of $\mathbb{R}^p$, and ν_K the restriction of ν to K. If

$$k_2 * \nu(x) = 0 \qquad \text{almost everywhere on } K,$$

then $\nu_K \equiv 0$. Moreover, if

$$k_2 * \nu(x) = 0 \qquad \text{almost everywhere on } \mathbb{R}^p,$$

then $\nu \equiv 0$, and consequently,

$$k_2 * \nu(x) = 0 \qquad \text{for every } x \in \mathbb{R}^p.$$

Proof.

It is sufficient to show that $\nu(f) = 0$ for every three times continuously differentiable function f with compact support $S(f) \subset K$. This is a consequence of the fact that the set of all such functions f is dense in $C_c(K)$.

For any function g defined on $\mathbb{R}^p$, we denote by $\check{g}$ the function defined by $\check{g}(x) = g(-x)$ for every $x \in \mathbb{R}^p$. Then we set

$$\Phi(x) = -\Delta f(x) \qquad \text{for every } x \in \mathbb{R}^p.$$

It follows from Lemma 2.2 that

$$\check{f}(x) = k_2 * \check{\Phi}(x) \qquad \text{for every } x \in \mathbb{R}^p.$$

We obtain from Proposition 1.14 the result that

$$\nu(f) = \nu * \check{f}(0) = \nu * (k_2 * \check{\Phi})(0) = (k_2 * \nu) * \check{\Phi}(0)$$

$$= \int_{S(\Phi)} k_2 * \nu(x)\Phi(x)\, dm_p(x) = 0,$$

since $k_2 * \nu(x) = 0$ almost everywhere on $S(\Phi) \subset S(f) \subset K$. Therefore, $\nu_K \equiv 0$.

Theorem 2.32.

The M. Riesz kernel of order α on $\mathbb{R}^p$ does not satisfy the principle of domination for $2 < \alpha < p$.

Proof.

We will show that the M. Riesz kernel of order α on $\mathbb{R}^p$ does not satisfy the principle of balayage for $2 < \alpha < p$.

Suppose that the kernel k_α satisfies the principle of balayage for $2 < \alpha \le 4$. Let B be a closed ball in $\mathbb{R}^p$ and $\varepsilon_x^{(B)}$ the Green measure for B relative to the pole $x \notin B$. Then

$$k_\alpha * \varepsilon_x^{(B)}(y) = k_\alpha * \varepsilon_x(y) \qquad \text{for every } y \in B$$

and

$$k_\alpha * \varepsilon_x^{(B)}(y) \le k_\alpha * \varepsilon_x(y) \qquad \text{for every } y \in \mathbb{R}^p$$

(c.f. (2.18) and (2.19)).

Since $k_\alpha = k_2 * k_{\alpha-2}$ by Theorem 2.1, we obtain from Proposition 1.13 the result that

$$\begin{aligned}(k_2*(k_{\alpha-2}*\varepsilon_x))(y) &= ((k_2*k_{\alpha-2})*\varepsilon_x)(y) = k_\alpha*\varepsilon_x(y)\\ &= k_\alpha*\varepsilon_x^{(B)}(y) = ((k_2*k_{\alpha-2})*\varepsilon_x^{(B)}(y)\\ &= (k_2*(k_{\alpha-2}*\varepsilon_x^{(B)}))\ (y)\end{aligned}$$

for every $y \in B$.

Let $\nu = k_{\alpha-2} * \varepsilon_x - k_{\alpha-2} * \varepsilon_x^{(B)} \in \mathcal{M}$. Then

$$k_2 * \nu(y) = 0 \qquad \text{for every } y \in B,$$

and consequently, $\nu(B) = 0$ by Theorem 2.31. Therefore,

$$k_{\alpha-2} * \varepsilon_x^{(B)}(y) = k_{\alpha-2} * \varepsilon_x(y) \qquad \text{for every } y \in B.$$

Since $0 < \alpha-2 \leq 2$, it follows from the principle of domination that

$$k_{\alpha-2} * \varepsilon_x^{(B)}(y) \leq k_{\alpha-2} * \varepsilon_x(y) \qquad \text{for every } y \in \mathbb{R}^p.$$

Hence $\nu \in \mathcal{M}^+$, which implies that $k_2 * \nu$ is a nonnegative superharmonic function on $\mathbb{R}^p$ which vanishes on B (c.f. Theorem 2.5). By Proposition 1.18,

$$k_2 * \nu(y) = 0 \qquad \text{for every } y \in \mathbb{R}^p,$$

and consequently, $\nu \equiv 0$ by Theorem 2.31. Therefore,

$$k_{\alpha-2} * \varepsilon_x^{(B)} = k_{\alpha-2} * \varepsilon_x(y) \qquad \text{for every } y \in \mathbb{R}^p,$$

and as a result, $\varepsilon_x^{(B)} \equiv \varepsilon_x$ by Theorem 2.8. But this contradicts the definition of $\varepsilon_x^{(B)}$.

The result follows inductively for $2n < \alpha \leq 2(n+1)(n \in \mathbb{N})$.

§6. Superharmonic Functions of Fractional Order on a Region in $\mathbb{R}^p (p \geq 2)$

We shall assume that $0 < \alpha \leq 2$ ($0 < \alpha < 2$ if $p = 2$) throughout this section. If F is a closed subset of $\mathbb{R}^p$ and $x \notin F$, then we denote by $\varepsilon_x^{(F)}$ the Green measure for F relative to the pole x.

We have already defined what it means for a function to be α-superharmonic on $\mathbb{R}^p$. If we wish to extend the notion of α-superharmonicity to functions which are defined on a region, it is necessary to employ a definition which depends on the region. We refer to Frostman [21] for the following results.

Let D be a region in $\mathbb{R}^p$ and set $Y = \mathbb{R}^p \setminus D$. Then the Green function of order α for D is given by

$$G_\alpha(x,y) = \frac{1}{|x-y|^{p-\alpha}} - \int_Y \frac{d\varepsilon_x^{(y)}(z)}{|y-z|^{p-\alpha}} \quad \text{for } x \in \mathbb{R}^p,\ y \in D.$$

Definition 2.22.

Let ν be a charge distribution on $\mathbb{R}^p$. The extended real valued function $G_\alpha\nu$ defined on $\mathbb{R}^p$, which is given by

$$G_\alpha\nu(x) = \int_D G_\alpha(x,y)d\nu(y) \qquad \text{for } x \in \mathbb{R}^p,$$

is called the Green potential of order α of the charge distribution ν.

Let F be a closed subset of D and $x \in D \setminus F$. We will show that the Green function of order α for D satisfies the principle of sweeping the Dirac measure ε_x onto F; that is, there exists a mass distribution $\mu_x^{(F)}$ with $S(\mu_x^{(F)}) \subset F$ such that

$$G_\alpha \mu_x^{(F)}(y) = G_\alpha(y,x) \qquad \alpha\text{-quasi-everywhere on } F \qquad (2.45)$$

and

$$G_\alpha \mu_x^{(F)}(y) \leq G_\alpha(y,x) \qquad \text{for every } y \in \mathbb{R}^p.$$

In particular, (2.45) holds for every regular point $y \in F$.

Let $\varepsilon_x^{(F\cup Y)}$ be the Green measure for $F \cup Y$ relative to the pole x and $\mu_x^{(F)}$ the restriction of $\varepsilon_x^{(F\cup Y)}$ to F. It follows from (2.22), (2.18) and Fubini's theorem that

$$\begin{aligned}
G_\alpha \mu_x^{(F)}(y) &= \int_F G_\alpha(y,z)\,d\mu_x^{(F)}(z) \\
&= \int_F G_\alpha(y,z)\,d\varepsilon_x^{(F\cup Y)}(z) \\
&= \int_{F\cup Y} G\ (y,z)\,d\varepsilon_x^{(F\cup Y)}(z) \\
&= \int_{F\cup Y} \frac{d\varepsilon_x^{(F\cup Y)}(z)}{|y-z|^{p-\alpha}} - \int_{F\cup Y} d\,\varepsilon_x^{(F\cup Y)}(z) \int_Y \frac{d\varepsilon_x^{(Y)}(u)}{|z-u|^{p-\alpha}}. \\
&= \frac{1}{|y-x|^{p-\alpha}} - \int_Y d\varepsilon_y^{(Y)}(u) \int_{F\cup Y} \frac{d\varepsilon_x^{(F\cup Y)}(z)}{|z-u|^{p-\alpha}} \\
&= \frac{1}{|y-x|^{p-\alpha}} - \int_Y \frac{d\varepsilon_y^{(Y)}(u)}{|x-u|^{p-\alpha}} \\
&= G_\alpha(y,x) \qquad (2.46)
\end{aligned}$$

α-quasi-everywhere on F. This result holds for any regular point $y \in F$. Since the Green measure $\varepsilon_y^{(Y)}$ for Y relative to the pole y is zero on all subsets of Y with outer α-capacity zero (c.f. Remark 2.4), it follows from (2.19) and (2.18) that

$$G_\alpha \mu_x^{(F)}(y) = \int_{F \cup Y} \frac{d\varepsilon_x^{(F\cup Y)}(z)}{|y-z|^{p-\alpha}} - \int_{F\cup Y} d\varepsilon_x^{(F\cup Y)}(z) \int_Y \frac{d\varepsilon_y^{(Y)}(u)}{|z-u|^{p-\alpha}}$$

$$\leq \frac{1}{|y-x|^{p-\alpha}} \int_Y d\varepsilon_y^{(Y)}(u) \int_{F\cup Y} \frac{d\varepsilon_x^{(F\cup Y)}(z)}{|z-u|^{p-\alpha}} \tag{2.47}$$

$$\leq \frac{1}{|y-x|^{p-\alpha}} - \int_Y \frac{d\varepsilon_y^{(Y)}(u)}{|x-u|^{p-\alpha}} = G_\alpha(y,x)$$

for every $y \in D$. By Corollary 2.22.2, $\mu_x^{(F)}$ is unique.

Definition 2.22.

An extended real valued function f defined on the region $D \subset \mathbb{R}^p$ is called superharmonic of order α (or simply, α-superharmonic) on D iff it satisfies the following three conditions:

(a) $f \not\equiv +\infty$;

(b) f is lower semicontinuous on D;

(c) For every closed set $F \subset D$ and $x \in D \setminus F$;

$$f(x) \geq \int_F f(y)\, d\mu_x^{(F)}(y).$$

We set

$$f^{(F,D)}(x) = \int_F f(y)\, d\mu_x^{(F)}(y) \quad \text{for } x \in D \setminus F.$$

The set of all α-superharmonic functions on D forms a convex cone containing the nonnegative constant functions on D.

The α-superharmonic functions on D have analogous properties to the α-superharmonic functions on $\mathbb{R}^p$. Indeed, every α-superharmonic function f on D is nonnegative and there exists a monotone nondecreasing sequence of continuous α-superharmonic function on D converging to f.

If ν is a mass distribution on D, then $G_\alpha \nu$ is either identically equal to infinity or is an α-superharmonic function on D. This follows from the facts that $G_\alpha \nu$ is lower semicontinuous on D (c.f. Proposition 2.1) and for every closed set $F \subset D$ and $x \in D \setminus F$,

$$\begin{aligned}
(G_\alpha \nu)^{(F,D)}(x) &= \int_F d\mu_x^{(F)}(y) \int_{S(\nu)} G_\alpha(y,z)d\nu(z) \\
&= \int_{S(\nu)} d\nu(z) \int_F G_\alpha(y,z)d\mu_x^{(F)}(y) \\
&\leq \int_{S(\nu)} G_\alpha(x,z)d\nu(z) = G_\alpha \nu(x)
\end{aligned}$$

by Fubini's theorem and (2.47).

Let f be an α-superharmonic function on $\mathbb{R}^p$. Since

$$\mu_x^{(F)} \leq \varepsilon_x^{(F)}$$

by (2.28), we obtain the result that

$$f(x) \geq f^{(F)}(x) = \int_F f(y)d\varepsilon_x^{(F)}(y) \geq \int_F f(y)d\mu_x^{(F)}(y) = f^{(F,D)}(x),$$

and consequently, f is α-superharmonic on D.

The fundamental representation theorem for superharmonic functions of fractional order on a region in $\mathbb{R}^p$ takes the following form.

Theorem 2.33.

Let K be a compact subset of the region $D \subset \mathbb{R}^p$ with α-capacity $C_\alpha(K) > 0$ and f an α-superharmonic function on D. Then there exists a unique mass distribution $\mu^{(K)}$ with $S(\mu^{(K)}) \subset K$ such that

$$G_\alpha \mu^{(K)}(x) = f(x) \qquad \text{for every regular point } x \in K$$

and

$$G_\alpha \mu^{(K)}(x) \leq f(x) \qquad \text{for every } x \in D.$$

The proof of Theorem 2.33 is analogous to the one given above for $D = \mathbb{R}^p$. We first consider f to be continuous and minimize the functional

$$J'_\alpha(\mu) = \frac{1}{2} \iint_K G_\alpha(x,y)d\mu(x)d\mu(y) - \int_K f(x)d\mu(x),$$

where $\mu \in \mathcal{E}^+_\alpha$ and $S(\mu) \subset K$. The unique solution of this problem gives the required mass distribution. As in Theorem 2.29, we extend this result to a discontinuous α-superharmonic function f on D by considering f as the limit of a nondecreasing sequence $\{f_n\}_{n \in \mathbb{N}}$ of continuous α-superharmonic functions on D. These functions f_n can be written as Green potentials of mass distributions with support contained in K. Hence, they form a nondecreasing sequence of Green potentials of mass distribution which tend to a Green potential of a mass distribution with support contained in K.

We obtain the following result in an analogous way to that of

Theorem 2.29 (see Frostman [21], p. 12).

Theorem 2.34.

Let f be an α-superharmonic function on the region $D \subset \mathbb{R}^p$. Then there exists a mass distribution μ with $S(\mu) \subset D$ and a constant $A \geq 0$ such that

$$f(x) = G_\alpha \mu(x) + A \qquad \text{for every } x \in D. \tag{2.48}$$

The pair (μ, A) in the representation (2.48) of f is uniquely determined.

CHAPTER 3

THE M. RIESZ KERNEL AS THE POTENTIAL KERNEL FOR THE SYMMETRIC STABLE SEMIGROUP

This chapter is devoted to the study of the general potential theory introduced by G.A. Hunt [24] : We will define the potential kernel on a locally compact abelian group to be the vague integral of a transient convolution semigroup of mass distributions. Then we will demonstrate that the M. Riesz kernel of order α on $\mathbb{R}^p$, considered as an absolutely continuous mass distribution with respect to Lebesgue measure, is the potential kernel for the symmetric stable semigroup of order α in the cases $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$. We will follow the book of Berg and Forst [1] throughout this chapter.

§1. Some Results from Fourier Analysis on Groups

We refer to Rudin [37] for the material presented in this section.

Let G be a locally compact abelian group, where addition is the group operation and 0 denotes the neutral element.

We denote by $C(G)$ the set of all continuous complex valued functions on G and endow it with the topology of compact convergence. By $C_c(G)$ we denote the set of all functions in $C(G)$ with compact support and endow it with the inductive limit topology.

The topological dual space of $C_c(G)$ is the space of charge distributions on G, denoted by $\mathcal{M}(G)$. We denote by $\mathcal{M}_b(G)$ the space of all bounded charge distributions on G; that is, the set of all charge distributions ν on G for which the total variation $|\nu|(G)$ is finite.

