DECOMPOSITIONS OF COMPLETE MULTIPARTITE GRAPHS AND GROUP DIVISIBLE DESIGNS INTO ISOMORPHIC FACTORS

By

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A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree
Doctor of Philosophy

McMaster University
April 1994

DECOMPOSITIONS

OF COMPLETE MULTIPARTITE

GRAPHS AND *GDD*'S

DOCTOR OF PHILOSOPHY (1994) McMASTER UNIVERSITY
(Mathematics) Hamilton, Ontario

TITLE: Decompositions of complete multipartite graphs and group divisible designs into isomorphic factors

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NUMBER OF PAGES: viii, 115

ABSTRACT

A multipartite graph $K_{m_1,m_2,\ldots,m_r}$ (group divisible design GDD) is (t,d)-*decomposable* if it can be decomposed into t factors with the same diameter d. The graph $K_{m_1,m_2,\ldots,m_r}$ (design GDD) is (t,d)-*isodecomposable* if the factors are moreover isomorphic. $K_{m_1,m_2,\ldots,m_r}$ (GDD) is *admissible* for a given t if its number of edges (or blocks) is divisible by t. $f_r(t,d)$ or $g_r(t,d)$, respectively, is the minimum number of vertices of a (t,d)-decomposable or (t,d)-isodecomposable complete r-partite graph, respectively. $g'_r(t,d)$ is the minimum number such that for every $p \geq g'_r(t,d)$ there exists a (t,d)-isodecomposable r-partite graph with p vertices, and $h_r(t,d)$ is the minimum number such that all admissible r-partite graphs with $p \geq h_r(t,d)$ vertices are (t,d)-isodecomposable.

We completely determine the spectrum of all bipartite and tripartite $(2,d)$-isodecomposable graphs. We show that $f_2(2,d) = g_2(2,d) = g'_2(2,d) = h_2(2,d)$ and $f_3(2,d) = g_3(2,d) = g'_3(2,d)$ for each d, that is possible, while $h_3(2,2) = \infty$ (i.e., for any given p, there is an admissible graph with more than p vertices which is not $(2,2)$-isodecomposable), $h_3(2,3) = g'_3(2,3) + 2$, $h_3(2,4) = g'_3(2,4)$ and $h_3(2,5) = g'_3(2,5) + 1$.

For complete four-partite graphs we completely determine the spectrum of $(2,d)$-isodecomposable graphs with at most one odd part. For the remaining admissible graphs, namely for those with all odd parts, we show that there is no

such $(2,5)$-isodecomposable graph. For $d = 2,3,4$ we solve the problem in this class completely for the graphs $K_{n,n,n,m}$ and $K_{n,n,m,m}$.

For all $r \geq 5$ we determine smallest $(2,d)$-isodecomposable r-partite graphs for all possible diameters and show that also in these cases always $g_r(2,d) = g'_r(2,d)$. Some values of $h_r(2,d)$ are also determined.

We furthermore prove that if a GDD with $r \geq 3$ groups is $(2,d)$-isodecomposable, then $d \leq 4$ or $d = \infty$. We show that for every admissible n there exists a $(2,3)$- and $(2,4)$-isodecomposable $3 - GDD(n,3)$, i.e., a GDD with 3 groups of cardinality n and block size 3.

Finally, we determine the spectrum of the designs $3 - GDD(n,3)$ which are decomposable into unicyclic factors.

ACKNOWLEDGEMENT

I would like to express my thanks to my supervisor, Dr. Alexander Rosa, for introducing me to the topics discussed in this thesis and encouraging me to study them, and for his many helpful suggestions and advice during the preparation of this thesis. I am also thankful for his careful and critical review of the manuscript.

My thanks are also due to Dr. Jozef Širáň for many useful discussions and comments that helped me to improve the final version of this thesis.

I am grateful to McMaster University for awarding me the Ashbaugh Graduate Scholarship and the C. W. Sherman Scholarship and to the Department of Mathematics and Statistics for the Departmental Scholarship. I also wish to thank NSERC and Dr. J. Csima, Dr. C. Riehm and Dr. A. Rosa, whose NSERC Operating Grants made the Departmental Scholarship possible.

I thank the Graduate Advisor Dr. Manfred Kolster and Dr. Frantisek Franek from the Department of Computer Science for their assistance, especially in the initial stages of my stay at McMaster University.

Finally, I wish to express my deepest thanks to Dr. Alexander Rosa and his wife Nadja, whose broad support helped me to get over many difficulties during my first months in Canada.

Dedicated to the memory of

Professor Štefan Znám

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1. Introduction

After years of exhausting competition and unsuccessful battles against increasingly discriminating regulations which preferred regular airlines, two major charter companies from Beland and Deland decided to reach an agreement. The agreement should enable them not only to continue their international operations more effectively, but also to avoid the rules which prohibited charters to operate domestic flights in both Beland and Deland. The top executives from Beland B-ways and Deland D-lines agreed on the following conditions.

1. Each line will be operated only by one of the companies.
2. To reach any destination, a passenger travelling with any one of the two companies will not have to change more than twice.
3. The networks served by the two companies will have the same structure up to the lengths of single lines.

They signed the agreement and appointed a joint committee to specify the new networks. Unfortunately, the committee was unable to do this and was holding meetings for weeks. Finally one of the senior members of the committee was so bored that he resigned and a recent graduate was appointed to fill the vacancy. At her first meeting she realized that she has already heard about a similar problem, so she dug up old notes from her graph theory course and, as a result, one of the authors of the paper [6] was approached. Being too busy, he recommended his student for the job. The student solved the problem, got a lot of money and free lifetime tickets from both companies and did not have to do an applied research any more.

Although not all details of the story are completely true, the fact is that in this thesis we solve the above mentioned problem. Translated to the language of graph theory, this is the problem of decompositions of complete bipartite graphs into two isomorphic factors with diameter 3, and the cited article [6] is dealing with decompositions of complete graphs into factors with given diameters.

A *factor* F of a graph $G = G(V, E)$ is a subgraph of G having the same vertex set V. A *decomposition* of a graph $G(V, E)$ into factors $F_1(V, E_1), F_2(V, E_2), \dots, F_t(V, E_t)$ is a t-tuple of factors such that $E_i \cap E_j = \emptyset$ for any $1 \le i < j \le t$ and $\bigcup_{i=1}^{t} E_i = E$. A decomposition of G is called *isomorphic* if $F_1 \cong F_2 \cong \cdots \cong F_t$. If $t = 2$ then an isomorphism $\phi : F_1 \to F_2$ is also called a *self-complementing isomorphism* and the factors F_1 and F_2 the *selfcomplementary factors*. The *diameter* $\operatorname{diam} G$ of a connected graph G is the maximum of the set of distances $\operatorname{dist}_G(x, y)$ among all pairs of vertices of G. If G is disconnected, then $\operatorname{diam} G = \infty$.

We already mentioned twice the first paper on decompositions of complete graphs into factors with given diameters that was presented by J. Bosák, A. Rosa and Š. Znám [6] in 1966. The paper was published in 1968 and started an extensive research in this area. Many authors studied the problem [see, e.g., 4, 5, 17, 21, 31], some of them also for directed graphs [26, 29, 30]. In 1975 first paper on decompositions of complete graphs into isomorphic factors with given diameters by A. Kotzig and A. Rosa [16] appeared, followed by others [see, e.g., 15, 17, 24]. E. Tomová [27] published first results on decompositions of complete bipartite graphs into factors with given diameters in 1977 and later some others [28]. Decompositions of complete multipartite graphs into selfcomplementary factors were studied by T. Gangopadhyay and S. P. Rao Hebbare [11]. T. Gangopadhyay [10] then published a paper dealing with decompositions of complete multipartite graphs into factors with given diameters. F. Harary, R. W. Robinson and N. C. Wormald studied isomorphic factorizations of multipartite graphs [13], and S. J. Quinn [20] studied isomorphic factorizations of a special class of multipartite graphs, namely the graphs with all parts of the same cardinality (called *equipartite graphs*).

Although some of the graphs presented in the papers [26, 27, 10] are self-complementary, isomorphic factorizations per se were not considered. On the other hand, the authors of the paper on selfcomplementary factors [11] were not interested in diameters of the factors. This thesis therefore joins both concepts. We study decompositions of complete r-partite graphs, for all $r \geq 2$, into two isomorphic factors with a given diameter. We always assume that the number of vertices of an r-partite graph is at least $r+1$, i.e., the graph is not a complete graph K_r.

E. Tomová [27] proved that a complete bipartite graph $K_{n,m}$ decomposable into two factors with the same finite diameter d exists if and only if $d = 3, 4, 5$ or 6, and presented smallest decomposable graphs for each of the diameters. T.

Gangopadhyay [10] proved that there exists a complete r-partite graph for $r \geq 3$ decomposable into two factors with the same finite diameter d if and only if $d = 2, 3, 4$ or 5. He also presented the smallest numbers of vertices of such decomposable graphs.

A complete r-partite graph is (t, d)-*decomposable* if it is decomposable into t factors with the same finite diameter d. If we in addition require all factors to be mutually isomorphic, we say that the graph is (t, d)-*isodecomposable*. We denote a complete r-partite graph with r parts having $m_1, m_2, \ldots, m_r$ elements, respectively, by $K_{m_1, m_2, \ldots, m_r}$. Or we denote the complete r-partite graph having k_i parts of cardinality n_i for $i = 1, 2, \ldots, s$ by $K_{n_1^{k_1} n_2^{k_2} \ldots n_s^{k_s}}$. In this case we always assume that $k_1 + k_2 + \cdots + k_s = r$ and $n_i \neq n_j$ for $i \neq j$.

Decompositions of more general combinatorial objects were also studied, though not as extensively as graph decompositions. Zs. Baranyai [1] in 1975 and P. Tomasta [23, 24, 25] in 1976 and later studied decompositions of complete k-uniform hypergraphs. Relatively recently three papers on decompositions of designs into two factors appeared. A. Hartman [14] considered halving complete designs into two factors with the same number of blocks, while P. K. Das and A. Rosa [8] were halving Steiner triple systems into selfcomplementary factors. K. Phelps [19] studied decompositions of complete designs with block size 4.

We are interested in decompositions of group divisible designs into self-complementary factors with given diameter and into smallest connected factors. A *group divisible design* $k - GDD(n, r)$ is a triple $(V, \mathcal{G}, \mathcal{B})$ where V is a set of elements, $\mathcal{G}$ is a partition of V into r subsets of cardinality n called *groups* and $\mathcal{B}$ is a collection of subsets of V of cardinality k called *blocks* such that $|G \cap B| \leq 1$ for any group $G \in \mathcal{G}$ and any block $B \in \mathcal{B}$ and for any two elements x, y from distinct groups there is exactly one block containing both x and y. *Factors* are defined analogically as in the case of graphs.

A decomposition of a GDD is, in fact, equivalent to a decomposition of a multipartite complete graph satisfying an additional condition. If E is a factor of a $k - GDD(n, r)$ $(V, \mathcal{G}, \mathcal{B})$ then the *underlying graph* of E is the r-partite graph $U(E)$ with the vertex set V in which two vertices x, y are adjacent if and only if the elements x, y are adjacent in E, i.e., if they belong to the same block of E. A decomposition of a $k - GDD(n, r)$ is then equivalent to the decomposition of the complete r-partite graph with parts of cardinality n into factors whose edge sets can be partitioned into complete graphs K_k, where each K_k corresponds to one block of E.

In Chapter 2 we study decompositions of complete multipartite graphs into two isomorphic factors with given diameters. In Sections 2.0 and 2.1 we give the definitions and some necessary preliminary results. In particular, we define two important classes of graphs. A complete multipartite graph $K_{n_1^{k_1} n_2^{k_2} \ldots n_s^{k_s}}$ is *admissible* if it has an even number of edges, and is *strongly admissible* if there is at most one odd number n_i having an odd exponent k_i.

In Section 2.2 we prove that every strongly admissible complete multipartite graph is decomposable into two isomorphic disconnected factors and present the smallest numbers of vertices of the complete r-partite graphs that are decomposable into two isomorphic disconnected factors for every $r > 1$.

The method of extensions of factors given in Section 2.3 is later used in many constructions.

Section 2.4 deals with bipartite and tripartite graphs. An r-partite complete graph is $(2,d)$-*isodecomposable* if it can be decomposed into two isomorphic factors with the diameter d. In this section we completely determine the spectrum of all bipartite and tripartite $(2,d)$-isodecomposable graphs for all possible finite diameters d.

Section 2.5 has two parts. In the first part we completely determine all $(2,d)$-isodecomposable complete four-partite graphs with at most one odd part. The second part contains results on the remaining class of admissible four-partite graphs, i.e., the graphs K_{m_1,m_2,m_3,m_4} with all odd parts. We prove that there is no $(2,5)$-isodecomposable complete four-partite graph with all odd parts and completely solve the problem of $(2,d)$-isodecomposability for $d = 2,3,4$ in the case that the parts are of at most two different cardinalities.

Finally, the complete r-partite graphs with $r \geq 5$ are studied in Section 2.6. We determine smallest $(2,d)$-isodecomposable r-partite complete graphs for each possible finite d and each $r \geq 5$. We also prove that if such a smallest $(2, d)$-isodecomposable r-partite complete graph has p_0 vertices, then for every $p > p_0$ there exists an $(2, d)$-isodecomposable r-partite complete graph with p vertices.

In Chapter 3 we study isomorphic decompositions of group divisible designs. In Section 3.0 we give the necessary definitions.

In Section 3.1 we prove that if a GDD is isodecomposable into two connected factors, then the diameter of the factors is at most 4.

It is obvious that a $k-GDD(n, r)$ is isodecomposable into two factors only if the number of the blocks of the design is even. In particular, a $3-GDD(n, 3)$ is not isodecomposable for any odd n. In Section 3.2 we construct a $(2, d)$-isodecomposable $3 - GDD(n, 3)$ for $d = 3, 4$ and every even $n \geq 4$. An example of an (2,d)-isodecomposable $3 - GDD(4, 4)$ for $d = 3, 4$ and ∞ is presented in Section 3.3.

In Section 3.4 we prove that there is no $3 - GDD(n, 3)$ isodecomposable into connected acyclic factors and that the smallest possible connected factors giving an isomorphic decomposition are unicyclic, namely cycles. We prove that a decomposition of a $3 - GDD(n, 3)$ into isomorphic connected unicyclic factors is possible only if $n \equiv 0 (\mathrm{mod}\, 6)$ and for each such n we construct the $3 - GDD(n, 3)$ having the required property.

2. Decompositions of multipartite graphs into selfcomplementary factors

2.0. Introductory notes and definitions

In this chapter we study decompositions of finite complete multipartite graphs into two isomorphic factors with a prescribed diameter. A *factor* F of a graph $G = G(V, E)$ is a subgraph of G having the same vertex set V. A *decomposition* of a graph $G(V, E)$ into two factors $F_1(V, E_1)$ and $F_2(V, E_2)$ is a pair of factors such that $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E$. A decomposition of G is called *isomorphic* if $F_1 \cong F_2$. An isomorphism $\phi : F_1 \to F_2$ is then also called a *self-complementing isomorphism*, *self-complementing permutation* or *complementing permutation* and the factors F_1 and F_2 the *selfcomplementary factors with respect to G* or simply the *selfcomplementary factors*. The *diameter* $\operatorname{diam} G$ of a connected graph G is the maximum of the set of distances $\operatorname{dist}_G(x, y)$ among all pairs of vertices of G. If G is disconnected, then $\operatorname{diam} G = \infty$. The *order* of a graph G is the number of vertices of G while the *size* of G is the number of its edges. For terms not defined here, see [2].

A. Kotzig and A. Rosa [16] and later P. Tomasta [24], D. Palumbíny [18], and P. Híc and D. Palumbíny [15] studied decompositions of complete graphs into isomorphic factors with a given diameter. E. Tomová [27] studied decompositions of complete bipartite graphs into two factors with given diameters and determined all possible pairs of diameters of such factors. T. Gangopadhyay [10] studied decompositions of complete r-partite graphs ($r \geq 3$) into two factors with given diameters and determined also all possible pairs of diameters of such factors.

In this thesis we join both concepts. We study decompositions of complete r-partite graphs, for all $r \geq 2$ into two isomorphic factors with a given diameter. We always assume that the number of vertices of an r-partite graph is at least $r+1$, i.e. the graph is not a complete graph K_r.

It is well known that a complete graph K_n is decomposable into two isomorphic factors (called *selfcomplementary graphs*) if and only if $n \equiv 0$ or $1 (\mathrm{mod}\, 4)$ and the diameter of such a factor is either 2 or 3. The decomposition exists for every $n \geq 4$ ($n \geq 5$ in the case of diameter 2), $n \equiv 0$ or $1 (\mathrm{mod}\, 4)$. The decomposition into an odd number m of isomorphic factors with a given diameter is studied in [16] for $m = 3$ and in [24] for $m > 3$.

E. Tomová [27] proved that a complete bipartite graph $K_{n,m}$ decomposable into two factors with the same finite diameter d exists if and only if $d = 3, 4, 5$ or 6 and determined the smallest decomposable graphs for each of the diameters. T. Gangopadhyay [10] proved that a complete r-partite graph for $r \geq 3$ decomposable into two factors with the same finite diameter d exists if and only if $d = 2, 3, 4$ or 5. He also determined the smallest numbers of vertices of such decomposable graphs.

A complete r-partite graph is (t, d)-*decomposable* if it is decomposable into t factors with the same finite diameter d. If we in addition require all factors to be mutually isomorphic, we say that the graph is (t, d)-*isodecomposable*. We also often say that a graph G is *isodecomposable* if it is $(2, d)$-isodecomposable for a finite diameter d which we do not determine specifically. We show that there are $(2, d)$-isodecomposable complete r-partite graphs for each of the above mentioned diameters for any $r \geq 2$. In all cases we also present smallest isodecomposable graphs.

2.1. Preliminary theorems

We denote a complete r-partite graph with r parts having $m_1, m_2, \ldots, m_r$ elements, respectively, by $K_{m_1, m_2, \ldots, m_r}$. Or, especially if there are more parts having the same cardinality, we denote the complete r-partite graph having k_i parts of cardinality n_i for $i = 1, 2, \ldots, s$ by $K_{n_1^{k_1} n_2^{k_2} \ldots n_s^{k_s}}$. In this case we always suppose that $k_1 + k_2 + \cdots + k_s = r$ and $n_i \neq n_j$ for $i \neq j$.

Let $f_r(t,d)$ denote the smallest number of vertices of a complete r-partite (t,d)-decomposable graph. If such a number does not exist, then we define $f_r(t,d) = \infty$. Let $f'_r(t,d)$ denote the smallest integer such that for every $m \geq f'_r(t,d)$ there exist (t,d)-decomposable graph of order m. We again put $f'_r(t,d) = \infty$ if such a number does not exist. We can see from the following theorems that always $f_r(t,d) = f'_r(t,d)$. J. Bosák, A. Rosa and Š. Znám [6] proved the following result:

Theorem 2.1.1. (Bosák, Rosa, Znám) *If a complete graph K_n $(n \geq 2)$ is decomposable into m factors with diameters $d_1, d_2, \ldots, d_m$, then for any $N > n$ the complete graph K_N is also decomposable into m factors with the diameters $d_1, d_2, \ldots, d_m$.*

An analogue for r-partite graphs is due to E. Tomová [27].

Theorem 2.1.2. (Tomová) *If a complete r-partite graph $K_{n_1, n_2, \ldots, n_r}$ $(r \geq 2)$ is decomposable into m factors with diameters $d_1, d_2, \ldots, d_m$ (where $d_i \geq 2$ for $i = 1, 2, \ldots, m$), then for any $N_1 \geq n_1, N_2 \geq n_2, \ldots, N_r \geq n_r$ the graph $K_{N_1, N_2, \ldots, N_r}$ is also decomposable into m factors with the diameters $d_1, d_2, \ldots, d_m$.*

It is obvious that any $(2,d)$-isodecomposable complete r-partite graph $K_{m_1, m_2, \ldots, m_r}$ must have an even number of edges and hence the number of parts having odd cardinalities must be 0 or 1 (mod 4). A graph with this property, as well as the corresponding r-tuple $m_1, m_2, \ldots, m_r$, is called *admissible.*

We can similarly introduce $g_r(t,d)$ as the smallest number of vertices of a complete (t,d)-isodecomposable r-partite graph. We also define $g'_r(t,d)$ as the smallest integer with the property that for any $n \geq g'_r(t,d)$ there is a complete r-partite (t,d)-isodecomposable graph with n vertices. Finally, we define $h_r(t,d)$ as the smallest integer such that any admissible complete r-partite graph with at least $h_r(t,d)$ vertices is (t,d)-isodecomposable. If such numbers do not exist, we again put $g_r(t,d) = \infty$, $g'_r(t,d) = \infty$ or $h_r(t,d) = \infty$, respectively. It is obvious that

$$f_r(t,d) \leq g_r(t,d) \leq g'_r(t,d) \leq h_r(t,d).$$

We show now that the first and last inequality can be in some cases sharp. For instance, Gangopadhyay [10] proved that $f_4(2,3) = 5$, but there is no admissible four-partite graph with 5 vertices and so $g_4(2,3) \geq 6$. The last inequality can be sharp as well: $f_2(2,4) = g_2(2,4) = g'_2(2,4) = 8$, but $h_2(2,4) = \infty$ since no graph $K_{2,m}$ can be decomposed into two factors with diameter 4, as has been proved by Tomová [27]. On the other hand, the difference between $h_r(t,d)$ and $g'_r(t,d)$ can be small. There is only one admissible complete tripartite graph with at least 7 vertices which is not (2,5)-isodecomposable, namely $K_{2,2,3}$. Because no graph with less than 7 vertices is (2,5)-isodecomposable either, we can see that $h_3(2,5) = g'_3(2,5)+1 = 8$.

In the following paragraphs we prove some lemmas which will be useful in constructions of classes of graphs with given parameters. From now on, we always assume that the number of parts, r, is at least 2 and the number of vertices is at least $r+1$.

The following example shows that not all admissible complete multipartite graphs are isodecomposable into connected factors.

Example 2.1.3. Consider a graph $G \cong K_{3,3,3,11}$ with nine vertices $v_1, v_2, \ldots, v_9$ of degree 17 and eleven vertices $u_1, u_2, \ldots, u_{11}$ of degree 9. Suppose that the graph is decomposable into two connected isomorphic factors F_1 and F_2. Denote a_i and b_i the degrees of the vertex v_i in F_1 and F_2, and c_j and d_j the degrees of a vertex u_j in F_1 and F_2, respectively. Because $a_i + b_i = 17$ for $i = 1, 2, \ldots, 9$ and $c_j + d_j = 9$ for $j = 1, 2, \ldots, 11$, obviously $a_i \neq b_i$ and $c_j \neq d_j$. Since the number of vertices of degree 17 in G is odd, there is a value, say t, which appears more times in the sequence $a_1, a_2, \ldots, a_9$ than in $b_1, b_2, \ldots, b_9$. If t appears k times in $a_1, a_2, \ldots, a_9$ and $a_{i_0} = r$, then since $a_{i_0} + b_{i_0} = 17$, we may assume without loss of generality that $t = a_{i_0} > b_{i_0}$ and hence $t \geq 9$. Now there are k vertices of the set $\{v_1, v_2, \ldots, v_9\}$ which are of degree t in F_1 while less than k vertices of this set are of degree t in F_2. But then there must be a vertex u_{j_0} such that $d_{j_0} = t \geq 9$ and hence $c_{j_0} = 0$, i.e. v_{j_0} is an isolated vertex in F_1, which is impossible because F_1 is connected. So we have shown that $K_{3,3,3,11}$, although admissible, is not $(2, d)$-isodecomposable for any finite d.

This leads us to introduce another class of graphs for which we prove that its members are always decomposable into two connected isomorphic factors. A multipartite complete graph $K_{n_1^{k_1} n_2^{k_2} \ldots n_s^{k_s}}$, where $n_1, n_2, \ldots, n_p$ are odd and $n_{p+1}, n_{p+2}, \ldots n_s$ are even is *strongly admissible* if it is admissible and at most one of the numbers $k_1, k_2, \ldots, k_p$ is odd (i.e., if the number of parts of the same odd cardinality is always even with at most one exception).

First we deal with the bipartite case.

Lemma 2.1.4. *A strongly admissible bipartite graph $K_{n,m}$ is decomposable into two connected isomorphic factors if and only if $n, m > 2$.*

Proof. If $n = 1$ then the graph is a star and every proper factor is disconnected. If $n = 2$ then $K_{2,m}$ has $m + 2$ vertices and a connected factor has to have at least $m + 1$ edges, which is impossible, because $K_{2,m}$ has size $2m$. Since every admissible bipartite graph has at least one partite set with an even number of vertices, we may assume that we have a graph $K_{2n,m}$ with $n \geq 2$ and $m \geq 3$. Let the partite sets be $\{u_1, u_2, \ldots, u_{2n}\}$ and $\{v_1, v_2, \ldots, v_m\}$. We construct a connected factor F_1 containing edges u_iv_1 for $i = 1, 2, \ldots, n$, u_iv_2 for $i = n+1, n+2, \ldots, 2n$ and $u_{2i}v_j$ for $i = 1, 2, \ldots, n$; $j = 3, \ldots, m$. The other factor F_2 contains all remaining edges of $K_{2n,m}$ and is, clearly, isomorphic to F_1. □

Having the bipartite case solved, we can prove the general theorem.

Theorem 2.1.5. *Every strongly admissible multipartite graph other than $K_{1,2m}$ or $K_{2,m}$ is decomposable into two connected isomorphic factors.*

Proof. Suppose that we have a strongly admissible graph $K_{n_1^{2k_1} n_2^{2k_2} \ldots n_t^{2k_t}}$ where all $n_1, n_2, \ldots, n_t$ are odd. Then $\sum_{i=1}^{t} k_i$ is even and $\sum_{i=1}^{t} 2k_i n_i = 4n$. Take a complete graph K_{4n} with vertices $v_{11}, \ldots, v_{1n}, v_{21}, \ldots, v_{2n}, v_{31}, \ldots, v_{3n}, v_{41}, \ldots, v_{4n}$ and decompose it into 2 isomorphic factors F_1 and F_2 as follows: F_1 contains K_{2n} induced by the vertices $v_{11}, \ldots, v_{1n}, v_{21}, \ldots, v_{2n}$ and all edges $v_{1i}v_{4j}$ and $v_{2i}v_{3j}$, $i, j = 1, 2, \ldots, n$. Then choose k_1 mutually disjoint subsets of cardinality n_1 of the set $\{v_{11}, \ldots, v_{1n}, v_{21}, \ldots, v_{2n}\}$ and delete from F_1 all edges having both end-vertices in the same subset. Repeat this for k_2 subsets of cardinality n_2 and so on such that no two subsets of any cardinality have a vertex in common. Finally, delete an edge $v_{3i}v_{4j}$ from F_2 if and only if $v_{1i}v_{2j}$ has been deleted from F_1. The remainder of the original graph K_{4n} is now isomorphic to $K_{n_1^{2k_1} n_2^{2k_2} \ldots n_t^{2k_t}}$ and F_1

is clearly isomorphic to F_2. Now take any complete $(z+1)$-partite complete graph $K_{2m_1,2m_2,\ldots,2m_z,m_{z+1}}$ (which is, of course, strongly admissible) with the partite sets $U_i = \{u_{ij}|j = 1,\ldots,2m_i\}$, $i = 1,\ldots,z+1$ and decompose it into two isomorphic factors F_1' and F_2' as follows: For any $i < l \leq z$ and any $s = 1,\ldots,2m_l$ an edge $u_{ij}u_{ls}$ belongs to F_1' if $j \in \{1,\ldots,m_i\}$ and to F_2' if $j \in \{m_i+1,\ldots,2m_i\}$. For any $i \leq z$ and any $s = 1,\ldots,m_{z+1}$ an edge $u_{ij}u_{z+1\ s}$ belongs to F_1' if $j \in \{m_i+1,\ldots,2m_i\}$ and to F_2' if $j \in \{1,\ldots,m_i\}$.

If $t = 0$ and $z \geq 2$, the proof is finished. If $t = 0$ and $z = 1$, the result is given by Lemma 2.1.4.

If $t > 0$ we can join every vertex of F_1' (even if $z = 0$ – in this case F_1' is just a set of isolated vertices) to all vertices $v_{11},\ldots,v_{1n},v_{21},\ldots,v_{2n}$ of F_1 and every vertex of F_2' to all vertices $v_{31},\ldots,v_{3n},v_{41},\ldots,v_{4n}$ to obtain two isomorphic factors F_1'' and F_2'' of a strongly admissible multipartite graph with $2k_1$ parts of an odd cardinality n_1, $2k_2$ parts of an odd cardinality n_2,...,$2k_t$ parts of an odd cardinality n_t, z parts of even cardinalities (not necessarily different) $2m_1, 2m_2,\ldots,2m_z$ and a part of cardinality m_{z+1}. Because the cardinality m_{z+1} is arbitrary, the theorem is proved. □

Now we determine the spectrum of orders of strongly admissible graphs. Because every strongly admissible graph is isodecomposable, as it follows from Theorem 2.1.5., we will at the same time see that once we have an r-partite isodecomposable graph of order p then there exists an isodecomposable r-partite graph for all orders greater than p.

Lemma 2.1.6. *Let $G \cong K_{m_1,m_2,\ldots m_r}$ be a strongly admissible complete r-partite graph containing a vertex of an even degree. Then there is a number $i \in \{1,2,\ldots,r\}$ such that the graph $G' \cong K_{m_1,\ldots,M_i,\ldots,m_r}$ is strongly admissible for any $M_i \geq m_i$.*

Proof. First suppose that the number of vertices of G is odd. Then it follows from the definition that there is an odd number among $m_1, m_2, \ldots, m_r$ which appears an odd number of times in the r-tuple $m_1, m_2, \ldots, m_r$. Let $G \cong K_{n_1^{k_1} n_2^{k_2} \ldots n_s^{k_s}}$. Let $n_1, \ldots, n_p$ be odd and $n_{p+1}, \ldots, n_s$ even numbers. We may assume without loss of generality that $m_i = n_1$ and hence k_1 is odd. Obviously, any vertex belonging to a partite set of an odd cardinality has an even degree in G and, by definition, $k_2, \ldots, k_p$ are even.

Now consider the graph G'. It contains $k'_1 = k_1 - 1 \equiv 0 (\bmod 2)$ parts of cardinality n_1. If $M_i = n_j \in \{n_2, \ldots, n_p\}$, then there are $k'_j = k_j + 1 \equiv 1 (\bmod 2)$ parts of cardinality n_j, $k'_1 \equiv \cdots \equiv k_{j-1} \equiv k_{j+1} \equiv \cdots \equiv k_p \equiv 0 (\bmod 2)$ and G' is strongly admissible. If M_i is an odd number not belonging to $\{n_2, \ldots, n_p\}$, then there is just one part having M_i vertices, $k'_1, k_2, \ldots, k_p$ are even and G' is strongly admissible. If M_i is an even number, then again $k'_1, k_2, \ldots, k_p$ are even and G' is strongly admissible, too.

Now suppose that the order of G is even. Then, by the definition of strongly admissible graphs, for any even m_j there is an even number of parts having cardinality m_j. Since all vertices belonging to parts of odd cardinality have odd degrees, in order for G to contain a vertex of an even degree, there must be at least one part of an even cardinality. We can denote the even cardinality by m_i and follow the first part of the proof to obtain the desired result. □

The following corollary is an immediate consequence of Theorem 2.1.5 and Lemma 2.1.6.

Corollary 2.1.7. *Let G be an isodecomposable complete r-partite graph of order p containing a vertex of an even degree. Then there exists an isodecomposable complete r-partite graph of order q for any $q \geq p$.*

Lemma 2.1.6 and Corollary 2.1.7 do not deal with strongly admissible graphs with all vertices of odd degrees. We can remedy this by making the following simple observation.

Proposition 2.1.8. *Let $K_{m_1,m_2,\ldots,m_r}$ be a strongly admissible graph of order p with all vertices of odd degrees. Then there exists a strongly admissible r-partite graph of the same order p having a vertex of an even degree.*

Proof. If a strongly admissible r-partite graph $K_{m_1,m_2,\ldots,m_r}$ has all vertices of odd degrees then r is an even number not less than 4, p is even, every partite set is of an odd cardinality and each cardinality appears in the r-tuple $m_1, m_2, \ldots, m_r$ an even number of times. Therefore we can without loss of generality suppose that $m_1 = m_2 \geq m_3 = m_4 \geq \cdots \geq m_{r-1} = m_r$. Moreover, because we always assume that $p \geq r+1$ we may assume that $m_1 = m_2 \geq 3$. Then $m_1 + 1$ and $m_2 - 1$ are both even, $m_2 - 1 \geq 2$ and one can easily see that the r-tuple $m_1 + 1, m_2 - 1, m_3, \ldots, m_r$ is strongly admissible and so is $K_{m_1+1,m_2-1,m_3,\ldots,m_r}$ whose order is p. The degree of every vertex belonging to the partite set of cardinality $m_1 + 1$ is now $p - m_1 - 1$, which is clearly an even number. □

Proposition 2.1.8 together with Corollary 2.1.7 instantly yields the following.

Lemma 2.1.9. *Let G be an isodecomposable complete r-partite graph of order p. Then there exists an isodecomposable complete r-partite graph of order q for any $q \geq p$.*

Now we can determine the smallest orders of isodecomposable complete r-partite graphs.

