PSEUDO-DIFFERENTIAL OPERATORS WITH ROUGH COEFFICIENTS

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Abstract

The theory of pseudo-differential operators is one of the most important tools in modern mathematics. It has found important applications in many mathematical developments. It was used in a crucial way in the proof of the Atiyah-Singer Index theorem in [AtSi] and in the regularity of elliptic differential equations. In the theory of several complex variables, pseudo-differential operators are indispensable in studying the $\overline{\partial}$−Neumann problem. The theory of subelliptic and hypoelliptic differential operators achieved its current satisfactory state largely because of pseudo-differential operators. In the solution to the local solvability problem for differential equations by Beals-Fefferman [BeFe], pseudo-differential operators played the key role. Many boundary value problems for differential equations can be reduced to pseudo-differential equations, see for example, Hörmander [Hor2]. Roughly speaking, almost everything involving pseudodifferential operators can be reduced to two parts: the mapping properties and the compositions of the associated special pseudo-differential operators.

In this thesis, we consider the mapping properties and symbolic calculus of an important class of pseudo-differential operators, the symbolic class of Hörmander type with rough coefficients. We will prove some new results for these operators. These operators arise naturally from problems in nonlinear

partial differential equations. After the introduction of the classical symbol class $S^m_{1,0}$ in [KohNi], Hörmander considered symbolic class $S^m_{\rho,\delta}$ in [Hor1]. Eventually, such classes of pseudodifferential operators played a key role in the local solvability problem for differential operators (see Beals-Fefferman [BeFe]). It is observed by Guan-Sawyer in [GuSa1] that the oblique derivative problem can be reduced to the problem of pseudodifferential equations on the boundary with a parametrix in the class $S^m_{1,\frac{1}{2}}$. That discovery led them to establish complete optimal regularity for the oblique derivative problem with smooth data. Later, they used the class $C^v S^m_{1,\delta}$ to study some nonlinear oblique derivative problems in [GuSa2]. While observing that the symbols arising here lie in the symbol class $C^\lambda S^0_{1,\frac{1}{2}}$, P. Guan and E. Sawyer [GuSa1] discovered that such symbols actually behave much better than $C^\lambda S^0_{1,\frac{1}{2}}$.

Indeed,

$$\begin{aligned}\partial_x \tau(x,\xi) &= -\left\{\int_{t'}^t (\partial_x a)(x,\theta)Q(x,\theta,\xi)d\theta + \int_{t'}^t a(x,\theta)(\partial_x Q)(x,\theta,\xi)d\theta\right\} e^{-\int_{t'}^t a(x,\theta)Q(x,\theta,\xi)d\theta} \\ &= \tau_1 + \tau_2\end{aligned}$$

where $\tau_1 \in C^\lambda S^{\frac{1}{2}}_{1,\frac{1}{2}}$, and $\tau_2 \in C^{\lambda-1} S^0_{1,\frac{1}{2}}$. Thus τ decomposes into two pieces, one term having order worse by $\frac{1}{2}$ but no loss in smoothness, another term having 1 degree less smoothness but no loss of order. Moreover this property persists for each of the symbols τ_1 and τ_2, etc., resulting in such symbols enjoying the mapping properties of the better behaved class $C^\lambda S^0_{1,0}$.

There have been many developments regarding the mapping properties and compositions of symbols in the class $S^m_{\rho,\delta}$. Specifically, the works of C. Fefferman, C. Fefferman and E. Stein, A. Calderòn and R. Vaillancourt, R. Coifman-Y. Meyer, A. Miyachi, we refer to [St2] for the complete references. Our results in this paper can be viewed as a further step in this direction.

In this thesis mapping properties of pseudo-differential operator are studied in various symbol classes.

In the first result (Theorem 2.3.1) we consider the symbol class $C^{k_1,k_2}S_{0,0}^{m_1,m_2}$ and obtain L^2 results extending those of [CoMe].

For the symbols in the class $C^vS^m_{\rho,\delta}$, mapping properties are obtained for H^r_p sobolev spaces (Theorem 2.3.2) and finally we consider pseudo-differential operators of symbol class $C^v_\infty S^m_{\rho,\delta}$, and prove that they have better mapping properties (Theorem 2.3.3).

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Notations and Definitions

We introduce here some necessary standard notations.

- $\mathbf{R}^n$ is an n-dimensional Euclidean space of points with coordinates $x = (x_1, ..., x_n)$.
- $|x| = [(x_1)^2 + ... + (x_n)^2]^{\frac{1}{2}}$,
- $x^\alpha = (x_1)^{\alpha_1} ... (x_n)^{\alpha_n}$, if $\alpha = (\alpha_1, ..., \alpha_n), \alpha_j \in \mathbf{Z}_+$; $\mathbf{Z}_+$ is the set of non-negative integers.
- Let $dx = dx_1 ... dx_n$ is the Lebesgue measure on $\mathbf{R}^n$.
- Let ξ is the dual of x; it consists of points ξ with coordinates $(\xi_1, ..., \xi_n)$. Similar, we let
- $|\xi| = [(\xi_1)^2 + ... + (\xi_n)^2]^{\frac{1}{2}}$,
- $\xi^\alpha = (\xi_1)^{\alpha_1} ... (\xi_n)^{\alpha_n}$, if $\alpha = (\alpha_1, ..., \alpha_n), \alpha_j \in \mathbf{Z}_+; \mathbf{Z}_+$ is the set of non-negative integers.
- $d\xi = d\xi_1 ... d\xi_n$ is the Lebesgue measure on $\mathbf{R}^n$.
- If $\alpha = (\alpha_1, ..., \alpha_n), \alpha_j \in \mathbf{Z}_+$. We denote $|\alpha| = \alpha_1 + ... + \alpha_n$; $\alpha! = \alpha_1! ... \alpha_n!$; $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}$.

- $\langle\xi\rangle = \left(1+|\xi|^2\right)^{\frac{1}{2}}$

- For $k \in \mathbf{R}$, $[k]$ denote the integer part of k.

- Fourier transform $\widehat{f}(\xi) = \frac{1}{(2\pi)^n}\int e^{-iy\cdot\xi} f(y)\,dy$

- Inverse Fourier transform $\check{f}(y) = \int e^{iy\cdot\xi} f(\xi)\,dy$

- $\sigma(x,D)f(x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\,\widehat{f(\xi)}d\xi$

- We let $L^p(\mathbf{R}^n) = \left\{f : \|f\|_p = \left(\int_{\mathbf{R}^n}|f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty\right\}$, $1 < p < \infty$, be the standard L^p space.

- We let $H^s(\mathbf{R}^n) = \left\{f : \|f\|^s = \left(\int_{\mathbf{R}^n}\left|\left(1+|\xi|^2\right)^{\frac{s}{2}}\widehat{f}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} < \infty\right\}$ be the standard Sobolev space H^s.

- Sobolev space $H^s_p(\mathbf{R}^n)$, where s is a positive integer. $f \in H^s_p(\mathbf{R}^n)$, if $f \in L^p(\mathbf{R}^n)$ and the partial derivatives $\partial^\alpha_x f$, taken in the sense of distributions, belong to $L^p(\mathbf{R}^n)$, whenever $0 \le |\alpha| \le s$. The norm in $H^s_p(\mathbf{R}^n)$ is given by

$$\|f\|^s_p = \sum_{|\alpha|\le s}\|\partial^\alpha_x f\|_{L^p}.$$

- Sobolev space $H^s_p(\mathbf{R}^n)$, where $s \in \mathbf{R}$. A distribution $f \in H^s_p(\mathbf{R}^n)$, if $(1-\Delta)^{\frac{s}{2}} f \in L^p(\mathbf{R}^n)$ and here $(1-\Delta)^{\frac{s}{2}}$ is the one-parameter family of pseudo-differential operators with symbols $a(x,\xi) = \left(1+|\xi|^2\right)^{\frac{s}{2}}$. The norm in $H^s_p(\mathbf{R}^n)$ is given by

$$\|f\|^s_p = \left\|(1-\Delta)^{\frac{s}{2}} f\right\|_{L^p}.$$

- We let $H^p(\mathbf{R}^n)$, $0 < p \le \infty$, denote the Hardy space. $H^p(\mathbf{R}^n) = \{f : f$ is an distribution, there is a $\Phi \in \mathcal{S}$ with $\int \Phi dx \neq 0$ so that $M_\Phi f \in L^p(\mathbf{R}^n)\}$ where M_Φ is the $\Phi-$Maximal function.

- $BMO(\mathbf{R}^n)$ denote the space of functions of bounded mean oscillation. A locally integralble function f will be said to belong to BMO if the inequality

$$\frac{1}{|B|}\int_B |f(x) - f_B|\,dx \le A$$

hold for all balls B; here $f_B = |B|^{-1}\int_B f dx$ denotes the mean value of f over the ball B. $\|f\|_{BMO}$ =smallest A.

- Local Hardy space $h^p(\mathbf{R}^n)$ (see [St2] P.134-136).

- Maximal function

$$Mf(x) = \sup_{r>0}\frac{c_n}{r^n}\int_{|y|\le r} |f(x-y)|\,dy. \tag{0.0.1}$$

- L^2 Maximal function M_2

$$M_2 f(x) = \sup_{r>0}\left(\frac{c_n}{r^n}\int_{|y|\le r} |f(x-y)|^2\,dy\right)^{\frac{1}{2}}. \tag{0.0.2}$$

- Sharp function

$$f^{\#}(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y) - f_Q|\,dy \tag{0.0.3}$$

where

$$f_Q = \frac{1}{|Q|}\int_Q f(y)\,dy \tag{0.0.4}$$

- If Q is cube in $\mathbf{R}^n$, then $|Q|$ =volume.

- If Q is cube in $\mathbf{R}^n$, then $l(Q)$ = edge length.

Contents

Chapter 1

Introduction

The study of pseudo-differential operators is a major part of mathematical analysis with important applications in applied mathematics and physics. Although the study of pseudo-differential operators goes back to the last century its modern foundation is perhaps due to the fundamental work in 1965 of Kohn-Nirenberg [KohNi] (see also Seeley [Se] and Unterberger-Bokobza [UnBo]). It was used in a crucial way in the proof of the Atiyah-Singer Index theorem in [AtSi] and in the regularity of elliptic differential equations. In the theory of several complex variables, pseudo-differential operators are indispensable in studying the $\overline{\partial}-$Neumann problem. The theory of subelliptic and hypoelliptic differential operators achieved its current satisfactory state largely because of pseudo-differential operators. In the solution to the local solvability problem for differential equations by Beals-Fefferman[BeFe], pseudo-differential operators played the key role. Many boundary value problems for differential equations can be reduced to pseudo-differential equations, see for example, Hörmander [Hor2]. Roughly speaking, almost everything involving pseudo-differential operators can be reduced to two parts: the mapping properties and the compositions of the associated special pseudo-differential operators.

In this paper, we consider the mapping properties and symbolic calculus of an important class of pseudo-differential operators, the symbolic class of Hörmander type with rough coefficients. We will prove some new results for these operators. These operators arise naturally from problems in nonlinear partial differential equations. After the introduction of the classical symbol class $S^m_{1,0}$ in [KohNi], Hörmander considered the symbol class $S^m_{\rho,\delta}$ in [Hor1]. Eventually, such classes of pseudo-differential operators played a key role in the local solvability problem for differential operators (see Beals-Fefferman [BeFe]). In an important result on the oblique derivative problem by [GuSa1], it is shown that the solution of the problem can be reduced to a problem of pseudo-differential operators on the boundary.

There have been many developments regarding the mapping properties and compositions of symbols in the class $S^m_{\rho,\delta}$. Specifically, C. Fefferman, C. Fefferman and E. Stein, A. Calderón and R. Vaillancourt, R. Coifman-Y. Meyer and A. Miyachi have made fundamental contributions to the subject, we refer to [St2] for the complete references. The results in this paper can be viewed as a further step in this direction.

We now briefly demonstrate through some examples here how we arrive at pseudo-differential operators. Let's start with a constant coefficient differential equation $P(D)\,u(x) = f(x)$. If one takes the Fourier transform on both sides, one obtains

$$P(\xi)\,\widehat{u}(\xi) = \widehat{f}(\xi),$$

and if $P(\xi) \neq 0$, we can divide both sides by $P(\xi)$ and then take the inverse Fourier transform to get

$$u(x) = \int e^{ix\cdot\xi} P(\xi)^{-1}\,\widehat{f}(\xi)\,d\xi. \tag{1.0.1}$$

Since the Fourier transform is given by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^n} \int e^{-iy\cdot\xi} f(y)\, dy, \tag{1.0.2}$$

it follows that

$$\begin{aligned} u(x) &= \int e^{ix\cdot\xi} \frac{1}{(2\pi)^n} \int e^{-iy\cdot\xi} f(y)\, dy P(\xi)^{-1}\, d\xi \\ &= \int \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} P(\xi)^{-1}\, d\xi f(y)\, dy \\ &= \int (\widetilde{P(\xi)^{-1}})(x-y) f(y)\, dy, \end{aligned} \tag{1.0.3}$$

where we have denoted

$$(\widetilde{P(\xi)^{-1}})(x-y) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} P(\xi)^{-1}\, d\xi.$$

Thus (1.0.1) and (1.0.3) are two expressions for the formal solution of the differential equation $P(D)\,u(x) = f(x)$. In order for these formal calculations to be justified, one requires restrictions on both $f(x)$ and $P(x)$. To illustrate one of the motivations for the study of pseudo-differential operators we present the underlying heuristics in terms of what may be called the "freezing principle".

Consider the solutions of the classical elliptic second-order equation

$$(Lu)(x) = \sum a_{i,j}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x), \tag{1.0.4}$$

where $\{a_{i,j}(x)\}$ is a matrix which is assumed to be real, symmetric and positive definite. Let P be the inverse of L(or more precisely), $LP = I + E$, where I is the identity operator and E is an error term which is "small" in an appropriate sense.

To do this, fix an arbitrary point x_0, and "freeze" the operator L at x_0, that is

$$L_{x_0} = \sum a_{i,j}(x_0) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{1.0.5}$$

which may be considered a differential operator with constant coefficients. Hence

$$L_{x_0} u(x) = \sum a_{i,j}(x_0) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x), \tag{1.0.6}$$

and applying the Fourier transform on both sides of (1.0.6) one obtains

$$(\widehat{L_{x_0} u})(\xi) = \sum a_{i,j}(x_0) \xi_i \xi_j = \hat{f}(\xi).$$

By inversion of the Fourier transform we obtain

$$u(x) = \int_{\mathbf{R}^n} \left(\sum_{i,j} a_{i,j}(x_0) \xi_i \xi_j \right)^{-1} \hat{f}(\xi) e^{ix\cdot\xi} d\xi,$$

where $\left(\sum_{i,j} a_{i,j}(x_0)\xi_i\xi_j\right)^{-1}$ is the multiplier. To avoid the awkward (and largely irrelevant) singularity of this multiplier at $\xi = 0$, we introduce a new cut-off function η that vanishes near the origin and equals 1 for large ξ. We then define the operator P_{x_0} by

$$(P_{x_0} f)(x) = \int_{\mathbf{R}^n} \left(\sum_{i,j} a_{i,j}(x_0) \xi_i \xi_j \right)^{-1} \eta(\xi) \hat{f}(\xi) e^{ix\cdot\xi} d\xi \tag{1.0.7}$$

and observe that here the error term is actually an infinitely smoothing operator.

It is reasonable to suppose that the P we are looking for should be well approximated by P_{x_0} when x is near x_0 . To make this precise, we unfreeze x_0 and define P by $(Pf)(x) = (P_x f)(x)$, i.e. ,

$$(Pf)(x) = \int_{\mathbf{R}^n} \left(\sum_{i,j} a_{i,j}(x) \xi_i \xi_j \right)^{-1} \eta(\xi) \hat{f}(\xi) e^{ix\cdot\xi} d\xi \tag{1.0.8}$$

The operator P so given is a prototype of a pseudo-differential operator. Moreover, one has $LP = I + E_1$, where the error operator E_1 is "smoothing of order 1". That this is indeed the case is the main point of the symbolic calculus.

Here we briefly indicate several of the reasons why the classes $S_{\rho,\rho}$ (in particular $S_{\frac{1}{2},\frac{1}{2}}$) are of interest.

1.1 The Heat Equation

Our underlying space will be $\mathbf{R}^{n+1}$ with points $x = (x_0, x_1, ..., x_n)$. We also write (t, x'), where $t = x_0$ and $x' = (x_1, ..., x_n)$. Similarly we split the dual variable ξ as $\xi = (\tau, \xi')$, with $\tau = \xi_0$ dual to $t = x_0$ and $\xi' = (\xi_1, ... \xi_n)$ dual to x', We consider the operator L given by

$$L(u) = \frac{\partial u}{\partial t} - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} \quad , \tag{1.1.9}$$

and try to solve the problem $Lu = f$ by constructing an approximate inverse P so that $LP = I + E$, with an appropriately small error term E. We do this by setting $P = T_a$, where the symbol a is essentially the reciprocal of the characteristic polynomial of (1.1.9) and is given by

$$a(x, \xi) = a(\xi) = \left(i\tau + |\xi'|^2\right)^{-1} \eta(\xi) \ ; \tag{1.1.10}$$

here η is a smooth cut-off function that vanishes near the origin and equals 1 for large $\xi = (\tau, \xi')$. Closely connected to the symbol a are the symbols $a_{i,j}$, given by

$$a_{i,j}(\xi) = \xi_i \xi_j a(\xi), \qquad for \qquad 1 \leq i, j \leq n. \tag{1.1.11}$$

Then

$$|a(\xi)| \leq A(1 + |\xi|)^{-1}, \tag{1.1.12}$$

upon considering ξ of the form (τ, ξ'), with $|\xi'|^2 \leq |\tau|$, $\tau \to \infty$. When differentiating (1.1.10) with the respect to any of the ξ' variables, the gain is only the factor $(1+|\xi|)^{-\frac{1}{2}}$ (and not $(1+|\xi|)^{-1}$), as is evident from taking $\xi = (\tau, \xi')$, with $|\xi'|^2 \sim |\tau| \to \infty$. Thus the best that can be said for the symbol $a(x, \xi)$ is that it belong to the class $S^{-1}_{\frac{1}{2},0}$, and similarly the symbols $a_{i,j}$ belong to $S^{0}_{\frac{1}{2},0}$. Finally, if we transform the underlying space by a smooth change of variables (which transforms the heat equation into another "parabolic" equation), the best that we could expect from the change of variables argument is that the corresponding symbols would be in the classes $S^{-1}_{\frac{1}{2},\frac{1}{2}}$ and $S^{0}_{\frac{1}{2},\frac{1}{2}}$. Indeed, for $\sigma(x, \xi) \in S^{m}_{\rho,\delta}$, with the change variable $x = \phi(y)$, the new symbol is congruent to

$$\tilde{\sigma}(y, \eta) = \sigma\left(\phi^{-1}(y), {}^{t}\Phi^{-1}(y)\,\eta\right)$$

modulo a lower order term, where Φ denotes the Jacobian matrix of the diffeomorphism $\phi^{-1} : \Omega' \to \Omega$ (see [Trv]) and ${}^{t}\Phi^{-1}$ the transpose of Φ^{-1}.