For any function f on G, $\check{f}$ denotes the function defined by

$$\check{f}(x) = f(-x) \qquad \text{for every } x \in G.$$

For any $\nu \in \mathcal{M}(G)$, $\check{\nu}$ denotes the charge distribution on G defined by

$$\check{\nu}(f) = \nu(\check{f}) \qquad \text{for } f \in C_c(G).$$

For any $\nu \in \mathcal{M}(G)$ and $f \in C_c(G)$,

$$(\nu * f)(x) = \int_G f(x-y)\,d\nu(y) \qquad \text{for } x \in G,$$

and hence

$$(\check{\nu} * f)(x) = \int_G f(x+y)\,d\nu(y) \qquad \text{for } x \in G.$$

Since G is a locally compact abelian group, there exists a Haar measure dx on G (c.f. Halmos ([22], p. 254)). We denote by $L^p(G)$ the space of all complex valued functions f defined on G for which the norm

$$\| f \|_p = \left(\int_G |f(x)|^p dx \right)^{1/p}$$

is finite ($1 \leq p < \infty$). If multiplication is defined by the convolution

$$(f * g)(x) = \int_G f(x-y)g(y)dy \qquad \text{for } f, g \in L^1(G) \text{ and } x \in G,$$

then $L^1(G)$ is a commutative Banach algebra (c.f. Rudin ([37], p. 6)).

Definition 3.1.

A character on G is defined to be a complex valued function γ on G which satisfies $|\gamma(x)| = 1$ for all $x \in G$ and

$$\gamma(x+y) = \gamma(x)\gamma(y) \qquad \text{for every } x, y \in G;$$

i.e. γ is a group homomorphism from G into the circle group of complex numbers with absolute value one.

Definition 3.2.

The dual group of G is defined to be the set Γ of all continuous characters on G together with a binary operation + defined by

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x) \cdot \gamma_2(x) \qquad \text{for } \gamma_1, \gamma_2 \in \Gamma \text{ and } x \in G.$$

The neutral element in Γ is the character 0 defined by $0(x) = 1$ for all $x \in G$ and the inverse of any element $\gamma \in \Gamma$ is the character $-\gamma$ defined by

$$(-\gamma)(x) = \overline{\gamma(x)} = (\gamma(x))^{-1} \qquad \text{for all } x \in G.$$

We endow Γ with the relativized topology from $C(G)$. Then Γ becomes a locally compact abelian group (c.f. Rudin ([37], p. 10)).

Definition 3.3.

For each $f \in L^1(G)$, the complex valued function $\hat{f}$ defined on Γ, which is given by

$$\hat{f}(\gamma) = \int_G \overline{\gamma(x)} \; f(x)dx \qquad \text{for } \gamma \in \Gamma,$$

is called the Fourier transform of f.

We remark that $L^1(G)$ is a subalgebra of $\mathcal{M}_b(G)$ (c.f. (1.12)). Then the Fourier transform can be extended from $L^1(G)$ to $\mathcal{M}_b(G)$ by the next definition.

Definition 3.4.

For each $\nu \in \mathcal{M}_b(G)$, the complex valued function $\hat{\nu}$ defined on Γ, which is given by

$$\hat{\nu}(\gamma) = \int_G \overline{\gamma(x)} d\nu(x) \qquad \text{for } \gamma \in \Gamma,$$

is called the Fourier-Stieltjes transform of ν.

The following theorem contains some of the most important properties of the Fourier-Stieltjes transform. We refer the reader to Rudin ([37], pp. 15-17) for the proof.

Theorem 3.1.

(a) For each $\nu \in \mathcal{M}_b(G)$, the Fourier-Stieltjes transform $\hat{\nu}$ is a uniformly continuous and bounded function from Γ to $\mathbb{C}$.

(b) For each pair $\nu, \rho \in \mathcal{M}_b(G)$ and all $\alpha, \beta \in \mathbb{C}$, we have the result that

$$(\alpha\nu + \beta\rho)^\wedge = \alpha\hat{\nu} + \beta\hat{\rho} \text{ and } \widehat{\nu * \rho} = \hat{\nu} \cdot \hat{\rho} .$$

(c) The Fourier-Stieltjes transform is an injective mapping from $\mathcal{M}_b(G)$ to $\mathbb{C}$; that is, if $\nu \in \mathcal{M}_b(G)$ and $\hat{\nu} \equiv 0$, then $\nu \equiv 0$.

Since Γ is a locally compact abelian group, Γ has a dual group $\hat{\Gamma}$. We define a mapping $\alpha : G \to \hat{\Gamma}$ by

$$\alpha_x(\gamma) = \gamma(x) \qquad \text{for } x \in G \text{ and } \gamma \in \Gamma$$

and call it the canonical injection of G into its bi-dual.

Theorem 3.2. (Pontryagin's Duality Theorem).

The canonical injection α is a topological group isomorphism of G onto its bi-dual $\hat{\Gamma}$.

We refer to Rudin ([37], p. 28) for the proof of Theorem 3.2. In view of the duality between G and Γ, we shall write (x,γ) in place of $\gamma(x)$ for each $x \in G$ and $\gamma \in \Gamma$.

§2. Positive Definite Functions

The following results can be found in the books of Rudin ([37], pp. 17-21) and Berg and Forst ([1], pp. 11-17).

Let G be a locally compact abelian group.

Definition 3.5.

A complex valued function Φ defined on G is called positive definite iff the inequality

$$\sum_{i,j=1}^{n} \Phi(x_i - x_j) c_i \overline{c_j} \geq 0 \tag{3.1}$$

holds for every $n \in \mathbb{N}$, for every $x_1, \ldots, x_n \in G$, and for every $c_1, \ldots, c_n \in \mathbb{C}$.

We denote by P(G) the set of all positive definite functions on G and by CP(G) the set of all continuous positive definite functions on G.

Proposition 3.1.

If $\Phi \in P(G)$, then the following three relations hold:

(a) $\Phi(-x) = \overline{\Phi(x)}$ for $x \in G$;

(b) $|\Phi(x)| \leq \Phi(0)$ for $x \in G$;

(c) $|\Phi(x) - \Phi(y)|^2 \leq 2\Phi(0)(\Phi(0) - \mathrm{Re}\,\Phi(x-y))$ for $x, y \in G$.

Proof.

To prove properties (a) and (b), we take $n = 2$, $x_1 = 0$, $x_2 = x$, $c_1 = 1$, and $c_2 = c$ in inequality (3.1) to obtain the following inequality

$$(1 + |c|^2)\,\Phi(0) + c\Phi(x) + \overline{c}\Phi(-x) \geq 0. \tag{3.2}$$

If $c = 1$, then $\Phi(x) + \Phi(-x)$ is real, and consequently, $\operatorname{Im}\Phi(x) = -\operatorname{Im}\Phi(-x)$. If $c = i$, then $i(\Phi(x) - \Phi(-x))$ is real, and consequently, $\operatorname{Re}\Phi(x) = \operatorname{Re}\Phi(-x)$. Therefore, $\overline{\Phi(x)} = \Phi(-x)$ for $x \in G$.

If we choose c so that $c\Phi(x) = -|\Phi(x)|$, then (3.2) becomes $2\Phi(0) - 2|\Phi(x)| \geq 0$, and hence $|\Phi(x)| \leq \Phi(0)$ for $x \in G$.

To prove property (c), we suppose that $\Phi(x) \neq \Phi(y)$. Then for $\lambda \in \mathbb{R}$, we take $n = 3$, $x_1 = 0$, $x_2 = x$, $x_3 = y$, $c_1 = 1$, $c_2 = \frac{\lambda|\Phi(x) - \Phi(y)|}{\Phi(x) - \Phi(y)}$, and $c_3 = -c_2$ in (3.1) to obtain the following inequality

$$\Phi(0)\ (1+2\lambda^2) + 2\lambda|\Phi(x) - \Phi(y)| - 2\lambda^2 \operatorname{Re}\Phi(x-y) \geq 0 ,$$

or equivalently,

$$2\lambda^2(\Phi(0) - \operatorname{Re}\Phi(x-y)) + 2\lambda|\Phi(x) - \Phi(y)| + \Phi(0) \geq 0. \qquad (3.3)$$

Therefore, the quadratic polynomial in λ on the left hand side of inequality (3.3) has discriminant

$$4|\Phi(x) - \Phi(y)|^2 - 8\,\Phi(0)(\Phi(0) - \operatorname{Re}\Phi(x-y)) \leq 0,$$

and consequently,

$$|\Phi(x) - \Phi(y)|^2 \leq 2\Phi(0)(\Phi(0) - \operatorname{Re}\Phi(x-y)) \qquad \text{for } x, y \in G.$$

We mention the following result which is an immediate consequence of Proposition 3.1.

Proposition 3.2.

Let $\Phi \in P(G)$. Then $\Phi(0) \geq 0$, Φ is bounded and $\sup_{x \in G} |\Phi(x)| = \Phi(0)$. If Φ is continuous at 0, then Φ is uniformly continuous on G.

It is possible to prove the following properties of the sets P(G) and CP(G).

Proposition 3.3.

(a) Both P(G) and CP(G) are convex cones.

(b) If $\Phi \in P(G)$, then $\overline{\Phi} \in P(G)$ and $\mathrm{Re}\,\Phi \in P(G)$.

(c) The nonnegative constant functions are elements of CP(G).

Example 3.1.

Every character γ on G is positive definite. This follows from the fact that

$$\sum_{i,j=1}^{n} (x_i - x_j, \gamma) c_i \overline{c_j} = \sum_{i,j=1}^{n} (x_i, \gamma)(-x_j, \gamma) c_i \overline{c_j}$$

$$= \sum_{i,j=1}^{n} (x_i, \gamma) \overline{(x_j, \gamma)} c_i \overline{c_j}$$

$$= \left| \sum_{i=1}^{n} (x_i, \gamma) c_i \right|^2 \geq 0.$$

We will now state the characterization theorem for continuous positive definite functions on G. The proof can be found in Rudin ([37], pp. 19-21).

Theorem 3.3. (Bochner's Theorem).

A continuous function Φ on G is positive definite iff there exists a mass distribution $\mu \in \mathcal{M}_b(\Gamma)$ such that

$$\Phi(x) = \int_\Gamma (x,\gamma)d\mu(\gamma) \qquad \text{for } x \in G. \tag{3.4}$$

Moreover, the mass distribution μ in the integral representation (3.4) of a continuous positive definite function Φ is unique.

Let $\Phi \in CP(G)$ and let μ be the bounded mass distribution on Γ which corresponds to Φ by equation (3.4). From Proposition 3.1(a), we obtain the result that

$$\Phi(x) = \int_\Gamma (x,\gamma)d\mu(\gamma) = \int_\Gamma \overline{(x,\gamma)}d\check{\mu}(\gamma) = (\check{\mu})^\wedge(x) \quad \text{for } x \in G,$$

and hence Φ has a unique representation as the Fourier-Stieltjes transform of a bounded mass distribution on Γ.

§3. Negative Definite Functions

The results in this section can be found in Berg and Forst ([1], pp. 39-48).

Let G be a locally compact abelian group with dual group Γ.

Definition 3.6.

A complex valued function Ψ defined on Γ is called negative definite iff the inequality

$$\sum_{i,j=1}^{n} \left(\Psi(\gamma_i) + \overline{\Psi(\gamma_j)} - \Psi(\gamma_i - \gamma_j)\right) c_i \overline{c_j} \geq 0 \tag{3.5}$$

holds for every $n \in \mathbb{N}$, for every $\gamma_1, \ldots \gamma_n \in \Gamma$, and for every $c_1, \ldots c_n \in \mathbb{C}$.

We denote by $N(\Gamma)$ the set of all negative definite functions on Γ and by $CN(\Gamma)$ the set of all continuous negative definite functions on Γ.

Proposition 3.4.

If $\Psi \in N(\Gamma)$, then the following three relations hold:

(a) $\Psi(0) \geq 0$;

(b) $\Psi(-\gamma) = \overline{\Psi(\gamma)}$ for $\gamma \in \Gamma$;

(c) $\mathrm{Re}\ \Psi(\gamma) \geq \Psi(0)$ for $\gamma \in \Gamma$.

Proof.

To prove property (a), we take $n = 1$, $\gamma_1 = 0$, and $c_1 = 1$ in (3.5) to obtain the inequality $\Psi(0) \geq 0$.

To prove properties (b) and (c), we take $n = 2$, $\gamma_1 = \gamma$, $\gamma_2 = 0$, $c_1 = 1$, and $c_2 = c$ in (3.4) to obtain the inequality

$$(|c|^2 + \text{Re } c - 1)\Psi(0) + \text{Re } \Psi(\gamma) + c(\overline{\Psi(\gamma)} - \Psi(-\gamma) \geq 0. \tag{3.6}$$

If $c = 1$, then $\overline{\Psi(\gamma)} - \Psi(-\gamma)$ is real, and consequently, $\text{Re } \overline{\Psi(\gamma)} = \text{Im } \Psi(-\gamma)$. If $c = i$, then $i(\overline{\Psi(\gamma)} - \Psi(-\gamma))$ is real, and consequently, $\text{Re } \overline{\Psi(\gamma)} = \text{Re } \Psi(-\gamma)$. Therefore, $\overline{\Psi(\gamma)} = \Psi(-\gamma)$ for $\gamma \in \Gamma$.

If $c = 0$, then inequality (3.6) implies that $\text{Re } \Psi(\gamma) \geq \Psi(0)$ for $\gamma \in \Gamma$.

We mention the following properties of the sets $N(\Gamma)$ and $CN(\Gamma)$.

<u>Proposition 3.5.</u>

(a) Both $N(\Gamma)$ and $CN(\Gamma)$ are convex cones.

(b) If $\Psi \in N(\Gamma)$, then $\overline{\Psi} \in N(\Gamma)$ and $\text{Re } \Psi \in N(\Gamma)$.

(c) The nonnegative constant functions are elements of $CN(\Gamma)$.

<u>Proposition 3.6.</u>

A complex valued function Ψ defined on Γ is negative definite iff the following three conditions are satisfied:

(a) $\Psi(0) \geq 0$;

(b) $\Psi(-\gamma) = \overline{\Psi(\gamma)}$ for $\gamma \in \Gamma$;

(c) For every $n \in \mathbb{N}$, for every $\gamma_1, \ldots, \gamma_n \in \Gamma$, and for every $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{i=1}^{n} c_i = 0 \text{ implies that } \sum_{i,j=1}^{n} \Psi(\gamma_i - \gamma_j) c_i \overline{c_j} \leq 0.$$

Proof.