Theorem 2.1.10. *There exists an isodecomposable complete r-partite graph of order $p > r$ if and only if*

$r = 2, p \geq 7$, *or*

$r \equiv 0 (\bmod 4)$ *and* $p \geq r + 3$, *or*

$r \equiv 1 (\bmod 4)$ *and* $p \geq r + 1$, *or*

$r \equiv 2 (\bmod 4), r > 2$ *and* $p \geq r + 1$, *or*

$r \equiv 3 (\bmod 4)$ *and* $p \geq r + 2$.

Proof. The case $r = 2$ follows immediately from Lemma 2.1.4.

The orders stated in the other parts of the theorem are the smallest respective orders of admissible r-partite graphs. This proves the necessity.

To prove sufficiency, one can observe that an r-partite graph $K_{2,2,2,1,\ldots,1}$ is strongly admissible for all $r \equiv 0 (\bmod 4)$. Similarly, $K_{2,1,1,\ldots,1}$ is strongly admissible for all $r \equiv 1$ or $2 (\bmod 4)$ and $K_{2,2,1,1,\ldots,1}$ is strongly admissible for all $r \equiv 3 (\bmod 4)$. Therefore all these graphs are isodecomposable by Theorem 2.1.5 which together with Lemma 2.1.9 yields our result. □

2.2. Decomposition into disconnected factors

First we show that strong admissibility is a sufficient condition for $(2, \infty)$-isodecomposability of an r-partite complete graph for every $r \geq 2$.

Theorem 2.2.1. *Every strongly admissible complete r-partite graph is decomposable into two isomorphic disconnected factors.*

Proof. Case 1: $r = 2$.

Every bipartite strongly admissible graph has a partite set of an even cardinality. Hence we may assume that we have a graph $K_{2n,m}$ with partite sets $V = \{v_1, v_2, \ldots, v_{2n}\}$ and $U = \{u_1, u_2, \ldots, u_m\}$. The factor F_1 then contains edges $v_i u_j$ for all $i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$ while F_2 contains the remaining edges. Both factors are clearly disconnected and $F_1 \cong F_2$.

Case 2: $r > 2$, at least one part is even.

Suppose we have a strongly admissible r-partite graph G with $r > 2$ and a partite set V of an even cardinality $2m_1$. Let $G \cong K_{2m_1, m_2, \ldots, m_r}$ and $V = \{v_1, v_2, \ldots, v_{2n}\}$. Then the $(r-1)$-partite graph $G' \cong K_{m_2, m_3, \ldots, m_r}$ with the vertex set $U = \{u_1, u_2, \ldots, u_q\}$, where $q = m_2 + m_3 + \cdots + m_r$, is also strongly admissible. Now we have to distinguish two subcases:

(i) $r = 3$. In this case at least two parts are of even cardinalities, hence we can decompose G' into factors F_1' and F_2' as in Case 1 and then add the set V and all edges $v_i u_j, i = 1, 2, \ldots, m_1; j = 1, 2, \ldots, q$ to F_1'. The resulting factor F_1 of G is disconnected and isomorphic to F_2 which contains the set V, the factor F_2' and the edges $v_i u_j$ for $i = m_1 + 1, m_1 + 2, \ldots, 2m_1; j = 1, 2, \ldots, q$.

(ii) $r \geq 4$. Here we can decompose G' into isomorphic (and connected) factors F_1' and F_2' using the construction from Theorem 2.1.5 and then add the set V and edges $v_i u_j$ as in the subcase (i) to obtain again mutually isomorphic disconnected factors F_1 and F_2.

Case 3: $r > 2$, all parts are odd.

In this case $r \equiv 0$ or $1 (\mathrm{mod}\, 4)$ as follows from admissibility of G. Since $G \cong K_{2m_1+1, 2m_2+1, \ldots, 2m_r+1}$ is strongly admissible, we may assume without loss of generality that $m_1 = m_2 \geq m_3 = m_4 \geq \cdots \geq m_{r-1}$. We again distinguish two subcases.

(i) $m_1 = 0$. Then $m_1 = m_2 = \cdots = m_{r-1} = 0$ which yields $m_r \geq 1$ (otherwise G is a complete graph K_r). Therefore $G \cong K_{1,1,\ldots,1,2m_r+1}$. Let the vertex set of G consist of the part $V = \{v_1, v_2, \ldots, v_{2m_r+1}\}$ and $r-1$ vertices $u_1, u_2, \ldots, u_{r-1}$. Let $G' \cong K_{1,1,\ldots,1,2m_r-1}$ have vertices $v_3, v_4, \ldots, v_{2m_r+1}, u_1, u_2, \ldots, u_{r-1}$. If $m_r \geq 2$, then G' is a strongly admissible graph of order greater than r and can be decomposed, by Theorem 2.1.5, into two isomorphic factors F_1' and F_2'. If $m_r = 1$, then $G' \cong K_r$ and $r \equiv 0$ or $1 \pmod 4$. It is well known that then G' can be also decomposed into two isomorphic factors, say F_1'' and F_2''. We can now construct the factor F_1 of G by joining v_1 to all vertices of F_1' (or F_1'') and v_2 to all vertices of F_2' (or F_2''). Then F_1 contains the isolated vertex v_1 and is isomorphic to F_2.

(ii) $m_1 \geq 1$. Because $m_1 = m_2$, G contains at least two non-trivial parts $V = \{v_1, v_2, \ldots, v_{2m_1+1}\}$ and $W = \{w_1, w_2, \ldots, w_{2m_1+1}\}$ and the remaining vertices $u_1, u_2, \ldots, u_q$. Now let $G' \cong K_{2m_1-1,2m_2-1,\ldots,2m_r+1}$ have the vertices $v_3, \ldots, v_{2m_1+1}, w_3, \ldots, w_{2m_1+1}, u_1, \ldots, u_q$. As in (i), G' is always decomposable into two isomorphic factors F_1' and F_2' and we can again extend F_1' to F_1, joining v_1 and w_1 to all vertices of F_1' and v_2 to w_2. Similarly we join v_2 and w_2 to all vertices of F_2' and v_1 to w_1 to obtain F_2. The factors F_1 and F_2 are mutually isomorphic and disconnected — both contain a component isomorphic to K_2, induced by the vertices v_2 and w_2 or v_1 and w_1, respectively. This completes the proof. □

Since every bipartite and tripartite admissible graph is strongly admissible, the following is evident.

Theorem 2.2.2. *Every admissible bipartite and tripartite graph is decomposable into two isomorphic disconnected factors. In particular,*

(a) $f_2(2,\infty) = g_2(2,\infty) = g_2'(2,\infty) = h_2(2,\infty) = 3,$

(b) $f_3(2,\infty) = 4, g_3(2,\infty) = g_3'(2,\infty) = h_3(2,\infty) = 5.$

Now we present the smallest $(2,\infty)$-isodecomposable graphs.

Although for $r \geq 4$ we are not able to determine all complete r-partite $(2,\infty)$-isodecomposable graphs, we show that for every order greater than the minimum one there exists a $(2,\infty)$-isodecomposable graph.

Theorem 2.2.3. *For every $r \geq 2$ one of the following holds:*

(a) if $r \equiv 0 (\mathrm{mod}\, 4)$ then $f_r(2,\infty) = r+1, g_r(2,\infty) = g_r'(2,\infty) = r+3$,

(b) if $r \equiv 1 (\mathrm{mod}\, 4)$ then $f_r(2,\infty) = g_r(2,\infty) = g_r'(2,\infty) = r+1$,

(c) if $r \equiv 2 (\mathrm{mod}\, 4)$ then $f_r(2,\infty) = g_r(2,\infty) = g_r'(2,\infty) = r+1$,

(d) if $r \equiv 3 (\mathrm{mod}\, 4)$ then $f_r(2,\infty) = r+1, g_r(2,\infty) = g_r'(2,\infty) = r+2$.

Proof. Every r-partite graph of order $r+1$ is decomposable into factors $F_1 \cong K_1 \cup K_r$ and $F_2 \cong K_1 \cup K_{1,r-1}$. Hence $f_r(2,\infty) = r+1$ for every $r > 1$. The factors of the minimal $(2,\infty)$-isodecomposable graphs are shown in Figure 2.2.1 and the corresponding isomorphisms are the following. We denote by $F_1(r)$ the disconnected selfcomplementary factor of the minimal $(2,\infty)$-isodecomposable complete r-partite graph.

(a) For $r = 4$ the factor $F_1(4)$ with the parts $W = \{w_0\}, V_i = \{v_{i1}, v_{i2}\}, i = 1,2,3$. The cycles of the isomorphism $\phi : F_1 \to F_2$ are $(w_0), (v_{11}v_{12}), (v_{21}v_{22}), (v_{31}v_{32})$. For any $r \equiv 0 (\mathrm{mod}\, 4), r > 4$ we get the factor $F_1(r)$ such that we "blow up"the vertex w_0 such that we put into the vertex any selfcomplementary graph of

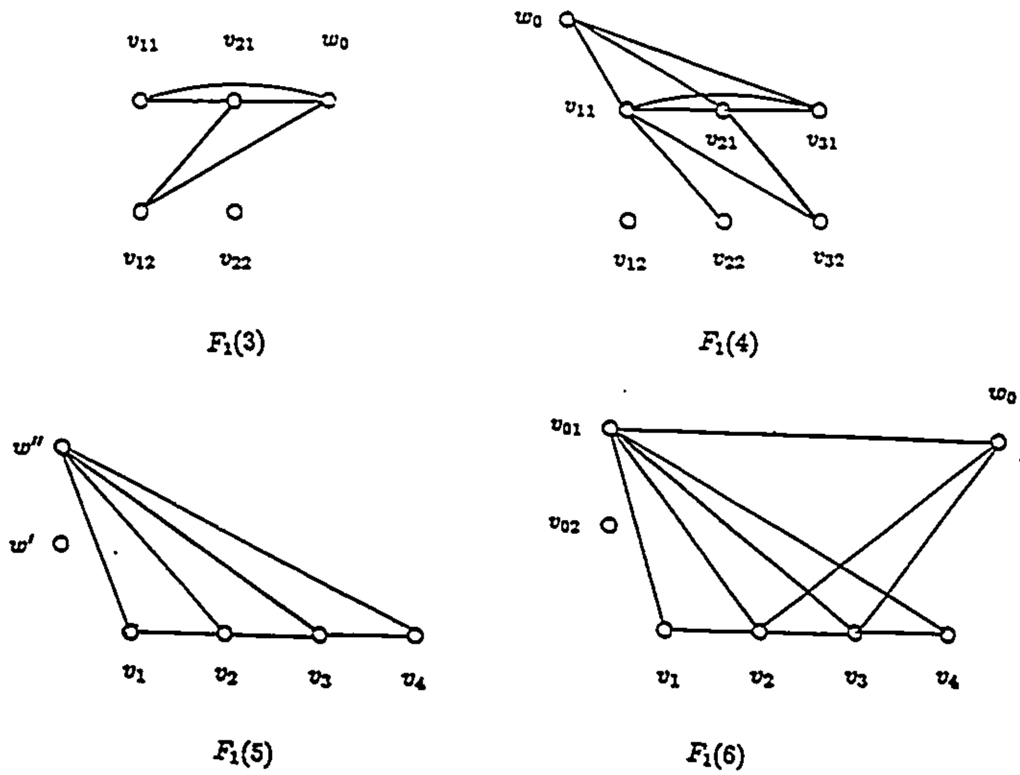


Figure 2.2.1

order $r-3$ with vertices $w_0, u_1, u_2, \ldots, u_{4r-4}$ and the complementing isomorphism ψ with $\psi(w_0) = w_0$. Then every vertex $u_i, 0 < i \leq 4r-4$ is adjacent to v_{jk} if and only if w_0 is adjacent to v_{jk}. The resulting factor G_0' has isomorphism induced by ϕ and ψ. To construct a factor of any order greater than $r+3$, say $r+3+p$, we simply add p new vertices $w_1, w_2, \ldots, w_p$ to the partite set W and join each of them to all neighbours of w_0. Then the isomorphism takes every w_i onto itself.

(b) For $r = 5$ the factor $F_1(5)$ with the parts $W = \{w', w''\}, V_i = \{v_i\}$ for $i = 1, 2, 3, 4$. The cycles of the isomorphism ϕ are $(w'w''), (v_1v_3v_4v_2)$. We construct a factor of any order $p+6$ by adding p vertices $w_1, w_2, \ldots, w_p$ to W and joining each of them to v_1 and v_4. The isomorphism now sends each w_i onto itself.

We construct the factor $F_1(9)$ with 10 vertices by adding 4 new parts $U_{11} = \{u_{11}\}, U_{12} = \{u_{12}\}, U_{13} = \{u_{13}\}, U_{14} = \{u_{14}\}$ to $F_1(5)$ with the edges

$u_{11}u_{12}, u_{12}u_{13}, u_{13}u_{14}$ and all edges $u_{11}v_i, u_{14}v_i, u_{11}w'', u_{14}w''$ for $i = 1,2,3,4$. The cycle of the isomorphism is $(u_{11}u_{13}u_{14}u_{12})$.

To increase the number of parts by any number $4r'$, we continue similarly, adding new parts $U_{jk}, j = 2,3,\ldots,r'; k = 1,2,3,4$ with edges and isomorphism exactly as above. To increase the order of an r-partite graph, we again add vertices $w_1, w_2, \ldots, w_p$ to the part W and edges w_tv_1, w_tv_4 for every $t = 1,2,\ldots,p$ and w_tu_{jk} whenever the edge w_tv_k exists. Again the isomorphism takes each w_t onto itself.

(c) For $r = 2$ the minimal graph is $K_{2,1}$ and the factors are $K_1 \cup K_2$. For $r = 6$ we consider the factor $F_1(6)$ with the parts $V_1 = \{v_{01}, v_{02}\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}, W = \{w_0\}$. The isomorphism is defined by $(w_0), (v_{01}v_{02}), (v_1v_3v_4v_2)$. We increase the number of parts by "blowing up" the vertex w_0 into selfcomplementary graph exactly as in part (a). We also increase the order of the factor by adding copies of the vertex w_0 to the partite set W.

(d) For $r = 3$ we consider the factor $F_1(3)$ with the parts $V_1 = \{v_{11}, v_{12}\}$, $V_2 = \{v_{21}, v_{22}\}, W = \{w_0\}$. The isomorphism is defined by $(v_{11}v_{12}), (v_{21}v_{22}), (w_0)$ and we increase the number of parts and the order of the factor exactly as in (a) and (c). □

We have not been able to determine $h_r(2,\infty)$ for $r > 3, r \equiv 1,2,3 (\mathrm{mod}\, 4)$. For $r \equiv 0 (\mathrm{mod}\, 4)$ we shall later show that $h_r(2,\infty) = \infty$.

2.3. Extensions of isomorphic factors preserving diameters

Although we have proved that every strongly admissible graph is isodecomposable, we have not been able yet to decompose a multipartite graph into isomorphic factors with a particular finite diameter. In this section we give the necessary tools for such decompositions.

Most of our constructions of isomorphic factors with a given diameter are based on "extensions" of smaller factors. In fact, we already used the "extension" in the proof of Theorem 2.2.3. Having a factor with a diameter d of an r-partite graph G, we construct factors with the same diameter by extending a partite set of G into sets of all strongly admissible cardinalities. The method is a special case of that used by Gangopadhyay and Rao Hebbare [11], although they studied just factors with given diameters not requiring their isomorphism. In the following lemmas we describe two different cases.

The first lemma allows us to extend a partite set of a cardinality m_i to any cardinality greater than m_i.

Lemma 2.3.1. *Let $G \cong K_{m_1,m_2,\ldots,m_r}$ be a complete r-partite graph decomposable into two isomorphic factors F_1 and F_2 with finite diameter d and let V_i be a partite set of G of cardinality m_i. Let there exist an isomorphism $\phi : F_1 \to F_2$ and a vertex $v_{ij} \in V_i$ such that $\phi(v_{ij}) = v_{ij}$. Then $G' \cong K_{m_1,\ldots,M_i,\ldots,m_r}$ is also decomposable into two isomorphic factors with the diameter d for any $M_i \geq m_i$.*

Proof. Let an r-partite graph G with partite sets $V_1, V_2, \ldots, V_r$ be decomposable into two isomorphic factors F_1 and F_2 with the diameter d. Suppose that the partite set V_1 of cardinality m_1 and its vertex v_{11} satisfy along with an isomorphism ϕ the assumptions of our lemma. Let G' be a complete multipartite graph with the parts $V'_1 = V_1 \cup \{v'_{1\ m_1+1}, \ldots, v'_{1\ M_1}\}, V_2, \ldots, V_r$. We construct factors F'_1 and F'_2 as follows:

F_1' (F_2', respectively) contains all edges belonging to F_1 (F_2) and moreover each vertex $v_{1i}'(i = m_1 + 1, \ldots, M_1)$ is adjacent to all neighbours of v_{11} in F_1 (F_2). Now we define an isomorphism $\phi' : F_1' \to F_2'$:

$$\phi'(v_{ij}) = \phi(v_{ij}) \text{ for } i = 1, 2, \ldots, r; j = 1, 2, \ldots, m_i$$

$$\phi'(v_{1j}') = v_{1j}' \text{ for } j = m_1 + 1, \ldots, M_1.$$

It is easy to see that ϕ' is really an isomorphism. For any v_{1j}' an edge $\phi'(v_{1j}')\phi'(v_{kl}) = v_{1j}'\phi(v_{kl})$ appears in F_2' if and only if $v_{11}\phi(v_{kl})$ exists in F_2'. But this occurs if and only if $v_{11}v_{kl}$ exists in F_1' and this is true if and only if v_{1j}' and v_{kl} are adjacent in F_1'. Hence $\phi'(v_{1j}')\phi'(v_{kl})$ appears in F_2' if and only if the edge $v_{1j}'v_{kl}$ exists in F_1'.

Furthermore, an edge $\phi'(v_{ij})\phi'(v_{kl})$ appears in F_2' if and only if $v_{ij}v_{kl}$ exists in F_1' because in this case $\phi'(v_{ij}) = \phi(v_{ij}), \phi'(v_{kl}) = \phi(v_{kl})$ and ϕ is an isomorphism.

Now let $\text{dist}_G(u, w)$ and $\text{dist}_{G'}(u, w)$ be the distances between vertices u and w in G and G', respectively. If $u, w \notin \{v_{11}, v_{1\,m_1+1}', \ldots, v_{1\,M_1}'\}$, then $\text{dist}_{G'}(u, w) = \text{dist}_G(u, w) \leq d$. If both $u, w \in \{v_{11}, v_{1\,m_1+1}', \ldots, v_{1\,M_1}'\}$, then $\text{dist}_{G'}(u, w) = 2 \leq d$. Finally, if u belongs to $\{v_{11}, v_{1\,m_1+1}', \ldots, v_{1\,M_1}'\}$ while w does not, then again $\text{dist}_{G'}(u, w) = \text{dist}_{G'}(v_{11}, w) = \text{dist}_G(v_{11}, w) \leq d$. Therefore F_1' is of diameter at most d. Because $\operatorname{diam} G = d$, there are vertices u_0 and w_0 in G such that $\text{dist}_G(u_0, w_0) = d$. Let P' be a $u_0 - w_0$ path containing any vertex v_{1j}' and let P be a $u_0 - w_0$ path which arises from P' by replacing v_{1j}' by v_{11}. Then P' and P are of the same length and hence $\text{dist}_{G'}(u_0, w_0) = \text{dist}_G(u_0, w_0) = d$. Therefore F_1' has diameter d, which completes the proof. □

However, Lemma 2.3.1 is not as universal as it seems to be. Since the "fixed" vertex required in the construction can appear in at most one part, we need another lemma which extends partite sets by even numbers of vertices.

Lemma 2.3.2. *Let $G \cong K_{m_1,m_2,\ldots,m_r}$ be a complete r-partite graph decomposable into two isomorphic factors F_1 and F_2 with finite diameter d and let V_i be a partite set of G of cardinality m_i. Let there exist an isomorphism $\phi : F_1 \to F_2$ and a pair of vertices $v_{ij}, v_{ik} \in V_i$ such that $\phi(v_{ij}) = v_{ik}$ and $\phi(v_{ik}) = v_{ij}$. Then $G' \cong K_{m_1,\ldots,M_i,\ldots,m_r}$ is also decomposable into two isomorphic factors with diameter d for any $M_i = m_i + 2m'$ where m' is an arbitrary positive integer.*

Proof. Suppose we have a factorization of G and an isomorphism with the required properties where $V_i = V_1$, $v_{ij} = v_{11}$ and $v_{ik} = v_{12}$. Let G' be the graph described in the proof of Lemma 2.3.1. We can construct factors F_1' and F_2' as follows:

F_1' contains all edges belonging to F_1 and in addition to it each vertex $v'_{1m_1+1}, v'_{1m_1+3}, \ldots, v'_{1M_1-1}$ is adjacent to all neighbours of v_{11} and each vertex $v'_{1m_1+2}, v'_{1m_1+4}, \ldots, v'_{1M_1}$ is adjacent to all neighbours of v_{12}.

Now we define an isomorphism $\phi' : F_1' \to F_2'$:

$$\phi'(v_{ij}) = \phi(v_{ij}) \text{ for } i = 1, 2, \ldots, r; j = 1, 2, \ldots, m_i$$

$$\phi'(v'_{1k}) = v'_{1\ k+1} \text{ for } k = m_1 + 1, m_1 + 3, \ldots, M_1 - 1$$

and

$$\phi'(v'_{1k}) = v'_{1\ k-1} \text{ for } k = m_1 + 2, m_1 + 4, \ldots, M_1.$$

One can easily verify that similarly as in Lemma 2.3.1, ϕ' is again an isomorphism. The distance between any two vertices of $\bigcup_{i=1}^{r} V_i$ remains in G' the same as in G, therefore $\operatorname{diam} G' \geq \operatorname{diam} G$. The distance between any v'_{1k} and any v_{st} in G' is the same as $\operatorname{dist}_G(v_{11}, v_{st})$ or $\operatorname{dist}_G(v_{12}, v_{st})$, according to parity of k and it is therefore always at most d. Finally, the distance $\operatorname{dist}_{G'}(v'_{1k}, v'_{1l})$ equals either 2 or

$\text{dist}_G(v_{11}, v_{12})$, according to the parity of $|k-l|$. Hence $\operatorname{diam} G' \leq \operatorname{diam} G$, which completes the proof. □

The following general theorem summarizes the techniques used in the lemmas above.

Theorem 2.3.3. *Let $K_{m_1,m_2,\ldots,m_r}$ with partite sets $V_i = \{v_{i1}, v_{i2}, \ldots, v_{im_i}\}, i = 1, 2, \ldots, r$ be $(2,d)$-isodecomposable into factors F_1 and F_2. Let $2 \leq q \leq r$ and $\phi : F_1 \to F_2$ be an isomorphism such that $\phi(v_{11}) = v_{11}, \phi(v_{i1}) = v_{i2}$ and $\phi(v_{i2}) = v_{i1}$ for $i = 2, 3, \ldots, q$. Then a graph $K_{M_1,M_2,\ldots,M_r}$ is $(2,d)$-isodecomposable for every admissible r-tuple $M_1, M_2, \ldots, M_r$ such that $M_1 = m_1 + m'_1$, $M_i = m_i + 2m'_i$ for $i = 2, 3, \ldots, q$ and $M_i = m_i$ for $i = q+1, q+2, \ldots, r$ where all m'_i's are arbitrary non-negative integers.*

Proof. Follows easily from repeated application of Lemmas 2.3.1 and 2.3.2. □

2.4. Bipartite and tripartite graphs

In this section we completely determine all complete bipartite and tripartite graphs decomposable into two isomorphic factors with a finite diameter.

Theorem 2.4.1. *A complete bipartite graph $K_{n,m}$ is $(2,d)$-isodecomposable for a finite diameter d if and only if at least one of the numbers n, m is even, $n \leq m$ and one of the following conditions applies:*

(a) $d = 3,\ n \geq 6, m \geq 6$;

(b) $d = 4,\ n \geq 4, m \geq 4$ or $n = 3, m \geq 6$;

(c) $d = 5,\ n \geq 3, m \geq 4$;

(d) $d = 6,\ n \geq 3, m \geq 4$.

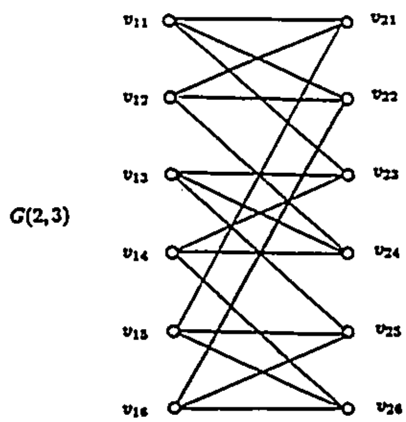


Figure 2.4.1

The necessity follows from Tomová's result [27]. We present the theorem in a restricted form stating only the parts relevant to our topic.

Theorem 2.4.2. (Tomová) *A complete bipartite graph $K_{n,m}$ is $(2,d)$-decomposable for a finite diameter d if and only if at least one of the numbers n, m is even and one of the conditions* (a)–(d) *of Theorem 2.4.1 applies.*

A *complementary permutation* of F_1 is an isomorphism between the factors F_1 and F_2 of a graph G. Isomorphism is often described in the form of cycles of the complementing permutation.

Proof of Theorem 2.4.1. Necessity has been shown above.

Sufficiency:

(a) Consider the factor $F_1 \cong G(2,3)$ of the graph $K_{6,6}$ of Figure 2.4.1 and the isomorphism

$$\phi_3 : F_1 \to F_2 : (v_{11})(v_{12})(v_{13}v_{16})(v_{14}v_{15})(v_{21}v_{25})(v_{22}v_{26})(v_{23}v_{24}).$$

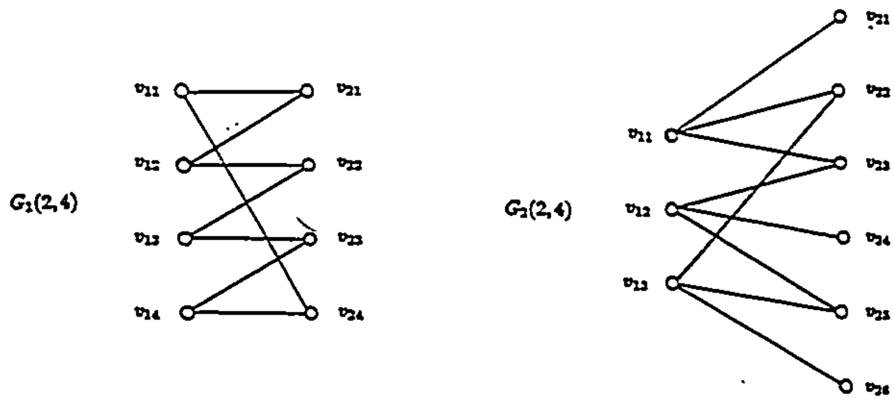


Figure 2.4.2

The distance $\operatorname{dist}(v_{11}, v_{26}) = 3$ and for any other pair of vertices $\operatorname{dist}(v_{ij}, v_{kl}) \leq 3$, hence $\operatorname{diam} F_1 = 3$.

(b) Consider the factor $F_1 \cong G_1(2,4)$ of $K_{4,4}$ of Figure 2.4.2 and

$$\phi_{41} : F_1 \to F_2 : (v_{11})(v_{12}v_{14}))(v_{13})(v_{21}v_{22})(v_{23}v_{24}).$$

Here $\operatorname{dist}(v_{11}, v_{13}) = 4$ and for any other pair of vertices $\operatorname{dist}(v_{ij}, v_{kl}) \leq 4$, hence $\operatorname{diam} F_1 = 4$.

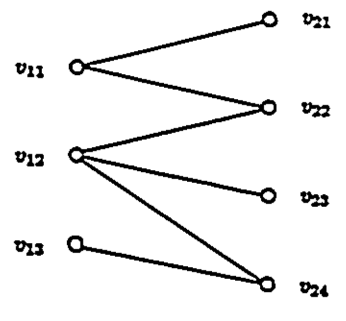
To complete this part one needs to see that $K_{3,6}$ is isodecomposable into factors $F_1 \cong F_2 \cong G_2(2,4)$ of Figure 2.4.2 where

$$\phi_{42} : F_1 \to F_2 : (v_{11})(v_{12})(v_{13})(v_{21}v_{25})(v_{22}v_{24})(v_{23}v_{26}).$$

Again $\operatorname{dist}(v_{21}, v_{26}) = 4$ and for any other pair of vertices $\operatorname{dist}(v_{ij}, v_{kl}) \leq 4$, hence $\operatorname{diam} F_1 = 4$.

(c) Consider the factor $F_1 \cong G(2,5)$ of $K_{3,4}$ of Figure 2.4.3 and

$$\phi_5 : F_1 \to F_2 : (v_{11})(v_{12}v_{13})(v_{21}v_{24})(v_{22}v_{23}).$$



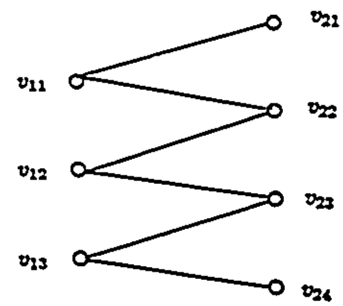


Figure 2.4.3

Now $\mathrm{dist}(v_{13}, v_{21}) = 5$ and for any other pair of vertices $\mathrm{dist}(v_{ij}, v_{kl}) \leq 5$, hence $\mathrm{diam}\, F_1 = 5$.

(d) Finally, consider the factor $F_1 \cong G(2,6)$ of $K_{3,4}$ of Figure 2.4.3 and

$$\phi_5 : F_1 \rightarrow F_2 : (v_{11})(v_{12})(v_{13})(v_{21}v_{23})(v_{22}v_{24}).$$

Here $\mathrm{dist}(v_{21}, v_{24}) = 6$ and for any other pair of vertices $\mathrm{dist}(v_{ij}, v_{kl}) \leq 6$, hence $\mathrm{diam}\, F_1 = 6$.

In all cases (a)–(d) the isomorphism ϕ_d satisfies the assumptions of Theorem 2.3.3 and all considered factors can be extended into factors of arbitrary admissible graphs $K_{N,M}$ for any $N \geq n$ and $M \geq m$, which are therefore $(2,d)$-isodecomposable. □

The following corollary follows easily from Theorem 2.4.1 and therefore the proof can be omitted. To see that always $h_2(2,d) = \infty$, one can notice that if $K_{n,m}$ is $(2,d)$-isodecomposable then the smaller partite set has at least 3 vertices. Since there is no $(2,d)$-isodecomposable graph $K_{2,m}$ for any finite d, we can for any

n_0 find an infinite class of graphs of order greater than n_0, namely all graphs $K_{2,n}$ with $n \geq n_0 - 1$, which are not isodecomposable.

Corollary 2.4.3.

(a) $f_2(2,3) = g_2(2,3) = g_2'(2,3) = 12, h_2(2,3) = \infty$,

(b) $f_2(2,4) = g_2(2,4) = g_2'(2,4) = 8, h_2(2,4) = \infty$,

(c) $f_2(2,5) = g_2(2,5) = g_2'(2,5) = 7, h_2(2,5) = \infty$,

(d) $f_2(2,6) = g_2(2,6) = g_2'(2,6) = 7, h_2(2,6) = \infty$.

Decomposability of r-partite graphs,$r \geq 3$, was proved by Gangopadhyay [10]. Here we present Gangopadhyay's results dealing with tripartite graphs.

Theorem 2.4.4. (Gangopadhyay) *A complete tripartite $(2, d)$-decomposable graph K_{m_1,m_2,m_3} with m vertices for a finite diameter d exists if and only if one of the following cases occurs:*

(a) $d = 2,\ m \geq 13$;

(b) $d = 3,\ m \geq 6$;

(c) $d = 4,\ m \geq 5$;

(d) $d = 5,\ m \geq 7$.

In the following theorems we show that for all the above mentioned cardinalities of minimal $(2, d)$-decomposable graphs there exist $(2, d)$-isodecomposable graphs, i.e. $f_3(2, d) = g_3(2, d)$ for $d = 2, 3, 4, 5$.

Lemma 2.4.5. *A complete tripartite graph K_{m_1,m_2,m_3} with $m_1 \leq m_2 \leq m_3$ is not $(2, 2)$-isodecomposable for $m_1 < 4$.*

Proof. Suppose there is a $(2, 2)$-isodecomposable graph K_{m_1,m_2,m_3} with the vertex sets $V_1 = \{v_{11}, \ldots, v_{1m_1}\}, V_2 = \{v_{21}, v_{22}, \ldots, v_{2m_2}\}, V_3 = \{v_{31}, v_{32}, \ldots, v_{3m_3}\}$, where $m_1 \leq 3$. First we show that if $m_1 > 1$ then no vertex of $V_2 \cup V_3$ can be adjacent to exactly one vertex of V_1 in either factor. Suppose, to the contrary, that $m_1 > 1$ and there is a vertex in $V_2 \cup V_3$ which is adjacent in one factor to exactly one vertex of V_1. We can assume without loss of generality that v_{31} is in F_1 adjacent to $v_{11}, v_{21}, v_{22}, \ldots, v_{2r}$. We can see that $r > 0$, otherwise v_{11} is the only neighbour of v_{31} and $\text{dist}_{F_1}(v_{31}, v_{12}) > 2$, which is impossible. If $r = m_2$, then v_{31} is in F_2 adjacent only to the vertices $v_{1l}, l > 1$, and $\text{dist}_{F_1}(v_{31}, v_{11}) > 2$. All vertices $v_{2\ r+1}, v_{2\ r+2}, \ldots, v_{2m_2}$ must be in F_1 adjacent to v_{11}, otherwise again $\text{dist}_{F_1}(v_{31}, v_{2m_2}) > 2$. But v_{31} is in F_2 adjacent to $v_{2\ r+1}, v_{2\ r+2}, \ldots, v_{2m_2}$ and the vertices $v_{1l}, l > 1$, and none of them is a neighbour of v_{11} there. Thus $\text{dist}_{F_2}(v_{31}, v_{11}) > 2$, which is a contradiction. Therefore every vertex of $V_2 \cup V_3$ is adjacent to all vertices of V_1 in one factor, and to none of them in the other factor. Assume that v_{31} is in F_1 not adjacent to any vertex of V_1. Then v_{31} must be in F_1 adjacent to all vertices of V_2, otherwise there is a vertex v_{2i} such that $\text{dist}_{F_2}(v_{2i}, v_{31}) > 2$. Hence v_{31} is in F_2 adjacent only to the vertices $v_{11}, \ldots, v_{1m_1}$ and every other vertex $v_{3j} \in V_3$ must be then in F_2 adjacent to one of the vertices $v_{11}, \ldots, v_{1m_1}$. Consequently, each vertex v_{3j} must be in F_2 adjacent to all vertices of V_1 and F_2 contains the complete bipartite graph $\langle V_1 \cup V_3 \rangle$.

