THE SCALING LIMIT OF LATTICE TREES ABOVE EIGHT DIMENSIONS

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Abstract

This work is concerned with the behavior of n-bond trees in a regular d dimensional lattice for large n. A lattice tree is, by definition, a connected cluster of bonds with no closed loops.

The results presented herein are for a 'spread-out' model in $d > 8$; this model differs from the nearest-neighbour model in that the 'bonds' are chosen uniformly from $\{x, y \in \mathbb{Z}^d : \max_{1\leq i\leq d} |x_i - y_i| \leq L\}$ where L will be taken large enough for a variant of the lace expansion (as adapted in [HS3 & 4]) to converge.

By way of comparison, it has been recently proved [HS1] via the lace expansion that, above 4 dimensions, n step self-avoiding walks on the hypercubic lattice $\mathbb{Z}^d$ converge (in distribution) to Brownian motion when space is scaled down by $n^{1/2}$, and n tends to infinity.

If we let $t_n(0, x)$ be the number of n bond trees connecting 0 and x, we can take its Fourier series $\hat{t}_n(k)$:

$$\hat{t}_n(k) \equiv \sum_{x\in\mathbb{Z}^d} t_n(0, x)e^{ix\cdot k},$$

where $k \in [-\pi, \pi]^d$. We prove that for L sufficiently large and $d > 8$ for the spread-out model,

$$\lim_{n\nearrow\infty} \frac{\hat{t}_n(k/\bar{D}n^{1/4})}{\hat{t}_n(0)} = \int_0^\infty dl\ le^{l^2/2-lk^2/2},$$

where $\bar{D}$ is related to the mean radius of gyration. This would correspond in x space to scaling space down by $n^{1/4}$ where n is the size of a tree as measured by the number of bonds it contains.

Similar calculations are carried out for trees connecting m points. The resulting distributions turn out to be exactly the characteristic functions of the measures D. Aldous conjectured in his 1993 J.S.P. paper regarding the embedding of random continuum trees in $\mathbb{R}^d$ — which are themselves related to variant of super Brownian motion known as integrated super-Brownian excursion (ISE).

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Thanks are also due to departmental computer gurus M. Valeriote, B. Hart, and A. Nicas for their invaluable computer expertise; many a student owes the timely completion of their thesis to their un-grudging help.

This work is dedicated:

To my family for their unconditional support.

To my teachers for believing in me.

To my friends and colleagues with whom I grew as a person
inside as well as outside of the class-room.

Contents

Chapter 1

Introduction

1.1 Overview of this work

Lattice trees can be viewed as models for branched polymers in theoretical chemistry, but more importantly from the mathematical physics stand-point, as models exhibiting features related to critical phenomena. Some of these features will be explained below, but the one feature which will be the focus of this work, is the existence of a scaling limit.

In general, the scaling limit will depend in an essential way on the spatial dimension in which one is working. For instance, it has recently been shown [HS1] that when the length n of a self-avoiding walk goes to infinity as space is scaled down by $n^{1/2}$, one gets convergence in distribution to Brownian motion if $d \geq 5$. For $d = 4$ the same is thought to occur (with a logarithmic correction to the $n^{1/2}$ power law), but for $d \leq 3$ the limit, if there is one, is not yet known.

The above is an example of a physical model which can be used to construct a probabilistic process – Brownian motion as it happens. The results in this work yield a connection with the scaling limit of lattice trees and a variant of super-processes called integrated super-Brownian excursion (ISE).

In particular, we show that under appropriate conditions, the scaling limit of lattice trees is distributed as ISE. The latter is a process which combines aspects of the spatial diffusion exhibited by Brownian motion, together with particle branching mechanisms [cf Chapter 5]. As a result of this work, lattice trees can now be viewed as a physical model whose scaling limit belongs to the class of super-processes known as ISE. It may be possible that the methods developed for this work, coupled with those in [BG1 & 2], may help in the study of the scaling limit of other models such as percolation. In fact T. Hara & G. Slade have conjectured in [DS] that for bond percolation, the scaling limit of the incipient infinite cluster is also distributed as ISE.

It should be mentioned that our investigation was greatly aided by a paper of D. Aldous [A1] (a probabilist), which contained the explicit distributions he conjectured should result from the scaling limit of lattice trees. That turned out to be a critical transfer of information since mathematical physicists had no concrete notion of what limit to expect, and probabilists had no technology to rigorously compute a scaling limit on the lattice. As will be described later, the technology that was up to the task turned out to be the lace expansion as adapted in [HS3&4].

Before describing our model, we should reiterate that the spatial dimension our model is cast in will play a crucial role. In fact, not only is it very difficult to construct a convergent expansion that can capture the behavior of lattice trees in high dimensions, but it is still difficult (even in lower dimensions) to construct algorithms to simulate them [RM] & [RJ]. The point is that even if there were a new expansion valid below $d = 8$ dimensions for lattice trees, one would have no *apriori* notion of what scaling limit to expect.

We will be working in the d-dimensional hypercubic lattice $\mathbb{Z}^d$, and will focus on a 'spread-out' model for $d > 8$, although we should mention that the results contained herein also apply to the nearest-neighbour (n.n.) model in sufficiently high dimensions. A lattice tree in $\mathbb{Z}^d$ is a connected cluster of bonds having no closed loops. In the spread-out model these bonds should be thought of as un-ordered pairs of sites, say $x, y \in \mathbb{Z}^d$, such that $\|x - y\|_\infty \leq L$; if $\|x - y\|_2 = 1$ we recover the nearest-neighbour model. We will say that a tree T contains the site 'x' if it is an endpoint of a bond in T, with the convention that $\{x\}$ is a zero bond tree.

The reason for looking at the spread-out model is that by taking L large enough, we can analyse it using a variant of the *lace* expansion if $d > 8$, the critical dimension above which mean-field behavior is expected (the concept of mean-field behavior will be discussed shortly). In fact, the main idea in this work is the adaptation of the lace expansion developed by Brydges and Spencer [BS], who originally used it to prove *mean-field* behavior for the weakly self-avoiding walk above four dimensions. The expansion was later improved by Hara and Slade [HS1] to deal with the fully self-avoiding walk (above four dimensions), and then adapted [HS2] to percolation, where the triangle condition [AN] was shown to hold in sufficiently high dimension (thus proving mean field behavior). Lately, the lace expansion was re-formulated in [HS3] and [HS4] to handle the problem of proving mean-field behavior for lattice trees and animals. There, the crucial bound was on the square diagram (to be discussed shortly), which was shown to be finite for $d > 8$ for the spread-out model, and in sufficiently high dimension for the nearest neighbour model. These adaptations, and some others to be presented later on, have ultimately enabled us to compute the scaling limit of of lattice trees under the above conditions.

Before discussing mean-field behavior for lattice trees we must define a parameter known as λ. Let a_n be the number of trees containing n bonds modulo translation in the lattice (thus only the shape of the tree matters). Then by sub-additivity arguments, it was shown [K] that there exists a constant λ such that

$$\sup_{n\geq 1} a_n^{1/n} \equiv \lambda = \lim_{n\to\infty} a_n^{1/n} \tag{1.1.1}$$

with $0 < \lambda < \infty$. It is thought that for $d \geq 2$,

$$a_n \overset{n\to\infty}{\sim} A\lambda^n n^{-\theta(d)} \tag{1.1.2}$$

except for a critical dimension (presumably $d = 8$ for lattice trees). Above this critical dimension, the *critical exponent* θ is believed to be constant. In fact, it was proven in [HS3&4] that if $d > d_0$ for some d_0 or if $L > L_0$ for some L_0 for spread-out trees above $d = 8$, then $\theta = 5/2$. Thus, the model exhibits the same qualitative behavior under the above conditions as for arbitrarily high d; the model is then said to exhibit mean-field behavior. If $d = 8$ the above expression is thought to need a logarithmic correction, but for $d < 8$, it is expected that θ will vary with d. The best bounds to date are

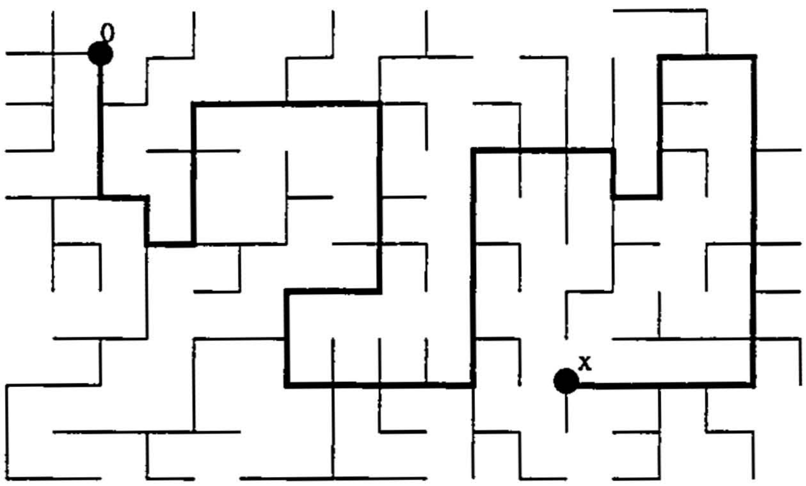
$$c_1 n^{-c_2 \log n} z_c^{-n} \leq a_n \leq c_3 n^{1/d-1} z_c^{-n},$$

which are proven in [JR] and [M] (respectively), and are probably not sharp.

There are other critical exponents for lattice trees which will be discussed in section 1.2, but one common to self-avoiding walks, percolation and lattice trees is related to the *susceptibility*. The susceptibility in turn is the sum over the lattice of the *2 point* function $G_z(0,x)$. The latter is the generating function for the number of n bond trees containing 0 and x. If $|T|$ denotes the number of bonds in a tree T, we have

$$G_z(0,x) = \sum_{T\ni 0,x} z^{|T|} \tag{1.1.3}$$

where the sum is over trees containing 0 and x as depicted below.



If we let $z_c \equiv \lambda^{-1}$, and

$$\Box(z) = \sum_{x,y,w\in\mathbf{Z}^d} G_z(0,x)G_z(x,y)G_z(y,w)G_z(w,0) - G_z(0,0)^4, \qquad (1.1.4)$$

we can state the square condition, that is $\lim_{z\nearrow z_c} \Box(z) < \infty$. If this condition holds it was shown [T,TH] that the susceptibility

$$\chi(z) \equiv \sum_{x\in\mathbf{Z}^d} G_z(0,x) \qquad (1.1.5)$$

is comparable to $(z_c - z)^{-1/2}$ for $z < z_c$, i.e. for some constants $0 < c_1 < c_2$,

$$c_1(z_c - z)^{-1/2} \leq \chi(z) \leq c_2(z_c - z)^{-1/2}. \qquad (1.1.6)$$

The singularity z_c is known as the *critical point.* It was proved in [HS4] that for $d > 8$ and L large enough for the spread out model (or d large in the nearest neighbour model):

$$\chi(z) = \frac{B}{(z_c - z)^\gamma} + \mathcal{E}(z), \qquad (1.1.7)$$

with $\gamma = 1/2$ and $|\mathcal{E}(z)| \leq$ const. $|z_c - z|^{\epsilon-1/2}$ for $|z| \leq z_c$, and any

$\epsilon < \min(1/2, (d-8)/4)$. There are corresponding expressions for $\chi(z)$ in percolation and self-avoiding walks, and the exponent governing their rates of divergence at their critical points is also denoted by γ.

We now give an overview of the main results in this work. In Chapter 3 we will be looking at $\chi_m(z)$, which we define as follows: let $\chi_1(z) \equiv g(z) = \sum_{T \ni 0} z^{|T|}$ be the generating function for trees containing the origin and $\chi_2(z) \equiv \chi(z)$, as in (1.1.5). Then, for $m \geq 3$, define

$$G_z(0, x_1, .., x_{m-1}) \equiv \sum_{T \ni 0, x_1, ..., x_{m-1}} z^{|T|} \tag{1.1.8}$$

to be the generating function for trees containing $\{0, x_1, .., x_{m-1}\}$. Now let $\chi_m(z) \equiv \sum_{x_1, ..., x_{m-1}} G_z(0, x_1, ..., x_{m-1})$. We now present the main result of chapter 3.

Theorem 1.1 *For $d > 8$ and L sufficiently large (or d sufficiently large for the nearest neighbour model), $m \geq 3$, and for $|z| < z_c$,*

$$\chi_m(z) = (2m-5)!!\chi^{2m-3}(z)v(z)^{m-2} + E_m(z), \tag{1.1.9}$$

where the un-subscripted $\chi(z)$'s were defined in (1.1.5). The following bounds also hold:

1) $|v(z)| \leq c$ for $|z| \leq z_c$

2) $\chi(z)^{2m-3}v(z)^{m-2} = \chi(z)^{2m-3}v(z_c)^{m-2} + \bar{E}(z)$ for $m \geq 3$, with $|\bar{E}(z)| \leq c|z_c - z|^{-m+3/2+\epsilon}$, for $|z| < z_c$ and any $\epsilon < \min(1/2, (d-8)/4)$.

3) $|E_m(z)| \leq c|z_c - z|^{-3/2} \cdot |z_c - |z||^{-m+3+\epsilon}$, for $|z| < z_c$ and $m \geq 4$. For $m = 3$, $|E_3(z)| \leq c|z_c - z|^{-1}$.

The main conclusion to be drawn from (1.1.9) is that trees connecting m distinct points, can be represented via a product of uncoupled generating functions mediated by vertex factors $v(z)$. In other words, starting from the most general structure for a trees' backbone, namely a binary backbone, one can connect m distinct points $\{0, x_1, .., x_{m-1}\}$ by $m-2$ internal nodes $\{y_1, .., y_{m-2}\}$; (1.1.9) indicates that via the vertex factors $v(z)$, one may connect these $2m-3$ nodes independently of each other. The $\chi(z)$s were the generating functions for trees connecting two points, and except for the coupling $v(z)$, there is no interaction between them.

The fact that $v(z)$ is finite on the disc $|z| \leq z_c$ is will be crucial for the continuum limit since, as we consider larger and larger trees, the effect of the $v(z)$ will be more and more local. In essence, the leading term in (1.1.9) should be viewed as $(2m-5)!!\chi(z)^{2m-3}v(z_c)^{m-2}$ as in the second conclusion of Theorem 1.1.

A more detailed explanation of $v(z)$ and the meaning of the error terms will be given in chapter 3, but for now, let us set the stage for the continuum limit by introducing another critical exponent called ν.

It was proved in [HS4] that if $t_n(0,x)$ is the number of n bond trees containing 0 and x, and $t_n = \sum_x t_n(0,x)$, then

$$\left(\frac{\sum_x t_n(0,x)\|x\|^2}{t_n}\right)^{1/2} \overset{n\to\infty}{\sim} Dn^{\nu}(1+O(n^{-\epsilon})), \tag{1.1.10}$$

with $\nu = 1/4$ for any $\epsilon < \min(1/2, (d-8)/4)$; here, $\|\cdot\|$ denotes the Euclidean metric. It turns out that ν governs the *mean radius of gyration* R_n, where

$$R_n^2 \equiv \frac{1}{t_n} \sum_{\substack{|T|=n \\ T\ni 0}} \sum_{x\in T} \|x - \bar{x}_T\|^2 = \frac{\sum_x t_n(0,x)\|x\|^2}{2t_n}, \tag{1.1.11}$$

with $\bar{x}_T = (n+1)^{-1}\sum_{x\in T} x$ being the centre of mass of the tree T. In view of the last equality in (1.1.11), it follows that

$$R_n \overset{n\to\infty}{\sim} \tilde{D}n^{1/4}(1+O(n^{-\epsilon})), \tag{1.1.12}$$

where $\tilde{D} = D/\sqrt{2}$. Heuristically, either result suggests that trees in 8 or more dimensions are inherently 4 dimensional objects in the sense that their 'mass' goes as the fourth power of their root-mean square radius. Thus, one should not expect that they intersect each other in more than 8 dimensions; this is why mean field behavior is expected if $d > 8$. In fact, (1.1.10) also says that a tree of size n containing 0 will have most of its mass distributed a distance $O(n^{1/4})$ away from 0. This last remark provides the impetus for looking at the following limit: $P(x) = \lim_{n\to\infty} P_n(x)$ where

$$P_n(x) = \frac{t_n(0, y(n))}{t_n^{(2)}} \tag{1.1.13}$$

with $y_i(n) = \lfloor n^{1/4} x_i \rfloor$ for $i = 1$ to d. Here $P_n(x)$ is the fraction of n bond trees which connect 0 to $y(n)$ in $\mathbb{Z}^d$. As will be discussed at the end of this section, it is the *momentum space analogue* of (1.1.13) which we will compute by looking at the Fourier transform $\hat{t}_n(k)$ of $t_n(0, x)$, where

$$\hat{t}_n(k) = \sum_x e^{ik\cdot x} t_n(0, x), \tag{1.1.14}$$

so that

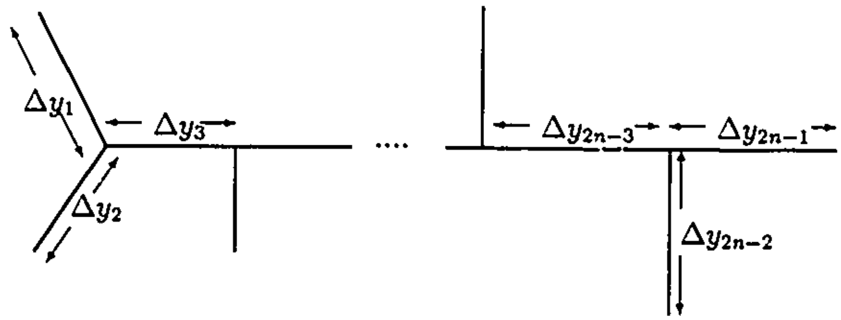
$$t_n(x) = \int_{[-\pi,\pi]^d} \hat{t}_n(k) e^{-ix\cdot k} \frac{d^d k}{(2\pi)^d} \tag{1.1.15}$$

with $k \in [-\pi, \pi]^d$. Since, $k/n^{1/4}$ will be dual to $n^{1/4}x$, we will compute:

$$P(k) \equiv \lim_{n\to\infty} \frac{\hat{t}_n(k/\bar{D}n^{1/4})}{\hat{t}_n(0)} = \int_0^\infty dl\; l e^{-l^2/2 - lk^2/2}, \tag{1.1.16}$$

where with the benefit of hind-sight we set $\bar{D}^2 = (2/\pi)^{1/2} D^2/d$. It should be pointed out that $(2/\pi)^{1/2} D^2/d = \alpha/\sqrt{1+b}$ where α is defined at the beginning of section 4.2 and b at the end of section 2.2. Another feature readily apparent to the cognoscenti is that $P(k)$ is related to parabolic cylinder functions [GR]; for those of us who are not special-function aficionados, suffice it to say it is transcendental.

To understand the continuum limit of lattice trees linking m points (which will be discussed below) we must bear in mind the binary 'growth' of these trees as discussed after Theorem 1.1. Thus, rather than specifying points $\{0, x_1.., x_{m-1}\}$, which are to be connected by an n bond tree, it will suffice to specify a binary backbone linking these points as illustrated below:



It is clear that to uniquely determine this backbone, it suffices to specify which 3 edges $\{\Delta y_{\sigma_1(i)}, \Delta y_{\sigma_2(i)}, \Delta y_{\sigma_3(i)}\}$ meet at each of the $m-2$ internal nodes in the m point tree's backbone. To simplify notation let $\{\sigma\}$ denote the set of $m-2$ triplets chosen from $\{1, .., 2m-3\}$ (which will represent the subscripts of the edge lengths Δy_i which are connected in the backbone's internal nodes).

Now, let $t_n^{(m)}(0, x_1, .., x_{m-1})$ be the number of n bond trees containing $\{0, x_1, .., x_{m-1}\}$, and let

$$t_n^{(m)} \equiv \sum_{x_1,...,x_{m-1}} t_n^{(m)}(0, x_1, ..., x_{m-1}) \tag{1.1.17}$$

(note $t_n^{(m)}$ is the n^{th} coefficient of the power series expansion of $\chi_m(z)$). If we let $T_n(\{\sigma\}; \Delta y_1, ..., \Delta y_{2m-3})$ be the number of n bond trees having the specified binary backbone structure, we can define

$$P_n(\{\sigma\}; \Delta y_1, ..., \Delta y_{2m-3}) = \frac{T_n(\{\sigma\}; \Delta y_1, ..., \Delta y_{2m-3})}{t_n^{(m)}}. \tag{1.1.18}$$

In essence, the above corresponds to the fraction of n bond trees which link the vectors Δy_i as per the backbone specified by σ.

We would like to investigate the behavior of $\lim_{n\to\infty} P_n(\{\sigma\}; n^{1/4}\Delta y_1, ..., n^{1/4}\Delta y_{2m-3})$, but we will do so by rescaling

$$\hat{P}_n(\{\sigma\}; k_1, .., k_{2m-3}) \equiv \sum_{y_1,...,y_{2m-3}} e^{i\sum_{j=1}^{2m-3}\Delta y_j k_j} P_n(\{\sigma\}; \Delta y_1, ..., \Delta y_{2m-3}),$$

(where as usual, $k \in [-\pi, \pi]^d$) so as to have

$$\hat{P}(\{\sigma\}; k_1, .., k_{2m-3}) \equiv \lim_{n\to\infty} \hat{P}_n(\{\sigma\}; k_1/n^{1/4}, .., k_{2m-3}/n^{1/4}). \tag{1.1.19}$$

To do so, we will find $\hat{T}_n(\{\sigma\}; k_1, .., k_{2m-3})$ by looking at the Fourier transform of its generating function. Put

$$\hat{G}_z(k) = \sum_x G_z(0,x)e^{ik\cdot x}, \tag{1.1.20}$$

it will turn out that,

$$\hat{T}_n(\{\sigma\}; k_1, .., k_{2m-3}) = \frac{1}{2\pi i}\oint_{|z|=\epsilon} \mathcal{T}_z(\{\sigma\}; k_1, .., k_{2m-3})\frac{dz}{z^{n+1}} \tag{1.1.21}$$

where

$$\begin{aligned}&\mathcal{T}_z(\{\sigma\}; k_1, .., k_{2m-3}) = \\ &\prod_{i=1}^{2m-3} \hat{G}_z(k_i) \prod_{j=1}^{m-2} \mathcal{B}_z(k_{\sigma_1(j)}, k_{\sigma_2(j)}, k_{\sigma_3(j)}) + E_m(z, \vec{k}),\end{aligned} \tag{1.1.22}$$

and $|E_m(z,\vec{k})|$ obeys the bounds of (3.1.2) on the disk $|z| < z_c$. As will become apparent later, $\mathcal{B}_z(k_{\sigma_1(j)}, k_{\sigma_2(j)}, k_{\sigma_3(j)})$ is a Fourier transform of the kernel $v(z)$ of (1.1.9) so that when $k_i \to k_i n^{-1/4}$, the "local effects" captured by $\mathcal{B}_z(k_{\sigma_1(j)}, k_{\sigma_2(j)}, k_{\sigma_3(j)})$ vanish as it converges to $v(z_c)$. In particular, all information contained in $\{\sigma\}$ is lost in the limit as $n \nearrow \infty$.

As with (1.1.16), we wish to analyse the distribution resulting from scaling the Δy_i's by $\bar{D}n^{1/4}$, by computing the fraction of the scaled trees that connect the specified distances in the limit. Again, we will compute the momentum space analogue of the latter limit by letting $k_i/\bar{D}n^{1/4}$ be the dual of $\bar{D}n^{1/4}\Delta y_i$.

It is now appropriate to draw a connection with Theorem 1.1, for in essence, the bulk of this thesis is devoted to showing that to "leading order", modulo certain scaling constants,

$$\mathcal{T}_n(\{\sigma\}; k_1, .., k_{2m-3}) \sim \prod_{i=1}^{2m-3} \frac{1}{k_i^2 + \sqrt{z_c - z}}. \tag{1.1.23}$$

Indeed, with the above result in hand, it is relatively easy to deduce the main result of chapter 4:

Theorem 1.2 *Put $k_i/\bar{D}n^{1/4} \equiv \kappa_i$ for $1 \leq i \leq 2m-3$ with $\bar{D}$ as defined below (1.1.16). Then, for $k_i \in \mathbb{R}^d$, $m \geq 2$, $d > 8$ and L sufficiently large (or d sufficiently large in the n.n. model),*

$$\mathcal{P}(\{\sigma\}; k_1, .., k_{2m-3}) \equiv \lim_{n\to\infty} \hat{P}_n(\{\sigma\}; \kappa_1, ...\kappa_{2m-3}) \tag{1.1.24}$$
$$= \int_0^\infty ... \int_0^\infty dl_1...dl_{2m-3}(\sum_{i=1}^{2m-3} l_i) \exp\left\{-(\sum_i^{2m-3} l_i)^2/2 - \sum_i^{2m-3} l_i k_i^2/2\right\}.$$

Though it may not be obvious, $\mathcal{P}(\{\sigma\}; k_1, ...k_{2m-3})$ is related to the characteristic function of a probability measure: if all the k_i are set to zero, we get $1/(2m-5)!!$. In fact, there are $(2m-5)!!$ ways of assembling $2m-3$ edges into a binary tree, so by summing over all configurations $\{\sigma\}$, we get 1.

If we revert to position space, now in $\mathbb{R}^d$, the distribution of the continuum trees becomes

$$\int_0^\infty ... \int_0^\infty dl_1...dl_{2m-3} \frac{(\sum_{i=1}^{2m-3} l_i)}{(\prod_i 2\pi l_i)^{d/2}} \exp\left\{-(\sum_i^{2m-3} l_i)^2/2 - \sum_i^{2m-3} (\Delta y_i)^2/2l_i\right\}. \tag{1.1.25}$$

This is the very distribution conjectured by D. Aldous in his 1993 J.S.P. [A0] paper in which he discussed various ways of obtaining integrated superBrownian excursion (ISE) as a limit of randomly embedded abstract trees.

We will postpone further discussion regarding the interpretation of (1.1.25) and its connection with ISE until chapter 5. Below is an overview of the contents of the following chapters.

Chapter 2 summarises the important machinery contained in [HS3] and [HS4]; the latter is the main reference upon which this work is based. Section 2.1 is a self contained explanation of the lace expansion as applied to our model; the principle behind the expansion is that one can recover a convolution equation for $G_z(0,x)$. By taking the Fourier transform of this convolution equation, one can obtain an algebraic equation for the Fourier transform $\hat{G}_z(k)$ of $G_z(0,x)$ involving another function, $\hat{\Pi}_z(k)$, which in turn can be bounded (essentially) by a geometric series in square diagrams [HS3]. Section 2.2 deals with $\hat{\Psi}_z(0)$, which turns out to be closely related to $\frac{d}{dz}z\hat{\Pi}_z(0)$, and will be used to construct the generating function for trees connecting 3 points. However, 2.2 can be omitted if reading only sections 3.1 through 3.3 but is essential to the analysis in sections 3.4 and 3.5.

Chapter 3 deals with the use of a differential operator to inductively calculate $\chi_m(z)$. The base case of the induction involves explicitly resumming the 3 point function, and this is done in section 3.2. For the 4 and higher point functions, the main difficulty in proving Theorem 1.1 will be to bound the derivatives of $\hat{\Psi}_z(0)$ resulting from the induction and this is done in sections 3.5 through 3.10. Although the results contained therein are of interest in and of themselves, their principal purpose is to bound error terms in the inductive proof of section 4.5; it should however be possible to read section 4.1 and 4.2 having only read section 2.1. In fact, section 4.1 follows straight from (1.1.9).

Chapter 4 deals with the continuum limit of trees connecting specified lattice points. Section 4.1 introduces some modified machinery from [MS] to estimate the size of the coefficients of the error terms in (1.1.9), and it is then shown that the leading term in *2)* Theorem 1.1 yields the leading order term behavior for $t_n^{(m)}$. In the next section, we tackle the scaling limit of the 2 point function; all calculations are carried out in terms of the Fourier transforms of the pertinent generating functions, and to achieve this, the leading order behavior of $\hat{G}_z(k)$ is isolated in section 4.2 so that the desired continuum limit may be calculated by a contour integral. Section 4.3 is the analogue of section 3.2 in that we explicitly resum the generating function for the 3 point function with the appropriate phase factors (mainly for the sake of those who may not have read section 2.2). Section 4.4 shows that $\mathcal{B}_z(k_{\sigma_1(j)}, k_{\sigma_2(j)}, k_{\sigma_3(j)})$ reduces to $v(z_c)$ upon scaling, and section 4.5 is where all previous results are pulled together in Lemma 4.7 to extract the leading order behavior of $\mathcal{T}_z(\{\sigma\}; k_1, .., k_{2m-3})$ of (1.1.22), and then its scaling limit in Theorem 4.8.