Suppose that $\Psi \in N(\Gamma)$. Then conditions (a) and (b) are satisfied (c.f. Proposition 3.4). In order to show that condition (c) is satisfied, let $n \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_n \in \Gamma$, and $c_1, \ldots, c_n \in \mathbb{C}$ such that $\sum_{i=1}^{n} c_i = 0$. Then

$$0 \le \sum_{i,j=1}^{n} (\Psi(\gamma_i) + \overline{\Psi(\gamma_j)} - \Psi(\gamma_i - \gamma_j))c_i\overline{c_j}$$

$$= \sum_{j=1}^{n} \overline{c_j}\left(\sum_{i=1}^{n} \Psi(\gamma_i)c_i\right) + \sum_{i=1}^{n} c_i\left(\sum_{j=1}^{n} \overline{\Psi(\gamma_j)}\ \overline{c_j}\right) - \sum_{i,j=1}^{n} \Psi(\gamma_i - \gamma_j)c_i\overline{c_j}$$

$$= -\sum_{i,j=1}^{n} \Psi(\gamma_i - \gamma_j)c_i\overline{c_j},$$

and consequently,

$$\sum_{i,j=1}^{n} \Psi(\gamma_i - \gamma_j)c_i\overline{c_j} \le 0.$$

Conversely, suppose that Ψ is a complex valued function defined on Γ which satisfies conditions (a), (b), and (c). Let $n \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_n \in \Gamma$, and $c_1, \ldots, c_n \in \mathbb{C}$. Set $c = -\sum_{i=1}^{n} c_i$. Then by applying condition (c) to $0, \gamma_1, \ldots, \gamma_n \in \Gamma$ and $c, c_1, \ldots, c_n \in \mathbb{C}$, we obtain the following inequality

$$\Psi(0)|c|^2 + \sum_{i=1}^{n} \Psi(\gamma_i)c_i\overline{c} + \sum_{j=1}^{n} \Psi(-\gamma_j)c\,\overline{c_j} + \sum_{i,j=1}^{n} \Psi(\gamma_i - \gamma_j)c_i\overline{c_j} \le 0,$$

and hence by conditions (a) and (b),

$$0 \le \Psi(0)|c|^2 \le (-\bar{c})\sum_{i=1}^{n}\Psi(\gamma_i)c_i + (-c)\sum_{j=1}^{n}\overline{\Psi(\gamma_j)}\bar{c}_j - \sum_{i,j=1}^{n}\Psi(\gamma_i-\gamma_j)c_i\overline{c_j}$$

$$= \sum_{i,j=1}^{n}(\Psi(\gamma_i) + \overline{\Psi(\gamma_j)} - \Psi(\gamma_i-\gamma_j))c_i\overline{c_j} .$$

Proposition 3.7.

Let $\Phi \in P(\Gamma)$ and let c be a nonnegative real number. Then the complex valued function Ψ defined on Γ, which is given by

$$\Psi(\gamma) = \Phi(0) - \Phi(\gamma) + c \qquad \text{for } \gamma \in \Gamma,$$

is negative definite.

Proof.

It is sufficient to show that Ψ satisfies conditions (a), (b) and (c) of Proposition 3.6. It is clear (c.f. Proposition 3.1) that Ψ satisfies (a) and (b). Let $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $c_1, \ldots, c_n \in \mathbb{C}$ such that $\sum_{i=1}^{n} c_i = 0$. Then

$$\sum_{i,j=1}^{n}(\Phi(0) - \Phi(\gamma_i-\gamma_j) + c)c_i\overline{c_j} = -\sum_{i,j=1}^{n}\Phi(\gamma_i-\gamma_j)c_i\overline{c_j} \le 0,$$

since $\Phi \in P(\Gamma)$. Therefore, Ψ satisfies (c).

We conclude this section by stating the result which establishes the connection between $N(\Gamma)$ and $P(\Gamma)$. We refer the reader to Berg and Forst ([1], p. 41) for the proof.

Theorem 3.4. (Schoenberg's Theorem).

A complex valued function Ψ defined on Γ is negative definite iff the following two conditions are satisfied:

(a) $\Psi(0) \geq 0$;

(b) The complex valued function Φ_t defined on Γ, which is given by

$$\Phi_t(\gamma) = \exp(-t\Psi(\gamma)) \qquad \text{for } \gamma \in \Gamma,$$

is an element of $P(\Gamma)$ for each $t > 0$.

§4. Convolution Semigroups

We refer to Berg and Forst ([1], pp. 48-61) for the results presented in this section.

Let G be a locally compact abelian group with dual group Γ.

Definition 3.7.

A family $\{\mu_t\}_{t>0}$ of bounded mass distributions on G is called a (vaguely continuous) sub-Markov convolution semigroup on G iff

(a)	$\mu_t(G) \leq 1$	for each $t > 0$;
(b)	$\mu_t * \mu_s = \mu_{t+s}$	for $t, s > 0$;
(c)	$\mu_t \to \varepsilon$	as $t \to 0$.

If condition (a) is replaced by the condition

(a')	$\mu_t(G) = 1$	for each $t > 0$,

then $\{\mu_t\}_{t>0}$ is called a (vaguely continuous) Markov convolution semigroup on G.

Remark 3.1.

The terms "sub-Markov" and "Markov" used in Definition 3.7 are taken from Meyer ([32], p. 173).

The concept of Markov convolution semigroups is closely related to the notion of Markov processes with translation invariant transition functions on a second countable locally compact abelian group. We refer to the book of Blumenthal and Getoor ([2], Chapter 1) for more information on these processes.

Let G be a second countable locally compact abelian group with Haar measure dx and let $\mathcal{B}$ denote the set of all Borel subsets of G.

Consider a Markov convolution semigroup $\{\mu_t\}_{t>0}$ on G. Then we define

$$P_t f(x) = P_t(x,f) = (\check{\mu}_t * f)(x) = \int_G f(x+y)d\mu_t(y) \qquad (3.7)$$

for each $t > 0$, $x \in G$, and $f \in C_c(G)$. If $B \in \mathcal{B}$, then we write

$$P_t(x,B) = (\check{\mu}_t * \chi_B)(x) = \mu_t(B-x)$$

in place of $P_t(x, \chi_B)$, where $B-x = \{y | x+y \in B\}$. For $t > 0$, $s > 0$, $x \in G$, and $f \in C_c(G)$, we obtain the result that

$$\begin{aligned} P_{t+s}(x,f) &= (\check{\mu}_{t+s} * f)(x) \\ &= (\check{\mu}_t * \check{\mu}_s * f)(x) \\ &= \iint_G f(x+y+z)\, d\mu_s(z)\, d\mu_t(y) \\ &= \int_G P_s(x+y, f)\, d\mu_t(y) \\ &= P_t(x, P_s f) \\ &= P_t P_s f(x) \\ &= (P_t P_s)(x,f), \end{aligned}$$

and hence for $B \in \mathcal{B}$,

$$P_{t+s}(x,B) = P_t P_s(x,B)$$
$$= \int_G P_t(x,dy)P_s(y,B) .$$

Therefore, P_t is a (temporally homogeneous) Markov transition function on the measurable space $(G,\mathcal{B})$; that is,

(a) $P_t(x,B)$, considered as a function of $B \in \mathcal{B}$, is a probability measure on $\mathcal{B}$ for each $t > 0$ and $x \in G$;

(b) $P_t(x,B)$, considered as a function of $x \in G$, is measurable with respect to $\mathcal{B}$ for each $t > 0$ and $B \in \mathcal{B}$;

(c) $P_{t+s}(x,B) = P_t P_s(x,B) = \int_G P_t(x,dy)P_s(y,B)$ for each $t > 0$, $s > 0$, $x \in G$, and $B \in \mathcal{B}$.

The semigroup property (c) is called the Chapman-Kolmogorov equation.

If we set $P_o(x,B) = \varepsilon_x(B)$ for each $x \in G$ and $B \in \mathcal{B}$, then there exists a Markov process $X = \{X_t\}_{t \geq 0}$ whose transition function is given by P_t (c.f. Blumenthal and Getoor ([2], p. 46)).

We mention that the Markov transition function P_t is translation invariant; that is, $P_t(x+y,B+y) = P_t(x,B)$ for each $x,y \in G$, and $B \in \mathcal{B}$.

Therefore, equation (3.7) establishes a one-to-one correspondence between translation invariant Markov transition functions on G and Markov convolution semigroups on G.

We refer to Berg and Forst ([1], p. 48) for the next result.

Proposition 3.8.

Let $\{\mu_t\}_{t>0}$ be a sub-Markov convolution semigroup on G and set $\mu_o = \epsilon$. Then for $t \geq 0$ and $t_o \geq 0$,

$$\mu_t \dot{\to} \mu_{t_o} \qquad \text{as } t \to t_o.$$

We will now state the theorem which connects sub-Markov convolution semigroups and continuous negative definite functions. We refer to Berg and Forst ([1], p. 49) for the proof.

Theorem 3.5.

If $\{\mu_t\}_{t>0}$ is a sub-Markov convolution semigroup on G, then there exists a function $\Psi \in CN(\Gamma)$ such that

$$\hat{\mu}_t(\gamma) = \exp(-t\Psi(\gamma)) \tag{3.8}$$

for each $t > 0$ and $\gamma \in \Gamma$. Conversely, if $\Psi \in CN(\Gamma)$, then equation (3.8) determines a sub-Markov convolution semigroup $\{\mu_t\}_{t>0}$ on G.

Definition 3.8.

If the sub-Markov convolution semigroup $\{\mu_t\}_{t>0}$ on G and $\Psi \in CN(\Gamma)$ correspond to each other by equation (3.8), then $-\Psi$ is called the <u>infinitesimal generator for the semigroup $\{\hat{\mu}_t\}_{t>0}$</u>.

Corollary 3.5.1.

Let $\{\mu_t\}_{t>0}$ be a sub-Markov convolution semigroup on G and $-\Psi$ the infinitesimal generator for the semigroup $\{\hat{\mu}_t\}_{t>0}$. Then

$$\mu_t(G) = \exp(-t\Psi(0)) \qquad \text{for each } t > 0.$$

Moreover, if $\Psi(0) = 0$, then $\{\mu_t\}_{t>0}$ is a Markov convolution semigroup on G.

Proof.

We obtain the result that

$$\exp(-t\Psi(0)) = \hat{\mu}_t(0) = \int_G \overline{(x,0)}\, d\mu_t(x) = \mu_t(G)$$

by Theorem 3.5.

Let $\{\mu_t\}_{t>0}$ be a sub-Markov convolution semigroup on G and $-\Psi$ the infinitesimal generator for the semigroup $\{\hat{\mu}_t\}_{t>0}$. It is clear that $\{\check{\mu}_t\}_{t>0}$ is also a sub-Markov convolution semigroup on G. By Proposition 3.4(b), the Fourier transform of $\check{\mu}_t$ is given by

$$\begin{aligned}
(\check{\mu}_t)\hat{}(\gamma) &= \check{\mu}_t(\overline{\gamma}) \\
&= \mu_t(\check{\overline{\gamma}}) \\
&= \mu_t(\overline{-\gamma}) \\
&= \hat{\mu}_t(-\gamma) \\
&= \exp(-t\Psi(-\gamma)) \\
&= \exp(-t\overline{\Psi(\gamma)})
\end{aligned}$$

for each $t > 0$ and $\gamma \in \Gamma$, and consequently, $-\overline{\Psi}$ is the infinitesimal generator for the semigroup $\{(\check{\mu}_t)\hat{}\}_{t>0}$.

Definition 3.9.

A sub-Markov convolution semigroup $\{\mu_t\}_{t>0}$ on G is called symmetric iff $\mu_t \equiv \check{\mu}_t$ for every $t > 0$.

It follows that a sub-Markov convolution semigroup $\{\mu_t\}_{t>0}$ on G is symmetric iff the infinitesimal generator for $\{\hat{\mu}_t\}_{t>0}$ is real valued.

Definition 3.10.

Let $\{\mu_t\}_{t>0}$ be a sub-Markov convolution semigroup on G. For $\lambda > 0$, we define the mass distribution ρ_λ on G by

$$\rho_\lambda(f) = \int_0^\infty \exp(-\lambda t)\mu_t(f)dt \qquad \text{for } f \in C_c(G).$$

The family $\{\rho_\lambda\}_{\lambda>0}$ is called the resolvent for $\{\mu_t\}_{t>0}$.

Let $\{\mu_t\}_{t>0}$ be a sub-Markov convolution semigroup on G and $-\Psi$ the infinitesimal generator for $\{\hat{\mu}_t\}_{t>0}$. Then the total mass of ρ_λ is given by

$$\begin{aligned}\rho_\lambda(G) &= \int_0^\infty \exp(-\lambda t)\mu_t(G)dt \\ &= \int_0^\infty \exp(-t(\lambda+\Psi(0)))dt \\ &= \frac{1}{\lambda+\Psi(0)} \\ &\leq \frac{1}{\lambda}\end{aligned}$$

by Corollary 3.5.1 and Proposition 3.4(a), and consequently, $\rho_\lambda \in \mathcal{M}_b(G)$. Furthermore, the Fourier-Stieltjes transform of ρ_λ is given by

$$\hat{\rho}_\lambda(\gamma) = \int_G \overline{(x,\gamma)}\,d\rho_\lambda(x)$$

$$= \int_0^\infty \exp(-\lambda t)\mu_t(\overline{\gamma})dt$$

$$= \int_0^\infty \exp(-\lambda t)\hat{\mu}_t(\gamma)dt$$

$$= \int_0^\infty \exp(-t(\lambda+\Psi(\gamma)))dt$$

$$= \frac{1}{\lambda+\Psi(\gamma)}$$

for $\gamma \in \Gamma$. Therefore,

$$\hat{\rho}_\lambda - \hat{\rho}_\beta = \frac{1}{\lambda+\Psi} - \frac{1}{\beta+\Psi}$$

$$= (\beta-\lambda)\,\frac{1}{\lambda+\Psi}\cdot\frac{1}{\beta+\Psi}$$

$$= (\beta-\lambda)\hat{\rho}_\lambda\cdot\hat{\rho}_\beta,$$

and as a result,

$$\rho_\lambda - \rho_\beta = (\beta-\lambda)\rho_\lambda * \rho_\beta \tag{3.9}$$

for $\lambda > 0$ and $\beta > 0$, by Theorem 3.1(c). Formula (3.9) is called the <u>resolvent equation</u>.

§5. Completely Monotone Functions and Bernstein Functions

We will now introduce the notions of completely monotone functions and Bernstein functions as presented in Berg and Forst ([1], pp. 61-72). We will observe that there is an analogy between the set of all completely monotone functions and the set of all continuous positive definite functions on $\mathbb{R}$ and an analogy between the set of all Bernstein functions and the set of all continuous negative definite functions on $\mathbb{R}$.

Definition 3.11.

An infinitely differentiable function $f:(o,\infty) \to \mathbb{R}$ is defined to be completely monotone iff $f \geq 0$ and $(-1)^n D^n f \geq 0$ for all $n \in \mathbb{N}$.

Proposition 3.9.

(a) A completely monotone function is decreasing and convex.

(b) The set of all completely monotone functions is a convex cone.

(c) The nonnegative constant functions are completely monotone.

(d) The product of two completely monotone functions is completely monotone.

Proof.

Properties (a), (b), and (c) follow immediately from Definition 3.11. Property (d) follows from the Liebniz formula $D^n(f\cdot g) = \sum_{k=o}^{n} \binom{n}{k} D^k f \cdot D^{n-k} g$.

Definition 3.12.

An infinitely differentiable function $f:(o,\infty) \to \mathbb{R}$ is defined to be a Bernstein function iff $f \geq 0$ and $(-1)^n D^n f \leq 0$ for all $n \in \mathbb{N}$.

The next proposition follows directly from Definition 3.12.

Proposition 3.10.

(a) A Bernstein function is increasing and concave.

(b) The set of all Bernstein functions is a convex cone.

(c) The nonnegative constant functions are Bernstein functions.

We will state a theorem analogous to Schoenberg's theorem. The proof can be found in Berg and Forst ([1], p. 62).

Theorem 3.6.

A function $f:(0,\infty) \to \mathbb{R}$ is a Bernstein function iff the following two conditions are satisfied:

(a) $f \geq 0$;

(b) The function $\exp(-tf)$ is completely monotone for each $t > 0$.

The following theorem expresses the integral representation of completely monotone functions. We refer to Meyer ([32], p. 237) for the proof.

Theorem 3.7. (Bernstein).