On the other hand, every vertex $v_{3j} \in V_3$ is in F_1 adjacent only to vertices of V_2 and therefore it must be adjacent to all of them. Thus F_1 contains the complete bipartite graph $\langle V_2 \cup V_3 \rangle$.

Because there is no edge $v_{2i}v_{3j}$ in F_2, every vertex v_{2i} must be adjacent to one of the vertices of V_1 (otherwise $\text{dist}_{F_2}(v_{2i}, v_{31}) > 2$) and therefore to all of them. Then F_2 contains also the complete bipartite graph $\langle V_1 \cup V_2 \rangle$ and the vertices $v_{11}, \ldots, v_{1m_1}$ are in F_1 isolated, which is a contradiction. □

The previous lemma shows that $h_3(2,2)=\infty$ and the only candidate for $(2,2)$-isodecomposability with $f_3(2,2)=13$ vertices is $K_{4,4,5}$. We further show that this graph is indeed $(2,2)$-isodecomposable and therefore $f_3(2,2)=g_3(2,2)=13$. In the other cases we also show that there is always exactly one $(2,d)$-isodecomposable graph with $f_3(2,d)$ vertices for each $d=3,4,5$. We also prove in these cases that every admissible complete tripartite graph with at least $g_3(2,d)$ vertices is $(2,d)$-isodecomposable, i.e. $f_3(2,d)=g_3(2,d)=g_3'(2,2)=h_3(2,d)$ for $d=3,4,5$.

First we exclude two graphs. Two sequences $B=b_1,b_2,\ldots,b_n$ and $C=c_1,c_2,\ldots,c_n$ are *isomorphic* if there exists a one-to-one mapping $\psi: N\to N$ such that $b_i=c_{\psi(i)}$. The *degree sequence* of a graph G with a vertex set $v_1,v_2,\ldots,v_n$ is the non-increasing sequence $A=a_1,a_2,\ldots,a_n$ where $a_i=\deg v_{j_i}$. The sequence is *isodecomposable* if there exist isomorphic sequences $B=b_1,b_2,\ldots,b_n$ and $C=c_1,c_2,\ldots,c_n$ (not necessarily non-increasing) such that $a_i=b_i+c_i$ for each $i\in N=\{1,2,\ldots,n\}$. Obviously, a graph G is isodecomposable only if the degree sequence of G is isodecomposable. Moreover, G is isodecomposable into two factors with a finite diameter only if the degree sequence of G is isodecomposable into two sequences with all positive entries.

Lemma 2.4.6. *$K_{1,2,4}$ is not $(2,3)$-isodecomposable.*

Proof. The degree sequence of $K_{1,2,4}$ is $A=6,5,5,3,3,3,3$. Let A be isodecomposable into $B=b_1,b_2,\ldots,b_7$ and $C=c_1,c_2,\ldots,c_7$. Let $B'=b_1',b_2',\ldots,b_7'$ be the re-ordered sequence corresponding to B such that $b_1'\geq b_2'\geq\cdots\geq b_7'$ and $C'=c_1',c_2',\ldots,c_7'$ be the re-ordered sequence corresponding to C such that if $b_i'=b_{j_i}$ then $c_i'=c_{j_i}$, i.e., $b_i'+c_i'=a_{j_i}$.

We start with $b'_1 = 5$. Here $b'_1 = b_1$, because $a_i \leq 5$ for each $i \geq 2$ and if $b_i = 5$ then $c_i = 0$, which is impossible. But if $b'_1 = b_1 = 5$, then there also has to be $c_i = 5$ for some $i > 1$, which is impossible as well.

Now let $b'_1 = 4$. If $B' = 4,4,2,1,1,1,1$ then neither c'_1 nor c'_2 can be 4, because $a_1 = 6$. Hence $c'_j = c'_k = 4$ for $j \neq k; j,k > 2$ and $b'_i + c'_i \geq 5$ for $i = 1,2,j,k$, which is impossible. If $B' = 4,3,3,1,1,1,1$ then either $c'_2 = 3$ or $c'_3 = 3$, say c'_2. Then $b'_3 + c'_3 = 4, 6$ or 7, which is not possible. If $B' = 4,3,2,2,1,1,1$ and $c'_1 = 2$, then necessarily $c'_2 = 2$ and $c'_3 = 3$ or $c'_4 = 3$, say c'_3. Then there is $i \geq 4$ such that $b'_i + c'_i \geq 5$, which is impossible. Therefore $c'_1 = 1$. If $c'_3 = 4$ or $c'_4 = 4$, this reduces to the previous case. Hence there must be $c'_2 = 3$ (otherwise $b'_i + c'_i \leq 5$ for each $i = 1,2,\ldots,7$) and $c'_3 = c'_4 = 1$ (otherwise $b'_i + c'_i = 6$ or 4 for $i = 3$ or 4, which is impossible). So $C' = 1,3,1,1,2,2,4$, and we examine factors with this degree sequence later.

Finally, consider $b'_1 = 3$. If $B' = 3,3,3,2,1,1,1$ then there must be $i \in \{1,2,3\}$ such that $c'_i = 3$, say $i = 1$. Clearly $c'_2, c'_3 < 3$ and then one of them, say c'_3, equals 1. Hence $c'_3 + b'_3 = 4$, which is impossible. If $B' = 3,3,2,2,2,1,1$ then there must be $C' = 3,2,3,1,1,2,2$ which we examine later. If $B' = 3,2,2,2,2,2,1$ then $c'_1 = 3$ and five of the entries $c'_2, c'_3, \ldots, c'_7$ are equal to 2. But then there is $i \in \{2,3,\ldots,7\}$ such that $c'_i + b'_i = 4$, which is again impossible.

So it remains to investigate factors with the degree sequences 4,3,2,2,1,1,1 and 3,3,2,2,2,1,1. If $K_{1,2,4}$ is $(2,3)$-isodecomposable, the factors F_1 and F_2 have both order and size 7 and are connected and therefore are unicyclic graphs. The longest possible cycle in F_1 is C_4, because otherwise we have less than 3 vertices of degree 1. If we have the cycle $C_4 = \langle v_1, v_2, v_3, v_4 \rangle$, one of the vertices, say v_1, must

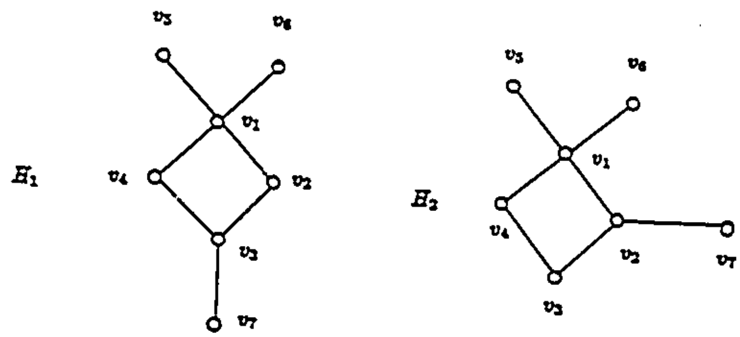


Figure 2.4.4

be of degree 4 in F_1. Since there are only 2 vertices of degree 2 in F_1, we have two possible graphs H_1 and H_2, shown in Figure 2.4.4.

But $\operatorname{diam} H_1 = 4$, which is a contradiction. Hence v_7 is adjacent to one of v_2, v_4, say v_2.

Now consider the graph $G = K_{1,2,4}$ with the partite sets $U_1 = \{u_{11}\}, U_2 = \{u_{21}, u_{22}\}, U_3 = \{u_{31}, u_{32}, u_{33}, u_{34}\}$. The degree sequence B corresponding to F_1 yields $v_2 = u_{11}$ and $v_1 = u_{2i}$, say u_{21}. Hence v_4 must be one of the vertices v_{3j}, say v_{31}, and $v_3 = u_{22}$. But if we compare it with the degree sequences A, B and C we can see that the degree of u_{22} in F_1 must be 1, which is a contradiction.

If the factor F_1 of G contains a cycle C_3, the vertices of the cycle must belong to different partite sets. Let the cycle be $\langle u_{11}, u_{21}, u_{31} \rangle$. The vertex u_{11} must be of degree 3 in F_1 and u_{31} of degree 4 as we have seen above. Then F_1 must be one of the graphs of Figure 2.4.5.

Since both have diameter 4, we get a contradiction and $K_{1,2,4}$ cannot be decomposed into two isomorphic factors with diameter 3 having the degree sequences 4,3,2,2,1,1,1.

Finally, let us examine the factors with the degree sequence 3,3,2,2,2,1,1. The factor F_1 is again unicyclic and cannot contain C_6. If F_1 contains C_5, then

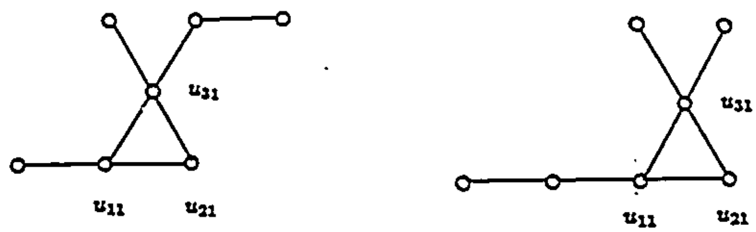


Figure 2.4.5

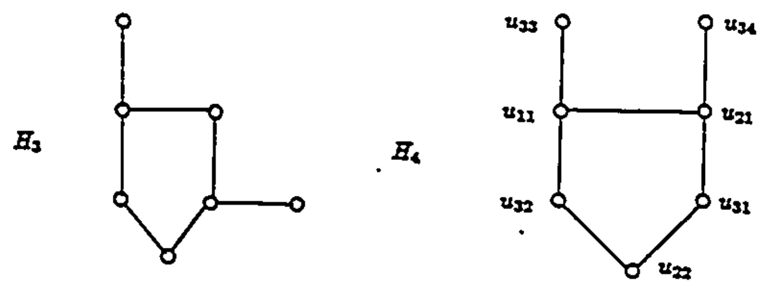


Figure 2.4.6

F_1 is one of the graphs H_3, H_4 of Figure 2.4.6. But $\operatorname{diam} H_3 = 4$, hence only H_4 remains. The cycle C_5 contains u_{11}, both u_{21} and u_{22} and two vertices of U_3, say u_{31} and u_{32} so that $C_5 = \langle u_{11}, u_{21}, u_{31}, u_{22}, u_{32} \rangle$. No vertex u_{3i} can be in F_1 of degree 3, otherwise it is isolated in F_2, which is impossible. Therefore u_{11} and u_{21} are the vertices of degree 3. Hence u_{11} is adjacent to u_{33}, say, while u_{21} is adjacent to u_{34}. Thus we have three mutually non-adjacent vertices u_{11}, u_{22}, u_{34}, and F_2 contains the triangle $\langle u_{11}, u_{22}, u_{34} \rangle$, which contradicts our assumption that the only cycle contained in F_2 is C_5.

If F_1 contains C_4, then F_1 must be one of the graphs H_5, H_6, H_7 of Figure 2.4.7, whose diameter is 4 (in the case of H_5 and H_6) or 5 (H_7).

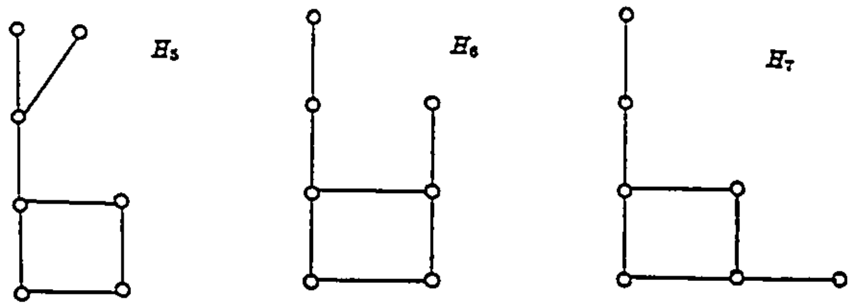


Figure 2.4.7

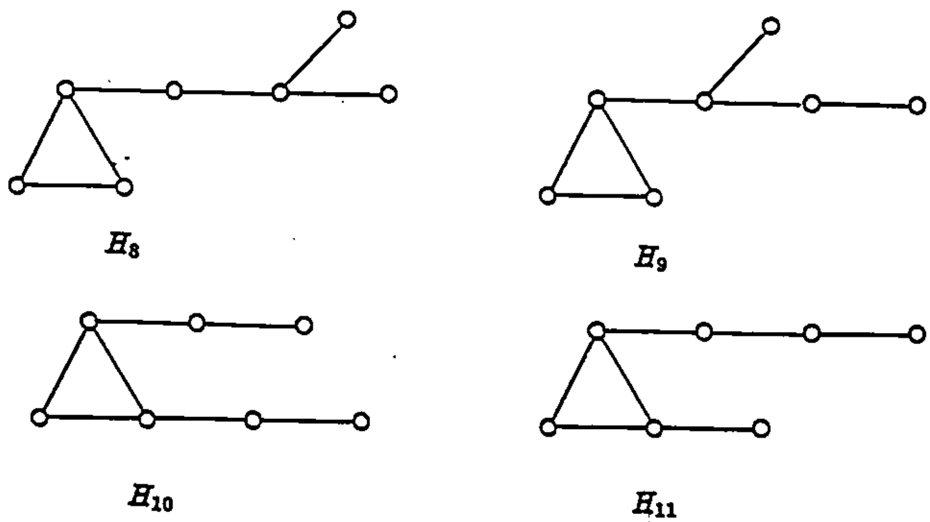


Figure 2.4.8

If F_1 contains C_3, then F_1 is one of the graphs H_8, H_9, H_{10}, H_{11} of Figure 2.4.8 and $\operatorname{diam} H_i = 4$ for $i = 8, 9$ or $\operatorname{diam} H_i = 5$ for $i = 10, 11$. Therefore $K_{1,2,4}$ cannot be decomposed into two isomorphic factors with diameter 3 having the degree sequences 4,3,2,2,1,1,1 and hence $K_{1,2,4}$ is not $(2,3)$-isodecomposable. □

Lemma 2.4.7. *$K_{2,2,3}$ is not $(2,5)$-isodecomposable.*

Proof. Suppose that $K_{2,2,3}$ is $(2,5)$-isodecomposable. Then the factor of $K_{2,2,3}$, F_1, must have 8 edges. The degree of a vertex in F_1 cannot exceed 4, otherwise

F_2 is disconnected. If there is a vertex v_0 of degree 4 in F_1 and v_1, v_2, v_3, v_4 are its neighbours in F_1, then the graph $\langle v_0, v_1, \ldots, v_4 \rangle$ has in F_1 diameter at most 2. Because F_1 contains only two more vertices, it is clear that then $\operatorname{diam} F_1 \leq 4$. If there is no vertex of degree 4 in F_1, then there must be at least two vertices of degree 3, say u and v, because F_1 is of size 8. But then u and v are at a distance at most 2 and F_1 must be again of a diameter at most 4, which is impossible. Finally, there is no graph with maximum degree 2 having 7 vertices and 8 edges and hence $K_{2,2,3}$ is not $(2,5)$-isodecomposable. □

We are now ready to characterize all $(2, d)$-isodecomposable graphs for a finite diameter d.

Theorem 2.4.8. *A complete tripartite graph K_{m_1,m_2,m_3} with $m_1 \leq m_2 \leq m_3$ is $(2, d)$-isodecomposable if and only if at most one of the numbers m_1, m_2, m_3 is odd and one of the following conditions holds:*

(a) $d = 2,\ m_1 \geq 4, m_2 \geq 4, m_3 \geq 5$;

(b) $d = 3,\ m_1 \geq 2, m_2 \geq 2, m_3 \geq 2$, or $m_1 = 1, m_2 \geq 4, m_3 \geq 4$;

(c) $d = 4,\ m_1 \geq 1, m_2 \geq 2, m_3 \geq 2$;

(d) $d = 5,\ m_1 \geq 1, m_2 \geq 2, m_3 \geq 4$.

Moreover, for each $d = 2,3,4,5$ there is a unique $(2, d)$-isodecomposable graph of the minimum order: $K_{4,4,5}$ for $d = 2$, $K_{2,2,2}$ for $d = 3$, $K_{1,2,2}$ for $d = 4$ and $K_{1,2,4}$ for $d = 5$.

Proof. The minimality of the orders follows again from Theorem 2.4.4. The uniqueness in the cases $d = 3$ and 4 is evident, because at most one partite set can have an odd cardinality. For $d = 2$, the graphs K_{m_1,m_2,m_3} with $m_1 < 4$ are not $(2,2)$-isodecomposable by Lemma 2.4.5. For $d = 5$, there is only one other admissible complete tripartite graph with 7 vertices, namely $K_{2,2,3}$, which is not $(2,5)$-isodecomposable by Lemma 2.4.7.

To prove that all admissible graphs larger than the minimal ones are also $(2,d)$-isodecomposable for $d = 2,3,4,5$, respectively, we consider the factors shown in Figures 2.4.9–2.4.12 and the corresponding isomorphisms $\phi_{di} : F_1 \to F_2$.

(a) We consider $F_1 \cong G_1(3,2)$ of Figure 2.4.9 with $\phi_{21} : F_1 \to F_2$.

$$\phi_{21} : (v_{11})(v_{13})(v_{15})(v_{12}v_{14})(v_{21}v_{22})(v_{23}v_{24})(v_{31}v_{34})(v_{32}v_{33}).$$

One can check that $\text{dist}(v_{ij}, v_{kl}) \leq 2$ for all pairs of vertices.

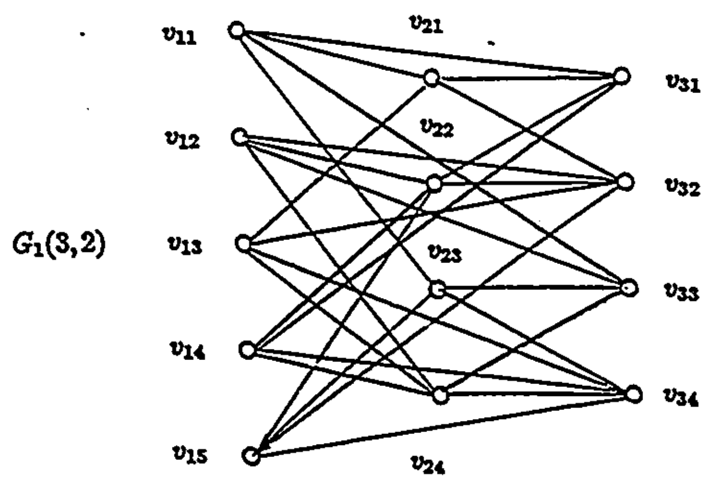
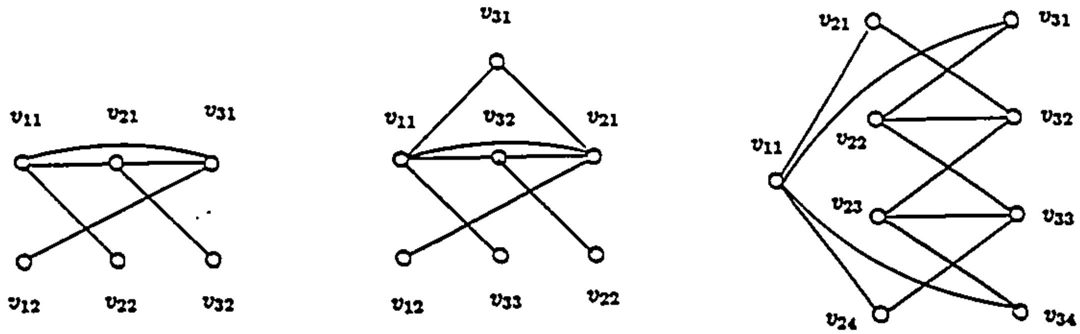


Figure 2.4.9

(b) The factors are shown in Figure 2.4.10. For $m_1 = m_2 = m_3 = 2$ consider the factor $F_1 \cong G_1(3,3)$ and

$$\phi_{31} : (v_{11}v_{12})(v_{21}v_{22})(v_{31}v_{32}).$$



$G_1(3,3)$ $G_2(3,3)$ $G_3(3,3)$

Figure 2.4.10

The distance $\text{dist}(v_{12}, v_{22}) = 3$; for all other pairs of vertices $\text{dist}(v_{ij}, v_{kl}) \leq 3$.

For $m_1 + m_2 + m_3 = 7$ consider first $m_1 \geq 2$ and $F_1 \cong G_2(3,3)$ with

$$\phi_{32} : (v_{11}v_{12})(v_{21}v_{22})(v_{31})(v_{32}v_{33}).$$

Here again $\text{dist}(v_{12}, v_{22}) = 3$ and $\text{dist}(v_{ij}, v_{kl}) \leq 3$ for all other pairs of vertices. It has been shown in Lemma 2.4.6 that $K_{1,2,4}$ is not $(2,3)$-isodecomposable, hence for $m_1 = 1$ we start with $m_2 = m_3 = 4$. We take $F_1 \cong G_3(3,3)$ and

$$\phi_{33} : (v_{11})(v_{21}v_{22})(v_{23}v_{24})(v_{31}v_{33})(v_{32}v_{34}).$$

Here $\text{dist}(v_{21}, v_{33}) = 3$ and $\text{dist}(v_{ij}, v_{kl}) \leq 3$ for all other pairs of vertices.

(c) The factors are shown in Figure 2.4.11. For $m_1+m_2+m_3 = 5$ consider the factor $F_1 \cong G_1(3,4)$ of the only admissible tripartite graph of order 5, $K_{1,2,2}$, and

$$\phi_{41} : (v_{11})(v_{21}v_{31}v_{22}v_{32}).$$

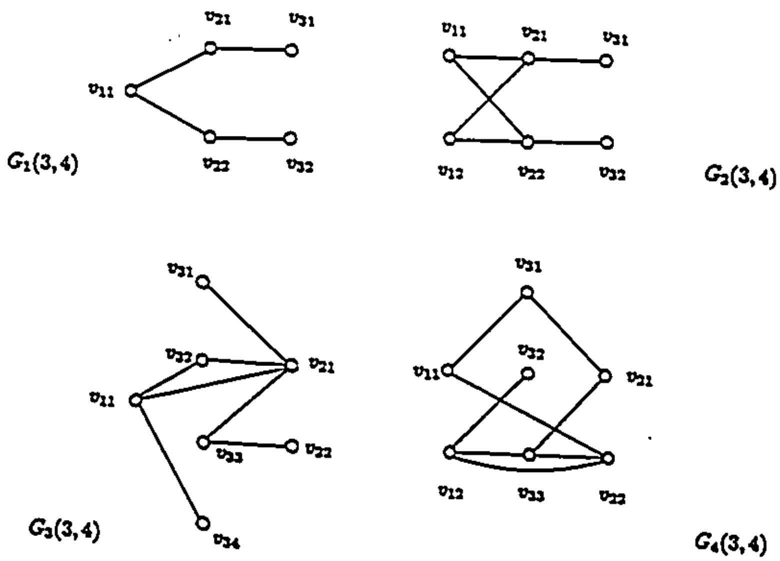


Figure 2.4.11

The distance $\text{dist}(v_{21}, v_{22}) = 4$; for all other pairs of vertices $\text{dist}(v_{ij}, v_{kl}) \leq 4$.

For $m_1 + m_2 + m_3 = 6$ consider the factor $F_1 \cong G_2(3,4)$ of the only admissible tripartite graph of order 6, $K_{2,2,2}$, and

$$\phi_{42} : (v_{11})(v_{12})(v_{21}v_{31}v_{22}v_{32}).$$

Here $\text{dist}(v_{31}, v_{32}) = 4$ and for all other pairs of vertices $\text{dist}(v_{ij}, v_{kl}) \leq 4$.

For $m_1 + m_2 + m_3 = 7$ consider first $F_1 \cong G_3(3,4)$ and

$$\phi_{43} : (v_{11})(v_{21}v_{22})(v_{31}v_{32})(v_{33}v_{34}).$$

The distance $\text{dist}(v_{34}, v_{22}) = 4$ and for all other pairs of vertices $\text{dist}(v_{ij}, v_{kl}) \leq 4$. By Theorem 2.3.3 we can extend V_2 to any even cardinality and V_3 to any even cardinality greater than 2, hence every admissible graph K_{1,m_2,m_3} with at least 7 vertices is $(2,4)$-isodecomposable.

$G_1(3,5)$

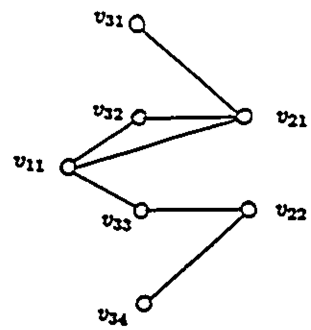


Figure 2.4.12

The other admissible tripartite graph of order 7 is $K_{2,2,3}$. We consider its factor $F_1 \cong G_4(3,4)$ and the isomorphism

$$\phi_{44} : (v_{11}v_{12})(v_{21}v_{22})(v_{31})(v_{32}v_{33}).$$

The distance $\text{dist}(v_{31}, v_{32}) = 4$; for all other pairs of vertices $\text{dist}(v_{ij}, v_{kl}) \leq 4$. The isomorphism ϕ_{44} clearly satisfies the assumptions of Theorem 2.3.3 so that we can extend V_1 and V_2 to any even cardinality and V_3 to any cardinality not less than 3. Therefore every admissible tripartite graph K_{m_1,m_2,m_3} with $2 \leq m_1 \leq m_2 \leq m_3$ is $(2,4)$-isodecomposable which together with the previous cases proves that every admissible tripartite graph K_{m_1,m_2,m_3} with at least 5 vertices is $(2,4)$-isodecomposable.

(d) As we have seen above, the only $(2,5)$-icodecomposable graph of order 7 is $K_{1,2,4}$. We take the factor $F_1 \cong G_1(3,5)$ of Figure 2.4.12 and the isomorphism

$$\phi_{51} : (v_{11})(v_{21}v_{22})(v_{31}v_{32})(v_{33}v_{34}).$$

The distance $\text{dist}(v_{31}, v_{34}) = 5$ and for all other pairs of vertices $\text{dist}(v_{ij}, v_{kl}) \leq 5$. It follows again from Theorem 2.3.3 that every admissible tripartite graph of order at least 7 with the single exception of $K_{2,2,3}$ is $(2,5)$-icodecomposable and the proof is now complete. □

From Theorem 2.4.8 we immediately have the following.

Corollary 2.4.9.

(a) $f_3(2,2) = g_3(2,2) = g_3'(2,2) = 13, h_3(2,2) = \infty$,

(b) $f_3(2,3) = g_3(2,3) = g_3'(2,3) = 6, h_3(2,3) = 8$,

(c) $f_3(2,4) = g_3(2,4) = g_3'(2,4) = h_3(2,4) = 5$,

(d) $f_3(2,5) = g_3(2,5) = g_3'(2,5) = 7, h_3(2,5) = 8$.

2.5. Four-partite graphs

This section has two parts. In the first part we completely determine all $(2,d)$-decomposable complete four-partite graphs with at most one partite set of an odd cardinality. In the second part we study the case of four odd partite sets. We prove that all $(2,d)$-decomposable graphs $K_{n,n,m,m}$ are also $(2,d)$-isodecomposable for $d = 3,4$. For $d = 2$ we show that all graphs with $n, m \geq 3$ are $(2,2)$-isodecomposable while the graphs $K_{1,1,m,m}$ are not $(2,2)$-isodecomposable. On the other hand, we prove that there are no $(2,5)$-isodecomposable four-partite graphs with all odd parts. We also prove (as a corollary of a more general result) that the graphs $K_{n,n,n,m}$ are not $(2,d)$-isodecomposable for any odd numbers n, m and for any d.

First we state a theorem of Gangopadhyay [10] dealing with decomposable graphs.

Theorem 2.5.1. (Gangopadhyay) *A complete four-partite $(2, d)$-decomposable graph of order n for a finite diameter d exists if and only if one of the following conditions applies:*

(a) $d = 2$ and $n \geq 7$,

(b) $d = 3$ and $n \geq 5$,

(c) $d = 4$ and $n \geq 6$,

(d) $d = 5$ and $n \geq 8$.

Let us remark that Gangopadhyay determined the minimum order for $d = 3$ as 4. But according to our definition of r-partite graphs, a four-partite graph has at least 5 vertices.

Theorem 2.5.2. *A complete four-partite graph $K_{m_1,m_2,m_3,m_4}, m_1 \leq m_2 \leq m_3 \leq m_4$, with at most one partite set of an odd cardinality is $(2, d)$-isodecomposable for a finite diameter d if and only if one of the following conditions applies:*

(a) $d = 2$ and $m_1 \geq 1, m_2, m_3, m_4 \geq 2$,

(b) $d = 3$ and $m_1 \geq 1, m_2, m_3, m_4 \geq 2$,

(c) $d = 4$ and $m_1 \geq 1, m_2, m_3, m_4 \geq 2$,

(d) $d = 5$ and $m_1 \geq 1, m_2, m_3 \geq 2, m_4 \geq 4$.

To prove the theorem, we need some preliminary results. First we present some simple observations on isodecomposable graphs and isomorphisms between their factors. Most of them can be found in [11].

A cycle $\phi^{(i)} = (v_1^i, v_2^i, \ldots, v_{s_i}^i)$ of an isomorphism $\phi : F_1 \rightarrow F_2$ is *pure* if all $v_{j_i}^i$ belong to one partite set of G. The subgraph $\langle \phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(t)} \rangle$ induced by the cycles $\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(t)}$ is the graph induced by all vertices $v_{j_i}^i, i = 1, 2, \ldots t; j_i = 1, 2, \ldots s_i$.

Observation 2.5.3.a. *Let G be a complete multipartite graph isodecomposable into isomorphic factors F_1 and F_2 and ϕ be an isomorphism from F_1 to F_2. If G', F_1' and F_2' are the subgraphs of G, F_1 and F_2, respectively, induced by cycles $\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(t)}$, then F_1' is isomorphic to F_2'.*

Observation 2.5.3.b. *Let $\phi^{(1)} = (v_1^1, v_2^1, \ldots, v_{s_1}^1)$ and $\phi^{(2)} = (v_1^2, v_2^2, \ldots, v_{s_2}^2)$ be pure cycles of odd lengths. Then all vertices $v_{j_i}^i, i = 1,2; j_i = 1, 2, \ldots s_i$ belong to the same partite set.*

Observation 2.5.3.c. *Let $\phi^{(1)} = (v_1^1, v_2^1)$ be a cycle of the isomorphism $\phi : F_1 \to F_2$. Then $\phi^{(1)}$ is pure.*

Observation 2.5.3.d. *Let $\phi^{(1)} = (v_{i1}, v_{i2})$ and $\phi^{(2)} = (v_{j1}, v_{j2})$ be pure cycles. Then $\langle \phi^{(1)}, \phi^{(2)} \rangle \cong K_{1,2} \cup K_1$.*

Observation 2.5.3.e. *Let $v_{ij}v_{kl}$ be an edge in F_2. Then the pre-images $\phi^{-1}(v_{ij})$ and $\phi^{-1}(v_{kl})$ are non-adjacent in F_1.*

Next we show that neither $K_{2,2,2,2}$ nor $K_{2,2,2,3}$ is $(2,5)$-isodecomposable.