Since $\sigma(x, \xi) \in S^{m}_{\frac{1}{2},0}$, the chain rule yields

$$\begin{aligned} \partial_y \tilde{\sigma}(y, \eta) &= \partial_x \sigma\left(\phi^{-1}(y), {}^{t}\Phi^{-1}(y)\,\eta\right) \partial_y \phi^{-1}(y) \qquad (1.1.13) \\ &\quad + \partial_\xi \sigma\left(\phi^{-1}(y), {}^{t}\Phi^{-1}(y)\,\eta\right) \partial_y^t \left({}^{t}\Phi^{-1}(y)\right) \eta. \end{aligned}$$

So

$$|\partial_y \tilde{\sigma}(y, \eta)| \leq C\,|1 + |\eta||^{m+\frac{1}{2}}$$

and we obtain $\tilde{\sigma}(y, \eta) \in S^{m}_{\frac{1}{2},\frac{1}{2}}$.

1.2 Cauchy-Szegö kernel

The Cauchy-Szegö kernel is a distribution represented, away from the origin, by the function

$$K(x) = c_n \left(it + |x'|^2\right)^{-n-1}, \qquad (t, x') \in \mathbf{R}^{n+1}; \tag{1.2.14}$$

the underlying complex space is $\mathbf{C}^{m+1}$ and $n = 2m$. The Fourier transform of K is the function

$$\widehat{K}(\xi) = \begin{cases} c_n e^{-\pi|\xi'|^2/2\tau} & if \quad \tau > 0; \\ 0 & if \quad \tau \leq 0. \end{cases} \tag{1.2.15}$$

where ξ' is the dual variable to x', τ is the dual variable to x and $\xi = (\tau, \xi')$.

The symbol corresponding to the Cauchy-Szegö projection is derived from the function $\widehat{K}(\xi) \cdot \eta(\xi)$, where η is a smooth cut-off function that vanishes near the origin and equals 1 for large $\xi = (\tau, \xi')$. This function belongs to $S^0_{\frac{1}{2},0}$. To pass from $\widehat{K}(\xi) \cdot \eta(\xi)$ to the actual symbol of Cauchy-Szegö projection forces us to consider an implicit change of variables in our formula. Thus again we get symbols in the class $S^0_{\frac{1}{2},\frac{1}{2}}$.

1.3 Oblique derivative problem

For example, let's consider the model problem

$$\begin{aligned} \Delta u &= f && in \quad \mathbf{R}^{n+2}_+ \\ \left(\frac{\partial}{\partial t} - a(x,t)\frac{\partial}{\partial r}\right) u &= g && on \ \partial\mathbf{R}^{n+2}_+ = \mathbf{R}^{n+1}, t \in \mathbf{R}, \\ u &= h && when \ t = 0, r = 0 \end{aligned} \tag{1.3.16}$$

where $\mathbf{R}_+^{n+2} = \{(x,t,r) : x \in \mathbf{R}^n,\ t \in \mathbf{R},\ r > 0\}$ and $ta(x,t) \geq 0$ for all $t \in \mathbf{R}$ When this problem is pushed to the boundary $\mathbf{R}^{n+1}$, via the Poisson integral, then microlocally, a parametrix for the resulting pseudo-differential equation in given by (when $f = 0$, $h = 0$)

$$\begin{aligned} u(x,t,0) &= \int_{\mathbf{R}^n} e^{ix\cdot\xi} \int_0^t e^{-|\xi|\int_{t'}^t a(x,\theta)d\theta} g(\xi,t')\,dt'd\xi \qquad (1.3.17) \\ &= \int_{\mathbf{R}^n} e^{ix\cdot\xi} \int_0^t \tau(x,\xi)\, g(\xi,t')\,dt'd\xi, \end{aligned}$$

here

$$\tau(x,\xi) = e^{-\int_{t'}^t a(x,\theta)d\theta|\xi|}. \qquad (1.3.18)$$

Now

$$\begin{aligned} \partial_x \tau(x,\xi) &= e^{-\int_{t'}^t a(x,\theta)d\theta|\xi|} \partial_x \left(-\int_{t'}^t a(x,\theta)d\theta\right)|\xi| \\ &= e^{A(x,t,t')|\xi|} \partial_x (A(x,t,t'))\,|\xi| \end{aligned} \qquad (1.3.19)$$

where $A(x,t,t') = -\int_{t'}^t a(x,\theta)d\theta$. For fixed t, t', we have $A(x,t,t') \geq 0$ and so

$$\begin{aligned} |\partial_x A(x,t,t')| &\leq CA(x,t,t')^{\frac{1}{2}} \left(\sup\left|\nabla^2 A(x,t,t')\right|\right)^{\frac{1}{2}} \\ &\leq CA(x,t,t')^{\frac{1}{2}}, \end{aligned} \qquad (1.3.20)$$

and we get

$$\begin{aligned} |\partial_x \tau(x,\xi)| &\leq e^{A(x,t,t')|\xi|} |\partial_x (A(x,t,t'))|\,|\xi| \\ &\leq Ce^{A(x,t,t')|\xi|} \left(A(x,t,t')^{\frac{1}{2}}\right)|\xi| \\ &\leq Ce^{A(x,t,t')|\xi|} (A(x,t,t')\,|\xi|)^{\frac{1}{2}}\,|\xi|^{\frac{1}{2}} \end{aligned}$$

$$\leq \ C\,|\xi|^{\frac{1}{2}}. \tag{1.3.21}$$

Here we used the fact that $y^{\alpha}e^{-y}$ is bounded, when $\alpha > 0$ and $y > 0$. Thus, we obtain $\tau(x,\xi) \in C^{\lambda}S^{m}_{1,\frac{1}{2}}$.

For a general domain, this operator has symbol

$$\tau(x,\xi) = e^{-\int_{t'}^{t} a(x,\theta)Q(x,\theta,\xi)d\theta} \tag{1.3.22}$$

where t and t' are parameters , $a \in C^{\lambda+2}$ is nonnegative and $Q \in C^{\lambda}$ is positive and homogeneous of degree 1 in ξ . Using the inequality $|\nabla_x a| \leq C\,|a|^{\frac{1}{2}}$, it is easy to see that $\tau(x,\xi) \in C^{\lambda}S^{m}_{1,\frac{1}{2}}$.

Finally, we point out that when P. Guan and E. Sawyer[GuSa1] checked that $\tau(x,\xi) \in C^{\lambda}S^{0}_{1,\frac{1}{2}}$, they discovered that such symbols actually behave much better than $C^{\lambda}S^{0}_{1,\frac{1}{2}}$. Indeed

$$\begin{aligned}\partial_x\tau(x,\xi) &= -\left\{\int_{t'}^{t}(\partial_x a(x,\theta))\,Q(x,\theta,\xi)\,d\theta + \int_{t'}^{t} a(x,\theta)\,(\partial_x Q(x,\theta,\xi))\,d\theta\right\} \\ &\quad \times e^{-\int_{t'}^{t} a(x,\theta)Q(x,\theta,\xi)d\theta} \\ &= \tau_1 + \tau_2 \end{aligned} \tag{1.3.23}$$

where $\tau_1 \in C^{\lambda}S^{\frac{1}{2}}_{1,\frac{1}{2}}$ and $\tau_2 \in C^{\lambda-1}S^{0}_{1,\frac{1}{2}}$. This means that $\partial_x\tau(x,\xi)$ decomposes into one term having order worse by $\frac{1}{2}$ but no loss in smoothness, and the other term having 1 degree less smoothness but no loss of order. This property persists for each of the symbols τ_1 and τ_2, etc., and such symbols τ are said to belong to the restricted symbol class $C^{v}_{\infty}S^{m}_{1,\frac{1}{2}}$. These symbols enjoy the mapping properties of the better behaved class $C^{\lambda}S^{0}_{1,0}$.

We conclude this introduction with the organization of the rest of the thesis. In the next section, we will introduce some necessary notations and state our main results. In Chapter 2 we will prove L^2 results. In Chapter 3 we will consider the symbol in class $C^v S^m_{\rho,\delta}$, prove some L^p results. In Chapter 4 we will deal with the restricted symbol classes $C^v_\infty S^m_{\rho,\delta}$, and prove that they have better mapping properties than $C^v S^m_{\rho,\delta}$.

Chapter 2

Review of Known Results and Statements of New Theorems

We begin this chapter by establishing our notations and definitions in the first section. We then recall the relevant known theorems in the second section, and conclude with statements of our new theorems in the third.

2.1 Definitions and Preliminaries

So far, we have used the notations $S^m_{\rho,\delta}$ and $C^v S^m_{\rho,\delta}$ freely without specifying their definitions. To state our results, we introduce here some necessary standard notations.

- Difference operator δ^h_x

$$\delta^h_x f(x) = f(x+h) - f(x). \tag{2.1.1}$$

- Fractional Integral

$$I_\theta f(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\theta}} dy. \tag{2.1.2}$$

- Identity

$$(1-\Delta_\xi)^N \left(\frac{1}{1+|x|^2}\right)^N e^{ix\cdot\xi} = e^{ix\cdot\xi}. \tag{2.1.3}$$

- Identity

$$\frac{\delta_x^h}{e^{ih\cdot(\xi-\eta)}-1} e^{ix\cdot(\xi-\eta)} = e^{ix\cdot(\xi-\eta)}. \tag{2.1.4}$$

$$f^\#(x) = \sup_Q \left\{ \inf_c \frac{1}{|Q|} \int_Q |f-c| \right\} \tag{2.1.5}$$

Definition 2.1.1 *A function σ belongs to the symbol class $S^m_{\rho,\delta}$, if $\sigma = \sigma(x,\xi)$ is smooth for $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n$ and*

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C_{\alpha,\beta} \langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|} \qquad x,\xi \in \mathbf{R}^n,\ \delta \leq \rho. \tag{2.1.6}$$

for all multi-indexes α , β and some fixed ρ and δ, here $\langle\xi\rangle = \left(1+|\xi|^2\right)^{\frac{1}{2}}$.

There are some special important cases .

(1) $S^0_{1,1}$ is the largest class whose corresponding operators T have kernel representations (in an appropriate sense)

$$(Tf)(x) = \int_{\mathbf{R}^n} K(x,y) f(y) dy, \tag{2.1.7}$$

where K satisfies inequalities of Calderon-Zygmund type required for the singular integrals. For these operators , one has in fact that

$$\left|\partial_x^\alpha \partial_y^\beta K(x,y)\right| \leq C_{\alpha,\beta} |x-y|^{-n-|\alpha|-|\beta|} \tag{2.1.8}$$

for all multi-indexes α and β. But unfortunately, such operators are not (in general) bounded on L^2. This is given in the following example in [St2].

Example 2.1.1 *We shall construct our example in* $\mathbf{R}^1$.

Let $\widehat{\Psi} \in S$ *with* $\widehat{\Psi}(\xi)$ *supported in* $2^{-\frac{1}{2}} \leq |\xi| \leq 2^{\frac{1}{2}}$ *and* $\widehat{\Psi}(\xi) = 1$ *for* $2^{-\frac{1}{4}} \leq |\xi| \leq 2^{\frac{1}{4}}$. *Choose* $a_j(x) = e^{-2\pi i 2^j x}$, *and set*

$$\sigma(x,\xi) = \sum_{j=1}^{\infty} e^{-2\pi i 2^j x} \widehat{\Psi}(2^{-j}\xi). \tag{2.1.9}$$

Noting $\widehat{\Psi}(2^{-j}\xi)$ *is supported in the set where* $2^{j-\frac{1}{2}} \leq |\xi| \leq 2^{j+\frac{1}{2}}$, *we see that, for each* ξ, *at most one term in above formula is nonzero. Since*

$$\left|\partial_x^{\beta} e^{-2\pi i 2^j x}\right| \leq A_{\beta} 2^{j|\beta|}, \tag{2.1.10}$$

it's easy to see $\sigma(x,\xi) \in S^0_{1,1}$.

Next we choose f_0 *to be a nonzero element of the Schwarz class* $\mathcal{S}$ *and whose Fourier transform is supported in the set* $|\xi| \leq \frac{1}{2}$, *and let*

$$f_N(x) = \sum_{j=4}^{N} (\frac{1}{j}) e^{2\pi i 2^j x} f_0(x). \tag{2.1.11}$$

By Plancherel's theorem, we see that

$$\|f_N\|_{L^2} = \left(\sum_{j=4}^{N} (\frac{1}{j^2})\right) \|f_0\|_{L^2} \leq c, \ \ as N \to \infty, \tag{2.1.12}$$

and

$$\sigma(x,D) f_N = \left(\sum_{j=4}^{N} (\frac{1}{j})\right) f_0, \tag{2.1.13}$$

giving $\|\sigma(x,D) f_N\|_{L^2} \geq c \log N$; *hence* $\sigma(x,D)$ *is not bounded on* L^2.

(2) $S^m_{\frac{1}{2},\frac{1}{2}}$, as we mentioned above in Chapter 1. Various operators arising in the study of the local solvability problem and the subelliptic problem (e.g. , heat equation and certain operators on the Heisenberg group) are of this kind.

(3) $S^m_{1,\frac{1}{2}}$ arising from the oblique derivative problem (see [GuSa1]).

2.2 L^p mapping properties

We now summarize some known results. While operators with symbols in $S^0_{0,0}$ are easily seen to be bounded on L^2, in general operators in $S^0_{1,1}$ are not bounded on L^2. However for $0 \leq \rho < 1$, Calderòn-Vaillancourt in 1972 [CaVa] proved that operators with symbols in $S^0_{\rho,\rho}$ are bounded on L^2.

Theorem 2.2.1 *Let $\sigma(x,\xi) \in S^0_{\rho,\rho}$, $0 \leq \rho < 1$, and $\sigma(x,\xi) : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow C$ is a continuous function whose derivatives $\partial^\alpha_x \partial^\beta_\xi \sigma(x,\xi)$ in distribution sense satisfy*

$$\left|\partial^\alpha_x \partial^\beta_\xi \sigma(x,\xi)\right| \leq C_{\alpha,\beta} \langle\xi\rangle^{\rho|\alpha|-\rho|\beta|} \qquad x,\xi \in \mathbf{R}^n, \tag{2.2.14}$$

for all multi-indexes α and β ,where $\langle\xi\rangle = \left(1+|\xi|^2\right)^{\frac{1}{2}}$, then $\sigma(x,D)$ is continuous from $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$.

For results related to $S^m_{\rho,\rho}$, we have the following (See E. Stein [St2], C. Fefferman [Fef], A. Miyachi [Miya2]):

When $\sigma(x,\xi) \in S^m_{\rho,\delta}$ and $0 < \rho \leq 1$, $0 \leq \delta \leq \rho$, $\delta < 1$, $m = \frac{n}{2}(1-\rho)$. We have that $\sigma(x,D)$ is weak-type $(1,1)$; also $\sigma(x,D) : H^1 \longrightarrow L^1$ and $\sigma(x,D) : L^\infty \to BMO$. Also, if $1 < p < \infty$, and $|p^{-1} - 2^{-1}| \leq \frac{m}{[n(\rho-1)]}$, then $\sigma(x,D) : L^p \to L^p$.

R. Coifman and Y. Meyer [CoMe] obtained the following boundness result for $S^m_{0,0}$. If $\sigma(x,\xi) \in S^{-\frac{n}{2}}_{0,0}$, then $\sigma(x,D) : H^1 \longrightarrow L^1$;and $\sigma(x,D) : L^\infty \to BMO$. Also, if $\sigma(x,\xi) \in S^m_{0,0}$, $1 < p < \infty$, and $|p^{-1} - 2^{-1}| \leq -\frac{m}{n}$, then $\sigma(x,D) : L^p \to L^p$.

In applications, one is required to work with operators with limited smoothness in the variables x. It is very important to know the minimal smoothness assumptions required on x, in particular, for the applications to nonlinear problems (in most cases, they are very crucial). So we have the following definition.

Definition 2.2.1 *A symbol* $\sigma \in C^k S^m_{\rho,\delta}$ *,*$k \geq 0$*, if* $\sigma(x,\xi) : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{C}$ *is a continuous function whose derivatives* $\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)$ *in the distribution sense satisfy the following conditions:*

There is a constant $C > 0$*, such that, for* α *,* $\beta \in \mathbf{N}^n$ *,* $|\alpha| \leq [k]$*, and* $x, \xi, h \in \mathbf{R}^n$*, we have*

(1) If $|\alpha| \leq [k]$*, then*

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C \langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|} \tag{2.2.15}$$

(2) If $|\alpha| = [k]$*, and* $|h| \leq 1$ *then*

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x+h,\xi) - \partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C \langle\xi\rangle^{m+\delta|k|-\rho|\beta|} |h|^{k-[k]} \tag{2.2.16}$$

where $\langle\xi\rangle = \left(1+|\xi|^2\right)^{\frac{1}{2}}$*,*

As for $C^v S^m_{\rho,\delta}$, there exists a vast literature. We record here some major developments. If $\sigma(x,\xi) \in C^k S^0_{\rho,\delta}, k > \frac{n}{2}$, then $\sigma(x,D) : L^2 \to L^2$, See R. Coifman and Y. Meyer [CoMe]. The introduction of the symbol smoothing method is very useful in obtaining mapping properties of pseudo-differential operators. One tries to write a symbol $\sigma(x,\xi) \in C^v S^m_{1,0}$ as a sum of a smooth symbol and a remainder of lower order. The smooth part will not belong to $S^m_{1,0}$, but rather to one of the Hörmander classes $S^m_{1,\delta}$. Using this method and the result of G. Bourdaud (see [Bou2]), following the pioneering work of E. Stein, various continuity results were obtained (see [Trv]). Here we only mention G. Bourdaud's result in [Bou2]and a very useful extension.