A function $f:(o,\infty) \to \mathbb{R}$ is completely monotone iff there exists a mass distribution μ on $[o,\infty)$ such that

$$f(x) = \int_0^\infty \exp(-xt)d\mu(t) \qquad \text{for } x \in (o,\infty). \tag{3.10}$$

Moreover, the mass distribution μ in the integral representation (3.10) of a completely monotone function f is unique.

Definition 3.13.

A mass distribution μ on $[o,\infty)$ is called the representing mass distribution for a completely monotone function f iff μ and f correspond to each other by equation (3.10).

Remark 3.2.

Let f be a completely monotone function with representing mass distribution μ. Since f is a decreasing function (c.f. Proposition 3.9(a)), $\lim\limits_{x\to o+} f(x)$ exists (although it may equal $+\infty$), and

$$\lim_{x\to o+} f(x) = \lim_{x\to o+} \int_0^\infty \exp(-xt)d\mu(t) = \mu([o,\infty)) \leq +\infty,$$

by the Lebesgue monotone convergence theorem. Therefore, $\lim\limits_{x\to o+} f(x) < +\infty$ iff μ is a bounded mass distribution.

The representation theorem for Bernstein functions follows from Bernstein's theorem. We refer to Berg and Forst ([1], p. 64) for the proof.

Theorem 3.8.

A function $f:(o,\infty) \to \mathbb{R}$ is a Bernstein function iff there exists nonnegative constants a and b, and a mass distribution μ on $[o,\infty)$ satisfying

$$\int_0^\infty \frac{t}{1+t}\, d\mu(t) < \infty ,$$

such that

$$f(x) = a + bx + \int_0^\infty (1-\exp(-xt))d\mu(t) \quad \text{for } x \in (o,\infty). \qquad (3.11)$$

Moreover, the triple (a, b, μ) in the representation (3.11) of a Bernstein function f is unique.

We will now explain the analogy between the set of all completely monotone functions and the set of all continuous negative definite functions.

Definition 3.14.

The Laplace integral of a charge distribution ν on $[o,\infty)$ is defined to be the Lebesgue-Stieltjes integral

$$\int_0^\infty \exp(-xt)d\nu(t) \qquad \text{for } x \in (o,\infty),$$

if the integral converges.

The Laplace integral defines a transformation in the following way.

Definition 3.15.

If ν is a charge distribution on $[o,\infty)$ for which

$$\int_0^\infty \exp(-xt)d|\nu|(t) < \infty \qquad \text{for every } x \in (o,\infty),$$

then the function $\mathcal{L}\nu$ defined by

$$\mathcal{L}\nu(x) = \int_0^\infty \exp(-xt)d\nu(t) \qquad \text{for } x \in (o,\infty),$$

is called the Laplace transform of ν.

Let ν be a charge distribution on $[o,\infty)$ for which

$$\int_0^\infty \exp(-xt)d|\nu|(t) < \infty \qquad \text{for every } x \in (o,\infty).$$

Since $|\exp(-zt)| = \exp(-t(\text{Re } z))$ for each $z \in \mathbb{C}$ and $t \in [o,\infty)$, the integral

$$\int_0^\infty \exp(-zt)d\nu(t)$$

is absolutely convergent for Re $z > 0$. Therefore, the Laplace transform $\mathcal{L}\nu$ extends to an analytic function in the open half-plane Re $z > 0$, which is defined by

$$\mathcal{L}\nu(z) = \int_0^\infty \exp(-zt)d\nu(t). \qquad \text{for Re } z > 0 \tag{3.12}$$

(see Doetsch ([16], p. 15)). If in addition ν is bounded, then the Laplace transform $\mathcal{L}\nu$ extends to a continuous function in the closed half-plane Re $z \geq 0$, which is also defined by (3.12) (see Doetsch ([16], p. 13)).

By (3.12), we obtain the following result

$$\mathcal{L}\nu(x+iy) = \int_0^\infty \exp(-(x+iy)t)d\nu(t)$$

$$= \int_0^\infty \exp(-iyt)\exp(-xt)d\nu(t)$$

$$= \hat{\mu}(y), \tag{3.13}$$

for $x \in (o,\infty)$ and $y \in R$, where μ is the charge distribution having density function $\exp(-xt)$ with respect to ν. If ν is bounded, then (3.13) also holds for $x = 0$.

As an immediate consequence of (3.13) and Theorem 3.1(c), we obtain the result that the Laplace tranform is an injective mapping; that is, if $\mathcal{L}\nu(x) = 0$ for every $x \in (o,\infty)$, then $\nu \equiv 0$. Therefore, we obtain the following corollary to Bernstein's theorem.

Corollary 3.7.1.

The Laplace transform $\mathcal{L}$ is a bijection from the set of all mass distributions μ on $[o,\infty)$ for which

$$\int_0^\infty \exp(-xt)d\mu(t) < \infty \qquad \text{for every } x \in (o,\infty)$$

onto the set of all completely monotone functions. Moreover, the Laplace transform $\mathcal{L}$ is a bijection from the set of all bounded mass distributions on $[o,\infty)$ onto the set of all completely monotone functions f which are bounded.

Therefore, the analogy between the set of all completely monotone functions and the set of all continuous positive definite functions on $\mathbb{R}$ is explained by Corollary 3.7.1 and Bochner's theorem together with identity (3.13). That is, a bounded completely monotone function f has a unique representation as the Laplace transform of a bounded mass distribution on $[o,\infty)$, whereas a continuous positive definite function Φ defined on $\mathbb{R}$ has a unique representation as the Fourier-Stieltjes transform of a bounded mass distribution μ with $S(\mu) \subset [o,\infty)$, and hence

$$\Phi(x) = \hat{\mu}(x) = \mathcal{L}\mu(ix) \qquad \text{for } x \in \mathbb{R}. \tag{3.14}$$

In a similar way, the analogy between the set of all Bernstein functions and the set of all continuous negative definite functions on $\mathbb{R}$ is explained by the next result and Theorem 3.5 together with identity (3.13).

<u>Definition 3.16.</u>

A sub-Markov convolution semigroup $\{\eta_t\}_{t>0}$ on $\mathbb{R}$ is defined to be <u>supported by $[o,\infty)$</u> iff the support $S(\eta_t) \subseteq [o,\infty)$ for every $t > 0$.

<u>Theorem 3.9.</u>

If $\{\eta_t\}_{t>0}$ is a sub-Markov convolution semigroup on $\mathbb{R}$ supported by $[o,\infty)$, then there exists a Bernstein function $f:(o,\infty) \to \mathbb{R}$ such that

$$\mathcal{L}\eta_t(x) = \exp(-tf(x)) \tag{3.15}$$

for each $t > 0$ and $x \in (o,\infty)$. Conversely, if $f:(o,\infty) \to \mathbb{R}$ is a Bernstein function, then equation (3.15) determines a sub-Markov convolution semigroup $\{\eta_t\}_{t>0}$ on $\mathbb{R}$ supported by $[o,\infty)$.

Definition 3.17.

If the sub-Markov convolution semigroup $\{\eta_t\}_{t>0}$ on $\mathbb{R}$ supported by $[o,\infty)$ and the Bernstein function f correspond to each other by equation (3.15), then -f is called the infinitesimal generator for the semigroup $\{\mathcal{L}\eta_t\}_{t>0}$.

Remark 3.3.

Let $f:(o,\infty) \to \mathbb{R}$ be a Bernstein function. Then by Theorem 3.8, there exists nonnegative constants a and b, and a mass distribution μ on $[o,\infty)$ satisfying

$$\int_0^\infty \frac{t}{1+t}\,d\mu(t) < \infty\ , \tag{3.16}$$

such that

$$f(x) = a + bx + \int_0^\infty (1-\exp(-xt))\,d\mu(t) \text{ for } x \in (o,\infty). \tag{3.17}$$

It is clear that condition (3.16) is satisfied iff

$$\int_0^1 t\ d\mu(t) < \infty \text{ and } \int_1^\infty d\mu(t) < \infty. \tag{3.18}$$

For $t \in (o,\infty)$ and $z \in \mathbb{C}$ such that $\mathrm{Re}\ z \geq 0$, we obtain the inequalities

$$|1-\exp(-zt)| \leq |z|t$$

and

$$|1-\exp(-zt)| \leq 1 + \exp(-t(\mathrm{Re}\ z)) \leq 2.$$

It follows (c.f. (3.18)) that

$$\int_0^\infty |1-\exp(-zt)|d\mu(t) < |z| \int_0^1 t\, d\mu(t) + 2 \int_1^\infty d\mu(t) < \infty$$

for Re $z \geq 0$. Therefore, the representation (3.17) of f extends to the function

$$f(z) = a + bz + \int_0^\infty (1-\exp(-zt))d\mu(t) \text{ for Re } z \geq 0, \quad (3.19)$$

which is continuous on the closed half-plane Re $z \geq 0$ and analytic on the open half-plane Re $z > 0$.

Let $\{\eta_t\}_{t>0}$ be the sub-Markov convolution semigroup on $\mathbb{R}$ supported by $[0,\infty)$ for which $-f$ is the infinitesimal generator for the semigroup $\{\mathcal{L}\eta_t\}_{t>0}$. Let G be a locally compact abelian group with dual group Γ, $\{\mu_t\}_{t>0}$ a sub-Markov convolution semigroup on G, and $-\Psi$ the infinitesimal generator for $\{\hat{\mu}_t\}_{t>0}$. By Proposition 3.4, Re $\Psi(\gamma) \geq 0$ for every $\gamma \in \Gamma$, and consequently by (3.19),

$$f(\Psi(\gamma)) = a + b\Psi(\gamma) + \int_0^\infty (1-\exp(-\Psi(\gamma)t)d\mu(t) \quad \text{for } \gamma \in \Gamma. \ (3.20)$$

We obtain from Theorem 3.4 the result that the complex valued function Φ_t defined on Γ, which is given by

$$\Phi_t(\gamma) = \exp(-\Psi(\gamma)t) \qquad \text{for } \gamma \in \Gamma,$$

is positive definite for every $t > 0$. It follows (c.f. Propositions 3.4 and 3.7) that the function $1 - \Phi_t$ is negative definite for every $t > 0$. From formula (3.20), we obtain the following equation

$$f(\Psi(\gamma)) = a + b\Psi(\gamma) + \int_0^\infty (1-\Phi_t(\gamma))d\mu(t) \quad \text{for } \gamma \in \Gamma,$$

and consequently by Proposition 3.4, $f \circ \Psi \in CN(\Gamma)$. We denote by $\{\mu_t^f\}_{t>0}$ the sub-Markov convolution semigroup on G for which $-(f \circ \Psi)$ is the infinitesimal generator for the semigroup $\{\hat{\mu}_t^f\}_{t>0}$. This leads us to the next definition.

Definition 3.18.

The sub-Markov convolution semigroup $\{\mu_t^f\}_{t>0}$ on G defined in Remark 3.3 is called the sub-Markov convolution semigroup subordinated to $\{\mu_t\}_{t>0}$ by means of $\{\eta_t\}_{t>0}$.

Proposition 3.11.

Let $\{\mu_t^f\}_{t>0}$ be the sub-Markov convolution semigroup on G subordinated to $\{\mu_t\}_{t>0}$ by means of $\{\eta_t\}_{t>0}$. Then for each $t > 0$, μ_t^f is given by the vague integral

$$\mu_t^f = \int_0^\infty \mu_s \, d\eta_t(s). \tag{3.21}$$

Proof.

For $t > 0$, define

$$\nu_t(g) = \int_0^\infty \mu_s(g) \, d\eta_t(s) \qquad \text{for } g \in C_c(G) \tag{3.22}$$

Then ν_t is a nonnegative linear functional on $C_c(G)$, and consequently, ν_t is a mass distribution on G. Now,

$$\nu_t(G) = \int_0^\infty \mu_s(G) \, d\eta_t(s) \le \eta_t([0,\infty)) \le 1 ,$$

and the Fourier transform of ν_t is given by

$$\begin{aligned}\hat{\nu}_t(\gamma) &= \int_0^\infty \hat{\mu}_s(\gamma)\, d\eta_t(s)\\ &= \int_0^\infty \exp(-s\Psi(\gamma))\, d\eta_t(s)\\ &= \mathcal{L}\eta_t(\Psi(\gamma))\\ &= \exp(-tf(\Psi(\gamma))\\ &= \hat{\mu}_t^f(\gamma)\end{aligned}$$

for $\gamma \in \Gamma$. Therefore, $\nu_t \equiv \mu_t^f$ by Theorem 3.1(c), and consequently by (3.22), μ_t^f is given by the vague integral (3.21).

§6. Examples of Markov Convolution Semigroups on $\mathbb{R}^p (p \geq 1)$

In this section, we shall present some examples of continuous negative definite functions on $\mathbb{R}^p (p \geq 1)$ and Bernstein functions together with the corresponding Markov convolution semigroups. We refer to Meyer ([33], pp. 66-69), and Berg and Forst ([1], pp. 72-74) for the examples given.

We will begin with a remark due to Feller ([18], p. 76).

Remark 3.4.

The solution of the initial value problem for the classical diffusion equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) \qquad u(0,x) = f(x)$$

is given by

$$u(t,x) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} f(y) \exp\left(- \frac{(x-y)^2}{4t}\right) dy$$

$$= f * g_t(x),$$

where

$$g_t(x) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \exp\left(- \frac{x^2}{4t}\right)$$

is the normal density function with respect to Lebesgue measure dx on $\mathbb{R}$, $f \in C(\mathbb{R})$, $t > 0$, and $x \in \mathbb{R}$. Since the Fourier transform of the density function g_t is given by

$$\hat{g}_t(y) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \exp(-ixy)\exp\left(-\frac{x^2}{4t}\right)dx$$

$$= \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x^2}{4t} + ixy\right)\right) dx$$

$$= \frac{1}{\pi^{\frac{1}{2}}} \exp(-ty^2) \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x}{2t^{\frac{1}{2}}} + i\, t^{\frac{1}{2}}y\right)^2\right) dx$$

$$= \exp(-ty^2) \qquad \text{for } y \in \mathbb{R}, \tag{3.23}$$

the normal density is a special case of the stable densities introduced by Paul Lévy. An absolutely continuous mass distribution with respect to Lebesgue measure is defined to be <u>stable</u> iff the Fourier transform of its density function is of the form

$$\Phi(z) = \exp\left(-t|z|^\alpha\left(1 + i\gamma\frac{z}{|z|} \tan\frac{\pi\alpha}{2}\right)\right),$$

where $z, \gamma \in \mathbb{R}$, $|\gamma| \le 1$, $t > 0$, and $0 < \alpha \le 2$. This is equivalent to saying that a density function g is stable iff the convolution of $g(ax)$ and $g(bx)$ is of the form $g(cx)$ for any positive constants a, b, and c.

<u>Example 3.2.</u>

The function $f:(o,\infty) \to \mathbb{R}$, defined by

$$f(x) = x^\alpha \qquad \text{for } x \in (o,\infty),$$

is a Bernstein function for $\alpha \in [o,1]$. It follows from Theorem 3.9 that there exists a sub-Markov convolution semigroup $\{\sigma_t^\alpha\}_{t>0}$ on $\mathbb{R}$ supported by $[o,\infty)$ such that

$$\mathcal{L}(\sigma_t^\alpha)(x) = \exp(-tx^\alpha)$$

for each $t > 0$ and $x \in (0,\infty)$. Therefore, $\{\sigma_t^\alpha\}_{t>0}$ is a symmetric, Markov convolution semigroup on $\mathbb{R}$ (c.f. Corollary 3.5.1 and (3.14)) and is called the <u>one-sided symmetric stable semigroup of order α on $\mathbb{R}$</u> (c.f. Remark 3.4 and (3.14)).