Lemma 2.5.4. *$K_{2,2,2,2}$ is not $(2,5)$-isodecomposable.*

Proof. Suppose, to the contrary, that $K_{2,2,2,2}$ is $(2,5)$-isodecomposable into factors F_1 and F_2. Obviously, F_1 cannot contain a vertex of degree 5, otherwise $\operatorname{diam} F_1 \leq 4$. Hence every vertex is of degree at most 4 in F_1 and therefore of degree at least 2 in F_2. Consequently, every vertex is of degree at least 2 in F_1. Because $\operatorname{diam} F_1 = 5$, for every pair of vertices u, v whose distance in F_1 is 5 there is an induced path P_5 of length 5, $\langle u = x_0, x_1, \ldots x_5 = v \rangle$, from u to v. Since $\deg_{F_1} x_0 \geq 2$, x_0 must be adjacent to one of the remaining vertices, say y_1. Then y_1 cannot be adjacent to any of x_3, x_4, x_5, otherwise $\operatorname{diam} F_1 \leq 4$. For the same reasons x_5 is adjacent to $y_2 \neq y_1$ and y_2 is adjacent to none of x_0, x_1, x_2. Now F_1 contains at most 11 edges: 5 edges in $\langle x_0, x_1, \ldots x_5 \rangle$ and at most 6 edges $y_i x_j$. The size of F_1 has to be 12 and there is only one other possible edge—$y_1 y_2$. This yields $\operatorname{dist}(x_0, x_5) = 3$, which is a contradiction. □

Lemma 2.5.5. *$K_{2,2,2,3}$ is not $(2,5)$-isodecomposable.*

Proof. Let $K_{2,2,2,3}$ with the partite sets $V_1 = \{v_{11}, v_{12}\}, V_2 = \{v_{21}, v_{22}\}, V_3 = \{v_{31}, v_{32}\}, V_4 = \{v_{41}, v_{42}, v_{43}\}$ be $(2,5)$-isodecomposable into factors F_1 and F_2. Obviously, F_1 cannot contain a vertex of degree 6, because then $\operatorname{diam} F_1$ would be less than 5. Now we show that if there is a vertex of degree 5 in F_1, then there must be exactly one other vertex of degree 1. Let x_0 and x_5 be at distance 5 in F_1 and let $\langle x_0, x_1, \ldots x_5 \rangle$ be an induced path. Suppose, to the contrary, that there is no vertex of degree 1 in F_1. Then x_0 has to be adjacent to one of the remaining vertices y_1, y_2, y_3, say y_1, and x_5 to another, say y_2. (Of course, x_0 and x_5 cannot have a common neighbour, since $\operatorname{dist}(x_0, x_5) = 5$.) If the vertex of degree 5 is x_i, where $i \in \{1, 2, 3, 4\}$, then it is adjacent to both y_1 and y_2, which is a contradiction, because then $\operatorname{dist}(x_0, x_5) = 4$.

None of the vertices y_1, y_2, y_3 can be adjacent to more than 3 vertices of $x_0, x_1, \ldots, x_5$, otherwise $\operatorname{dist}(x_0, x_5) \leq 4$. We have seen that no vertex x_i can be of degree 5. If $\deg y_3 = 5$ then y_3 must be adjacent to both y_1 and y_2 and $\operatorname{dist}(x_0, x_5) = 4$, which is impossible. If $\deg y_1 = 5$ or $\deg y_2 = 5$ then y_1 must be adjacent to y_2 and $\operatorname{dist}(x_0, x_5) = 3$, which is again impossible. Hence there is a vertex of degree 1 in F_1. If there were two or more vertices of degree 1 in each factor, at least one of them would have to be of degree 6 in the other factor, because only 3 vertices of G have degree 6. Hence there is precisely one vertex of degree 1 in each factor.

The vertex of degree 1 in F_2 must be one of the vertices having degree 6 in G, i.e. v_{41}, v_{42}, v_{43}, otherwise we have a vertex of degree 6 in F_1. Suppose then that it is the vertex v_{43}, which is of degree 5 in F_1. For the same reason, the vertex having degree 1 in F_1 must be one of v_{41}, v_{42}, say v_{41}. As we have seen, v_{42} is of degree at least 2. We may assume without loss of generality that v_{43} is adjacent to the vertices $v_{11}, v_{12}, v_{21}, v_{22}, v_{31}$. If the only neighbour of v_{41} in F_1 is different from v_{32}, say v_{21}, the subgraph of F_1 induced by all edges incident to vertices v_{41} and v_{43}, H, has diameter 3. None of the other vertices v_{42}, v_{32} is in adjacent in F_1 to v_{41} or to v_{43}. Because v_{32} is of degree at least 2 in F_1, it must be adjacent to at least one of the vertices $v_{11}, v_{12}, v_{21}, v_{22}$ and $\text{dist}(v_{32}, v_{ij}) \leq 4$ for any vertex $v_{ij} \in H$. Similarly v_{42}, which is also of degree at least 2 in F_1, must be adjacent to at least one of the vertices $v_{11}, v_{12}, v_{21}, v_{22}, v_{31}$ and $\text{dist}(v_{32}, v_{ij}) \leq 4$ for any vertex $v_{ij} \in H$. Both v_{32} and v_{42} are now at distance at most 2 from v_{43} and then $\text{dist}(v_{32}, v_{42}) \leq 4$. Thus $\text{diam}\, F_1 \leq 4$, which is a contradiction. So we can suppose that the only neighbour of v_{41} in F_1 is v_{32}. Because now v_{41} has the same neighbours in F_1 as v_{43} in F_2 and vice versa, we may assume without loss of generality that v_{32} is in F_1 adjacent to v_{42}. If v_{32} is adjacent to one of the vertices $v_{11}, v_{12}, v_{21}, v_{22}$, then $\text{dist}(v_{43}, v_{ij}) \leq 3$ for any vertex $v_{ij}, i \neq 4$ and $\text{dist}(v_{32}, v_{ij}) = 1$ for v_{41} and v_{42}. This yields $\text{diam}\, F_1 \leq 4$ and we have to examine the only remaining case.

In this case v_{32} is adjacent in F_1 only to v_{41} and v_{42}, and hence its neighbours in F_2 are $v_{11}, v_{12}, v_{21}, v_{22}, v_{43}$. Then in F_2 all vertices but v_{42} belong to the graph H induced by all edges incident to the vertices v_{41} and v_{32}. The only vertices in H having eccentricity 4 are v_{43} and v_{31}. Because the degree of v_{42} in F_2 is at least 2 and it is not adjacent to v_{43}, one of its neighbours has eccentricity less than 4. Therefore $\operatorname{diam} F_2 \leq 4$, which is again a contradiction and $K_{2,2,2,3}$ is not $(2,5)$-isodecomposable into factors containing a vertex of degree 5.

To prove that there are no isomorphic factors with diameter 5 having the highest degree 4 we start with the observation that no such factor F_1 can have more that four vertices of degree 4, otherwise F_2 has less vertices of degree 4 than F_1.

There are two possible degree sequences satisfying the assumption: 4, 4, 4, 4, 3, 3, 3, 3, 2 and 4, 4, 4, 3, 3, 3, 3, 3, 3. We first examine the former. Let u, v be an arbitrary pair of vertices having distance 5 in F_1. Then there is an induced path $P_5 = \langle u = x_0, x_1, \ldots, x_5 = v \rangle$ and 3 other vertices y_1, y_2, y_3. As follows from the degree sequence, $\deg x_0 + \deg x_5 \geq 5$. On the other hand, $\deg x_0 + \deg x_5 \leq 5$, otherwise x_0 and x_5 have a common neighbour. Thus $\deg x_0 + \deg x_5 = 5$, which is possible only if one of the vertices, say x_0, is of degree 3 and the other, x_5, of degree 2. Then x_0 is adjacent to, say, y_1 and y_2 while x_5 is adjacent to y_3. Neither y_1 nor y_2 can be adjacent to y_3, otherwise $\operatorname{dist}(x_0, x_5) = 3$. We have now in F_1 5 edges $x_i x_{i+1}$, 3 edges $x_i y_j$ and at most one edge $y_j y_k$, namely $y_1 y_2$. Because F_1 is of size 15 and altogether we have 8 or 9 edges, there must be 6 or 7 more edges $x_i y_j$ in F_1. No vertex y_j can be adjacent to more than 3 vertices of the path $\langle u = x_0, x_1, \ldots, x_5 = v \rangle$, because then there is another path between x_0 and x_5 of length less than 5, which is impossible. In addition, the neighbours of any y_j can be at distance at most 2 apart on the path P_5. This excludes the possibility that there are 7 other edges $x_i y_j$.

Hence we have to suppose that there is the edge $y_1 y_2$ in F_1 and 6 other edges $x_i y_j$. As we have shown above, the neighbours of x_0, y_1 and y_2, must be

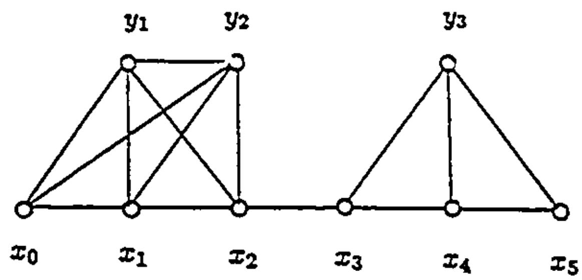


Figure 2.5.1

adjacent to x_1 and x_2 while the neighbour of x_5, y_3, must be adjacent to x_3 and x_4. Now F_1 contains a subgraph isomorphic to K_4, induced by the vertices $x_0, x_1,, y_1, y_2$. Because F_2 is isomorphic to F_1, it has to contain the graph K_4 as well. Hence there must be in F_1 at the same time at least 4 mutually non-adjacent vertices.

One can check Figure 2.5.1 to see that there are only two mutually non-adjacent vertices among x_0, x_1, x_2, y_1, y_2, namely x_0 and x_2, and two mutually non-adjacent vertices among x_3, x_4, x_5, y_3, namely x_3 and x_5. But the vertices x_2 and x_3 are adjacent and thus F_2 cannot contain K_4. Hence $K_{2,2,2,3}$ is not $(2,5)$-isodecomposable into factors with the degree sequence 4,4,4,4,3,3,3,3,2.

Finally, it is easy to see that $K_{2,2,2,3}$ is not $(2,5)$-isodecomposable into factors with the degree sequence 4,4,4,3,3,3,3,3,3 either. In this case both end-vertices of any induced path $\langle x_0, x_1, \ldots, x_5 \rangle$ are of degree at least 3, therefore one of the remaining vertices must be adjacent to both x_0 and x_5. Hence $\text{dist}(x_0, x_5) = 2$ and $\text{diam}\, F_1 < 5$, which is a contradiction completing the proof. □

Having the exceptions excluded, we can now prove Theorem 2.5.2.

Proof of Theorem 2.5.2. The minimality in the cases (a)–(c) is obvious. The graph $K_{1,2,2,2}$ is in all the cases the smallest admissible four-partite graph of this class

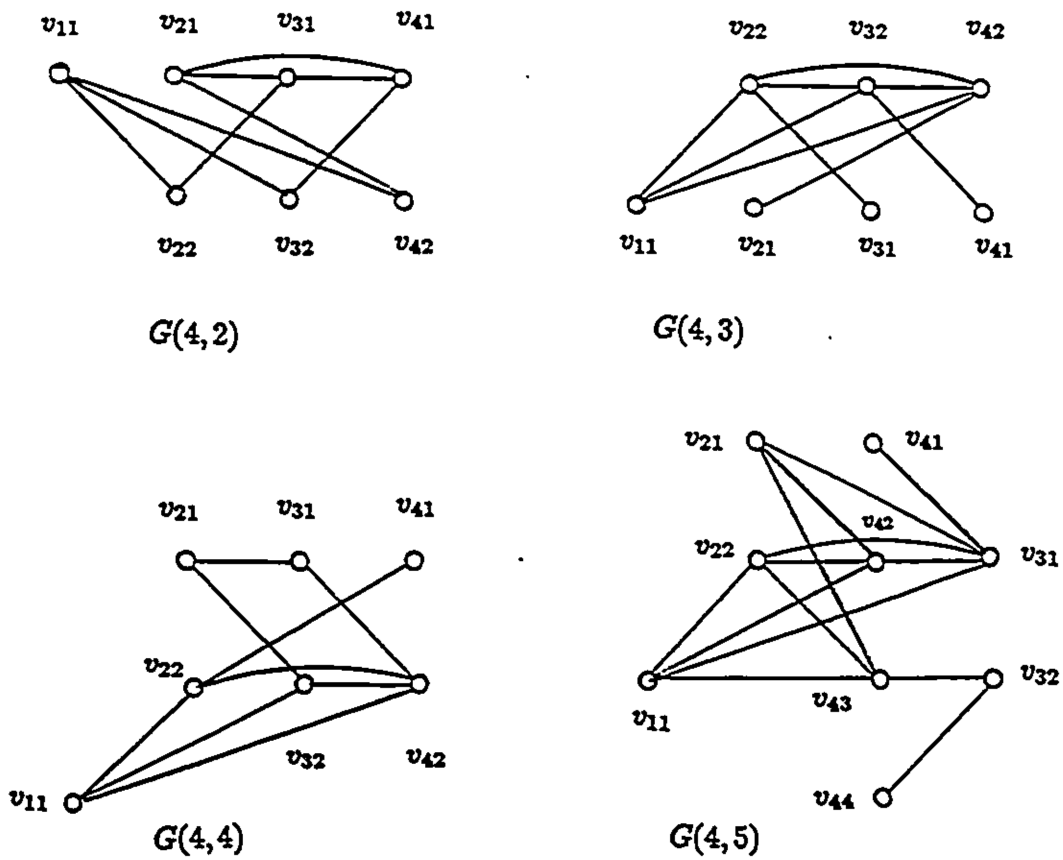


Figure 2.5.2

of order not less than the corresponding value of $f_4(2,d)$ shown by Gangopadhyay [10]. In the case (d) $f_4(2,5) = 8$ but Lemma 2.5.4 shows that the only admissible four-partite graph with at most one odd part, $K_{2,2,2,2}$, is not $(2,5)$-isodecomposable. Since Lemma 2.5.5 excludes one of the two graphs of order 9 with one odd partite set, $K_{2,2,2,3}$, only $K_{1,2,2,4}$ remains.

To prove sufficiency we show that for every of the above minimal graphs there is an isomorphism $\phi : F_1 \rightarrow F_2$ satisfying assumptions of Theorem 2.3.3.

(a) Consider the factor $F_1 \cong G(4,2)$ of the graph $K_{1,2,2,2}$ of Figure 2.5.2 and the isomorphism

$$\phi_2 : F_1 \to F_2 : (v_{11})(v_{21}v_{22})(v_{31}v_{32})(v_{41}v_{42}).$$

The distance for any pair of vertices $\mathrm{dist}(v_{ij}, v_{kl}) \le 2$, hence $\mathrm{diam}\, F_1 = 2$.

(b) Consider the factor $F_1 \cong G(4,3)$ of the graph $K_{1,2,2,2}$ of Figure 2.5.2 and the isomorphism ϕ_2 from (a). The distance $\mathrm{dist}(v_{21}, v_{41}) = 3$ and for any other pair of vertices $\mathrm{dist}(v_{ij}, v_{kl}) \le 3$, hence $\mathrm{diam}\, F_1 = 3$.

(c) Consider the factor $F_1 \cong G(4,4)$ of the graph $K_{1,2,2,2}$ of Figure 2.5.2 and again the isomorphism ϕ_2 from (a). The distance $\mathrm{dist}(v_{21}, v_{41}) = 4$ and for any other pair of vertices $\mathrm{dist}(v_{ij}, v_{kl}) \le 3$, hence $\mathrm{diam}\, F_1 = 4$.

(d) Consider the factor $F_1 \cong G(4,5)$ of the graph $K_{1,2,2,4}$ of Figure 2.5.2 and the isomorphism

$$\phi_5 : F_1 \to F_2 : (v_{11})(v_{21}v_{22})(v_{31}v_{32})(v_{41}v_{42})(v_{43}v_{44}).$$

The distance $\mathrm{dist}(v_{41}, v_{42}) = 5$ and for any other pair of vertices $\mathrm{dist}(v_{ij}, v_{kl}) \le 4$, hence $\mathrm{diam}\, F_1 = 5$.

Because both ϕ_2 and ϕ_5 satisfy conditions of Theorem 2.3.3, we can in all cases extend the minimal factor to a factor of any complete four-partite graph with zero or one odd partite set having more than 7 (in the cases (a)–(c)) or 9 (in (d)) vertices. □

The case of graphs with all odd parts is more complicated. The only diameter for which we completely solve the problem of isodecomposability in this case is $d = 5$. We prove that no graphs with all odd parts are $(2,5)$-isodecomposable.

For $d = 2, 3, 4$ we solve the problem only for special classes of graphs with all odd parts. We start with the following.

Theorem 2.5.6. *Let $l, m, r, s, r \neq s$ be odd numbers. Then the graph $K_{r^l s^m}$ is not $(2, d)$-isodecomposable for any d.*

Proof. The degree sequence of $K_{r^l s^m}$ is $p, p, \ldots, p, q, q, \ldots, q$ where both numbers $p = (l-1)r + ms$ and $q = lr + (m-1)s$ are odd and both appear in the sequence an odd number of times, namely p appears $lr = t$ times and q appears $ms = n - t$ times. Suppose, to the contrary, that $K_{r^l s^m}$ is isodecomposable. We may assume without loss of generality that $p < q$. Let $A = a_1, a_2, \ldots, a_n$ and $B = b_1, b_2, \ldots, b_n$ be isomorphic sequences such that $a_i + b_i = p$ for $i = 1, 2, \ldots, t$ and $a_i + b_i = q$ for $i = t+1, t+2, \ldots, n$. Let $\alpha(i)$ ($\beta(i)$) for $i = 0, 1, \ldots, p$ be the number of terms of $a_1, a_2, \ldots, a_t$ ($b_1, b_2, \ldots, b_t$) which are equal to i and $\alpha'(j)$ ($\beta'(j)$) for $j = 0, 1, \ldots, q$ be the number of terms of $a_{t+1}, a_{t+2}, \ldots, a_n$ ($b_{t+1}, b_{t+2}, \ldots, b_n$) which are equal to j. Obviously $\alpha(i) = \beta(p - i)$ and $\alpha'(i) = \beta'(q - i)$.

Because t is odd, there must be i such that $\alpha(i) > \beta(i)$. Let i_0 be the smallest number i such that $\alpha(i_0) > \beta(i_0)$. Denote $k = \alpha(i) - \beta(i)$. As the sequences A and B are isomorphic, i_0 must appear in $b_{t+1}, b_{t+2}, \ldots, b_n$ k-times more than in $a_{t+1}, a_{t+2}, \ldots, a_n$, i.e., $\beta'(i_0) - \alpha'(i_0) = k$. Then $\alpha'(q - i_0) - \beta'(q - i_0) = k$, i.e., $q - i_0$ appears more often in $a_{t+1}, a_{t+2}, \ldots, a_n$ than in $b_{t+1}, b_{t+2}, \ldots, b_n$. Hence $q - i_0$ must appear in $b_1, b_2, \ldots, b_t$ k more times than in $a_1, a_2, \ldots, a_t$, which yields $\beta(q - i_0) - \alpha(q - i_0) = k$. This is equivalent to $\alpha(i_0 + p - q) - \beta(i_0 + p - q) = k$. Because $k > 0$, we have $\alpha(i_0 + p - q) > \beta(i_0 + p - q)$. From the minimality of i_0 it follows that $i_0 + p - q \geq i_0$, which contradicts our assumption that $p < q$ and therefore $K_{r^l s^m}$ is not isodecomposable. □

We state the special case for four-partite graphs as corollary.

Corollary 2.5.7. *If r, s are odd with $r \neq s$ then the complete four-partite graph $K_{r,r,r,s}$ is not $(2,d)$-isodecomposable for any d.*

Using our earlier results, we can determine $h_r(2,d)$ for any $r \equiv 0 \pmod 4$ and any d, including $d = \infty$.

Theorem 2.5.8. *$h_r(2,d) = \infty$ for any $r \equiv 0 \pmod 4$ and any d.*

Proof. Given any $r \equiv 0 \pmod 4$ and any order n, we can construct an infinite class of graphs with order greater than n, for instance all graphs $K_{2p+1,4p+1,4p+1,\ldots,4p+1}$, where $p > n$. This graph is not $(2,d)$-isodecomposable for any finite or infinite d by Theorem 2.5.6. □

Now we attempt to prove that four-partite graphs with all odd parts are not $(2,5)$-isodecomposable. In fact, we prove a more general result, namely that there are no $(2,5)$-isodecomposable r-partite graphs with all odd parts. First we show some interesting properties of $(2,5)$-isodecomposable graphs.

Lemma 2.5.9. *Let $K_{m_1,m_2,\ldots,m_r}, r \geq 3$, be $(2,5)$-isodecomposable into factors F_1 and F_2. Let x be a vertex with eccentricity 5 in F_1. Then all vertices having in F_1 distance 5 from x belong to the same part as x.*

Proof. For convenience we assign a colour to each part. Let a white vertex w_0 have $\mathrm{ex}_{F_1} w_0 = 5$. Let U_i be the set of vertices having distance i from w_0 in F_1. A vertex belonging to U_i is denoted by c_i, where c is a colour. If a vertex belongs to a union $U_i \cup U_j \cup U_k$, then it is often denoted by c_{i-j-k}.

It is obvious that U_1 contains only non-white vertices and $U_4 \cup U_5$ contains vertices of two different colours, because the vertices of U_5 are in F_1 adjacent only to those of U_4. From now on we always assume that U_1 contains a blue vertex b_1.

We proceed by contradiction. Let us suppose that U_5 contains a non-white vertex. Let u_{0-1-2} and v_{0-1-2} be a pair of vertices both in $U_0 \cup U_1 \cup U_2$. Then no matter what their colours are, the distance between them in F_2 is always less than 5. If u and v have the same colour, then $\text{dist}_{F_2}(u, v) = 2$ because $U_4 \cup U_5$ contains vertices of at least two different colours and one of them must differ from the colour of u and v. If one of them, say u, is of a colour different from those in $U_4 \cup U_5$, then $U_4 \cup U_5$ contains a vertex x whose colour differs from that of v. Then in F_2 the vertices u and v have the common neighbour x and $\text{dist}_{F_2}(u, v) \leq 2$. If at least one of the vertices u, v is neither blue nor white, say red, and the other is of any colour c, and $U_4 \cup U_5$ contains only vertices of colour c and red, then $\text{dist}_{F_2}(r_{0-1-2}, c_{0-1-2}) \leq 4$, because F_2 contains a path P_5 of length 4 $r_{0-1-2} - c_{4-5} - w_0 - r_{4-5} - c_{0-1-2}$ if c is blue and $r_{0-1-2} - c_{4-5} - b_1 - r_{4-5} - c_{0-1-2}$ if c is white. Finally, if u is white, v is blue and there are only blue and white vertices in $U_4 \cup U_5$, the non-white vertex of U_5 must be blue. We have two possibilities. First, there is a vertex in $U_0 \cup U_1 \cup U_2$ which is neither blue nor white, say red. Then we have a path $u = w_{0-1-2} - b_5 - r_{0-1-2} - w_{4-5} - b_{0-1-2} = v$ of length 4 and $\text{dist}_{F_2}(u, v) \leq 4$. Secondly, there are no other vertices than white and blue in $U_0 \cup U_1 \cup U_2$. Because $U_0 = w_0$, we see that all vertices of U_1 are blue while all of U_2 are white. Now we have $u = w_{0-1-2} - b_5 - r_3 - b_2 = v$, a path of length 3 and $\text{dist}_{F_2}(u, v) \leq 3$. For similar reasons, $\text{dist}_{F_2}(u, v) \leq 4$ for any $u, v \in U_3 \cup U_4 \cup U_5$. If one of u, v is neither blue nor white then as above $\text{dist}_{F_2}(u, v) \leq 2$. The same holds if both u and v have the same colour. If $u = w_{3-4-5}, v = b_{3-4-5}$ and there is a vertex r_{3-4-5}, we have in F_2 a path $u = w_{3-4-5} - b_1 - r_{0-1-2} - w_0 - b_{4-5-6} = v$ of length 4. If there is no vertex other than white or blue in $U_3 \cup U_4 \cup U_5$ and U_1 contains a vertex r_1, then again $\mathrm{dist}_{F_2}(u,v) \leq 2$. If there are only blue vertices in U_1, then all vertices which are neither blue nor white belong to U_2. According to our assumption, there is a blue vertex $b_5 \in U_5$. If now $u = w_{4-5} \in U_4 \cup U_5$ and $v = b_{3-4-5}$ then $\mathrm{dist}_{F_2}(u,v) \leq 3$ because there is a path $u = w_{4-5} - r_2 - w_0 - b_{3-4-5}$ in F_2. If $u = w_3 \in U_3$ then we have a path $u = w_3 - b_5 - w_0 - b_{3-4-5}$ in F_2 and $\mathrm{dist}_{F_2}(u,v) \leq 4$.

Now we look at the pairs u, v where $u \in U_0 \cup U_1 \cup U_2$ and $v \in U_3 \cup U_4 \cup U_5$. First we consider both u, v having the same colour. Suppose the colour is different from both white and blue, say red. Because $U_4 \cup U_5$ contains a vertex c_{4-5} whose colour is different from red, we have in F_2 either a path $u = r_{1-2} - c_{4-5} - w_0 - r_{3-4-5} = v$ if c is blue or $u = r_{1-2} - c_{4-5} - b_1 - r_{3-4-5} = v$ if c is white. If both u, v are blue, and either U_1 or $U_4 \cup U_5$ contains a vertex which is neither white nor blue, say red, we have in F_2 a path $u = b_{1-2} - r_{4-5} - w_0 - b_{3-4-5} = v$ or $u = b_{1-2} - w_{4-5} - r_{2-3} - b_{3-4-5} = v$ and in both cases $\mathrm{dist}_{F_2}(u,v) \leq 3$. If there are no other vertices in $U_0 \cup U_1 \cup U_4 \cup U_5$ than white and blue, all other vertices must belong to $U_2 \cup U_3$. Hence we have in F_2 a path $u = b_1 - r_3 - w_0 - b_{3-4-5} = v$ if there is a vertex r_3 or $u = b_{1-2} - w_{4-5} - r_2 - w_0 - b_{3-4-5} = v$ if there are only white or blue vertices in U_3. In both cases $\mathrm{dist}_{F_2}(u,v) \leq 4$.

If both u, v are white and there is a vertex other than white or blue in $U_0 \cup U_1 \cup U_4 \cup U_5$ then the case is essentially similar to the previous one. If $U_0 \cup U_1 \cup U_4 \cup U_5$ contains only white and blue vertices, then according to our assumption there is a vertex $b_5 \in U_5$ and a red vertex in $U_2 \cup U_3$. If there is $r_3 \in U_3$ then we have in F_2 a path $u = w_{0-1-2} - b_5 - r_3 - b_1 - w_{3-4-5} = v$. If there is $r_2 \in U_2$ then we have in F_2 either $u = w_{0-1-2} - b_5 - w_3 = v$ or $u = w_{0-1-2} - b_5 - r_2 - w_{4-5} = v$ which in both cases yields $\mathrm{dist}_{F_2}(u,v) \leq 3$.

Finally, we investigate the pairs u, v, where $u \in U_0 \cup U_1 \cup U_2$, $v \in U_3 \cup U_4 \cup U_5$ and u and v have different colours. If $u \in U_i, v \in U_j$ and $j - i > 1$ then obviously the edge uv is in F_2. So we have the only possibility $u \in U_2, v \in U_3$. If none of them is white, we have in F_2 the path $u_2 - w_0 - v_3$. According to our assumption, there is a non-white vertex $c_5 \in U_5$. Hence if either u or v is white, we have in F_2 either $u = w_2 - c_5 - w_0 - v_3$ or $u_2 - w_0 - c_5 - w_3 = v_3$ and $\mathrm{dist}_{F_2}(u, v) \leq 3$.

Thus we have shown that if U_5 contains a non-white vertex, then the diameter of F_2 is always less than 5, which is a contradiction. Therefore U_5 contains only white vertices. □

Let us suppose now that a factor F_1 of a (2,5)-isodecomposable graph with more than two parts contains a vertex of eccentricity 5 which is adjacent to vertices of two different parts. Let $\mathrm{ex}_{F_1} w_0 = 5$ and let r_1 and b_1 be adjacent to w_0 in F_1. Using the notation of the previous lemma, we can see that there are vertices of at least two different colours in $U_4 \cup U_5$, say a_{4-5} and c_{4-5}. Any two vertices u_{3-4-5}, v_{3-4-5} of $U_3 \cup U_4 \cup U_5$ have in F_2 distance at most 2, because there are vertices of 3 different colours in $U_0 \cup U_1$. Similarly, if we have $u_{0-1-2}, v_{0-1-2} \in U_0 \cup U_1 \cup U_2$, and one of the colours u, v, say u, differs from a and c, then there is in F_2 a path $u_{0-1-2} - a_{4-5} - v_{0-1-2}$ or $u_{0-1-2} - c_{4-5} - v_{0-1-2}$, depending on the colour v.

Suppose now that $u_{0-1-2} \in U_0 \cup U_1 \cup U_2$ and $u_{3-4-5} \in U_3 \cup U_4 \cup U_5$ have the same colour. Of course u differs from one of a, c, say c, and one of w, r, b differs from both u and c. Then c_{4-5} is a neighbour of u_{0-1-2} in F_2 and u_{3-4-5} and c_{4-5} have in F_2 a common neighbour $U_0 \cup U_1 \cup U_2$. Hence $\mathrm{dist}_{F_2}(u_{0-1-2}, u_{3-4-5}) \leq 3$ for any two vertices of the same colour. If we have $u_{0-1-2} \in U_0 \cup U_1 \cup U_2$ and

$v_{3-4-5} \in U_3 \cup U_4 \cup U_5$ of different colours such that $u_i \in U_i$ and $v_j \in U_j$ and $j - i > 1$, they are adjacent in F_2. If $i = 2$ and $j = 3$, then one of the colours a, c, say c, differs from u and one of w, b, r differs from both v and c. Hence v_{3-4-5} is in F_2 at distance at most 3 apart from u_{0-1-2}, and $\operatorname{diam} F_2 \leq 4$. This is impossible, and therefore we can state another lemma.

The *neighbourhood* of a vertex x in a graph G, denoted $N_G(x)$, is a set of all vertices adjacent to x in G. If A is a set of vertices of G, then $N_G(A)$ is the union of neighbourhoods of all vertices of A.

Lemma 2.5.10. *Let $K_{m_1,m_2,\ldots,m_r}, r \geq 3$, be $(2,5)$-isodecomposable into factors F_1 and F_2. Let x be a vertex with eccentricity 5 in F_1 and U_i the set of all vertices having distance i from x in F_1. Then $U_0 \cup U_5$ is a subset of one partite set of $K_{m_1,m_2,\ldots,m_r}$ and $U_1 \cup U_4$ is a subset of another partite set.*

Proof. The first part of the assertion is restated Lemma 2.5.9. Above we have shown that U_1 consists of vertices of just one colour. We can repeat the consideration and instead of having vertices of two different colours in U_1 suppose that U_4 contains vertices of two colours different from white. We arrive at the same conclusion, that in this case $\operatorname{diam} F_2 \leq 4$, which is impossible. □

Now we are ready to show that once a factor F_i contains a white vertex of eccentricity 5, then all vertices with eccentricity 5 in either factor must be white. Consequently, the vertices adjacent in either factor to a vertex with eccentricity 5 also belong all to one part.

Lemma 2.5.11. *Let $K_{m_1,m_2,\ldots,m_r}, r \geq 3$, be $(2,5)$-isodecomposable into factors F_1 and F_2. Let A_i be the set of all vertices of eccentricity 5 in F_i. Then $A_1 \cap A_2 = \emptyset$,*

$A_1 \cup A_2 \subset V_j$ and $N_{F_1}(A_1) \cup N_{F_2}(A_2) \subset V_k$, where V_j, V_k are partite sets of the graph $K_{m_1,m_2,\ldots,m_r}$.

Proof. We again assume that there are vertices w_0, w_5 such that $\text{dist}_{F_1}(w_0, w_5) = 5$ and that w_0 is adjacent to a blue vertex b_1. Let u, v be vertices such that $\text{dist}_{F_2}(u, v) = 5$. It follows from Lemma 2.5.10. that u, v have the same colour. Let us assume that we have two red vertices r, r'. Then again from Lemma 2.5.10 we have $r, r' \in U_2 \cup U_3$ and they are both adjacent to w_0 in F_2. If we have two blue vertices, both in U_1, then they have the common neighbour w_5 in F_2. There is no blue vertex in U_2 and any two blue vertices of $U_3 \cup U_4$ have the common neighbour w_0 in F_2. Finally, let b_1 belong to U_1 and b_{3-4} to $U_3 \cup U_4$. Because there is a red vertex $r_{2-3} \in U_2 \cup U_3$, the path $b_1 - w_5 - r_{2-3} - w_0 - b_{3-4}$ yields $\text{dist}_{F_2}(b_1, b_{3-4}) \leq 4$. Hence the only vertices that can have eccentricity 5 in F_2 are white.

But this means that A_2 contains vertices of just one partite set, namely white. If we repeat the proof for the factor F_2 instead, we get the same for the set A_1. Thus $A_1 \cup A_2 \subset V_j$. Now we want to show that $A_1 \cap A_2 = \emptyset$. To do so, we check $\text{dist}_{F_2}(w_0, w')$ for each w'. For $w' \in U_2$ we have in F_2 a path $w_0 - b_4 - w_2'$. We know that there is a vertex $r_{2-3} \in U_2 \cup U_3$. Therefore for $w' \in U_3$ we have in F_2 a path $w_0 - r_{2-3} - w_5 - b_1 - w_3'$ and for $w' \in U_5$ a path $w_0 - r_{2-3} - w_5$. Thus $\text{ex}_{F_2} w_0 < 5$ and $A_1 \cap A_2 = \emptyset$.

Since every vertex w with $\text{ex}_{F_2} w = 5$ belongs to $U_2 \cup U_3$ and is in F_2 adjacent either to all blue vertices of U_1 or to all blue vertices of U_4, by Lemma 2.5.10 all neighbours of w in F_2 are blue. Hence $N_{F_1}(A_1) \cup N_{F_2}(A_2) \subset V_k$, which completes the proof. □

The following is an immediate consequence of the lemma.