Theorem 2.2.2 *[Bou2] If $v > 0$ and $1 < p < \infty$, and if $\sigma(x,\xi) \in C^v S^m_{1,1}$, then*

$$\sigma(x,D) : H_p^{s+m} \to H_p^s$$

provided $0 < s < v$. Furthermore , under these hypotheses ,

$$\sigma(x,D) : C^{s+m} \to C^s.$$

Theorem 2.2.3 *[Bou2] If $\sigma(x,\xi) \in C^v S^0_{1,\delta}$ with $v > 0, \delta \in (0,1)$, then, for $1 < p < \infty$,*

$$\sigma(x,D) : H_p^s \to H_p^s \quad \text{for} \quad s \in (-(1-\delta)v, v).$$

Furthermore, under these hypotheses,

$$\sigma(x,D) : C^s \to C^s.$$

Also an examination of G. Bourdaud's proof shows that the smoothness condition on the variable ξ can be reduced . We need a new definition to illustrate this.

Definition 2.2.2 *Let $k_1, k_2 \geq 0$, $0 \leq \rho, \delta \leq 1$ and $m \in \mathbf{R}$. A symbol $\sigma \in C^{k_1,k_2} S^m_{\rho,\delta}$, if $\sigma(x,\xi) : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{C}$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)$ in distribution sense satisfy the following conditions:*

There is a constant $C > 0$ such that for α, $\beta \in N^n$, $|\alpha| \leq [k_1]$, $|\beta| \leq [k_2]$ and $x, \xi, h, v \in \mathbf{R}^n$, we have

(1) If $|\alpha| \leq [k_1]$, $|\beta| \leq [k_2]$ then

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C \langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|} \tag{2.2.17}$$

(2) If $|\alpha| = [k_1]$, $|\beta| \leq [k_2]$ and $|h| \leq 1$ then

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x+h,\xi) - \partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C \langle\xi\rangle^{m+\delta k_1-\rho|\beta|} |h|^{k_1-[k_1]} \tag{2.2.18}$$

(3) If $|\alpha| \leq [k_1]$, $|\beta| = [k_2]$ and $|v| \leq 1$ then

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi + v) - \partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)\right| \leq C \langle \xi \rangle^{m+\delta|\alpha|-\rho k_2} |v|^{k_2-[k_2]} \qquad (2.2.19)$$

(4) If $|\alpha| = [k_1]$, $|\beta| = [k_2]$, $|h| \leq 1$ and $|v| \leq 1$ then

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x+h, \xi + v) - \partial_x^\alpha \partial_\xi^\beta \sigma(x+h, \xi) - \partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi + v) + \partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)\right|$$

$$\leq C \langle \xi \rangle^{m+\delta k_1-\rho k_2} |v|^{k_2-[k_2]} |h|^{k_1-[k_1]}. \qquad (2.2.20)$$

First, A. Calderón and R. Vaillancourt [CaVa] proved:

Theorem 2.2.4 *Let $\sigma(x, \xi) \in C^{k_1,k_2} S^0_{0,0}$. If k_1, k_2 are sufficiently large real numbers and $\sigma(x, \xi) : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow C$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)$ in the distribution sense satisfy*

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)\right| \leq C_{\alpha,\beta} \langle \xi \rangle^m \qquad x, \xi \in \mathbf{R}^n, \qquad (2.2.21)$$

for all multi-indices α and β ,where $\langle \xi \rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}}$, then $\sigma(x, D)$ is continuous from $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$.

For $1 < p < \infty$ case, R. Coifman and Y. Meyer [CoMe] proved:

Theorem 2.2.5 *Let $\sigma(x, \xi) \in C^{k_1,k_2} S^m_{0,0}$,and $m = -n\left|\frac{1}{p} - \frac{1}{2}\right|$ if k_1, k_2 are sufficiently large real numbers and $\sigma(x, \xi) : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow C$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)$ in distribution sense satisfy*

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)\right| \leq C_{\alpha,\beta} \langle \xi \rangle^m \qquad x, \xi \in \mathbf{R}^n, \qquad (2.2.22)$$

for all multi-indexes α and β, where $\langle \xi \rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}}$, then $\sigma(x, D)$ is continuous from $L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$,$1 < p < \infty$.

A. Calderón and R. Vaillancourt [CaVa] reduced the condition on k_1, k_2, so that $\alpha, \beta \in \{1,2,3\}^n$, and proved the following:

Theorem 2.2.6 *Let $\sigma(x,\xi) \in C^{k_1,k_2} S^m_{0,0}$, and $m = -n\left|\frac{1}{p} - \frac{1}{2}\right|$ and $\sigma(x,\xi)$: $\mathbf{R}^n \times \mathbf{R}^n \longrightarrow C$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)$ in the distribution sense satisfy*

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C_{\alpha,\beta} \langle \xi \rangle^m \qquad x,\xi \in \mathbf{R}^n;\ \alpha,\beta \in \{1,2,3\}^n \tag{2.2.23}$$

where $\langle \xi \rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}}$, then $\sigma(x,D)$ is continuous from $L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$, $1 < p < \infty$.

Coifman-Meyer further reduced the condition on k_1, k_2,and proved the above result for k_1, $k_2 \geq 2n$. Cordes weakened the condition (for the case $p = 2$) to $|\alpha|, |\beta| \leq [\frac{n}{2}]+1$. For the $0 \leq \delta \leq \rho < 1$ case , Calderón-Vaillancourt established the following:

Theorem 2.2.7 *Let $\sigma(x,\xi) \in C^{k_1,k_2} S^0_{\rho,\delta}, 0 \leq \delta \leq \rho < 1$. If k_1, k_2 are sufficiently large real numbers and $\sigma : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta \sigma$ in the distribution sense satisfy*

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C_{\alpha,\beta} \langle \xi \rangle^{\delta|\alpha| - \rho|\beta|} \qquad x,\xi \in \mathbf{R}^n,\ \delta \leq \rho, \tag{2.2.24}$$

for all multi-indexes α and β, where $\langle \xi \rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}}$, with $|\alpha| \leq k_1$ and $|\beta| \leq k_2$, then $\sigma(x,D)$ is continuous from $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$.

Then C. Fefferman extended this result to $p \neq 2$ as follows:

Theorem 2.2.8 *Let $\sigma(x,\xi) \in C^{k_1,k_2} S^m_{\rho,\delta}$, $0 \leq \delta \leq \rho < 1$ and $m = -n(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|$. If k_1, k_2 are sufficiently large real numbers and $\sigma : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta \sigma$ in the distribution sense satisfy*

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C_{\alpha,\beta} \langle \xi \rangle^{m + \delta|\alpha| - \rho|\beta|} \qquad x,\xi \in \mathbf{R}^n,\ \delta \leq \rho. \tag{2.2.25}$$

for all multi-indexes α *and* β *,where* $\langle\xi\rangle = \left(1+|\xi|^2\right)^{\frac{1}{2}}$ *, with* $|\alpha| \le k_1$ *and* $|\beta| \le k_2$*, then* $\sigma(x,D)$ *is continuous from* $L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$, $1 < p < \infty$.

Calderón-Vaillancourt specified the condition on k_1, k_2, they proved it for $|\alpha| \le 2m'$ with $m' \in N$ and $m'(1-\rho) \ge \frac{5n}{4}$ and $|\beta| \le 2[\frac{n}{2}]+n$. Coifman-Meyer showed that the result is true (for the case $p=2$) for $|\alpha|$, $|\beta| \le m'$ with $m' \in N$ and $m' \ge [\frac{n}{2}]+1$.

After some further improvements by Kato, Cordes, Beals, Nagase and Hwang, Miyachi [Miya2] proved the following theorem, which gives the sharpest results. See also references given in [Miya2] and [St2].

Theorem 2.2.9 *Let* $0 \le \delta \le \rho < 1$ *and* $m = -n(1-\rho)\left|\frac{1}{p}-\frac{1}{2}\right|$

(1) If $0 < p \le 1, \rho = 0, k_1 > \frac{n}{2}, k_2 > \frac{n}{p}$ *and* $\sigma(x,\xi) \in C^{k_1,k_2}S^m_{\rho,\delta}$*,then* $\sigma(x,D)$ *is continuous from* $H^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ *,where* H^p *are the Hardy spaces.*

(2) If $0 < p < 1, k_1 > \frac{n}{2}, k_2 > \frac{n}{p}$ *and* $\sigma(x,\xi) \in C^{k_1,k_2}S^m_{\rho,\delta}$*,then* $\sigma(x,D)$ *is continuous from* $h^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ *,where* h^p *are the local Hardy spaces.*

(3) If $1 < p \le 2, k_1 > \frac{n}{2}, k_2 > \frac{n}{p}$ *and* $\sigma(x,\xi) \in C^{k_1,k_2}S^m_{\rho,\delta}$*, then* $\sigma(x,D)$ *is continuous from* $L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$.

(4) If $2 < p < \infty, k_1 > \frac{p}{n}, k_2 > \frac{n}{2}$ *and* $\sigma(x,\xi) \in C^{k_1,k_2}S^m_{\rho,\delta}$*, then* $\sigma(x,D)$ *is continuous from* $L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$.

2.2.1 Mapping Properties for Restricted Symbols

When P. Guan and E. Sawyer extended their results on the oblique derivative problem [GuSa1] to nonsmooth domains and operators [GuSa2], they introduced a new class of symbols which have some special properties. These operators have better mapping properties than the corresponding operators in the usual symbol classes. The further study of these operators was carried out by E. Sawyer in [Sa]. The following is the definition of the operators they considered.

Definition 2.2.3 *(Guan and Sawyer)[Sa]Define symbol classes $C_k^v S_{\rho,\delta}^m$ by induction on k. Let $C_0^v S_{\rho,\delta}^m = C^v S_{\rho,\delta}^m$. For $\lambda \geq 0$, set $l = [\lambda]$ and $\theta = \lambda - [\lambda]$. Assuming $C_{k-1}^v S_{\rho,\delta}^m$ has been defined, we say σ is in the symbol class $C_k^v S_{\rho,\delta}^m$ if $\sigma \in C_{k-1}^v S_{\rho,\delta}^m$,*

$$\nabla_x^l \sigma(x,\xi) \in \sum_{i+j=l} C_{k-1}^{v-i} S_{\rho,\delta}^{m+j\delta} \tag{2.2.26}$$

and

$$|h|^{-\theta} \{\nabla_x^l \sigma(x+h,\xi) - \nabla_x^l \sigma(x,\xi)\} \in \sum_{i+j=\lambda} C_{k-1}^{v-i} S_{\rho,\delta}^{m+j\delta}. \tag{2.2.27}$$

Then set

$$C_\infty^v S_{1,\delta}^m = \cap_{k=0}^\infty C_k^v S_{1,\delta}^m.$$

We quote one of the results in [Sa] here, which shows that symbols in $C_\infty^v S_{1,\delta}^m$ enjoy the same mapping properties as those in $C^v S_{1,0}^m$ (which are strictly better than those in $C^v S_{1,\delta}^m$).

Theorem 2.2.10 *If $\sigma(x,\xi) \in C_\infty^v S_{1,\delta}^m, 0 \leq \delta \leq \frac{1}{2}$, $v > 0$ then $\sigma(x,D) : H_p^{s+m} \to H_p^s$, for $1 < p < \infty$, $s \in (-v, v)$.*

2.3 Statements of main Results

Finally, we state our main results. In this thesis, we observe an interesting symmetric phenomenon between the number of derivatives in the x variables and the decay in the ξ variables, and vice versa, for a symbol $\sigma(x,\xi)$. We now give the following definition:

Definition 2.3.1 *Let m_1, $m_2 \in \mathbf{R}$, $0 \le \delta \le \rho < 1$, k_1, $k_2 > 0$. We define $C^{k_1,k_2} S^{m_1,m_2}_{\rho,\delta}$ to be the collection of continuous functions $\sigma : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ whose derivatives $\partial_x^\alpha \partial_\xi^\beta \sigma$ in the distribution sense satisfy the following conditions:*

There is a constant $C > 0$ such that for α , $\beta \in \mathbf{N}^n$, $|\alpha| \le [k_1]$, $|\beta| \le [k_2]$ and $x, \xi, h, v \in \mathbf{R}^n$, we have

(1) If $|\alpha| \le [k_1]$, $|\beta| \le [k_2]$ then

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \le C \langle x\rangle^{m_2} \langle \xi\rangle^{m_1+\delta|\alpha|-\rho|\beta|} \tag{2.3.28}$$

(2) If $|\alpha| = [k_1]$, $|\beta| \le [k_2]$ and $|h| \le 1$ then

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x+h,\xi) - \partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \le C \langle x\rangle^{m_2} \langle \xi\rangle^{m_1+\delta|k_1|-\rho|\beta|} |h|^{k_1-[k_1]} \tag{2.3.29}$$

(3) If $|\alpha| \le [k_1]$, $|\beta| = [k_2]$ and $|v| \le 1$ then

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi+v) - \partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \le C \langle x\rangle^{m_2} \langle \xi\rangle^{m_1+\delta|\alpha|-\rho|k_2|} |v|^{k_2-[k_2]} \tag{2.3.30}$$

(4) If $|\alpha| = [k_1]$, $|\beta| = [k_2]$, $|h| \le 1$ and $|v| \le 1$ then

$$\begin{aligned}&\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x+h,\xi+v) - \partial_x^\alpha \partial_\xi^\beta \sigma(x+h,\xi) - \partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi+v) + \partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \\ &\qquad \le C \langle x\rangle^{m_2} \langle \xi\rangle^{m_1+\delta|k_1|-\rho|k_2|} |v|^{k_2-[k_2]} |h|^{k_1-[k_1]} .\end{aligned} \tag{2.3.31}$$

Note: The above definition is symmetric in x and ξ in the sense that as a function of ξ, σ behaves like a symbol in $C^{k_2} S^{m_2}_{\rho,\delta}$, and as a function of x, it behaves like a symbol in $C^{k_1} S^{m_1}_{0,0}$.

First, for the L^2–boundness of the operators in the symbol class $C^{k_1,k_2} S^{m_1,m_2}_{\rho,\delta}$, we have:

Theorem 2.3.1 *Let $\sigma \in C^{k_1,k_2}S^{m_1,m_2}_{0,0}$, and suppose $k_1 - m_1 > \frac{n}{2}$, $k_2 - m_2 > \frac{n}{2}$ and that $\partial^\alpha_x \partial^\beta_\xi \sigma(x,\xi)$ in the distribution sense satisfies the conditions in (2.3.28), (2.3.29), (2.3.30), (2.3.31). Then $\sigma(x,D)$: $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is continuous.*

The following two results have been established for the operators in the symbol classes $C^v S^m_{\rho,\delta}$ and $C^v_\infty S^m_{\rho,\delta}$ with $\rho < 1$ respectively. (For the case when $\rho = 1$ see [Sa]). The restriction to $p \geq 2$ in these theorems is required by our proof, see the proof of Theorem 2.3.2 in Chapter 4. (This remains an open question for $1 < p < 2$).

Theorem 2.3.2 *If $\sigma(x,\xi) \in C^v S^m_{\rho,\delta}$,$\rho > \delta$,$m < -\frac{n}{2}(1-\rho), v > \frac{n}{2}$, and $p \geq 2$, then*

$$\sigma(x,D) : H^{s+m_1}_p \to H^s_p, \tag{2.3.32}$$

for $s \in (-(1-\delta)v_1, v_1)$, where $v_1 = v - \frac{n}{2}$, $m_1 = m + \frac{n}{2}(1-\rho)$.

Theorem 2.3.3 *If $\sigma(x,\xi) \in C^v_\infty S^m_{\rho,\delta}$, $\rho > \delta$, $m < -\frac{n}{2}(1-\rho), v > \frac{n}{2}$, and $p \geq 2$, then*

$$\sigma(x,D) : H^{s+m_1}_p \to H^s_p, \tag{2.3.33}$$

for $s \in (-v_1, v_1)$,where $v_1 = v - \frac{n}{2}$, $m_1 = m + \frac{n}{2}(1-\rho)$.

Note that in Theorem 2.3.3 the hypothesis $\sigma(x,\xi) \in C^v_\infty S^m_{\rho,\delta}$ leads to the conclusion obtained in Theorem 2.3.2 for the symbol class $C^v S^m_{\rho,0}$.

Chapter 3

$\mathbf{L}^2$ Results

In order to obtain L^2 results, the Fourier transform is a most convenient tool due to Plancherel's theorem, as evidenced by its success for multiplier operators. Pseudo-differential operators, as we mentioned in the introduction, are defined in terms of the Fourier transform of the function, and it looks like a multiplier operator, so it is natural to think we can get L^2 results if we use the method as in the multiplier operators case. However, the method fails on pseudo-differential operators, because the Fourier transform no longer suffices to guarantee L^2 results, and must be supplanted by a more general approach.