Appendix A was included for completeness as the estimates contained therein are central to the analysis in section 3.3; it would have been somewhat unsatisfactory to quote them from Riesz' work in [R] as those results were explicitly stated only in four dimensions — yet we needed them for all $d > 8$.

Chapter 2

The lace expansion as applied to lattice trees

2.1 The 2 point generating function

As advertised, we will be looking at 'spread-out' lattice trees. The purpose of this artifice is to ensure convergence of the lace expansion above the critical dimension ($d > 8$) and should not qualitatively change the model's behavior.

We now proceed to describe the model. Consider a regular d-dimensional lattice, and henceforth view a tree in this lattice as a set of connected bonds with no closed loops (one could view trees as sets of sites in the lattice, but this would prove inconvenient when setting up the lace expansion). The object of our attention will be the generating function for the number of trees containing the origin, and a given site 'x':

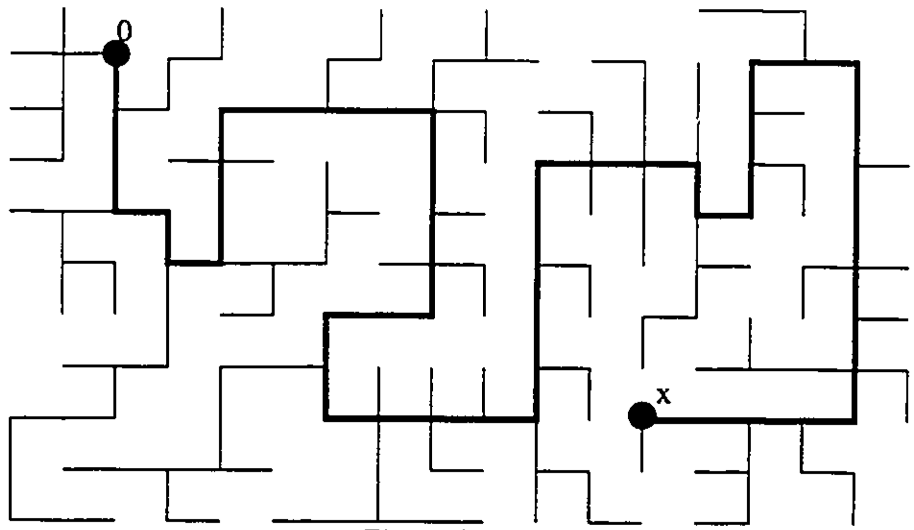


Fig 1 of a tree

$$G_z(0,x) = \sum_{T \ni 0,x} z^{|T|} = \sum_{n=0}^{\infty} t_n(0,x) z^n, \qquad (?.1.1)$$

(recall that $t_0(0,x) = \delta_{0,x}$)

where the first sum is over trees T which contain 0 and x, and $|T|$ denotes the number of bonds it contains; note that the no. of sites in a tree is always one greater than the number of bonds. We will also look at the associated quantity

$$\chi(z) = \sum_{x \in \mathbf{Z}^d} G_z(0,x) = \sum_{T \ni 0} (|T|+1) z^{|T|}. \qquad (2.1.2)$$

The idea behind the expansion in [HS3] is to decompose a tree into a backbone (i.e a walk ω of length $|\omega|$) connecting 0 and x, and a set of 'ribs' (sub-trees) $\vec{R} = \{R_0, ..., R_{|\omega|}\}$ so that the rib R_j 'grows' from the j^{th} step $\omega(j)$. The task is then to ensure that these 'ribs' do not intersect each other (thereby also ensuring ω is self-avoiding). This is accumplished by the use of the interaction $U_{st}(\vec{R})$, where:

$$U_{st} = \begin{cases} 0 & \text{if } R_s \cup R_t = \emptyset \\ -1 & \text{if } R_s \cup R_t \neq \emptyset \end{cases}, \qquad (2.1.3)$$

and the kernel $K[0,|\omega|]$ where

$$K[0,|\omega|] = \prod_{0 \leq s < t \leq |\omega|} (1 + U_{s,t}(\vec{R})) \tag{2.1.4}$$

It is therefore clear that if we sum over random walks ω, we may recover equation (2.1.1) by writing:

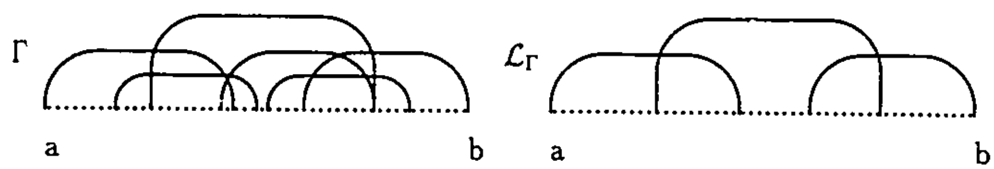
$$G_z(0,x) = \sum_{\omega:0 \to x} z^{|\omega|} \left(\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|\omega|} \right) K[0,|\omega|]. \tag{2.1.5}$$

In the above sum, ω is a walk which may take steps in *any* set Ω which *does not intersect the origin and also respects the symmetries of the lattice.* The set used to carry out the analysis for the convergence of the expansion [HS3] was the set $\{x \in Z^d \setminus \{0\} : ||x||_\infty \leq L\}$.

In what is to follow we will recall the derivation of the expansion for the 2-point function as in [HS3], and some results regarding the norms of certain quantities via fractional derivatives (cf. appendix C) as in [HS4].

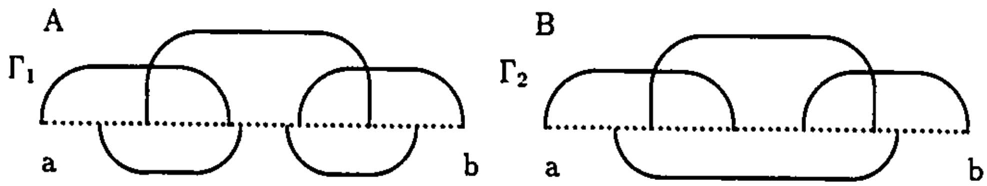
As in [HS3], given an interval $[a,b]$ with $a, b \geq 0$ define any pair $st \subset [a,b]$ with $s < t$ to be an edge. A collection of edges is defined to be a graph, and a graph Γ will be defined as connected on $I = [a,b]$ if both a and b are endpoints of edges in Γ and if for any $c \in (a,b)$ there is at least one edge $st \in \Gamma$ such that $s \leq c \leq t$ (this definition of connectedness differs from the one used for self-avoiding walk, but is a natural one to deal with rib-rib interactions). A lace is a minimally connected graph (i.e. the removal of any edge of a lace on $[a,b]$ would result in a disconnected graph). Furthermore, given a connected graph Γ on [a,b], we can associate to it a *unique* lace $\mathcal{L}_\Gamma$ via the following prescription:

Given Γ, $\mathcal{L}_\Gamma$ will consist of a set $\{s_i t_i\}$, where $s_1 = a$ and $t_1 = \max\{t : at \in \Gamma\}$ and subsequently, $t_{i+1} = \max\{t : st \in \Gamma, s \leq t_i\}$ with $s_i = \min\{s : st_i \in \Gamma\}$. The above concepts are illustrated below:



Definition 2.1.1 *Given a lace L, the set of edges $st \notin L$ such that $\mathcal{L}_{L\cup\{st\}} = L$ will be denoted by $\mathcal{C}(L)$. This will be called the set of compatible edges of L.*

To illustrate these definitions let $L = \mathcal{L}_\Gamma$ (with Γ as in the previous diagram), and consider diagrams A and B. In A, we have added 2 new edges e_1, e_2 so that $\Gamma_1 = L \cup_i e_i$. Since $\mathcal{L}_{\Gamma_1} = L$, e_1 and $e_2 \in \mathcal{C}(L)$. In B $\Gamma_2 = L \cup e_3$, but $\mathcal{L}_{\Gamma_2} \neq L$ so e_3 is *incompatible* with L.



Let us denote the set of all graphs on $[a, b]$ by $\mathcal{B}(a, b)$, and the subset of all connected graphs in $\mathcal{B}(a, b)$ by $\mathcal{G}(a, b)$. With these definitions, we can write:

$$K[a, b] = \prod_{a \le s < t \le b} (1 + U_{st}) = \sum_{\Gamma \in \mathcal{B}(a,b)} \prod_{st \in \Gamma} U_{st}, \tag{2.1.6}$$

where the sum on an empty graph is equal to one. We seek a convolution equation involving $G_z(0, x)$. The first step in deriving this convolution equation is to resum the interaction $K[a, b]$ as follows: for $n \geq 1$ let

$$K[0, n] = K[1, n] + \sum_{i=1}^{n} \sum_{\Gamma \in \mathcal{G}[0,i]} \prod_{st \in \Gamma} U_{st} K[i + 1, n]. \tag{2.1.7}$$

The above can be seen by breaking up the sum of all graphs on $[0, n]$ into those whose edges contain 0, and those that do not. Those which contain 0 may themselves be resummed by looking at those graphs which connect 0 to

i and then multiplying these by $K[i+1,n]$ thus giving us a 'bridge' from 0 to i and then all other possibilities from $i+1$ onwards (this gap is required because the above definition of connectedness).

The sum over connected graphs can also be resummed as:

$$\begin{aligned}\sum_{\Gamma\in\mathcal{G}[0,n]}\prod_{st\in\Gamma}U_{st} &= \sum_{L\in\mathcal{L}[0,n]}\sum_{\Gamma:\mathcal{L}_\Gamma=L}\prod_{st\in L}U_{st}\prod_{s't'\in\Gamma\setminus L}U_{s't'} \\ &= \sum_{L\in\mathcal{L}[0,n]}\prod_{st\in L}U_{st}\prod_{s't'\in\mathcal{C}(L)}(1+U_{s't'})\equiv J[0,n]. \quad (2.1.8)\end{aligned}$$

Now, equation (2.1.7) may be written as:

$$K[0,n]=K[1,n]+\sum_{i=1}^{n}J[0,i]K[i+1,n] \qquad (2.1.9)$$

where $K[n,n]$ and $K[n+1,n]$ are to taken as equal to 1 in the above sum. For convenience let us separate the $i=n$ term in the above sum so as to get:

$$K[0,n]=K[1,n]+\sum_{i=1}^{n-1}J[0,i]K[i+1,n]+J[0,n] \qquad (2.1.10)$$

We are now in a position to write our convolution equation.

$$\begin{aligned}G_z(0,x) &= \sum_{R_0\ni 0}z^{|R_0|}\delta_{0,x} \\ &+ \sum_{\substack{\omega:0\to x\\ |\omega|\geq 1}} z^{|\omega|}\left(\prod_{i=0}^{|\omega|}\sum_{R_i\ni\omega(i)}z^{|R_i|}]K[1,|\omega|]\right) \\ &+ \sum_{\substack{\omega:0\to x\\ |\omega|\geq 2}} z^{|\omega|}[\prod_{R_i\ni\omega(i)}z^{|R_i|}]\sum_{j=1}^{|\omega|-1}J[0,j]K[j+1,|\omega|] \\ &+ \sum_{\substack{\omega:0\to x\\ |\omega|\geq 1}} z^{|\omega|}[\prod_{i=0}^{|\omega|}\sum_{R_i\ni\omega(i)}z^{|R_i|}]J[0,|\omega|]. \qquad (2.1.11)\end{aligned}$$

Let us define

$$g(z)\equiv\sum_{T\ni 0}z^{|T|}=G_z(0,0) \qquad (2.1.12)$$

and explain the 'lace expansion' by writing $J[0,n] = \sum_{m\geq 1} J_m[0,n]$, where

$$J_m[0,n] = \sum_{L\in\mathcal{L}_m[0,n]} \prod_{st\in L} \mathcal{U}_{st} \prod_{s't'\in\mathcal{C}(L)} (1+\mathcal{U}_{s't'}) \qquad (2.1.13)$$

($\mathcal{L}_m[0,n]$ denotes the set of laces consisting of m edges on $[0,n]$).

If we now define

$$\Pi_z(0,x) = \sum_{\substack{\omega:0\to x\\ |\omega|\geq 1}} z^{|\omega|} \left(\prod_{i=0}^{|\omega|} \sum_{R_i\ni\omega(i)} z^{|R_i|}\right) J[0,|\omega|], \qquad (2.1.14)$$

it can be seen that $\Pi_z(0,x) = \sum_{m\geq 1} \Pi_z^{(m)}(0,x)$, where

$$\Pi_z^{(m)}(0,x) = \sum_{\substack{\omega:0\to x\\ |\omega|\geq 1}} z^{|\omega|} \left(\prod_{i=0}^{|\omega|} \sum_{R_i\ni\omega(i)} z^{|R_i|}\right) J_m[0,|\omega|]. \qquad (2.1.15)$$

Now we re-write $G_z(0,x)$ in what will be its final form:

$$G_z(0,x) = \delta_{0,x} g(z) + \Pi_z(0,x) + z g(z) \sum_{(0,u)} G_z(u,x) + \sum_{(u,v)} z\Pi_z(0,u) G_z(v,x) \qquad (2.1.16)$$

In the above sums, $(0,u)$ is a step taken in Ω. Similarly, (u,v) is such that $(v-u)\in\Omega$. The last two terms in (2.1.16) could be written in terms of convolutions with the indicator function $I_{[\Omega]}(x)$ of the set Ω. The normalised Fourier transform of the latter will be denoted as

$$\hat{D}(k) \equiv \frac{1}{\Omega} \sum_{x\in\Omega} e^{ik\cdot x}. \qquad (2.1.17)$$

Finally it should be mentioned that the factor $g(z)$ in the third term is necessary because the walk in its corresponding term in (2.1.11) was non-trivial (it consisted of at least one step). Now, if we assume that all the functions in the above convolution are summable [HS3], we can Fourier transform both sides of equation (2.1.16) to obtain:

$$\hat{G}_z(k) = \frac{g(z) + \hat{\Pi}_z(k)}{1 - z\Omega\hat{D}(k)(g(z)+\hat{\Pi}_z(k))} \qquad (2.1.18)$$

We will now discuss some analytic bounds on $\hat{G}_z(k)$. Let

$$\hat{F}_z(k) \equiv 1 - z\Omega\hat{D}(k)(g(z) + \hat{\Pi}_z(k)). \tag{2.1.19}$$

By the bounds in [HS4] p. 1015, we know that $\hat{F}_{z_c}(0) = 0$, where $z_c = \lambda^{-1}$ [cf. (1.1.1)] and that

$$\hat{F}_z(0)^2 = B_1^2(z_c - z) + E(z),$$

where $|E(z)| \leq c|z_c - z|^{1+\epsilon}$ for $|z| \leq z_c$, and B_1 is a constant which depends on L and will be described more precisely in section 4.1. Using the above facts, we can show that

$$\hat{F}_z(0) = B_1(z_c - z)^{1/2} + \tilde{E}(z), \tag{2.1.20}$$

where $|\tilde{E}(z)| \leq c|z_c - z|^{1/2+\epsilon}$. Simply put $\tilde{E}(z) = [B_1^2(z_c - z) + E(z)]^{1/2} - B_1(z_c - z)^{1/2}$ and $B(z) = E(z)/B_1^2(z_c - z)$ (note that $|B(z)| \leq c|z_c - z|^{\epsilon}$). Then by the binomial theorem

$$\begin{aligned} |\tilde{E}(z)| &= |z_c - z|^{1/2}|(1 + B(z))^{1/2} - 1| \\ &\leq |z_c - z|^{1/2}(1 + B(z)^2/2 + \ldots - 1). \end{aligned} \tag{2.1.21}$$

2.2 The derivation of $\hat{\Psi}_z(0)$

Next we recall the derivation of $\hat{\Psi}_z(0)$ (as per [HS4]) defined by the equation:

$$\frac{d}{dz}(z\hat{\Pi}_z(0)) = \hat{\Psi}_z(0)(z\Omega\chi + 1). \tag{2.2.1}$$

This generating function will be of fundamental importance when calculating the behavior of higher order connectivity functions.

Let us carry out the differentiation in (2.1.14). Put $\gamma_j = \sum_{R_j \ni \omega(j)} z^{|R_j|}$. Then,

$$\frac{d}{dz}(z\hat{\Pi}_z(0)) = \frac{d}{dz}\sum_{|\omega|\geq 1}\prod_{j=1}^{|\omega|}\left(\sum_{R_j \ni \omega(j)} z^{|R_j|+1}\right) J[0, |\omega|]$$

$$= \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \left\{ \prod_{j:j\neq i} \gamma_j \right\} \sum_{R_i \ni \omega(i)} (|R_i|+1) z^{|R_i|} J[0,|\omega|]$$

$$= \sum_y \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \left(\prod_{j:j\neq i} \gamma_j \right) \sum_{R_i \ni \omega(i), y} z^{|R_i|} J[0,|\omega|], \qquad (2.2.2)$$

since $|R_i|+1$ is the number of sites in the i^{th} rib.

Roughly speaking $\hat{\Psi}(0)$ involves two simultaneous lace expansions: the first sums over back-bones with one lace expansion so that the ribs intersect according to the sum over laces $L \in \mathcal{L}[0,|\omega|]$ and the other sums over the ribs so that the rib R_i in (2.2.2) becomes a back-bone with its own ribs (we will denote these with R'_k); these primed ribs then interact according to a sum over $L' \in \mathcal{L}[0,|\omega'|]$. The problem is to keep track of the interaction between these primed and un-primed ribs; this will require several interactions to mediate. The desired interaction in a given expression will be specified by appending it to the kernel it is acting within. For example, the term corresponding to the rib at $\omega(i)$ is defined by:

$$\sum_{R_i \ni \omega(i), y} z^{|R_i|} = \sum_{\omega' : \omega(i) \to y} z^{|\omega'|} \left(\prod_{k=0}^{|\omega'|} \gamma'_k \right) K[0,|\omega'|;\mathcal{U}'], \qquad (2.2.3)$$

where $\gamma'_k = \sum_{R'_k \ni \omega(i)} z^{|R'_k|}$,

$$K[0,|\omega'|;\mathcal{U}'] = \prod_{0\leq s<t\leq|\omega'|} (1+\mathcal{U}'_{st}), \qquad (2.2.4)$$

and

$$\mathcal{U}'_{st} = \begin{cases} 0 & \text{if } R'_s \cap R'_t = \emptyset \\ -1 & \text{if } R'_s \cap R'_t \neq \emptyset \end{cases} \qquad (2.2.5)$$

Let us also introduce $\mathcal{V}'_{st}$:

$$\mathcal{V}'_{st} = \begin{cases} 0 & \text{if } R_s \cap R'_t = \emptyset \\ -1 & \text{if } R_s \cap R'_t \neq \emptyset \end{cases} \qquad (2.2.6)$$

As an intermediary we define $1+\mathcal{V}'_l$:

$$1+\mathcal{V}'_l = \prod_{s:si\in\mathcal{C}(L)} (1+\mathcal{V}'_{sl}), \tag{2.2.7}$$

with which we can define $\mathcal{X}_{kl}$:

$$1+\mathcal{X}_{kl} = \begin{cases} 1+\mathcal{U}'_{kl}, & 0<k<l \\ (1+\mathcal{U}'_{0l})(1+\mathcal{V}'_l), & 0=k<l \end{cases}. \tag{2.2.8}$$

If $0<k<l$, $\mathcal{X}_{kl}$ will only involve primed rib interactions, whereas $\mathcal{X}_{0l}$ will be non-zero only if $\mathcal{U}'_{0l}\neq 0$ or if R'_l intersects some rib R_s with $si\in\mathcal{C}(L)$, where i is our distinguished rib.

If we use the convention that si represents the edge is if $i<s$, we can single out a distinguished rib by writing

$$J[0,|\omega|;\mathcal{U}] = \sum_{L\in\mathcal{L}[0,|\omega|]} \mathcal{U}(L,i)\left(\prod_{s:si\in L}\mathcal{U}_{si}\prod_{t:ti\in\mathcal{C}(L)}(1+\mathcal{U}_{ti})\right), \tag{2.2.9}$$

where

$$\mathcal{U}(L,i) = \prod_{\substack{st\in L\\ s,t\neq i}} \mathcal{U}_{st} \prod_{\substack{s't'\in\mathcal{C}(L)\\ s',t'\neq i}} (1+\mathcal{U}_{s't'}). \tag{2.2.10}$$

The product over compatible bonds in (2.2.9) may be combined with $K[0,|\omega'|;\mathcal{U}']$:

$$K[0,|\omega'|;\mathcal{U}'] \prod_{t:ti\in\mathcal{C}(L)} (1+\mathcal{U}_{ti}) = (1+\mathcal{V}'_0)K[0,|\omega'|;\mathcal{X}]. \tag{2.2.11}$$

Thus (2.2.2) may be re-cast as:

$$\begin{aligned}\frac{d}{dz}(z\hat{\Pi}_z(0)) = \sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|}\left(\prod_{j:j\neq i}\gamma_j\right)\sum_{L\in\mathcal{L}[0,|\omega|]}\mathcal{U}(L,i) \\ \times\sum_{|\omega'|\geq 0} z^{|\omega'|}\left(\prod_{k=0}^{|\omega|}\gamma'_k\right)(1+\mathcal{V}'_0)K[0,|\omega'|;\mathcal{X}](\prod_{s:si\in L}\mathcal{U}_{si}).\end{aligned} \tag{2.2.12}$$

Let us consider the case where $|\omega'| = 0$ and write that as:

$$\hat{\Psi}_z^1(0) = \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \left(\prod_{j:j\neq i}^{|\omega|} \gamma_j \right) J[0, |\omega|] \tag{2.2.13}$$

(Recall that $\prod_{si\in C(L)}(1+\mathcal{U}_{si})$ is embedded in $(1+\mathcal{V}_0'))$. Now we consider the case $|\omega'| \geq 1$. First we define a kernel:

$$\mathcal{H} = \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \left(\prod_{j:j\neq i} \gamma_j \right) \sum_{L\in\mathcal{L}[0,|\omega|]} \mathcal{U}(L,i) \sum_{|\omega'|\geq 1} z^{|\omega'|} \left(\prod_{k=0}^{|\omega'|} \gamma_k' \right) (1+\mathcal{V}_0'). \tag{2.2.14}$$

It is clear from (2.2.13) that the summations within $\mathcal{H}$ must still act upon $\prod_{s:si\in L} \mathcal{U}_{si}$ and $K[0, |\omega'|; \mathcal{X}]$. By resumming we obtain

$$K[0, |\omega'|; \mathcal{X}] = K[1, |\omega'|; \mathcal{U}'] + \sum_{m=1}^{|\omega'|} J[0, m; \mathcal{X}] K[m+1, |\omega'|; \mathcal{U}'] \tag{2.2.15}$$

since $\mathcal{X} = \mathcal{U}'$ for laces without an edge at $\omega(i)$. If $|\omega'| \geq 1$ in (2.2.13), we must allow for the eventuality that i be in an edge of the lace L. If this is not the case however, the empty product $\prod_{s:si\in L} \mathcal{U}_{si} = 1$.

Now fix $L \in \mathcal{L}[0, |\omega'|]$. The case where $i \in L$ will be dealt with as follows: given a configuration such that $\mathcal{U}_{si} = -1$, let $k(s,i)$ be the smallest k such that $R_s \cap R_i' \neq \emptyset$ (there may be at most two edges in L with common end-point i) and put $k(L,i) = \max_{s:si\in L} k(s,i)$. Then $k(L,i)$ gives us the 'furthest' point on ω' where a rib growing from ω' is intersected by an un-primed rib. This shall be encoded into $\mathcal{W}_l$ which is defined by

$$\prod_{s:si\in L} \mathcal{U}_{si} \equiv \sum_{l=0}^{|\omega'|} \mathcal{W}_l = \pm \sum_{l=0}^{|\omega'|} I_{[k(L,i)=l]}. \tag{2.2.16}$$

The $\pm$ sign is -1 if there is only a single edge with end-point i and +1 if there are two such edges.

To take into account the 'point of last intersection' between primed and un-primed ribs, we define $\mathcal{Y}_{st}^{(l)}$ by:

$$1+\mathcal{Y}_{st}^{(l)}=\begin{cases}1+\mathcal{U}'_{st}, & l<s<t\\ \prod_{k=0}^{l}(1+\mathcal{X}_{kt}), & l=s<t\end{cases} \tag{2.2.17}$$

So again, $\mathcal{Y}_{st}^{(l)}\neq 0$ if either there is a non-empty primed rib — primed rib intersection, or when $s=l$ and $R'_t\cap R_s\neq\emptyset$ with $l: li\in\mathcal{C}(L)$.

We now seek to split the interaction $K[0,|\omega|]$ into two pieces. As in [HS4] we have for $0\leq l\leq|\omega'|$,

$$K[0,|\omega'|;\mathcal{X}]=K[0,l;\mathcal{X}]K[l,|\omega'|;\mathcal{Y}^{(l)}] \tag{2.2.18}$$

If $l=0$ or $l=|\omega'|$ the above is trivial. For $0<l<|\omega'|$ we obtain (2.2.18) by separating the $\mathcal{Y}_{lt'}^{(l)}$ terms so that the R.H.S. of (2.2.18) can be written as

$$\begin{aligned}
&\prod_{0\leq s<t\leq l}(1+\mathcal{X}_{st})\prod_{l<t'\leq|\omega'|}(1+\mathcal{Y}_{lt'}^{(l)})\prod_{l<s'<t'\leq|\omega'|}(1+\mathcal{Y}_{s't'}^{(l)}) \qquad (2.2.19)\\
&=\prod_{t\leq l}(1+\mathcal{X}_{0t})\prod_{0<s<t\leq l}(1+\mathcal{U}'_{st})\times\\
&\prod_{l<t'\leq|\omega'|}\left(\prod_{k=0}^{l}(1+\mathcal{X}_{kt'})\right)(1+\mathcal{U}'_{lt'})\prod_{l<s'<t'\leq|\omega'|}(1+\mathcal{U}'_{s't'})\\
=&\prod_{0<t\leq l}\left((1+\mathcal{U}'_{0t})\prod_{s:si\in\mathcal{C}(L)}(1+\mathcal{V}'_{0})\right)\prod_{0<s<t\leq l}(1+\mathcal{U}'_{st})\times\\
&\prod_{l<t'\leq|\omega'|}\left\{(1+\mathcal{U}'_{0t'})\prod_{s:si\in\mathcal{C}(L)}(1+\mathcal{V}'_{st'})\prod_{k=1}^{l}(1+\mathcal{U}'_{kt'})\right\}\prod_{l<s'<t'\leq|\omega'|}(1+\mathcal{U}'_{s't'})
\end{aligned}$$

(the first line of the last equality involves only ribs *up to* the l^{th} vertebra; the first product in the last line involves interactions *between* a rib before and a rib after the l^{th} vertebra, whereas the last product involves only ribs *after* the l^{th} vertebra). The above effectively yields $K[0,|\omega'|;\mathcal{X}]$ as per the L.H.S. of (2.2.18). Combining (2.2.18) with (2.2.15) gives us

$$K[0,|\omega'|;\mathcal{X}]=K[0,l;\mathcal{X}](K[l+1,|\omega'|;\mathcal{U}']+\sum_{j=l+1}^{|\omega'|}J[l,j;\mathcal{Y}^{(l)}]K[j+1,|\omega'|;\mathcal{U}']) \tag{2.2.20}$$

(by the definitions of the interactions.)