Corollary 2.5.12. *Let $K_{m_1,m_2,\ldots,m_r}$ be $(2,5)$-isodecomposable and let $r \geq 3; m_1 \geq m_{\geq} \cdots \geq m_r$. Then $m_1 \geq 4$ and $m_2 \geq 2$.*

Proof. A factor F_i contains at least 2 vertices with eccentricity 5, hence $|A_i| \geq 2$. Because $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 \subset V_j$, then $m_1 \geq m_j \geq 4$. Obviously $|N_{F_1}(A_1)| \geq 2$ and hence $N_{F_1}(A_1) \subset V_k$ yields $m_2 \geq 2$. □

Although the following lemma could be included in the proof of the main result of this section, we prefer to state it on its own, because we use it explicitly later in the case of r-partite graphs where $r \geq 5$. First we show that $N_{F_1}(A_1) = N_{F_2}(A_2)$, i.e., that an isomorphism $\phi : F_1 \to F_2$ takes the set of neighbours of vertices with eccentricity 5 onto itself. Then we prove that the isomorphism takes the whole partite set containing the neighbours of the vertices with eccentricity 5 onto itself.

Lemma 2.5.13. *Let $K_{m_1,m_2,\ldots,m_r}, r \geq 3$, be $(2,5)$-isodecomposable into factors F_1 and F_2. Let A_i and $N_{F_i}(A_i)$ be defined as above and $\phi : F_1 \to F_2$ be an isomorphism. Then $N_{F_1}(A_1) = N_{F_2}(A_2)$, or equivalently $\phi(N_{F_1}(A_1)) = N_{F_1}(A_1)$. Moreover, if V_k is the partite set containing $N_{F_1}(A_1)$, then $\phi(V_k) = V_k$.*

Proof. We again denote the vertices of A_1 (and, consequently, of A_2) as white, and their neighbours, i.e. the vertices of $N_{F_1}(A_1)$ and $N_{F_2}(A_2)$, as blue. Let w_0 be a vertex of eccentricity 5 in F_1 and U_i be again the sets of vertices at distance i apart from w_0 in F_1. Let $w_5 \in U_5$, i.e., $\text{dist}_{F_1}(w_0, w_5) = 5$. The set U_1 consists only of blue vertices. All vertices of $U_0 \cup U_5$ are of eccentricity less than 5 in F_2 by Lemma 2.5.11. Because all vertices of U_2 are non-blue, they have in F_2 common neighbours in U_4 and therefore their mutual distance in F_2 is at most 2. The same holds for all white vertices of U_3. By Lemma 2.5.11 all vertices having in F_2 eccentricity 5 are white and none of them belongs to $U_0 \cup U_5$. At the same time $U_1 \cup U_4$ contains only blue vertices. Therefore for every pair of vertices w', w'' such that $\mathrm{dist}_{F_2}(w', w'') = 5$ one of them, say w', belongs to U_2 while the other, w'', belongs to U_3. Obviously $U_1 \subset N_{F_2}(w'')$. Since $U_1 = N_{F_1}(w_0)$, we have $N_{F_1}(w_0) \subset N_{F_2}(A_2)$. This yields $N_{F_1}(A_1) \subset N_{F_2}(A_2)$, as the vertex w_0 was chosen arbitrarily. We can repeat our considerations for any vertex $w_0' \in A_2$ to obtain $N_{F_2}(A_2) \subset N_{F_1}(A_1)$, which yields $N_{F_1}(A_1) = N_{F_2}(A_2)$.

Now we show that $\phi(V_k) = V_k$. Suppose it is not the case. Then there exists a non-blue vertex $c \notin V_k$ such that $\phi(c) = b_k \in V_k$. Since $\phi(N_{F_1}(A_1)) = N_{F_1}(A_1)$, we can see that c is not adjacent in F_2 to any vertex of $N_{F_1}(A_1)$. Hence it is adjacent to all vertices of $N_{F_1}(A_1)$ in the factor F_1. Then for any vertex $w \in A_1$ we have $\mathrm{dist}_{F_1}(w, c) = 2$ which yields $\mathrm{dist}_{F_1}(w, w') = 4$ for any pair of vertices of A_1. This is impossible, since the vertices $w_0, w_5 \in A_1$ have in F_1 mutual distance 5. This contradiction shows that there is no vertex $c \notin V_k$ such that $\phi(c) \in V_k$ and therefore $\phi(V_k) = V_k$. □

The general theorem now follows easily.

Theorem 2.5.14. *Let $r \equiv 0 \pmod 4$ and $K_{m_1,m_2,\ldots,m_r}$ be (2,5)-isodecomposable. Then at least 3 of the cardinalities $m_1, m_2, \ldots, m_r$ must be even.*

Proof. We need to show only that one of the numbers $m_1, m_2, \ldots, m_r$ must be even, because if just one or two of them are even, then $K_{m_1,m_2,\ldots,m_r}$ has an odd number of edges. Let $N_{F_1}(A_1) \subset V_r$. If $|V_r|$ is even, we are done. From Lemma 2.5.13 it follows that $\phi(V_r) = V_r$ and hence by Observation 2.5.3.a the graph $K_{m_1,m_2,\ldots,m_{r-1}}$ is isodecomposable. This is possible only if the number of odd parts is either 0 or $1 \pmod 4$ which implies that at least two of the numbers $m_1, m_2, \ldots, m_{r-1}$ must be even. But $|V_r|$ was odd and hence the actual number of even cardinalities among $m_1, m_2, \dots, m_{r-1}$ must be at least 3. □

As one of the main results of this section, we state the case of four-partite graphs on its own. The assertion follows easily from the previous theorem for $r = 4$.

Theorem 2.5.15. *K_{m_1,m_2,m_3,m_4} is not $(2,5)$-isodecomposable for any odd numbers m_1, m_2, m_3, m_4.*

Thus we can already determine the parameters $f_4(2,d)$, $g_4(2,d)$, $g_4'(2,d)$ and $h_4(2,d)$ for all $d = 2,3,4,5$.

Theorem 2.5.16.

(a) $f_4(2,2) = g_4(2,2) = g_4'(2,2) = 7, h_4(2,2) = \infty$,

(b) $f_4(2,3) = 5, g_4(2,3) = g_4'(2,3) = 7, h_4(2,3) = \infty$,

(c) $f_4(2,4) = 6, g_4(2,4) = g_4'(2,4) = 7, h_4(2,4) = \infty$,

(d) $f_4(2,5) = 8, g_4(2,5) = g_4'(2,5) = 9, h_4(2,5) = \infty$.

Proof. From Corollary 2.5.7 it follows that $h_4(2,d) = \infty$ for any d.

(a) The values f, g, g' follow directly from part (a) of Theorem 2.5.2.

(b), (c) There is no admissible graph with 5 vertices, and only one with 6 vertices, namely $K_{1,1,1,3}$, which is not isodecomposable again by Corollary 2.5.7. Thus the values follow from parts (b) and (c) of Theorem 2.5.2.

(d) Since $K_{1,1,1,5}$ is not isodecomposable by Corollary 2.5.7, the result follows from part (d) of Theorem 2.5.2. □

As we have seen above, there is no isodecomposable graph with odd parts $K_{r,r,r,s}$ for $r \neq s$. On the other hand, we prove in the following paragraphs that all graphs with odd parts of the class $K_{r,r,s,s}$ are $(2,d)$-isodecomposable for $d = 2,3,4$ with the following exception.

Lemma 2.5.17. *A complete four-partite graph $K_{1,1,r,r}$ is not* $(2,2)$*-isodecomposable for any* $r \geq 2$.

Proof. If r is even, then $1,1,r,r$ is not an admissible quadruple. Hence we may assume that we have a $(2,2)$-isodecomposable graph $K_{1,1,r,r}$ with an odd $r \geq 3$. Let $V_1 = \{v_{11}\}, V_2 = \{v_{21}, v_{22}, \ldots v_{2r}\}, V_3 = \{v_{31}, v_{32}, \ldots v_{3r}\}, V_4 = \{v_{41}\}$ be the partite sets and F_1 and F_2 the isomorphic factors with diameter 2. We may assume without loss of generality that the edge $v_{11}v_{41}$ belongs to F_1. Then in F_2 there must be a vertex, say v_{3r}, adjacent to both v_{11} and v_{41}, otherwise $\mathrm{dist}_{F_2}(v_{11}, v_{41}) > 2$. Because v_{3r} is not adjacent in F_1 to any of $v_{11}, v_{31}, v_{32}, \ldots, v_{3\ r-1}, v_{41}$, the distance $\mathrm{dist}_{F_1}(v_{3r}, v_{2i})$ can never be 2. Hence v_{3r} must be adjacent to all $v_{21}, v_{22}, \ldots, v_{2r}$, otherwise $\mathrm{diam}\, F_1 > 2$.

Therefore v_{3r} is of degree 2 in F_2. If v_{ij} is any vertex of $V_2 \cup V_3$ then it must be adjacent in F_2 either to v_{11} or to v_{41}. In the opposite case $\mathrm{dist}_{F_2}(v_{3r}, v_{ij}) > 2$ because v_{3r} has no other neighbours than v_{11} and v_{41}. Since v_{3r} is in F_1 adjacent neither to v_{11} nor to v_{41}, at least one vertex of V_2, say v_{21}, must be adjacent to v_{41}, and one vertex of V_2 must be adjacent to v_{11}. As we saw above, it cannot be v_{21}, otherwise $\mathrm{dist}_{F_2}(v_{3r}, v_{21}) > 2$. Thus we can without loss of generality suppose that v_{11} is in F_1 adjacent to v_{2r}. Now we are going to show that every vertex $v_{ij}, i = 2,3$ with the exception of v_{3r} is adjacent in F_2 (and consequently in F_1) to exactly one of v_{11}, v_{41}. We first observe that if F_1 contains a vertex u of degree 1, then the only neighbour of u, say w, must be in F_1 adjacent to all other vertices, otherwise $\mathrm{ex}_{F_1}\, u > 2$. But then w is isolated in F_2, which is impossible. Suppose then, to the contrary, that there is another vertex than v_{3r} adjacent in F_2 to both v_{11} and v_{41}. Then $\deg_{F_2} v_{11} + \deg_{F_2} v_{41} \geq 2r + 2$. We distinguish two cases:

(i) $\deg_{F_2} v_{11} = \deg_{F_2} v_{41} = r+1$. Clearly, F_1 now must contain two vertices of degree $r+1$. Because $\deg_{F_1} v_{11} = \deg_{F_1} v_{41} = r$, there must be another vertex $v_{ij}, i \neq 1,4$ such that $\deg_{F_1} v_{ij} = r+1$. But then $\deg_{F_2} v_{ij} = 1$, which is impossible.

(ii) One of the vertices v_{11}, v_{41}, say V_{11}, is of degree $r+k \geq r+2$ in F_2. There is only one vertex which could possibly be of degree $r+k$ in F_1, namely v_{41}, because all other vertices are only of degree $r+2$ in $K_{1,1,r,r}$. But in this case $\deg_{F_2} v_{41} = 2r+1-(r+k) = r-k+1$, which yields $\deg_{F_2} v_{11} + \deg_{F_2} v_{41} = (r+k)+(r-k+1) = 2r+1$. This contradicts our assumption and hence each vertex $v_{21}, v_{22}, \ldots, v_{2r}, v_{31}, v_{32}, \ldots, v_{3r}$ is in F_2 (and in F_1, too) adjacent to just one of v_{11}, v_{41}. As we have seen above, $\deg_{F_2} v_{3r} = 2$ and F_1 must also contain a vertex of degree 2. Suppose it is one of v_{11}, v_{41}, say v_{11}. Then $\deg_{F_2} v_{11} = 2r-1$ and F_1 contains a vertex of degree $2r-1$. Apparently, it must be v_{41}. Then $\deg_{F_2}, v_{41} = 2$ and F_2 contains two vertices of degree 2 and so does F_1. Hence there is at least one vertex of degree 2 in F_2 different from v_{11}, v_{41}. Let it be v_{2i}. It is in F_1 adjacent to v_{3r} and one of v_{11}, v_{41}, say v_{11}. Then each vertex $v_{3j}, j \in \{1, 2, \ldots, r-1\}$ is adjacent to v_{11}, otherwise $\text{dist}_{F_1}(v_{2i}, v_{3j}) > 2$. So in F_2, v_{11} is adjacent only to v_{3r} and some vertices $v_{2t}, t \neq i$. But none of the vertices v_{2t} is a neighbour of v_{2i} in F_2 and $\text{dist}_{F_2}(v_{11}, v_{2i}) > 2$, which is a contradiction.

Now suppose that $\deg_{F_1} v_{3i} = 2, i < r$. Each vertex of V_3 different from v_{3r} is adjacent to exactly one of v_{11}, v_{41} in F_1 so that we may assume without loss of generality that v_{3i} is adjacent to v_{11} and some v_{2j}. Now each vertex $v_{2s}, s \neq j$ must be adjacent to v_{11}, otherwise $\text{dist}_{F_1}(v_{2s}, v_{3i}) > 2$. As we have seen, v_{2r} is adjacent in F_1 to v_{41} and therefore $j = r$. So v_{2r} is in F_2 adjacent to v_{11} while v_{3i}

to v_{41}. But neither the edge $v_{2r}v_{3i}$ belongs to F_2 nor v_{2r} and v_{3i} have a common neighbour in F_2 which yields $\mathrm{dist}_{F_2}(v_{2r}, v_{3i}) > 2$. Then $\mathrm{diam}\, F_1 = \mathrm{diam}\, F_2 > 2$, which is a contradiction completing the proof. □

Let us recall now the decomposition of the graph $K_{n_1^{2k_1} n_2^{2k_2} \ldots n_p^{2k_p}}$ with all $n_1, n_2, \ldots, n_p$ odd from the first part of Theorem 2.1.5. If we put $p = 2$ and $k_1 = k_2 = 1$, we see that we have decomposed the complete four-partite graph $K_{2m_1+1,2m_1+1,2m_2+1,2m_2+1}$ into two isomorphic factors with diameter 3. The main idea, due to J. Širáň [22], is the following. We take the complete graph K_4 and decompose it into two paths P_4. Then we "blow up" the paths so that we replace each original vertex by $m_1 + m_2 + 1$ vertices and replace the original edges with all possible edges between the new vertices. Then we add all edges between vertices belonging to the original inner vertices of the path. Finally, we remove all edges of a subgraph K_{2m_1+1} from one of the sets corresponding to an original inner vertex and all edges of K_{2m_2+1} induced by all other vertices corresponding to original inner vertices.

To illustrate the method more clearly we decompose $K_{3,3,5,5}$ into factors with diameter 3.

Example 2.5.18. Take the complete graph K_4 with vertices v_1, v_2, v_3, v_4 and decompose it into two paths: v_1, v_2, v_3, v_4 and v_2, v_4, v_1, v_3. The former gives rise to the factor F_1, the latter to F_2. Now replace each vertex v_i with 4 vertices $v_{i1}, v_{i2}, v_{i3}, v_{i4}$ and each edge v_iv_j with 16 edges $v_{ik}v_{jl}; k, l = 1, 2, 3, 4$. In F_1 for $i = 2, 3$ add edges $v_{ik}v_{il}$ for all $k \neq l$; $k, l = 1, 2, 3, 4$. In F_2 for $i = 1, 4$ add edges $v_{ik}v_{il}$ for all $k \neq l$; $k, l = 1, 2, 3, 4$. The factors of K_{16} are, of course, isomorphic and the diameter is 3. To get the graph $K_{3,3,5,5}$ and its factors, we have to remove a factor $2K_3 \cup 2K_5$.

So we remove from F_1 all edges of K_3 induced by v_{21}, v_{22}, v_{23} and all edges of K_5 induced by $v_{24}, v_{31}, v_{32}, v_{33}, v_{34}$. From F_2 we remove the edges induced by v_{41}, v_{42}, v_{43} and all edges of K_5 induced by $v_{44}, v_{11}, v_{12}, v_{13}, v_{14}$. F_1 and F_2 are now factors of the graph $K_{3,3,5,5}$ with partite sets $V_1 = \{v_{21}, v_{22}, v_{23}\}, V_2 = \{v_{41}, v_{42}, v_{43}\}, V_3 = \{v_{24}, v_{31}, v_{32}, v_{33}, v_{34}\}, V_3 = \{v_{44}, v_{11}, v_{12}, v_{13}, v_{14}\}$. The isomorphism $\phi : F_1 \to F_2$ is defined as follows: $\phi(v_{1i}) = v_{2i}, \phi(v_{2i}) = v_{4i}, \phi(v_{3i}) = v_{1i}, \phi(v_{4i}) = v_{3i}$ for $i = 1,2,3,4$. To prove that $\operatorname{diam} F_1 = \operatorname{diam} F_2 = 3$ we can observe that in F_1 the distance $\operatorname{dist}_{F_1}(v_{ij}, v_{ik}) \leq 2$ because they have always a common neighbour. The vertices v_{1j} are adjacent to all vertices v_{2l}, similarly each v_{2j} is adjacent to all v_{1l}, each v_{4j} is adjacent to all v_{3l} and each v_{3j} is adjacent to all v_{4l}. The distance $\operatorname{dist}_{F_1}(v_{1j}, v_{kl}) = k - 1 \leq 3$ because there is for $k = 3,4$ always a path v_{1j}, v_{21}, v_{3l} or $v_{1j}, v_{21}, v_{31}, v_{4l}$. Similarly, for $j = 1,2,3$ we can see that $\operatorname{dist}_{F_1}(v_{2j}, v_{3l}) = 1$ and $\operatorname{dist}_{F_1}(v_{2j}, v_{4l}) = 2$. This also yields $\operatorname{dist}_{F_1}(v_{24}, v_{3l}) = 2$ and $\operatorname{dist}_{F_1}(v_{24}, v_{4l}) = 3$ because v_{24} is adjacent to the other vertices v_{2j}. Finally, $\operatorname{dist}(v_{3j}, v_{4l}) = 1$ for all $j, l = 1,2,3,4$ and hence $\operatorname{diam} F_1 = 3$. □

We repeatedly use modifications of the idea to prove $(2, d)$-isodecomposability of the graphs $K_{2m_1+1,2m_1+1,2m_2+1,2m_2+1}$ for $d = 2,3,4$.

Theorem 2.5.19. *Let r, s be odd integers. A complete four-partite graph $K_{r,r,s,s}$ is $(2, d)$-isodecomposable for a finite diameter d if and only if*

(a) $d = 2$ and $r, s \geq 3$, or

(b) $d = 3$ and $r \geq 1, s \geq 3$ or

(c) $d = 4$ and $r \geq 1, s \geq 3$.

Proof. The only other finite diameter for which there exist isodecomposable four-partite graphs, 5, is excluded by Theorem 2.5.15. Necessity in (a) follows from

Lemma 2.5.17. In the cases (b) and (c) it follows from our definition of multipartite graphs that $1 < s = 3$. To prove sufficiency, we start with one of the simpler cases, (c). We use a construction slightly different from that in Theorem 2.1.5 which can be easily modified for the cases (a) and (b).

Take a complete graph $K_{2(r+s)}$ and partition its vertex set into 8 subsets $X_1, \ldots, X_4, Y_1, \ldots, Y_4$ where for each $i = 1,2,3,4$, $|X_i| = r$ and $|Y_i| = (s-r)/2 = t$. Let $X_i = \{x_{i1}, x_{i2}, \ldots, x_{ir}\}$ and $Y_i = \{y_{i1}, y_{i2}, \ldots, y_{it}\}$ for $i = 1,2,3,4$. First we construct isomorphic factors F_1 and F_2 as follows: F_1 contains all edges $x_{ij}x_{i+1,k}$, where $i = 1,2,3$ and $j,k = 1,2,\ldots,r$, all edges $y_{ij}y_{i+1,k}$, where $i = 1,2,3$ and $j,k = 1,2,\ldots,t$, and all edges $y_{1i}x_{jk}$ and $y_{4i}x_{jk}$, where $i = 1,2,\ldots,t$ and $j = 1,\ldots,4; k = 1,2,\ldots,r$. Furthermore, F_1 contains all edges $x_{2i}x_{2j}$ and $x_{3i}x_{3j}$ where $i \neq j$; $i,j = 1,2,\ldots,r$, $y_{2i}y_{2j}$ and $y_{3i}y_{3j}$ where $i \neq j$; $i,j = 1,2,\ldots,t$. One can verify that $\phi : F_1 \to F_2$ is an isomorphism with cycles $(x_{4i}x_{2i}x_{1i}x_{3i})$ and $(y_{4j}y_{2j}y_{1j}y_{3j})$ for $i = 1,2,\ldots,r$ and $j = 1,2,\ldots,t$. Now we can remove from the complete graph $K_{2(r+s)}$ all edges of its complete subgraphs $\langle x_{21}, x_{22}, \ldots, x_{2r}\rangle \cong K_r$, $\langle x_{41}, x_{42}, \ldots, x_{4r}\rangle \cong K_r$, $\langle y_{21}, y_{22}, \ldots, y_{2t}, y_{31}, y_{32}, \ldots, y_{3t}, x_{31}, x_{32}, \ldots, x_{3r}\rangle \cong K_s$ and $\langle y_{11}, y_{12}, \ldots, y_{1t}, y_{41}, y_{42}, \ldots, y_{4t}, x_{11}, x_{12}, \ldots, x_{1r}\rangle \cong K_s$ to obtain $K_{r,r,s,s}$. If we remove the edges also from the factors F_1, F_2 of $K_{2(r+s)}$, we have factors F_1', F_2' of $K_{r,r,s,s}$. The isomorphism $\phi' : F_1' \to F_2'$ is then induced by the isomorphism $\phi : F_1 \to F_2$.

The factor F_1' has now the edges $x_{ij}x_{i+1\ k}$ with $i = 1,2,3$; $j,k = 1,2,\ldots,r$; $y_{1j}y_{2k}$ and $y_{3j}y_{4k}$ with $j,k = 1,2,\ldots,t$; $y_{1i}x_{jk}$ with $i = 1,2,\ldots,t$; $j = 2,3,4$; $k = 1,2,\ldots,r$; and $y_{4i}x_{jk}$ again with $i = 1,2,\ldots,t$; $j = 2,3,4$; $k = 1,2,\ldots,r$. One can verify that $\operatorname{dist}_{F_1}(y_{2i}, y_{3j}) = 4$ for any $i,j = 1,2,\ldots,t$. Therefore from Theorems 2.5.1 and 2.5.15 it follows that $\operatorname{diam} F_1 = 4$.

In the case (b) we take again the graph $K_{2(r+s)}$ with the vertex set having the same subsets $X_1, \ldots, X_4, Y_1, \ldots, Y_4$ as in part (c) and construct the factors F_1 and F_2 in a slightly different way. F_1 now contains the edges $x_{ij}x_{i+1\ k}$, where $i = 1,2,3$ and $j,k = 1,2,\ldots,r$, all edges $y_{ij}y_{i+1\ k}$, where $i = 1,2,3$ and $j,k = 1,2,\ldots,t$, and all edges $x_{2i}y_{jk}$ and $x_{3i}y_{jk}$, where $i = 1,2,\ldots,r; j = 1,\ldots,4; k = 1,2,\ldots,t$. Furthermore, F_1 contains all edges $x_{2i}x_{2j}$ and $x_{3i}x_{3j}$ where $i \neq j$; $i,j = 1,2,\ldots,r$; $y_{2i}y_{2j}$ and $y_{3i}y_{3j}$ where $i \neq j$; $i,j = 1,2,\ldots,t$. We now remove from F_1 all edges of induced complete subgraphs $\langle x_{31}, x_{32}, \ldots, x_{3r}\rangle \cong K_r$ and $\langle y_{21}, y_{22}, \ldots, y_{2t}, y_{31}, y_{32}, \ldots, y_{3t}, x_{21}, x_{22}, \ldots, x_{2r}\rangle \cong K_s$. From F_2 we remove the edges of the subgraphs $\langle x_{41}, x_{42}, \ldots, x_{4r}\rangle \cong K_r$ and $\langle y_{11}, y_{12}, \ldots, y_{1t}, y_{41}, y_{42}, \ldots, y_{4t}, x_{11}, x_{12}, \ldots, x_{1r}\rangle \cong K_s$. The resulting graphs F_1', F_2' are certainly factors of the graph $K_{r,r,s,s}$ with the partite sets $V_1 = X_3, V_2 = X_4, V_3 = X_2 \cup Y_2 \cup Y_3, V_4 = X_1 \cup Y_1 \cup Y_4$. The isomorphism between them is again defined as above, i.e. $\phi' : F_1' \to F_2'$ with the cycles $(x_{3i}x_{1i}x_{2i}x_{4i})$ and $(y_{3j}y_{1j}y_{2j}y_{4j})$ for $i = 1,2,\ldots,r$ and $j = 1,2,\ldots,t$.

The factors have diameter 3. For instance, $\text{dist}_{F_1}(x_{1i}, x_{4j}) = 3$ for any $i,j = 1,2,\ldots,r$ and for any pair of vertices it does not exceed 3.

The construction of factors with diameter 2 (case (a)) is quite similar. We partition the vertex set of $K_{2(r+s)}$ into sets $X_1, \ldots, X_4, Y_1, \ldots, Y_4$ such that $|X_i| = (r+s-4)/2 = t$ and $|Y_i| = 2$ for $i = 1,2,3,4$ and construct the factor F_1 similarly to that of part (c). We have then edges $x_{ij}x_{i+1\ k}$ for $i = 1,3,4$; $j,k = 1,2,\ldots,t$; $y_{ij}y_{i+1\ k}$ for $i = 1,2,3$ and $j,k = 1,2$; $y_{1i}x_{jk}$ and $y_{4i}x_{jk}$ for $i = 1,2$; $j = 1,2,3,4$ and $k = 1,2,\ldots,t$. Furthermore we have again all edges $x_{2i}x_{2j}$ and $x_{3i}x_{3j}$ for $i \neq j$; $i,j = 1,2,\ldots,t$ and $y_{21}y_{22}$ with $y_{31}y_{32}$. The factor F_1 is of diameter 2, but now we have to be more careful while removing edges to obtain $K_{r,r,s,s}$ because we must preserve the diameter. If one removes all edges between any two of the sets X_i and Y_j, the diameter increases to at least 3. Hence we have to leave edges between any two sets to preserve a path of length 2 between any two vertices.

So we remove from $K_{2(r+s)}$ the edges of the complete graphs $\langle x_{21}, x_{22}, \ldots, x_{2\ r-2}, y_{21}, y_{31}\rangle \cong K_r$, $\langle x_{41}, x_{42}, \ldots, x_{4\ r-2}, y_{11}, y_{41}\rangle \cong K_r$, $\langle x_{2\ r-1}, x_{2r}, x_{2\ r+1}, \ldots, x_{2t}, x_{31}, x_{32}, \ldots, x_{3t}, y_{22}, y_{32}\rangle \cong K_s$ and $\langle x_{4\ r-1}, x_{4r}, x_{4\ r+1}, \ldots, x_{4t}, x_{11}, x_{12}, \ldots, x_{1t}, y_{12}, y_{42}\rangle \cong K_s$. The remaining edges form a factor F_1' of the complete graph $K_{r,r,s,s}$ with the partite sets $V_1 = \{x_{21}, x_{22}, \ldots, x_{2\ r-2}, y_{21}, y_{31}\}$, $V_2 = \{x_{41}, x_{42}, \ldots, x_{4\ r-2}, y_{11}, y_{41}\}$, $V_3 = \{x_{2\ r-1}, x_{2r}, x_{2\ r+1}, \ldots, x_{2t}, x_{31}, x_{32}, \ldots, x_{3t}, y_{22}, y_{32}\}$ and $V_4 = \{x_{4\ r-1}, x_{4r}, x_{4\ r+1}, \ldots, x_{4t}, x_{11}, x_{12}, \ldots, x_{1t}, y_{12}, y_{42}\}$. The isomorphism is again $\phi' : F_1' \to F_2'$ with the cycles $(x_{3i}x_{1i}x_{2i}x_{4i})$ and $(y_{3j}y_{1j}y_{2j}y_{4j})$ for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, t$.

Finally we verify that $\operatorname{diam} F_1' = 2$: $\operatorname{dist}_{F_1}(y_{1i}, y_{2j}) = 1$; this yields $\operatorname{dist}_{F_1}(y_{11}, y_{12}) = 2$ and $\operatorname{dist}_{F_1}(y_{1i}, y_{3k}) = 2$ because there are the edges $y_{21}y_{32}$ and $y_{22}y_{31}$; $\operatorname{dist}_{F_1}(y_{1i}, y_{4l}) = 2$ because there is always a path y_{1i}, x_{3k}, y_{4l}; similarly $\operatorname{dist}_{F_1}(y_{11}, x_{1i}) = 1$ and $\operatorname{dist}_{F_1}(y_{12}, x_{1i}) = 2$ since there is always a path $y_{12}, x_{2j}x_{1i} (j \leq r-2)$; $\operatorname{dist}_{F_1}(y_{1i}, x_{2j}) = \operatorname{dist}_{F_1}(y_{1i}, x_{3j}) = 1$; $\operatorname{dist}_{F_1}(y_{1i}, x_{4j}) \leq 2$ since there are all edges $y_{1i}x_{3k}$ and $x_{3k}y_{4j}$; $\operatorname{dist}_{F_1}(y_{21}, y_{22}) = \operatorname{dist}_{F_1}(y_{21}, y_{32}) = \operatorname{dist}_{F_1}(y_{22}, y_{31}) = 2$, hence $\operatorname{dist}_{F_1}(y_{21}, y_{31}) = \operatorname{dist}_{F_1}(y_{22}, y_{32}) = \operatorname{dist}_{F_1}(y_{2i}, y_{4j}) = 2$; $\operatorname{dist}_{F_1}(y_{2i}, x_{jk}) = 2$ because both y_{21} and y_{22} are adjacent to both y_{11}, y_{12} and x_{jk} is adjacent to at least one of y_{11}, y_{12}; $\operatorname{dist}_{F_1}(y_{3i}, x_{jk}) = 2$ by the same argument (via y_{41} and y_{42}); $\operatorname{dist}_{F_1}(y_{31}, y_{32}) = \operatorname{dist}_{F_1}(y_{3i}, y_{4j}) = 1$, $\operatorname{dist}_{F_1}(y_{41}, y_{42}) = 2$, $\operatorname{dist}_{F_1}(y_{4i}, x_{jk}) \leq 2$ since there are all edges $y_{4i}x_{2l}$ and $x_{2l}x_{1m}$ and all edges $y_{4i}x_{3l}$ and $x_{3l}x_{4m}$; $\operatorname{dist}_{F_1}(x_{1i}, x_{2j}) = 1$ and hence $\operatorname{dist}_{F_1}(x_{1i}, x_{1j}) = \operatorname{dist}_{F_1}(x_{2i}, x_{2j}) = 2$,

$\mathrm{dist}_{F_1}(x_{1i}, x_{3j}) = 2$ because all x_{1i} and x_{3j} are adjacent to y_{11}; $\mathrm{dist}_{F_1}(x_{1i}, x_{4j}) \leq 2$ because each x_{1i} is adjacent to all $x_{41}, \ldots, x_{4\ r-2}$ while all $x_{4\ r-1}, \ldots, x_{4t}$ are adjacent to y_{41} which is a neighbour of all x_{1i}; $\mathrm{dist}_{F_1}(x_{2i}, x_{3j}) = \mathrm{dist}_{F_1}(x_{2i}, x_{4k}) = 2$ since each vertex x_{4j} is adjacent to one of y_{41}, y_{42} and they are both neighbours of all x_{2i} and x_{3j}; and finally $\mathrm{dist}_{F_1}(x_{3i}, x_{4j}) = 1$ and therefore $\mathrm{dist}_{F_1}(x_{3i}, x_{3j}) = \mathrm{dist}_{F_1}(x_{4i}, x_{4j}) = 2$. □

2.6. r-PARTITE GRAPHS WITH $r \geq 5$

In this section we determine smallest $(2, d)$-isodecomposable complete r-partite graphs for every $r \geq 5$ and every possible finite d, i.e., the values of $g_r(2, d)$. We also prove that $g_r(2, d) = g_r'(2, d)$ for any possible pair r, d. In other words, we prove that for every $p \geq g_r(2, d)$ there is a complete r-partite $(2, d)$-isodecomposable graph with p vertices. We start with Gangopadhyay's result [10] on decomposability into factors (not necessarily isomorphic) with the same finite diameter.

Theorem 2.6.1. (Gangopadhyay) *Let a complete r-partite graph $K_{m_1, m_2, \ldots, m_r}$ with more than 4 parts be $(2, d)$-decomposable for a finite diameter d. Then $d = 2, 3, 4$ or 5 and*

(a) $m_1 + m_2 + \cdots + m_r \geq r + 1$ *if* $d = 2$;

(b) $m_1 + m_2 + \cdots + m_r \geq r + 1$ *if* $d = 3$;

(c) $m_1 + m_2 + \cdots + m_r \geq r + 2$ *if* $d = 4$;

(d) $m_1 + m_2 + \cdots + m_r \geq r + 4$ *if* $d = 5$.

We prove later that for every $r > 4, r \equiv 2 \pmod 4$ and each $d = 2, 3, 4, 5$, every $r \equiv 1 \pmod 4$ and $d = 2, 3$, and every $r \equiv 3 \pmod 4$ and $d = 4, 5$ all $(2, d)$-decomposable complete r-partite graphs are also $(2, d)$-isodecomposable. However, the following lemmas show that it is not true in general.