Let G be a locally compact abelian group with dual group Γ, $\{\mu_t\}_{t>0}$ a sub-Markov convolution semigroup on G, and $-\Psi$ the infinitesimal generator for $\{\hat{\mu}_t\}_{t>0}$. Then the sub-Markov convolution semigroup $\{\mu_t^\alpha\}_{t>0}$ on G subordinated to $\{\mu_t\}_{t>0}$ by means of $\{\sigma_t^\alpha\}_{t>0}$ is defined by

$$\mu_t^\alpha = \int_0^\infty \mu_s \, d\sigma_t^\alpha(s) \qquad \text{for } t > 0 \qquad (3.24)$$

(c.f. Proposition 3.11), and $-\Psi^\alpha$ is the infinitesimal generator for $\{\widehat{\mu_t^\alpha}\}_{t>0}$.

<u>Example 3.3.</u>

For $t > 0$, let $g_t : \mathbb{R}^p \to \mathbb{R}$ $(p \geq 1)$ be the function defined by

$$g_t(x) = \frac{1}{(4\pi t)^{p/2}} \exp\left(-\frac{|x|^2}{4t}\right) \qquad \text{for } x \in \mathbb{R}^p, \qquad (3.25)$$

and μ_t the mass distribution on $\mathbb{R}^p$ having density function g_t with respect to Lebesgue measure on $\mathbb{R}^p$.

We shall demonstrate that the family $\{\mu_t\}_{t>0}$ of mass distributions is a Markov convolution semigroup on $\mathbb{R}^p$.

The Fourier transform of the density function g_t is given by

$$\hat{g}_t(y) = \int_{\mathbb{R}^p} \exp(-ix\cdot y) \frac{1}{(4\pi t)^{p/2}} \exp\left(-\frac{|x|^2}{4t}\right) dm_p(x)$$

$$= \frac{1}{(4\pi t)^{p/2}} \prod_{j=1}^{p} \int_{-\infty}^{+\infty} \exp\left(-\left(\frac{x_j^2}{4t} + ix_j y_j\right)\right) dx_j$$

$$= \exp(-t|y|^2) \qquad \text{for } y \in \mathbb{R}^p, \qquad (3.26)$$

(c.f. (3.23)), where $x = (x_1, \ldots, x_p)$, $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$. It follows that

$$\mu_t(\mathbb{R}^p) = 1$$

by Corollary 3.5.1, and

$$\hat{\mu}_t \cdot \hat{\mu}_s = \hat{\mu}_{t+s} \qquad \text{for } t, s > 0,$$

and therefore,

$$\mu_t * \mu_s = \mu_{t+s} \qquad \text{for } t, s > 0$$

by Theorem 3.1. Furthermore,

$$\lim_{t\to 0} \hat{g}_t(y) = 1 \qquad \text{for } y \in \mathbb{R}^p,$$

and hence by the continuity theorem of Paul Lévy (c.f. Berg and Forst [1], p. 17),

$$\mu_t \xrightarrow{\cdot} \varepsilon \qquad \text{as } t \to 0.$$

Therefore, $\{\mu_t\}_{t>0}$ is a Markov convolution semigroup on $\mathbb{R}^p$. Moreover, if we define $\Psi:\mathbb{R}^p \to \mathbb{R}$ by

$$\Psi(y) = |y|^2 \qquad \text{for } y \in \mathbb{R}^p, \tag{3.27}$$

then $-\Psi$ is the infinitesimal generator for $\{\hat{\mu}_t\}_{t>0}$ (c.f. (3.26)).

Let us consider the Markov transition function P_t corresponding to the Markov convolution semigroup $\{\mu_t\}_{t>0}$ by equation (3.7). It follows (c.f. Remark 3.1 and (3.25)) that the Markov process $X = \{X_t\}_{t>0}$ having translation invariant transition function P_t is a process of normalized Brownian motion; that is $X = \{X_t\}_{t>0}$ is a stochastic process satisfying

(a) $X_o = 0$

(b) $X_t = X(t)$ is continuous on (o,∞) for each point in the sample space,

(c) X has independent increments (i.e. for $0 < t_1 < \ldots < t_n$, the random variables $X_{t_1}, X_{t_2}-X_{t_1}, \ldots, X_{t_n}-X_{t_{n-1}}$ are independent), and

(d) X_t-X_s has a normal distribution with mean 0 and variance $t-s$ for $s < t$.

Therefore, it is natural to call the Markov convolution semigroup $\{\mu_t\}_{t>0}$ the <u>Brownian motion semigroup on $\mathbb{R}^p$</u>.

<u>Example 3.4.</u>

We denote by $\{\mu_t^\alpha\}_{t>0}$ the Markov convolution semigroup on $\mathbb{R}^p (p \geq 1)$ subordinated to the Brownian motion semigroup $\{\mu_t\}_{t>0}$ on $\mathbb{R}^p$ by means of the one-sided stable semigroup $\{\sigma_t^{\alpha/2}\}_{t>0}$ of order $\alpha/2$ on $\mathbb{R}$ for $\alpha \in (o,2]$. We obtain from (3.24) the result that

$$\mu_t^{\alpha} = \int_0^{\infty} \mu_s \, d\sigma_t^{\alpha/2}(s) \qquad \text{for } t > 0 \text{ and } \alpha \in (o,2].$$

Furthermore, if Ψ is defined by (3.27), then $-\Psi^{\alpha/2}$ is the infinitesimal generator for $\{\widehat{\mu_t^{\alpha}}\}_{t>0}$, and consequently, the Fourier transform of μ_t^{α} is given by

$$\widehat{\mu_t^{\alpha}}(y) = \exp(-t\Psi^{\alpha/2}(y)) = \exp(-t|y|^{\alpha}) \tag{3.28}$$

for $t > 0$, $\alpha \in (o,2]$, and $y \in \mathbb{R}^p$. Therefore, the Markov convolution semigroup $\{\mu_t^{\alpha}\}_{t>0}$ is symmetric and is called the <u>symmetric stable semigroup of order α on $\mathbb{R}^p$</u> (c.f. (3.28) and Remark 3.4).

We note that the symmetric stable semigroup of order 2 on $\mathbb{R}^p$ is the Brownian motion semigroup on $\mathbb{R}^p$.

§7. Transient Convolution Semigroups and the M. Riesz Kernel

We refer to Berg and Forst ([1], pp. 97- 136) for much of the material presented in this section.

Let G be a locally compact abelian group with dual group Γ. Let $\{\mu_t\}_{t>0}$ be a sub-Markov convolution semigroup on G with resolvent $\{\rho_\lambda\}_{\lambda>0}$. For $f \in C_c^+(G)$, the function $h:(o,\infty) \to \mathbb{R}$, defined by

$$h(\lambda) = \rho_\lambda(f) = \int_0^\infty \exp(-\lambda t)\mu_t(f)dt \quad \text{for } \lambda \in (o,\infty),$$

is decreasing, and hence by the Lebesgue monotone convergence theorem,

$$\lim_{\lambda\to o} \rho_\lambda(f) = \int_0^\infty \mu_t(f)dt \leq + \infty .$$

If $\lim_{\lambda\to o} \rho_\lambda(f) < + \infty$ for every $f \in C_c^+(G)$, then the mapping $\kappa:C_c(G) \to \mathbb{R}$, defined by

$$\kappa(f) = \lim_{\lambda\to o} \rho_\lambda(f) = \int_0^\infty \mu_t(f)dt \qquad \text{for } f \in C_c(G), \tag{3.29}$$

is a nonnegative linear functional on $C_c(G)$, and therefore, κ is a mass distribution on G.

Definition 3.19.

A sub-Markov convolution semigroup $\{\mu_t\}_{t>0}$ on G with resolvent $\{\rho_\lambda\}_{\lambda>0}$ is defined to be transient iff

$$\lim_{\lambda\to o} \rho_\lambda(f) = \int_0^\infty \mu_t(f)dt < + \infty$$

for each $f \in C_c^+(G)$.

Definition 3.20.

A sub-Markov convolution semigroup $\{\mu_t\}_{t>0}$ on G is defined to be recurrent iff $\{\mu_t\}_{t>0}$ is not transient.

Definition 3.21.

If $\{\mu_t\}_{t>0}$ is a transient sub-Markov convolution semigroup on G, then the mass distribution κ on G defined by (3.29) is defined to be the potential kernel for $\{\mu_t\}_{t>0}$.

Definition 3.22.

A sub-Markov convolution semigroup $\{\mu_t\}_{t>0}$ on G is defined to be integrable iff $\{\mu_t\}_{t>0}$ is transient and the potential kernel for $\{\mu_t\}_{t>0}$ tends to zero at infinity.

We refer to Berg and Forst ([1], p. 110) for the next theorem.

Theorem 3.10.

Let $\{\mu_t\}_{t>0}$ be a symmetric sub-Markov convolution semigroup on G and let $-\Psi$ be the infinitesimal generator for $\{\hat{\mu}_t\}_{t>0}$. Then the following three properties are equivalent:

(a) $\{\mu_t\}_{t>0}$ is integrable.

(b) $\{\mu_t\}_{t>0}$ is transient.

(c) $\frac{1}{\Psi}$ is locally integrable on Γ.

Example 3.5.

Let us consider the Brownian motion semigroup $\{\mu_t\}_{t>0}$ on $\mathbb{R}^p (p \geq 1)$. If we define $\Psi: \mathbb{R}^p \to \mathbb{R}$ by

$$\Psi(y) = |y|^2 \qquad \text{for } y \in \mathbb{R}^p,$$

then $-\Psi$ is the infinitesimal generator for $\{\hat{\mu}_t\}_{t>0}$. By changing to spherical coordinates, it is possible to show that the function $\frac{1}{\Psi}$ is locally Lebesgue integrable on $\mathbb{R}^p$ iff $p \geq 3$. Since $\{\mu_t\}_{t>0}$ is symmetric, it follows from Theorem 3.10 that the Brownian motion semigroup $\{\mu_t\}_{t>0}$ on $\mathbb{R}^p$ is recurrent if $p = 1, 2$ and integrable if $p \geq 3$.

The mass distribution μ_t has the density function

$$g_t(x) = \frac{1}{(4\pi t)^{p/2}} \exp\left(-\frac{|x|^2}{4t}\right) \qquad \text{for } x \in \mathbb{R}^p$$

with respect to Lebesgue measure on $\mathbb{R}^p$. Then for $p \geq 3$ and $x \neq 0$,

$$\int_0^\infty g_t(x)dt = \int_0^\infty \frac{1}{(4\pi t)^{p/2}} \exp\left(-\frac{|x|^2}{4t}\right) dt, \tag{3.30}$$

and hence by substituting $s = \frac{|x|^2}{4t}$ in the right hand side of (3.30), we obtain the following equations

$$\int_0^\infty g_t(x)dt = \frac{|x|^{2-p}}{4\pi^{p/2}} \int_0^\infty s^{(p-2)/2 - 1} \exp(-s)ds$$

$$= \frac{\Gamma\left(\frac{p-2}{2}\right)}{4\pi^{p/2}} |x|^{2-p}$$

$$= k_2(x).$$

Therefore, the potential kernel for the Brownian motion semigroup $\{\mu_t\}_{t>0}$ on $\mathbb{R}^p (p \geq 3)$ is the Newtonian kernel k_2 on $\mathbb{R}^p$.

We refer to Berg and Forst ([1], p. 121) for the following theorem.

<u>Theorem 3.11.</u>

Let $\{\eta_t\}_{t>0}$ be a sub-Markov convolution semigroup on $\mathbb{R}$ supported by $[0,\infty)$ and let $-f$ be the infinitesimal generator for $\{\mathcal{L}\eta_t\}_{t>0}$. Then exactly one of the following two cases holds:

(a) f is identically equal to zero, in which case $\eta_t = \varepsilon$ for every $t > 0$.

(b) $f(x) > 0$ for every $x \in (0,\infty)$. Then $\{\eta_t\}_{t>0}$ is transient and the Laplace transform of the potential kernel τ for $\{\eta_t\}_{t>0}$ is given by

$$\mathcal{L}\tau(x) = \frac{1}{f(x)} \qquad \text{for } x \in (0,\infty).$$

<u>Example 3.6.</u>

Consider the one-sided symmetric stable semigroup $\{\sigma_t^\alpha\}_{t>0}$ of order $\alpha \in [0,1]$ on $\mathbb{R}$. If we define the function $f:(0,\infty) \to \mathbb{R}$ by

$$f(x) = x^\alpha \qquad \text{for } x \in (0,\infty),$$

then $-f$ is the infinitesimal generator for $\{\mathcal{L}\sigma_t^\alpha\}_{t>0}$.

For $\alpha > 0$, we define the function $g:\mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} & \text{for } x \in (0,\infty), \\ 0 & \text{for } x \in (-\infty,0], \end{cases} \tag{3.31}$$

Then g is locally Lebesgue integrable on $\mathbb{R}$, and consequently, we set $\tau_0 = \varepsilon$ and let τ_α be the mass distribution on $\mathbb{R}$ with $S(\tau_\alpha) = [0,\infty)$ which has density function g with respect to Lebesgue measure on $\mathbb{R}$. The Laplace transform of τ_α is given by

$$\mathcal{L}\tau_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \exp(-xs)dx \tag{3.32}$$

for $x \in (o,\infty)$. By substituting $t = xs$ in the right hand side of (3.32), we obtain the result that

$$\mathcal{L}\tau_\alpha(x) = \frac{x^{-\alpha}}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \exp(-t)dt$$

$$= x^{-\alpha}$$

for $x \in (o,\infty)$. We obtain from Theorem 3.11 the result that τ_α is the potential kernel for the one-sided symmetric stable semigroup $\{\sigma_t^\alpha\}_{t>0}$ of order $\alpha \in [o,1]$ on $\mathbb{R}$.

We refer to Itô ([27], Theorem 3) for the following theorem.

Theorem 3.12.

Let $\{\mu_t\}_{t>0}$ be a transient sub-Markov convolution semigroup on G with potential kernel κ and let $\{\eta_t\}_{t>0}$ be a sub-Markov convolution semigroup on $\mathbb{R}$ supported by $[o,\infty)$ satisfying $\eta_t \neq \varepsilon$ for $t > 0$ and with potential kernel τ. Then the sub-Markov convolution semigroup subordinated to $\{\mu_t\}_{t>0}$ by means of $\{\eta_t\}_{t>0}$ is transient with potential kernel

$$\kappa_\tau = \int_0^\infty \mu_t d\tau(t).$$

Example 3.7.

Let $\{\mu_t\}_{t>0}$ be a transient sub-Markov convolution semigroup on G with potential kernel κ and let $-\Psi$ be the infinitesimal generator for $\{\hat{\mu}_t\}_{t>0}$. If we let $-f$ be the infinitesimal generator for $\{\mathcal{L}\sigma_t^\alpha\}_{t>0}$, where $\alpha \in [o,1]$, then the sub-Markov convolution semigroup $\{\mu_t^f\}_{t>0}$ subordinated to $\{\mu_t\}_{t>0}$ by means of $\{\sigma_t^\alpha\}_{t>0}$ is given by

$$\mu_t^f = \int_0^\infty \mu_s d\sigma_t^\alpha(s)$$

for $t > 0$ (c.f. (3.24)). It follows from Theorem 3.12 that the sub-Markov convolution semigroup $\{\mu_t^f\}_{t>0}$ is transient with potential kernel

$$\kappa^\alpha = \begin{cases} \varepsilon & \text{for } \alpha = 0, \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\infty \mu_t t^{\alpha-1} dt & \text{for } \alpha \in (0,1), \\ \kappa & \text{for } \alpha = 1 \end{cases} \tag{3.33}$$

(c.f. (3.31)).