Finally we can substitute (2.2.20) and (2.2.16) into (2.2.13) for the case where $i \in L$ and (2.2.15) for the case $i \notin L$. By separating the $l = 0$ term from the rest of the sum (2.2.16) we have

$$\begin{aligned}
&\frac{d}{dz} z\hat{\Pi}_z(0) = \qquad\qquad (2.2.21)\\
&\hat{\Psi}^1_z(0) + \\
&\mathcal{H}\left(I_{[i\notin L]}K[1,|\omega'|;\mathcal{U}'] + I_{[i\in L]}\mathcal{W}_0 K[1,|\omega'|;\mathcal{U}']\right) + \\
&\mathcal{H}I_{[i\notin L]}\sum_{a=1}^{|\omega'|} J[0,a;\mathcal{X}]K[a+1,|\omega'|;\mathcal{U}'] + \\
&\mathcal{H}I_{[i\in L]}\left\{\sum_{l=1}^{|\omega'|}\mathcal{W}_l K[0,l;\mathcal{X}]K[l+1,|\omega'|;\mathcal{U}']\right\} + \\
&\mathcal{H}I_{[i\in L]}\left\{\sum_{l=0}^{|\omega'|-1}\mathcal{W}_l K[0,l;\mathcal{X}]\sum_{a=l+1}^{|\omega'|} J[l,a;\mathcal{Y}^{(l)}]K[a+1,|\omega'|;\mathcal{U}']\right\}
\end{aligned}$$

(the term $l = |\omega'|$ in the second line of the last term would have yielded an empty sum, and hence was dropped). The second term in the above sum can be resummed to give:

$$\hat{\Psi}^1_z(0) z\Omega\chi(z), \qquad (2.2.22)$$

which when combined with $\hat{\Psi}^1_z(0)$ from the $|\omega'| = 0$ case gives us:

$$\hat{\Psi}^1_z(0)(1 + z\Omega\chi(z)). \qquad (2.2.23)$$

As in the case of $\Psi_1(0,x)$ of [HS4] (4.28), let us extract the $l = |\omega'|$ term of the first line in the last term of (2.2.21) as well as the $a = |\omega'|$ term in the last line in (2.2.21). The result is:

$$\begin{aligned}
&\hat{\Psi}^2_z(0) = \mathcal{H}I_{[i\notin L]}J[0,|\omega'|;\mathcal{X}] + \\
&\mathcal{H}I_{[i\in L]}\sum_{l=0}^{|\omega'|}\mathcal{W}_l K[0,l;\mathcal{X}]J[l,|\omega'|;\mathcal{Y}^{(l)}]. \qquad (2.2.24)
\end{aligned}$$

This leaves us with:

$$\begin{aligned}
&\mathcal{H}I_{[i\notin L]}\sum_{a=1}^{|\omega'|-1} J[0,a;\mathcal{X}]K[a+1,|\omega'|;\mathcal{U}'] + \\
&\mathcal{H}I_{[i\in L]}\left\{\sum_{l=1}^{|\omega'|-1} \mathcal{W}_l K[0,l;\mathcal{X}]K[l+1,|\omega'|;\mathcal{U}']\right\} + \\
&\mathcal{H}I_{[i\in L]}\left\{\sum_{l=0}^{|\omega'|-2} \mathcal{W}_l K[0,l;\mathcal{X}]\sum_{a=l+1}^{|\omega'|-1} J[l,a;\mathcal{Y}^{(l)}]K[a+1,|\omega'|;\mathcal{U}']\right\},
\end{aligned} \tag{2.2.25}$$

which can be resummed to yield:

$$\hat{\Psi}_z^2(0)z\Omega\chi(z). \tag{2.2.26}$$

Letting $\hat{\Psi}_z(0) = \hat{\Psi}_z^1(0) + \hat{\Psi}_z^2(0)$ produces (2.2.1).

It will be important to recall from [HS4] Lemma 2.1 (*iii*) that

$$b + \zeta(z) \equiv z\Omega\hat{\Psi}_z(0), \tag{2.2.27}$$

where $\lim_{L\to\infty} b = 0$ and $|\zeta(z)| \leq c(\epsilon)|z_c - z|^\epsilon$ (c is independent of L) for any $\epsilon < \min\{1/2, (d-8)/4\}$. To be more precise, $b = z_c\Omega\hat{\Psi}_{z_c}(0)$.

Chapter 3

The n-point connectivity functions

3.1 The n-point functions.

Let $\chi_1(z) = g(z)$ as per (2.1.12), and $\chi_2(z) = \chi(z)$. Then, for $n \geq 3$, define $\chi_n(z)$ to be $\sum_{x_1,\dots,x_{n-1}} G_z(0, x_1, \dots, x_{n-1})$, where

$$G_z(0, x_1, \dots, x_{n-1}) = \sum_{T \ni 0, x_1, \dots, x_{n-1}} z^{|T|}, \tag{3.1.1}$$

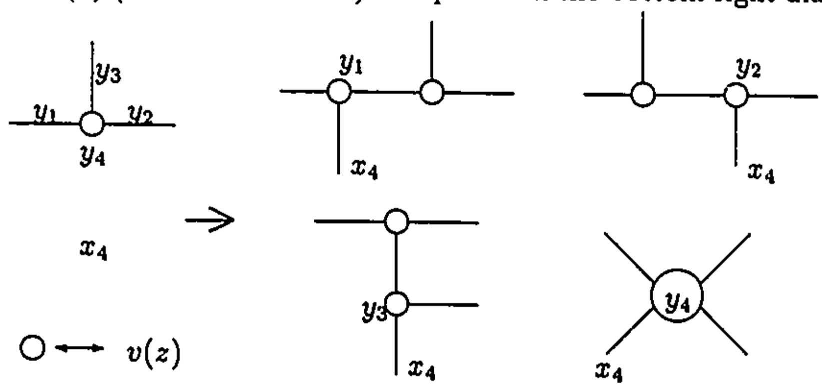
is the generating function for trees connecting $\{0, x_1, .., x_{n-1}\}$. We seek to express this quantity in terms of $\chi(z)$, and thus to control the asymptotics of $\chi_n(z)$ in terms of powers of $z_c - z$. The main result in this chapter is that for $n \geq 3$

$$\chi_n(z) = (2n-5)!!\chi^{2n-3}(z)v(z)^{n-2} + E_n(z), \tag{3.1.2}$$

where $v(z) \equiv (z\Omega)^2(1+z\Omega\hat{\Psi}_z(0))$ [cf. (1.1.9)] and $E_n(z)$ is an analytic function on the disc $|z| < z_c$, where it satisfies $|E_n(z)| \leq c_n|z_c - |z||^{-n+3+\epsilon} \cdot |z_c - z|^{-3/2}$ for $n \geq 4$, with $\epsilon < \min(1/2, (d-8)/4)$, and $|z_c - z|^{-1}$ for $n = 3$.

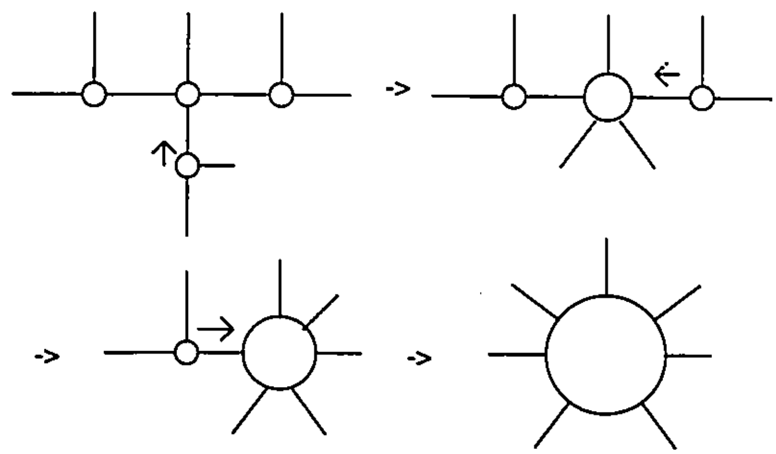
The reason for the strange looking bound on $|E_n(z)|$ is that it involves estimating derivatives of $\hat{\Psi}_z(0)$; however, the latter can only be estimated for real z. Astonishingly, these bounds will suffice to estimate the asymptotics of $E_n(z)$'s coefficients (with the machinery of section 4.1).

We now explain the meaning of the results (3.1.2). Each factor of $\chi(z)$ in (1.1.9) corresponds to a tree linking 2 points; these trees are then themselves linked in groups of 3 by a factor $v(z)$ which operates by looking at the case where the trees do not avoid each other and then subtracting the cases where they do intersect via the function $\hat{\Psi}_z(0)$. This binary structure accounts for the $v(z)^{m-2}$ and the double factorial in (1.1.9) as will be explained below. The degenerate case where the 'extra' point is connected to the tree via the vertex factor $v(z)$ (i.e. an error term) is depicted in the bottom right diagram below:

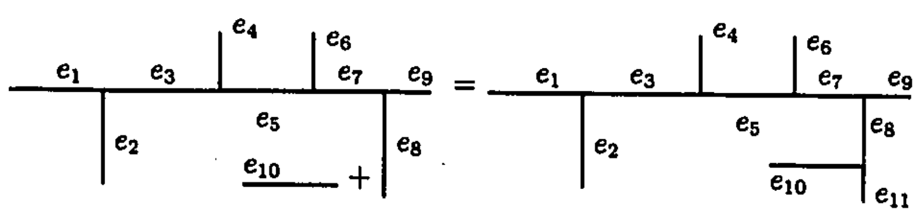


The above illustrates the 4 ways of connecting x_4 to a tree connecting 3 points.

The internal vertex in the *error* term is no longer finite at z_c, but rather, for real z, behaves as $\chi(z)^{1/2}$ (as indicated by the bloated blob) so that the overall behavior of the generating function corresponding to that diagram is $O(\chi(z)^{4+1/2})$ as per the error bound in (1.1.9). There are many ways for the 'internal edges' of a diagram corresponding to $\chi_m(z)$ to collapse, but in all cases, the generating functions they produce are $O(\chi^{2m-7/2})$. As an example, consider a tree diagram corresponding to $\chi_7(z)$. Some of the possible degenerate diagrams which result are shown below.



The $(2n-5)!!$ in (3.1.2) comes from the number of ways of assembling trees with vertex coordination number 3 so that they connect n distinct external end-points. The factor $v(z)$ accounts for the fact that such a tree will have precisely $n-2$ internal branch-points. The diagram below explains how such trees are inductively assembled.



n edges+ another (n +2) edges

By adding another edge, we increase the number of edges by two; the number of possible edges to graft e_{10} onto, is 9 (if we added another edge to the new tree, there would be 11 branches to choose from). In general, for a binary tree connecting n external end-points we can inductively see that there would be $(2n-5)!!$ ways of assembling it.

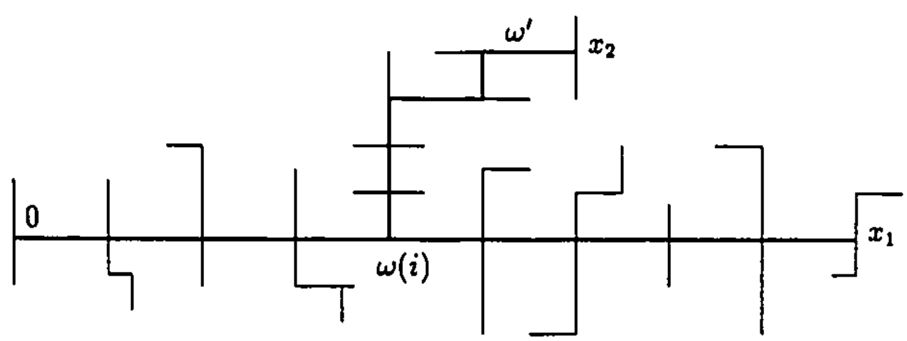
3.2 The 3 point function.

The expression for $\chi_n(z)$ in (3.1.2) will follow by induction on n. Let us start with the base case ($n = 3$). As will become apparent later, we could compute $\chi_3(z)$ by means of a differential operator, or resum it as we shall do below. While the former method has the advantage of being more mechanical, it has the draw-back of not yielding an expression in terms of $v(z)$ and $\chi(z)$ and is therefore not being amenable to capturing all the $(2m-5)!!$ leading order terms in (3.1.2) by the methods of the next section. The technique below however, will yield an *identity* in $\chi(z)$ and $v(z)$ which will later be used to construct the Fourier transform of the 3-point function.

We can write an expression for $\chi_3(z)$ by summing over all its sites — twice. Once by summing over all end-points x_1 and a second time by summing over the sites x_2 on the i^{th} branch, and then summing over all the branches.

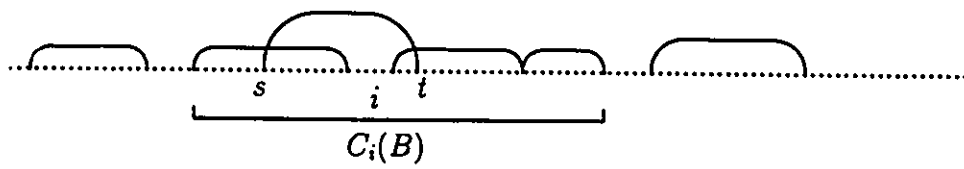
$$\chi_3(z) = \sum_{x_1,x_2} \sum_{\omega:0\to x_1} z^{|\omega|} \sum_{i=0}^{|\omega|} \left(\prod_{j:j\neq i} \sum_{R_j \ni \omega(j)} z^{|R_j|} \right) \sum_{R_i \ni \omega(i), x_2} z^{|R_i|} K[0, |\omega|] \quad (3.2.1)$$

This is illustrated below.



We now introduce one last piece of lace machinery from [MS] (because of our definition of connectedness for graphs, we produce a slightly different expression than in [MS]).

Definition 3.2.1 *Consider a graph $B \in \mathcal{B}[0,n]$. If $i \in [0,n]$, and B contains an edge, say st, such that $s \leq i \leq t$ then put $C_i(B) \equiv [I_1, I_2] = I$ where I_1 is the smallest endpoint of the left-most edge connected to st and I_2 is the largest endpoint of the right-most edge connected to st (the definition of connected was defined on p.13). If is no such edge st exists, let $C_i(B) = \{i\}$.*



Now we can introduce:

Lemma 3.1 *The interaction $K[0, |\omega|]$ may be resummed as follows: given $i \in [0, |\omega|]$,*

$$K[0, |\omega|] = \sum_{I \ni i} K[0, I_1 - 1] J[I_1, I_2] K[I_2 + 1, |\omega|], \qquad (3.2.2)$$

where the sum is restricted to $I = [I_1, I_2]$ s.t $I \subseteq [0, |\omega|]$.

Proof.

We see that

$$\begin{aligned} K[0, |\omega|] &= \sum_{I \ni i} \sum_{B : C_i(B) = I} \prod_{st \in B} U_{st} & (3.2.3) \\ &= \sum_{I \ni i} K[0, I_1 - 1] \sum_{\Gamma \in \mathcal{G}(I)} \prod_{st \in \Gamma} U_{st} K[I_2 + 1, |\omega|] \\ &= \sum_{I \ni i} K[0, I_1 - 1] J[I_1, I_2] K[I_2 + 1, |\omega|], \end{aligned}$$

if we use the conventions $J[i,i] = 1$, $K[0,-1] = K[|\omega| + 1, |\omega|] = 1$. ...■

For convenience we will let

$$\mathcal{K}(I) = K[0, I_1 - 1] J[I_1, I_2] K[I_2 + 1, |\omega|]. \qquad (3.2.4)$$

Before using this resummation, we will first recall a result from [HS4]:

(as before, $\gamma_i = \sum_{R_i \ni \omega(i)} z^{|R_i|}$)

$$
\begin{aligned}
&\frac{d}{dz}(z\Pi_z(0,x)) \\
= \;& \sum_y \sum_{\omega:0\to x;|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \left(\prod_{j:j\neq i} \gamma_j\right) \sum_{R_i \ni \omega(i),y} z^{|R_i|} J[0,|\omega|] \\
= \;& \hat{\Psi}_z(0,x)(z\Omega\chi + 1),
\end{aligned}
\tag{3.2.5}
$$

where $|\hat{\Psi}_z(0)| \leq c$ for $|z| \leq z_c$ as shown in [HS4]. The latter bound on $\hat{\Psi}_z(0)$ will later play a crucial role when calculating the continuum limit of these generating functions, but for now, the idea is to use Lemma 3.1 to exploit the independence between the 3 interactions and thereby isolate a product of 3 $\chi(z)$s.

The only nuisance in the application of (3.2.2) is that various special 'lower order' terms are produced:

$$
\begin{aligned}
\chi_3(z) \;=\;& \sum_{x_2} \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \sum_{I\ni i} \left(\prod_{j:j\neq i} \gamma_i\right) \sum_{R_i \ni \omega(i),x_2} z^{|R_i|} \mathcal{K}(I,i) \qquad (3.2.6) \\
=\;& \sum_{x_2} \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \sum_{\substack{I\ni i \\ I_1\neq 0; I_2\neq|\omega|}} \left(\prod_{j:j\neq i} \gamma_i\right) \sum_{R_i \ni \omega(i),x_2} z^{|R_i|} \mathcal{K}(I,i) \\
&+ \sum_{x_2} \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \sum_{\substack{I\ni i \\ I=[0,I_2]}} \left(\prod_{j:j\neq i} \gamma_i\right) \sum_{R_i \ni \omega(i),x_2} z^{|R_i|} J[0,I_2] K[I_2+1,|\omega|] \\
&+ \sum_{x_2} \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \sum_{\substack{I\ni i \\ I=[I_1,|\omega|]}} \left(\prod_{j:j\neq i} \gamma_i\right) \sum_{R_i \ni \omega(i),x_2} z^{|R_i|} K[0,I_1-1] J[I_1,|\omega|] \\
&+ \sum_{x_2} \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \sum_{\substack{I\ni i \\ I=[0,|\omega|]}} \left(\prod_{j:j\neq i} \gamma_i\right) \sum_{R_i \ni \omega(i),x_2} z^{|R_i|} J[0,|\omega|].
\end{aligned}
$$

By (3.2.5), the last term yields $(1 + z\Omega\chi(z))\hat{\Psi}_z(0)$. We further split the remaining terms in (3.2.6) into sub-cases. The first term may be split as:

$$\sum_{x_2}\sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|}\sum_{\substack{I\ni i\\ I_1\neq 0;\, I_2\neq|\omega|}}\left(\prod_{j:j\neq i}\gamma_i\right)\sum_{R_i\ni\omega(i),x_2} z^{|R_i|}\mathcal{K}(I,i) \quad (3.2.7)$$

$$= \sum_{x_2}\sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|}\sum_{\substack{I\ni i\\ 0\neq I_1<I_2\neq|\omega|}}\left(\prod_{j:j\neq i}\gamma_i\right)\sum_{R_i\ni\omega(i),x_2} z^{|R_i|}\mathcal{K}(I,i)$$

$$+ \sum_{x_2}\sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|}\left(\prod_{j:j\neq i}\gamma_i\right)\sum_{R_i\ni\omega(i),x_2} z^{|R_i|}K[0,i-1]K[i+1,|\omega|].$$

($J[i,i] = 1$ was used in the last line since $I_1 = I_2 = i$). The two last terms yield $(z\Omega\chi(z))^2(1 + z\Omega\chi(z))\hat{\Psi}_z(0)$ and $(z\Omega)^2\chi(z)^3$ respectively.

We may also split the second term as shown below:

$$\sum_{x_2}\sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|}\sum_{\substack{I\ni i\\ I=[0,I_2]}}\left(\prod_{j:j\neq i}\gamma_i\right)\sum_{R_i\ni\omega(i),x_2} z^{|R_i|}J[0,I_2]K[I_2+1,|\omega|]$$

$$= \sum_{x_2}\sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|}\sum_{\substack{I\ni i\\ 0<I_2;\, I=[0,I_2]}}\left(\prod_{j:j\neq i}\gamma_i\right)\sum_{R_i\ni\omega(i),x_2} z^{|R_i|}J[0,I_2]K[I_2+1,|\omega|]$$

$$+ \sum_{x_2}\sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{0}\left(\prod_{j:j\neq i}\gamma_i\right)\sum_{R_i\ni\omega(i),x_2} z^{|R_i|}K[0+1,|\omega|] \qquad (I=i=0),$$

since $J[0,0] = 1$, thereby giving us $(1 + z\Omega\chi(z))\hat{\Psi}_z(0)\cdot z\Omega\chi(z)$ and $z\Omega\chi(z)^2$ respectively. The counter-part of the previous term is dealt with similarly below:

$$\sum_{x_2}\sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|}\sum_{\substack{I\ni i\\ I=[I_1,|\omega|]}}\left(\prod_{j:j\neq i}\gamma_i\right)\sum_{R_i\ni\omega(i),x_2} z^{|R_i|}K[0,I_1-1]J[I_1,|\omega|]$$

$$= \sum_{x_2}\sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|} \sum_{\substack{I \ni i \\ |I_1| < |\omega|; I = [I_1, |\omega|]}} \left(\prod_{j:j\neq i}\gamma_i\right) \sum_{R_i \ni \omega(i), x_2} z^{|R_i|} K[0, I_1 - 1] J[I_1, |\omega|]$$

$$+ \sum_{x_2}\sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=|\omega|}^{|\omega|} \left(\prod_{j:j\neq i}\gamma_i\right) \sum_{R_i \ni \omega(i), x_2} z^{|R_i|} K[0, |\omega| - 1] \qquad (I = i = |\omega|)$$

which yields (as per the last term): $(1 + z\Omega\chi(z))\hat{\Psi}_z(0) \cdot z\Omega\chi(z)$ and $z\Omega\chi(z)^2$ respectively.

Upon collecting terms it can be seen that (3.2.6) gives

$$\begin{aligned} \chi_3(z) &= \chi(z)^3(z\Omega)^2(z\Omega\hat{\Psi}_z(0) + 1) \\ &+ 3\hat{\Psi}_z(0)(z\Omega\chi)^2 + 2z\Omega\chi(z)^2 + 3z\Omega\hat{\Psi}_z(0)\chi(z) + \hat{\Psi}_z(0). \end{aligned} \tag{3.2.8}$$

Remark 3.2.1 *As a check, we obtain the leading and next-to-leading term in (3.2.8) (asymptotically as $z \nearrow z_c$) by computing (the derivation immediately below will be explained in more detail in the next section):*

$$\sum_{x_1, x_2} G_z(0, x_1, x_2) = \sum_{T \ni 0} (|T| + 1)^2 z^{|T|} = \frac{d}{dz}(z\chi(z)). \tag{3.2.9}$$

The above is the main ingredient, but we must also use the following identities from [HS4]:

$1 - z_c\Omega(g(z_c) + \hat{\Pi}_{z_c}(0)) = 0$, $\chi(z) = \hat{h}_z(0)/\hat{F}_z(0)$, $\hat{h}_{z_c}(0) = 1/(z_c\Omega)$,

$$\chi(z) = \frac{\hat{h}_z(0)}{\hat{F}_z(0)} = \frac{g(z) + \hat{\Pi}_z(0)}{1 - z\Omega(g(z) + \hat{\Pi}_z(0))}, \tag{3.2.10}$$

and $\frac{d}{dz}z(g(z) + \hat{\Pi}_z(0)) = \chi(z) + (z\Omega\chi(z) + 1)\hat{\Psi}_z(0)$. We then have

$$\begin{aligned} \frac{d}{dz}z\chi(z) &= \frac{\chi(z)(1 + z\Omega\hat{\Psi}_z(0)) + \hat{\Psi}_z(0)}{\hat{F}_z(0)} \\ &+ \frac{z\Omega\hat{h}_z(0)[\chi(z)(1 + z\Omega\hat{\Psi}_z(0)) + \hat{\Psi}_z(0)]}{\hat{F}_z(0)^2} \end{aligned}$$

But since $z\Omega\hat{h}_z(0) = 1 - \hat{F}_z(0)$, the above reduces to

$$\frac{\chi(z)(1 + z\Omega\hat{\Psi}_z(0)) + \hat{\Psi}_z(0)}{\hat{F}_z(0)^2} \tag{3.2.11}$$

Now, $\hat{F}_z(0)^{-2} = \chi(z)^2\hat{h}_z(0)^{-2}$ and $\hat{h}_z(0)^{-2} = (z\Omega)^2/(1 - \hat{F}_z(0))^2$. For z near z_c, $\hat{F}_z(0)$ is small and thus:

$$\hat{h}_z(0)^{-2} \overset{z\nearrow z_c}{\sim} (z\Omega)^2(1 + 2\hat{F}_z(0) + ...) \tag{3.2.12}$$

From [HS4] we know that $\hat{F}_z(0) \overset{z\nearrow z_c}{\sim} (z\Omega\chi(z))^{-1} + O(z_c - z)^{1/2+\epsilon}$. Thus,

$$\hat{h}_z(0)^{-2} \overset{z\nearrow z_c}{\sim} (z\Omega)^2[1 + (z\Omega\chi(z))^{-1} + O((z_c - z)^{-1/2+\epsilon})] \tag{3.2.13}$$

Then using $\hat{F}_z(0)^{-2} = \chi(z)^2\hat{h}_z(0)^{-2}$ in (3.2.11) we deduce that

$$\begin{aligned} \chi_3(z) &= [\chi(z)(1 + z\Omega\hat{\Psi}_z(0)) + \hat{\Psi}_z(0)]\chi(z)^2\hat{h}_z(0)^{-2} \\ &\overset{z\nearrow z_c}{\sim} (z\Omega)^2(1 + z\Omega\hat{\Psi}_z(0))\chi(z)^3 \\ &+ \chi(z)^2[3\hat{\Psi}_z(0)(z\Omega)^2 + z\Omega] + O((z_c - z)^{-1+\epsilon}) \end{aligned} \tag{3.2.14}$$

as per the resummation method. Unfortunately, we cannot capture the next lower order term $\chi(z)$ since we have already made an error of order $(z_c - z)^{-1+\epsilon}$.

All we really need is good control over the leading and next-to-leading order terms for $\chi(z)$ as these will be recursively fed into $\chi_m(z)$ by the iterative machinery which will be explained in the next section.

3.3 The method for computing higher order functions.

Recall from the previous section that $\chi_3(z) = \chi^3(z\Omega)^2(1 + z\Omega\hat{\Psi}_z(0))$ to highest order in χ; to proceed by induction, we note that since the number of sites in an n bond tree is $n+1$,

$$\chi_m(z) = \sum_n t_n^{(m)} z^n = \sum_{T \ni 0} (|T|+1)^{m-1} z^{|T|} \tag{3.3.1}$$

where

$$t_n^{(m)} = \sum_{x_1,..,x_{m-1}} t_n(0, x_1, .., x_{m-1}). \tag{3.3.2}$$

This leads us to the relation

$$\chi_{m+1}(z) = \mathcal{D}\chi_m(z) \tag{3.3.3}$$

for $m \geq 1$, where $\mathcal{D}(\chi_m(z)) \equiv \frac{d}{dz}(z\chi_m(z))$. Thus in general, for $m \geq 2$, $\chi_m(z) = \mathcal{D}^{m-2}\chi(z)$. The only problem in writing down a general formula is the fact that for any differentiable functions f and g,

$$\mathcal{D}(fg) = (\mathcal{D}f)g + (\mathcal{D}g)f - fg. \tag{3.3.4}$$

In other words, although $\mathcal{D}$ is linear, it is not quite a derivation. This is merely a technical nuisance however, since we are only interested in the leading order behavior of a product. For a true derivation $D(\cdot) \equiv \frac{d}{dz}(\cdot)$ we would have

$$D^m \prod_{i=1}^n f_i = \sum_{i_1+..+i_n=m} c(i_1, .., i_n; m) D^{i_1} f_1 ... D^{i_n} f_n. \tag{3.3.5}$$

For convenience let us introduce:

Definition 3.3.1 *For $m \geq 1$, and smooth $\{f_i\}_{i=1}^n$ let*

$$P_m(n) \equiv \sum_{i_1+..+i_n=m} c(i_1, .., i_n; m) \mathcal{D}^{i_1} f_1 ... \mathcal{D}^{i_n} f_n,$$

with $c(i_1, .., i_n; m)$ as in (3.3.5).