Lemma 2.6.2. *Let $r > 4, r \equiv 0$ or $3 \pmod 4$ and $d = 2$ or 3. Then there is no $(2, d)$-isodecomposable graph $K_{m_1, m_2, \ldots, m_r}$ with $r + 1$ vertices.*

Proof. Obviously, the only graph $K_{m_1, m_2, \ldots, m_r}$ with $r + 1$ vertices is $K_{2,1,1,\ldots,1}$, which is not admissible for $r \equiv 0$ or $3 \pmod 4$. □

Lemma 2.6.3. *Let $r > 4, r \equiv 0 \pmod 4$. Then there is no $(2, d)$-isodecomposable graph $K_{m_1, m_2, \ldots, m_r}$ with $r + 2$ vertices for any d.*

Proof. There are only two possible complete r-partite graphs with $r + 2$ vertices, namely $K_{2,2,1,1,\ldots,1}$ and $K_{3,1,1,\ldots,1}$. The former is not admissible, while the latter is not isodecomposable by Theorem 2.5.6. □

Lemma 2.6.4. *Let $r > 4, r \equiv 0 \pmod 4$. Then there is no $(2, 5)$-isodecomposable graph $K_{m_1, m_2, \ldots, m_r}$ with $r + 4$ vertices.*

Proof. By Corollary 2.5.12, every $(2, 5)$-isodecomposable r-partite complete graph has one part of cardinality at least 4 and another of cardinality at least 2. The only r-partite graph with $r + 4$ vertices, satisfying this condition, is $K_{4,2,1,1,\ldots,1}$. But the number of the odd parts of this graph is $r - 2 \equiv 2 \pmod 4$ and therefore the graph is not admissible. □

Lemma 2.6.5. *Let $r > 4, r \equiv 1 (\mathrm{mod}\, 4)$. Then there is no $(2,4)$-isodecomposable graph $K_{m_1,m_2,\ldots,m_r}$ with $r+2$ vertices.*

Proof. There are two graphs $K_{m_1,m_2,\ldots,m_r}$ with $r+2$ vertices. $K_{2,2,1,1,\ldots,1}$, which is not admissible, and $K_{3,1,1,\ldots,1}$. Let us suppose that there is $r \equiv 1 (\mathrm{mod}\, 4)$ such that the r-partite graph $K_{3,1,1,\ldots,1}$ is $(2,4)$-isodecomposable into factors F_1 and F_2. Let $U = \{u_1, u_2, u_3\}$ be one part and $V_i = \{v_i\}, i = 1, 2, \ldots, r-1$ the other parts, and let $V = V_1 \cup V_2 \cup \cdots \cup V_{r-1}$.

We first assume that there is a pair of vertices u_i, u_j, say u_1, u_2, such that $\mathrm{dist}_{F_1}(u_1, u_2) = 4$. Obviously, $N_{F_1}(u_1) \cup N_{F_1}(u_2) \subset V$ and $N_{F_1}(u_1) \cap N_{F_1}(u_2) = \emptyset$. Furthermore, there is no edge between $N_{F_1}(u_1)$ and $N_{F_1}(u_2)$. Let $M = V \backslash N_{F_1}(u_1) \backslash N_{F_1}(u_2)$. Then in F_2 all vertices of $N_{F_1}(u_1)$ are adjacent to u_2, all vertices of $N_{F_1}(u_2)$ are adjacent to u_1, and each vertex of $N_{F_1}(u_1)$ is adjacent to all vertices of $N_{F_1}(u_2)$. If the vertices u_1 and u_2 have no common neighbour in F_2, i.e., if $M = \emptyset$, the diameter of the graph $\langle V \cup u_1 \cup u_2 \rangle_{F_2} = F_2 - u_3$ is 3 and the only vertices having eccentricity 3 in this graph are u_1 and u_2. Since u_3 is not adjacent to either of them, we can see that $\mathrm{ex}_{F_2}\, u_3 \leq 3$, which yields $\mathrm{diam}\, F_2 \leq 3$. If $M \neq \emptyset$, then the diameter of the graph $\langle V \cup u_1 \cup u_2 \rangle_{F_2} = F_2 - u_3$ is 2 and therefore again $\mathrm{diam}\, F_2 \leq 3$. Thus if $\mathrm{dist}_{F_i}(x, y) = 4$, at least one of the vertices x, y belongs to V.

Now we show that if $\mathrm{dist}_{F_i}(x, y) = 4$ and $x = v_i \in V$ then $y \notin V$. Suppose it is not the case and there are vertices of V, say v_1, v_2, such that $\mathrm{dist}_{F_1}(v_1, v_2) = 4$. Denote F_i' the subgraph of F_i induced by the vertices of V. Then clearly $\mathrm{diam}\, F_1' \geq 4$. It is well known that if a factor of a complete graph K_n has diameter greater than 3, then its complement (with respect to K_n) has diameter at most 2. Because $\langle V \rangle = K_{r-1}$, the diameter of F_2' is at most 2. Then all vertices with eccentricity 4 in F_2 belong to U, which is impossible by the preceding paragraph.

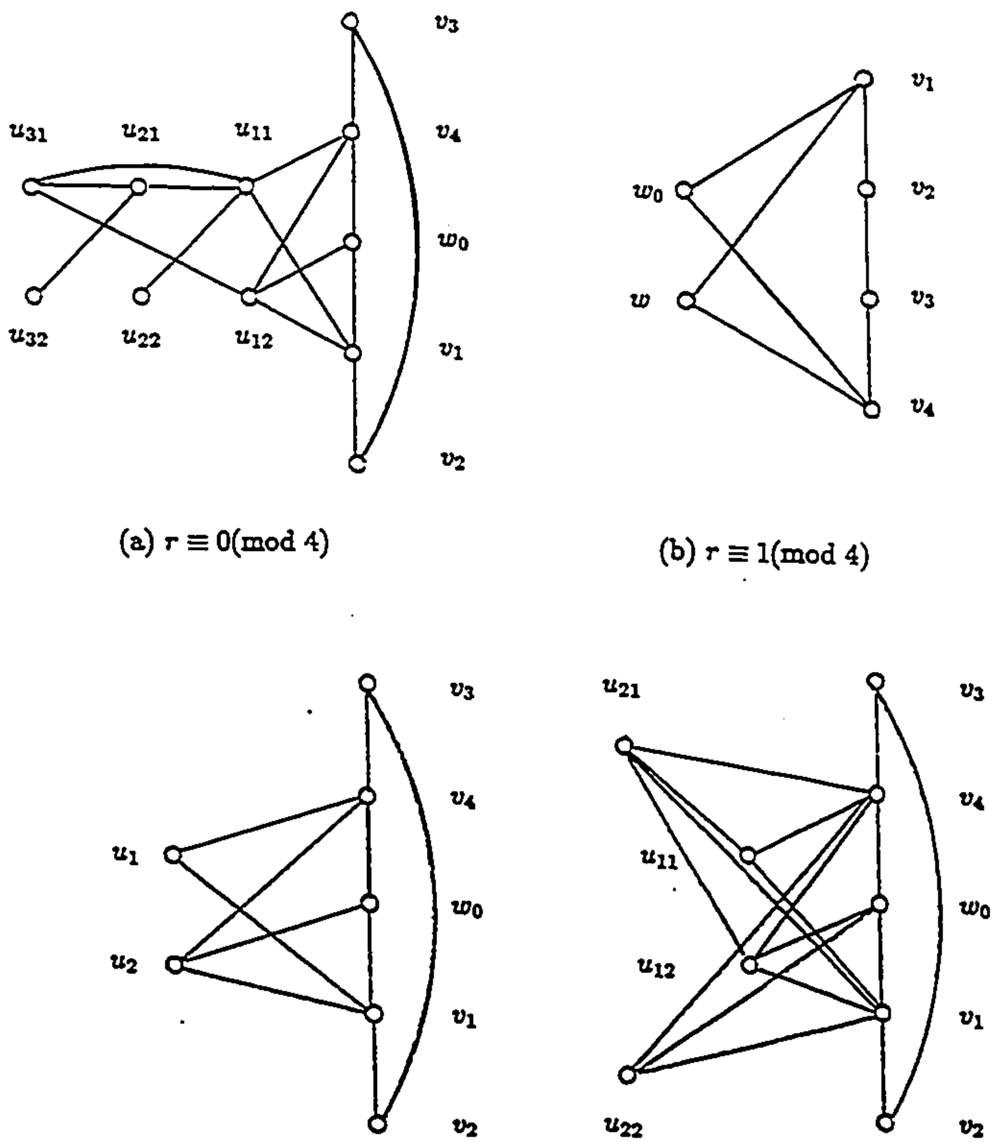
Thus we have only one possibility left, namely that there are vertices u_i and v_j, say u_1, v_1, such that $\text{dist}_{F_1}(u_1, v_1) = 4$. Then $\langle V \cup u_1 \rangle \cong K_r$ and the graph $\langle V \cup u_1 \rangle_{F_2}$ has diameter at most 2, because $\text{diam}\langle V \cup u_1 \rangle_{F_1} \geq 4$. Hence the only vertices which could have eccentricity 4 in F_2 are u_2 and u_3. Then $\text{dist}_{F_2}(u_2, u_3) = 4$, which is a contradiction completing the proof. □

Lemma 2.6.6. *Let $r > 4, r \equiv 1 (\text{mod}\, 4)$. Then there is no $(2,5)$-isodecomposable graph $K_{m_1,m_2,\ldots,m_r}$ with less than $r+6$ vertices.*

Proof. By Corollary 2.5.12 every $(2,5)$-isodecomposable graph $K_{m_1,m_2,\ldots,m_r}$ contains $K' = K_{4,2,1,1,\ldots,1}$. This graph has $r+4$ vertices and is not admissible for $r \equiv 1 (\text{mod}\, 4)$.

There are only 3 graphs or order $r+5$, containing K'. The first one, $K_{4,2,2,1,\ldots,1}$, is not admissible. Let us investigate then the graph $K_{5,2,1,1,\ldots,1}$ and denote the part with 5 vertices by V_1, and the part with 2 vertices by V_2. It follows from Lemma 2.5.11 that the vertices which have eccentricity 4 in either factor belong to V_2. By Lemma 2.5.13 the self-complementing isomorphism, ϕ, takes V_2 onto itself. Hence, by Observation 2.5.3.a, the r-partite graph $K_{5,2,1,1,\ldots,1}$ is isodecomposable only if the $(r-1)$-partite graph $K_{5,1,1,\ldots,1}$ is isodecomposable. But $K_{5,1,1,\ldots,1}$ has $r-2$ trivial parts, which is an odd number, and therefore is not $(2,d)$-isodecomposable for any d by Theorem 2.5.6.

The last case, $K_{4,3,1,1,\ldots,1}$, is similar. By the same arguments as above, ϕ takes the part with 3 vertices onto itself and $K_{4,3,1,1,\ldots,1}$ is isodecomposable only if the $(r-1)$-partite graph $K_{4,1,1,\ldots,1}$ is isodecomposable, too. But for $r \equiv 1 (\text{mod}\, 4)$ the graph $K_{4,1,1,\ldots,1}$ with $r-2$ parts of cardinality 1 is not admissible, and therefore $K_{4,3,1,1,\ldots,1}$ is not $(2,5)$-isodecomposable. □



(a) $r \equiv 0 (\mathrm{mod}\ 4)$

(b) $r \equiv 1 (\mathrm{mod}\ 4)$

(c) $r \equiv 2 (\mathrm{mod}\ 4)$

(d) $r \equiv 3 (\mathrm{mod}\ 4)$

Figure 2.6.1

Now we present smallest $(2,2)$-isodecomposable graphs for each $r \geq 5$.

Construction 2.6.7. *(a)* Case $r \equiv 0 \pmod 4$. For $r = 8$ we take the graph shown in Figure 2.6.1.a. To get a selfcomplementary factor of $K_{2,2,2,1,\dots,1}$ with parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}\}, U_2 = \{u_{21}, u_{22}\}, U_3 = \{u_{31}, u_{32}\}, V_i = \{v_i\}, i = 1, 2, 3, 4$, we add all edges $u_{21}x$ and $u_{31}x$ for $x \in \{w_0, v_1, v_2, v_3, v_4\}$ whenever the edge $u_{11}x$ exists and all edges $u_{22}x$ and $u_{32}x$ whenever the edge $u_{12}x$ exists. The self-complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12}), (u_{21}u_{22}), (u_{31}u_{32}), (v_1v_3v_4v_2)$. For any $r = 4k + 8, k \geq 1$, we add parts $V_5, V_6, \dots, V_{4k+4}$, where $V_j = \{v_j\}$. Then for every quadruple $v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}$ we add the edges of the path $P_4 = \langle v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4} \rangle$, i.e., $v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3}, v_{4i+3}v_{4i+4}$, and join the end-vertices v_{4i+1} and v_{4i+4} of P_4 to all "preceding" vertices, i.e., to the vertices $u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}, w_0, v_1, v_2, \dots, v_{4i}$. The new cycles of ϕ are then $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \dots, k$.

(b) Case $r \equiv 1 \pmod 4$. For $r = 5$ we take the selfcomplementary factor shown in Figure 2.6.1.b. The parts of $K_{2,1,1,1,1}$ are $W = \{w, w_0\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$, the self-complementing permutation ϕ is determined by the cycles $(w_0), (w), (v_1v_3v_4v_2)$. For any $r = 4k + 5, k \geq 1$, we add again vertices $v_5, v_6, \dots, v_{4k+4}$ (or, more precisely, parts $V_j = \{v_j\}$) and for every quadruple $v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}$ we add the edges of the path $P_4 = \langle v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4} \rangle$, i.e., $v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3}, v_{4i+3}v_{4i+4}$, and join the end-vertices v_{4i+1} and v_{4i+4} of P_4 to all "preceding" vertices, i.e., to the vertices $w_0, w, v_1, v_2, \dots, v_{4i}$. The new cycles of ϕ are now again $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \dots, k$.

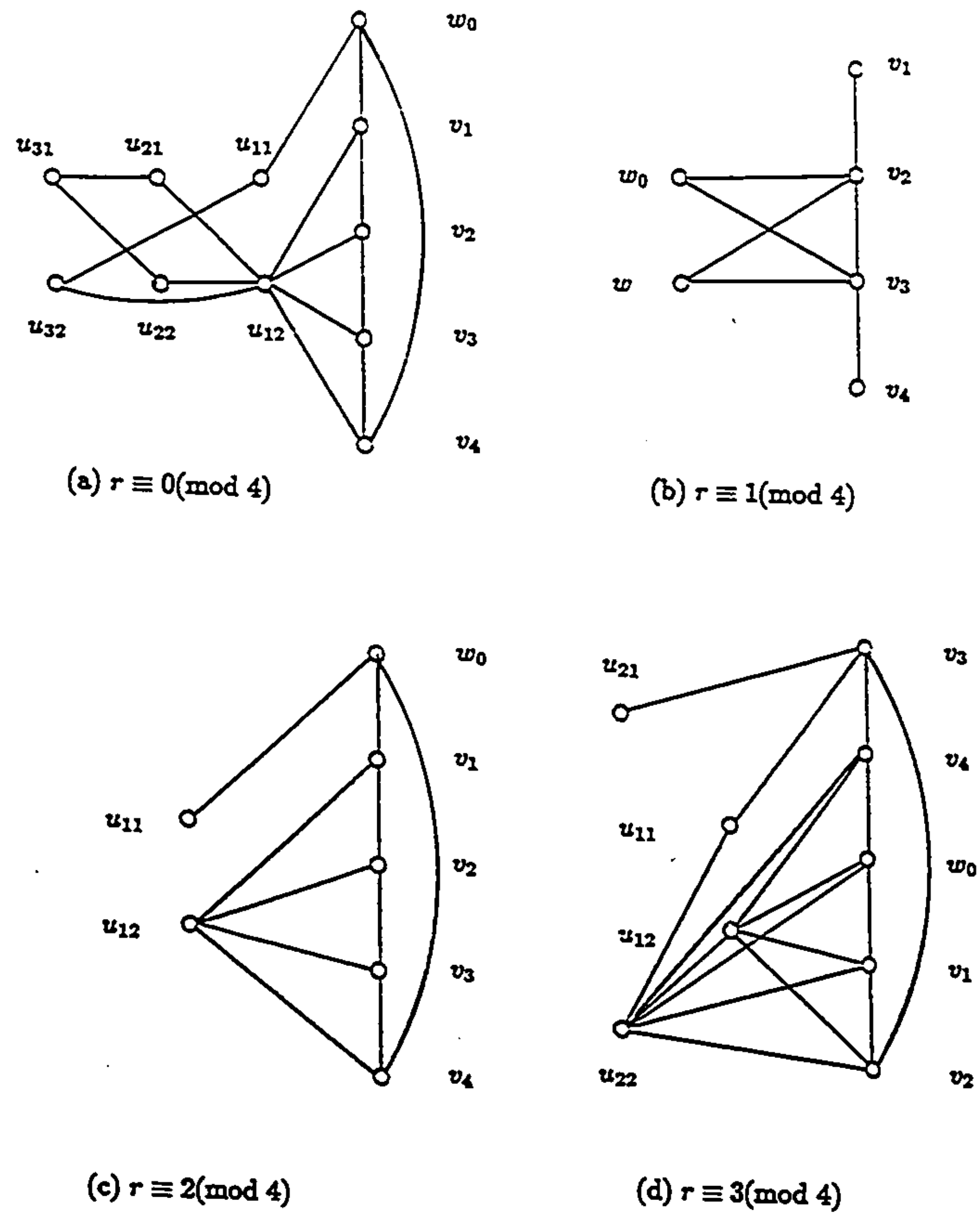
(c) Case $r \equiv 2 \pmod 4$. For $r = 6$ we take the selfcomplementary factor shown in Figure 2.6.1.c. The self-complementing permutation ϕ is determined by

the cycles $(w_0),(u_1u_2),(v_1v_3v_4v_2)$. For any $r=4k+6, k\geq 1$, we add again vertices (i.e., parts,) $v_5,v_6,\ldots,v_{4k+4}$ and all the edges as in the case (b). The new cycles of ϕ are again $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i=1,2,\ldots,k$.

(d) Case $r\equiv 3(\mathrm{mod}\,4)$. For $r=7$ we take the selfcomplementary factor shown in Figure 2.6.1.d. The self-complementing permutation ϕ is determined by the cycles $(w_0),(u_{11}u_{12}),(u_{21}u_{22}),(v_1v_3v_4v_2)$. For any $r=4k+7, k\geq 1$, we again add the vertices, edges and permutation cycles as in the previous cases. □

We continue with smallest (2,3)-isodecomposable graphs for each $r\geq 5$. The construction is in all cases very similar to the previous one. We again take first the r-partite factors for $r=5,6,7,8$ and extend them by adding paths P_4, but we join to the "preceding" vertices the inner vertices of P_4 rather than the end-vertices.

Construction 2.6.8. *(a)* Case $r\equiv 0(\mathrm{mod}\,4)$. For $r=8$ we take the graph shown in Figure 2.6.2.a. To get a selfcomplementary factor of $K_{2,2,2,1,\ldots,1}$ with parts $W=\{w_0\},U_1=\{u_{11},u_{12}\},U_2=\{u_{21},u_{22}\},U_3=\{u_{31},u_{32}\},V_i=\{v_i\},i=1,2,3,4$, we add all edges $u_{21}x$ and $u_{31}x$ for $x\in\{w_0,v_1,v_2,v_3,v_4\}$ whenever the edge $u_{11}x$ exists and all edges $u_{22}x$ and $u_{32}x$ whenever the edge $u_{12}x$ exists. The self-complementing permutation ϕ is determined by the cycles $(w_0),(u_{11}u_{12}),(u_{21}u_{22})$, $(u_{31}u_{32}),(v_1v_3v_4v_2)$. For any $r=4k+8, k\geq 1$, we add parts $V_5=\{v_5\},V_6=\{v_6\},\ldots,V_{4k+4}=\{v_{4k+4}\}$. Then for every quadruple $v_{4i+1},v_{4i+2},v_{4i+3},v_{4i+4}$ we add the edges of the path $P_4=\langle v_{4i+1},v_{4i+2},v_{4i+3},v_{4i+4}\rangle$, namely $v_{4i+1}v_{4i+2}$, $v_{4i+2}v_{4i+3}$, $v_{4i+3}v_{4i+4}$, and join the vertices $v_{4i+1},v_{4i+2},v_{4i+3},v_{4i+4}$ to the vertices v_{12},v_{22},v_{32} and, furthermore, the inner vertices v_{4i+2} and v_{4i+3} of P_4 to the vertices $w_0,v_1,v_2,\ldots,v_{4i}$. The new cycles of ϕ are then $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i=1,2,\ldots,k$. The vertices at distance 3 apart are always u_{31} and u_{32}.



w_0
v_1
u_{31}
u_{21}
u_{11}
v_2
u_{32}
u_{22}
u_{12}
v_3
v_4
(a) $r \equiv 0 (\mathrm{mod}\ 4)$
v_1
w_0
v_2
w
v_3
v_4
(b) $r \equiv 1 (\mathrm{mod}\ 4)$
w_0
v_1
u_{11}
v_2
u_{12}
v_3
v_4
(c) $r \equiv 2 (\mathrm{mod}\ 4)$
u_{21}
v_3
v_4
u_{11}
w_0
u_{12}
v_1
u_{22}
v_2
(d) $r \equiv 3 (\mathrm{mod}\ 4)$

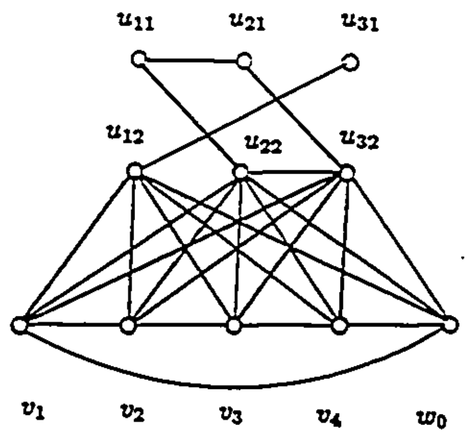
Figure 2.6.2

(b) Case $r \equiv 1 (\mathrm{mod}\, 4)$. For $r = 5$ we take the selfcomplementary factor shown in Figure 2.6.2.b. The parts of $K_{2,1,1,1,1}$ are $W = \{w, w_0\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$, the self-complementing permutation ϕ is determined by the cycles $(w_0), (w), (v_1 v_3 v_4 v_2)$. For any $r = 4k+5, k \geq 1$, we add again vertices $v_5, v_6, \ldots, v_{4k+4}$ (i.e., parts $V_j = \{v_j\}$) and for every quadruple $v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}$ we add again the edges of $P_4 = \langle v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4} \rangle$, i.e., $v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3}, v_{4i+3}v_{4i+4}$, and join the inner vertices v_{4i+2} and v_{4i+3} of P_4 to all "preceding" vertices, i.e., to the vertices $w_0, w, v_1, v_2, \ldots, v_{4i}$. The new cycles of ϕ are now again $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$. The vertices having mutual distance 3 are v_{k+1} and v_{4k+4}.

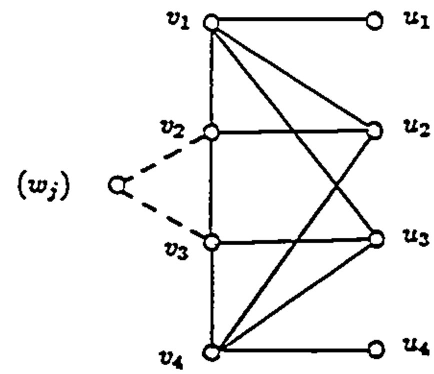
(c) Case $r \equiv 2 (\mathrm{mod}\, 4)$. For $r = 6$ we take the selfcomplementary factor shown in Figure 2.6.2.c. The self-complementing permutation ϕ is determined by the cycles $(w_0), (u_1 u_2), (v_1 v_3 v_4 v_2)$. For any $r = 4k+6, k \geq 1$, we add again vertices (parts) $v_5, v_6, \ldots, v_{4k+4}$ and all the edges as in the case (b). The new cycles of ϕ are again $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$. The vertices u_{11} and u_{12} are always at distance 3.

(d) Case $r \equiv 3 (\mathrm{mod}\, 4)$. For $r = 7$ we take the selfcomplementary factor shown in Figure 2.6.2.d. The self-complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12}), (u_{21}u_{22}), (v_1 v_3 v_4 v_2)$. For any $r = 4k+7, k \geq 1$, we again add the vertices, edges and permutation cycles as in the previous cases. □

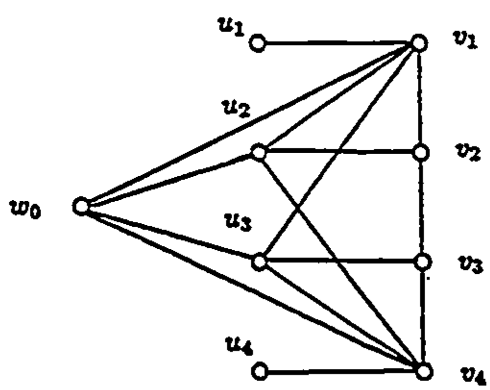
In constructions of factors with diameters 4 and 5 we use a different approach. To increase the number of parts, we "blow up" the path P_4 induced by vertices belonging to different trivial parts similarly as in Section 5, e.g., in Example 2.5.18 or Theorem 2.5.19. First we construct smallest selfcomplementary factors with diameter 4 of the complete r-partite graphs for each $r \geq 5$.



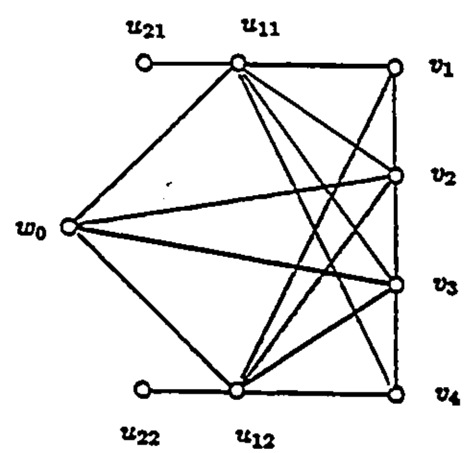
(a) $r \equiv 0 \pmod 4$



(b) $r \equiv 1 \pmod 4$



(c) $r \equiv 2 \pmod 4$



(d) $r \equiv 3 \pmod 4$

Figure 2.6.3

Construction 2.6.9. *(a)* Case $r \equiv 0 (\mathrm{mod}\, 4)$. We start with decomposition of the 8-partite graph $K_{2,2,2,1,\ldots,1}$ with the parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}\}, U_2 = \{u_{21}, u_{22}\}, U_3 = \{u_{31}, u_{32}\}, V_i = \{v_i\}, i = 1,2,3,4$. The selfcomplementary factor is shown in Figure 2.6.3.a. The self-complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12}), (u_{21}u_{22}), (u_{31}u_{32}), (v_1v_3v_4v_2)$. The vertices having distance 4 are u_{11} and u_{31}. For any $r = 4k+8, k \geq 1$, we add parts $V_j = \{v_j\}, i = 5,6,\ldots,4k+4$. Now we "blow up" the path $P_4(0) = \langle v_1, v_2, v_3, v_4 \rangle$. We add the edges of the paths $P_4(i)$, namely $v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3}, v_{4i+3}v_{4i+4}$ for $i = 1,2,\ldots,k$, all edges $v_{4i+1}v_{4l+2}, v_{4i+2}v_{4l+3}, v_{4i+3}v_{4l+4}$ and all edges $v_{4i+2}v_{4l+2}$ and $v_{4i+3}v_{4l+3}$ for all pairs $i, l \in \{0,1,\ldots,k\}, i \neq l$. We also add the edges $v_{4i+r}x$ for all $i = 1,2,\ldots,k$ and $r = 1,2,3,4$ whenever the edge v_rx exists. Here x is any vertex of $W \cup U_1 \cup U_2 \cup U_3$. In other words, we take the path $P_4(0) = \langle v_1, v_2, v_3, v_4 \rangle$, put the vertices $v_{4i+r}, i = 0,1,\ldots,k; r = 1,2,3,4$ "into" the vertex v_r and substitute the original edge v_rv_{r+1} for all possible edges $v_{4i+r}v_{4l+r+1}$. The vertices v_{4i+2} and $v_{4i+3}, i = 0,1,\ldots,k$ induce complete graphs K_{k+1}, while the vertices v_{4i+1} and $v_{4i+1}, i = 0,1,\ldots,k$ remain mutually non-adjacent. Finally, every vertex v_{4i+r} has the same neighbours in $W \cup U_1 \cup U_2 \cup U_3$ as the vertex v_r. One can check that u_{11} and u_{31} are at distance 4. The new cycles of ϕ are now $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1,2,\ldots,k$.

(b) Case $r \equiv 1 (\mathrm{mod}\, 4)$. We first decompose the graph $K_{4,1,1,1,1}$ with parts $U = \{u_1, u_2, u_3, u_4\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ into factors isomorphic to the one shown in Figure 2.6.3.b. (The indicated vertex w_j appears later in the construction of graphs of greater orders.) The self-complementing permutation is determined by the cycles $(u_1u_3u_4u_2)$ and $(v_1v_3v_4v_2)$. For any $r = 4k+5, k \geq 1$,

we add the parts $V_5 = \{v_5\}, V_6 = \{v_6\}, \ldots, V_{4k+4} = \{v_{4k+4}\}$ and "blow up" the path $P_4(0) = \langle v_1, v_2, v_3, v_4 \rangle$ exactly as in part (a). The new cycles of ϕ are again $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$. The vertices having mutual distance 4 are u_1 and u_4.

(c) Case $r \equiv 2(\bmod 4)$. We start with the graph $K_{4,1,1,1,1}$ with parts $W = \{w_0\}, U = \{u_1, u_2, u_3, u_4\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ and decompose it into factors isomorphic to the factor shown in Figure 2.6.3.c. The self-complementing permutation ϕ is determined by the cycles $(w_0), (u_1u_3u_4u_2)$ and $(v_1v_3v_4v_2)$. For any $r = 4k + 6, k \geq 1$, we again "blow up" the path $P_4(0) = \langle v_1, v_2, v_3, v_4 \rangle$ exactly as in part (a), adding the parts $V_5 = \{v_5\}, V_6 = \{v_6\}, \ldots,$ $V_{4k+4} = \{v_{4k+4}\}$ and the corresponding edges. The new cycles of ϕ are again $(v_{4i+1}$ $v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$. The vertices at distance 4 apart are u_1 and u_4.

(d) Case $r \equiv 3(\bmod 4)$. For $r = 7$ we decompose the graph $K_{2,2,1,1,1,1,1}$ with parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}\}, U_2 = \{u_{21}, u_{22}\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ into factors isomorphic to the graph in Figure 2.6.3.d. The self-complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{22}), (u_{12}u_{21})$ and $(v_1v_3v_4v_2)$. We increase the number of parts for any $r = 4k + 3$ as in the previous cases. The vertices having mutual distance 4 are u_{21} and u_{22}. □

Finally, we construct factors of smallest $(2, 5)$-isodecomposable complete r-partite graphs for each $r \geq 5$.

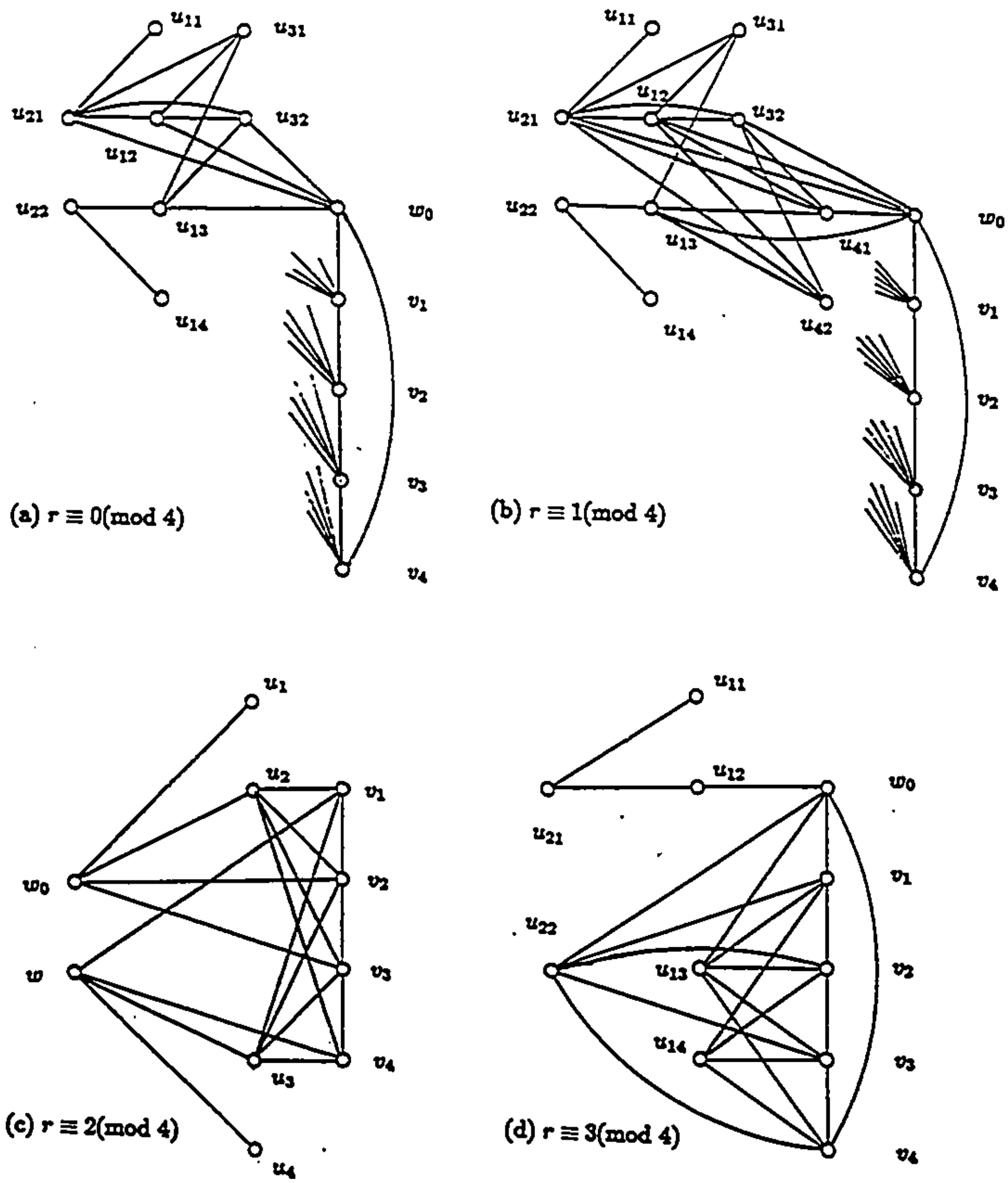
Construction 2.6.10. In this construction we present only the factors of smallest $(2, 5)$-isodecomposable complete r-partite graphs with $r = 5, 6, 7, 8$ and 9 parts. The factors of smallest graphs for any $r \geq 10$ can be obtained exactly as in Construction 2.6.9—by "blowing up" the path $P_4(0) = \langle v_1, v_2, v_3, v_4 \rangle$.