3.1 Almost Orthogonality Methods

One of the most effective methods to obtain regularity results is via the Cotlar-Stein Lemma. It says that an operator T is bounded on L^2 if it can be decomposed as a sum $T = \sum T_j$, in which the components are uniformly bounded, and the different T_j are mutually "almost orthogonal ". Using this method Calderón and Vaillancourt[CaVa] proved:

Theorem 3.1.1 *Let $\sigma(x,\xi) \in S^0_{\rho,\rho}$, $0 \leq \rho < 1$, and suppose $\sigma(x,\xi)$: $\mathbf{R}^n \times \mathbf{R}^n \longrightarrow C$ is a continuous function whose derivatives $\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)$ in the distribution sense satisfy*

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C_{\alpha,\beta} \langle \xi \rangle^{m+\rho|\alpha|-\rho|\beta|} \qquad x,\xi \in \mathbf{R}^n, \tag{3.1.1}$$

for all multi-indices α and β, where $\langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}}$. Then $\sigma(x,D)$ is continuous from $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$.

Later R. Coifman and Y. Meyer [CoMe] used further types of almost-orthogonality methods to prove

Theorem 3.1.2 *Suppose $\sigma(x,\xi) \in C^k S^0_{\rho,\delta}$, $0 \leq \delta \leq \rho < 1$, $k > \frac{n}{2}$, then $\sigma(x,D) : L^2 \to L^2$.*

3.2 A Dual Symmetry

Theorem 2.3.1 generalizes Theorem 3.1.2 in the case when $\rho = \delta = 0$ by exploiting symmetry in x and ξ. It is known that there is a relation between the derivative of x and decay of ξ in the symbol $\sigma(x,\xi)$. In the proof of Theorem 2.3.1 one must show that the L^2 norms of

$$H(x,\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\eta} h(\eta) \left(\frac{1}{1+|\xi-\eta|^2}\right)^M d\eta \tag{3.2.2}$$

and

$$F(x,\xi) = \int_{\mathbf{R}^n} e^{-iy\cdot\xi} f(y) \left(\frac{1}{1+|x-y|^2}\right)^N dy \tag{3.2.3}$$

are controlled by the L^2 norms of h and f respectively. Now using the relation between derivative of x and decay of ξ, derivative of ξ and decay of x in symbol $\sigma(x,\xi)$, we have following result.

Theorem 3.2.1 *Let $\sigma(x,\xi) \in C^{k_1,k_2}S_{0,0}^{m_1,m_2}$, and suppose $k_1 - m_1 > \frac{n}{2}$, $k_2 - m_2 > \frac{n}{2}$, $k_1, k_2 \geq 0$ are even integers and that $\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)$ exists in the distribution sense and satisfies the above conditions (2.3.28), (2.3.29), (2.3.30), (2.3.31). Then $\sigma(x,D)$ is continuous on $L^2(\mathbf{R}^n)$.*

To illustrate the ideas involved in the proof , we begin by considering two special cases in subsection 3.2.1 and 3.2.1 below. In order to prove that T is bounded on L^2, we only need to prove

$$\left|\int_{\mathbf{R}^n} g(x)\,Tf(x)dx\right| \leq C\ \|f\|_2\ \|g\|_2 \quad f,g \in L^2. \tag{3.2.4}$$

First we need some formulas. Recall

$$Tf(x) = \sigma(x,D)f(x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\,\widehat{f(\xi)}d\xi. \tag{3.2.5}$$

Using the definition of Fourier transform, we have

$$\int_{\mathbf{R}^n} \widehat{h}(x)\,Tf(x)\,dx \ = (\frac{1}{2\pi})^n \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \widehat{h}(x)\ \widehat{f}(\xi)\,\{e^{ix\cdot\xi}\sigma(x,\xi)\}dxd\xi \tag{3.2.6}$$

or equivalently

$$\int_{\mathbf{R}^n} \widehat{h}(x)\,Tf(x)\,dx \ = (\frac{1}{2\pi})^{2n} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} h(\eta)f(y)\{\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i[x\cdot\xi - x\cdot\eta - y\cdot\xi]}\sigma(x,\xi)\,dxd\xi\}dyd\eta. \tag{3.2.7}$$

Let's examine some special extreme cases.

3.2.1 The Case of Pure Decay

Consider first $\sigma(x,\xi) \in C^{0,0}S_{\rho,\delta}^{m_1,m_2}$, where $m_1, m_2 > \frac{n}{2}$, i.e.

$$|\sigma(x,\xi)| \leq c(1+|\xi|)^{-\frac{n}{2}-\epsilon}(1+|x|)^{-\frac{n}{2}-\epsilon}. \tag{3.2.8}$$

For some $\epsilon > 0$. We just use formula (3.2.6), together with the fact that $\sigma \in L^2(\mathbf{R}^n \times \mathbf{R}^n)$ as follows:

$$
\begin{aligned}
\left| \int_{\mathbf{R}^n} \widehat{h}(x)\, Tf(x)\, dx \right| &= \left| (\tfrac{1}{2\pi})^n \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \widehat{h}(x)\, \widehat{f}(\xi) e^{ix\cdot\xi} \sigma(x,\xi)\, dx d\xi \right| \\
&= \left| (\tfrac{1}{2\pi})^n \int_{\mathbf{R}^n} \widehat{h}(x) \int_{\mathbf{R}^n} \widehat{f}\,(\xi) e^{ix\cdot\xi} \sigma(x,\xi)\, d\xi dx \right| \\
&\leq (\tfrac{1}{2\pi})^n \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \left| \widehat{h}(x) \widehat{f}\,(\xi) \sigma(x,\xi) \right| dx d\xi \\
&= (\tfrac{1}{2\pi})^n \int_{\mathbf{R}^n} \left| \widehat{h}(x) \right| \int_{\mathbf{R}^n} \left| \widehat{f}(\xi) \sigma(x,\xi) \right| d\xi dx \\
&\leq c \int_{\mathbf{R}^n} \left| \widehat{h}(x) \right| \cdot \left\| \widehat{f} \right\|_2 \left(\int_{\mathbf{R}^n} |\sigma(x,\xi)|^2 d\xi \right)^{\frac{1}{2}} dx \\
&\leq c \left\| \widehat{f} \right\|_2 \left\| \widehat{h} \right\|_2 \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\sigma(x,\xi)|^2 d\xi dx = c \|f\|_2 \|h\|_2
\end{aligned}
\tag{3.2.9}
$$

where the last equality follows (3.2.8).

3.2.2 The Case of Pure Smoothness

Consider secondly that $\sigma(x,\xi) \in C^{k_1,k_2} S^{0,0}_{0,0},\ k_1, k_2 > \frac{n}{2}$, i.e.

$$\left| \partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi) \right| \leq C, \quad |\alpha| \leq \frac{n}{2} + \epsilon, \quad |\beta| \leq \frac{n}{2} + \epsilon. \tag{3.2.10}$$

We also assume here that k_1 and k_2 are even integers. We use formula (3.2.7), together with integration by parts in order to trade off the smoothness in x for the decay in ξ, and then use the symmetric property. By (3.2.7) and identity (2.1.3) on page 12,

$$\left| \int_{\mathbf{R}^n} \widehat{h}(x)\, Tf(x)\, dx \right| = \left| (\frac{1}{2\pi})^n \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} h(\eta)\, f(y) \{ \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i[x\cdot\xi - x\cdot\eta - y\cdot\xi]} \sigma(x,\xi)\, dx d\xi \} dy d\eta \right|$$

$$= \left(\tfrac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} h(\eta)\, f(y) \left(\tfrac{1}{1+|\xi-\eta|^2}\right)^M \times$$

$$\{\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i[x\cdot\xi - x\cdot\eta - y\cdot\xi]} (1-\Delta_x)^M (1-\Delta_\xi)^N \left(\tfrac{1}{1+|x-y|^2}\right)^N \sigma(x,\xi)\, dx d\xi dy d\eta$$

$$= \left(\tfrac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} (1-\Delta_x)^M (1-\Delta_\xi)^N \sigma(x,\xi) \times$$

$$\left(\int_{\mathbf{R}^n} e^{-ix\cdot\eta} h(\eta) \left(\tfrac{1}{1+|\xi-\eta|^2}\right)^M d\eta\right) \left(\int_{\mathbf{R}^n} e^{-iy\cdot\xi} f(y) \left(\tfrac{1}{1+|x-y|^2}\right)^N dy\right) dx d\xi + ..., \tag{3.2.11}$$

where the dots indicate the remaining terms in commuting $(1-\Delta_x)^N$ with $\left(\frac{1}{1+|x-y|^2}\right)^N$. The arguments for these remaining terms is essentially the same as that for the principal term, and hence we omit the argument. Now we continue to treat the principal term, which is

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \left((1-\Delta_x)^M (1-\Delta_\xi)^N \sigma(x,\xi)\right) H(x,\xi)\, F(x,\xi)\, dx d\xi, \tag{3.2.12}$$

where

$$H(x,\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\eta} h(\eta) \left(\frac{1}{1+|\xi-\eta|^2}\right)^M d\eta \tag{3.2.13}$$

and

$$F(x,\xi) = \int_{\mathbf{R}^n} e^{-iy\cdot\xi} f(y) \left(\frac{1}{1+|x-y|^2}\right)^N dy. \tag{3.2.14}$$

We now claim that $H(x,\xi)$, $F(x,\xi)$ are in $L^2(\mathbf{R}^n\times\mathbf{R}^n)$. Set $\eta(x)=1$ when $|x|\le 1$, and $\eta(x)=0$, when $|x|\ge 2$. If R is sufficiently large then

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|F(x,\xi)|^2\eta(\frac{\xi}{R})dxd\xi$$

$$=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}F(x,\xi)\,\overline{F(x,\xi)}\eta(\frac{\xi}{R})dxd\xi$$

$$=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\eta(\frac{\xi}{R})e^{i(y-z)\cdot\xi}f(y)\overline{f(z)}$$

$$\times\left(\frac{1}{1+|x-y|^2}\right)^N\left(\frac{1}{1+|x-z|^2}\right)^N dxd\xi dydz$$

$$=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}f(y)\overline{f(z)}\left(\int_{\mathbf{R}^n}\eta(\frac{\xi}{R})e^{i(y-z)\cdot\xi}d\xi\right) \tag{3.2.15}$$

$$\times\left(\int_{\mathbf{R}^n}\left(\frac{1}{1+|x-y|^2}\right)^N\left(\frac{1}{1+|x-z|^2}\right)^N dx\right)dydz.$$

Where we substituted the expression for F and $\bar{F}$. Now, if $N>\frac{n}{2}$, then as a function of x

$$\left(\frac{1}{1+|x-y|^2}\right)^N,\ \left(\frac{1}{1+|x-z|^2}\right)^N\in L^2, \tag{3.2.16}$$

and so after a change of variable

$$\int_{\mathbf{R}^n}\left(\frac{1}{1+|x-y|^2}\right)^N\left(\frac{1}{1+|x-z|^2}\right)^N dx \tag{3.2.17}$$

is a bounded function of $y-z$. Also, since $\eta(\frac{\xi}{R})\in\mathcal{S}$, we have

$$\int_{\mathbf{R}^n}\eta(\frac{\xi}{R})e^{iz\cdot\xi}d\xi=CR^n\hat{\eta}(Rx)\in L^1 \tag{3.2.18}$$

and so altogether we obtain that

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} | F(x,\xi)|^2 \eta(\frac{\xi}{R})dxd\xi$$

$$\begin{aligned}
&= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(y)\overline{f(z)} \left(\int_{\mathbf{R}^n} \eta(\frac{\xi}{R})e^{i(y-z)\cdot\xi}d\xi\right) \\
&\quad \left(\int_{\mathbf{R}^n} \left(\frac{1}{1+|x-y|^2}\right)^N \left(\frac{1}{1+|x-z|^2}\right)^N dx\right) dydz \\
&= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(y)\overline{f(z)}K(y-z)dydz \qquad (3.2.19) \\
&\leq \frac{1}{2}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K(y-z)\left(|f(y)|^2+|f(z)|^2\right)dydz \\
&= C\,\|f\|_{L^2}^2 ,
\end{aligned}$$

where we have used the facts that

$$K(y-z) = \left(\int_{\mathbf{R}^n} \eta(\frac{\xi}{R})e^{i(y-z)\cdot\xi}d\xi\right) \times \left(\int_{\mathbf{R}^n} \left(\frac{1}{1+|x-y|^2}\right)^N \left(\frac{1}{1+|x-z|^2}\right)^N dx\right) \tag{3.2.20}$$

is an integrable function of $y-z$, and

$$f(y)\overline{f(z)} \leq \frac{1}{2}\left(|f(y)|^2+|f(z)|^2\right). \tag{3.2.21}$$

Thus $\|F\|_{L^2} \leq \|f\|_{L^2}$ and $\|G\|_{L^2} \leq \|g\|_{L^2}$ and since $(1-\Delta_x)^M (1-\Delta_\xi)^N \sigma(x,\xi)$ is bounded by hypothesis, if we take $M=\frac{k_1}{2}$, $N=\frac{k_2}{2}$, then we have proved (3.2.4) with $\widehat{h}$ in place of g.

Hence we have shown the pure smoothness case.

3.3 Proof of the L^2 theorem

Now we give the full proof of Theorem 2.3.1.

Let

$$Tf(x) = \sigma(x, D)f(x) = (\frac{1}{2\pi})^n \int_{\mathbf{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\widehat{f(\xi)}d\xi. \tag{3.3.22}$$

We use formula (3.2.7), and the following identity for N an integer:

$$(1-\Delta_\xi)^N\left(\frac{1}{1+|x|^2}\right)^N e^{ix\cdot\xi} = e^{ix\cdot\xi}. \tag{3.3.23}$$

If N is not integer, let $[N]$ denote the integer part, use identity (3.3.23) for $[N]$, and for the fractional part $N-[N]$, use the identity as in [Sa]:

$$\frac{\delta_x^h}{e^{ih\cdot(\xi-\eta)}-1}e^{ix\cdot(\xi-\eta)} = e^{ix\cdot(\xi-\eta)}. \tag{3.3.24}$$

Therefore we have

$$\left|\int_{\mathbf{R}^n}\widehat{h}(x)\,Tf(x)\,dx\right| = \left|(\frac{1}{2\pi})^n\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\widehat{h}(x)\,\widehat{f}(\xi)e^{ix\cdot\xi}\sigma(x,\xi)\,dxd\xi\right|$$

$$= \left|(\frac{1}{2\pi})^n\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}h(\eta)\,f(y)\{\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}e^{i[x\cdot\xi-x\cdot\eta-y\cdot\xi]}\sigma(x,\xi)\,dxd\xi\}dyd\eta\right|$$

$$= (\tfrac{1}{2\pi})^n\textstyle\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}h(\eta)\,f(y)\left(\frac{1}{1+|\xi-\eta|^2}\right)^M\times$$

$$\{\textstyle\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}e^{i[x\cdot\xi-x\cdot\eta-y\cdot\xi]}(1-\Delta_x)^M(1-\Delta_\xi)^N\left(\frac{1}{1+|x-y|^2}\right)^N\sigma(x,\xi)\,dxd\xi dyd\eta$$

$$= (\tfrac{1}{2\pi})^n\textstyle\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}e^{ix\cdot\xi}(1-\Delta_x)^M(1-\Delta_\xi)^N\sigma(x,\xi)\times$$

$$\left(\textstyle\int_{\mathbf{R}^n}e^{-ix\cdot\eta}h(\eta)\left(\frac{1}{1+|\xi-\eta|^2}\right)^M d\eta\right)\left(\textstyle\int_{\mathbf{R}^n}e^{-iy\cdot\xi}f(y)\left(\frac{1}{1+|x-y|^2}\right)^N dy\right)dxd\xi+\dots. \tag{3.3.25}$$

Again, we treat only the principal term, the other terms in the commutator of $(1-\Delta_x)^M$ and $\left(\frac{1}{1+|x-y|^2}\right)^N$ being similar and hence omitted. The principal term is

$$C(\frac{1}{2\pi})^n \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \widetilde{\sigma(x,\xi)} H(x,\xi)\, F(x,\xi)\, dx d\xi, \tag{3.3.26}$$

where

$$\widetilde{\sigma(x,\xi)} = \left((1-\Delta_x)^M (1-\Delta_\xi)^N \sigma(x,\xi) + \ldots\right)^{M_1} |\xi|^{M_1} |x|^{N_1} \in L^\infty(x,\xi); \tag{3.3.27}$$

Let

$$H(x,\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\eta} h(\eta) \left(\frac{1}{1+|\xi-\eta|^2}\right)^M \left(\frac{1}{|\xi|}\right)^{M_1} d\eta\ , \tag{3.3.28}$$

where $M + M_1 > \frac{n}{2}$; and

$$F(x,\xi) = \int_{\mathbf{R}^n} e^{-iy\cdot\xi} f(y) \left(\frac{1}{1+|x-y|^2}\right)^N \left(\frac{1}{|x|}\right)^{N_1} dy\ , \tag{3.3.29}$$

where $N + N_1 > \frac{n}{2}$.