In fact, to leading order, $\mathcal{D}$ does behave like D as shown by

Lemma 3.2 *For any sufficiently smooth functions $\{f_i\}_{i=1}^n$, $n \geq 2$ and $m \geq 1$,*

$$\mathcal{D}^m \prod_{i=1}^n f_i = \sum_{i=0}^m \binom{m}{i} (-1)^i P_{m-i}(n)(n-1)^i.$$

Before proving Lemma 3.2, we introduce a proposition:

Proposition 3.3.1 *For $n \geq 2$ and sufficiently smooth f_i,*

$$\mathcal{D}\prod_{i=1}^{n} f_i = \sum_{i=1}^{n}\left(\prod_{j:j\neq i} f_j(\mathcal{D}f_i)\right) - (n-1)\prod_{j=1}^{n} f_j. \tag{3.3.6}$$

Proof of proposition 3.3.1. By induction [the $n = 2$ case is the content of (3.3.4)] we can use (3.3.4) on $\prod_{i=1}^{n} f_i$:

$$\begin{aligned} \mathcal{D}(f_{n+1}\prod_{i=1}^{n} f_i) &= f_{n+1}\left(\sum_{i=1}^{n}(\prod_{j:j\neq i} f_j)\mathcal{D}f_i - (n-1)\prod_{j=1}^{n} f_j\right) \\ &+ (\prod_{i=1}^{n} f_i)\mathcal{D}f_{n+1} - \prod_{j=1}^{n+1} f_j \\ &= \sum_{i=1}^{n+1}(\prod_{j:j\neq i} f_j)\mathcal{D}f_i - n\prod_{j=1}^{n+1} f_j. \end{aligned} \tag{3.3.7}$$

$\ldots\blacksquare$

Proof of Lemma 3.2. Now we can understand the action of $\mathcal{D}$ on

$$P_m(n) = \sum_{i_1+..+i_n=m} c(i_1,..,i_n;m)\mathcal{D}^{i_1}f_1\ldots\mathcal{D}^{i_n}f_n,$$

since by proposition 3.3.1,

$$\begin{aligned} &\mathcal{D}\sum_{i_1+..+i_n=m} c(i_1,..,i_n;m)\mathcal{D}^{i_1}f_1\ldots\mathcal{D}^{i_n}f_n \\ =\ &\sum_{j=1}^{n}\sum_{i_1+..+i_n=m} c(i_1,..,i_n;m)(\prod_{k:k\neq j}\mathcal{D}^{i_k}f_k)\mathcal{D}^{i_j+1}f_j \\ -\ &(n-1)\sum_{i_1+..+i_n=m} c(i_1,..,i_n;m)\mathcal{D}^{i_1}f_1\ldots\mathcal{D}^{i_n}f_n. \end{aligned} \tag{3.3.8}$$

One can now proceed by brute force. The $c(i_1,..,i_n;m)$s are really the multinomial coefficients

$$\frac{m!}{i_1!\cdots i_n!} \tag{3.3.9}$$

Keeping this in mind,

$$\sum_{j=1}^{n}\sum_{i_1+..+i_n=m} c(i_1,..,i_n;m)(\prod_{k:k\neq j}\mathcal{D}^{i_k}f_k)\mathcal{D}^{i_j+1}f_j \tag{3.3.10}$$
$$=\sum_{i_1+..i_n=m+1}\sum_{j=1}^{n} c(i_1,.,i_j-1,.,i_n;m)(\prod_{k:k\neq j}\mathcal{D}^{i_k}f_k)\mathcal{D}^{i_j}f_j$$

by letting $i_j \to i_j - 1$. However, since the $c(i_1,..,i_n;m)$ s are multinomial coefficients, $c(i_1,.,i_j-1,.,i_n;m) = i_j c(i_1,..,i_n;m)$. Since $\sum_{k=1}^{n} i_k = m+1$, it follows that

$$\sum_{j=1}^{n} i_j c(i_1,..,i_n;m) = c(i_1,..,i_n;m+1) \tag{3.3.11}$$

as can be seen from (3.3.9). Inserting this last identity in the last line of (3.3.10) yields

$$\sum_{i_1+..+i_n=m+1} c(i_1,..i_n;m+1)\mathcal{D}^{i_1}f_1...\mathcal{D}^{i_n}f_n \equiv P_{m+1}(n). \tag{3.3.12}$$

It therefore follows from (3.3.10), (3.3.8) that for $n \geq 2$ and $m \geq 0$,

$$\mathcal{D}P_m(n) = P_{m+1}(n) - (n-1)P_m(n). \tag{3.3.13}$$

Using (3.3.13) the Lemma follows by induction on m [the base case is the content of proposition 3.3.1],

$$\mathcal{D}^{m+1}\prod_{i=1}^{n} f_i = \mathcal{D}\sum_{i=0}^{m}\binom{m}{i}(-1)^i P_{m-i}(n)(n-1)^i \tag{3.3.14}$$
$$= \sum_{i=0}^{m}\binom{m}{i}(-1)^i P_{m-i+1}(n)(n-1)^i - \sum_{j=0}^{m}\binom{m}{j}(-1)^j P_{m-j}(n)(n-1)^{j+1}$$
$$= P_{m+1}(n) + \sum_{i=1}^{m-1}(-1)^i\left(\binom{m}{i}+\binom{m}{i-1}\right)P_{m+1-i}(n)(n-1)^i$$
$$+ (-1)^{m+1}(n-1)^{m+1}P_0(n) = \sum_{i=0}^{m+1}\binom{m+1}{i}(-1)^i P_{m+1-i}(n)(n-1)^i$$

This concludes the proof of Lemma 3.2. ...■

Remark 3.3.1 *The $c(i_1, .., i_n; m)$'s are constants whose values we will not need since the total number of leading order terms produced (counting multiplicities) by the above differentiation is n^m since the 'usual' multinomial expansion for derivatives produces terms in a 1 - 1 correspondence with the expansion of $(a_1 + .. + a_n)^m$.*

As preparation for the proof of (3.1.2) we introduce:

Lemma 3.3 *Suppose that $|\mathcal{D}^i\chi(z)| = c_i|z_c - z|^{-3/2}|z_c - |z||^{-i+1}$ for $1 \leq i \leq m-4$ and $|z| < z_c$. Then for $|z| < z_c$, and $1 \leq j \leq m-3$,*

$$|\mathcal{D}^j \hat{\Psi}_z| \leq c_m |z_c - |z||^{-j+\epsilon}, \tag{3.3.15}$$

where $\epsilon < \min(1/2, (d-8)/4$ (w.l.o.g. the same result holds if we replace $\hat{\Psi}_z(0)$ with $v(z)$).

The proof of Lemma 3.3 proper will be deferred until section 3.6.

Remark 3.3.2 *One really must assume an apriori bound on $\mathcal{D}^i\chi(z)$ to say anything about $\mathcal{D}^{i+1}\chi(z)$ since we do not know anything about the i^{th} derivative of $v(z)$ when using (3.3.3) on (3.2.8) as will be explained later on. It will turn out that the latter will only involve derivatives of $\chi(z)$ up to order i.*

3.4 The body of the proof

Let it be understood that from here on in, $\epsilon < \min(1/2, (d-8)/4)$.

Theorem 3.4 *For $m \geq 4$ we have*

$$\chi_m(z) = (2m-5)!!\chi(z)^{2m-3}v(z)^{m-2} + E_m(z), \tag{3.4.1}$$

where $|E_m(z)| \leq c_m|z_c - z|^{-3/2} \cdot |z_c - |z||^{-m+3+\epsilon}$ for $|z| < z_c$.

Proof of Theorem 3.4.

The idea is that the leading term in (3.2.8) will yield the main term in (3.4.1). By (3.2.8), $\chi_3(z) = \chi(z)^3 v(z) + E(z)$ where

$$E(z) = (3\hat{\Psi}_z(0)z\Omega + 2)(z\Omega\chi(z)^2) + \hat{\Psi}_z(0)(1 + 3z\Omega\chi(z)), \tag{3.4.2}$$

and $v(z) = (z\Omega)^2(1 + z\Omega\hat{\Psi}_z(0))$. In other words, after a little algebra

$$\begin{aligned} \chi_4(z) &= \mathcal{D}\chi_3(z) = \mathcal{D}(\chi(z)^3 v(z) + E(z)) \\ &= 3\chi(z)^2 v(z)(\chi(z)^3 v(z) + E(z)) \qquad (3.4.3) \\ &+ \chi(z)^3 \mathcal{D}v(z) - 3\chi(z)^3 v(z) + \mathcal{D}E(z). \qquad (3.4.4) \end{aligned}$$

By Lemma 3.3, $|\mathcal{D}v(z)| \leq c|z_c - |z||^{-1+\epsilon}$ so that

$$|\chi(z)^3 \mathcal{D}v(z)| \leq c|z_c - z|^{-3/2} \cdot |z_c - |z||^{-1+\epsilon}.$$

By similar arguments, $|\mathcal{D}E(z)|$ can be dominated by the error term above. Thus, we have proven (3.4.1) for $m = 4$ (the base case of the induction).

The point is that the leading order term is produced when $\mathcal{D}$ operates *exclusively* on the $\chi(z)$; as soon as a $\mathcal{D}$ operates on $v(z)$, we obtain a lower order term. To prove the general result, we proceed by induction and use Lemma 3.2 to view $\mathcal{D}$ as an ordinary derivative and inductively drop any resulting lower order terms when using (3.2.8), in the expression for $\chi_m(z)$ below:

$$\begin{aligned} \chi_m(z) &= \mathcal{D}^{m-3}\chi_3(z) = \mathcal{D}^{m-3}(\chi(z)^3 v(z) + E(z)) \qquad (3.4.5) \\ &= \sum_{i=0}^{m-3} \binom{m-3}{i} \mathcal{D}^i v(z) \mathcal{D}^{m-3-i}\chi^3(z) \qquad (3.4.6) \\ &+ \sum_{j=1}^{m-4} (-1)^{j+1} \binom{m-4}{j} \sum_{i=0}^{m-3-j} \binom{m-3-j}{i} \mathcal{D}^{m-3-j}\chi(z)^3 \mathcal{D}^j v(z) \qquad (3.4.7) \\ &+ (-1)^m \chi(z)^3 v(z) + \mathcal{D}^{m-3}E(z). \qquad (3.4.8) \end{aligned}$$

For the induction, we will assume the hypotheses of Lemma 3.3; in particular these *imply* that:

$$|\mathcal{D}^j\chi(z)| \leq c_j|z_c - z|^{-1/2} \cdot |z_c - |z||^{-j}, \tag{3.4.9}$$

for $1 \leq j \leq m-4$ and $|z| \leq z_c$. Now suppose that $m \geq 5$. By harnessing the content of (3.4.9) using Lemma 3.2 with $P_m(3)$ and $f_k = \chi(z)$ for $k = 1, 2, 3$, together with Lemma 3.3, we can bound[1] *any* term in (3.4.6) with $1 \leq i \leq m-3$ by

$$\begin{aligned}
& |z_c - |z||^{-i+\epsilon} \sum_{i_1+i_2+i_3=m-3-i} |\mathcal{D}^{i_1}\chi(z)| \ldots |\mathcal{D}^{i_3}\chi(z)| \\
\leq\ & c_m|z_c - |z||^{-i+\epsilon} \sum_{i_1+i_2+i_3=m-3-i} \prod_{j=1}^{3}(|z_c - z|^{-1/2}|z_c - |z||^{-i_j}) \\
\leq\ & c_m|z_c - |z||^{-i+\epsilon} \cdot (|z_c - z|^{-3/2} \cdot |z_c - |z||^{-(m-3-i)}) \\
=\ & c_m|z_c - |z||^{-m+3+\epsilon} \cdot |z_c - z|^{-3/2}.
\end{aligned} \tag{3.4.10}$$

If however, $i = 0$, in (3.4.6), we can recover the leading term in (3.4.1). Before we do so, note that by exactly the same procedure as above, the $i = 0$ term in (3.4.6) could be bounded by $|z_c - z|^{-3/2} \cdot |z_c - |z||^{-m+3}$ [cf Lemma 3.3]. This would then imply that the lower order terms in (3.4.7) could all be bounded by $|z_c - z|^{-3/2} \cdot |z_c - |z||^{-m+4}$, since for these terms, the m in $P_m(n)$ of Lemma 3.2 is effectively knocked down by at least 1. The same bound can be obtained for $|\mathcal{D}^{m-3}E(z)|$ in (3.4.8).

At the risk of belabouring the above, it follows by the induction hypothesis that $|\mathcal{D}^{m-3}\chi_3(z)| \leq c|z_c - z|^{-3/2}|z_c - |z||^{-m+3}$. The only way that the leading term may come about is by letting $\mathcal{D}$ operate only on $\chi(z)$ and ignoring all other terms produced (remember that by Lemma 3.2, $\mathcal{D}$ can be viewed to leading order as a derivation). This is now illustrated:

[1] it clearly suffices to consider the leading term in Lemma 3.2

$$
\begin{aligned}
\chi_m(z) &\equiv \mathcal{D}^{m-3}v(z)\chi(z)^3 &(3.4.11)\\
&= v(z)\mathcal{D}^{m-4}\{3\chi(z)^2(\chi(z)^3v(z)+E(z))\}+l.o.t.\\
&= 3v(z)^2\mathcal{D}^{m-5}\{5\chi(z)^4(\chi(z)^3v(z)+E(z))\}+l.o.t.\\
&\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&= (2m-7)!!v(z)^{m-3}\{(2m-5)\chi(z)^{2m-6}(\chi(z)^3v(z)+E(z))\}+l.o.t.\\
&= (2m-5)!!v(z)^{m-2}\chi(z)^{2m-3}+l.o.t.
\end{aligned}
$$

By the arguments preceeding this computation, any $l.o.t.$ may be bounded by (3.4.10), since $\mathcal{D}$ would have operated on at least one $v(z)$, or better still, an $E(z)$. ...■

The rest of the chapter will be devoted to proving Lemma 3.3.

3.5 The structure of derivatives of $\hat{\Psi}_z(0)$.

Before starting the proof of Lemma 3.3 in earnest, we first discuss the structure of $\hat{\Psi}_z$ and how it will be affected by successive applications of $\mathcal{D}$. Recall that $\hat{\Psi}_z$ is given by:

$$
\begin{aligned}
\hat{\Psi}_z(0) &= \sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|}\left(\prod_{j:j\neq i}\gamma_j\right)J[0,|\omega|] &(3.5.1)\\
&+ \sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|}\left(\prod_{j:j\neq i}\gamma_j\right)\sum_{L\in\mathcal{L}[0,|\omega|]}\mathcal{U}(L,i)\\
&\times \left(\sum_{|\omega'|\geq 1} z^{|\omega'|}\left(\prod_{k=0}^{|\omega'|}\gamma'_k\right)(1+\mathcal{V}'_0)I_{[j\notin L]}J[0,|\omega'|;\mathcal{X}]\right.\\
&+ \left.\sum_{|\omega'|\geq 1} z^{|\omega'|}\left(\prod_{k=0}^{|\omega'|}\gamma'_k\right)(1+\mathcal{V}'_0)I_{[j\in L]}\sum_{l=0}^{|\omega'|}K[0,l;\mathcal{X}]J[l,|\omega'|;\mathcal{Y}^{(l)}]\right)
\end{aligned}
$$

We start with the first term in (3.5.1), and define $\Phi_z = \sum_{n\geq 1}\Phi_n(z)$ where

$$\Phi_m(z) = \sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|}\left(\prod_{j:j\neq i}\gamma_j\right) J_m[0,|\omega|]. \tag{3.5.2}$$

We next deal with the second part of (3.5.1).

Define $T(z) = \sum_{m,n\geq 1} T^{(1)}_{m,n}(z) + T^{(2)}_{m,n}(z)$. Where $T^{(1)}_{m,n}(z)$ and $T^{(2)}_{m,n}(z)$ comprise the cases $j \notin L$ and $j \in L$ respectively. Thus,

$$\begin{aligned} T^{(1)}_{m,n}(z) &= \sum_{|\omega|\geq 1} z^{|\omega|}\sum_{j=0}^{|\omega|}\left(\prod_{i:j\neq i}\gamma_i\right)\sum_{L\in\mathcal{L}_n[0,|\omega|]}\mathcal{U}(L,i) \\ &\times \sum_{\substack{|\omega'|\geq 1\\ \omega'(0)=\omega(j)}} z^{|\omega'|}\left(\prod_{k=0}^{|\omega'|}\gamma'_k\right)(1+\mathcal{V}'_0)I_{[j\notin L]}\sum_{L\in\mathcal{L}_m[0,|\omega'|]} J_m[0,|\omega'|;\mathcal{X}]. \end{aligned} \tag{3.5.3}$$

Diagrammatically, $T^{(1)}_{m,n}(z)$ will be a sum of 2^{m+n-2} $(m+n)$ loop 'ladders' as will be explained later. Similarly for the case where $j\in L$ we write the corresponding term

$$\begin{aligned} T^{(2)}_{m,n}(z) &= \sum_{|\omega|\geq 1} z^{|\omega|}\sum_{j=0}^{|\omega|}\left(\prod_{i:j\neq i}\gamma_i\right)\sum_{L\in\mathcal{L}_n[0,|\omega|]}\mathcal{U}(L,i) \\ &\times \sum_{\substack{|\omega'|\geq 1\\ \omega'(0)=\omega(j)}} z^{|\omega'|}\left(\prod_{k=0}^{|\omega'|}\gamma'_k\right)(1+\mathcal{V}'_0) \\ &\times I_{[j\in L]}\sum_{l=0}^{|\omega'|}\mathcal{W}_l K[0,l;\mathcal{X}]J_m[l,|\omega'|;\mathcal{Y}^{(l)}] \end{aligned} \tag{3.5.4}$$

Before proceeding, we introduce one last inequality involving $J_n[0,|\omega|;U]$ and the so called compatible edges $\mathcal{C}(L)$ of definition 2.1.1. Recall that

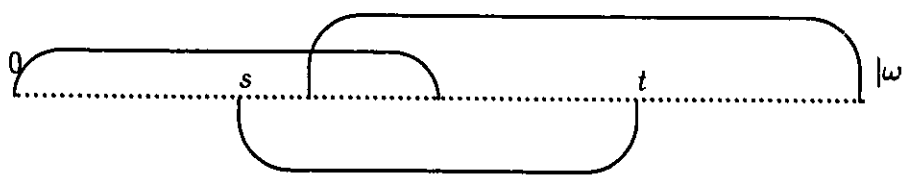
$$\mathcal{C}(L) = \{st\notin L : \mathcal{L}_{L\cup\{st\}} = L\}.$$

We will now modify the above so that

Definition 3.5.1 *Given a lace* $L = \cup_{i=1}^{n}(s_i t_i) \in \mathcal{L}_n[0, |\omega|]$,

$$\tilde{C}(L) = \{st \notin L : s_i \leq s < t \leq t_i, \textit{ for some } i\}.$$

This new definition amounts to weakening the avoidance restrictions imposed by the edges compatible with a given lace, and doing away with the alternating sign inherent in the product over laces. In terms of ladder diagrams (to be described below), all we are giving up is the avoidance -imposed by the compatible edges- of the branches of certain sides of one loop with its neighbouring loops. An example of a lace L and an edge $st \in C(L)$ such that $st \notin \tilde{C}(L)$ is given below (st is the lower edge):



Thus,

$$\begin{aligned} |J_n[0,|\omega|;U]| &= |\sum_{L \in \mathcal{L}_n[0,|\omega|]} \prod_{st \in L} U_{st} \prod_{s't' \in C(L)} (1 + U_{s't'})| \qquad (3.5.5) \\ &\leq \sum_{L \in \mathcal{L}_n[0,|\omega|]} \prod_{st \in L} |U_{st}| \prod_{s't' \in \tilde{C}(L)} (1 + U_{s't'}) \equiv P_n[0,|\omega|;U] \end{aligned}$$

for any any walk ω, any set of ribs $\{R_0, ..., R_{|\omega|}\}$, and any interaction $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{X}$, or $\mathcal{Y}^{(l)}$.

Now we can state:

Definition 3.5.2 *Let* $\tilde{\Phi}_m(z)$, *and* $\tilde{T}^{(i)}_{m,n}(z)$ *(*$i = 1, 2$*), be the functions that result by first replacing* $J_n[0,|\omega|]$ *by* $P_n[0,|\omega|]$ *in the definitions of* $\Phi_m(z)$ *and* $T^{(i)(z)}_{m,n}$ *respectively. Also, let*

$$\tilde{\Psi}_z(0) \equiv \sum_{m \geq 1} \tilde{\Phi}_m(z) + \sum_{i=1}^{2} \sum_{m,n \geq 1} \tilde{T}^{(i)}_{m,n}(z). \qquad (3.5.6)$$

Remark 3.5.1 *We can then majorise — for positive $z < z_c$ — the derivatives of $\Phi_m(z)$ and $T^{(i)}_{m,n}(z)$ in terms of those of $\tilde{\Phi}_m(z)$ and $\tilde{T}^{(i)}_{m,n}(z)$ respectively, since the* coefficients *in the series expansion of the* $\tilde{(\cdot)}$ *functions dominate those of the original functions by* construction. *Now, since*

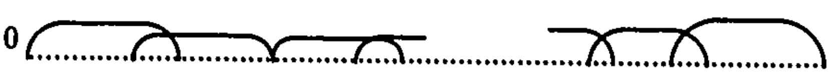
$$\hat{\Psi}_z(0) = \sum_{m\geq 1} \Phi_m(z) + \sum_{i=1}^{2} \sum_{m,n\geq 1} T^{(i)}_{m,n}(z),$$

and since the coefficients of the functions in definition 3.5.2 are positive, *it follows that for any $|z| < z_c$ and any $m \geq 0$,*

$$|\mathcal{D}^m \hat{\Psi}_z(0)| \leq c \left(\sum_{m\geq 1} \mathcal{D}^m \tilde{\Phi}_m(|z|) + \sum_{i=1}^{2} \sum_{m,n\geq 1} \mathcal{D}^m \tilde{T}^{(i)}_{m,n}(|z|) \right). \tag{3.5.7}$$

In view of remark 3.5.1, any bound proven for $\mathcal{D}^m \tilde{\Psi}_\zeta(0)$ with $0 < \zeta < z_c$ $[m \geq 0]$ will carry over for $|\mathcal{D}^m \hat{\Psi}_\xi(0)|_{\xi=|z|}$ | on the disc $|z| \leq \zeta < z_c$.

The functions in definition 3.5.2 can be expressed (and again slightly majorised) for $0 < z < z_c$ in terms of convolutions of 2-point functions. These convolutions may take on certain 'topologies' depending how the laces $L \in \mathcal{L}_n(0, |\omega|)$ are configured.

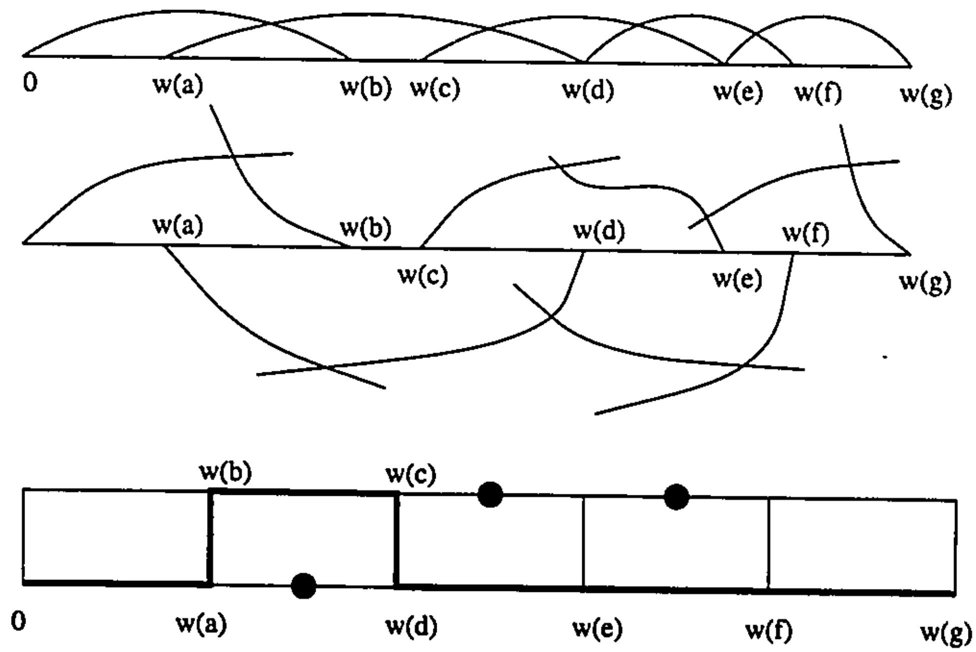
0  $|\omega|$

Consider the above n edge lace. There are 2^{n-1} generic possibilities for edges to be configured, since the right end-point of the i^{th} edge may or may not coincide with the left end-point of the $i+1^{st}$ edge.

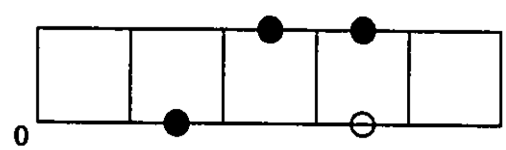
Consider for example the function below:

$$\Phi_5(z) = \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \left(\prod_{j:j\neq i} \gamma_j \right) P_5[0, |\omega|; U]. \tag{3.5.8}$$

Shown below is a typical lace $L \in \mathcal{L}_5[0, |\omega|]$, and below that, the typical rib intersections it would induce.



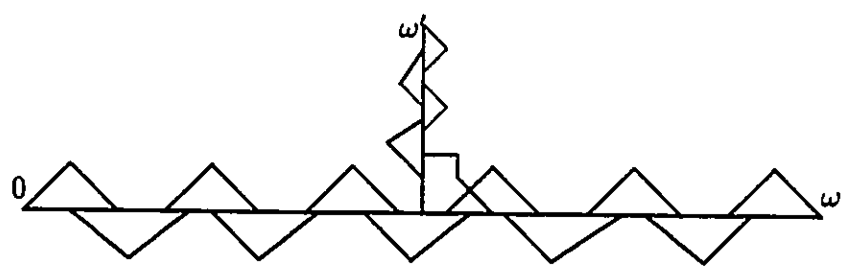
The last of the three diagrams represents the convolutions of 2-point functions that would bound such a diagram; the thick line represents the tree's backbone, and a solid dot represents a convolution of two 2-point functions. This last diagram *does not* take into account the sum over backbone sites in (3.5.8) however. This can be achieved diagrammatically by a sum of diagrams each of which has a line in its backbone replaced by a convolution of 2-point functions as depicted by the hollow circle in the single diagram below (we will use the convention that hollow vertices are *added* vertices).



In general, any of the generic 2^{n-1} loop[2] diagrams will have no more than $\lceil (n+1)/2 \rceil$ lines in its backbone. Thus there will be at most $\lceil (n+1)/2 \rceil \times 2^{n-1}$ diagrams to consider when bounding a member of $\Phi_n(z)$.

[2]These loops may not collapse to a point.

Similar considerations hold for terms comprising $\tilde{T}_{m,n}(z)$; there will be on the order of 2^{m+n} generic ways for the laces in $\tilde{T}_{m,n}(z)$ to be configured, and thus, there will be no more than a multiple of 2^{m+n} generic diagrams generated. The detailed description of terms comprising $\tilde{T}_{m,n}(z)$ will be dealt with in later analysis, but for now, the picture to keep in mind is that of two backbones ω and ω' with ribs intersecting as shown below. In general, there will be one rib belonging to ω' that intersects ω no further than 3 loops[3] away from ω'.



The above diagram illustrates a generic diagram corresponding to $\tilde{T}_{5,11}(z)$.

Eventually, we will have to bound the derivatives of these convolved functions, so we must begin by understanding what happens when $\mathcal{D}^m$ operates on a term $G_z(x,y)$: (recall that $\gamma_i = \sum_{T\ni\omega(i)} z^{|T|}$)

$$\begin{aligned}
\mathcal{D}^m G_z(x,y) &= \mathcal{D}^m \sum_{\omega:x\to y} z^{|\omega|} \prod_{i=0}^{|\omega|} \gamma_i K[0,|\omega|] \qquad (3.5.9)\\
&= \mathcal{D}^{m-1}\left\{\frac{d}{dz}\sum_{\omega:x\to y}\prod_{i=0}^{|\omega|}\sum_{R_i\ni\omega(i)} z^{|R_i|+1} K[0,|\omega|]\right\}\\
&= \sum_{\omega:x\to y}\sum_{p=1}^{m}\sum_{j_1+..+j_p=m} z^{|\omega|}\sum_{0\le i_1<..<i_p\le|\omega|} c_{i_1,\ldots,i_p}\times\\
&\quad \left(\prod_{j:j\neq i_1,\ldots,i_p}\gamma_j\right)\mathcal{D}^{j_1}\gamma_{i_1}\ldots\mathcal{D}^{j_p}\gamma_{i_p}K[0,|\omega|]
\end{aligned}$$

The second equality follows by using the z in $\mathcal{D}$ to obtain $|\omega|+1$ for the number of sites in the back-bone. If we assume that for $z < z_c$, and $1 \le j_k \le m-3$,

[3]Loops formed by the ribs of ω.