Example 3.8.

Let us consider the symmetric stable semigroup $\{\mu_t^\alpha\}_{t>0}$ of order $\alpha \in (0,2]$ on $\mathbb{R}^p (p \geq 1)$. If we let $-\Psi$ be the infinitesimal generator for the Brownian motion semigroup $\{\mu_t\}_{t>0}$ on $\mathbb{R}^p$, then $-\Psi^{\alpha/2}$ is the infinitesimal generator for $\{\hat{\mu}_t^\alpha\}_{t>0}$. The function $\frac{1}{\Psi^{\alpha/2}}$, which is defined by

$$\frac{1}{\Psi^{\alpha/2}}(y) = \frac{1}{|y|^\alpha} \qquad \text{for } y \in \mathbb{R}^p,$$

is locally Lebesgue integrable on $\mathbb{R}^p$ in the cases $0 < \alpha < 1$ for $p = 1$, $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$. It follows from Theorem 3.10 that $\{\mu_t^\alpha\}_{t>0}$ is integrable for these values of p and α. Furthermore, $\{\mu_t^\alpha\}_{t>0}$ is recurrent for $\alpha \in [1,2]$, $p = 1$ and for $\alpha = 2$, $p = 2$.

We obtain from (3.25) and (3.33) the result that the potential kernel for $\{\mu_t^\alpha\}_{t>0}$ on $\mathbb{R}^p$ has density function given by

$$\frac{1}{\Gamma(\alpha/2)} \int_0^\infty \frac{1}{(4\pi t)^{p/2}} \exp\left(-\frac{|x|^2}{4t}\right) t^{\alpha/2 -1}\, dt$$

$$= \frac{|x|^{\alpha-p}}{2^{\alpha+1}\pi^{p/2}\Gamma(\alpha/2)} \int_0^\infty s^{(p-\alpha)/2 -2} \exp(-s)ds$$

(by substituting $s = \frac{|x|^2}{4t}$)

$$= \frac{|x|^{\alpha-p}}{2^{\alpha+1}\pi^{p/2}\Gamma(\alpha/2)} \Gamma\left(\frac{p-\alpha}{2} - 1\right)$$

$$= \frac{\Gamma\left(\frac{p-\alpha}{2}\right)}{2^{\alpha}\pi^{p/2}\Gamma(\alpha/2)} |x|^{\alpha-p}$$

$$= k_\alpha(x)$$

for $x \neq 0$ in the cases $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$. Therefore, the potential kernel for the symmetric stable semigroup $\{\mu_t^\alpha\}_{t>0}$ of order α on $\mathbb{R}^p$ is the M. Riesz kernel of order α on $\mathbb{R}^p$ in the cases $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$.

CHAPTER 4

THE α-GREEN KERNEL FOR A HALF-SPACE

In this chapter, we will study the properties of the α-Green kernel for a half-space of $\mathbb{R}^p (p \geq 2)$ introduced by Essén and Jackson [17]. We will show that the α-Green kernel is a Hunt convolution kernel on the half-space in the cases $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$. Then we will show that the α-Green kernel satisfies the principle of domination in these cases only.

§1. The Definition and Basic Properties of the α-Green Kernel for a Half-space and the α-Green potential

Let $\mathcal{H} = \{x = (x_1, \ldots, x_p) \in \mathbb{R}^p | x_1 > 0\}$ be a half-space in $\mathbb{R}^p (p \geq 2)$ and let $\partial\mathcal{H} = \{x = (x_1, \ldots, x_p) \in \mathbb{R}^p | x_1 = 0\}$ be the Euclidean boundary of $\mathcal{H}$. If $x \in \mathcal{H}$, then we denote by x' the reflection of x through the boundary $\partial\mathcal{H}$ of the half-space $\mathcal{H}$ (i.e. if $x = (x_1, \ldots, x_p) \in \mathcal{H}$, then $x' = (-x_1, \ldots, x_p)$).

Definition 4.1.

The extended real valued function $\mathcal{G}_\alpha$ defined on $\mathcal{H}$ which is given by

$$\mathcal{G}_\alpha(x,y) = k_\alpha(x-y) - k_\alpha(x-y') \qquad \text{for } (x,y) \in \mathcal{H} \times \mathcal{H},$$

where $0 < \alpha < p$, is called the α-Green kernel for the half-space $\mathcal{H}$.

We remark that

$$\mathcal{G}_\alpha(x,y) = k_\alpha * \iota_y(x) - k_\alpha * \iota_{y'}(x)$$

$$= (\varepsilon_y,\varepsilon_x)_\alpha - (\varepsilon_{y'},\varepsilon_x)_\alpha$$

$$= (\varepsilon_y-\varepsilon_{y'},\varepsilon_x)_\alpha \tag{4.1}$$

for $(x,y) \in \mathcal{H}\times\mathcal{H}$. We will show later that the α-Green kernel for the half-space differs from the Green function of order α for $\mathcal{H}$ defined in Chapter 2, §5.

It is clear that the kernel $\mathcal{G}_\alpha$ is positive and symmetric on $\mathcal{H}\times\mathcal{H}$ (i.e. $\mathcal{G}_\alpha(x,y) = \mathcal{G}_\alpha(y,x)$ for $(x,y) \in \mathcal{H}\times\mathcal{H}$). For any $x \in \mathcal{H}$ fixed, the function $\mathcal{G}_\alpha(x,y)$ is continuous on $\mathcal{H}\setminus\{x\}$. If we let

$$\mathcal{G}_\alpha^{(1/n)}(x,y) = \begin{cases} k_\alpha(x-y) - k_\alpha(x-y') & \text{for } |x-y| > 1/n, \\ H(p,\alpha)n^{p-\alpha} - k_\alpha(x-y') & \text{for } |x-y'| \le 1/n \end{cases} \tag{4.2}$$

for each $n \in \mathbb{N}$ and $x \in \mathcal{H}$ fixed, then $\mathcal{G}_\alpha(x,y)$ is the limit of the monotone increasing sequence $\{\mathcal{G}_\alpha^{(1/n)}(x,y)\}_{n\in\mathbb{N}}$ of continuous truncated kernels, and hence $\mathcal{G}_\alpha(x,y)$ is lower semicontinuous on $\mathcal{H}\setminus\{x\}$ (c.f. Proposition 1.8).

<u>Theorem 4.1.</u>

Let $x \in \mathcal{H}$ be fixed. For $2 < \alpha \le p$, $\mathcal{G}_\alpha(x,y)$ is superharmonic on $\mathcal{H}$. It is not harmonic on $\mathcal{H}$.

For $\alpha = 2$, $\mathcal{G}_\alpha(x,y)$ is superharmonic on $\mathcal{H}$ and harmonic on $\mathcal{H}\setminus\{x\}$.

For $0 < \alpha < 2$, $\mathcal{G}_\alpha(x,y)$ is subharmonic on $\mathcal{H}\setminus\{x\}$.

The results of Theorem 4.1 follow immediately from the fact that

$$\Delta \mathcal{G}_\alpha(x,y) = H(p,\alpha)(p-\alpha)(2-\alpha)\left(\frac{1}{|x-y|^{p-\alpha+2}} - \frac{1}{|x-y'|^{p-\alpha+2}}\right)$$

for $y \in \mathcal{H}$.

Definition 4.2.

Let ν be a charge distribution on the half-space $\mathcal{H}$ and $0 < \alpha < p$. The extended real valued function $\mathcal{G}_\alpha * \nu$ defined on $\mathcal{H}$, which is given by

$$\mathcal{G}_\alpha\nu(x) = \int_{S(\nu)} \mathcal{G}_\alpha(x,y)d\nu(y) \qquad \text{for } x \in \mathcal{H},$$

is called the <u>α-Green potential</u> due to the charge distribution ν.

For any $\nu \in \mathcal{E}_\alpha$ and $x \in \mathcal{H}$, we obtain the result that

$$\mathcal{G}_\alpha\nu(x) = \int_{S(\nu)} \mathcal{G}_\alpha(x,y)d\nu(y)$$

$$= \int_{S(\nu)} \mathcal{G}_\alpha(y,x)d\nu(y)$$

$$= \int_{S(\nu)} k_\alpha * \varepsilon_x(y)d\nu(y) - \int_{S(\nu)} k_\alpha * \varepsilon_{x'}(y)d\nu(y)$$

$$= (\varepsilon_x,\nu)_\alpha - (\varepsilon_{x'},\nu)_\alpha$$

$$= (\varepsilon_x - \varepsilon_{x'}, \nu)_\alpha$$

by the fact that $\mathcal{G}_\alpha$ is symmetric on $\mathcal{H} \times \mathcal{H}$ and (4.1).

The following results are obtained in an analogous way to those of Proposition 2.1.

Proposition 4.1.

Let ν be a mass distribution on $\mathcal{H}$ with compact support. Then the following properties hold:

(a) $\mathcal{G}_\alpha \nu$ is lower semicontinuous on $\mathcal{H}$ and for any point $x \in \mathcal{H}$, we have the stronger condition that

$$\mathcal{G}_\alpha \nu(x) = \liminf_{\begin{cases} y \to x \\ y \in \mathcal{H} \end{cases}} \mathcal{G}_\alpha \nu(y);$$

(b) For any point $x \in \mathcal{H}$,

$$0 \leq \mathcal{G}_\alpha \nu(x) \leq +\infty.$$

Moreover,

$$0 < \mathcal{G}_\alpha \nu(x) \leq +\infty, \qquad \text{if } \nu \not\equiv 0;$$

(c) The set $\{x \in \mathcal{H} \mid \mathcal{G}_\alpha \nu(x) = +\infty\}$ has Lebesgue measure zero.

In the remainder of this section, we will also consider α-Green potentials due to mass distributions or charge distributions ν without compact support. However, we shall assume that both the potentials $\mathcal{G}_\alpha \nu^+$ and $\mathcal{G}_\alpha \nu^-$ are finite almost everywhere on $\mathcal{H}$ in all cases. This will always happen unless $\mathcal{G}_\alpha \nu^+ \equiv +\infty$ or $\mathcal{G}_\alpha \nu^- \equiv +\infty$. Then

$$\mathcal{G}_\alpha \nu(x) = \mathcal{G}_\alpha \nu^+(x) - \mathcal{G}_\alpha \nu^-(x)$$

is defined almost everywhere on $\mathcal{H}$.

Theorem 4.2. (The Principle of Descent).

If $\{\mu_n\}_{n \in \mathbb{N}}$ is a sequence of mass distributions on $\mathcal{H}$ such that $\mu_n \dot{\to} \mu$, then

$$\mathcal{G}_\alpha \mu(x) \leq \liminf_{n \to \infty} \mathcal{G}_\alpha \mu_n(x) \qquad \text{for any } x \in \mathcal{H}.$$

The proof of Theorem 4.2 is analogous to the proof of Theorem 2.3, which is found in Landkov [30], p. 62.

We obtain from Theorems 1.11 and 4.1 the following results.

Theorem 4.3.

Let μ be a mass distribution on $\mathcal{H}$. Then the following three properties hold:

(a) For $2 < \alpha < p$, $\mathcal{G}_\alpha \mu$ is superharmonic on $\mathcal{H}$;

(b) For $\alpha = 2$, $\mathcal{G}_\alpha \mu$ is superharmonic on $\mathcal{H}$ and harmonic on $\mathcal{H} \setminus S(\mu)$;

(c) For $\alpha < 2$, $\mathcal{G}_\alpha \mu$ is subharmonic on $\mathcal{H} \setminus S(\mu)$.

§2. The Definition and Basic Properties of α-Green Energy

The following definitions and results are analogous to those given in Chapter 2, §2.

Definition 4.3.

Let $0 < \alpha < p$ and let ν and ρ be charge distributions on the half-space $\mathcal{H}$ such that

$$\int_{\mathcal{H}} \mathcal{G}_\alpha \rho^+(x)\,d\nu^-(x) < +\infty \text{ and } \int_{\mathcal{H}} \mathcal{G}_\alpha \rho^-(x)\,d\nu^+(x) < +\infty .$$

We define the mutual α-Green energy of the charge distributions ν,ρ to be the integral

$$I_{\mathcal{G}_\alpha}(\nu,\rho) = \iint_{\mathcal{H}} \mathcal{G}_\alpha(x,y)\,d\nu(x)\,d\rho(y).$$

Definition 4.4.

Let $0 < \alpha < p$ and ν be a charge distribution in the half-space $\mathcal{H}$ such that

$$\int_{\mathcal{H}} \mathcal{G}_\alpha \nu^+(x)\,d\nu^-(x) < +\infty \text{ and } \int_{\mathcal{H}} \mathcal{G}_\alpha \nu^-(x)\,d\nu^+(x) < +\infty .$$

We define the α-Green energy of the charge distribution ν to be the integral

$$I_{\mathcal{G}_\alpha}(\nu) = \iint_{\mathcal{H}} \mathcal{G}_\alpha(x,y)\,d\nu(x)\,d\nu(y).$$

Theorem 4.4.

If $\mu_n \overset{\bullet}{\to} \mu$ and $\lambda_n \overset{\bullet}{\to} \lambda$, where μ_n and λ_n are mass distributions on the half-space $\mathcal{H}$, then

$$I_{\mathcal{G}_\alpha}(\mu,\lambda) \leq \liminf_{n \to \infty} I_{\mathcal{G}_\alpha}(\mu_n,\lambda_n) .$$

Moreover,

$$I_{\mathcal{G}_\alpha}(\mu) \leq \liminf_{n \to \infty} I_{\mathcal{G}_\alpha}(\mu_n).$$

The mass distributions $\mu_n \times \lambda_n$ on $\mathcal{H} \times \mathcal{H}$ converge vaguely to $\mu \times \lambda$, and hence Theorem 4.4 is an immediate consequence of Propositions 1.11 and 4.1.

We denote by $\mathcal{E}_{\mathcal{G}_\alpha}$ the set of all charge distributions on $\mathcal{H}$ having finite α-energy and by $\mathcal{E}^+_{\mathcal{G}_\alpha}$ the set of all mass distributions on $\mathcal{H}$ having finite α-energy.

§3. The Principle of Domination for the α-Green Kernel for the Half-Space $\mathcal{H}$

In this section, we will show that the α-Green kernel for the half-space $\mathcal{H}$ is a Hunt convolution kernel on $\mathcal{H}$ in the cases $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$. Then we will show that the α-Green kernel for the half-space $\mathcal{H}$ satisfies the principle of domination in these cases only. The proof is based on an argument due to Itô [28].

We will begin by introducing the notion of a Hunt convolution kernel. Some of the important properties of a Hunt convolution kernel will be given. The proofs will be omitted, but references to Deny ([14] and [15]) will be given.

Let G be a σ-compact abelian group. We denote by $\mathcal{M}_c(G)$ the set of all charge distributions on G having compact support.

Definition 4.5.

Let N be a mass distribution on G. The mapping $N: \mathcal{M}_c(G) \to \mathcal{M}(G)$, defined by

$$N(\nu) = N * \nu \qquad \text{for } \nu \in \mathcal{M}_c(G),$$

is called a convolution kernel on G. The charge distribution $N * \nu$ on G is called the potential of ν with respect to the convolution kernel N.

Definition 4.6.

A family $\{\mu_t\}_{t \geq 0}$ of mass distributions on G is called a (vaguely continuous) convolution semigroup on G iff

(a) $\mu_o = \varepsilon$;

(b) $\mu_t * \mu_s = \mu_{t+s}$ for $t, s \geq 0$;

(c) $\mu_t \to \mu_{t_o}$ as $t \to t_o$, for $t, t_o \geq 0$.