If we can prove $H(x,\xi)$, $F(x,\xi) \in L^2(\mathbf{R}^n \times \mathbf{R}^n)$ and that both

$$\|H(x,\xi)\|_{L^2} \leq \|h\|_{L^2}\ , \tag{3.3.30}$$

$$\|F(x,\xi)\|_{L^2} \leq \|f\|_{L^2}\,, \tag{3.3.31}$$

we are done. Since $H(x,\xi)$, $F(x,\xi)$ are symmetric, we only need to prove one of them. In fact, if we set $\eta(x) = 1$ when $|x| \leq 1$, $\eta(x) = 0$, when $|x| \geq 2$, and suppose R is large enough. Then

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\ F(x,\xi)|^2 \eta(\frac{\xi}{R}) dx d\xi = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} F(x,\xi)\ \overline{F(x,\xi)} \eta(\frac{\xi}{R}) dx d\xi$$

$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \eta(\frac{\xi}{R})e^{i(y-z)\cdot\xi} f(y)\overline{f(z)} \left(\frac{1}{1+|x-y|^2}\right)^N \left(\frac{1}{|x|}\right)^{N_1}$$

$$\times \left(\frac{1}{1+|x-z|^2}\right)^N \left(\frac{1}{|x|}\right)^{N_1} dx d\xi dy dz$$

$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(y)\overline{f(z)} \left(\int_{\mathbf{R}^n} \eta(\frac{\xi}{R})e^{i(y-z)\cdot\xi} d\xi\right) \tag{3.3.32}$$

$$\times \left(\int_{\mathbf{R}^n} \left(\frac{1}{1+|x-y|^2}\right)^N \left(\frac{1}{|x|}\right)^{N_1} \left(\frac{1}{|x|}\right)^{N_1} \left(\frac{1}{1+|x-z|^2}\right)^N dx\right) dydz. \tag{3.3.33}$$

Since $N + N_1 > \frac{n}{2}$, we have both

$$\left(\frac{1}{1+|x-y|^2}\right)^N \left(\frac{1}{|x|}\right)^{N_1}, \left(\frac{1}{1+|x-z|^2}\right)^N \left(\frac{1}{|x|}\right)^{N_1} \in L^2 \tag{3.3.34}$$

as functions of x, and so as a function of (y, z)

$$\int_{\mathbf{R}^n} \left(\frac{1}{1+|x-y|^2}\right)^N \left(\frac{1}{|x|}\right)^{N_1} \left(\frac{1}{|x|}\right)^{N_1} \left(\frac{1}{1+|x-z|^2}\right)^N dx \in L^\infty, \tag{3.3.35}$$

and as before

$$\int_{\mathbf{R}^n} \eta(\frac{\xi}{R})e^{ix\cdot\xi} d\xi = R^n\hat{\eta}(Rx) \in L^1. \tag{3.3.36}$$

Therefore

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |\ F(x,\xi)|^2 \eta(\frac{\xi}{R}) dx d\xi = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} F(x,\xi)\ \overline{F(x,\xi)}\eta(\frac{\xi}{R}) dx d\xi$$

$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(y)\overline{f(z)} \left(\int_{\mathbf{R}^n} \eta(\frac{\xi}{R})e^{i(y-z)\cdot\xi} d\xi\right)$$

$$\times \left(\int_{\mathbf{R}^n} (\frac{1}{1+|x-y|^2})^N \left(\frac{1}{|x|}\right)^{N_1} \left(\frac{1}{|x|}\right)^{N_1} (\frac{1}{1+|x-z|^2})^N dx\right) dydz$$

$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(y)\overline{f(z)}K(y,z)dydz \tag{3.3.37}$$

$$\leq \frac{1}{2}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K(y,z)\left(|f(y)|^2+|f(z)|^2\right)dydz$$

$$= C\,\|f\|_{L^2},$$

where we have used the facts that

$$K(y,z) = \left(\int_{\mathbf{R}^n}\eta(\frac{\xi}{R})e^{i(y-z)\cdot\xi}d\xi\right) \tag{3.3.38}$$

$$\times\left(\int_{\mathbf{R}^n}\left(\frac{1}{1+|x-y|^2}\right)^N\left(\frac{1}{|x|}\right)^{N_1}\left(\frac{1}{|x|}\right)^{N_1}\left(\frac{1}{1+|x-z|^2}\right)^N dx\right) \tag{3.3.39}$$

is an integrable function of y for fixed z, and an integrable function of z for fixed y,

$$f(y)\overline{f(z)} \leq \frac{1}{2}\left(|f(y)|^2+|f(z)|^2\right). \tag{3.3.40}$$

This completes the proof of the case $\rho=\delta=0$ with k_1 and k_2 are even integers.

When k_1 and k_2 are not even integers, we conjecture the result still is true. But you need to use the identity (2.1.4). Even for the case $\rho,\delta>0$, the result may still be true, but the method here is not applicable.

The result is sharp, let

$$\sigma(x,\xi) = \frac{e^{-ix\cdot\xi}}{(1+|x|^2)^{\frac{n}{2}}(1+|\xi|^2)^{\frac{n}{2}}} \in C^{k_1,k_2}S^{m_1,m_2}_{0,0}, \tag{3.3.41}$$

and

$$k_1+m_1=\frac{n}{2},\ k_2+m_2=\frac{n}{2}. \tag{3.3.42}$$

Let $f, g \in L^2$, then

$$
\begin{aligned}
(\sigma(x,D)f,g) &= \int \sigma(x,D)f(x)\overline{g(x)}dx \qquad (3.3.43)\\
&= \int\int e^{ix\cdot\xi}\sigma(x,\xi)\,\widehat{f(\xi)}\overline{g(x)}dxd\xi\\
&= \int\int e^{ix\cdot\xi}\frac{e^{-ix\cdot\xi}}{\left(1+|x|^2\right)^{\frac{n}{2}}\left(1+|\xi|^2\right)^{\frac{n}{2}}}\widehat{f(\xi)}\overline{g(x)}dxd\xi\\
&= \int \frac{1}{\left(1+|\xi|^2\right)^{\frac{n}{2}}}\widehat{f(\xi)}d\xi \int \frac{1}{\left(1+|x|^2\right)^{\frac{n}{2}}}\overline{g(x)}dx.
\end{aligned}
$$

With proper choice of $f, g \in L^2$ the last two integrals will diverge. So $\sigma(x,D)$ is not bounded on L^2.

Chapter 4

Mapping Properties

Historically, in the development of classical analysis, singular integral operators appeared first, followed by pseudo-differential operators. Singular integral operators are well understood with respect to L^p results. It is natural to try to realize pseudo-differential operators in singular integral form, and then to use known singular integral results. This is what we will do here. First we state our result.

4.1 L^p Sobolev Spaces

Theorem 4.1.1 *If $0 \le \delta \le \rho \le 1$ and $\sigma(x,\xi) \in C^v S^m_{\rho,\delta}$, $m < -\frac{n}{2}(1-\rho), v > \frac{n}{2}$, then $\sigma(x,D) : H^s_p \to H^s_p$, $p \ge 2$, $s \in (-(1-\delta)v_1, v_1)$, where $v_1 = v - \frac{n}{2}$.*

The result is known when $s = 0$. When $s \ge 0, \rho = 1$ and $p \ge 2$, there exist many different proofs. Here we prove the result for $p \ge 2$; for $p < 2$ our method doesn't work, since we couldn't control the diagonal terms (see term I below). The proof of the Theorem 2.3.2 explicitly involves the Fourier transform, makes use of the division of the dual (frequency) space into "dyadic" spherical shells. This decomposition–whose ideas originated in the work of Bernstein, Littlewood and Paley, and others will now be described in the form most suitable for us.

We begin by fixing η, a C^∞ function of compact support, defined in the ξ–space $\mathbf{R}^n$, with the properties that $\eta(\xi) = 1$ for $|\xi| < 1$ and $\eta(\xi) = 0$ for $|\xi| > 2$. Together with η , we define another function δ , by $\delta(\xi) = \eta(\xi) - \eta(2\xi)$. Then we have the following two "partitions of unity" of ξ-space:

$$1 = \eta(\xi) + \sum_{j=1}^{\infty} \delta\left(2^{-j}\xi\right), \qquad all \;\; \xi, \tag{4.1.1}$$

and

$$1 = \sum_{-\infty}^{\infty} \delta\left(2^{-j}\xi\right), \qquad all \;\; \xi \neq 0. \tag{4.1.2}$$

In fact ,

$$\eta(\xi) + \sum_{j=1}^{l} \delta\left(2^{-j}\xi\right) = \eta(2^{-l}\xi) \to 1, \tag{4.1.3}$$

as $l \to \infty$, for all ξ ; while

$$\sum_{j=-l'}^{j=l} \delta\left(2^{-j}\xi\right) = \eta\left(2^{-l}\xi\right) - \eta\left(2^{-l'+1}\xi\right) \to 1, \tag{4.1.4}$$

if $l \to \infty$, $l' \to \infty$, and $\xi \neq 0$. Note also that $\delta(\xi)$ is supported in the shell $\frac{1}{2} \leq |\xi| \leq 2$, so that the $\delta(2^{-j}\xi)$ are supported in the shells $2^{-j-1} \leq |\xi| \leq 2^{-j+1}$. It follows that for each ξ there are at most two nonzero terms in the sums.

Let

$$I = \sum_{k=0}^{\infty} \phi_k \overline{\phi_k}, \quad 1 = \sum_{k=0}^{\infty} \left|\widehat{\phi_k}(\xi)\right|^2, \tag{4.1.5}$$

where

$$\widehat{\phi_k}(\xi) = \widehat{\phi_1}(2^{-k}\xi), \tag{4.1.6}$$

$\widehat{\phi_1}$ is supported in $\{\xi : \frac{3}{4} \leq |\xi| \leq \frac{5}{2}\}$. $\widehat{\phi_0}$ is supported in $\{\xi : |\xi| \leq \frac{3}{2}\}$. Then for s real, we have

$$
\begin{aligned}
\|\sigma(x,D)f(x)\|_{H_p^s} &= \left\| \left(\sum_{l=0}^{\infty} \left| 2^{ls}\phi_l \circ \sigma f(x) \right|^2 \right)^{\frac{1}{2}} \right\|_p \\
&= \left\| \left(\sum_{l=0}^{\infty} \left| \sum_{k=0}^{\infty} 2^{ls}\phi_l \circ \sigma \circ 2^{-ks}\phi_k (2^{ks}\phi_k f) \right|^2 \right)^{\frac{1}{2}} \right\|_p \\
&\le \left\| \left(\sum_{l=0}^{\infty} \left| \sum_{|k-l|\le 5} 2^{ls}\phi_l \circ \sigma \circ 2^{-ks}\phi_k (2^{ks}\phi_k f) \right|^2 \right)^{\frac{1}{2}} \right\|_p \\
&+ \left\| \left(\sum_{l=0}^{\infty} \left| \sum_{|k-l|\ge 5} 2^{ls}\phi_l \circ \sigma \circ 2^{-ks}\phi_k (2^{ks}\phi_k f) \right|^2 \right)^{\frac{1}{2}} \right\|_p \\
&= I + II
\end{aligned}
\tag{4.1.7}
$$

Let

$$
\begin{aligned}
T_{l,k}f(y) &= \phi_l \circ \sigma \circ \phi_k f(y) = \int_{\mathbf{R}^n} e^{iy\cdot\eta}\widehat{\phi_l}(\eta)(\widehat{\sigma \circ \phi_k f})(\eta)d\eta \\
&= c\int_{\mathbf{R}^n} e^{iy\cdot\eta}\widehat{\phi_l}(\eta)\int_{\mathbf{R}^n} e^{-ix\cdot\eta}\left(\sigma\circ\phi_k f\right)(x)dxd\eta \\
&= c\int_{\mathbf{R}^n} e^{iy\cdot\eta}\widehat{\phi_l}(\eta)\int_{\mathbf{R}^n} e^{-ix\cdot\eta}\int_{\mathbf{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\,\widehat{\phi_k f}(\xi)d\xi dxd\eta \\
&= c\int_{\mathbf{R}^n} e^{iy\cdot\eta}\widehat{\phi_l}(\eta)\int_{\mathbf{R}^n} e^{-ix\cdot\eta}\int_{\mathbf{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\,\widehat{\phi_k}(\xi)\int_{\mathbf{R}^n} e^{-ix'\cdot\xi}f(x')dx'd\xi dxd\eta \\
&= c\int_{\mathbf{R}^n} f(x')\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(y-x)\cdot\eta}e^{i(x-x')\cdot\xi}\widehat{\phi_l}(\eta)\sigma(x,\xi)\,\widehat{\phi_k}(\xi)d\xi dxd\eta dx' \\
&= c\int_{\mathbf{R}^n} f(x')\left(\phi_l\circ\sigma\circ\phi_k\right)(y,x')dx',
\end{aligned}
\tag{4.1.8}
$$

where

$$\phi_{l'} \circ \sigma \circ \phi_k(y, x') = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(y-x)\cdot\eta} e^{i(x-x')\cdot\xi} \widehat{\phi_l}(\eta)\sigma(x,\xi)\,\widehat{\phi_k}(\xi)d\xi dx d\eta. \tag{4.1.9}$$

4.2 Diagonal Terms and the Maximal Operator M_2

To control the diagonal terms of I, we will use the square integrable maximal operator M_2. We will prove $(T_{l,l}f)^{\#}(x) \leq cM_2 f(x)$ for all x and l. Without loss of generality, assume $x = 0$. Let Q be the cube centered at x with edge length equal to $l(Q)$ and $Q^{1-\theta}$ be the cube centered at x with edge length equal to $l(Q)^{1-\theta}$. Also let $2Q$ be the cube centered at x with edge length equal to $2l(Q)$. Let

$$f = f_1 + f_2, \tag{4.2.10}$$

where

$$f_1 = \chi_{2Q^{1-\theta}} f,\ \ f_2 = f - f_1, \theta = 1 - \rho. \tag{4.2.11}$$

We have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T_{l,l}f - T_{l,l}f_2(0)| \\
\leq\ & \frac{1}{|Q|}\int_Q |T_{l,l}f_1| + \frac{1}{|Q|}\int_Q |T_{l,l}f_2 - T_{l,l}f_2(0)| \\
=\ & I + II.
\end{aligned}$$

To estimate term I we note

$$T_{l,l}f_1 = |D|^{-\frac{n}{2}\theta}|D|^{\frac{n}{2}\theta}T_{l,l}f_1 \tag{4.2.12}$$

$$= \quad I_{\frac{n}{2}\theta}(|D|^{\frac{n}{2}\theta} T_{l,l} f_1)$$

$$= \quad I_{\frac{n}{2}\theta}(g),$$

where I_θ is as in (2.1.2) on page 11 and

$$g \quad = \quad |D|^{\frac{n}{2}\theta} T_{l,l} f_1 \tag{4.2.13}$$

$$= \quad |D|^{\frac{n}{2}\theta} \phi_l \circ \sigma \circ \phi_l f_1 \,.$$

If we define q so that $\frac{1}{q} = \frac{1}{2} - \frac{\frac{n}{2}\theta}{n} = \frac{1}{2} - \frac{\theta}{2}$, then the Sobolev embedding Theorem yields

$$\left\| I_{\frac{n}{2}\theta}(g) \right\|_q \le C \left\| g \right\|_2 . \tag{4.2.14}$$

As for the term

$$g = |D|^{\frac{n}{2}\theta} T_{l,l} f_1 = |D|^{\frac{n}{2}\theta} \phi_l 2^{-l\frac{n}{2}\theta} 2^{l\frac{n}{2}\theta} \circ \sigma \circ \phi_l f_1, \tag{4.2.15}$$

we have

$$|\xi|^{\frac{n}{2}\theta} \widehat{\phi_l} 2^{-l\frac{n}{2}\theta} \in C^\infty S^0_{1,0},\ 2^{l\frac{n}{2}\theta} \sigma \widehat{\phi}_l \in C^{\frac{n}{2}+\epsilon} S^0_{\rho,\delta}, \tag{4.2.16}$$

by the hypothesis on σ and therefore

$$\|g\|_2 = \left\| |D|^{\frac{n}{2}\theta} T_{l,l} f_1 \right\|_2 \le C \left\| f_1 \right\|_2 , \tag{4.2.17}$$

by the Theorem of Coifman and Meyer ([CoMe]).

Thus term I can be estimated as follows:

$$\frac{1}{|Q|} \int_Q |T_{l,l} f_1| \tag{4.2.18}$$

$$
\begin{aligned}
&\leq \left(\frac{1}{|Q|}\int_Q |T_{l,l}f_1|^q\right)^{\frac{1}{q}} \\
&\leq C\left(\frac{1}{|Q|}\right)^{\frac{1}{q}} \left\||D|^{\frac{n}{2}\theta}T_{l,l}f_1\right\|_2 && (4.2.19)\\
&\leq C\left(\frac{1}{|Q|}\right)^{\frac{1}{2}} \|f_1\|_2 && (4.2.20)\\
&\leq C\left(\frac{1}{|Q|}\right)^{\frac{1}{q}} \left(\int_{2Q^{1-\theta}} |f|^2\right)^{\frac{1}{2}} \leq CM_2f(0)\,.
\end{aligned}
$$

For $T_{l,l}f_2$, let

$$\phi_l \circ \sigma \circ \phi_l(y, x') = K_l(y, x'). \tag{4.2.21}$$

We get

$$
\begin{aligned}
&|T_{l,l}f_2(y) - T_{l,l}f_2(0)| \\
&\leq |\textstyle\int_{\mathbf{R}^n} (K_l(y,x') - K_l(0,x'))\, f_2(x')dx'| \\
&\leq \left|\textstyle\int_{y\in(2Q^{1-\theta})^c} (K_l(y,x') - K_l(0,x'))\, f(x')dx'\right| \\
&= \left|\textstyle\sum_{k=1}^{\infty} \int_{2^{k+1}Q^{1-\theta}\setminus 2^kQ^{1-\theta}} (K_l(y,x') - K_l(0,x'))\, f(x')dx'\right| \\
&\leq \textstyle\sum_{k=1}^{\infty} \left(\int_{2^{k+1}Q^{1-\theta}\setminus 2^kQ^{1-\theta}} |K_l(y,x') - K_l(0,x')|^2 \left|x'\right|^{n+\epsilon} dx'\right)^{\frac{1}{2}} \times \\
&\left(\int_{2^{k+1}Q^{1-\theta}\setminus 2^kQ^{1-\theta}} \frac{\left|f(x')\right|^2}{\left|x'\right|^{n+\epsilon}} dx'\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \sum_{k=1}^{\infty}\left(\int_{2^{k+1}Q^{1-\theta}\backslash 2^kQ^{1-\theta}}|K_l(y,x')-K_l(0,x')|^2\left|x'\right|^{n+\epsilon}dx'\right)^{\frac{1}{2}}\times \tag{4.2.22}$$

$$\left(2^{k+1}\right)^{-\frac{\epsilon}{2}}\left|L^{1-\theta}\right|^{-\frac{\epsilon}{2}}M_2f(0)\,,\ (where\ L\ is\ the\ lenth\ of\ Q).$$

Next we will estimate

$$\left(\int_{2^{k+1}Q^{1-\theta}\backslash 2^kQ^{1-\theta}}|K_l(y,x')-K_l(0,x')|^2|x'|^{n+\epsilon}dx'\right)^{\frac{1}{2}}. \tag{4.2.23}$$

Using Plancherel's formula, we have

$$\left(\int_{2^{k+1}Q^{1-\theta}\backslash 2^kQ^{1-\theta}}|K_l(y,x')-K_l(0,x')|^2|x'|^{n+\epsilon}dx'\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{\mathbf{R}^n}\left|\partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}}\widetilde{K_l(y,\xi)}-\partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}}\widetilde{K_l(0,\xi)}\right|^2d\xi\right)^{\frac{1}{2}}$$

$$=\left(\int_{\mathbf{R}^n}\left|\partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\left(e^{i(y-x)\cdot\eta}-e^{i(0-x)\cdot\eta}\right)e^{ix\cdot\xi}\widehat{\phi_l}(\eta)\sigma(x,\xi)\,\widehat{\phi_l}(\xi)dxd\eta\right|^2d\xi\right)^{\frac{1}{2}}. \tag{4.2.24}$$

Here

$$K_l(y,x') = \phi_l\circ\sigma\circ\phi_l(y,x') \tag{4.2.25}$$

$$= \int_{\mathbf{R}^n}e^{-ix'\cdot\xi}\left(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}e^{i(y-x)\cdot\eta}e^{ix\cdot\xi}\widehat{\phi_l}(\eta)\sigma(x,\xi)\,\widehat{\phi_l}(\xi)dxd\eta\right)d\xi.$$

In order to estimate

$$\partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(e^{i(y-x)\cdot\eta}-e^{i(0-x)\cdot\eta})e^{ix\cdot\xi}\widehat{\phi_l}(\eta)\sigma(x,\xi)\,\widehat{\phi_l}(\xi)dxd\eta, \tag{4.2.26}$$

we note that:

- If ∂_ξ hits $e^{ix\cdot\xi}$, this brings down ix; we can then use $ixe^{ix\cdot\eta}=\partial_\eta e^{ix\cdot\eta}$ and integration by parts to transfer $\partial_\xi e^{ix\cdot\xi}$ to $\partial_\eta\widehat{\phi_l}(\eta)$ with a gain of $\left(2^l\right)^{-1}$;

- If ∂_ξ hits $\sigma(x,\xi)$, we gain $\left(2^l\right)^{-\rho}$;
- If ∂_ξ hits $\widehat{\phi_l}(\xi)$, we gain $\left(2^l\right)^{-1}$.