$\mathcal{D}^{j_k}\gamma_{i_k}(z) \leq c(z_c - z)^{-j_k+1/2}$, (as per the hypotheses of Lemma 3.3)[4] then the last expression above can be majorised by

$$
\begin{aligned}
& c\sum_{p=1}^{m}(G_z * \cdots * G_z)_{p+1}(x-y)\prod_{k=1}^{p+1}\chi^{2j_k-1} \\
\leq\ & c\sum_{p=1}^{m}(G_z * \cdots * G_z)_{p+1}(x-y)\chi^{2m-p}, \qquad (3.5.10)
\end{aligned}
$$

where $(G_z * \cdots * G_z)_p(x-y)$ denotes the p fold convolution of 2 point functions, and $(f*g)(x)$ is defined in the usual way,

$$(f*g)(x) = \sum_{y\in\mathbb{Z}^d} f(x-y)g(y). \qquad (3.5.11)$$

Case (1)

Let the $4n$ 2-point functions involved in the spatial convolution for $\tilde{\Phi}_n(z)$ be labeled $G_z^1,..,G_z^{4n}$. Then, interchanging the order of convolution and differentiation, and using the the formulæ regarding the action of $\mathcal{D}$ on products of functions, we have:

$$
\begin{aligned}
\mathcal{D}^m\tilde{\Phi}_n(z) &= \sum_{(\text{s.c.})}\mathcal{D}^m\prod_{i=1}^{4n}G_z^i = \sum_{(\text{s.c.})}\ \sum_{i_1+..+i_{4n}=m} c(i_1,..,i_{4n};m)\mathcal{D}^{i_1}G_z^1..\mathcal{D}^{i_{4n}}G_z^{4n} \\
&+ \sum_{(\text{s.c.})}\sum_{j=1}^{m-1}\left\{m(-(n-1))^{m-j}\sum_{i_1+..+i_{4n}=m-j} c(i_1,..,i_{4n};m-1)\mathcal{D}^{i_1}G_z^1..\mathcal{D}^{i_{4n}}G_z^{4n}\right\} \\
&+(-1)^m(n-1)^m\sum_{(\text{s.c.})}\prod_{i=1}^{4n}G_z^i \qquad (3.5.12)
\end{aligned}
$$

(where s.c. denotes the spatial 'convolution arguments'). Bearing in mind that every additional convolution in (3.5.12) corresponds to adding an additional vertex in the spatially convolved 2 point functions, we see, using (3.5.9) and

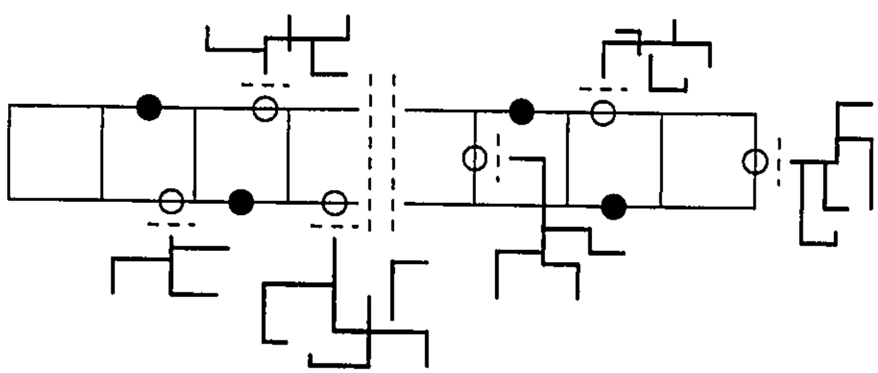
[4]note that it takes one $\mathcal{D}$ to get from $\gamma_{j_k}(z)$ to $\chi(z)$

(3.5.10), that the leading term of (3.5.12) is:

$$\mathcal{D}^m \tilde{\Phi}_n(z) \leq c(4nm)(4n)^m \sum_{\text{c.a.}} \sum_{j=1}^{m} \prod_{i=1}^{4n+j} G_z^i \chi^{2m-j} \leq c(4nm)(4n)^m \sum_{j=1}^{m} \tilde{\Phi}_{n;j}(z) \chi^{2m-j} \tag{3.5.13}$$

The factor $4nm$ accounts for the fact that each term $\mathcal{D}^{i_k} G_z^k$ may give rise to at most m terms, and it is only at this level that the above bound is not sharp in its n dependence (cf. remark 3.3.1).

In terms of diagrams, $\tilde{\Phi}_{n;m}(z)$ has the pictorial representation below,



where the resulting 'ladder' diagrams have been pruned of their extra branches (i.e. the $\chi(z)^{j_k-1}$s) and have extra vertices at the site of the 'pruning' (the solid dots represent vertices which were present before any differentiation). In diagrammatic notation, $\tilde{\Phi}_{n;p}(z)$ is essentially a sum of $\tilde{\Phi}_n(z)$s with p distinctly distributed *extra* vertices.

Case 2.

By exactly the same methods we can deduce that for $i = 1$ or 2

$$\mathcal{D}^p \tilde{T}^{(i)}_{m,n}(z) \leq c(4(n+m)p)(m+n)^p \sum_{\text{c.a.}} \sum_{j=1}^{p} \prod_{i=1}^{m+n+j} G_z^i(\cdot) \chi^{2p-j}$$

$$\equiv \quad c(4(n+m)p)(m+n)^p \sum_{j=1}^{p} \tilde{T}^{(i)}_{m,n;j}(z)\chi^{2p-j}, \tag{3.5.14}$$

where again, $\tilde{T}^{(i)}_{m,n;p}(z)$ is a sum of diagrams such as $\tilde{T}^{(i)}_{m,n}(z)$ with p randomly added extra vertices.

3.6 Proof of Lemma 3.3.

In view of remark 3.5.1, Lemma 3.3 follows from

Lemma 3.5 *For $0 < z < z_c$, and L sufficiently large,*

$$\tilde{T}^{(i)}_{m,n;p}(z) \leq c(n,m)|z_c - z|^{-p/2+\epsilon} \tag{3.6.1}$$

for $i = 1, 2$ and,

$$\tilde{\Phi}_{n;p}(z) \leq c(n)|z_c - z|^{-p/2+\epsilon} \tag{3.6.2}$$

with

$$c(n,m;p) = \begin{cases} c(\frac{1}{2+\delta})^{m+n-p} & \text{for } m+n > p \\ c & \text{for } m+n \leq p \end{cases}, \tag{3.6.3}$$

for some $\delta > 0$, and $c(n;p) \equiv c(n,0;p)$.

Proof of Lemma 3.3. It follows from (3.5.13) (under the same hypotheses as for Lemma 3.3) that

$$\begin{aligned} \mathcal{D}^p\tilde{\Phi}(z) &= \mathcal{D}^p \sum_{n\geq 1} \tilde{\Phi}_n(z) \leq c \sum_{n\geq 1} 2^n(4np)(4n)^p \sum_{j=1}^{p} \tilde{\Phi}_{n;j}(z)\chi^{2p-j} \\ &\leq \sum_{n\geq 1} 2^n(4np)(4n)^p c(n;p) \sum_{j=1}^{p} |z_c - z|^{-j/2+\epsilon} \cdot |z_c - z|^{-p+j/2} \\ &\leq c|z_c - z|^{-p+\epsilon} \end{aligned} \tag{3.6.4}$$

(the 2^n corresponds to the number of generic diagrams produced by the 2^n generic configurations of n laces). Similarly,

$$\mathcal{D}^p\tilde{T}(z) = \mathcal{D}^p \sum_{n,m\geq 1} \sum_{i=1}^{2} \tilde{T}^{(i)}_{n,m}(z) \tag{3.6.5}$$

$$\begin{aligned}
&\leq c \sum_{n,m\geq 1} \sum_{i=1}^{2} 2^{n+m}(4(n+m)p)(m+n)^p \sum_{j=1}^{p} \tilde{T}^{(i)}_{n,m;j}(z)\chi^{2p-j} \\
&\leq c \sum_{n,m\geq 1} 2^{n+m}(4(n+m)p)(m+n)^p c(n,m;p) \times \\
&\quad \sum_{j=1}^{p} |z_c - z|^{-j/2+\epsilon} \cdot |z_c - z|^{-p+j/2} \leq c|z_c - z|^{-p+\epsilon}
\end{aligned}$$

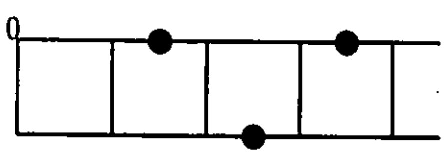
(again there are on the order of 2^{m+n} generic diagrams to consider).

This concludes the proof of Lemma 3.3. ...■

3.7 The proof of Lemma 3.5

This section will deal with the estimates needed to bound $\tilde{\Phi}_{m;p}(z)$ and later $\tilde{T}^{(i)}_{m,n;p}(z)$. Roughly speaking, the idea is to break-up the convolutions comprising these functions into a product of terms, some of which are small (and therefore responsible for the geometric convergence when summing on m and n), and some of which diverge as $z \nearrow z_c$. A later Lemma will provide a universal estimates for any divergent diagram.

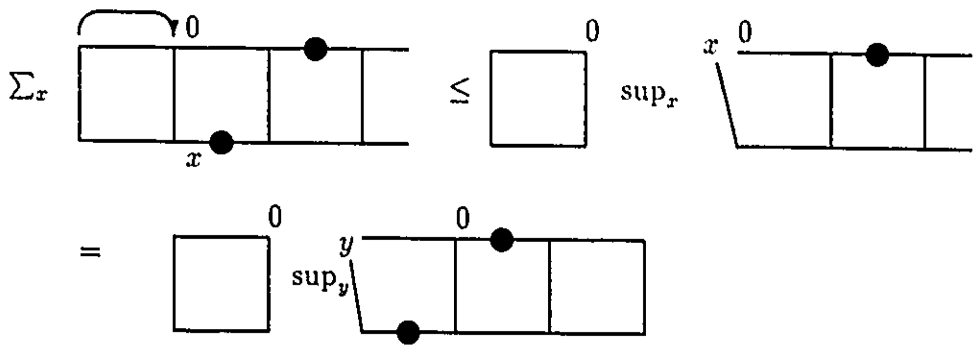
The ingredient in what is to follow, is the translation invariance and symmetry of the 2 point function. As a consequence of this, the diagram below



may be changed so that it is pinned down with the zero being at any vertex of our choosing. Starting from left to right, we then repeatedly use the inequality

$$|\sum_x f(x)g(x)| \leq \sup_y |f(y)| \sum_x |g(x)|. \tag{3.7.1}$$

The result of the first application of this procedure is shown below.

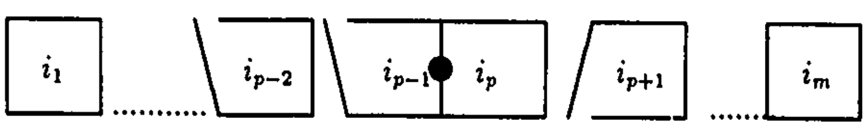


where y is the vector separating the broken end-points.

Proof of Lemma 3.3 Using the above procedure, we start by majorising $\tilde{\Phi}_{m;n}(z)$. Diagrammatically, we have:

where $i_1 + i_2 + ... + i_m = n + 1$ (extra 1 accounts for the 'hollow' vertex that takes care of counting the number of sites in the backbone) and i_j counts the number of extra vertices on the top, bottom, and right lines of the j^{th} loop — with the obvious caveat that i_1 include the vertices on the left-most line. By repeated use of (3.7.1), we arrive at

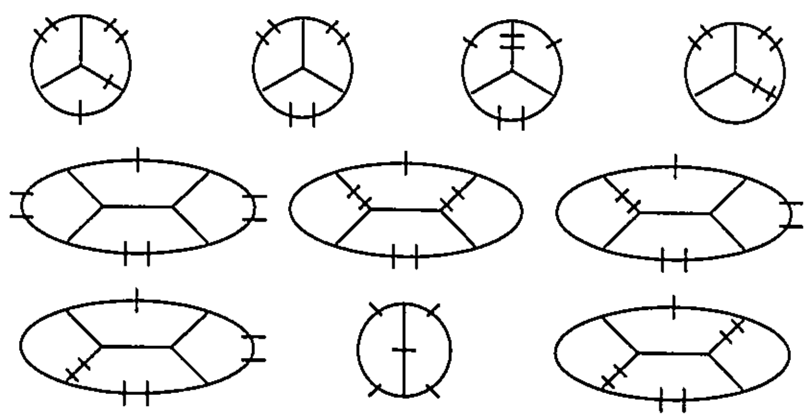
or



depending on whether any of the extra vertices fall on of the top or bottom lines. If any do, we can use *one* of these *extra* vertices to yield single loop diagrams (i.e. the top case); if none do, we must be left with one double loop diagram with an added vertex as shown. In *either* event, we now have

$\sum i_k = n$ (the extra 1 was used to make a square or to 'dot' the 2 loop diagram). By possibly re-labeling, let $\{\bar{i}_1, .., \bar{i}_q\}$ be the subset of non-zero i_ks for either distribution of vertices (with $\bar{i}_q \equiv i_p + i_{p-1}$ if $i_p + i_{p-1} \geq 1$ in the second case). We begin with a definition that will be used in this and other sections:

Definition 3.7.1 *Let $\mathcal{I}(n,z)$ be the function which dominates for $0 < z < z_c$ the modulus of the square diagram and any of the 10 diagrams below when n vertices are added, for $d > 8$ (independently of where the vertices are added).*



The following Lemma will allow us to deal with the 2 loop diagram and the other indecomposable diagrams which will result when using (3.7.1) to bound $\tilde{T}_{m,n}(z)$.

Lemma 3.6 *For $0 < z < z_c$, and L sufficiently large,*

$$\mathcal{I}(n,z) \leq c|z_c - z|^{-n/2+\epsilon} \tag{3.7.2}$$

for $n \geq 1$ and $d > 8$ independent of the distribution of vertices [the 1 loop diagram being the worst offender].

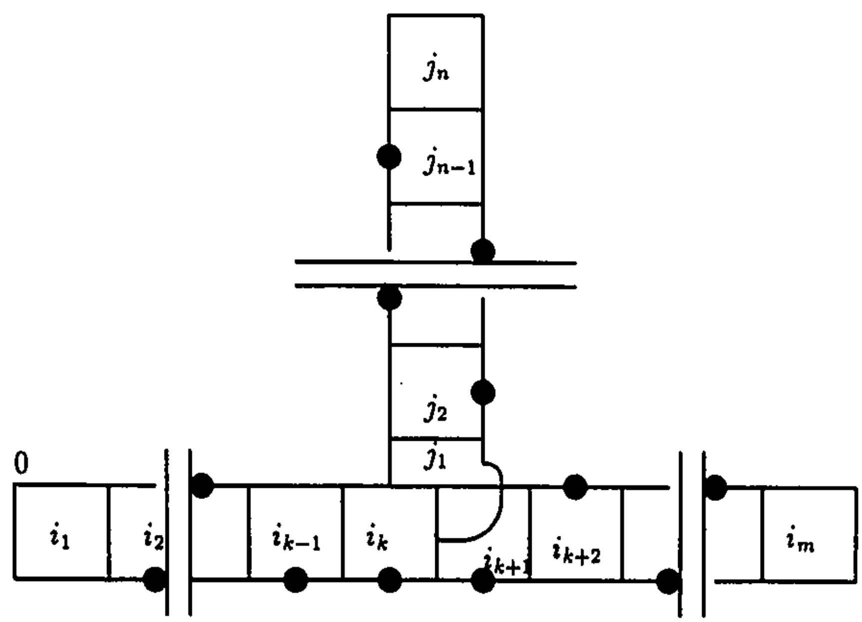
We now use (3.7.2), which in particular gives bounds on the value of a one or two loop graph with n extra vertices:

$$\prod_{k=1}^{q} \mathcal{I}(\tilde{n}_k, z) \;\; \leq \;\; c \prod_{i=1}^{q} |z_c - z|^{-\tilde{n}_i/2+\epsilon} \tag{3.7.3}$$

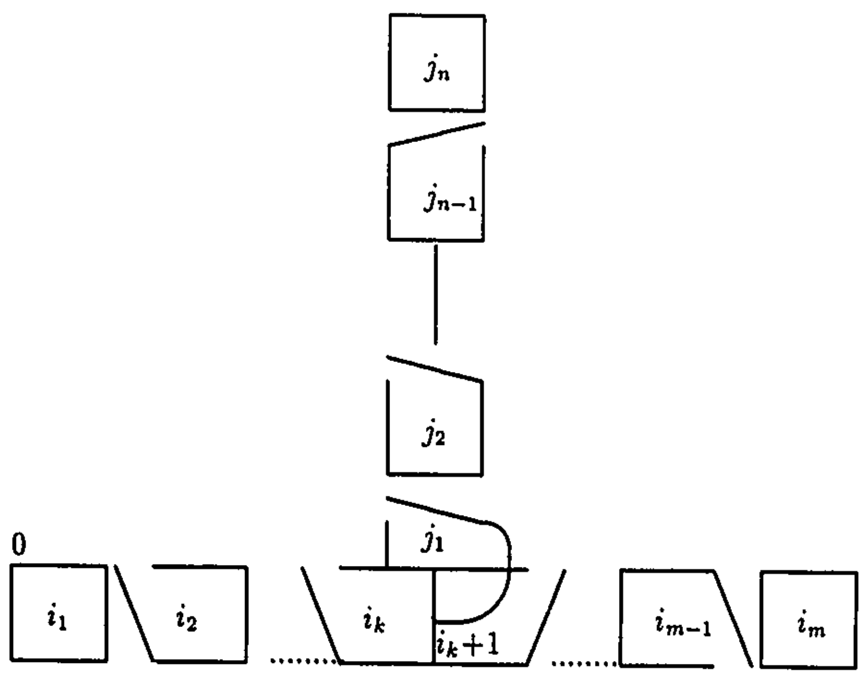
$$\leq \; c|z_c - z|^{-n/2+q\epsilon} \leq c|z_c - z|^{-n/2+\epsilon}$$

for $0 < z < z_c$ as per Lemma 3.3 by remark 3.5.1. If $m > n$ there will be $m - n$ zero i_ks; these 'square' diagrams could have been no more than $\frac{1}{2+\delta}$ for the original expansion to converge. This is what provides us with the bound for $c(m;n)$ in (3.6.1). This concludes the analysis of terms coming from derivatives of $\tilde{\Phi}_{m;n}(z)$. ...■

We must now deal with the terms corresponding to the bounding of $\tilde{T}_{m,n;p}(z)$ by the same procedure as for $\tilde{\Phi}_{m;p}(z)$. The obstacle in this case is that not only will the secondary backbone ω' intersect ω, but so too will one of its ribs (via the compatible bonds of associated with ω's lace expansion). These intersections will lead to 10 irreducible diagrams which will be dealt with separately.



The following conventions will be used: i_j is the number of added vertices on the top, bottom and right-most edge of the j^{th} loop (with the exception of i_1 where the left edge is also considered) for $j \leq k$, k being the first loop intersected by ω' or a primed rib [cf section 2.2]. For $j > k$ the convention is reversed in the sense that i_j then encompasses the extra vertices on the top, bottom, and left-most (except for i_{k+1}) edges of the j^{th} loop (again i_m also includes any added vertices on the right-most edge of the diagram). Similarly, j_k counts the number of added vertices on the left, right and bottom of the k^{th} loop (with the obvious caveat for the top loop and the bottom edge for j_1). The after-math of majorising the above diagram is:



In general, after having bounded a diagram, there may be up to 3 consecutive i_ks as well as a j_1 (see diagram) which contribute to an irreducible diagram. Let n_0 be the sum of these i_ks and j_1, then let $\{n_1, ..., n_f\}$ be a re-labeling of all the non-zero i_ks and j_ks which do not contribute to an irreducible diagram. Clearly $\sum_{i=0}^{f} n_i = \sum_k i_k + \sum_k j_k = p$, and thus by using (3.7.2),

$$\prod_{k=0}^{f} \mathcal{I}(n_k, z) \leq |z_c - z|^{-p/2+(f+1)\epsilon} \leq |z_c - z|^{-p/2+\epsilon} \tag{3.7.4}$$

for $0 < z < z_c$ as per Lemma 3.3 in view of remark 3.5.1. (i.e. the worst bound results when all the extra vertices are piled onto a single diagram). Now again, if $p < m + n$, there will surely be some zero i_ks and j_ks which will yield $m + n - p$ 'square' diagrams. Each of these will be at most $\frac{1}{2+\delta}$ for L at least as large as in [HS3], thus yielding the constant $c(m, n; p)$ of (3.6.2). This concludes the proof of Lemma 3.5 assuming Lemma 3.6. $\ldots\blacksquare$

The next section will deal with the 10 diagrams which result from the intersection of ω' and the rib - 'primed' rib interactions. These are the diagrams of definition 3.7.1, which are then used in Lemma 3.6.

3.8 The 10 spurious diagrams

The spurious diagrams *without* the p added vertices (of definition 3.7.1) arise from the primed-rib rib interactions mediated by $\mathcal{X}_{kl}$ in (2.2.8).

In the 4 diagrams below, one should view the long straight line in the above diagram as the back-bone ω of the tree, and the other black line with the dotted lines re-intersecting the back-bone as the back-bone of ω' (the dotted lines represent the t in $ti \in \mathcal{C}(L)$ of (2.2.9) - i.e. the position of the rib R_t which intersects ω' via some other rib R'_l).

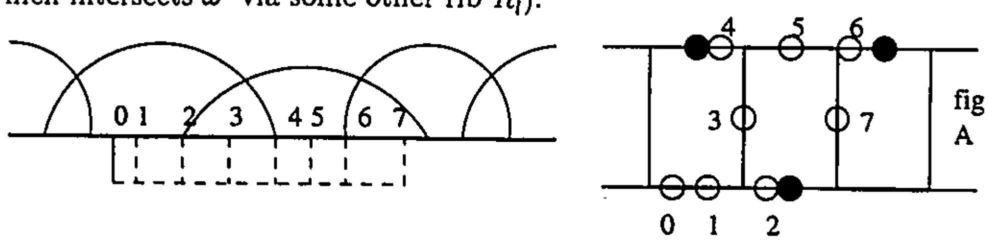


fig A

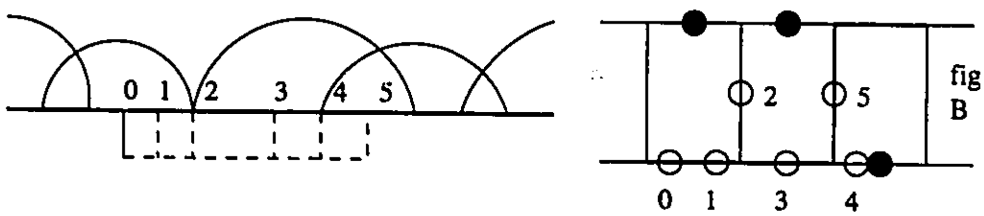


fig B

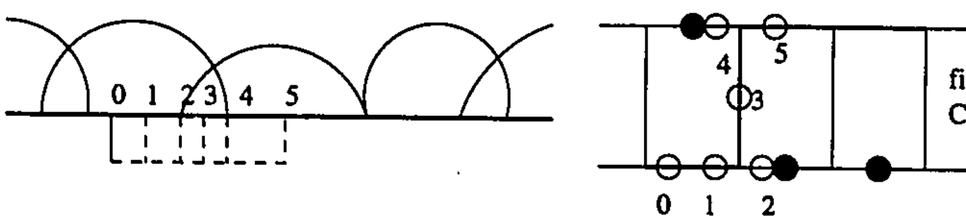


fig C

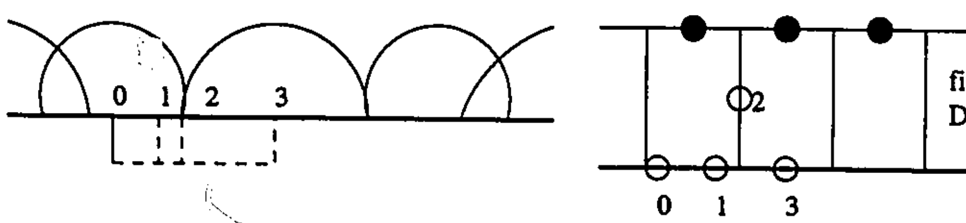
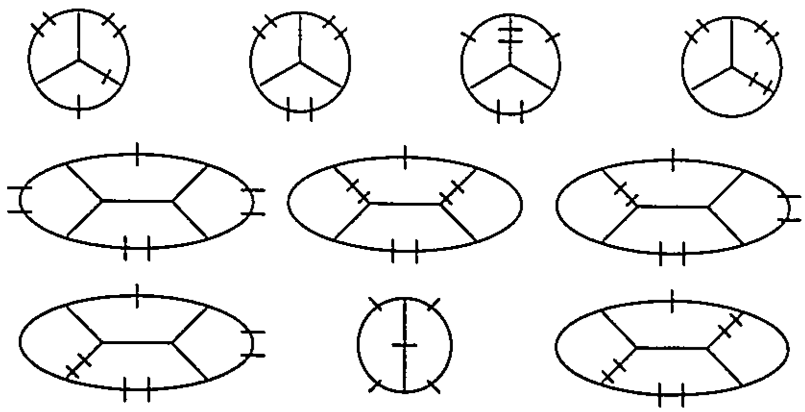


fig D

The four diagrams account for all generic lace configurations, and the dotted diagrams account for all the possible intersections via compatible edges.

The diagrams below correspond to the possible intersections, and were isolated in [HS4].



3.9 The bounds on the ten irreducible diagrams

Having isolated the 11 fundamental diagrams arising from the double lace expansion for $\hat{\Psi}_z(0)$, we set about estimating their behavior for $0 < z < z_c$ after adding n extra vertices to any of them, as per the content of Lemma 3.6.

The idea is to bound these diagrams in momentum space by using a theorem of T. Riesz [R] dealing with lattice Feynman diagrams; the propagators in *our* diagrams will be the Fourier transform of the 2-point function. Since our estimates need only be valid for z positive, we will use the following lower bound.[5]

Lemma 3.7 *For $d > 8$ and L large enough, $\exists\ c > 0$ such that*

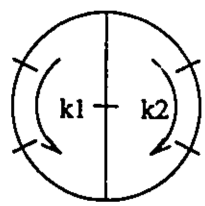
$$|\hat{G}_z(k)| \leq \frac{c}{k^2 + |z_c - z|^{1/2}}$$

for $z \in [(5\Omega)^{-1}, z_c)$ and $k \in [-\pi, \pi]^d$.

[5] In fact, the Lemma 3.7 could be strengthened to hold for *all* $|z| < z_c$.

Proof. See [HS4] p. 1020 (3.4) ...■

We now turn to the task of estimating the value of the above diagrams by computing their value in momentum space. Setting-up these integrals is a standard exercise in mathematical physics. The number of integration variables will correspond to the number of loops; the momenta flowing through any line are the super-position of the individual loop momenta in the case where a line is shared by two loops. As an example consider the 2 loop diagram. The integral corresponding to the diagram below



(Here there are no added vertices).

A 2 loop diagram

is,

$$\int_{[-\pi,\pi]^{2d}} d^d k_1 d^d k_2 \hat{G}_z(k_1)^3 \hat{G}_z(k_2+k_1)^2 \hat{G}_z(k_2)^3 \qquad (3.9.1)$$

since the middle line has 2 propagators with the shared momenta k_1 and k_2, and the other lines have 3 propagators.

To clear-up some of the above lingo we introduce:

Definition 3.9.1

1) A propagator corresponds to the Fourier transform of a 2-point function.
2) A 'line' is a set of propagators with exactly the same momenta.
(Note that in all our diagrams, a 'line' will be a line with vertices of coordination number 3 at its end-points).

For convenience we will put $\lambda = |z_c - z|^{-1/2}$; using (3.7), we obtain

$$\int_{[-\pi,\pi]^{2d}} d^d k_1 d^d k_2 \frac{c}{(k_1^2+\lambda^{-1})^3((k_1+k_2)^2+\lambda^{-1})^2(k_2^2+\lambda^{-1})^3}. \qquad (3.9.2)$$

(appendix B works out the $U.V.$[6] divergence of the above example using the method in [R]).

Proof of lemma 3.6.