(a) Case $r \equiv 0 \pmod 4$. The 8-partite graph $K_{4,2,2,1,\ldots,1}$ with the parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}, u_{13}, u_{14}\}, U_2 = \{u_{21}, u_{22}\}, U_3 = \{u_{31}, u_{32}\}, V_i = \{v_i\}$, $i = 1,2,3,4$, is (2,5)-isodecomposable into the selfcomplementary factors shown in Figure 2.6.4.a. The vertices $v_1, \ldots, v_4$ are adjacent to the neighbours of the vertex w_0, i.e., to $u_{12}, u_{13}, u_{21}, u_{32}$. The self-complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12}), (u_{13}u_{14}), (u_{21}u_{22}), (u_{31}u_{32}), (v_1v_3v_4v_2)$. The vertices having mutual distance 5 are u_{11} and u_{14}.

(b) Case $r \equiv 1 \pmod 4$. The 5-partite graph $K_{4,2,2,2,1}$ with the parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}, u_{13}, u_{14}\}, U_2 = \{u_{21}, u_{22}\}, U_3 = \{u_{31}, u_{32}\}, U_4 = \{u_{41}, u_{42}\}$ is (2,5)-isodecomposable into the selfcomplementary factors isomorphic to the subgraph of the graph shown in Figure 2.6.4.b induced by the above mentioned parts. The vertices $v_1, \ldots, v_4$ are adjacent to the same vertices u_{ij} as the vertex w_0. The self-complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12})$, $(u_{13}u_{14}), (u_{21}u_{22}), (u_{31}u_{32}), (u_{41}u_{42})$. The vertices at mutual distance 5 are u_{11} and u_{14}.

To obtain the selfcomplementary factor of the 9-partite graph $K_{4,2,2,2,1,\ldots,1}$, we have to add to the graph in Figure 2.6.4.b the parts $V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ and edges v_iu_{jl} for each $i = 1,2,3,4$ whenever the edge w_0u_{jl} exists. The permutation ϕ contains now one more cycle, $(v_1v_3v_4v_2)$.

(c) Case $r \equiv 2 \pmod 4$. The factor of the 6-partite graph $K_{4,2,1,1,1,1}$ with the parts $W = \{w, w_0\}, U = \{u_1, u_2, u_3, u_4\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ is shown in Figure 2.6.4.c. The cycles of ϕ are $(w), (w_0), (u_1u_3u_4u_2), (v_1v_3v_4v_2)$ and the vertices at distance 5 are u_1 and u_4.



u11
u31
u21
u32
u12
u22
u13
w0
v1
u14
v2
v3
(a) r ≡ 0(mod 4)
v4
u11
u31
u12
u21
u32
u22
w0
u13
u41
v1
u14
u42
v2
v3
(b) r ≡ 1(mod 4)
v4
u1
u2
v1
w0
v2
w
v3
u3
v4
(c) r ≡ 2(mod 4)
u4
u11
u12
w0
u21
v1
u22
u13
v2
u14
v3
(d) r ≡ 3(mod 4)
v4

Figure 2.6.4

(d) Case $r \equiv 3 \pmod 4$. The 7-partite graph $K_{4,2,1,\ldots,1}$ with the parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}, u_{13}, u_{14}\}, U_2 = \{u_{21}, u_{22}\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ is (2,5)-isodecomposable into the factors isomorphic to that in Figure 2.6.4.d. The cycles of ϕ are $(w_0), (u_{11}u_{12}), (u_{13}u_{14}), (u_{21}u_{22}), (v_1v_3v_4v_2)$ and the vertices at distance 5 are u_{11} and u_{14}. □

We can summarize the results given in this section as follows.

Theorem 2.6.11. *Let* $r \geq 5$. *Then*
$g_r(2,2) = g_r(2,3) = g_r(2,4) = r+3,\ g_r(2,5) = r+5$ *if* $r \equiv 0 \pmod 4$,
$g_r(2,2) = g_r(2,3) = r+1,\ g_r(2,4) = r+3,\ g_r(2,5) = r+6$ *if* $r \equiv 1 \pmod 4$,
$g_r(2,2) = g_r(2,3) = r+1,\ g_r(2,4) = r+2,\ g_r(2,5) = r+4$ *if* $r \equiv 2 \pmod 4$, *and*
$g_r(2,2) = g_r(2,3) = g_r(2,4) = r+2,\ g_r(2,5) = r+4$ *if* $r \equiv 3 \pmod 4$.

Proof. Apply Lemmas 2.6.2–2.6.6 and Constructions 2.6.7–2.6.10. □

P. Das [7] introduced the following classes of graphs. A complete graph without one edge, $\tilde{K}_n = K_n - e$, is called an *almost complete graph.* A graph G with n vertices is *almost selfcomplementary* if the graph $\tilde{K}_n$ can be decomposed into two factors that are both isomorphic to G. Obviously, if a graph with n vertices is selfcomplementary, then $n \equiv 2$ or $3 \pmod 4$. Since the graph $\tilde{K}_n$ is the complete $(n-1)$-partite graph $K_{2,1,1,\ldots,1}$, which appears among the smallest isodecomposable graphs in the previous theorem, the following re-phrasing of the results dealing with almost selfcomplementary graphs may be of some interest.

Theorem 2.6.12. (Das) *An almost complete graph* $\tilde{K}_n$ *is decomposable into two connected isomorphic factors with diameter* d *if and only if* $n \equiv 2$ *or* $3 \pmod 4$ *and* $d = 2$ *or* 3.

In the previous sections we have seen that for $r = 2,3,4$, we always have $g_r(2,d) = g'_r(2,d)$ for any d. Constructions 2.6.7—2.6.10 provide the necessary tools to prove that the equality holds for any finite r and any d.

Theorem 2.6.13. *Let $2 \le r < \infty$. Then $g_r(2,d) = g'_r(2,d)$ for any d.*

Proof. $g_r(2,d) = \infty$ for $r > 2$ and $d = 1$ or $5 < d < \infty$, hence the result is immediate. The same holds for $r = 2$ and $d = 1,2$ or $6 < d < \infty$. All other cases for $2 \le r \le 4$ follow from Theorem 2.2.3 ($d = \infty$), Corollary 2.4.3 ($r = 2$), Corollary 2.4.9 ($r = 3$) and Theorem 2.5.16 ($r = 4$). To prove the assertion for any $r \ge 5$ we need to show that for a given $d, 2 \le d \le 5$ and any $p \ge g_r(2,d)$ there is a complete r-partite $(2,d)$-isodecomposable graph with p vertices. Let $p = g_r(2,d) + q$. For $d = 4$ and $r \equiv 1 (\mathrm{mod}\, 4)$ we take the factor constructed in part (b) of Construction 2.6.9, add q vertices $w_1, w_2, \ldots, w_q$ into part U and join each of them in the factor F_1 to all vertices $v_{4i+2}, v_{4i+3}, i = 1,2,\ldots,k$. Then $\phi(w_j) = w_j$ for each $j = 1,2,\ldots,q$ and obviously $F_1 \cong F_2$. In all other cases one can see that $\phi(w_0) = w_0$. Therefore we can always add q vertices $w_1, w_2, \ldots, w_q$ into part W and join in F_1 each of them to all neighbours of w_0. Then again $\phi(w_j) = w_j$ for each $j = 1,2,\ldots,q$ and $F_1 \cong F_2$. □

3. Decompositions of group divisible designs

3.0. Introductory notes and definitions

A *group divisible design* $k-GDD(n,r)$ is a triple $(V,\mathcal{G},\mathcal{B})$ where V is a set of elements, $\mathcal{G}$ is a partition of V into r subsets $G_1, G_2, \ldots, G_r$ of the same cardinality n called *groups* and $\mathcal{B}$ is a collection of subsets of V of cardinality k called *blocks* such that $|G_l \cap B| \leq 1$ for any group $G_l \in \mathcal{G}$ and any block $B \in \mathcal{B}$ and for any two elements x, y from distinct groups there is exactly one block containing both x and y. (Our definition is somewhat more restrictive than that given usually in the literature, cf., e.g., [3]). A *transversal design* $k-TD(n)$ is a group divisible design $k-GDD(n,k)$, i.e., $|G_l \cap B| = 1$ for any group $G_l \in \mathcal{G}$ and any block $B \in \mathcal{B}$. A *factor* E of a $k-GDD(n,r)$ is a triple $(V,\mathcal{G},\mathcal{D})$ where $\mathcal{D}$ is a subset of $\mathcal{B}$. A *decomposition* of a $k-GDD(n,r)$ is an m-tuple of factors $E_i = (V,\mathcal{G},\mathcal{D}_i), i = 1,2,\ldots,m$ such that $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ and $\bigcup_{i=1}^{m} \mathcal{D}_i = \mathcal{B}$. Two factors E_i and E_j are *isomorphic* (denoted $E_i \cong E_j$) if there exists a one-to-one mapping ϕ_{ij} of V onto itself such that $D' = \{\phi_{ij}(x_1), \phi_{ij}(x_2), \ldots, \phi_{ij}(x_k)\} \in \mathcal{D}_j$ if and only if $D = \{x_1, x_2, \ldots, x_k\} \in \mathcal{D}_i$. A decomposition is *isomorphic* if $E_i \cong E_j$ for every pair $1 \leq i < j \leq m$. If $m = 2$, the isomorphism $\phi : E_1 \to E_2$ is also called a *self-complementing isomorphism*, *self-complementing permutation* or *complementing permutation* and the factors E_1 and E_2 the *selfcomplementary factors with respect to* $k-GDD(n,r)$ or simply the *selfcomplementary factors*.

A *path* of length q, P_q, is a sequence $x_0 - B_1 - x_1 - B_2 - x_2 - \cdots - B_q - x_q$ of elements and blocks such that for each $i = 1,2,\ldots,q$ the elements x_{i-1} and x_i belong to the block B_i and no block and no element appears more than once.

Since each pair of elements in a $k-GDD$ belongs to at most one block, the path is uniquely determined by the elements and we usually use the simpler notation $P_q = x_0 - x_1 - \cdots - x_q$. A *cycle* of length q, C_q, is a sequence $x_0 - B_1 - x_1 - B_2 - x_2 - \cdots - B_q - x_q$ (or simply $x_0 - x_1 - \cdots - x_q$) of elements and blocks such that $x_0 = x_q$, for each $i = 1, 2, \ldots, q$ the elements x_{i-1} and x_i belong to the block B_i and no block or element appears more than once. A *distance* between elements x and y in a factor E, denoted $\mathrm{dist}_E(x, y)$, is the length of the shortest path from x to y. A factor E is *connected*, if for each pair $x, y \in V$ there is a path from x to y; otherwise, it is *disconnected*. A *diameter* of a connected factor E, $\mathrm{diam}\, E$, is the maximum of the set of distances $\mathrm{dist}_E(x, y)$ among all pairs of elements of E. If E is disconnected, we define $\mathrm{diam}\, E = \infty$.

Hartman [14], Das and Rosa [8], and Phelps [19] studied decompositions of designs into two factors. We are interested in decompositions of GDD's into two isomorphic factors with a given diameter and also in isomorphic decompositions of GDD's into smallest connected factors. A GDD is *(t, d)-decomposable* if it can be decomposed into t factors with diameter d each. A GDD is *(t, d)-isodecomposable* if it can be decomposed into t mutually isomorphic factors with diameter d.

3.1. Diameters of selfcomplementary factors of group divisible designs

There is an obvious similarity between decompositions of GDD's and multipartite complete graphs. If E is a factor of a $k - GDD(n, r)$ $(V, \mathcal{G}, \mathcal{B})$ then the *underlying graph* or *underlying factor* of E is the r-partite graph $U(E)$ with the vertex set V and the parts $G_1, G_2, \ldots, G_r$ in which two vertices x, y are adjacent if and only if the elements x, y are adjacent in E, i.e., if they belong to the same block of E. Clearly, $\text{dist}_E(x, y) = \text{dist}_{U(E)}(x, y)$ an hence $\text{diam}\, E = \text{diam}\, U(E)$. If $U(E)$ is the underlying graph of a factor E of $k - GDD(n, r)$ then the edge set of $U(E)$ can be partitioned into complete graphs K_k, where each K_k corresponds to one block of E. We say that a complete r-partite graph $K_{n,n,\ldots,n} = K_{n^r}$ is $K_k - (t, d)$*-decomposable* if it is (t, d)-decomposable and the edge set of each factor can be partitioned into complete graphs K_k. Similarly, K_{n^r} is $K_k - (t, d)$*-isodecomposable* if it is $K_k - (t, d)$-decomposable and the factors are mutually isomorphic.

The necessary and sufficient conditions for decomposability of group divisible designs now follow easily so that the proof can be omitted.

Theorem 3.1.1. *A (t, d)-decomposable group divisible design $k - GDD(n, r)$ exists if and only if the underlying graph K_{n^r} is $K_k - (t, d)$-decomposable, and (t, d)-isodecomposable group divisible design $k - GDD(n, r)$ exists if and only if the graph K_{n^r} is $K_k - (t, d)$-isodecomposable.*

Remark 3.1.2. It is easy to see that a similar theorem holds also in a more general case when we consider the groups and blocks having different sizes $n_1, n_2, \ldots, n_r$ and $k_1, k_2, \ldots, k_s$, respectively.

Since for $k = 2$ the group divisible design is isomorphic to its underlying graph, K_{n^r}, we always suppose that $r, k > 2$. As we have seen in the previous sections, connected selfcomplementary factors of complete r-partite graphs with $r > 2$ can have diameters 2,3,4 or 5. Therefore, no selfcomplementary factor of a $k - GDD(n, r)$ can have diameter greater than 5 either. In fact, in the case of selfcomplementary factors of GDD's even the diameter 5 is not possible. We could prove this applying Lemmas 2.5.9 and 2.5.10, but we are looking for a more general result. We prove that if a $k-GDD(n,r)$ is decomposable into two connected factors E_1 and E_2, not necessarily isomorphic, then one of them is of diameter at most 4. We could prove even this applying Lemmas 2.5.9 and 2.5.10, or more precisely, the proofs of the lemmas, because we never used the isomorphism of the factors in the proofs of the lemmas. But the proof in the case of GDD's is more compact and therefore we prefer to present the modified form.

We prove the result first for the underlying factors. The main result is then an easy corollary. A *clique* of a graph F is a maximal complete subgraph of G.

Theorem 3.1.3. *Let a complete r-partite graph be decomposable into two connected factors F_1 and F_2 such that the smallest clique in any of the factors has at least 3 vertices. Let* $\operatorname{diam} F_1 \geq 5$. *Then* $\operatorname{diam} F_2 \leq 3$.

Proof. For convenience we again assign a colour to each part. Let a white vertex w_0 have in F_1 eccentricity $d = \operatorname{diam} F_1 \geq 5$. Let V_i be the set of all vertices having in F_1 distance i from w_0. Since every clique of F_1 is of order at least 3, w_0 has at least two neighbours of different colours, say red and blue. For the same reason, if a_d, a vertex of colour a, is in F_1 at distance d from w_0, then $V_{d-1} \cup V_d$ contains vertices of two other colours, say b and c. The distance in F_2 between any two vertices $x_i, y_j \in V_3 \cup V_4 \cup \cdots \cup V_d$ is at most 2, because one of the red, blue or white vertices of $V_0 \cup V_1$ has a colour different from both x, y and hence is in F_2 adjacent to both x_i, y_j. If $x_i, y_j \in V_0 \cup V_1 \cup V_2$ then again $\operatorname{dist}_{F_2}(x_i, y_j) \leq 2$ because $V_{d-1} \cup V_d$ contains vertices of 3 different colours.

If $x_i \in V_i$ and $y_j \in V_j$ have different colours, $i \leq 2, j \geq 3$ and $j - i > 1$, then x_i and y_j are adjacent in F_2; if they have different colours and $i = 2$ and

$j = 3$, then x_2 is adjacent in F_2 to vertices of at least two colours of $V_{d-1} \cup V_d$, say a_p, b_q. At the same time y_j is adjacent in F_2 to a vertex of the set $V_0 \cup V_1 \cup V_2$. But every vertex of $V_0 \cup V_1 \cup V_2$ is, according to its colour, adjacent in F_2 to at least one of a_p, b_q. Then F_2 contains a path of length 3 between x_i and y_j and $\text{dist}_{F_2}(x_i, y_j) \leq 3$. Finally, if $x_i \in V_i$ and $x_j \in V_j$ have the same colour, x, and $i \leq 2$ and $j \geq 3$, then similarly as above x_i is in F_2 adjacent to vertices of at least two colours of $V_{d-1} \cup V_d$, say a_p, b_q, while x_j has in F_2 a neighbour in $V_0 \cup V_1$. This neighbour is in F_2 adjacent to at least one of a_p, b_q which yields $\text{dist}_{F_2}(x_i, x_j) \leq 3$. Thus for any two vertices x, y the distance between them in F_2 is at most 3 and $\text{diam}\, F_2 \leq 3$. □

An equivalent result for group divisible designs now follows easily.

Theorem 3.1.4. *Let a group divisible design with at least 3 groups be decomposable into two connected factors E_1 and E_2. If every block of the GDD is of size at least 3 and $\text{diam}\, E_1 \geq 5$ then $\text{diam}\, E_2 \leq 3$.*

Proof. Because the minimal blocks have at least 3 elements, the underlying factors $U(E_1)$ and $U(E_2)$ satisfy the conditions of Theorem 3.1.3. Since $\text{diam}\, E_i = \text{diam}\, U(E_i)$, we can repeat the previous proof to show that if there is a vertex w_0 with $\text{ex}_{E_1} w_0 = \text{ex}_{U(E_1)} w_0 \geq 5$, then $\text{diam}\, U(E_2) \leq 3$. Therefore $\text{diam}\, E_2 \leq 3$, which completes the proof. □

If we require the factors to be isomorphic, we immediately have the following.

Corollary 3.1.5. *Let a complete r-partite graph be isodecomposable into two connected factors $F_1 \cong F_2$ such that the smallest clique in the factors has at least 3 vertices. Then $\text{diam}\, F_1 = \text{diam}\, F_2 \leq 4$.*

Corollary 3.1.6. *Let a GDD with at least 3 groups be isodecomposable into two connected factors $E_1 \cong E_2$. If every block of the GDD is of size at least 3, then* $\operatorname{diam} E_1 = \operatorname{diam} E_2 \leq 4$.

3.2. Selfcomplementary factors of $3 - TD$'s

In this section we prove that for every even size $2n \geq 4$ and $d = 3, 4$ and ∞ there exists a $(2, d)$-isodecomposable transversal design $3 - TD(2n)$, i.e., a group divisible design with 3 groups of size $2n$ and blocks of size 3. Obviously, such a decomposition of a $3 - TD(2n+1)$ is not possible because the number of its blocks is odd. Since we do not know an example of a $(2,2)$-isodecomposable $3 - TD(2n)$ but, at the same time, we are unable to prove that such a TD does not exist, we leave the case $d = 2$ in doubt. For both $d = 3$ and 4 we construct factors of a $3 - TD(2n)$ arising from the additive group $\mathbb{Z}_{2n}$. The groups of the design are $G_1 = \{0_1, 1_1, 2_1, \ldots, (2n-1)_1\}, G_2 = \{0_2, 1_2, 2_2, \ldots, (2n-1)_2\}, G_3 = \{0_3, 1_3, 2_3, \ldots, (2n-1)_3\}$ and its blocks are the triples $(x_1, y_2, (x+y)_3)$ for $x, y = 0, 1, 2, \ldots, 2n-1$. It is well known that a $3 - TD(2n)$ is equivalent to a latin square of order $2n$. If we assign the numbers $0, 1, 2, \ldots, 2n-1$ to both the rows and the columns then a triple (x_1, y_2, z_3) belongs to the $3 - TD(2n)$ if and only if the entry z appears in the x-th row and y-th column, i.e., in our case, if and only if $x + y \equiv z (\operatorname{mod} 2n)$.

Construction 3.2.1. $(d = 3)$ The factor E_1 of the $3 - TD(2n)$ above contains the blocks

$(0_1, 0_2, 0_3), (0_1, 1_2, 1_3), (0_1, 2_2, 2_3), \ldots, (0_1, (n-1)_2, (n-1)_3)$,

$(2_1, 0_2, 2_3), (2_1, 1_2, 3_3), (2_1, 2_2, 4_3), \ldots, (2_1, (n-1)_2, (n+1)_3),$

...

...

$((2n-2)_1, 0_2, (2n-2)_3), ((2n-2)_1, 1_2, (2n-1)_3), \ldots, ((2n-2)_1, (n-1)_2, (n-3)_3),$

$(1_1, n_2, (n+1)_3), (1_1, (n+1)_2, (n+2)_3), \ldots, (1_1, (2n-1)_2, 0_3),$

$(3_1, n_2, (n+3)_3), (3_1, (n+1)_2, (n+4)_3), \ldots, (3_1, (2n-1)_2, 2_3),$

...

...

$((2n-1)_1, n_2, (n-1)_3), ((2n-1)_1, (n+1)_2, n_3), \ldots, ((2n-1)_1, (2n-1)_2, (2n-2)_3).$

The factor E_2 contains all blocks not contained in E_1 and the self-complementing permutation is $\phi(x_1) = x_1, \phi(x_2) = (n+x)_2, \phi(x_3) = (n+x)_3$.

Now we have to show that $\operatorname{diam} E_1 = 3$. To see this, we present the factor E_1 as the sub-array of the array of the additive group $\mathbb{Z}_{2n}$ shown in Figure 3.2.1 for n odd and in Figure 3.2.2 for n even. The elements of the group G_1 are assigned to the rows, the elements of G_2 are assigned to the columns and the elements of G_3 are the entries. If there is no block containing both elements x_1 and y_2 we leave the space in the x-th row and y-th column blank.

We start with n odd. First we check the distances from the elements of G_1 to all others. For every $x_1, y_1 \in G_1, x_1 \neq y_1$ the distance $\operatorname{dist}_{E_1}(x_1, y_1) = 2$ because each of the two rows contains $\lfloor \frac{n}{2} \rfloor + 1$ even entries and hence they have at least one even entry, say z, in common. Thus x_1 and y_2 have in E_1 a common neighbour z_3.

For $y_2 \in G_2$ we have $\operatorname{dist}_{E_1}(x_1, y_2) = 1$ if x is even and $y < n$ or x is odd and $y \geq n$ because the entries in the x-th row and y-th columns are non-blank; if x is even and $y \geq n$ or x is odd and $y < n$ then $\operatorname{dist}_{E_1}(x_1, y_2) = 2$ because

odd	0	1	2	...	n-1	n	n+1	n+2	...	2n-1
0	0	1	2	...	n-1	-	-	-	-	-
1	-	-	-	-	-	n+1	n+2	n+3	...	0
2	2	3	4	...	n+1	-	-	-	-	-
3	-	-	-	-	-	n+3	n+4	n+5	...	2
.										
.										
n-1	n-1	n	n+1	...	2n-2	-	-	-	-	-
n	-	-	-	-	-	0	1	2	...	n-1
n+1	n+1	n+2	n+3	...	0	-	-	-	-	-
n+2	-	-	-	-	-	2	3	4	...	n+1
.										
.										
2n-2	2n-2	2n-1	0	...	n-3	-	-	-	-	-
2n-1	-	-	-	-	-	n-1	n	n+1	...	2n-2

Figure 3.2.1

each column contains either all even or all odd entries and each row contains n consecutive entries and therefore at least one of them is even and one is odd. Thus every row contains at least one entry in common with each column which means that every element of G_1 has a common neighbour with each element of G_2.

Finally, for $y_3 \in G_3$, $\text{dist}_{E_1}(x_1, y_3) = 1$ if x is even and $y \in \{x, x+1, x+$

even	0	1	2	...	n-1	n	n+1	n+2	...	2n-1
0	0	1	2	...	n-1	-	-	-	-	-
1	-	-	-	-	-	n+1	n+2	n+3	...	0
2	2	3	4	...	n+1	-	-	-	-	-
3	-	-	-	-	-	n+3	n+4	n+5	...	2
.										
.										
n-1	-	-	-	-	-	2n-1	0	1	...	n-2
n	n	n+1	n+2	...	2n-1	-	-	-	-	-
n+1	-	-	-	-	-	1	2	3	...	n
n+2	n+2	n+3	n+4	...	1	-	-	-	-	-
.										
.										
2n-2	2n-2	2n-1	0	...	n-3	-	-	-	-	-
2n-1	-	-	-	-	-	n-1	n	n+1	...	2n-2

Figure 3.2.2

$2, \ldots, x+n-1\}$ or x is odd and $y \in \{x+n, x+n+1, \ldots, x+2n-1\}$ because all the mentioned entries y appear in the x-th row and the corresponding elements belong to the same block; if x is even and $y \in \{x+n, x+n+1, \ldots, x+2n-1\}$ or x is odd and $y \in \{x, x+1, x+2, \ldots, x+n-1\}$ then $\mathrm{dist}_{E_1}(x_1, y_3) = 2$ since every even x_1 is adjacent to 0_2 and 1_2 (there is a non-blank entry in the first two columns in each even row) and two neighbouring columns together contain all entries $0, 1, 2, \ldots, 2n-1$—hence every element z_3 is adjacent to one of the neighbours of x_1, either 0_2 or 1_2. Similarly, if x_1 is odd then it is adjacent to n_2 and $(n+1)_2$ and every z_3 is adjacent to one of them.

Thus we have shown that $\mathrm{dist}_{E_1}(x_1, y_i) \leq 2$ for any $x_1 \in G_1$ and any $y_i \in G_2 \cup G_3$. Now let $y_i, z_j \in G_2 \cup G_3$. Since E_1 is connected, then z_j is adjacent to an element x_1^0. Because $\mathrm{dist}_{E_1}(x_1, y_i) \leq 2$ and, in particular, $\mathrm{dist}_{E_1}(x_1^0, y_i) \leq 2$, we immediately have $\mathrm{dist}_{E_1}(y_i, z_j) \leq \mathrm{dist}_{E_1}(x_1^0, z_j) + \mathrm{dist}_{E_1}(x_1^0, y_i) \leq 3$ for any $y_i, z_j \in G_2 \cup G_3$, which yields $\operatorname{diam} E_1 \leq 3$. To prove that $\operatorname{diam} E_1 = 3$, we have to find a pair of elements whose distance is greater than 2. One of such pairs is $0_2, (n+1)_2$, because the neighbourhood of 0_2, $N_{E_1}(0_2)$, contains elements $0_1, 2_1, \ldots, (2n-2)_1, 0_3, 2_3, \ldots, (2n-2)_3$ while $N_{E_1}((n+1)_2) = \{1_1, 3_1, \ldots, (2n-1)_1, 1_3, 3_3, \ldots, (2n-1)_3\}$. Thus $N_{E_1}(x_2) \cap N_{E_1}(y_2) = \emptyset$ and therefore $\mathrm{dist}_{E_1}(0_2, (n+1)_2) \geq 3$, which completes the case of n odd.

Now we consider n even. Then $\mathrm{dist}_{E_1}(x_1, y_1) = 2$ for any $x_1, y_1 \in G_1$, because if the difference $x - y$ is even then x_1 and y_1 have n common neighbours in G_2—$0_2, 1_2, \ldots, (n-1)_2$ for x, y even, $n_2, (n+1)_2, \ldots, (2n-1)_2$ for x, y odd; if the difference $x - y$ is odd then exactly one of x, y, say x, is even and $N_{E_1}(x_1)$ contains elements $x_3, (x+1)_3, \ldots, (x+n-1)_3$ while $N_{E_1}(y_1)$ contains elements $(y+n)_3, (y+n+1)_3, \ldots, (y-1)_3$. Then $N_{E_1}(y_1)$ contains either x_3 or $(x+n-1)_3$ and in any case $N_{E_1}(x_1) \cap N_{E_1}(y_1) \neq \emptyset$.

For any $x_1 \in G_1, y_2 \in G_2$ we have $\mathrm{dist}_{E_1}(x_1, y_2) \leq 2$, because every $x_1 \in G_1$ has $\frac{n}{2}$ odd and $\frac{n}{2}$ even neighbours $z_3 \in G_3$ and each $y_2 \in G_2$ is adjacent either to all odd or all even elements of G_3.

Also $\text{dist}_{E_1}(x_1, y_3) \leq 2$ for any $x_1 \in G_1, y_3 \in G_3$, because if x is even then x_1 is adjacent to 0_2 and 1_2, while at the same time $N_{E_1}(0_2)$ contains $0_3, 2_3, \ldots, (2n-2)_3$ and $N_{E_1}(1_2)$ contains $1_3, 3_3, \ldots, (2n-1)_3$. Thus $N_{E_1}(0_2) \cup N_{E_1}(1_2) = G_3$, which yields the desired inequality. If x is odd then x_1 is adjacent to n_2 and $(n+1)_2$, and again $N_{E_1}(n_2) \cup N_{E_1}((n+1)_2) = G_3$, which completes the case.

We have shown again that $\text{dist}_{E_1}(x_1, y_i) \leq 2$ for any $x_1 \in G_1$ and any $y_i \in G_2 \cup G_3$. Similarly as in the case of n odd it follows now that $\text{dist}_{E_1}(y_i, z_j) \leq 3$ for any $y_i, z_j \in G_2 \cup G_3$, because every vertex z_3 is adjacent to a vertex x_1^0 and $\text{dist}_{E_1}(x_1^0, y_i) \leq 2$, which yields the inequality above. Hence $\text{dist}_{E_1}(x_i, y_j) \leq 3$ for any $x_i, y_j \in V$. The elements at distance 3 are again 0_2 and $(n+1)_2$ as in the previous case, since again $N_{E_1}(0_2) = \{0_1, 2_1, \ldots, (2n-2)_1, 0_3, 2_3, \ldots, (2n-2)_3\}$ while $N_{E_1}((n+1)_2) = \{1_1, 3_1, \ldots, (2n-1)_1, 1_3, 3_3, \ldots, (2n-1)_3\}$. Thus $N_{E_1}(x_2) \cap N_{E_1}(y_2) = \emptyset$ and $\text{dist}_{E_1}(0_2, (n+1)_2) \geq 3$, which completes the construction. □

For $d = 4$ we consider again the $3-TD(2n)$ from the additive group $\mathbb{Z}_{2n}$. It is much easier now to show that the factors have diameter 4. Since from Corollary 3.1.6 it follows that a selfcomplementary factor of a GDD can have diameter at most 4, we only need to show that there is a pair of elements whose distance is 4.

Construction 3.2.2. $(d = 4)$ The factor E_1 consists of blocks

$(0_1, 1_2, 1_3), (0_1, 2_2, 2_3), (0_1, 3_2, 3_3), \ldots, (0_1, (2n-1)_2, (2n-1)_3)$,

$(1_1, 0_2, 1_3), (1_1, 1_2, 2_3), (1_1, 2_2, 3_3), \ldots, (1_1, (n-2)_2, (n-1)_3), (1_1, n_2, (n+1)_3)$,

$(1_1, (n+1)_2, (n+2)_3), \ldots, (1_1, (2n-1)_2, 0_3)$,

...

...

$(i_1, 0_2, i_3), (i_1, 1_2, (i+1)_3), (i_1, 2_2, (i+2)_3), \ldots, (i_1, (n-i-1)_2, (n-1)_3), (i_1, (n-i+1)_2, (n+1)_3), (1_1, (n-i+2)_2, (n+2)_3), \ldots, (i_1, (2n-1)_2, (2n+i-1)_3)$,

...

...

$((n-1)_1, 0_2, (n-1)_3), ((n-1)_1, 2_2, (n+1)_3), ((n-1)_1, 3_2, (n+2)_3), \ldots, ((n-1)_1, (2n-1)_2, (n-2)_3)$, and

$(n_1, 0_2, n_3), ((n+1)_1, (n-1)_2, 0_3), ((n+2)_1, (n-2)_2, 0_3) \ldots, ((2n-1)_1, 1_2, 0_3)$.

We again present the factor E_1 in Figure 3.2.3 as the sub-array of the array of $\mathbb{Z}_{2n}$.