From the identity,

$$\frac{2^{-2l}-\Delta_\eta}{2^{-2l}+|x-y|^2}e^{i(y-x)\cdot\eta}=e^{i(y-x)\cdot\eta}, \tag{4.2.27}$$

we get the following

$$\left|\partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(e^{i(y-x)\cdot\eta}-e^{i(0-x)\cdot\eta})e^{ix\cdot\xi}\widehat{\phi_l}(\eta)\sigma(x,\xi)\widehat{\phi_l}(\xi)dxd\eta\right|$$

$$\leq C(2^l)^{(-\frac{n}{2}(1-\rho)-(\frac{n}{2}+\frac{\epsilon}{2})\rho)}\min\left\{|y|\,2^l,1\right\}\int_{\mathbf{R}^n}\left(\frac{2^{-2l}}{2^{-2l}+|x-y|^2}\right)^N\left(2^l\right)^n\ \left|\widehat{\phi_l}(\xi)\right|dx$$

$$\leq C(2^l)^{(-\frac{n}{2}(1-\rho)-(\frac{n}{2}+\frac{\epsilon}{2})\rho)}\left|\widehat{\phi_l}(\xi)\right|\min\left\{|y|\,2^l,1\right\},\quad (for N>\tfrac{n}{2}) \tag{4.2.28}$$

and

$$\left(\int_{\mathbf{R}^n}\left|\partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(e^{i(y-x)\cdot\eta}-e^{i(0-x)\cdot\eta})e^{ix\cdot\xi}\widehat{\phi_l}(\eta)\sigma(x,\xi)\widehat{\phi_l}(\xi)dxd\eta\right|^2d\xi\right)^{\frac{1}{2}}$$

$$\leq C(2^l)^{(-\frac{n}{2}(1-\rho)-(\frac{n}{2}+\frac{\epsilon}{2})\rho)+\frac{n}{2}}\min\left\{|y|\,2^l,1\right\}\leq C(2^l)^{-\frac{\epsilon}{2}\rho}\min\left\{|y|\,2^l,1\right\}. \tag{4.2.29}$$

Thus we have

$$.|T_{l,l}f_2(y) - T_{l,l}f_2(0)|$$

$$\leq C \textstyle\sum_{k=1}^{\infty} \min\left\{|y|\,2^l, 1\right\}(2^l)^{-\frac{\epsilon\rho}{2}}\left|2^{k+1}L^{1-\theta}\right|^{-\frac{\epsilon}{2}} M_2f(0)$$

$$(here\ L = l(Q)\ is\ the\ lenth\ of\ Q) \tag{4.2.30}$$

$$\leq C \min\left\{|y|\,2^l, 1\right\}(2^l)^{-\frac{\epsilon\rho}{2}}|L^\rho|^{-\frac{\epsilon}{2}} M_2f(0)\ \ (since\ \ 1-\theta=\rho)$$

$$\leq CM_2f(0)\min\left\{(L2^l)^{1-\frac{\epsilon\rho}{2}}, (L2^l)^{-\frac{\epsilon\rho}{2}}\right\}$$

$$\leq CM_2f(0).$$

Altogether, we have

$$\frac{1}{|Q|}\int_Q |T_{l,l}f - T_{l,l}f_2(0)| \leq CM_2f(0)$$

and taking the supremum over Q we obtain

$$(T_{l,l}f)^{\#}(0) \leq CM_2f(0).$$

Here we use (2.1.5).

Similarly it can be shown that

$$(T_{l,k}f)^{\#}(x) \leq CM_2f(x) \quad forall\ |k-l| \leq 5.$$

So

$$
\begin{aligned}
I. &= \left\|\Big(\sum_{l=0}^{\infty}\Big|\sum_{|k-l|\le 5} 2^{ls}\phi_l \circ \sigma \circ 2^{-ks}\phi_k(2^{ks}\phi_k f)\Big|^2\Big)^{\frac{1}{2}}\right\|_p \\
&\le C\left\|\Big(\sum_{l=0}^{\infty}\Big|\sum_{|k-l|\le 5} M_2(2^{ks}\phi_k f)\Big|^2\Big)^{\frac{1}{2}}\right\|_p \\
&\le C\left\|\left(\sum_{k=0}^{\infty}\left|M_2\left(2^{ks}\phi_k f\right)\right|^2\right)^{\frac{1}{2}}\right\|_p \le C\left\|\left(\sum_{k=0}^{\infty}\left|2^{ks}\phi_k f\right)\right|^2\right)^{\frac{1}{2}}\right\|_p \\
&= C\,\|f\|_{H_p^s} \quad for \quad p > 2.
\end{aligned}
\tag{4.2.31}
$$

Here we use the vector-valued version for M_2, $p > 2$ (see [FefSt2]).

4.3 The Off-Diagonal Terms

For the term of II ,we need an estimate of the kernel of $\phi_l \circ \sigma \circ \phi_k$ using the smoothness of σ in the x variable as well. We follow closely the argument in [Sa]. Our goal is to get the estimate

$$
|\phi_l \circ \sigma \circ \phi_k \circ g| \le C 2^{-v(k\vee l)} 2^{\delta v k} M g, \tag{4.3.32}
$$

here $(k \vee l)$ stands for $\max\{k, l\}$.

If v is an integer, since

$$
\frac{\partial_x}{(\xi-\eta)} e^{ix\cdot(\xi-\eta)} = e^{ix\cdot(\xi-\eta)} \quad for \quad \xi \neq \eta, \tag{4.3.33}
$$

we obtain

$$
\phi_l \circ \sigma \circ \phi_k(y, x') = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(y-x)\cdot\eta} e^{i(x-x')\cdot\xi}
$$

$$
\times \left\{\widehat{\phi_l}(\eta)\left(\tfrac{\partial_x}{(\xi-\eta)}\right)^v \sigma(x,\xi)\widehat{\phi_k}(\xi)\right\} d\xi\, dx\, d\eta.
$$

$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(y-x)\cdot\eta} e^{i(x-x')\cdot\xi} \left(\frac{1-2^{2l}\Delta_\eta}{1+2^{2l}|y-x|^2}\right)^N$$

$$\times \left(\frac{1-2^{2k\rho}\Delta_\xi}{1+2^{2k\rho}|x-x'|^2}\right)^N \left\{\widehat{\phi_l}(\eta)\left(\frac{\partial_x}{(\xi-\eta)}\right)^v \sigma(x,\xi)\widehat{\phi_k}(\xi)\right\} d\xi dx d\eta.$$

$$= \int_{\mathbf{R}^n} \left(\frac{1}{1+2^{2l}|y-x|^2}\right)^N \left(\frac{1}{1+2^{2k\rho}|x-x'|^2}\right)^N \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(y-x)\cdot\eta} e^{i(x-x')\cdot\xi} \left(1-2^{2l}\Delta_\eta\right)^N$$

$$\times \left(1-2^{2k\rho}\Delta_\xi\right)^N \left\{\widehat{\phi_l}(\eta)\left(\frac{\partial_x}{(\xi-\eta)}\right)^v \sigma(x,\xi)\widehat{\phi_k}(\xi)\right\} d\xi d\eta dx. \tag{4.3.34}$$

Now if we note that:

- When $\frac{\partial_x}{(\xi-\eta)}$ hits $\sigma(x,\xi)$, we gain $2^{-(k\vee l)}2^{\delta k}$;
- When $\left(1-2^{2k\rho}\Delta_\xi\right)$ hits $\left(\frac{1}{(\xi-\eta)}\right)^v \sigma(x,\xi)\widehat{\phi_k}(\xi)$, we gain nothing;
- When $\left(1-2^{2l}\Delta_\eta\right)$ hits $\widehat{\phi_l}(\eta)\left(\frac{1}{(\xi-\eta)}\right)^v$, we gain nothing.

Consider $|\eta| \approx 2^l$, $|\xi| \approx 2^k$, we have

$$|\phi_l \circ \sigma \circ \phi_k \circ g| \le C 2^{-v_1(k\vee l)} 2^{\delta v_1 k} Mg\ , \quad v_1 = v - \frac{n}{2}, \tag{4.3.35}$$

when ξ, η are not too close.

If v is not a positive integer, for $[v]$ we use this method, for fractional part $v-[v]$, we may use similar methods as in [Sa] with the identity

$$\frac{\delta_x^h}{e^{ih\cdot(\xi-\eta)}-1} e^{ix\cdot(\xi-\eta)} = e^{ix\cdot(\xi-\eta)}. \tag{4.3.36}$$

For this we introduce a further decomposition. Let $\{\widehat{\psi}_\alpha\}_{\alpha\in A}$ be a smooth finite partition of unity on the sphere S^{n-1} and let $\widehat{\phi}_{k,\alpha}(\xi) = \widehat{\phi}_k(\xi)\widehat{\psi}_\alpha(\frac{\xi}{|\xi|})$. We choose the partition so fine that there exists constants c, C with $0 < c < C < 2\pi$ such that for all k, l with $|k-l| \ge 5$ and all $\alpha, \beta \in A$, there are points $h_{\alpha,\beta}^{l,k} \in \mathbf{R}^n$ such that

$$0 < c \le (\xi-\eta)\cdot h_{\alpha,\beta}^{l,k} \le C < 2\pi\ , \qquad for\ \ \widehat{\phi}_{k,\alpha}(\xi)\widehat{\phi}_{l,\beta}(\eta) \ne 0. \tag{4.3.37}$$

Since

$$\phi_l \circ \sigma \circ \phi_k = \sum_{\alpha,\beta \in A} \phi_{l,\beta} \circ \sigma \circ \phi_{k,\alpha} \tag{4.3.38}$$

and so it suffices to estimate the kernel of $\phi_{l,\beta} \circ \sigma \circ \phi_{k,\alpha}$. We have

$$\phi_{l,\beta} \circ \sigma \circ \phi_{k,\alpha}(y,x')$$

$$= \int_{R^n}\int_{R^n}\int_{R^n} e^{i(x-y)\cdot\eta} e^{i(x-x')\cdot\xi} \left\{ \widehat{\phi_{l,\beta}}(\eta)\sigma(x,\xi)\widehat{\phi_{k,\alpha}}(\xi) \right\} d\xi dx d\eta. \tag{4.3.39}$$

We now use the fractional smoothness of σ in x via the eigenfunction identity

$$\frac{\delta_x^h}{e^{ih\cdot(\xi-\eta)}-1} e^{ix\cdot(\xi-\eta)} = e^{ix\cdot(\xi-\eta)}, \tag{4.3.40}$$

where δ_x^h is the first difference operator given in (2.1.1) on page 11, and $h = h_{\alpha,\beta}^{l,k}$. Note that $|h| = \left|h_{\alpha,\beta}^{l,k}\right| \approx 2^{-(k\vee l)}$. Applying this identity to $\phi_{l,\beta} \circ \sigma \circ \phi_{k,\alpha}(y,x')$ and noting that the transpose of δ_x^h is δ_x^{-h} , we obtain

$$\phi_{l,\beta} \circ \sigma \circ \phi_{k,\alpha}(y,x')$$

$$= \int_{R^n}\int_{R^n}\int_{R^n} e^{i(y-x)\cdot\eta} e^{i(x-x')\cdot\xi} \left\{ \widehat{\phi_{l,\beta}}(\eta) \frac{\delta_x^{-h}}{e^{ih\cdot(\xi-\eta)}-1} \left(\frac{\partial_x}{(\xi-\eta)}\right)^{[v]} \sigma(x,\xi)\widehat{\phi_{k,\alpha}}(\xi) \right\} d\xi dx d\eta. \tag{4.3.41}$$

Since $e^{i(y-x)\cdot\eta}$ and $e^{i(x-x')\cdot\xi}$ are eigenfunctions of $\frac{1-2^{2l}\Delta_\eta}{1+2^{2l}|y-x|^2}$ and $\frac{1-2^{2k\rho}\Delta_\xi}{1+2^{2k\rho}|x-x'|^2}$ respectively , we further obtain that

$$\phi_{l,\beta} \circ \sigma \circ \phi_{k,\alpha}(y,x') = \int_{R^n}\int_{R^n}\int_{R^n} e^{i(y-x)\cdot\eta} e^{i(x-x')\cdot\xi} \left(\frac{1-2^{2l}\Delta_\eta}{1+2^{2l}|y-x|^2}\right)^N \times$$

$$\left(\frac{1-2^{2k\rho}\Delta_\xi}{1+2^{2k\rho}|x-x'|^2}\right)^N \left\{ \widehat{\phi_{l,\beta}}(\eta) \frac{\delta_x^{-h}}{e^{ih\cdot(\xi-\eta)}-1} \left(\frac{\partial_x}{(\xi-\eta)}\right)^{[v]} \sigma(x,\xi)\widehat{\phi_{k,\alpha}}(\xi) \right\} d\xi dx d\eta. \tag{4.3.42}$$

Note the following;

$$\left|\frac{\delta_x^{-h}}{e^{ih\cdot(\xi-\eta)}-1}\left(\frac{\partial_x}{(\xi-\eta)}\right)^{[v]}\partial_\xi^\beta\sigma(x,\xi)\right| \le C_{\beta,v}\left|h_{\alpha,\beta}^{l,k}\right|^{v-[v]}\left(\frac{1}{|\xi-\eta|}\right)^{[v]}(1+|\xi|^2)^{(\delta v-|\beta|)/2}$$

$$\le C_{\beta,v}2^{-(k\vee l)v}(1+|\xi|^2)^{(\delta v-|\beta|)/2}, \tag{4.3.43}$$

since $\left|e^{ih\cdot(\xi-\eta)}-1\right| \ge c$. On the other hand , we also have

$$\left|\partial_\xi^\beta\partial_\eta^\gamma\frac{\delta_x^{-h}}{e^{ih\cdot(\xi-\eta)}-1}\right| \le C\left|h_{\alpha,\beta}^{l,k}\right|^{(|\beta|+|\gamma|)}$$

$$\le C2^{-(k\vee l)(|\beta|+|\gamma|)} \tag{4.3.44}$$

and

$$\left|\partial_\xi^\beta\partial_\eta^\gamma\left(\frac{1}{(\xi-\eta)}\right)^{[v]}\right| \le C\left(\frac{1}{|\xi-\eta|}\right)^{[v]+(|\beta|+|\gamma|)}. \tag{4.3.45}$$

If we consider support conditions on $\widehat{\phi}_{k,\alpha}(\xi)$ and $\widehat{\phi}_{l,\beta}(\xi)$, and integrating in ξ and η, we obtain as in [Sa], that

$$|\phi_l\circ\sigma\circ\phi_k\circ g| \le C2^{-v_1(k\vee l)}2^{\delta v_1 k}Mg\ ,\ \ v_1 = v-\frac{n}{2}. \tag{4.3.46}$$

So,

$$\begin{aligned} II &= \left\|(\textstyle\sum_{l=0}^\infty\left|\sum_{|k-l|\ge 5}2^{ls}\phi_l\circ\sigma\circ 2^{-ks}\phi_k(2^{ks}\phi_k f)\right|^2)^{\frac12}\right\|_p \\ &\le C\left\|(\textstyle\sum_{l=0}^\infty\left|\sum_{|k-l|\ge 5}2^{-v_1(k\vee l)}2^{\delta v_1 k}2^{s(l-k)}M(2^{ks}\phi_k f)\right|^2)^{\frac12}\right\|_p \\ &\le C\left\|(\textstyle\sum_{k=0}^\infty\left|M(2^{ks}\phi_k f)\right|^2)^{\frac12}\right\|_p \\ &\le C\left\|(\textstyle\sum_{k=0}^\infty\left|2^{ks}\phi_k f\right|^2)^{\frac12}\right\|_p = C\left\|f\right\|_{H_p^s} \end{aligned} \tag{4.3.47}$$

provided $-v_1(k \vee l) + \delta v_1 k + s(l-k) < 0$, i.e. for $\delta v_1 < s < v_1$ and $-v_1(1-\delta) < s < 0$, as in [Sa]. Now we use interpolation to get the full range

$$-(1-\delta)v_1 < s < v_1. \tag{4.3.48}$$

Chapter 5

Restricted Rough Symbol Classes

Here we consider the symbol classes $C^v_\infty S^m_{\rho,\delta}$, which are modelled on the case $\rho = 1$ in [Sa]. In [GuSa2] the mapping properties of the better behaved class $\sigma(x,\xi) \in C^v S^m_{1,\delta}$ are considered.