We start by obtaining (3.7.2) for the square diagram. For convenience, let

$$\lambda \equiv |z_c - z|^{-1/2}. \qquad (3.9.3)$$

By the same philosophy as in [HS4], and using Lemma 3.7 the square diagram with n added vertices can be bounded by

$$\int_{[-\pi,\pi]^d} d^d k \frac{c}{(k^2+\lambda^{-1})^{4+n}}. \qquad (3.9.4)$$

Since $\lambda^{-1} \searrow 0$ as $z \nearrow z_c$, it is clear that the above integral will be I.R.[7] singular for d *small* enough. Let $k = \lambda^{-1/2}k'$, and scale out the λs to get

$$\mathcal{I}(n,d,\lambda) = \lambda^{(8-d)/2+n} \int_{[-\pi\lambda^{1/2},\pi\lambda^{1/2}]^d} d^d k \frac{c}{(k^2+1)^{4+n}}. \qquad (3.9.5)$$

We have now turned the problem inside-out. As $z \nearrow z_c$, the domain of integration now extends to all of $\mathbb{R}^d$ (growing as $\lambda^{1/2}$), and the integral is now I.R. finite (since the propagators are now massive), but U.V. singular for d *large* enough. The key to turning the problem right-side-out is the $\lambda^{(8-d)/2+n}$ which was scaled out:

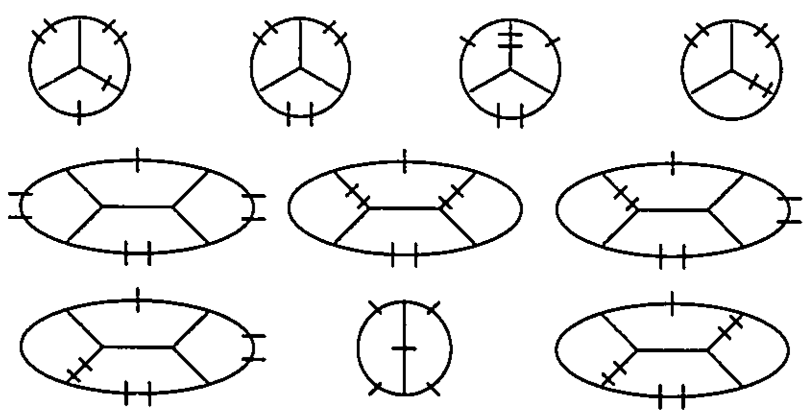
$$\begin{aligned} \mathcal{I}(n,d,\lambda) &\le \lambda^{(8-d)/2+n} \begin{cases} 1 & \text{if } (d-8)-2n<0 \\ \lambda^{(d-8)/2-n}\log(\lambda) & \text{if } (d-8)-2n \ge 0 \end{cases} \\ &\le \begin{cases} \lambda^{(8-d)/2+n} & \text{if } (d-8)-2n<0 \\ \log(\lambda) & \text{if } (d-8)-2n \ge 0 \end{cases}. \end{aligned} \qquad (3.9.6)$$

[6]an Ultra Violet divergence is one which occurs for large values of k.
[7]Infra Red divergences are those that occur for small values of k.

The worst case is $d = 9$, so that $\mathcal{I}(n,d,\lambda) \lesssim \lambda^{n-1/2}$ for $n \geq 1$. For $d \geq 10$ and $n = 1$ we will use the bound $\log(\lambda) \leq c|z_c - z|^{-\delta}$ for any $\delta > 0$. Thus, for $d \geq 9$, we can write the uniform estimate $\mathcal{I}(n,d,\lambda) \leq c|z_c - z|^{-n/2+\epsilon}$ for any $\epsilon < \min(1/2, (d-8)/4)$.

The above will be the tactic to estimate the 2, 3 and 4 loop diagrams. By substituting $k_i = \lambda^{-1/2}k_i'$ for the integration variables, and by observing that for all 10 of the original diagrams the number of propagators is 4 times the number of loops, we obtain integrals of the form considered in the appendix multiplied by a factor of $\lambda(z)^{l(8-d)/2+n}$ (n is the number of added vertices). The volume cut-off for these integrals will of course be $\lambda^{1/2}$.

It is now time to invoke the results in the appendix to deal with remaining 10 graphs:



The results of [R] state that the overall u.v. divergence of a graph with l loops and volume cutoff ρ can be bounded above by $\rho^{\omega(H)}\log(\rho)^l$ if $\omega(H) \geq 0$, where

$$\omega(H) = \max_{S \subseteq P}(\omega(S)) \tag{3.9.7}$$

and $\omega(S)$ and P are defined as follows: let N be the number of lines [cf. def. 3.9.1] on the graph, and let n_i ($1 \leq i \leq N$) be the number of propagators on the i^{th} line. Now let $P = \{Q \subset \{1, ..., N\} : |Q| \leq l-1\}$ where $|Q|$ denotes the cardinality of Q. For any $S \subseteq P$, let $\omega(S) = d(l - |S|) - 2(\sum_{i=1}^{N} n_i - \sum_{j\in S} n_j)$.

The bounds for the 1 loop case have already been worked out; to deal with the remaining diagrams it suffices to have:

Proposition 3.9.1 *$\omega(H) \leq \max(l(d-8) - 2n, (l-1)(d-8) - 2)$ for $n \geq 1$, for any distribution of extra vertices, and for any of the 10 diagrams shown in definition 3.7.1.*

(Proof of proposition following Lemma)

We henceforth assume $l \geq 2$. If we let $\mathcal{G}(n, l, d, \lambda^{1/2})$ denote the value of an $l \leq 2$ loop diagram with n added vertices in d dimensions with volume cutoff $\lambda^{1/2}$, the bound on $\omega(H)$ in proposition 3.9.1 will give us

$$\mathcal{G}(n,l,d,\lambda) \leq c \begin{cases} 1 & \text{if } \omega(H) < 0 \\ \lambda^{\omega(H)/2}(\log\lambda)^4 & \text{if } \omega(H) \geq 0. \end{cases} \tag{3.9.8}$$

Multiplying by the overall factor of $\lambda^{l(8-d)/2+n}$ that was scaled out in the beginning (when turning an I.R. problem into a U.V. problem), yields

$$\lambda^{l(8-d)/2+n}\mathcal{G}(n,l,d,\lambda) \leq c \begin{cases} \lambda^{l(8-d)/2+n} & \text{if } \omega(H) < 0 \\ \lambda^{l(8-d)/2+n+\omega(H)/2}(\log\lambda)^l & \text{if } \omega(H) \geq 0. \end{cases} \tag{3.9.9}$$

By proposition 3.9.1, $\omega(H) < 0$ if $d < 8 + \max(2/(l-1), 2n/l)$, and for these cases, we obtain the bound $c|z_c - z|^{l(d-8)/4-n/2}$ in (3.9.9). The next worst case is if $\omega(H) \geq 0$ and $l(d-8) - 2n \leq (l-1)(d-8) - 2$, i.e. if $d \leq 8 + 2(n-1)$, which yields $c|z_c - z|^{l(d-8)/4-(n-1)/2} \log|z_c - z|$ in (3.9.9). If on the other hand $\omega(H) \geq 0$ and $l(d-8) - 2n > (l-1)(d-8) - 2$, we can bound (3.9.9) by $c(\log|z_c - z|)^4$ for $d > 8 + 2(n-1)$.

By combining these bounds with those for the square diagram (which is clearly the worst case), we obtain the universal estimate:

$$\mathcal{I}(n,z) \leq c|z_c - z|^{-n/2+\epsilon} \qquad (3.9.10)$$

(with $\epsilon < \min(1/2, (d-8)/4)$) for $0 < z < z_c$ and $d \geq 9$. This concludes the proof of Lemma 3.6. ...■

Proof of proposition 3.9.1. To show that
$\omega(H) = \max(l(d-8) - 2n, (l-1)(d-8) - 2)$ we first make some topological observations about the original 11 diagrams:

1) For each diagram #vertices = 4 #loops. Thus $\omega(\emptyset) = l(d-8) - 2n$.

2) $\omega(S)$ can be gotten by deleting p lines $p \leq l-1$ (a line has 2 vertices of coordination no.3 at its end pts.) and then applying the formula:
$\omega(S) = l(d-8) - d$(#deleted lines)+2(#(vertices on the deleted line)+1).

Now let us suppose that the overall degree of divergence is not the worst, i.e. that $\omega(\emptyset) \leq \omega(H)$. Then by inspection, for all 10 diagrams (the 1 loop case has no sub-graphs), the worst case scenario is that all the n extra vertices fall on the same line. Furthermore, no line had more than 3 propagators on it to begin with (before the addition of extra vertices). Let m be the number of loops erased from the original graph when implementing the procedure described in 2). Thus, for any sub-graph,

$$\omega(S) \leq l(d-8) - 2n + m(-d + 2\cdot 3) + 2n = (l-m)(d-8) - 2m, \quad (3.9.11)$$

since the degree of divergence of the original graph is $l(d-8) - 2n$ and we are considering a sub-graph in which $2n$ *extra* vertices are added. The factor $m(-d + 2\cdot 3)$ takes into account the number of erased loops; the factor of $2\cdot 3$

stems from the fact that each line had at most 3 propagators originally, and each propagator behaves as k^2. The worst case in (3.9.11) is $m = 1$. $\ldots\blacksquare$

Chapter 4

The scaling limit of trees

4.1 The asymptotics of $t_n^{(m)}$

Before taking the continuum limits described in (1.1.16) and (1.1.24), we must first obtain precise estimates for the normalising terms $t_n^{(m)}$.

Recall that $t_n^{(m)}$ is the number of n bond trees linking m (not necessarily distinct points). If $t_n^{(m)}(0, x_1, ..., x_{m-1})$ is the number of n bond trees connecting $\{0, x_1, ..., x_{m-1}\}$,

$$\begin{aligned} t_n^{(m)} &= \sum_{x_1,...,x_{m-1}} t_n^{(m)}(0, x_1, ..., x_{m-1}) \\ &= (n+1)^{m-2} \sum_{x_1} t_n^{(2)}(0, x_1) = (n+1)^{m-2} t_n^{(2)}. \end{aligned} \tag{4.1.1}$$

From [HS4],

$$\chi(z) = \sum_{n=0}^{\infty} t_n^{(2)} z^n = \frac{\hat{h}_{z_c}(0)}{B_1(z_c - z)^{1/2}} + \mathcal{E}(z), \tag{4.1.2}$$

where for $i = 0, 1$ $|\frac{d^i}{dz^i}\mathcal{E}(z)| \le c(i)|z_c - z|^{-(2i+1)/2+\epsilon}$ on the disc $|z| \le z_c$ with $\epsilon < \min((d-8)/4, 1/2)$. Also $\hat{h}_{z_c}(0) = (\Omega z_c)^{-1}$, and $B_1^2 = 2(1+b)/z_c$ with b as defined in (2.2.27).

Now we introduce two separate but complementary results. The first is a standard result about algebraic functions. The n^{th} coefficient of the Maclaurin series of $(1-z)^\beta$ is asymptotic (as $n \nearrow \infty$) to

$$n^{-\beta-1}/\Gamma(-\beta) \tag{4.1.3}$$

for $\beta \notin \mathbb{N}$ (see [W] p.150 for example). The other result is Lemma 6.3.3 of [MS] which we reproduce below with certain modifications as:

Lemma 4.1 *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence greater or equal to $R > 0$.*

(i) Suppose that for $|z| < R$, and for some $b_1 \geq 1$ and $b_2 \geq 0$,

$$|f(z)| \leq c|R-z|^{-b_1} \cdot |R-|z||^{-b_2}. \tag{4.1.4}$$

Then,

$$|a_n| \leq O(R^{-n} n^{p-1}) \tag{4.1.5}$$

for any $p > b_1 + b_2$.

(ii) Let $i \geq 1$.

If for some $b_1 \geq 1$ and $b_2 \geq 0$, $|\frac{d^i}{dz^i} f(z)| \leq c|R-z|^{-b_1} \cdot |R-|z||^{-b_2}$ for $|z| < R$, then

$$|a_n| \leq O(R^{-n} n^{-(1+i)+p}) \tag{4.1.6}$$

for any $p > b_1 + b_2$.

Because of the condition $b_1 \geq 1$ it will not be until section 4.5 that we will be in a position to use part (i) of the above Lemma.

Proof of Lemma 4.1

(i) By the result on fractional derivatives in the appendix,

$$n^{-\alpha} a_n = \frac{1}{2\pi i} \oint \delta_z^{-\alpha} f(z) \frac{dz}{z^{n+1}}, \tag{4.1.7}$$

where the integral is around a circle centred at the origin of radius $r < R$. By Lemma A.5

$$n^{-\alpha}|a_n| \leq cr^{-n}\int_{-\pi}^{\pi} d\theta \int_0^{\infty} d\lambda |f(re^{i\theta}e^{-\lambda^{1/\alpha}}) - f(0)|. \qquad (4.1.8)$$

Since $|f(z) - f(0)| = O(|z|)$ for small z, the contribution to (4.1.8) is finite for $\lambda \in [1, \infty]$. By our assumptions on $f(z)$,

$$n^{-\alpha}|a_n| \leq cr^{-n}\left(1 + \int_{-\pi}^{\pi} d\theta \int_0^1 d\lambda |R - re^{-\lambda^{1/\alpha}}|^{-b_2} \cdot |R - re^{i\theta}e^{-\lambda^{1/\alpha}}|^{-b_1}\right). \qquad (4.1.9)$$

Now let $r \nearrow R$ in the above (this gives us our R^{-n}). By symmetry, it suffices to consider the θ integral on the interval $[0, 1]$, and show that

$$R^{-b_1-b_2}\int_0^1 d\theta \int_0^1 d\lambda |1 - e^{-\lambda^{1/\alpha}}|^{-b_2} \cdot |1 - e^{i\theta}e^{-\lambda^{1/\alpha}}|^{-b_1} < \infty. \qquad (4.1.10)$$

By letting $u = \lambda^{1/\alpha}$ and using the obvious bounds for the squares of the real and imaginary parts of the second factor in the integrand leads to the upper bound

$$\int_0^1 d\theta \int_0^1 du\ u^{\alpha-1}[1 - e^{-u}]^{-b_2} \cdot [(1 - e^{-u})^2 + \theta^2 e^{-2u}]^{-b_1/2}. \qquad (4.1.11)$$

Using the change of variables $w = \theta e^{-u}/(1 - e^{-u})$ on the θ integral in the above expression yields:

$$\int_0^1 du\ u^{\alpha-1}\frac{1 - e^{-u}}{e^{-u}}(1 - e^{-u})^{-b_1-b_2}\int_0^{e^{-u}/(1-e^{-u})} dw(1 + w^2)^{-b_1/2}. \qquad (4.1.12)$$

The w integral is finite for $b_1 > 1$ and $O(|\log(u)|)$ for $b_1 = 1$. Hence if $b_1 \geq 1$, (4.1.10) is bounded by a multiple of

$$\int_0^1 du\ u^{\alpha-b_1-b_2}|\log(u)|, \qquad (4.1.13)$$

which is finite for $\alpha > b_1 + b_2 - 1$.

(ii) By the bounds on the derivative, it follows from (i) that

$$|n(n-1)\cdots(n-i+1)a_n| \leq O(R^{-n}n^{p-1})$$

for any $p > b_1 + b_2$. ...■

We now show that the leading order term in Theorem 1.1 captures the leading order behavior of $t_n^{(m)}$.

By the first result, the n^{th} coefficient of $(z_c - z)^{-1/2}$ is asymptotic to $z_c^{-n}n^{-1/2}/\sqrt{z_c\pi}$ and we can therefore deduce from (4.1.2) that

$$t_n^{(2)} \overset{n\to\infty}{\sim} n^{-1/2}\lambda^n \frac{1/(\Omega z_c)}{\sqrt{2\pi(1+b)}}(1+O(n^{-\epsilon})) \tag{4.1.14}$$

since (by Lemma 4.1 part (ii)) the n^{th} coefficient of $\mathcal{E}(z)$ is $O(z_c^{-n}n^{-1/2-\epsilon})$ as per the bound $|\mathcal{E}'(z)| \leq c|z_c - z|^{-3/2+\epsilon}$ proven in [HS4] $p.$1012 (1.11). From (4.1.1), we obtain

$$t_n^{(m)} \overset{n\to\infty}{\sim} \frac{1/(\Omega z_c)}{\sqrt{2\pi(1+b)}} n^{m-5/2}\lambda^n(1+O(n^{-\epsilon})) \tag{4.1.15}$$

for any $\epsilon < \min(1/2, (d-8)/4)$.

However if we define $\tilde{t}_n^{(m)}$ through the leading term for $\chi_m(z)$ as in *2)* Theorem 1.1, we deduce the following asymptotics:

$$\tilde{t}_n^{(m)} \overset{n\to\infty}{\sim} \frac{z_c^{-n}}{(\Omega z_c\sqrt{1+b})^{(2m-3)}} \frac{1}{\sqrt{2\pi}} v(z_c)^{m-2}(1+O(n^{-\epsilon})). \tag{4.1.16}$$

The $(2m-5)!!$ pre-factor in the leading term for (3.1.2) is cancelled by the $1/\Gamma(m-3/2)$ of (4.1.3), and Lemma 4.1 allows us to control the error terms. By (2.2.27) $v(z_c) = (z_c\Omega)^2(1+b)$, so the above is in fact asymptotic to the leading term in (4.1.15) since by Lemma 4.1, the n^{th} coefficient of the error term $\bar{E}_m(z)$ of (3.1.2) will be $O(\lambda^n n^{m-5/2-\epsilon})$ for any $\epsilon < \min(1/2, (d-8)/4)$.

4.2 The 2 point function scaling limit

In this section we will develop most of the techniques and bounds we will need for the scaling limit of the m point functions. However, we only work out the scaling limit of the 2 point function (section 4.5 contains all the calculations for the m point functions).

The results from the previous section will now enable us to investigate the following scaling limit:

$$\hat{P}(k) = \lim_{n\to\infty} \frac{t_n^{(2)}(k/n^{\frac{1}{4}})}{t_n^{(2)}} = \lim_{n\to\infty} \frac{1}{2\pi i} \oint_{|z|=\epsilon} \frac{\hat{G}_z(k/n^{\frac{1}{4}})}{t_n^{(2)}} \frac{dz}{z^{n+1}}. \tag{4.2.1}$$

Our first step will be to derive an explicit algebraic expression for the leading order behavior of $\hat{G}_z(k)$ and the ensuing error terms. The estimates we will derive for the error terms will be valid only on the disc $|z| \leq z_c$ and will be dealt with by using Lemma 4.1.

Recall that

$$\hat{G}_z(k) = \frac{\hat{h}_z(k)}{\hat{F}_z(k)}, \tag{4.2.2}$$

where $\hat{h}_z(k) = g(z) + \hat{\Pi}_z(k)$, and $\hat{F}_z(k) = 1 - z\Omega\hat{D}(k)\hat{h}_z(k)$.

To extract $\hat{G}_z(k)$'s leading order behavior we will first need

Lemma 4.2

$$\hat{F}_z(k) = \alpha k^2/2 + \beta(z_c - z)^{1/2} + B(z,k), \tag{4.2.3}$$

where $\alpha = -\nabla^2(\hat{D}(0) + z_c\Omega\hat{\Pi}_{z_c}(0))/d$, $\beta^2 = 2\Omega(1+b)\hat{h}_{z_c}(0)$ *and the error term* $B(z,k) = \sum_{i=1}^{3} E_i(z,k)$. *Also, for* $|z| < z_c$ *and* $\kappa = kn^{-1/4}$,

$$|E_1(z,\kappa)| \leq cn^{-1/2-\epsilon}, \tag{4.2.4}$$

$$|E_2(z,\kappa)| \leq c\kappa^2|z_c - z|^{\epsilon}, \tag{4.2.5}$$

$$|E_3(z)| \leq c|z_c - z|^{1/2+\epsilon}, \tag{4.2.6}$$

with $\epsilon < \min(1/2, (d-8)/4)$, *and*

$$|\frac{d}{dz}B(z,\kappa)| \leq c|z_c - z|^{-1/2}(n^{-\epsilon/4} + |z_c - z|^{\epsilon}). \tag{4.2.7}$$

In order to deal with the denominator in (4.2.2) we will need:

Lemma 4.3 *For* $|z| < z_c$, *there is some* $c > 0$ *so that*

$$|\frac{d}{dz}\hat{F}_z(k)| \leq c|z_c - z|^{-1/2}, \tag{4.2.8}$$

and for $|z| \leq z_c$ *and* n *large enough,*

$$|\hat{F}_z(kn^{-1/4})| \geq c'|z_c - z|^{1/2}. \tag{4.2.9}$$

As for the numerator of (4.2.2) we will use:

Lemma 4.4

$$\hat{h}_z(k) = \hat{h}_{z_c}(0) + \mathcal{E}(z,k), \tag{4.2.10}$$

where $\mathcal{E}(z,k) = \hat{h}_z(0) - \hat{h}_{z_c}(0) + \hat{\Pi}_z(k) - \hat{\Pi}_z(0)$.
With ϵ *as in (4.2.6) and* κ *as in Lemma 4.2, this new error term obeys*

$$|\mathcal{E}(z, kn^{-1/4})| \leq c|z_c - z|^{\epsilon} + cn^{-\epsilon/4}, \tag{4.2.11}$$

and

$$|\frac{d}{dz}\mathcal{E}(z, kn^{-1/4})| \leq c|z_c - z|^{-1/2} \tag{4.2.12}$$

for $|z| < z_c$.

The idea is that by dividing (4.2.3) by $(\alpha k^2/2 + \beta(z_c - z)^{1/2})\hat{F}_z(k)$, we can write

$$\frac{1}{\hat{F}_z(k)} = \frac{1}{\alpha k^2/2 + \beta(z_c - z)^{1/2}} - \frac{B(z,k)}{(\alpha k^2/2 + \beta(z_c - z)^{1/2})\hat{F}_z(k)}. \tag{4.2.13}$$

Finally, by using (4.2.13) and (4.2.10) we have

$$\begin{aligned}\hat{G}_z(k) &= \frac{\hat{h}_z(k)}{\hat{F}_z(k)} \qquad (4.2.14)\\ &= \frac{\hat{h}_{z_c}(0)+\mathcal{E}_z(k)}{\alpha k^2/2+\beta(z_c-z)^{1/2}} - \frac{B(z,k)(\hat{h}_{z_c}(0)+\mathcal{E}_z(k))}{(\alpha k^2/2+\beta(z_c-z)^{1/2})\hat{F}_z(k)}.\end{aligned}$$

In essence, the only term contributing to the numerator of (4.2.14) is $\hat{h}_{z_c}(0)$.

Let us first deal with the first term in (4.2.14) when taking the limit in (4.2.1):

$$\frac{1}{2\pi i}\oint_{|z|=\epsilon}\frac{n^{\frac{1}{2}}\hat{h}_{z_c}(0)}{(\beta(z_c-z)^{\frac{1}{2}}+\alpha k^2/2n^{1/2})}\frac{dz}{z_c(\frac{z}{z_c})^{n+1}} \qquad (4.2.15)$$

We start by choosing a branch cut[1] for $\sqrt{z-z_c} \equiv \sqrt{w}$ on the real axis to the *left* of z_c with $-\pi < \arg w < \pi$. If we now look at $z_c - z$, i.e. $-w$, the branch cut will now be rotated by π about 0 in the w plane, and lie to the *right* of 0 on the real axis with $\arg(-w) = -\pi$ on the *top* half of the branch, and $\arg(z_c - z) = +\pi$ on the *bottom* half of the branch.

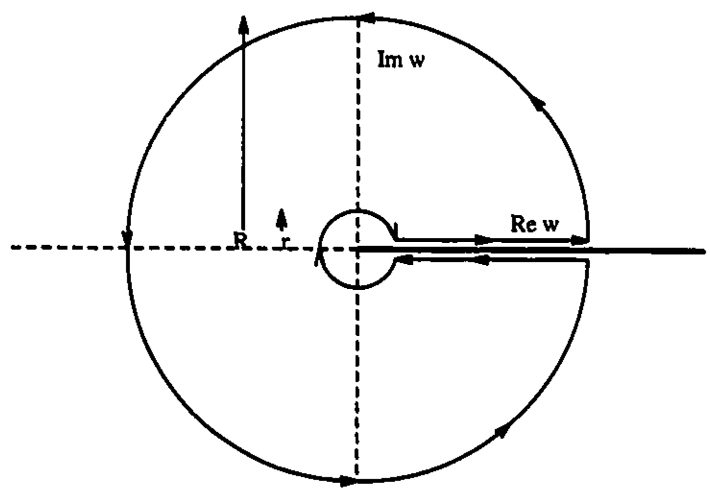


Fig 2 : The pack-man contour

[1]If we had not been careful in choosing our branch-cut, we would have wound up with an added contribution from a simple pole as well as the wrong sign from the integration around the branch-cut.

We now deform the contour around $z = 0$ in (4.2.15) to a 'pack-man' type contour (in the w plane) which swallows the branch-cut as its mouth closes (the contributions from the arcs around $w = re^{i\theta}$ and $w = Re^{i\theta}$ both vanish as $R \nearrow \infty$, and $r \searrow 0$). We can make a further change of variables so that $z_c - z = -\frac{w}{n}$, and thus, our new expression becomes:

$$\begin{aligned} I(n,k) &= \frac{1}{2\pi i}\int_0^\infty \frac{\hat{h}_{z_c}(0)n^{\frac{1}{2}}}{n^{-\frac{1}{2}}[\alpha k^2/2 - \beta i w^{\frac{1}{2}}]}\frac{dw}{n(1+\frac{w}{z_c n})^{n+1} z_c} \\ &- \frac{1}{2\pi i}\int_0^\infty \frac{\hat{h}_{z_c}(0)n^{\frac{1}{2}}}{n^{-\frac{1}{2}}[\alpha k^2/2 + \beta i w^{\frac{1}{2}}]}\frac{dw}{n(1+\frac{w}{z_c n})^{n+1} z_c} \\ &= \frac{1}{\pi}\int_0^\infty \mathcal{I}m\left\{\frac{\hat{h}_{z_c}(0)}{\alpha k^2/2 - \beta i w^{\frac{1}{2}}}\right\}\frac{dw}{(1+\frac{w}{z_c n})^{n+1} z_c}. \end{aligned} \quad (4.2.16)$$

We now observe that for any $x \geq 0$ and $2m \ll n$,

$$(1 + x/n)^n \geq 1 + \binom{n}{m}(x/n)^m \geq 1 + x^m 2^{-m}/m!, \quad (4.2.17)$$

since

$$\frac{n\cdot(n-1)\cdots(n-m+1)}{m!\; n^m} \geq \left(\frac{n-m}{n}\right)^m /m! \geq 2^{-m}/m!$$

if $2m \ll n$. By (4.2.17), we may use Lebesgue's dominated convergence theorem to conclude that:

$$\hat{P}(k) = \frac{1}{\pi z_c}\int_0^\infty e^{-w/z_c}\mathcal{I}m\left\{\frac{\hat{h}_{z_c}(0)}{\alpha k^2/2 - \beta i w^{\frac{1}{2}}}\right\} dw. \quad (4.2.18)$$

We could at this point use the parabolic cylinder function D_2 to recover the $m = 2$ case of Theorem (1.2) since $\mathcal{P}(k) = e^{k^4/16}D_2(k^2/2)$ as per [GR] (3.462.1). However, we will handle all cases $m \geq 2$ in one fell swoop in section 4.5 via more direct means.

We will now deal with the error terms in (4.2.14). We would be done if we could show that

$$\frac{1}{2\pi i}\oint_{|z|=\epsilon}\frac{n^{1/2}B(z,k/n^{1/4})}{\hat{F}_z(k)(\alpha k^2 n^{-1/2}/2 + \beta(z_c - z)^{1/2})}\frac{dz}{(z/z_c)^{n+1}} = o(n), \quad (4.2.19)$$

and

$$\frac{1}{2\pi i}\oint_{|z|=\epsilon}\frac{n^{1/2}\mathcal{E}(z,k/n^{1/4})}{\alpha k^2 n^{-1/2}/2+\beta(z_c-z)^{1/2}}\frac{dz}{(z/z_c)^{n+1}}=o(n). \qquad (4.2.20)$$

By our choice of branch-cut $(z_c-z)^{1/2}$, will always have positive real part, and thus, by Lemma 4.3,

$$|\alpha k^2/2+\beta(z_c-z)^{1/2}||\hat{F}_z(k)|\geq|z_c-z| \qquad (4.2.21)$$

and similarly,

$$|\alpha k^2/2+\beta(z_c-z)^{1/2}|\geq c|z_c-z|^{1/2} \qquad (4.2.22)$$

both for $|z|\leq z_c$. Using the above, Lemma 4.3 and the bounds of Lemma 4.2 part (ii), we find that the derivative of the integrand in (4.2.19) satisfies

$$\begin{aligned} & n^{1/2}z_c^n\left|\frac{d}{dz}\frac{B(z,kn^{-1/4})}{\hat{F}_z(k)(\alpha k^2 n^{-1/2}/2+\beta(z_c-z)^{1/2})}\right| \qquad (4.2.23)\\ \leq\; & cn^{1/2}z_c^n(n^{-1/2-\epsilon/4}|z_c-z|^{-2}+n^{-1/2}|z_c-z|^{-2+\epsilon})\\ +\; & cn^{1/2}z_c^n(n^{-\epsilon/4}|z_c-z|^{-3/2}+|z_c-z|^{-3/2+\epsilon})\end{aligned}$$

on the disc $|z|<z_c$.