If $\{\mu_t\}_{t>0}$ is a sub-Markov convolution semigroup on G and if $\mu_o = \varepsilon$, then it follows from Proposition 3.8 that $\{\mu_t\}_{t \geq 0}$ is a convolution semigroup on G.

Definition 4.7.

A convolution kernel N on G is defined to be a Hunt convolution kernel on G iff there exists a convolution semigroup $\{\mu_t\}_{t\geq 0}$ on G such that N is given by the vague integral

$$N = \int_0^\infty \mu_t dt \tag{4.3}$$

It follows that the potential kernel for a transient sub-Markov convolution semigroup on G is a Hunt convolution kernel on G. In particular, the M. Riesz kernel of order α on $\mathbb{R}^p$ is a Hunt convolution kernel on $\mathbb{R}^p$ in the cases $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$ (c.f. Example 3.7)

We refer to Deny ([15], pp. 649-650) for the following proposition.

Proposition 4.2.

Let N be a Hunt convolution kernel on G. Then the convolution semigroup $\{\mu_t\}_{t\geq 0}$ corresponding to the Hunt convolution kernel N by equation (4.3) is unique.

Theorem 4.5.

Every Hunt convolution kernel N on G satisfies the principle of balayage onto an open set; that is, for any open and relatively compact set $D \subset G$ and any mass distribution μ on G for which $N * \mu$ is finite, there exists a mass distribution μ' on G with $S(\mu') \subset \overline{D}$ such that

$$N * \mu'(x) = N * \mu(x) \qquad \text{for every } x \in D$$

and

$$N * \mu'(x) \leq N * \mu(x) \qquad \text{for every } x \in G.$$

The proof of Theorem 4.5 can be found in Deny ([15], p. 660).

Therefore, the M. Riesz kernel of order α on $\mathbb{R}^p$ satisfies the principle of balayage onto an open set in the cases $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$.

We refer the reader to Deny ([14], p. 8-03) for the next theorem.

Theorem 4.6. (The Principle of Domination for a Hunt Convolution Kernel).

Let N be a Hunt convolution kernel on G. For any $f, g \in C_c^+(G)$, if

$$Nf(x) \leq Ng(x) \qquad \text{for every } x \in S(f),$$

then

$$Nf(x) \leq Ng(x) \qquad \text{for every } x \in G.$$

Example 4.1.

Let us consider the symmetric stable semigroup $\{\mu_t^\alpha\}_{t>0}$ of order α on $\mathbb{R}^p$ in the cases $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$. Set

$$\nu_t^\alpha(x,y) = \mu_t^\alpha(x-6) - \mu_t^\alpha(x-y') \quad \text{for each } t > 0 \text{ and } (x,y) \in \mathcal{H} \times \mathcal{H},$$

where y' is the reflection of y through the boundary $\partial\mathcal{H}$ of the half-space $\mathcal{H}$.

It follows from (3.28) that the mass distribution μ_t^α is spherically symmetric and $\mu_t^\alpha(x)$ decreases as $|x|$ increases. Therefore,

$$\mu_t^\alpha(x-y') = \mu_t^\alpha(x'-y), \tag{4.4}$$

$$\mu_t^\alpha(x-y) = \mu_t^\alpha(x'-y'), \tag{4.5}$$

and

$$\mu_t^\alpha(x-y) > \mu_t^\alpha(x-y') \tag{4.6}$$

for each $t > 0$ and $(x,y) \in \mathcal{H} \times \mathcal{H}$. Also,

$$\mu_t^\alpha(x-y) = \mu_t^\alpha(x'-y) \tag{4.7}$$

for each $t > 0$, $x \in \mathcal{H}$, and $y \in \partial\mathcal{H}$.

For each t, $s > 0$ and $(x,y) \in \mathcal{H} \times \mathcal{H}$, we obtain the following semigroup property for $\{\nu_t^\alpha\}_{t>0}$

$$\begin{aligned}
\nu_t^\alpha * \nu_s^\alpha(x,y) &= \int_{\mathcal{H}} \nu_t^\alpha(x,z)\nu_s^\alpha(z,y)\,dm_p(z) \\
&= \int_{\mathcal{H}} \mu_t^\alpha(x-z)\mu_s^\alpha(z-y)\,dm_p(z) + \int_{\mathcal{H}} \mu_t^\alpha(x-z')\mu_s^\alpha(z-y')\,dm_p(z) \\
&\quad - \left(\int_{\mathcal{H}} \mu_t^\alpha(x-z)\mu_s^\alpha(z-y')\,dm_p(z) + \int_{\mathcal{H}} \mu_t^\alpha(x-z')\mu_s^\alpha(z-y)\,dm_p(z)\right) \\
&= \int_{\mathcal{H}} \mu_t^\alpha(x-z)\mu_s^\alpha(z-y)\,dm_p(z) + \int_{\mathcal{H}} \mu_t^\alpha(x-z')\mu_s^\alpha(z'-y)\,dm_p(z) \\
&\quad - \left(\int_{\mathcal{H}} \mu_t^\alpha(x'-z')\mu_s^\alpha(z'-y)\,dm_p(z) + \int_{\mathcal{H}} \mu_t^\alpha(x'-z)\mu_s^\alpha(z-y)\,dm_p(z)\right) \\
&= \int_{\mathbb{R}^p} \mu_t^\alpha(x-z)\mu_s^\alpha(z-y)\,dm_p(z) - \int_{\mathbb{R}^p} \mu_t^\alpha(x'-z)\mu_s^\alpha(z-y)\,dm_p(z) \\
&= \mu_t^\alpha * \mu_s^\alpha(x-y) - \mu_t^\alpha * \mu_s^\alpha(x'-y) \\
&= \mu_{t+s}^\alpha(x-y) - \mu_{t+s}^\alpha(x'-y) \\
&= \mu_{t+s}^\alpha(x-y) - \mu_{t+s}^\alpha(x-y') \\
&= \nu_{t+s}^\alpha(x,y)
\end{aligned}$$

by (4.4), (4.5), and (4.7). It follows from (4.6) that ν_t^α is strictly positive on $\mathcal{H} \times \mathcal{H}$. Hence, if we set $\nu_0^\alpha = \varepsilon$, then $\{\nu_t^\alpha\}_{t\geq 0}$ is a convolution semigroup on $\mathcal{H}$.

We obtain from Example 3.8 the result that the α-Green kernel for the half-space $\mathcal{H}$ is given by the vague integral

$$\begin{aligned}
\mathcal{G}_\alpha(x,y) &= k_\alpha(x-y) - k_\alpha(x-y') \\
&= \int_0^\infty \mu_t^\alpha(x-y)\,dt - \int_0^\infty \mu_t^\alpha(x-y')\,dt \\
&= \int_0^\infty \nu_t^\alpha(x-y)\,dt \qquad \text{for } (x,y) \in \mathcal{H} \times \mathcal{H}
\end{aligned}$$

in the cases $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$. Therefore, the α-Green kernel $\mathcal{G}_\alpha$ is a Hunt convolution kernel on the half-space $\mathcal{H}$ in the cases $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$.

We obtain the following result from Theorem 4.5.

Theorem 4.7.

The α-Green kernel for the half-space $\mathcal{H}$ satisfies the principle of balayage onto an open set in the cases $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$.

The next result is a direct consequence of Theorem 4.6.

Theorem 4.8.

Let $0 < \alpha < 2$ for $p = 2$ and $0 < \alpha \leq 2$ for $p \geq 3$. For any $f, g \in C_c^+(\mathcal{H})$, if

$$\mathcal{G}_\alpha f(x) \leq \mathcal{G}_\alpha g(x) \qquad \text{for every } x \in S(f),$$

then

$$\mathcal{G}_\alpha f(x) \leq \mathcal{G}_\alpha g(x) \qquad \text{for every } x \in \mathcal{H},$$

where $\mathcal{G}_\alpha f(x) = \int_{\mathcal{H}} \mathcal{G}_\alpha(x,y) f(y)\, dm_p(y)$.

Theorem 4.9. (The Principle of Domination).

Let λ be a mass distribution on $\mathcal{H}$ and $\mu \in \mathcal{E}^+_{\mathcal{G}_\alpha}$, where $0 < \alpha < 2$ for $p = 2$, and $0 < \alpha \leq 2$ for $p \geq 3$. If

$$\mathcal{G}_\alpha \mu(x) \leq \mathcal{G}_\alpha \lambda(x) \qquad \text{for every } x \in S(\mu),$$

then

$$\mathcal{G}_\alpha \mu(x) \leq \mathcal{G}_\alpha \lambda(x) \qquad \text{for every } x \in \mathcal{H} .$$

Proof.

Let $\mathcal{M}^+_{\mathcal{H}}$ denote the set of all mass distributions with support contained in $\mathcal{H}$. It follows from Theorem 4.2 that $\mathcal{G}_\alpha$ is lower semicontinuous on $\mathcal{M}^+_{\mathcal{H}}$. Then for any $\delta > 0$, there exists a neighbourhood $N(\lambda)$ of λ with respect to the vague topology such that for any $\nu \in N(\lambda)$,

$$\mathcal{G}_\alpha \mu(x) \leq (1 + \tfrac{\delta}{2}) \ \mathcal{G}_\alpha \nu(x) \qquad \text{for every } x \in S(\mu)$$

(c.f. Proposition 1.4).

We recall that for any $f \in C_c^+(\mathbb{R}^p)$, f is the density of an absolutely continuous mass distribution with respect to Lebesgue measure on $\mathbb{R}^p$. We will denote an absolutely continuous mass distribution and its density by the same letter.

It follows from Theorem 4.2 and Proposition 4.1 that $\mathcal{G}_\alpha$ is lower semicontinuous on $\mathcal{M}^+_{\mathcal{H}} \times \mathcal{H}$. Then there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions in $C_c^+(\mathbb{R}^p)$ such that $\lambda * f_n \in N(\lambda)$ for all $n \in \mathbb{N}$, $f_n \to \varepsilon$ as n tends to infinity, and

$$\mathcal{G}_\alpha(\mu * f_n)(x) \leq (1+\delta)\mathcal{G}_\alpha(\lambda * f_n)(x) \qquad \text{for every } x \in S(\mu * f_n)$$

for all $n \in \mathbb{N}$ (c.f. Proposition 1.3). Hence we obtain from Theorem 4.8 the result that

$$\mathcal{G}_\alpha(\mu * f_n)(x) \leq (1+\delta) \ \mathcal{G}_\alpha(\lambda * f_n)(x) \text{ for every } x \in \mathcal{H} .$$

for all $n \in \mathbb{N}$. Since $\delta > 0$ is arbitrary and taking the limit as n tends to infinity, we have the result that

$$\mathcal{G}_\alpha\mu(x) \leq \mathcal{G}_\alpha\lambda(x) \qquad \text{for every } x \in \mathcal{H}.$$

This completes the proof.

Cartan and Deny ([8], p. 88) proved that the balayage principle and the principle of domination are equivalent for kernels such as the α-Green kernel for the half-space $\mathcal{H}$. Furthermore, both principles are equivalent to the principle that for any $\mu, \lambda \in \mathcal{E}^+_{\mathcal{G}_\alpha}$,

$$\inf\{\mathcal{G}_\alpha\mu, \mathcal{G}_\alpha\lambda\}$$

is the α-Green potential of a mass distribution in $\mathcal{E}^+_{\mathcal{G}_\alpha}$.

We are now ready to prove that the α-Green kernel for the half-space $\mathcal{H}$ does not satisfy the principle of domination for $2 < \alpha < p$.

<u>Theorem 4.10.</u>

Let ν be a charge distribution on $\mathcal{H}$ $(p \geq 3)$ such that $\mathcal{G}_2\nu$ is well-defined, K a compact subset of $\mathcal{H}$, and ν_K the restriction of ν to K. If

$$\mathcal{G}_2\nu(x) = 0 \qquad \text{almost everywhere on } K,$$

then $\nu_K \equiv 0$. Moreover, if

$$\mathcal{G}_2\nu(x) = 0 \qquad \text{almost everywhere on } \mathcal{H},$$

then $\nu \equiv 0$, and consequently,

$$\mathcal{G}_2\nu(x) = 0 \qquad \text{for every } x \in \mathcal{H}.$$

This result follows from Theorem 2.31 and the fact that

$$\mathcal{G}_2\nu(x) = k_2 * (\nu-\nu')(x), \tag{4.8}$$

where $d\nu'(y') = d\nu(y)$ for $y \in \mathcal{H}$.

Theorem 4.11.

The α-Green kernel for the half-space $\mathcal{H}$ does not satisfy the principle of domination for $2 < \alpha < p$.

Proof.

We will show that the α-Green kernel for the half-space $\mathcal{H}$ does not satisfy the principle of balayage for $2 < \alpha < p$.

Suppose that the kernel $\mathcal{G}_\alpha$ satisfies the principle of balayage for $2 < \alpha \le 4$. Let B be a closed ball in $\mathcal{H}$, $y \in \mathcal{H} \setminus B$, and $\varepsilon_y^{(B)}$ the mass distribution obtained by sweeping the Dirac measure ε_y with repect to $\mathcal{G}_\alpha$ onto B. Then

$$\mathcal{G}_\alpha\varepsilon_y^{(B)}(x) = \mathcal{G}_\alpha\varepsilon_y(x) \qquad \text{for every } x \in B.$$

Since $k_\alpha = k_2 * k_{\alpha-2}$ by Theorem 2.1, we obtain the result that for any $(x,y) \in \mathcal{H} \times \mathcal{H}$,

$$\begin{aligned}\int_{\mathcal{H}} \mathcal{G}_2(x,z)\mathcal{G}_{\alpha-2}(z,y)dm_p(z) &= \int_{\mathcal{H}} k_2(x-z)k_{\alpha-2}(z-y)dm_p(z) + \int_{\mathcal{H}} k_2(x-z')k_{\alpha-2}(z-y')dm_p(z) \\ &\quad - \left(\int_{\mathcal{H}} k_2(x-z')k_{\alpha-2}(z-y)dm_p(z) + \int_{\mathcal{H}} k_2(x-z)k_{\alpha-2}(z-y')dm_p(z)\right) \\ &= \int_{\mathbb{R}^p} k_2(x-z)k_{\alpha-2}(z-y)dm_p(z) - \int_{\mathbb{R}^p} k_2(x-z)k_{\alpha-2}(z-y')dm_p(z) \\ &= k_2 * k_{\alpha-2}(x-y) - k_2 * k_{\alpha-2}(x-y')\end{aligned}$$

$$= k_\alpha(x-y) - k_\alpha(x-y')$$

$$= \mathcal{G}_\alpha(x,y),$$

and consequently,

$$\begin{aligned}
(\mathcal{G}_2(\cdot\ \mathcal{G}_{\alpha-2}\varepsilon_y))(x) &= \mathcal{G}_\alpha(x,y) \\
&= \mathcal{G}_\alpha\varepsilon_y(x) \\
&= \mathcal{G}_\alpha\varepsilon_y^{(B)}(x) \\
&= \int_{\mathcal{H}} \mathcal{G}_\alpha(x,z)d\varepsilon_y^{(B)}(z) \\
&= \iint_{\mathcal{H}} \mathcal{G}_2(x,u)\,\mathcal{G}_{\alpha-2}(u,z)dm_p(u)d\varepsilon_y^{(B)}(z) \\
&= \iint_{\mathcal{H}} \mathcal{G}_2(x,u)\,\mathcal{G}_{\alpha-2}(u,z)d\varepsilon_y^{(B)}(z)dm_p(u) \\
&= \int_{\mathcal{H}} \mathcal{G}_2(x,u)\,\mathcal{G}_{\alpha-2}\varepsilon_y^{(B)}(u)dm_p(u) \\
&= \mathcal{G}_2(\cdot\,\mathcal{G}_{\alpha-2}\varepsilon_y^{(B)})(z)
\end{aligned}$$

for every $x \in B$, by Fubini's theorem.