Definition 5.0.1 *We define symbol classes $C^v_k S^m_{\rho,\delta}$ by induction on k. Let $C^v_0 S^m_{\rho,\delta} = C^v S^m_{\rho,\delta}$, for $v \geq 0$,*

set l $=[v]$ and $\theta = v - [v]$. Assuming $C^v_{k-1} S^m_{\rho,\delta}$ has been defined,we say σ is in the symbol

class $C^v_k S^m_{\rho,\delta}$ if $\sigma \in C^v_{k-1} S^m_{\rho,\delta}$,

$$\nabla^l_x \sigma(x,\xi) \in \sum_{i+j=l} C^{v-i}_{k-1} S^{m+j\delta}_{\rho,\delta} \tag{5.0.1}$$

and

$$|h|^{-\theta} \{\nabla^l_x \sigma(x+h,\xi) - \nabla^l_x \sigma(x,\xi)\} \in \sum_{i+j=\lambda} C^{v-i}_{k-1} S^{m+j\delta}_{\rho,\delta} \tag{5.0.2}$$

5.1 L^p Sobolev Spaces

Theorem 5.1.1 *If* $0 \le \delta \le \rho \le 1$ *and* $\sigma(x,\xi) \in C^v_\infty S^m_{\rho,\delta}$, $m < -\frac{n}{2}(1-\rho), v > \frac{n}{2}$, *then* $\sigma(x,D) : H^s_p \to H^s_p$, $p \ge 2$, $\rho > \delta$, $s \in (-v_1, v_1)$, *where* $v_1 = v - \frac{n}{2}$.

We follow the argument used in the previous chapter, incorporating the changes necessary to obtain improved estimates for restricted symbols as in [Sa]. The main difference occurs after 5.3.22. Let

$$I = \sum_{k=0}^{\infty} \phi_k \overline{\phi_k}, \qquad 1 = \sum_{k=0}^{\infty} \left|\widehat{\phi_k}(\xi)\right|^2,$$

where $\widehat{\phi_k}(\xi) = \widehat{\phi_1}(2^{-k}\xi)$, $\widehat{\phi_1}$ is supported in $\left\{\xi : \frac{3}{4} \le |\xi| \le \frac{5}{2}\right\}$, $\widehat{\phi_0}$ is supported in $\left\{\xi : |\xi| \le \frac{3}{2}\right\}$. Then for s real, we have

$$\begin{aligned}
&\|\sigma(x,D)f(x)\|_{H^s_p} \\
&= \left\|(\textstyle\sum_{l=0}^{\infty} \left|2^{ls}\phi_l \circ \sigma f(x)\right|^2)^{\frac{1}{2}}\right\|_p \\
&= \left\|(\textstyle\sum_{l=0}^{\infty} \left|\sum_{k=0}^{\infty} 2^{ls}\phi_l \circ \sigma \circ 2^{-ks}\phi_k(2^{ks}\phi_k f)\right|^2)^{\frac{1}{2}}\right\|_p \\
&\le \left\|(\textstyle\sum_{l=0}^{\infty} \left|\sum_{|k-l|<5} 2^{ls}\phi_l \circ \sigma \circ 2^{-ks}\phi_k(2^{ks}\phi_k f)\right|^2)^{\frac{1}{2}}\right\|_p \\
&+ \left\|(\textstyle\sum_{l=0}^{\infty} \left|\sum_{|k-l|\ge 5} 2^{ls}\phi_l \circ \sigma \circ 2^{-ks}\phi_k(2^{ks}\phi_k f)\right|^2)^{\frac{1}{2}}\right\|_p \\
&= I + II
\end{aligned} \tag{5.1.3}$$

Let

$$
\begin{aligned}
T_{l,k}f(y) &= \phi_l \circ \sigma \circ \phi_k f(y) = \int_{\mathbf{R}^n} e^{iy\cdot\eta}\widehat{\phi_l}(\eta)(\widehat{\sigma \circ \phi_k f})(\eta)d\eta \\
&= c\int_{\mathbf{R}^n} e^{iy\cdot\eta}\widehat{\phi_l}(\eta)\int_{\mathbf{R}^n} e^{-ix\cdot\eta}\left(\sigma \circ \phi_k f\right)(x)dxd\eta \\
&= c\int_{\mathbf{R}^n} e^{iy\cdot\eta}\widehat{\phi_l}(\eta)\int_{\mathbf{R}^n} e^{-ix\cdot\eta}\int_{\mathbf{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\widehat{\phi_k f}(\xi)d\xi dxd\eta \\
&= c\int_{\mathbf{R}^n} e^{iy\cdot\eta}\widehat{\phi_l}(\eta)\int_{\mathbf{R}^n} e^{-ix\cdot\eta}\int_{\mathbf{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\widehat{\phi_k}(\xi)\int_{\mathbf{R}^n} e^{-ix'\cdot\xi}f(x')dx'd\xi dxd\eta \\
&= c\int_{\mathbf{R}^n} f(x')\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(y-x)\cdot\eta}e^{i(x-x')\cdot\xi}\widehat{\phi_l}(\eta)\sigma(x,\xi)\widehat{\phi_k}(\xi)d\xi dxd\eta dx' \\
&= c\int_{\mathbf{R}^n} f(x')\left(\phi_l \circ \sigma \circ \phi_k\right)(y,x')dx',
\end{aligned}
\tag{5.1.4}
$$

where,

$$
\phi_l \circ \sigma \circ \phi_k.(y,x') = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(y-x)\cdot\eta}e^{i(x-x')\cdot\xi}\widehat{\phi_l}(\eta)\sigma(x,\xi)\widehat{\phi_k}(\xi)d\xi dxd\eta.
\tag{5.1.5}
$$

5.2 Diagonal Terms and the Maximal Operator M_2

To control the diagonal terms I, we will use the square integrable maximal operator M_2. We will prove $(T_{l,l}f)^{\#}(x) \le cM_2 f(x)$ for all x and l. Without loss of generality, assume $x = 0$. Let Q be the cube centered at x with edge length equal to $l(Q)$ and $Q^{1-\theta}$ be the cube centered at x with edge length equal to $l(Q)^{1-\theta}$. Also let $2Q$ be the cube centered at x with edge length equal to $2l(Q)$.

Let

$$f = f_1 + f_2,$$

where

$$f_1 = \chi_{2Q^{1-\theta}} f,\ f_2 = f - f_1,\ \theta = 1 - \rho$$

We have

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |T_{l,l} f - T_{l,l} f_2\,(0)| \\ \le\ & \frac{1}{|Q|} \int_Q |T_{l,l} f_1| + \frac{1}{|Q|} \int_Q |T_{l,l} f_2 - T_{l,l} f_2\,(0)| \\ =\ & I + II \end{aligned}$$

To estimate term I we note

$$\begin{aligned} T_{l,l} f_1 &= |D|^{-\frac{n}{2}\theta}\, |D|^{\frac{n}{2}\theta}\, T_{l,l} f_1 \\ &= I_{\frac{n}{2}\theta}(|D|^{\frac{n}{2}\theta}\, T_{l,l} f_1) \\ &= I_{\frac{n}{2}\theta}(g) \end{aligned}$$

where I_θ is as in (2.1.2) on page 11 and

$$\begin{aligned} g &= |D|^{\frac{n}{2}\theta}\, T_{l,l} f_1 \\ &= |D|^{\frac{n}{2}\theta}\, \phi_l \circ \sigma \circ \phi_l f_1 \,. \end{aligned}$$

If we define q so that $\frac{1}{q} = \frac{1}{2} - \frac{\frac{n}{2}\theta}{n} = \frac{1}{2} - \frac{\theta}{2}$, then the Sobolev embedding Theorem yields

$$\left\| I_{\frac{n}{2}\theta}(g) \right\|_q \le C\, \|g\|_2 \,.$$

As for the term

$$g = |D|^{\frac{n}{2}\theta}\, T_{l,l} f_1 = |D|^{\frac{n}{2}\theta}\, \phi_l 2^{-l\frac{n}{2}\theta} 2^{l\frac{n}{2}\theta} \circ \sigma \circ \phi_l f_1,$$

we have

$$|\xi|^{\frac{n}{2}\theta}\, \widehat{\phi_l} 2^{-l\frac{n}{2}\theta} \in C^\infty S^0_{1,0},\ 2^{l\frac{n}{2}\theta} \sigma \widehat{\phi}_l \in C^{\frac{n}{2}+\epsilon} S^0_{\rho,\delta},$$

by the hypothesis on σ and therefore

$$\|g\|_2 = \left\| |D|^{\frac{n}{2}\theta}\, T_{l,l} f_1 \right\|_2 \leq C \|f_1\|_2\,,$$

by the Theorem of Coifman and Meyer ([CoMe]).

Thus term I can be estimated as follows:

$$\begin{aligned}
& \frac{1}{|Q|} \int_Q |T_{l,l} f_1| \\
\leq\ & \left(\frac{1}{|Q|} \int_Q |T_{l,l} f_1|^q \right)^{\frac{1}{q}} \\
\leq\ & C \left(\frac{1}{|Q|} \right)^{\frac{1}{q}} \left\| |D|^{\frac{n}{2}\theta}\, T_{l,l} f_1 \right\|_2 \\
\leq\ & C \left(\frac{1}{|Q|} \right)^{\frac{1}{2}} \|f_1\|_2 \\
\leq\ & C \left(\frac{1}{|Q|} \right)^{\frac{1}{q}} \left(\int_{2Q^{1-\theta}} |f|^2 \right)^{\frac{1}{2}} \leq C M_2 f\,(0)\,.
\end{aligned}$$

For $T_{l,l} f_2$, let

$$\phi_l \circ \sigma \circ \phi_l(y, x') = K_l(y, x'), \tag{5.2.6}$$

We get

$$|T_{l,l} f_2(y) - T_{l,l} f_2(0)| \ \leq\ \left| \int_{\mathbf{R}^n} (K_l(y, x') - K_l(0, x')) f_2(x') dx' \right|$$

$$\leq \left|\int_{y\in(2Q^{1-\theta})^c} (\acute{K}_l(y,x') - \acute{K}_l(0,x'))f(x')dx'\right|$$

$$= \left|\sum_{k=1}^{\infty} \int_{2^{k+1}Q^{1-\theta}\setminus 2^kQ^{1-\theta}} (\acute{K}_l(y,x') - \acute{K}_l(0,x'))f(x')dx'\right| \tag{5.2.7}$$

$$\leq \sum_{k=1}^{\infty} \left(\int_{2^{k+1}Q^{1-\theta}\setminus 2^kQ^{1-\theta}} |\acute{K}_l(y,x') - \acute{K}_l(0,x')|^2 |x'|^{n+\epsilon} dx'\right)^{\frac{1}{2}}$$

$$\times \left(\int_{2^{k+1}Q^{1-\theta}\setminus 2^kQ^{1-\theta}} \frac{\left|f(x')\right|^2}{\left|x'\right|^{n+\epsilon}} dx'\right)^{\frac{1}{2}}$$

$$\leq \sum_{k=1}^{\infty} \left(\int_{2^{k+1}Q^{1-\theta}\setminus 2^kQ^{1-\theta}} |\acute{K}_l(y,x') - \acute{K}_l(0,x')|^2 \left|x'\right|^{n+\epsilon} dx'\right)^{\frac{1}{2}}$$

$$\times \left(2^{k+1}\right)^{-\frac{\epsilon}{2}} \left|L^{1-\theta}\right|^{-\frac{\epsilon}{2}} M_2 f(0), \tag{5.2.8}$$

where L is the length of Q.

Next we will estimate

$$\left(\int_{2^{k+1}Q^{1-\theta}\setminus 2^kQ^{1-\theta}} |\acute{K}_l(y,x') - \acute{K}_l(0,x')|^2 |x'|^{n+\epsilon} dx'\right)^{\frac{1}{2}}. \tag{5.2.9}$$

Using Plancherel's formula, we have

$$\left(\int_{2^{k+1}Q^{1-\theta}\setminus 2^kQ^{1-\theta}} |\acute{K}_l(y,x') - \acute{K}_l(0,x')|^2 |x'|^{n+\epsilon} dx'\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{\mathbf{R}^n} \left|\partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}} K_l\widetilde{(y,\xi)} - \partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}} K_l\widetilde{(0,\xi)}\right|^2 d\xi\right)^{\frac{1}{2}}$$

$$= \left(\int_{\mathbf{R}^n} \left|\partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} (e^{i(y-x)\cdot\eta} - e^{i(0-x)\cdot\eta}) e^{ix\cdot\xi} \widehat{\phi_l}(\eta)\sigma(x,\xi)\widehat{\phi_l}(\xi) dx d\eta\right|^2 d\xi\right)^{\frac{1}{2}}. \tag{5.2.10}$$

Here

$$K_l(y,x') \quad = \phi_l \circ \sigma \circ \phi_l(y,x')$$

$$= \int_{\mathbf{R}^n} e^{-ix'\cdot\xi} \left(\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i(y-x)\cdot\eta} e^{ix\cdot\xi} \widehat{\phi_l}(\eta)\sigma(x,\xi)\widehat{\phi_l}(\xi) dx d\eta\right) d\xi. \tag{5.2.11}$$

In order to estimate

$$\partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} (e^{i(y-x)\cdot\eta} - e^{i(0-x)\cdot\eta}) e^{ix\cdot\xi} \widehat{\phi_l}(\eta)\sigma(x,\xi)\widehat{\phi_l}(\xi) dx d\eta. \tag{5.2.12}$$

We note that:

- If ∂_ξ hits $e^{ix\cdot\xi}$, this brings down ix; we can then use $ixe^{ix\cdot\eta} = \partial_\eta e^{ix\cdot\eta}$ and integration by parts to transfer $\partial_\xi e^{ix\cdot\xi}$ to $\partial_\eta \widehat{\phi_l}(\eta)$ with a gain of $\left(2^l\right)^{-1}$;
- If ∂_ξ hits $\sigma(x,\xi)$, we gain $\left(2^l\right)^{-\rho}$;
- If ∂_ξ hits $\widehat{\phi_l}(\xi)$, we gain $\left(2^l\right)^{-1}$.

From the identity,

$$\frac{2^{-2l} - \Delta_\eta}{2^{-2l} + |x-y|^2} e^{i(y-x)\cdot\eta} = e^{i(y-x)\cdot\eta} \tag{5.2.13}$$

we get the following

$$\left|\partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} (e^{i(y-x)\cdot\eta} - e^{i(0-x)\cdot\eta}) e^{ix\cdot\xi} \widehat{\phi_l}(\eta)\sigma(x,\xi)\widehat{\phi_l}(\xi) dx d\eta\right|$$

$$\leq C(2^l)^{(-\frac{n}{2}(1-\rho)-(\frac{n}{2}+\frac{\epsilon}{2})\rho)} \min\left\{|y|\, 2^l, 1\right\} \int_{\mathbf{R}^n} \left(\frac{2^{-2l}}{2^{-2l}+|x-y|^2}\right)^N \left(2^l\right)^n \left|\widehat{\phi_l}(\xi)\right| dx$$

$$\leq C(2^l)^{(-\frac{n}{2}(1-\rho)-(\frac{n}{2}+\frac{\epsilon}{2})\rho)} \left|\widehat{\phi_l}(\xi)\right| \min\left\{|y|\, 2^l, 1\right\}, \quad (for N > \tfrac{n}{2}) \tag{5.2.14}$$

and

$$\left(\int_{\mathbf{R}^n}\left|\partial_\xi^{\frac{n}{2}+\frac{\epsilon}{2}}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(e^{i(y-x)\cdot\eta}-e^{i(0-x)\cdot\eta})e^{ix\cdot\xi}\widehat{\phi_l}(\eta)\sigma(x,\xi)\widehat{\phi_l}(\xi)dxd\eta\right|^2 d\xi\right)^{\frac{1}{2}}$$

$$\leq C(2^l)^{(-\frac{n}{2}(1-\rho)-(\frac{n}{2}+\frac{\epsilon}{2})\rho)+\frac{n}{2}}\min\left\{|y|\,2^l,1\right\}\leq C(2^l)^{-\frac{\epsilon}{2}\rho}\min\left\{|y|\,2^l,1\right\}, \quad (5.2.15)$$

Thus we have

$$|T_{l,l}f_2(y)-T_{l,l}f_2(0)| \leq C\sum_{k=1}^{\infty}\min\left\{|y|\,2^l,1\right\}(2^l)^{-\frac{\epsilon\rho}{2}}\left|2^{k+1}L^{1-\theta}\right|^{-\frac{\epsilon}{2}}M_2f(0) \quad (5.2.16)$$

(*here $L=l(Q)$ is the length of Q.*)

$$\leq C\min\left\{|y|\,2^l,1\right\}(2^l)^{-\frac{\epsilon\rho}{2}}|L^\rho|^{-\frac{\epsilon}{2}}M_2f(0) \qquad (\text{since}\quad 1-\theta=\rho)$$

$$\leq CM_2f(0)\min\left\{(L2^l)^{1-\frac{\epsilon\rho}{2}},(L2^l)^{-\frac{\epsilon\rho}{2}}\right\}M_2f(0) \quad (5.2.17)$$

$$\leq CM_2f\,(0). \quad (5.2.18)$$

Altogether, we have

$$\frac{1}{|Q|}\int_Q|T_{l,l}f-T_{l,l}f_2(0)|\leq CM_2f(0)$$

and taking the supremum over Q we obtain

$$(T_{l,l}f)^{\#}(0)\leq CM_2f(0).$$

Here we use (2.1.5).