Now we can use Lemma 4.1 to conclude that the n^{th} coefficient of each of the above terms is $O(n^{-\epsilon/4-2+2})$, $O(n^{-2+(2-\epsilon)})$, $O(n^{1/2-2-\epsilon/4+3/2})$, and $O(n^{1/2+-2+(3/2-\epsilon)})$ respectively. Technically, we should also consider the cross-term $B(z,k)\mathcal{E}(z,k)$ in (4.2.23) but it and its derivatives will yield the same bounds, so the corresponding analysis will be omitted.

The last step in the error analysis is to look at the contribution from (4.2.20). We now use (4.2.11) of Lemma 4.4 and the lower bound in (4.2.22) to conclude that

$$\begin{aligned} & n^{1/2}z_c^{-n}\left|\frac{d}{dz}\frac{\mathcal{E}_z(k/n^{1/4})}{\alpha k^2/2+\beta(z_c-z)^{1/2}}\right| \qquad (4.2.24)\\ \leq\; & cz_c^{-n}n^{1/2}|z_c-z|^{-3/2}(|z_c-z|^{\epsilon}+n^{-\epsilon/4}).\end{aligned}$$

Again, by Lemma (4.1), we see that the n^{th} coefficient of the above terms is $O(n^{1/2-2+(-\epsilon+3/2)})$ and $O(n^{1/2-\epsilon/4-2+3/2})$ respectively. Thus (4.2.19) and (4.2.20) behave as $n^{-\epsilon/4}$ for $\epsilon < \min(1/2,(d-8)/4)$.

Before proving Lemma 4.2, let us introduce some notation:

Definition 4.2.1 *Given a power series* $f(z) = \sum_{n=1}^{\infty} a_n z^n$, *let*

$$\|f(z)\| \equiv \sum_{n=1}^{\infty} |a_n||z|^n. \tag{4.2.25}$$

This definition will be used when dealing with fractional derivatives [cf Appendix C] in the proofs of the following Lemmas.

Proof of Lemma 4.2 Our task is to show that:

$$\hat{F}_z(k) = \beta(z_c - z)^{1/2} + \alpha k^2/2 + B(z,k) \tag{4.2.26}$$

with $\alpha = \nabla^2(\hat{D}(0) + z_c\Omega\hat{\Pi}_{z_c}(0))/d$, and $\beta^2 = 2\Omega((1+b)h_{z_c}(0))$. Before decomposing $\hat{F}_z(k)$, let

$$E(z,k) \equiv \hat{\Pi}_z(k) - \hat{\Pi}_z(0) - k^2\nabla^2\hat{\Pi}_z(0)/(2d), \tag{4.2.27}$$

and recall that $z\Omega\hat{h}_z(0) = 1 - \hat{F}_z(0)$. We will use these identities forth with:

$$\begin{aligned}
\hat{F}_z(k) &= 1 - z\Omega\hat{D}(k)(g(z) + \hat{\Pi}_z(0) + (\hat{\Pi}_z(k) - \hat{\Pi}_z(0))) && (4.2.28)\\
&= 1 - z\Omega\hat{D}(k)(\hat{h}_z(0) + E(z,k) + k^2\nabla^2\hat{\Pi}_z(0)/(2d)) \\
&= 1 - \hat{D}(k)(1 - \hat{F}_z(0)) && (4.2.29)\\
&- z\Omega\hat{D}(k)E(z,k) && (4.2.30)\\
&- \hat{D}(k)z\Omega k^2\nabla^2\hat{\Pi}_z(0)/(2d). && (4.2.31)
\end{aligned}$$

We begin with some bounds for term (4.2.30): let $\Pi_m(x)$ denote the coefficient of z^m in $\Pi_z(0,x)$. Then $|x|^\epsilon|\Pi_m(x)| \le m^\epsilon|\Pi_m(x)|$. We can then write (using symmetry)

$$E(z,k) = \sum_x \Pi_z(0,x)(\cos(k \cdot x) - 1 + (k \cdot x)^2/2). \tag{4.2.32}$$

Since $|\cos(y) - 1 + y^2/2| \leq O(y^{2+\epsilon})$, and using the fact that ([HS4] Lemma 2.1 (iv)) $\delta_z^\epsilon \|\nabla^2 \hat{\Pi}_{z_c}(0)\| \leq c$, we deduce that

$$|E(z, k/n^{\frac{1}{4}})| \leq cn^{-\frac{1}{2}-\epsilon/4} \tag{4.2.33}$$

for all $|z| \leq z_c$. As for term (4.2.31),

$$\begin{aligned}
- \quad & \hat{D}(k)k^2 \left(z_c \Omega \nabla^2 \hat{\Pi}_{z_c}(0)/(2d) + (z\Omega \nabla^2 \hat{\Pi}_z(0) - z_c \Omega \nabla^2 \hat{\Pi}_{z_c}(0))/(2d)\right) \\
= \quad & -k^2 z_c \Omega \nabla^2 \hat{\Pi}_{z_c}(0)/(2d) + O(k^4) \qquad (4.2.34) \\
& -\hat{D}(k)k^2(z\Omega \nabla^2 \hat{\Pi}_z(0) - z_c \Omega \nabla^2 \hat{\Pi}_{z_c}(0))/(2d).
\end{aligned}$$

To deal with (4.2.29) we use $\hat{D}(k) = 1 - k^2/2\nabla^2 \hat{D}(0)/d + O(k^4)$ so that,

$$\begin{aligned}
& (1 - \hat{D}(k)) + \hat{D}(k)\hat{F}_z(0) \\
= \quad & k^2/(2d)\nabla^2 \hat{D}(0) + \hat{F}_z(0)(1 - k^2 \nabla^2 \hat{D}(0)/(2d)) + O(k^4), \quad (4.2.35)
\end{aligned}$$

with the understanding that the term $O(k^4)$ may incorporate a function of z which is bounded on the disc $|z| \leq z_c$. By (2.1.20),

$$\hat{F}_z(0) = \beta(z_c - z)^{1/2} + E_3(z), \tag{4.2.36}$$

where $|E_3(z)| \leq c|z_c - z|^{1/2+\epsilon}$ as per (4.2.6), and β is as advertised in (4.2.26). Now define

$$E_1(z,k) \equiv -z\Omega \hat{D}(k) E(z,k) + O(k^4), \tag{4.2.37}$$

which satisfies (4.2.4). $E_1(z,k)$ incorporates the $O(k^4)$ error term from (4.2.34) and (4.2.35). We can now also define:

$$\begin{aligned}
E_2(z,k) \quad & \equiv \quad -\hat{D}(k)k^2/(2d)(z\Omega \nabla^2 \hat{\Pi}_z(0) - z_c \Omega \nabla^2 \hat{\Pi}_{z_c}(0)) \qquad (4.2.38) \\
& + \quad -k^2/(2d)\nabla^2 \hat{D}(0)\hat{F}_z(0),
\end{aligned}$$

(where $E_2(z,k)$ incorporates the error terms in (4.2.35) and (4.2.34)). Since $\delta_z^\epsilon \| z_c \Omega \nabla^2 \hat{\Pi}_{z_c}(0) \| \le c$, (C.1) shows that

$$|(z\Omega\nabla^2\hat{\Pi}_z(0) - z_c\Omega\nabla^2\hat{\Pi}_{z_c}(0))| \le c|z_c - z|^\epsilon$$

and [HS4] Lemma 2.1 (i), shows that $|\hat{F}_z(0)| \le c|z_c - z|^{1/2}$. It therefore follows that

$$|E_2(z, kn^{-1/4})| \le cn^{-1/2}|z_c - z|^\epsilon, \tag{4.2.39}$$

for $|z| \le z_c$ as per (4.2.5).

By combining the leading terms in (4.2.35) and (4.2.34) we recover the coefficient α of (4.2.26); (4.2.36) yields the coefficient β, thus completing the decomposition in (4.2.26).

Now, we deal with the derivatives of $B(z,k) = \sum_{i=1}^{3} E_i(z,k)$. By construction, $B(z,k) = \hat{F}_z(k) - \beta(z_c - z)^{1/2} - \alpha k^2/2$, so by (4.2.28), and the identity $\mathcal{D}\hat{\Pi}_z(k) = \hat{\Psi}_z(k)(1 + z\Omega\chi(z))$,

$$\begin{aligned} \frac{d}{dz}B(z,k) &= \frac{d}{dz}\left(\hat{F}_z(k) - \beta(z_c - z)^{1/2}\right) && (4.2.40)\\ &= -\hat{D}(k)\Omega\mathcal{D}(g(z) + \hat{\Pi}_z(0)) - \hat{D}(k)\Omega\mathcal{D}(\hat{\Pi}_z(k) - \hat{\Pi}_z(0)) \\ &\quad +\beta/2(z_c - z)^{-1/2} \\ &= -\hat{D}(k)\Omega(\chi(z)(1 + z\Omega\hat{\Psi}_z(0)) + \beta/2(z_c - z)^{-1/2} && (4.2.41)\\ &\quad -\hat{D}(k)\Omega\hat{\Psi}_z(0) - \hat{D}(k)\Omega(\hat{\Psi}_z(k) - \hat{\Psi}_z(0))(1 + z\Omega\chi(z)). \end{aligned}$$

Let $\psi_m(x)$ be the m^{th} coefficient of $\hat{\Psi}_z(x)$, and note that $|x|^\epsilon|\psi_m(x)| \le m^\epsilon|\psi_m(x)|$. Now, from Appendix C, we know that $\delta_{z_c}^\epsilon \|\hat{\Psi}_z(k)\| \le c$. Hence, for $|z| \le z_c$,

$$|\hat{\Psi}_z(kn^{-1/4}) - \hat{\Psi}_z(0)| \le cn^{-\epsilon/4}. \tag{4.2.42}$$

This takes care of the last term in (4.2.41). From (2.2.27), $(1 + z\Omega\hat{\Psi}_z(0)) = (1 + b) + K(z)$, with $|K(z)| \le c|z_c - z|^\epsilon$. By (4.1.2)

$$\chi(z) = \frac{(z_c - z)^{-1/2}}{\Omega(2z_c(1+b))^{1/2}} + K(z)(z_c - z)^{-1/2}, \tag{4.2.43}$$

however $\beta/2 = ((1+b)/2z_c)^{1/2}$. By using the above with the expansion $\hat{D}(kn^{-1/4}) = 1 + O(n^{-1/2})$ together with $|\chi(z)| \le c|z_c - z|^{-1/2}$ we can bound the first line of (4.2.41). The end result is that for $|z| < z_c$,

$$|\frac{d}{dz}B(z, kn^{-1/4})| \le c|z_c - z|^{-1/2}(n^{-\epsilon/4} + |z_c - z|^{\epsilon}) \tag{4.2.44}$$

This concludes the proof of Lemma 4.2. ...■

Proof of Lemma 4.3 By (3.3.3) and (2.2.1),

$$\frac{d}{dz}\hat{F}_z(k) = -\Omega\hat{D}(k)(\chi(z) + \hat{\Psi}_z(k)(1 + z\Omega\chi(z))). \tag{4.2.45}$$

thus we can find $c > 0$ such that

$$|\frac{d}{dz}\hat{F}_z(k)| \le c|z_c - z|^{-1/2}. \tag{4.2.46}$$

We will now prove a stronger version of what was claimed in the last part of the Lemma. Namely, that

$$|\hat{F}_z(k)| \ge c(k^2 + |z_c - z|^{1/2}) \tag{4.2.47}$$

for $|z| \le z_c$ and $k^2 \le \eta < (\pi/2)^2$ for some $\eta > 0$. Recall from (2.1.20) that

$$\hat{F}_z(0) = B_1(z_c - z)^{1/2} + \mathcal{E}(z), \tag{4.2.48}$$

with $|\mathcal{E}(z)| \le c|z_c - z|^{1/2+\epsilon}$, for $|z| \le z_c$, $\epsilon < \min(1/2, (d-8)/4)$, and with $B_1^2 = 2\Omega(1+b)\hat{h}_{z_c}(0)$ and $b \to 0$ as $L \to \infty$. From Lemma 2.1 (i) and Lemma 2.2 of [HS4], we have

$$c'|1 - z/z_c|^{1/2} \le |\hat{F}_z(0)| \le c|1 - z/z_c|^{1/2} \tag{4.2.49}$$

for $|z| \le z_c$.

To derive (4.2.47) we use

$$z\Omega = \frac{(1 - \hat{F}_z(0))}{\hat{h}_z(0)} \tag{4.2.50}$$

together with

$$\hat{F}_z(k) = 1 - \hat{D}(k)z\Omega\{\hat{h}_z(0) + \hat{\Pi}_z(k) - \hat{\Pi}_z(0)\} \tag{4.2.51}$$

to arrive at

$$\hat{F}_z(k) \;=\; 1 - \hat{D}(k) + \hat{D}(k)\hat{F}_z(0) + z\Omega\hat{D}(k)(\hat{\Pi}_z(k) - \hat{\Pi}_z(0)). \tag{4.2.52}$$

It follows from (3.11) [HS4] that for $k \in [-\pi, \pi]^d$

$$1 - \hat{D}(k) \geq ck^2 \tag{4.2.53}$$

(uniformly in L), and from the arguments immediately preceding it, that

$$|\hat{\Pi}_z(k) - \hat{\Pi}_z(0)| \leq cL^{3-d}k^2 \tag{4.2.54}$$

for $|z| \leq z_c$ (and $k \in [-\pi, \pi]^d$). Thus,

$$|z\Omega\hat{D}(k)(\hat{\Pi}_z(k) - \hat{\Pi}_z(0))| \leq cL^{3-d}k^2 \tag{4.2.55}$$

for $|z| \leq z_c$. Using these bounds we can estimate (4.2.52) as follows:

$$\begin{aligned}
&|\hat{F}_z(k)| \geq |(1 - \hat{D}(k)) + \hat{D}(k)\hat{F}_z(0)| - \\
&|z\Omega\hat{D}(k)(\hat{\Pi}_z(k) - \hat{\Pi}_z(0))| \geq \\
&|(1 - \hat{D}(k)) + \hat{D}(k)(B_1\sqrt{z_c - z} + \mathcal{E}(z))| - cL^{3-d}k^2 \geq \\
&c|1 - \hat{D}(k) + \hat{D}(k)B_1\sqrt{z_c - z}| - \tilde{c}|\mathcal{E}(z)| - \tilde{c}L^{3-d}k^2.
\end{aligned} \tag{4.2.56}$$

We now use $\hat{D}(k) \geq c$ for $k^2 \leq \eta < \pi/2$. Let $\sqrt{z_c - z} = x + iy$; by our choice of branch-cut in (4.2.15) we will always have $0 \leq |y| \leq x$ (since $\theta = \arg(\sqrt{z_c - z})$

will lie in the wedge $|\theta| \le \pi/4$). Therefore, $|1-\hat{D}(k)+\hat{D}(k)\sqrt{z_c-z}| \ge c(k^2+|z_c-z|^{1/2})$ for $|z| \le z_c$. Thus,

$$|\hat{F}_z(k)| \ge c(k^2+|z_c-z|^{1/2}) - \bar{c}|z_c-z|^{1/2+\epsilon} \tag{4.2.57}$$

for L sufficiently large, $k^2 \le \eta$. Now let $V = \{|z| \le z_c : |z_c-z| \le (c_0)^{1/\epsilon}\}$. Then for c_0 small enough and $z \in V$,

$$c|z_c-z|^{1/2}(1-\bar{c}|z_c-z|^{\epsilon}) \ge c|z_c-z|^{1/2}(1-\bar{c}c_0) \ge c'|z_c-z|^{1/2}. \tag{4.2.58}$$

We now have (4.2.47) for $z \in V$ and $k^2 \le \eta$. It remains to obtain

$$|\hat{F}_z(k)| \ge \tilde{c} \tag{4.2.59}$$

on $W \equiv \{|z| \le z_c \setminus V\}$.

Now, since $\hat{F}_z(k)$ is analytic for $|z| < z_c$ and continuous on the boundary of the disc $|z| \le z_c$, we can obtain (4.2.59) by the same reasoning as in [HS4]: by applying the maximum modulus principle to $1/\hat{F}_z(k)$ on ∂W. Before proceeding, we must show $|\hat{F}_z(k)| \ge c > 0$ on W. This follows by the identities in (4.2.56) together with the lower bound in (4.2.49):

$$\begin{aligned} |\hat{F}_z(k)| &\ge |(1-\hat{D}(k))+\hat{D}(k)\hat{F}_z(0)| \\ &\quad -|z\Omega\hat{D}(k)(\hat{\Pi}_z(k)-\hat{\Pi}_z(0))| \\ &\ge |D(k)|\cdot|\hat{F}_z(0)| - c\eta \\ &\ge c'|z_c-z|^{1/2} - c\eta > 0 \end{aligned} \tag{4.2.60}$$

for all $z \in W$ provided η is sufficiently small; since, $1/\hat{F}_z(k)$ in analytic in W (with k^2 sufficiently small) we may proceed to exploit the maximum modulus principle.

We already have a lower bound for $|\hat{F}_z(k)|$ on V and its boundary, so it suffices to show (4.2.59) for $z = z_c e^{i\phi}$, $\phi \in [\phi_0, 2\pi-\phi_0]$ with ϕ_0 independent of L (ϕ_0 is defined by the choice of V). By using (2.35) of [HS4] on

$$|\hat{F}_z(k)| \geq 1 - z_c\Omega|g(z)| - z_c\Omega|\hat{\Pi}_z(k)|, \tag{4.2.61}$$

we deduce that

$$|\hat{F}_z(k)| \geq c(\phi_0) - |\hat{\Pi}_z(k)| \geq c(\phi_o) - c_L > 0 \tag{4.2.62}$$

for $L \gg 1$. We therefore have (4.2.59) which yields

$$|\hat{F}_z(k)| \geq \frac{\tilde{c}}{1+d\pi^2}(k^2 + \frac{|z_c - z|^{1/2}}{|z_c - z|^{1/2}}) \geq c(k^2 + |z_c - z|^{1/2}), \tag{4.2.63}$$

for $k^2 < \eta < (\pi/2)^2$ if c is small enough.

We have now proved Lemma 4.3. ...■

Proof of Lemma 4.4. The procedure is a combination of that in Lemma 4.3 and Lemma 4.2 except that for

$$\hat{\Pi}(k) - \hat{\Pi}_z(0), \tag{4.2.64}$$

we will use $|x|^\epsilon|\Pi_m(x)| \leq m^\epsilon|\Pi_m(x)|$. That together with the bound on $\delta_z^\epsilon\|\hat{\Pi}_{z_c}(0)\|$, allows us to conclude that

$$|\hat{\Pi}_z(kn^{-1/4}) - \hat{\Pi}_z(0)| \leq cn^{-\epsilon/4}. \tag{4.2.65}$$

From [HS4] we know[2] that $|\hat{h}_{z_c}(0) - \hat{h}(0)| \leq c|z_c - z|^\epsilon$, so that

$$|\mathcal{E}(z, kn^{-1/4})| \leq c(|z_c - z|^\epsilon + n^{-\epsilon/4}) \tag{4.2.66}$$

It remains to deal with the bound on the derivative. Since $\frac{d}{dz}f(z) = (\mathcal{D}f - f)/z$ and $\hat{\Pi}_z(0)$ has no constant coefficient, we may as well work with $\mathcal{D}$ instead of $\frac{d}{dz}$.

$$\mathcal{D}\mathcal{E}(z,k) = \hat{\Psi}_z(k)(1 + z\Omega\chi(z)) + (\chi(z) - g(z))/z \tag{4.2.67}$$

(the last term has a removable singularity at $z = 0$). Thus $|\frac{d}{dz}\mathcal{E}(z,k)| \leq c|z_c - z|^{-1/2}$. ...■

[2]the bound could be improved from ϵ to $1/2$

4.3 The Fourier transform of the 3 point function

We now proceed in detail with the Fourier transform of the 3-point function. If a tree connects 3 distinct points, the backbone these points induce has no choice but to contain an internal branch-point of coordination number 3; if we then take the Fourier transform with respect to the lattice vectors corresponding to the distance between the each external point and internal node, we should recover (to leading order) a term reminiscent of $\chi_3(z)$ except for some phase factors. The Fourier transform just described can be expressed as:

$$\hat{P}_z(k_1,k_2,k_3) = \sum_{x_1,x_2,x_3} \sum_{\omega:0\to x_3} \sum_{i=0}^{|\omega|} I_{[\omega(i)=x_2]} e^{i\{k_1x_2+k_2(x_3-x_2)\}} z^{|\omega|} \times$$
$$\left(\prod_{j\neq i} \gamma_j\right) \left\{ e^{ik_3(x_1-x_2)} \sum_{R_i\ni\omega(i),x_1} z^{|R_i|} \right\} K[0,|\omega|], \tag{4.3.1}$$

where

$$\sum_{x_1} e^{ik_3(x_1-x_2)} \sum_{R_i\ni\omega(i),x_1} z^{|R_i|} = \sum_{x_1} e^{ik_3(x_1-x_2)} \sum_{\omega':x_2\to x_1} z^{|\omega'|} \left(\prod_{k=0}^{|\omega'|} \gamma'_k\right) K[0,|\omega'|]. \tag{4.3.2}$$

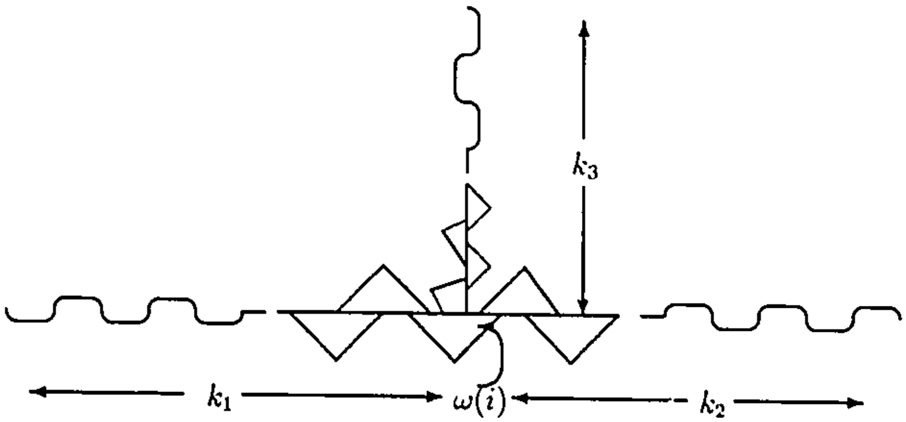
As in chapter 3 we will resum the interaction $K[0,|\omega|]$ by using

$$\mathcal{K}(I) = K[0, I_1-1] J[I_1, I_2] K[I_2+1, |\omega|], \tag{4.3.3}$$

where for any $i \in [0,|\omega|]$, $\sum_{I\ni i}\mathcal{K}(I) = K[0,|\omega|]$. Before resumming (4.3.1), define

$$\phi(k_1,k_2,k_3; I,x,y,i) = e^{ik_1\{\omega(i)-\omega(I_1)+\omega(I_1)-\omega(I_1-1)+\omega(I_1-1)-0\}} \times$$
$$e^{ik_2\{x-\omega(I_2+1)+\omega(I_2+1)-\omega(I_2)+\omega(I_2)-\omega(i)\}} \times e^{ik_3\{y-\omega(i)\}} \tag{4.3.4}$$

The diagram below illustrates the vectors these phase factors encompass;



as we shall show, by resumming the interaction $K[0,|\omega|]$ as in (3.2.2) we can extract 2 factors corresponding to $\hat{G}_z(k)$. These correspond to the wavy lines to the left and to the right of the central 'ladder' diagram. By the independence of the interactions we need not worry about the intersections between branches of the wavy lines and the the 'central ladder' diagram. The top wavy line however must be kept from intersecting the branches of the central 'ladder' whence it 'grew' from. This is the job of the kernel we formally knew as $\hat{\Psi}_z(0)$. The kernel we will replace it with will have to depend on k_1, k_2, k_3, and this is the only new twist. Now, let

$$\psi(k_1,k_2,k_3;y,i) = e^{ik_1\omega(i)+ik_2(\omega(|\omega|)-\omega(i))+ik_3(y-\omega(i))}. \tag{4.3.5}$$

With this, the machinery involved in the derivation of (2.2.21), and the independence between the interactions in (4.3.3), we can resum the 3-point function as follows:

$$\begin{aligned}
&\hat{P}_z(k_1,k_2,k_3) = \qquad\qquad (4.3.6)\\
&\sum_{x,y}\sum_{\omega:0\to x}\sum_{i=0}^{|\omega|} z^{|\omega|}\sum_{I\ni i}\phi(k_1,k_2,k_3;I,x,y,i)\Big(\prod_{j:j\neq i}\gamma_j\Big)\sum_{R_i\ni\omega(i),y} z^{|R_i|}\mathcal{K}(I)\\
&= \hat{G}_z(k_1)\hat{G}_z(k_2)(z\Omega)^2\hat{D}(k_1)\hat{D}(k_3)\times\\
&\left\{\hat{G}_z(k_3)+\left(\sum_{y}\sum_{|\omega|\geq 1} z^{|\omega|}\sum_{i=0}^{|\omega|}\Big(\prod_{j:j\neq i}\gamma_j\Big)\psi(k_1,k_2,k_3;y,i)\sum_{R_i\ni y,\omega(i)} z^{|R_i|}J[0,|\omega|]\right)\right\}
\end{aligned}$$

$$+ \quad E(z;k_1,k_2,k_3).$$

The term $E(z;k_1,k_2,k_3)$ encompasses precisely the same lower order terms encountered in chapter 3 [cf (3.2.6)], albeit with some decorating phase factors, and thus can be bounded by $E(z)$ as defined in (3.4.2).