The factor E_2 contains all blocks not contained in E_1 and the self-complementing permutation is $\phi(x_1) = (n+x)_1, \phi(x_2) = x_2, \phi(x_3) = (n+x)_3$. To prove that $\operatorname{diam} E_1 = 4$ we only need to observe that $\operatorname{dist}_{E_1}(n_1, (n+1)_1) > 3$. Because n_1 is adjacent only to 0_2 and n_3 while $(n+1)_1$ is adjacent to $(n-1)_2$ and 0_3, we can see that $\operatorname{dist}_{E_1}(n_1, (n+1)_1) > 2$. Moreover, there is no block in E_1 containing either the pair $0_2, 0_3$ or $(n-1)_2, n_3$. Since the $3-TD(2n)$ contains no block with two elements of the same group, no neighbour of n_1 is adjacent to any neighbour of $(n+1)_1$, which yields $\operatorname{dist}_{E_1}(n_1, (n+1)_1) > 3$. Because $\operatorname{diam} E_1 \leq 4$, the construction indeed yields factors of diameter 4. □

Construction 3.2.3. ($d = \infty$) The factor E_1 consists of all blocks (x_1, y_2, z_3) with $x = 1, 2, \ldots, n$ while E_2 contains the blocks (x_1, y_2, z_3) with $x = n+1, n+2, \ldots, 2n$. The self-complementing permutation is the same as in the previous construction: $\phi(x_1) = (n+x)_1, \phi(x_2) = x_2, \phi(x_3) = (n+x)_3$. The factor E_1 is indeed disconnected, because the elements $(n+1)_1, (n+2)_1, \ldots, (2n)_1$ are not contained in any block. □

Using the constructions, we can state the following result.

	0	1	2	...	n-2	n-1	n	n+1	n+2	...	2n-2	2n-1
0	-	1	2	...	n-2	n-1	n	n+1	n+2	...	2n-2	2n-1
1	1	2	3	...	n-1	-	n+1	n+2	n+3	...	2n-1	0
2	2	3	4	...	-	n+1	n+2	n+3	n+4	...	0	1
3	3	4	5	...	n+1	n+2	n+3	n+4	n+5	...	1	2
.												
.												
n-2	n-2	n-1	-	...	2n-4	2n-3	2n-2	2n-1	0	...	n-4	n-3
n-1	n-1	-	n+1	...	2n-3	2n-2	2n-1	0	1	...	n-3	n-2
n	n	-	-	-	-	-	-	-	-	-	-	-
n+1	-	-	-	-	-	0	-	-	-	-	-	-
n+2	-	-	-	-	0	-	-	-	-	-	-	-
n+3	-	-	-	-	-	-	-	-	-	-	-	-
.												
.												
2n-2	-	-	0	-	-	-	-	-	-	-	-	-
2n-1	-	0	-	-	-	-	-	-	-	-	-	-

Figure 3.2.3

Theorem 3.2.4. *For every* $n \geq 2$ *and* $d = 3, 4$ *and* ∞ *there exists a* $(2, d)$*-isodecomposable* $3 - TD(2n)$.

Let us remark that the $3 - TD(2)$ arising from $\mathbb{Z}_2$ (which is unique up to isomorphism), is isodecomposable only into disconnected factors.

3.3 An isodecomposable $3 - GDD(4, 4)$

Franek, Mathon and Rosa proved that there are exactly 23 nonisomorphic GDD's with 4 groups of size 4 and block size 3. We choose the most "symmetric" of them, with the largest automorphism group, and show that it is isodecomposable into factors with diameters 3,4, and ∞. The elements are $0, 1, 2, \ldots, 15$, the groups are $G_1 = \{0, 1, 2, 3\}$, $G_2 = \{4, 5, 6, 7\}$, $G_3 = \{8, 9, 10, 11\}$, $G_4 = \{12, 13, 14, 15\}$. The set of blocks, $\mathcal{B}$, consists of 32 blocks:

$(0, 4, 15), (0, 5, 8), (0, 6, 11), (0, 7, 12), (0, 9, 13), (0, 10, 14), (1, 4, 9), (1, 5, 15),$
$(1, 6, 12), (1, 7, 10), (1, 8, 13), (1, 11, 14), (2, 4, 14), (2, 5, 10), (2, 6, 9), (2, 7, 13),$
$(2, 8, 15), (2, 11, 12), (3, 4, 11), (3, 5, 14), (3, 6, 13), (3, 7, 8), (3, 9, 15), (3, 10, 12),$
$(4, 8, 12), (4, 10, 13), (5, 9, 12), (5, 11, 13), (6, 8, 14), (6, 10, 15), (7, 9, 14), (7, 11, 15).$

One of the 96 automorphisms, ϕ, is given by $\phi = (0\ 3)(1\ 2)(4\ 6)(5\ 7)(8)(9)(10)(11)$ $(12\ 14)(13\ 15)$.

The factor E_1 with the blocks

$(0, 4, 15), (0, 5, 8), (0, 6, 11), (0, 7, 12)(1, 8, 13), (1, 11, 14), (2, 4, 14), (2, 5, 10),$
$(2, 6, 9), (2, 7, 13), (3, 9, 15), (3, 10, 12), (4, 8, 12), (4, 10, 13), (5, 9, 12), (5, 11, 13)$

has diameter 3. Let $V_i(x)$ be a set of all elements having in E_1 distance i from x. Then $V_1(0) = \{4, 5, 6, 7, 8, 11, 12, 15\}$ and all elements not belonging to $V_1(0)$ have neighbours in $V_1(0)$: 1 and 13 belong to the block $(1, 8, 13)$, 2 and 14 belong to $(2, 4, 14)$, 3 and 9 to $(3, 9, 15)$ and 10 to $(3, 10, 12)$. So $V_2(0) = \{1, 2, 3, 9, 10, 13, 14\}$ and $V_1(0) \cup V_2(0) = V$. Therefore $\mathrm{ex}_{E_1} 0 = 2$ and all elements of $V_1(0)$ have in E_1 eccentricity at most 2 and we have only to check the distances $\mathrm{dist}_{E_1}(x, y)$ for all pairs $x, y \in V_2(0)$. The distance $\mathrm{dist}_{E_1}(1, 3) = 3$, because $V_{E_1}(1) = \{8, 11, 13, 14\}$, $V_{E_1}(3) = \{9, 10, 12, 15\}$ and 8 and 12 belong to the block $(4, 8, 12)$. Similarly,

$\text{dist}_{E_1}(1,9) = 3$, because $V_{E_1}(9) = \{2,3,5,6,12,15\}$, and 8 and 12 belong to the common block $(4,8,12)$. The other elements of $V_2(0)$ not belonging to $V_1(1)$ have a neighbour in $V_1(1)$, namely 2 belongs to $(2,4,14)$ and 10 belongs to $(4,10,13)$. Thus $\text{ex}_{E_1} 1 = 3$. Since the element 2 is adjacent to all elements of $V_2(0)$ with the exception of the elements 1 and 3, their mutual distance is at most 2. Because $\text{ex}_{E_1} 1 = 3$, we must show only that $\text{dist}_{E_1}(3,x) \leq 3$ for $x = 9,10,13,14$. This is true, since E_1 contains the blocks $(3,9,15)$ and $(3,10,12)$, which yields $\text{dist}_{E_1}(3,9) = \text{dist}_{E_1}(3,10) = 1$, and 13 is adjacent to 10 in $(4,10,13)$. Finally, there is the block $(4,8,12)$ which has elements in common with both $(3,10,12)$ and $(2,4,14)$ which yields $\text{dist}_{E_1}(3,14) = 3$. Thus $\text{ex}_{E_1} x \leq 3$ for every element $x \in V$ and $\text{diam}\, E_1 = 3$. The factor E_2 contains all blocks not contained in E_1 and one can check that the automorphism ϕ is the self-complementing isomorphism.

For the diameter 4, the case is simpler. Because we know that every connected selfcomplementary factor has diameter at most 4, we have to show only that the factor F_1 described below is connected, contains a pair of elements having distance 4 and its complement, F_2, is isomorphic to F_1. The factor F_1 contains the blocks

$(0,4,15),(0,5,8),(0,6,11),(0,7,12),(0,10,14),(1,11,14),(2,4,14),(2,5,10),$
$(2,6,9),(2,7,13),(2,8,15),(3,9,15),(4,8,12),(4,10,13),(5,9,12),(5,11,13)$

and $\text{dist}_{F_1}(1,3) = 4$. Really, $V_1(3) = \{9,15\}, V_1(1) = \{11,14\}, V_2(1) = \{0,2,4,5,6,10,13\}$ and therefore $\text{dist}_{F_1}(1,3) \geq 4$. Because there is the path $1-(1,11,14)-11-(0,6,11)-6-(2,6,9)-9-(3,9,5)-3$ of length 4, we can see that $\text{dist}_{F_1}(1,3) = 4$. To prove connectivity, we observe that $V_1(0) = \{4,5,6,7,8,10,11,12,14,15\}$. All other elements belong to $V_2(0)$, because the element 1 appears in the block $(1,11,14)$, 3 and 9 appear in $(3,9,15)$, and 2 and 13 appear in $(2,7,13)$. Thus F_1 is connected. The factor F_2 containing all triples that are not in F_1 is isomorphic to F_1—the isomorphism is again the automorphism ϕ described above.

The factor I_1 with the blocks
$(0,4,15),(0,5,8),(0,6,11),(0,7,12),(0,9,13),(0,10,14),(1,4,9),(1,5,15),$
$(1,6,12),(1,7,10),(1,8,13),(1,11,14),(4,8,12),(4,10,13),(5,9,12),(5,11,13),$
is clearly disconnected, because the elements 2 and 3 are not contained in any block and are therefore isolated. The isomorphism from I_1 to its complement I_2 is again ϕ, the automorphism of the $3-GDD(4,4)$.

The case of decomposition into two factors with diameter 2 remains in doubt.

3.4. Isomorphic decompositions of $3-TD$'s into small connected factors

In this section we study decompositions into the smallest possible isomorphic connected factors. It is not difficult to observe that the smallest connected factor is acyclic. If a $3-TD(n)$ has such a factor $E(s)$ with s blocks, it is obvious that it contains $2s+1$ elements and therefore the number of elements of the $3-TD(n)$, $3n$, must be equal to $2s+1$. Hence $s=\frac{3n-1}{2}$ and n must be an odd number. So we can state the following simple observation.

Proposition 3.4.1. *A $3-TD(n)$ has a connected acyclic factor only if n is odd.*

Let us suppose now that n is odd, say $2m+1$. Then the number of blocks of the factor $E(s)$ is $s=\frac{3n-1}{2}=\frac{3(2m+1)-1}{2}=3m+1$. Since the number of blocks of the $3-TD(2m+1)$ is $(2m+1)^2$, the $3-TD(2m+1)$ is decomposable into connected acyclic factors only if $3m+1 \mid (2m+1)^2$. Suppose it is the case. Then there is a positive number k such that $(2m+1)^2 = k(3m+1)$. We can write $k = tm+1$, where $0 \leq t \in \mathbb{Q}$. Then we have $4m^2+4m+1 = (tm+1)(3m+1) = 3tm^2+(t+3)m+1$, which yields $4(m+1) = 3tm+t+3$. Hence $4m-3tm = t-1$ and $m = \frac{t-1}{4-3t}$. Since m is a non-negative integer and the fraction is negative for all $t \neq 1$, we are left with $t = 1$, which yields $m = 0$. Then $n = 1$ and the following holds.

Proposition 3.4.2. *No $3-TD(n)$ with $n > 1$ is decomposable into connected acyclic factors.*

Let us consider now connected factors of $3-TD(2m+1)$'s with $3m+2$ blocks. The $3-TD(3)$ of the additive group $\mathbb{Z}_3$ with groups $G_1 = \{0_1, 1_1, 2_1\}, G_2 = \{0_2, 1_2, 2_2\}$ and $G_3 = \{0_3, 1_3, 2_3\}$ and blocks $(0_1, 0_2, 0_3), (0_1, 1_2, 1_3), (0_1, 2_2, 2_3), (1_1, 0_2, 1_3), (1_1, 1_2, 2_3), (1_1, 2_2, 0_3), (2_1, 0_2, 2_3), (2_1, 1_2, 0_3), (2_1, 2_2, 1_3)$ has a connected factor $E(5)$ with $3m+2$ blocks, e.g., $(0_1, 0_2, 0_3), (0_1, 1_2, 1_3), (1_1, 0_2, 1_3), (2_1, 0_2, 2_3), (2_1, 2_2, 1_3)$. The factor $E(5)$ contains two cycles: $0_1 - (0_1, 0_2, 0_3) - 0_2 - (1_1, 0_2, 1_3) - 1_3 - (0_1, 1_2, 1_3) - 0_1$ and $2_1 - (2_1, 0_2, 2_3) - 0_2 - (1_1, 0_2, 1_3) - 1_3 - (2_1, 2_2, 1_3) - 2_1$, and is therefore not the "simplest possible", i.e., unicyclic.

A necessary condition for decomposability into unicyclic factors follows.

Lemma 3.4.3. *If a $3-TD(n)$ is decomposable into unicyclic factors, then $n \equiv 0 \pmod 6$.*

Proof. Let $E(s)$ be a unicyclic factor with s blocks. The shortest cycle, C_3, consists of 3 blocks that contain together 6 elements. Since every other block contributes 2 to the number of elements, we have $s = \frac{3n}{2}$. Therefore n must be even. On the other hand, the number of blocks of the factor must divide the number of blocks of the $3-TD(n)$, i.e., $3\frac{n}{2} \mid n^2$. This yields $3 \mid n$ and hence $n \equiv 0 (\bmod 6)$. □

We show further that for every $n \equiv 0(\bmod 6)$ there is a decomposable $3-TD(n)$. We even show that the factors can be mutually isomorphic. But first we state the following.

Corollary 3.4.4. *If a $3-TD(n)$ is decomposable into t connected factors of size t, then $t \geq 3\frac{n}{2}$. The equality can hold only if $n \equiv 0(\bmod 6)$.*

Now we present constructions of $3-TD$'s that are isodecomposable into unicyclic factors, namely cycles. We start with the case $n \equiv 6(\bmod 12)$.

Construction 3.4.5. $n \equiv 6(\bmod 12)$. Let $n = 12m+6$. First we construct a Latin square A of order $6m+3$ as follows. The first row is $1, 3m+3, 2, 3m+4, 3, \ldots, 3m+1, 6m+3, 3m+2$. An entry in i-th row and j-th column, $a^{i,j}$, is then equal to $a^{1,i+j-1}$. Then we construct a Latin square C of order $12m+6$ with entries $c^{i,j} = a^{i,j}$ for $1 \leq i,j \leq 6m+3$, $c^{i,j} = a^{i-6m-3,j}$ for $6m+4 \leq i \leq 12m+6, 1 \leq j \leq 6m+3$, $c^{i,j} = a^{i,j-6m-3}$ for $1 \leq i \leq 6m+3$, $6m+4 \leq j \leq 12m+6$, and $c^{i,j} = a^{i-6m-3,j-6m-3}$ for $6m+4 \leq i,j \leq 12m+6$. The triples of the $3-TD(12m+6)$ are then $(i_1, j_2, c_3^{i,j})$. One can notice that the Latin square C is a multiplication array of a commutative half-idempotent quasigroup. An example of the Latin square C is shown in Figure 3.4.1. Since the third element of a triple is determined uniquely, we usually write just (i_1, j_2, c_3).

The factor E_0 contains the blocks (i_1, i_2, c_3) for $i = 1, 2, \ldots, 12m+6$, the block $(1_1, (12m+6)_2, c_3)$ and the blocks $(j_1, (j+6m+2)_2, c_3)$ for $j = 2, 3, \ldots, 6m+3$. Then E_0 is the cycle $1_1 - (1_1, 1_2, 1_3) - 1_3 - ((6m+4)_1, (6m+4)_2, 1_3) - (6m+4)_2 -$

	1	2	3	4	5	6
1	1	3	2	4	6	5
2	3	2	1	6	5	4
3	2	1	3	5	4	6
4	4	6	5	1	3	2
5	6	5	4	3	2	1
6	5	4	6	2	1	3

Figure 3.4.1

$(2_1,(6m+4)_2,c_3) - 2_1 - (2_1,2_2,2_3) - 2_3 - \cdots - i_1 - (i_1,i_2,i_3) - i_3 - ((6m+3+i)_1,(6m+3+i)_2,i_3) - (6m+3+i)_2 - ((i+1)_1,(6m+3+i)_2,c_3) - (i+1)_1 - ((i+1)_1,(i+1)_2,(i+1)_3) - (i+1)_3 - \cdots - ((12m+6)_1,(12m+6)_2,(6m+3)_3) - (12m+6)_2 - (1_1,(12m+6)_2,c_3) - 1_1$.

The factor E_1 is determined by the isomorphism $\psi_1 : E_0 \to E_1$ with $\psi_1(x_1) = x_1, \psi_1(y_2) = (y+6m+3)_2, \psi_1(z_3) = (z+6m+3)_3$.

E_2 is determined by $\psi_2 : E_0 \to E_2$, where

$\psi_2(1_1) = (6m+3)_2, \psi_2(2_1) = 1_2, \psi_2(3_1) = 2_2, \ldots, \psi_2((6m+3)_1) = (6m+2)_2$,

$\psi_2((6m+4)_1) = (12m+6)_2, \psi_2((6m+5)_1) = (6m+4)_2, \psi_2((6m+6)_1) = (6m+5)_2, \ldots, \psi_2((12m+6)_1) = (12m+5)_2$,

$\psi_2(1_2) = 2_1, \psi_2(2_2) = 3_1, \psi_2(3_2) = 4_1, \ldots, \psi_2((6m+3)_2) = 1_1$,

$\psi_2((6m+4)_2) = (6m+5)_1, \psi_2((6m+5)_2) = (6m+6)_1, \ldots, \psi_2((12m+6)_2) = (6m+4)_1$,

$\psi_2(z_3) = z_3$.

E_4 is determined by $\psi_4 : E_0 \to E_4$, where

$\psi_4(1_1) = 4_1, \psi_4(2_1) = 5_1, \psi_4(3_1) = 6_1, \ldots, \psi_4((6m+3)_1) = 3_1$,

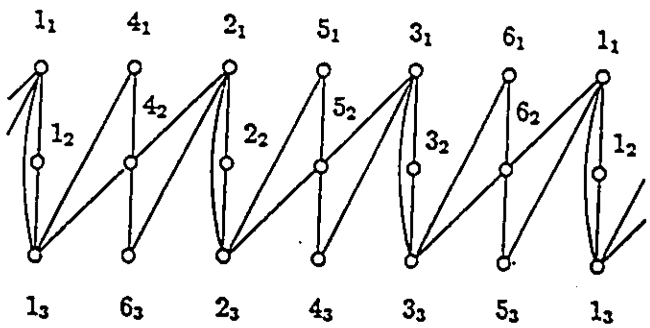


Figure 3.4.2

$\psi_4((6m+4)_1) = (6m+7)_1, \psi_4((6m+5)_1) = (6m+8)_1, \ldots, \psi_4((12m+6)_1) = (6m+6)_1$,

$\psi_4(y_2) = y_2$,

$\psi_4(1_3) = (3m+4)_3, \psi_4(2_3) = (3m+5)_3, \ldots, \psi_4((6m+3)_3) = (3m+3)_3$,

$\psi_4((6m+4)_3) = (9m+7)_3, \psi_4((6m+5)_3) = (9m+8)_3, \ldots, \psi_4((12m+6)_3) = (9m+6)_3$.

In general, a factor E_t, where $t = 4u+2v+w$, $1 \le t \le 12m+5 = n-1$, is determined by an isomorphism $\phi_t : E_0 \to E_t$, which is defined as the composition $\phi_t = \psi_4^u \circ \psi_2^v \circ \psi_1^w$, with $\psi_j^0 = id$.

For $n = 6$, the underlying factor $U(E_0)$ is shown in Figure 3.4.2 and the arrays corresponding to all factors are shown in Figure 3.4.3.

In the case $n \equiv 0 (\mathrm{mod}\, 12)$ we construct a Latin square corresponding to a non-commutative half-idempotent quasigroup.

Construction 3.4.6. $n \equiv 0(\mathrm{mod}\, 12)$. Let $n = 12m$. First we construct an array B of order $6m$. The main diagonal is defined by $b^{i,i} = i$, $i = 1, 2, \ldots, 6m$. The entries $b^{i,j}$, where $i - j \equiv 0(\mathrm{mod}\, 2)$ are defined as follows. Let $2l = i - j (\mathrm{mod}\, 6m)$, then $b^{i,j} = b^{j,j} + l$. To define the entries $b^{i,j}$, where $i - j \equiv 1(\mathrm{mod}\, 2)$, we define

E_0	1	2	3	4	5	6
1	1					5
2		2		6		
3			3		4	
4				1		
5					2	
6						3

E_1	1	2	3	4	5	6
1			2	4		
2	3				5	
3		1				6
4	4					
5		5				
6			6			

E_2	1	2	3	4	5	6
1		3				
2			1			
3	2					
4			5		3	
5	6					1
6		4		2		

E_3	1	2	3	4	5	6
1					6	
2						4
3				5		
4		6				2
5			4	3		
6	5				1	

Figure 3.4.3

$\hat{b}^{p,q}$ as the number of the set $\{1,2,\ldots,6m\}$ such that $b^{p,q} \equiv \hat{b}^{p,q} (\bmod 6m)$. Then $b^{i,j} = \hat{b}^{i-1,j} + 6m$, i.e., $b^{p,q} \in \{6m+1, 6m+2, \ldots, 12m\}$.

Then we construct a Latin square D of order $12m$ with entries $d^{i,j} = b^{i,j}$ for $1 \leq i,j \leq 6m$, $d^{i,j} = b^{i-6m,j}$ for $6m+1 \leq i \leq 12m, 1 \leq j \leq 6m$, $d^{i,j} = b^{i,j-6m}$ for $1 \leq i \leq 6m$, $6m+1 \leq j \leq 12m$, and $d^{i,j} = b^{i-6m,j-6m}$ for $6m+1 \leq i,j \leq 12m$. The triples of the $3-TD(12m)$ are then $(i_1, j_2, d_3^{i,j})$. An example of the Latin square D of order 12 is shown in Figure 3.4.4. We again write usually just (i_1, j_2, d_3) instead

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	7	5	8	6	9	4	10	2	11	3	12
2	10	2	8	6	9	1	7	5	11	3	12	4
3	2	11	3	9	1	10	5	8	6	12	4	7
4	11	3	12	4	10	2	8	6	9	1	7	5
5	3	12	4	7	5	11	6	9	1	10	2	8
6	12	4	7	5	8	6	9	1	10	2	11	3
7	4	10	2	11	3	12	1	7	5	8	6	9
8	7	5	11	2	12	4	10	2	8	6	9	1
9	5	8	6	12	4	7	2	11	3	9	1	10
10	8	6	9	1	7	5	11	3	12	4	10	2
11	6	9	1	10	2	8	3	12	4	7	5	11
12	9	1	10	2	11	3	12	4	7	5	8	6

Figure 3.4.4

of $(i_1, j_2, d_3^{i,j})$.

The factor E_0 contains the blocks (i_1, i_2, i_3) for $i = 1, 2, \ldots, 12m$, the block $(1_1, (12m)_2, d_3)$ and the blocks $(j_1, (j-1+6m)_2, (j-1+6m)_3)$ for $j = 2, 3, \ldots, 6m$. Then E_0 is the cycle $1_1 - (1_1, 1_2, 1_3) - 1_3 - ((6m+1)_1, (6m+1)_2, 1_3) - (6m+1)_2 - (2_1, (6m+1)_2, (6m+1)_3) - 2_1 - (2_1, 2_2, 2_3) - 2_3 - \cdots - i_1 - (i_1, i_2, i_3) - i_3 - ((6m+i)_1, (6m+i)_2, i_3) - (6m+i)_2 - ((i+1)_1, (6m+i)_2, (6m+i)_3) - (i+1)_1 - ((i+1)_1, (i+1)_2, (i+1)_3) - (i+1)_3 - \cdots - ((12m)_1, (12m)_2, (6m)_3) - (12m)_2 - (1_1, (12m)_2, (12m)_3) - 1_1$. The other factors are defined similarly as in the case $n \equiv 6 \pmod{12}$. For $n = 12$, the underlying factor $U(E_0)$ is shown in Figure 3.4.5. The factor E_1 is determined by the isomorphism $\psi_1 : E_0 \to E_1$ with $\psi_1(x_1) =$

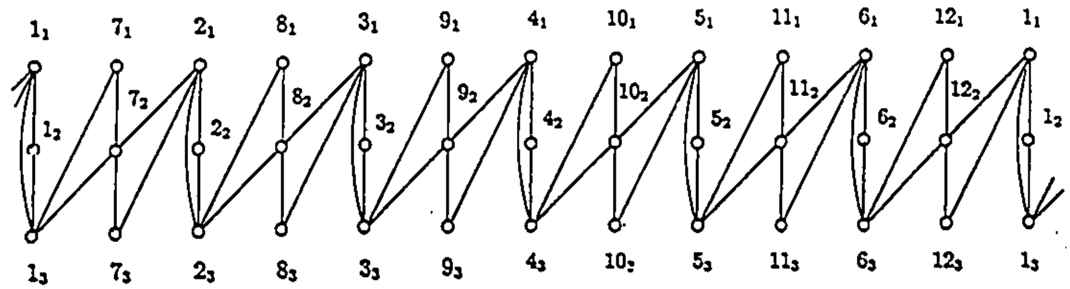


Figure 3.4.5

$x_1, \psi_1(y_2) = (y + 6m)_2, \psi_1(z_3) = (z + 6m)_3$.

E_2 is determined by $\psi_2 : E_0 \to E_2$, where $\psi_2(1_1) = (6m)_2, \psi_2(2_1) = 1_2, \psi_2(3_1) = 2_2, \ldots, \psi_2((6m)_1) = (6m-1)_2$,

$\psi_2((6m+1)_1) = (12m)_2, \psi_2((6m+2)_1) = (6m+1)_2, \psi_2((6m+3)_1) = (6m+2)_2, \ldots, \psi_2((12m)_1) = (12m-1)_2$,

$\psi_2(1_2) = 2_1, \psi_2(2_2) = 3_1, \psi_2(3_2) = 4_1, \ldots, \psi_2((6m)_2) = 1_1$,

$\psi_2((6m+1)_2) = (6m+2)_1, \psi_2((6m+2)_2) = (6m+3)_1, \ldots, \psi_2((12m)_2) = (6m+1)_1$,

$\psi_2(z_3) = z_3$.

E_4 is determined by $\psi_4 : E_0 \to E_4$, where $\psi_4(1_1) = 4_1, \psi_4(2_1) = 5_1$, $\psi_4(3_1) = 6_1, \ldots, \psi_4((6m)_1) = 3_1$,

$\psi_4((6m+1)_1) = (6m+4)_1, \psi_4((6m+2)_1) = (6m+5)_1, \ldots, \psi_4((12m)_1) = (6m+3)_1$,

$\psi_4(y_2) = y_2$,

$\psi_4(1_3) = (9m+2)_3, \psi_4(2_3) = (2m+3)_3, \ldots, \psi_4((6m)_3) = (9m+1)_3$,

$\psi_4((6m+1)_3) = (6m)_3, \psi_4((6m+2)_3) = 1_3, \ldots, \psi_4((12m)_3) = (6m-1)_3$.

In general, a factor E_t, where $t = 4u + 2v + w,\ 1 \le t \le 12m - 1 = n - 1$, is again determined by the isomorphism $\phi_t : E_0 \to E_t$, which is defined as the composition $\phi_t = \psi_4^u \circ \psi_2^v \circ \psi_1^w$, with $\psi_j^0 = id$.

E_0	1	2	3	4	5	6	7	8	9	10	11	12
1	1											12
2		2					7					
3			3					8				
4				4					9			
5					5					10		
6						6					11	
7							1					
8								2				
9									3			
10										4		
11											5	
12												6

E_2	1	2	3	4	5	6	7	8	9	10	11	12
1					6							
2						1						
3	2											
4		3										
5			4									
6				5								
7						12					6	
8	7											1
9		8					2					
10			9					3				
11				10					4			
12					11					5		

E_1	1	2	3	4	5	6	7	8	9	10	11	12
1						9	4					
2	10							5				
3		11							6			
4			12							1		
5				7							2	
6					8							3
7	4											
8		5										
9			6									
10				1								
11					2							
12						3						

E_3	1	2	3	4	5	6	7	8	9	10	11	12
1											3	
2												4
3							5					
4								6				
5									1			
6										2		
7					3							9
8						4	10					
9	5							11				
10		6							12			
11			1							7		
12				2							8	

Figure 3.4.6

	1	2	3	4	5	6	7	8	9	10	11	12
1	1				6	9	4				3	12
2	10	2				1	7	5				4
3	2	11	3				5	8	6			
4		3	12	4				6	9	1		
5			4	7	5				1	10	2	
6				5	8	6				2	11	3
7	4				3	12	1					
8	7	5				4	10	2				
9	5	8	6				2	11	3			
10		6	9	1				3	12	4		
11			1	10	2				4	7	5	
12				2	11	3				5	8	6

Figure 3.4.7

For $n = 12$, the arrays corresponding to the factors E_0, E_1, E_2, E_3 are shown in Figure 3.4.6. Figure 3.4.7 shows the array with the factors E_0, E_1, E_2, E_3 together.

Since we proved that for every $n \equiv 0 (\mathrm{mod}\, 12)$ there exists a $3 - TD(n)$ which is isodecomposable into cycles, the complete characterization of TD's that are isodecomposable into unicyclic factors follows immediately from the constructions and Lemma 3.4.3.

Theorem 3.4.7. *A transversal design with group size n and block size 3 isodecomposable into unicyclic factors exists if and only if $n \equiv 0 (\mathrm{mod}\, 6)$. Moreover, for each such n there exists a $3 - TD(n)$ isodecomposable into cycles.*

4. Conclusion

Decompositions of complete graphs and later complete multipartite graphs into factors with given diameters were studied since 1966, when the first paper [6] was presented. Decompositions of complete graphs have been studied very extensively, and many authors studied decompositions of complete graphs into isomorphic factors with given diameters.

There were also several papers on decompositions of complete multipartite graphs into factors with given diameters, but none of them considered isomorphic factors. Such decompositions are therefore studied in the present thesis. In particular, we are dealing with decompositions of complete multipartite graphs into two isomorphic factors with given diameters, either finite or ∞. The possible finite diameters of such factors were determined by Tomová [27] and Gangopadhyay [10].

Although we were mostly interested in decompositions into connected factors, some results for $d = \infty$ were also obtained. We proved that every strongly admissible multipartite graph (i.e., a graph $K_{m_1^{p_1}\dots m_s^{p_s} n_1^{q_1}\dots n_t^{q_t}}$, where each m_i is odd, each n_j is even and at most one of $p_1, \dots p_s$ is odd) is decomposable into two isomorphic disconnected factors. In particular, every bipartite complete graph with at least 3 vertices and tripartite graph with at least 5 vertices is decomposable in such factors, providing that the number of edges of the graph is even.

In the case of bipartite and tripartite graphs we also completely determined all graphs decomposable into two isomorphic factors for every possible finite diameter. The case of four-partite graphs with at most one odd part was also solved completely. The remaining case of graphs K_{m_1,m_2,m_3,m_4}, where all numbers m_1, m_2, m_3, m_4 are odd, splits into several subcases. No graph K_{m_1,m_2,m_3,m_4} with all odd parts is decomposable into 2 isomorphic factors with diameter 5. For the diameters 2,3 and 4 the subcase where $m_1 = m_2$ and $m_3 = m_4$ was solved completely, as well as the subcase $m_1 = m_2 = m_3$. The subcase where the set $\{m_1, m_2, m_3, m_4\}$ contains at least 3 different numbers remains for $d = 2, 3, 4$ open.

For r-partite graphs with $r \geq 5$ we determined smallest graphs decomposable into two isomorphic factors for every possible diameter. We also showed that if for a given diameter d there exists a complete r-partite graph with p_0 vertices isodecomposable into two factors with the diameter d, then for every number of vertices $p > p_0$ such a graph with p vertices exists, too.

Decompositions of hypergraphs into factors with given diameters were also studied. Recently several authors [8, 14, 19] published results on decompositions of designs into two isomorphic factors, but none of them was particularly interested in the diameters of the factors. We attempted to open two directions in the research of decompositions of designs into isomorphic factors.

The first area includes decompositions of group divisible designs into two isomorphic factors with given diameters. We proved that the diameter of the connected factors can be at most 4, providing each block has at least three elements. We also presented for any even number n and the diameters $d = 3, 4$ and ∞ a group divisible design with three groups of size n and blocks of size 3, i.e., a $3-GDD(n,3)$, that is decomposable into two isomorphic factors with the diameter d. The diameter 2 remains in doubt.

We also studied isomorphic decompositions of $3-GDD$'s into smallest connected factors. There is no $3-GDD(n,3)$ decomposable into mutually isomorphic connected acyclic factors. The decomposition into isomorphic unicyclic factors is also impossible unless $n \equiv 0 \pmod 6$. For each such n there exists decomposition into cycles.

In both of the above mentioned directions many other interesting questions remain open. For instance, one such question is, for which triples n, k, d does there exist $k-GDD(n,k)$ decomposable into two isomorphic factors with the diameter d. Another question concerns decomposability of GDD's into more than two factors with given diameters, including factors which are not necessarily isomorphic. The question of smallest mutually isomorphic connected factors decomposing $k-GDD(n,k)$ for $n \not\equiv 0 \pmod 6$ can be interesting as well. We can also decompose complete multipartite graphs with odd number of edges or GDD's with odd number of blocks into almost selfcomplementary factors, in analogy to the approach of Das [7] and Das and Rosa [8], respectively. Many other areas remain virtually untouched and may be challenging both for the author and the reader.

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