Let $\nu = \mathcal{G}_{\alpha-2}\varepsilon_y - \mathcal{G}_{\alpha-2}\varepsilon_y^{(B)}$. Then ν is a charge distribution on $\mathcal{H}$ such that

$$\mathcal{G}_2\nu(x) = 0 \qquad \text{for every } x \in B,$$

and hence, $\nu(B) = 0$ by Theorem 4.10. Therefore,

$$\mathcal{G}_{\alpha-2}\,\varepsilon_y^{(B)}(x) = \mathcal{G}_{\alpha-2}\varepsilon_y(x) \quad \text{for every } x \in B.$$

Since $0 < \alpha-2 \leq 2$, it follows from the principle of domination that

$$\mathcal{G}_{\alpha-2}\varepsilon_y^{(B)}(x) \leq \mathcal{G}_{\alpha-2}\varepsilon_y(x) \quad \text{for every } x \in \mathbb{R}^p.$$

Therefore, ν is a mass distribution on $\mathcal{H}$, and consequently, $\mathcal{G}_2\nu$ is a nonnegative superharmonic function on $\mathcal{H}$ which vanishes on B (c.f. Theorem 4.3). It follows from Proposition 1.18 that

$$\mathcal{G}_2\nu(x) = 0 \qquad \text{for every } x \in \mathcal{H},$$

and hence, $\nu \equiv 0$ by Theorem 4.10.

Let $\mu = \varepsilon_y - \varepsilon_y^{(B)}$. Then μ is a charge distribution on $\mathcal{H}$ such that

$$\mathcal{G}_{\alpha-2}\mu(x) = k_{\alpha-2} * (\mu-\mu')(x) = 0 \quad \text{for every } x \in \mathcal{H},$$

where $d\mu'(z') = d\mu(z)$ for $z \in \mathcal{H}$. We obtain from Theorem 2.8 the result that $\mu \equiv 0$, and therefore, $\varepsilon_y \equiv \varepsilon_y^{(B)}$. This leads to a contradiction.

The result follows inductively for $2n < \alpha \leq 2(n+1)$ $(n \in \mathbb{N})$.

§4. $\mathcal{G}_\alpha$-Superharmonic Functions of Fractional Order on the Half-space $\mathcal{H}$

We shall assume that $0 < \alpha < 2$ for $p = 2$ and $0 < \alpha \le 2$ for $p \ge 3$ throughout this section.

Since the α-Green kernel $\mathcal{G}_\alpha$ for $\mathcal{H}$ satisfies the balayage principle, we can define a new class of functions on $\mathcal{H}$ in an analogous way to the class of α-superharmonic functions on $\mathcal{H}$ defined by Frostman in Chapter 2, §6, where G_α is replaced by $\mathcal{G}_\alpha$.

Let F be a closed subset of $\mathcal{H}$ and $x \in \mathcal{H} \setminus F$. We denote by $\rho_x^{(F)}$ the mass distribution on F obtained by sweeping the Dirac measure ε_x onto F relative to $\mathcal{G}_\alpha$. Then

$$\mathcal{G}_\alpha \rho_x^{(F)}(y) = \mathcal{G}_\alpha(y,x) \qquad \alpha\text{-quasi-everywhere on } F, \quad (4.9)$$

and

$$\mathcal{G}_\alpha \rho_x^{(F)}(y) \le \mathcal{G}_\alpha(y,x) = \mathcal{G}_\alpha \varepsilon_x(y) \quad \text{for every } y \in \mathcal{H} \quad (4.10)$$

Definition 4.8.

An extended real valued function f defined on the half-space $\mathcal{H}$ of $\mathbb{R}^p$ shall be called $\mathcal{G}_\alpha$-superharmonic on $\mathcal{H}$ iff it satisfies the following three conditions:

(a) $f \not\equiv +\infty$;

(b) f is lower semicontinuous on $\mathcal{H}$;

(c) For any closed set $F \subset \mathcal{H}$ and $x \in \mathcal{H} \setminus F$,

$$f(x) \geq \int_F f(y) d\rho_x^{(F)}(y).$$

The $\mathcal{G}_\alpha$-superharmonic functions on $\mathcal{H}$ have analogous properties to the α-superharmonic functions on $\mathcal{H}$. Indeed, every $\mathcal{G}_\alpha$-superharmonic function f on $\mathcal{H}$ is nonnegative and there exists a monotone nondecreasing sequence of continuous $\mathcal{G}_\alpha$-superharmonic functions on $\mathcal{H}$ which converge to f.

If ν is a mass distribution on $\mathcal{H}$, then $\mathcal{G}_\alpha \nu$ is either identically equal to infinity or is a $\mathcal{G}_\alpha$-superharmonic function on $\mathcal{H}$. This follows from the facts that $\mathcal{G}_\alpha \nu$ is lower semicontinuous on $\mathcal{H}$ (c.f. Proposition 4.1) and for every closed set $F \subset \mathcal{H}$ and $x \in \mathcal{H} \setminus F$,

$$\begin{aligned}
\int_F \mathcal{G}_\alpha \nu(y) d\rho_x^{(F)}(y) &= \int_F d\rho_x^{(F)}(y) \int_{S(\nu)} \mathcal{G}_\alpha(y,z) d\nu(z) \\
&= \int_{S(\nu)} d\nu(z) \int_F \mathcal{G}_\alpha(y,z) d\rho_x^{(F)}(y) \\
&\leq \int_{S(\nu)} \mathcal{G}_\alpha(x,z) d\nu(z) = \mathcal{G}_\alpha \nu(x).
\end{aligned}$$

by Fubini's theorem and (4.10).

Let F be a closed subset of $\mathbb{R}^p (p \geq 2)$ and $x \in \mathbb{R}^p \setminus F$. We denote by $\varepsilon_x^{(F)}$ the Green measure for F relative to the pole x, and by G_α the Green function of order α for $\mathcal{H}$. It follows from (2.19) that

$$G_\alpha(x,y) = k_\alpha * \varepsilon_y(x) - k_\alpha * \varepsilon_y^{(\mathbb{R}^p \setminus \mathcal{H})}(x) \geq \mathcal{G}_\alpha(x,y)$$

for $(x,y) \in \mathcal{H} \times \mathcal{H}$. Furthermore,

$$G_\alpha(x,y) - \mathcal{G}_\alpha(x,y) = k_\alpha * \varepsilon_{y'}(x) - k_\alpha * \varepsilon_y^{(\mathbb{R}^p \setminus \mathcal{H})}(x)$$

$$= k_\alpha * \varepsilon_{y'}^{(\mathcal{H} \cup \partial\mathcal{H})}(x) - k_\alpha * \varepsilon_y^{(\mathbb{R}^p \setminus \mathcal{H})}(x)$$

$$= \int_{\mathcal{H} \cup \partial\mathcal{H}} \frac{1}{|x-y|^{p-\alpha}} \, d\varepsilon_{y'}^{(\mathcal{H} \cup \partial\mathcal{H})}(z)$$

$$- \int_{\mathbb{R}^p \setminus \mathcal{H}} \frac{1}{|x-z|^{p-\alpha}} \, d\varepsilon_y^{(\mathbb{R}^p \setminus \mathcal{H})}(z)$$

$$= \int_{\mathcal{H}} \frac{1}{|x-z|^{p-\alpha}} \, d\varepsilon_{y'}^{(\mathcal{H} \cup \partial\mathcal{H})}(z)$$

$$- \int_{\mathcal{H}} \frac{1}{|x-z'|^{p-\alpha}} \, d\varepsilon_{y'}^{(\mathcal{H} \cup \partial\mathcal{H})}(z')$$

$$= \mathcal{G}_\alpha \varepsilon_{y'}^{(\mathcal{H})}(z) \qquad \text{for } (x,y) \in \mathcal{H} \times \mathcal{H},$$

where $\varepsilon_{y'}^{(\mathcal{H})}$ is the restriction of the Green measure $\varepsilon_{y'}^{(\mathcal{H} \cup \partial\mathcal{H})}$ to $\mathcal{H}$, because of (2.18) and the symmetry of the Green measures for $\mathbb{R}^p \setminus \mathcal{H}$ and $\mathcal{H} \cup \partial\mathcal{H}$ relative to the poles y and y' respectively.

If ν is a charge distribution on $\mathcal{H}$, then

$$G_\alpha \nu(x) = \mathcal{G}_\alpha \nu(x) + \int_{S(\nu)} \mathcal{G}_\alpha \varepsilon_{y'}^{(\mathcal{H})}(x) \, d\nu(y)$$

$$= \mathcal{G}_\alpha \nu(x) + \iint_{\mathcal{H}} \mathcal{G}_\alpha(x,z) \, d\varepsilon_{y'}^{(\mathcal{H})}(z) \, d\nu(y)$$

$$= \mathcal{G}_\alpha \nu(x) + \iint_{\mathcal{H}} \mathcal{G}_\alpha(x,z) \, d\varepsilon_{y'}^{(\mathcal{H})}(z) \, d\nu'(y')$$

$$= \mathcal{G}_\alpha \nu(x) + \mathcal{G}_\alpha (\nu')^{(\mathcal{H})}(x)$$

$$= \mathcal{G}_\alpha(\nu + (\nu')^{(\mathcal{H})})(x) \qquad \text{for } x \in \mathcal{H}, \qquad (4.11)$$

where $d\nu'(y') = d\nu(y)$ for y $\mathcal{H}$, and $(\nu')^{(\mathcal{H})}$ is the charge distribution obtained by sweeping ν' onto $\mathcal{H}$ relative to G_α.

Concluding Remarks.

The set of all $\mathcal{G}_\alpha$-superharmonic functions on $\mathcal{H}$ forms a convex cone containing the nonnegative constant functions on $\mathcal{H}$.

We obtain from (4.11) and Theorem 2.34 the result that the convex cone of $\mathcal{G}_\alpha$-superharmonic functions on $\mathcal{H}$ contains the convex cone of α-superharmonic functions on $\mathcal{H}$. This inclusion is strict when $0 < \alpha < 2$, and results in equality when $\alpha = 2$, $p \geq 3$.

BIBLIOGRAPHY

1. Berg, C. and Forst, G., Potential Theory on Locally Compact Abelian Groups, Springer-Verlag, N.Y. (1975).

2. Blumenthal, R.M. and Getoor, R.K., Markov Processes and Potential Theory, Academic Press, N.Y. (1968).

3. Bourbaki, N., General Topology, Parts 1 and 2, Addison-Wesley, Reading, Mass. (1966).

4. Bourbaki, N., Éléments de Mathématique, Livre VI, Hermann & Co., Paris (1952).

5. Brelot, M., Éléments de la Théorie Classique du Potentiel, Les Cours de Sorbonne, Paris (1965).

6. Cartan, H., Sur les fondements de la théorie du potentiel, Bull. Soc. Math de France 69, pp. 71-96 (1941).

7. Cartan, H., Théorie du potentiel Newtonien: énergie, capacité, suite de potentiels, C.R. 73, pp. 74-106 (1945).

8. Cartan, H. and Deny, J., Le principe du maximum en théorie du potentiel et la notion de fonction surharmonique, Acta Szeged 12, pp. 81-100 (1951).

9. Choquet, G., Theory of capacities, Ann. Inst. Fourier 5, pp. 131-289 (1953).

10. Courrége, P., Le Processus Stochastique du Mouvement Brownien, Les Cours de Sorbonne, Paris (1967).

11. Deny, J., Sur la définition de l'énergie en théorie du potentiel, Ann. Inst. Fourier 2, pp. 83-99 (1950).

12. Deny, J., Les potentiel's d'énergie finie, Acta Math. 82, pp. 107-183 (1953).

13. Deny, J., Familles fondamentales. Noyaux associés, Ann. Inst. Fourier 3, pp. 73-101 (1951).

14. Deny, J., Éléments de théorie du potentiel par rapport à un noyau de Hunt, Séminaire Brelot-Choquet-Deny, Théorie du Potentiel, 5e année, pp. 8-01 - 8-08 (1960-1961).

15. Deny, J., Noyaux de convolution de Hunt et noyaux associés à une famille fondamentale, Ann. Inst. Fourier 12, pp. 643-667 (1962).

16. Doetsch, G., Introduction to the Theory and Application of the Laplace Transform, Springer-Verlag, Berlin (1974).

17. Essén, M. and Jackson, H.L., Sur les propriétés d'un ensemble effilé au sens minimal dans un demi-space, C.R. 279, pp. 865-867 (1974).

18. Feller, W., On a generalization of Marcel Riesz' potentials and the semi-groups generated by them, Comm. Sém. Math. Univ. Lund, Medd. Lunds Univ. Sem., Tome Supplémentaire, pp. 72-81 (1952).

19. Frostman, O., Sur le balayage des masses, Acta Szeged [illegible], pp. 43-51 (1938).

20. Frostman, O., La méthode de variation de Gauss et les fonctions sousharmonique, Acta Szeged 8, pp. 149-159 (1937).

21. Frostman, O., Sur les fonctions surharmoniques d'ordre fractionnaire, Arkiv för Mat., Astr., Fys. 26A, no. 16, pp. 1-16 (1938).

22. Halmos, P.R., Measure Theory, Van Nostrand, N.Y. (1950).

23. Helms, L.L., Introduction to Potential Theory, Krieger, Huntington, N.Y. (1975).

24. Hunt, G.A., Markov processes and potentials, I-III, Illinois J. Math. 1, pp. 44-93 and 316-396 (1957); 2, pp. 151-215 (1958).

25. Itô, M., Une caractérisation sur le principe relatif de domination pour les noyaux de Riesz-Frostman, Osaka J. Math. 10, pp. 265-270 (1973).

26. Itô, M., Sur le principe de domination pour les noyaux de convolution, Nagoya Math. J. 50, pp. 149-173 (1973).

27. Itô, M., Sur la famille sous-ordonnée au noyau de convolution de Hunt II, Nagoya Math. J. 53, pp. 115-126 (1974).

28. Itô, M., Private communication, Dept. of Math., Nagoya Univ. (1976).

29. Jackson, H.L., Sur la comparaison des deux types d'effilement, Séminaire Brelot-Choquet-Deny, Théorie du Potentiel, 14e-15e années, pp. 23-01 - 23-08 (1972).

30. Landkov, N.S., Foundations of Modern Potential Theory, Springer-Verlag, N.Y. (1972).

31. Loève, M., Probability Theory, Van Nostrand, Princeton (1960).

32. Meyer, P., Probability and Potentials, Blaisdell, Mass. (1966).

33. Meyer, P., Processus de Markov, Springer-Verlag, Berlin (1967).

34. Ninomiya, N., Sur un principe du maximum pour le potentiel de Riesz-Frostman, J. Math. Osaka City Univ. 13, pp. 57-62 (1962).

35. Riesz, M., Integrales de Riemann-Liouville et potentiels, Acta Szeged 9, pp. 1-42 (1938).

36. Royden, H.L., Real Analysis, MacMillan, N.Y. (1968).

37. Rudin, W., Fourier Analysis on Groups, Wiley, N.Y. (1962).