Similarly it can be shown that

$$(T_{l,k}f)^{\#}(x) \le C M_2 f(x) \quad for \;\; all \;\; |k-l| \le 5.$$

So

$$I = \left\| (\textstyle\sum_{l=0}^{\infty} \left| \sum_{|k-l|<5} 2^{ls} \phi_l \circ \sigma \circ 2^{-ks} \phi_k (2^{ks} \phi_k f) \right|^2)^{\frac{1}{2}} \right\|_p$$

$$\le C \left\| (\textstyle\sum_{l=0}^{\infty} \left| \sum_{|k-l|<5} M_2 (2^{ks} \phi_k f) \right|^2)^{\frac{1}{2}} \right\|_p \tag{5.2.19}$$

$$\le C \left\| (\textstyle\sum_{k=0}^{\infty} \left| M_2 (2^{ks} \phi_k f) \right|^2)^{\frac{1}{2}} \right\|_p \le C \left\| (\textstyle\sum_{k=0}^{\infty} \left| 2^{ks} \phi_k f) \right|^2)^{\frac{1}{2}} \right\|_p$$

$$= C \|f\|_{H_p^s} \;\; for \;\; p > 2.$$

Here we use the vector-valued version for $M_2 : L^p \to L^p,\ p > 2$ (*see*[FefSt2]).

5.3 Off-Diagonal Terms

For term II ,we need an estimate of the kernel of $\phi_l \circ \sigma \circ \phi_k$ using the smoothness of σ in the x variable as well. We follow closely the argument in [Sa]. Our goal is to get the estimate

$$|\phi_l \circ \sigma \circ \phi_k . g| \le C 2^{-v(k \vee l)} 2^{\delta v k} M g. \tag{5.3.20}$$

This is easy, if v is integer, since

$$\frac{\partial_x}{(\xi - \eta)} e^{ix\cdot(\xi-\eta)} = e^{ix\cdot(\xi-\eta)} \; for \;\; \xi \ne \eta. \tag{5.3.21}$$

We obtain

$$\phi_l \circ \sigma \circ \phi_k(y, x') = \textstyle\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i(y-x)\cdot\eta} e^{i(x-x')\cdot\xi} \times$$

$$\left\{ \widehat{\phi_l}(\eta) \left(\tfrac{\partial_x}{(\xi-\eta)} \right)^v \sigma(x, \xi) \widehat{\phi_k}(\xi) . \right\} d\xi dx d\eta.$$

$$= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i(y-x)\cdot\eta} e^{i(x-x')\cdot\xi} \left(\frac{1-2^{2l}\Delta_\eta}{1+2^{2l}|y-x|^2}\right)^N$$

$$\times \left(\frac{1-2^{2k\rho}\Delta_\xi}{1+2^{2k\rho}|x-x'|^2}\right)^N \left\{\widehat{\phi_l}(\eta)\left(\frac{\partial_x}{(\xi-\eta)}\right)^v \sigma(x,\xi)\widehat{\phi_k}(\xi).\right\} d\xi dx d\eta.$$

$$= \int_{\mathbf{R}^n} \left(\frac{1}{1+2^{2l}|y-x|^2}\right)^N \left(\frac{1}{1+2^{2k\rho}|x-x'|^2}\right)^N \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i(y-x)\cdot\eta} e^{i(x-x')\cdot\xi} \left(1-2^{2l}\Delta_\eta\right)^N$$

$$\times \left(1-2^{2k\rho}\Delta_\xi\right)^N \left\{\widehat{\phi_l}(\eta)\left(\frac{\partial_x}{(\xi-\eta)}\right)^v \sigma(x,\xi)\widehat{\phi_k}(\xi).\right\} d\xi d\eta dx. \tag{5.3.22}$$

Here is where we use the definition of restricted symbol to advantage. We note:

- When $\frac{\partial_x}{(\xi-\eta)}$ hits $\sigma(x,\xi)$, consider the definition of $\sigma(x,\xi) \in C^v_\infty S^m_{\rho,\delta}$,

$$\partial_x \sigma(x,\xi) = \sigma_1(x,\xi) + \sigma_2(x,\xi), \tag{5.3.23}$$

  $\sigma_1(x,\xi) \in C^v S^{m+\delta}_{\rho,\delta}$, $\sigma_2(x,\xi) \in C^{v-1} S^m_{\rho,\delta}$, for $\sigma_1(x,\xi)$, we use the method in previous chapter, here we need $\delta < \frac{1}{2}$, we gain $2^{-(k \vee l)}$;

- When $\left(1-2^{2k\rho}\Delta_\xi\right)$ hits $\left(\frac{1}{(\xi-\eta)}\right)^v \sigma(x,\xi)\widehat{\phi_k}(\xi)$, we gain nothing;

- When $\left(1-2^{2l}\Delta_\eta\right)$ hits $\widehat{\phi_l}(\eta)\left(\frac{1}{(\xi-\eta)}\right)^v$, we gain nothing.

Consider $|\eta| \approx 2^l$, $|\xi| \approx 2^k$, and let $N > \frac{n}{2}$, we have

$$|\phi_l \circ \sigma \circ \phi_k . g| \leq C 2^{-v_1 (k \vee l)} Mg \ , \quad v_1 = v - \frac{n}{2} \tag{5.3.24}$$

here we need ξ, η not too close.

If v is not a positive integer, for $[v]$ we use this method, for fractional part $v-[v]$, we may use similar methods as in [Sa] with the identity

$$\frac{\delta^h_x}{e^{ih\cdot(\xi-\eta)}-1} e^{ix\cdot(\xi-\eta)} = e^{ix\cdot(\xi-\eta)}. \tag{5.3.25}$$

For this we introduce a further decomposition. Let $\{\widehat{\psi}_\alpha\}_{\alpha\in A}$ be a smooth finite partition of unity on the sphere S^{n-1} and let $\widehat{\phi}_{k,\alpha}(\xi) = \widehat{\phi}_k(\xi)\widehat{\psi}_\alpha(\frac{\xi}{|\xi|})$. We choose the partition so fine that there exists constants c, C with $0 < c < C < 2\pi$ such that for all k, l with $|k-l| \geq 5$ and all $\alpha, \beta \in A$, there are points $h^{l,k}_{\alpha,\beta} \in \mathbf{R}^n$ such that

$$0 < c \leq (\xi - \eta) \cdot h^{l,k}_{\alpha,\beta} \leq C < 2\pi \ , \qquad for \quad \widehat{\phi}_{k,\alpha}(\xi)\widehat{\phi}_{l,\beta}(\eta) \neq 0. \tag{5.3.26}$$

Since

$$\phi_l \circ \sigma \circ \phi_k = \sum_{\alpha,\beta\in A} \phi_{l,\beta} \circ \sigma \circ \phi_{k,\alpha} \tag{5.3.27}$$

and so it suffices to estimate the kernel of $\phi_{l,\beta} \circ \sigma \circ \phi_{k,\alpha}$. We have

$$\begin{aligned} &\phi_{l,\beta} \circ \sigma \circ \phi_{k,\alpha}(y, x') \\ &= \int_{R^n} \int_{R^n} \int_{R^n} e^{i(y-x)\cdot\eta} e^{i(x-x')\cdot\xi} \left\{\widehat{\phi_{l,\beta}}(\eta)\sigma(x,\xi)\widehat{\phi_{k,\alpha}}(\xi)\right\} d\xi dx d\eta. \end{aligned} \tag{5.3.28}$$

We will use the fractional smoothness of σ in x via the eigenfunction identity

$$\frac{\delta^h_x}{e^{ih\cdot(\xi-\eta)} - 1} e^{ix\cdot(\xi-\eta)} = e^{ix\cdot(\xi-\eta)}, \tag{5.3.29}$$

where δ^h_x is the first difference operator given in (2.1.4) on page 11, and $h = h^{l,k}_{\alpha,\beta}$. Note that $|h| = \left|h^{l,k}_{\alpha,\beta}\right| \approx 2^{-(k\vee l)}$. Applying this identity to $\phi_{l,\beta} \circ \sigma \circ \phi_{k,\alpha}(y, x')$ and noting that the transpose of δ^h_x is δ^{-h}_x , we obtain

$$\phi_{l,\beta} \circ \sigma \circ \phi_{k,\alpha}(y, x')$$

$$= \int_{R^n} \int_{R^n} \int_{R^n} e^{i(y-x)\cdot\eta} e^{i(x-x')\cdot\xi} \left\{\widehat{\phi_{l,\beta}}(\eta) \frac{\delta^{-h}_x}{e^{ih\cdot(\xi-\eta)}-1} \left(\frac{\partial_x}{(\xi-\eta)}\right)^{[v]} \sigma(x,\xi)\widehat{\phi_{k,\alpha}}(\xi)\right\} d\xi dx d\eta. \tag{5.3.30}$$

Since $e^{i(y-x)\cdot\eta}$ and $e^{i(x-x')\cdot\xi}$ are eigenfunctions of $\frac{1-2^{2l}\Delta_\eta}{1+2^{2l}|y-x|^2}$ and $\frac{1-2^{2k\rho}\Delta_\xi}{1+2^{2k\rho}|x-x'|^2}$ respectively , we further obtain that

$$\phi_{l,\beta}\circ\sigma\circ\phi_{k,\alpha}(y,x') = \int_{R^n}\int_{R^n}\int_{R^n} e^{i(y-x)\cdot\eta}e^{i(x-x')\cdot\xi}\left(\frac{1-2^{2l}\Delta_\eta}{1+2^{2l}|y-x|^2}\right)^N \times$$

$$\left(\frac{1-2^{2k\rho}\Delta_\xi}{1+2^{2k\rho}|x-x'|^2}\right)^N \left\{\widehat{\phi_{l,\beta}}(\eta)\frac{\delta_x^{-h}}{e^{ih\cdot(\xi-\eta)}-1}\left(\frac{\partial_x}{(\xi-\eta)}\right)^{[v]}\sigma(x,\xi)\widehat{\phi_{k,\alpha}}(\xi)\right\}d\xi dx d\eta. \tag{5.3.31}$$

Note the following:

$$\left|\frac{\delta_x^{-h}}{e^{ih\cdot(\xi-\eta)}-1}\left(\frac{\partial_x}{(\xi-\eta)}\right)^{[v]}\partial_\xi^\beta\sigma(x,\xi)\right| \le C_{\beta,v}\left|h_{\alpha,\beta}^{l,k}\right|^{v-[v]}\left(\frac{1}{|\xi-\eta|}\right)^{[v]}(1+|\xi|^2)^{(\delta v-|\beta|)/2}$$

$$\le C_{\beta,v}2^{-(k\vee l)v}(1+|\xi|^2)^{(\delta v-|\beta|)/2} \tag{5.3.32}$$

since $\left|e^{ih\cdot(\xi-\eta)}-1\right| \ge c$. On the other hand , we also have

$$\left|\partial_\xi^\beta\partial_\eta^\gamma\frac{\delta_x^{-h}}{e^{ih\cdot(\xi-\eta)}-1}\right| \le C\left|h_{\alpha,\beta}^{l,k}\right|^{(|\beta|+|\gamma|)}$$

$$\le C2^{-(k\vee l)(|\beta|+|\gamma|)} \tag{5.3.33}$$

and

$$\left|\partial_\xi^\beta\partial_\eta^\gamma\left(\frac{1}{(\xi-\eta)}\right)^{[v]}\right| \le C\left(\frac{1}{|\xi-\eta|}\right)^{[v]+(|\beta|+|\gamma|)} \tag{5.3.34}$$

If we consider support conditions on $\widehat{\phi}_{k,\alpha}(\xi)$ and $\widehat{\phi}_{l,\beta}(\xi)$, and integrating in ξ and η , we obtain as in [Sa], that

$$|\phi_l\circ\sigma\circ\phi_k\circ g| \le C2^{-v_1(k\vee l)}2^{\delta v_1 k}Mg\ ,\ \ v_1 = v-\frac{n}{2}. \tag{5.3.35}$$

So

$$II \;=\; \left\|\left(\sum_{l=0}^{\infty}\left|\sum_{|k-l|\ge 5}2^{ls}\phi_l\circ\sigma\circ 2^{-ks}\phi_k(2^{ks}\phi_k f)\right|^2\right)^{\frac{1}{2}}\right\|_p$$

$$\leq C\left\|\left(\sum_{l=0}^{\infty}\left|\sum_{|k-l|\geq 5}2^{-v_1(k\vee l)}2^{s(l-k)}M(2^{ks}\phi_k f)\right|^2\right)^{\frac{1}{2}}\right\|_p$$

$$\leq C\left\|\left(\sum_{k=0}^{\infty}\left|M(2^{ks}\phi_k f)\right|^2\right)^{\frac{1}{2}}\right\|_p$$

$$\leq C\left\|\left(\sum_{k=0}^{\infty}\left|2^{ks}\phi_k f)\right|^2\right)^{\frac{1}{2}}\right\|_p \tag{5.3.36}$$

$$= C\,\|f\|_{H_p^s} \tag{5.3.37}$$

provided $-v_1(k \vee l) + s(l-k) < 0$, i.e. for $0 \leq s < v_1$ and $-v_1 < s < 0$, as in [Sa]. Now we use interpolation to get the full range

$$-v_1 < s < v_1. \tag{5.3.38}$$

This is the end of the proof.

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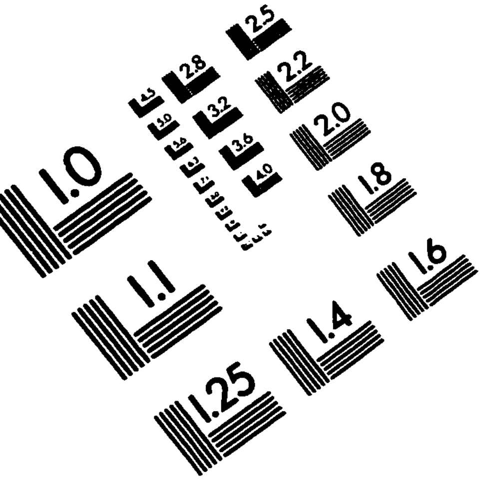
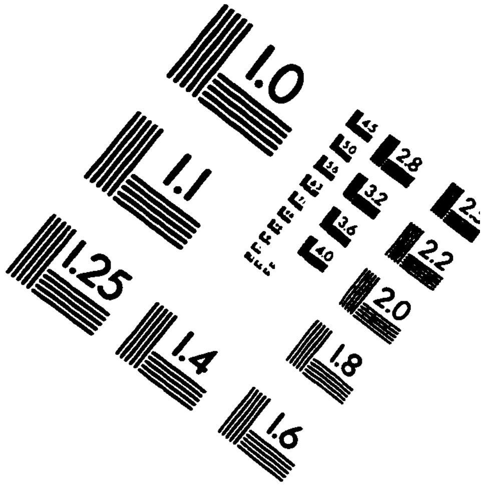
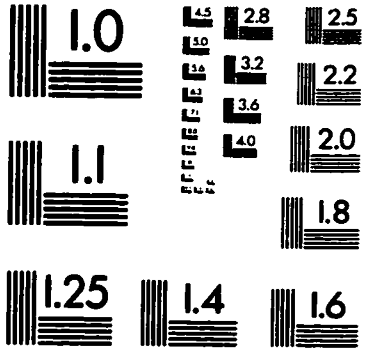
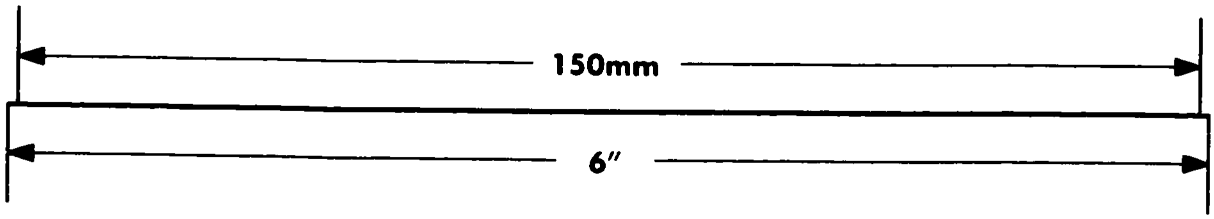
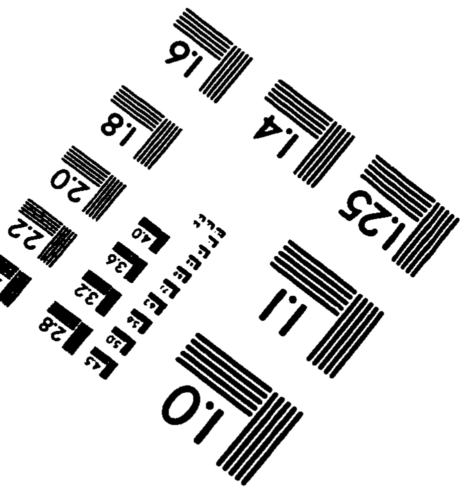
IMAGE EVALUATION TEST TARGET (QA-3)

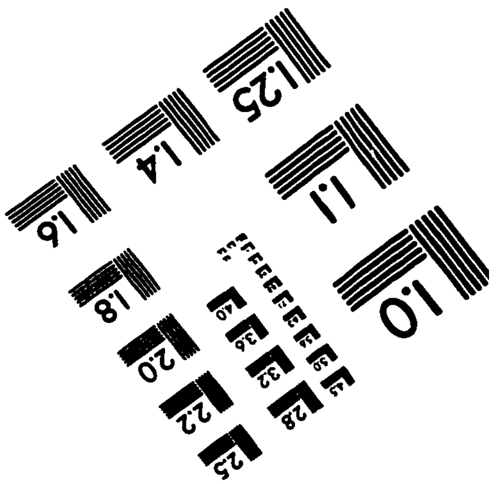












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