The term $\hat{G}_z(k_3)$ within (4.3.6) is a result of the case $I = \{i\}$ with $0 < I_1$ and $I_2 < |\omega|$ in (4.3.3) (recall that $J[i,i] = 1$). The remaining term in the braces results from the case where $|I| > 1$ with the same restriction upon I_1 and I_2 as above and may again be resummed using the machinery of section 2.2:

$$\sum_y \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} (\prod_{j:j\neq i} \gamma_j) \psi(k_1,k_2,k_3;y,i) \sum_{R_i \ni y,\omega(i)} z^{|R_i|} J[0,|\omega|] \tag{4.3.7}$$

$$= \sum_y \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \left(\prod_{j\neq i} \gamma_j \right) \sum_{L\in\mathcal{L}[0,|\omega|]} \mathcal{U}(L,i)$$

$$\times \sum_{\omega':\omega(i)\to y} z^{|\omega'|} \sum_{l=0}^{|\omega'|} \left(\prod_{l=0}^{|\omega'|} \gamma'_l \right) (1+\mathcal{V}'_0) K[0,|\omega'|;\mathcal{X}] \prod_{s:si\in L} \mathcal{U}_{si} \psi(k_1,k_2,k_3;y,i)$$

As before, we will split the above expression up into terms where $i \in L$, or $i \notin L$. For the latter case, we resum $K[0,|\omega'|;\mathcal{X}]$ using

$$K[0,|\omega'|;\mathcal{X}] = K[1,|\omega'|;\mathcal{U}'] + \sum_{m=1}^{|\omega'|} J[0,m;\mathcal{X}] K[m+1,|\omega'|;\mathcal{U}'], \tag{4.3.8}$$

whereas for the former, we will use

$$K[0,|\omega'|;\mathcal{X}] \tag{4.3.9}$$

$$= \sum_{l=0}^{|\omega'|} \mathcal{W}_l K[0,l;\mathcal{X}] \left\{ K[l+1,|\omega'|;\mathcal{U}'] + \sum_{j=l+1}^{|\omega'|} J[l,j;\mathcal{Y}^{(l)}] K[j+1,|\omega'|;\mathcal{U}'] \right\},$$

with the understanding that the above expression depends on i and L through $\mathcal{W}_l$. We further separate the case $|\omega'| = 0$ from the case $|\omega'| \geq 1$; in the latter case we will use the kernel:

$$\mathcal{H} = \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} \left(\prod_{j:j\neq i} \gamma_j \right) \sum_{L\in\mathcal{L}[0,|\omega|]} \mathcal{U}(L,i) \sum_{\substack{|\omega'|\geq 1 \\ \omega'(0)=\omega(i)}} z^{|\omega'|} \left(\prod_{k=0}^{|\omega'|} \gamma'_k \right) (1+\mathcal{V}'_0), \quad (4.3.10)$$

Performing the resummation of the last line of (4.3.6) using the above identities yields: (cf. (2.2.2))

$$\begin{aligned}
& \sum_y \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} (\prod_{j:j\neq i} \gamma_j) \psi(k_1,k_2,k_3;y,i) \sum_{R_i \ni y,\omega(i)} z^{|R_i|} J[0,|\omega|] && (4.3.11) \\
= \; & \mathcal{H} I_{[j\in L]} \psi(k_1,k_2,k_3;y,i) \sum_{l=1}^{|\omega'|} \mathcal{W}_l K[0,l;\mathcal{X}] K[l+1,|\omega'|;\mathcal{U}'] \\
+ \; & \mathcal{H} I_{[j\in L]} \psi(k_1,k_2,k_3;y,i) \sum_{l=1}^{|\omega'|} \mathcal{W}_l K[0,l;\mathcal{X}] \sum_{j=l+1}^{|\omega'|} J[l,j;\mathcal{Y}^{(l)}] K[j+1,|\omega'|;\mathcal{U}'] \\
+ \; & \mathcal{H} I_{[j\notin L]} \} \psi(k_1,k_2,k_3;y,i) K[1,|\omega'|;\mathcal{U}'] \\
+ \; & \mathcal{H} I_{[j\notin L]} \} \psi(k_1,k_2,k_3;y,i) \sum_{m=1}^{|\omega'|} J[0,m;\mathcal{X}] K[m+1,|\omega'|;\mathcal{U}'] \\
+ \; & \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} (\prod_{j:j\neq i} \gamma_j) \gamma_i e^{ik_1(\omega(i)+ik_2(\omega(|\omega|)-\omega(i))} J[0,|\omega|]
\end{aligned}$$

(the last term takes care of the case where $|\omega'| = 0$). Next we define

$$\Psi_z^{(2)}(k_1,k_2) = \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} (\prod_{j:j\neq i} \gamma_j) \gamma_i e^{ik_1(\omega(i)+ik_2(\omega(|\omega|)-\omega(i))} J[0,|\omega|], \quad (4.3.12)$$

and extract the first term of (4.3.8) and the $l = 0$ term of the of the first sum in (4.3.10) to yield

$$\begin{aligned}
& \mathcal{H}\{I_{[i\notin L]} K[1,|\omega'|] + I_{[i\in L]} \mathcal{W}_0 K[1,|\omega'|;\mathcal{U}']\} \psi(k_1,k_2,k_3;y,i) && (4.3.13) \\
= \; & \sum_{|\omega|\geq 1} z^{|\omega|} \sum_{i=0}^{|\omega|} (\prod_{j:j\neq i} \gamma_j) \gamma_i e^{ik_1(\omega(i)+ik_2(\omega(|\omega|)-\omega(i))} J[0,|\omega|] z\Omega \hat{D}(k_3) \hat{G}_z(k_3).
\end{aligned}$$

Combining this the last expression of with (4.3.14) yields

$$\Psi_z^{(2)}(k_1,k_2)(1+z\Omega\hat{D}(k_3))\hat{G}_z(k_3) \quad (4.3.14)$$

The remaining interaction terms are

$$
\begin{aligned}
& I_{[i\notin l]} \sum_{j=1}^{|\omega'|} J[0,j;\mathcal{X}]K[j+1,|\omega'|;\mathcal{U}'] \qquad (4.3.15)\\
+ \quad & I_{[i\in L]} \sum_{l=1}^{|\omega'|} \mathcal{W}_l K[0,l;\mathcal{X}]K[l+1,|\omega'|;\mathcal{U}'] \\
+ \quad & I_{[i\in L]} \sum_{l=1}^{|\omega'|} \mathcal{W}_l K[0,l;\mathcal{X}] \sum_{j=l+1}^{|\omega'|} J[l,j;\mathcal{Y}^{(l)}]K[j+1,|\omega'|;\mathcal{U}']
\end{aligned}
$$

However, we shall further extract $j = |\omega'|$ from the term $I_{[i\notin L]}$ and $j = |\omega'|$ from the term $I_{[i\in L]}$ with $l = |\omega'|$. Define

$$
\begin{aligned}
& \Psi_z^{(1)}(k_1,k_2,k_3) \qquad (4.3.16)\\
\equiv \quad & \mathcal{H}\{I_{[i\notin l]}J[0,|\omega'|;\mathcal{X}] + I_{[i\in l]}\sum_{l=0}^{|\omega'|}\mathcal{W}_l K[0,l;\mathcal{X}]J[l,|\omega'|;\mathcal{Y}^{(l)}]\}\psi(k_1,k_2,k_3;y,i)
\end{aligned}
$$

Then we are left with:

$$
\begin{aligned}
& \mathcal{H}\psi(k_1,k_2,k_3;y,i)I_{[j\notin L]} \sum_{a=1}^{|\omega'|-1} J[0,a;\mathcal{X}]K[a+1,|\omega'|;\mathcal{U}'] \qquad (4.3.17)\\
+ \quad & \mathcal{H}\psi(k_1,k_2,k_3;y,i)I_{[j\in l]} \sum_{l=1}^{|\omega'|-1} \mathcal{W}_l K[0,l;\mathcal{X}]K[l+1,|\omega'|;\mathcal{U}'] \\
+ \quad & \mathcal{H}\psi(k_1,k_2,k_3;y,i)I_{[j\in l]} \sum_{l=0}^{|\omega'|-2} \sum_{a=l+1}^{|\omega'|-1} \mathcal{W}_l K[0,l;\mathcal{X}]J[l,a;\mathcal{Y}^{(l)}]K[a+1,|\omega'|;\mathcal{U}']
\end{aligned}
$$

Again, as per the derivation following (2.2.21), combining the above expressions yields:

$$
\begin{aligned}
& \mathcal{H}(I_{[j\notin L]}J[0,|\omega'|;\mathcal{X}] + I_{[j\in L]}\sum_{l=0}^{|\omega'|}\mathcal{W}_l K[0,l;\mathcal{X}]J[l,|\omega'|;\mathcal{Y}^{(l)}]) \\
& \times\psi(k_1,k_2,k_3;y,i)z\Omega\hat{D}(k_3)\hat{G}_z(k_3)) \qquad (4.3.18)
\end{aligned}
$$

So we recover a factor

$$
\hat{\Psi}_z^{(1)}(k_1,k_2,k_3)\{z\Omega\hat{D}(k_2)\hat{G}_z(k_2)+1\} \qquad (4.3.19)
$$

Now, if we define

$$B_z(k_1,k_2,k_3) \equiv \Psi_z^{(1)}(k_1,k_2,k_3) + \Psi_z^{(2)}(k_1,k_2), \tag{4.3.20}$$

we have

$$\begin{aligned} P_z(k_1,k_2,k_3) = \hat{G}_z(k_1)\hat{G}_z(k_2)(z\Omega)^2\hat{D}(k_1)\hat{D}(k_2) \times \quad (4.3.21)\\ \left\{\hat{G}_z(k_3) + B_z(k_1,k_2,k_3)(1 + z\Omega\hat{D}(k_3)\hat{G}_z(k_3))\right\} + E(z;k_1,k_2,k_3) \end{aligned}$$

Thus we can now proceed to write $\hat{P}(k_1,k_2,k_3)$ in the same manner as $\hat{P}(k_1)$. To leading order in $\hat{G}_z(k)$ we have

$$(z\Omega)^2\hat{D}(k_1)\hat{D}(k_2)\left(\prod_{i=1}^{3}\hat{G}_z(k_i)\right)\{1 + z\Omega\hat{D}(k_3)B_z(k_1,k_2,k_3)\} \tag{4.3.22}$$

For convenience when dealing with error terms we introduce:

Definition 4.3.1 *Let*

$$\mathcal{B}_z(k_1,k_2,k_3) \equiv (z\Omega)^2\hat{D}(k_1)\hat{D}(k_2)(1 + z\Omega\hat{D}(k_3)B_z(k_1,k_2,k_3)) \tag{4.3.23}$$

It will be shown in section 4.5 that the leading order term for $P(k_1,k_2,k_3)$ is:

$$\lim_{n\to\infty}\frac{1}{2\pi i}\oint_{|z|=\epsilon}\frac{\hat{h}_{z_c}(0)^3(z_c\Omega)^2\{1 + z_c\Omega\hat{\Psi}_{z_c}(0)\}}{\prod_{j=1}^{3}\{\alpha^2k_j^2/2 + \beta\sqrt{z_c - z}\}t_n^{(3)}}\frac{dz}{z^{n+1}} \tag{4.3.24}$$

We will postpone evaluating this expression until we find the general expression for $\hat{P}(k_1,\ldots,k_{2m-1})$ in section 4.5. Next we show that the numerator of (4.3.24) is indeed the correct leading order term.

4.4 The leading order behavior of $\hat{P}(k_1,k_2,k_3)$

We already know the leading order behavior of $\hat{G}_z(k)$, so the only new twist compared to the analysis performed in section 4.2 is that we have to deal with the term $\mathcal{B}_z(k_1, k_2, k_3)$ much as we did with $\hat{h}_z(k)$. The point of this section is to show that to "leading order", $\mathcal{B}_z(k_1, k_2, k_3)$ behaves as $(z_c\Omega)^2(1+z_c\Omega\hat{\Psi}_{z_c}(0)) = v(z_c)$ as the momenta k_i are scaled to zero. The expression in (4.3.24) will seem more plausible in view of the following Lemma, which will be used in the next section.

Lemma 4.5 *Let $\kappa_i = k_i/n^{1/4}$, $\vec{\kappa} = (\kappa_1, \kappa_2, \kappa_3)$, and $v(z) = (z\Omega)^2(1+z\Omega\hat{\Psi}_z(0))$. Then*

$$
\begin{aligned}
\mathcal{B}_z(\vec{\kappa}) &= v(z_c) + \{\mathcal{B}_{z_c}(\vec{\kappa}) - v(z_c)\} \\
&+ \{\mathcal{B}_z(\vec{\kappa}) - \mathcal{B}_{z_c}(\vec{\kappa})\},
\end{aligned}
\tag{4.4.1}
$$

and the last two terms may be bounded by $cn^{-\epsilon/4}$ and $c|z_c - z|^\epsilon$ (respectively) for $|z| \le z_c$ and any $\epsilon < \min(1/2, (d-8)/4)$.

Proof.

We start by looking at the fractional derivative of $\mathcal{B}_z(k_1, k_2, k_3)$. To do so we make use of the fact that

$$
\delta_z^\epsilon \|\mathcal{B}_z(k_1, k_2, k_3)\| \le \delta_z^\epsilon \|(z\Omega)^2(1+z\Omega\hat{\Psi}_z(0))|_{z_c}\| \le c\delta_z^\epsilon \|\hat{\Psi}_z(0)|_{z_c}\| \le c', \tag{4.4.2}
$$

as shown in Appendix C. In view of the above, and (C.1), we can already conclude that

$$
|\mathcal{B}_z(\vec{\kappa}) - \mathcal{B}_{z_c}(\vec{\kappa})| \le c|z_c - z|^\epsilon, \tag{4.4.3}
$$

for $|z| \le z_c$ and any $\epsilon < \min(1/2, (d-8)/4)$.

We must now deal with the first term in (4.4.1). Recall from definition 4.3.1 that

$$
\mathcal{B}_z(\kappa_1, \kappa_2, \kappa_3) \equiv (z\Omega)^2 \hat{D}(\kappa_1)\hat{D}(\kappa_2)(1 + z\Omega\hat{D}(\kappa_3)\mathcal{B}_z(\kappa_1, \kappa_2, \kappa_3)), \tag{4.4.4}
$$

where $B_z(0,0,0) = \hat{\Psi}_z(0)$. Using $B_z(\vec{\kappa})$, we may recast $v(z_c) - B_{z_c}(\vec{\kappa})$ as

$$(z\Omega)^2(1 - \hat{D}(\kappa_1)\hat{D}(\kappa_2)) + (z\Omega)^3(\hat{\Psi}_z(0) - \prod_{i=1}^{3} \hat{D}(\kappa_i)B_z(\kappa_1, \kappa_2, \kappa_3)). \quad (4.4.5)$$

Clearly, the first term decays as $cn^{-1/2}$ for $|z| \leq z_c$. For the second term, we will use the fact that

$$|1 - \prod_{i=1}^{3} \hat{D}(\kappa_i)\cos(y_i k_i)| \leq c(\sum_{i=1}^{3} |y_i \kappa_i|^{\epsilon} + \sum_{j=1}^{3} |\kappa_j|^2) \quad (4.4.6)$$

for any $\epsilon < 2$ and $\kappa_i \in [-\pi, \pi]^d$, as can be seen by performing a series expansion of the L.H.S. and using $\hat{D}(k) = 1 + O(k^2)$.

Let $B_z(y_1, y_2, y_3)$ be the function whose Fourier transform is $B_z(\vec{k})$, and let $b_m(y_1, y_2, y_3)$ be the m^{th} coefficient of its Maclaurin series. It follows by (4.4.6), symmetry, and the inequality $|y_i|^{\epsilon} b_m(y_1, y_2, y_3) \leq m^{\epsilon} b_m(y_1, y_2, y_3)$, that for $|z| \leq z_c$,

$$\begin{aligned}
&|\hat{\Psi}_z(0) - \prod_{i=1}^{3} \hat{D}(\kappa_i)B_z(\vec{\kappa})| = \quad (4.4.7)\\
&|\sum_{m\geq 1} \sum_{y_1, y_2, y_3} b_m(y_1, y_2, y_3)\left(1 - \prod_{i=1}^{3} \hat{D}(\kappa_i)\cos(y_i\kappa_i)\right)| \\
\leq\; & \sum_{m\geq 1} \sum_{y_1, y_2, y_3} c(\sum_{i=1}^{3} |y_i \kappa_i|^{\epsilon} + |\kappa_i|^2)|b_m(y_1, y_2, y_3)| \\
\leq\; & \sum_{m\geq 1} \sum_{y_1, y_2, y_3} c(3n^{-\epsilon/4}\, m^{\epsilon} + \sum_{i=1}^{3} |\kappa_i|^2)|b_m(y_1, y_2, y_3)| \\
\leq\; & c' n^{-\epsilon/4} |\delta_z^{\epsilon} \hat{\Psi}_z(0)|_{z_c}| + c'' n^{-1/2} \leq c''' n^{-\epsilon/4}
\end{aligned}$$

for any $\epsilon < \min(1/2, (d-8)/4)$. This takes care of the last term of (4.4.5) and completes the proof of Lemma 4.5. ...■

4.5 The higher order finite dimensional distributions

In this section, we compute the general expression for $\hat{P}(\{\sigma\}; k_1, \dots k_{2m-3})$ as defined in (1.1.19).

Before proceeding, let us recap our results from chapter 3: we inductively calculated the generating function for trees connecting $m+1$ points from that of trees connecting m points by applying an operator $\mathcal{D}$ which counted the number of sites in a tree. For the 3 point function (the base case of the induction), this amounted to counting the number of sites in a rib, on all of the ribs of the tree's backbone. A resummation was performed to re-cast the 3 point function in terms of a convolution of 2 point functions (to leading order). In the general case, when summing over ribs, the same resummation could be used on the 2 point functions (by their independence), although when summing over ribs in the kernel $\hat{\Psi}_z(0)$, no resummation need be performed since even when the interaction between the rib and $\hat{\Psi}_z(0)$ is ignored (yielding an upper-bound), the resulting terms will be of lower order.

Presently, we will perform much the same drill although instead of counting the number of sites on a rib, we will Fourier transform them with respect the vertebra to which the rib was attached. By doing so, we will induce a new node, and hence a new backbone whose new internodal distances must be recorded in momentum space. As an upper bound for Fourier transforming ribs inside the kernel $B_z(k_{\sigma_1(i)}, k_{\sigma_2(i)}, k_{\sigma_3(i)})$, we will merely count the sites on these ribs as in chapter 3, where there were no phases. Indeed the estimates in the proof of (3.1.2) will be most valuable.

More recently, by the results of the 2 previous sections, we had

$$\hat{P}_z(k_1, k_2, k_3) = \prod_{j=1}^{3} \hat{G}_z(k_j) \mathcal{B}_z(k_1, k_2, k_3) + E(z), \tag{4.5.1}$$

where $\hat{P}_z(k_1, k_2, k_3)$ was defined in (4.3.1) and $|E(z)| \le c|z_c - z|^{-1}$ for $|z| \le z_c$. However, for trees connecting $m \ge 4$ points, we will have to worry about the backbone configuration via $\{\sigma\}$, which was defined to be a set of ordered triplets $\{(\sigma_1(j), \sigma_2(j), \sigma_3(j))\}_{j=1}^{m-2}$ chosen from $\{1, .., 2m-3\}$ which specify which momenta meet at $\mathcal{B}(k_{\sigma_1}, k_{\sigma_2}, k_{\sigma_3})$ in the j^{th} internal node. Keeping this in mind we can now prove:

Lemma 4.6 *The Fourier transform with respect to the internodal spacing in the backbone of trees connecting $m \ge 3$ points, given a backbone structure specified by $\{\sigma\}$, is*

$$\begin{aligned} \hat{P}_z(\{\sigma\}; k_1, ..., k_{2m-3}) = \prod_{i=1}^{2m-3} \hat{G}_z(k_i) \prod_{j=1}^{m-2} \mathcal{B}_z(k_{\sigma_1(j)}, k_{\sigma_2(j)}, k_{\sigma_3(j)}) \\ + E_m(z, \vec{k}), \end{aligned} \tag{4.5.2}$$

where for $|z| < z_c$,

$$|E_m(z, \vec{k})| \le c|z_c - z|^{-3/2} \cdot |z_c - |z||^{-m+3+\epsilon},$$

uniformly in the k_i. The error term accounts for trees which do not possess binary backbones.

Proof.

The case $m = 3$ has already been dealt with. Let us assume (4.5.2) for $3 \le k \le m-1$. Then, for $k = m$ by using a lower order generating function and 'splicing' its backbone, we have for any $1 \le i \le m-3$,

$$\prod_{i=1}^{2m-6} \hat{G}_z(k_i) \prod_{j=1}^{m-3} B_z(k_{\sigma_1(j)}, k_{\sigma_2(j)}, k_{\sigma_3(j)}) \tag{4.5.3}$$
$$\times \sum_{x_1} \sum_{|\omega|\geq 1} \sum_{i=0}^{|\omega|} e^{i\{k_{2m-5}\omega(i)+k_{2m-3}(\omega(|\omega|)-\omega(i))\}}$$
$$\times \; z^{|\omega|} \left(\prod_{j\neq i} \gamma_j\right) \left\{ e^{ik_{2m-4}(x_1-\omega(i))} \sum_{R_i \ni \omega(i), x_1} z^{|R_i|} \right\} K[0, |\omega|] + l.o.t.$$

where the lower order terms above may be bounded uniformly in the k_i by

$$\sum_{i=1}^{m-3} c_{m,i}(\mathcal{D}\{\mathcal{D}^i((1+z\Omega\hat{\Psi}_z(0))\}\mathcal{D}^{m-3-i}\chi(z)^3)$$
$$+c_m\{\mathcal{D}(1+z\Omega\hat{\Psi}_z(0))\}\chi(z)^{2m-5}.$$

The first term corresponds to previous generating functions where the sites on the ribs in $\hat{\Psi}_z(0)$ are Fourier transformed, and the second error term is an upper bound for the case where we Fourier transform the ribs in $B_z(k_{\sigma_1(j)}, k_{\sigma_2(j)}, k_{\sigma_3(j)})$ for some j. With the bounds in chapter 3, these error terms can be bounded for $|z| < z_c$ by $|z_c - z|^{-3/2}|z_c - |z||^{-m+3+\epsilon}$.

By resumming the principal term as in (4.3.21) we obtain

$$\begin{aligned} \hat{P}_z(\{\sigma\}; k_1, ..., k_{2m-3}) &= \prod_{i=1}^{2m-3} \hat{G}_z(k_i) \prod_{i=1}^{m-2} B_z(k_{\sigma_1(i)}, k_{\sigma_2(i)}, k_{\sigma_3(i)}) \\ &+ \tilde{E}_m(z, \vec{k}), \end{aligned} \tag{4.5.4}$$

where $|\tilde{E}_m(z,\vec{k})| \leq c|z_c - z|^{-3/2}|z_c - |z||^{-m+3+\epsilon}$ for $|z| < z_c$ uniformly in the k_i.

$\ldots\blacksquare$

Our original quest was to compute $\mathcal{P}(\{\sigma\}; k_1, .., k_{2m-1})$ as defined in Theorem 1.2. However, we will let $\kappa_i \equiv \tilde{k}_i/n^{1/4}$ [cf $\kappa_i = k_i/\tilde{D}n^{1/4}$ in (1.1.24)] and compute

$$\hat{P}(\{\sigma\}; \tilde{k}_1, .., \tilde{k}_{2m-1}) \equiv \lim_{n\to\infty} \frac{1}{2\pi i} \oint_{|z|=\epsilon} \frac{\hat{P}_z(\{\sigma\}; \kappa_1, .., \kappa_{2m-3})}{t_n^{(m)}} \frac{dz}{z^{n+1}}. \tag{4.5.5}$$

The scaling factor $\bar{D}$ will be dealt with at the end of the proof.

To compute (4.5.5) we must separate the wheat from the chaff and re-express (4.5.4) as:

$$\prod_{i=1}^{2m-3}\left\{\frac{\hat{h}_{z_c}(0)}{\alpha\kappa_i^2/2+\beta\sqrt{z_c-z}}+E_1(z,\kappa_i)+E_2(z,\kappa_i)\right\}\times \tag{4.5.6}$$
$$\prod_{i=1}^{m-2}\left((z_c\Omega)^2(1+z_c\Omega\hat{\Psi}_{z_c}(0)+E_3(z,\kappa_i))\right)+\tilde{E}_m(z,\vec{\kappa}),$$

where as usual:

$$E_1(z,\kappa_i) = \frac{B(z,\kappa_i)(\hat{h}_{z_c}(0)+\mathcal{E}_z(\kappa_i))}{(\alpha\kappa_i^2/2+\beta(z_c-z)^{1/2})\hat{F}_z(\kappa_i)}, \tag{4.5.7}$$
$$E_2(z,\kappa_i) = \frac{\mathcal{E}_z(\kappa_i)}{\alpha\kappa_i^2/2+\beta(z_c-z)^{1/2}}, \tag{4.5.8}$$
$$\begin{aligned} E_3(z,\vec{\kappa}) &= \{B_{z_c}(\vec{\kappa})-(z_c\Omega)^2(1+z_c\Omega\hat{\Psi}_{z_c}(0))\} \\ &+ \{B_z(\vec{\kappa}/n^{1/4})-B_{z_c}(\vec{\kappa}/n^{1/4})\}, \end{aligned}$$

with $B(z,k)$ and $\mathcal{E}(z,k)$ as defined in (4.2.3) and (4.2.10) respectively.

A typical term in the first product in (4.5.6) will be of the form:

$$\tilde{E}_m^{(\vec{p})}(z,\vec{\kappa})=\left(\prod_{i=1}^{p_1}\frac{\hat{h}_{z_c}(0)}{\alpha\kappa_i^2/2+\beta\sqrt{z_c-z}}\right)\left(\prod_{i=1}^{p_2}E_1(z,\kappa_i)\right)\left(\prod_{i=1}^{p_3}E_2(z,\kappa_i)\right), \tag{4.5.9}$$

where $\sum_i p_i = 2m-3$. A typical term in the second product in (4.5.6) will be of the form

$$\tilde{E}_m^{(\vec{q})}(z,\vec{\kappa})=\left((z_c\Omega)^2(1+z_c\Omega\hat{\Psi}_{z_c}(0))\right)^{q_1}\left(\prod_{j=1}^{q_2}E_3(z,\vec{\kappa}_{\sigma(j)})\right), \tag{4.5.10}$$

with $q_1+q_2=m-2$. Of course, the only term in (4.5.6) that will contribute to (4.5.5) is

$$\prod_{i=1}^{2m-3}\left\{\frac{\hat{h}_{z_c}(0)}{\alpha\kappa_i^2/2+\beta\sqrt{z_c-z}}\right\}\left((z_c\Omega)^2\{1+z_c\Omega\hat{\Psi}_{z_c}(0)\}\right)^{m-2}, \tag{4.5.11}$$

as shown by:

Lemma 4.7 *Let $m \geq 2$. Suppose $p_1 < 2m-3$ in (4.5.9), or $p_1 = 2m-3$ but $q_1 < m-2$ in (4.5.10). Then,*

$$\frac{1}{2\pi i}\oint_{|z|=\epsilon} \tilde{E}_m^{(\vec{p})}(z,\vec{\kappa})\tilde{E}_m^{(\vec{q})}(z,\vec{\kappa})\frac{dz}{t_n^{(m)}z^{n+1}} = O(n^{-\epsilon/4}), \tag{4.5.12}$$

for any $\epsilon < \min(1/2,(d-8)/4)$. The above result holds with the same ϵ if the integrand in (4.5.12) is replaced by any function obeying the bound on the error term in (4.5.4). (Recall that $t_n^{(m)} \overset{n\to\infty}{\sim} cn^{m-5/2}z_c^{-n}$).

Corollary 4.5.1 *All information contained in $\{\sigma\}$ is lost in the continuum limit, as can be seen from (4.5.11).*

We are finally in a position to prove (with the notation in (4.5.5)):

Theorem 4.8 *For $m \geq 2$,*

$$\begin{aligned}&\hat{P}(\{\sigma\};\tilde{k}_1,..,\tilde{k}_{2m+1}) \\ &= \int_0^\infty \dots \int_0^\infty dl_1 \dots dl_{2m+1}(\sum_{i=1}^{2m+1} l_i)e^{-(\sum_i^{2m+1} l_i)^2/2-\sum_j^{2m+1} \alpha l_j\tilde{k}_j^2/2\sqrt{1+b}},\end{aligned} \tag{4.5.13}$$

[cf (1.1.24) with $k_i/\tilde{D} = \tilde{k}_i$].

Proof. Define the auxiliary variables $p_i^2 \equiv \tilde{k}_i^2 + \delta$ for $1 \leq i \leq 2m-3$ and some $\delta > 0$. Also let

$$c(m) = v(z_c)^{m-2}\hat{h}_{z_c}(0)^{2m-3} = \left((z_c\Omega)^2\{1+z_c\Omega\hat{\Psi}_{z_c}(0)\}\right)^{m-2}(1/\Omega z_c)^{2m-3}. \tag{4.5.14}$$

By Lemma 4.7, it suffices to compute, for $m \geq 2$,

$$\lim_{n\to\infty}\lim_{\delta\searrow 0}\frac{1}{2\pi i}\oint_{|z|<\epsilon'}\prod_{i=1}^{2m-3}\frac{c(m)}{\alpha p_i^2/2n^{1/2}+\beta(z_c-z)^{1/2}}\frac{dz}{t_n^{(m)}z^{n+1}}, \tag{4.5.15}$$

where by (4.1.15),

$$t_n^{(m)} \overset{n\to\infty}{\sim} \frac{1/(z_c\Omega)}{\sqrt{2\pi(1+b)}}n^{m-5/2}\lambda^n(1+O(n^{-\epsilon})),$$

for any $\epsilon < \min(1/2, (d-8)/4)$. Proceeding exactly as for the 2-point function, we deform the contour around the origin to the 'pack-man' contour of fig 2, page 70, and make the change of variables $-w/n = z_c - z$ with the same branches as before. By the definition of the p_i, the integrand in (4.5.15) is bounded below uniformly in w on the cut plane. Thus, the contributions from the arcs of radius r and R of the pack-man contour vanish as $r \searrow 0$ and $R \nearrow \infty$; the latter because of the factor z^{n+1} in the denominator.

Now let

$$I(w,\vec{k}) = \prod_{i=1}^{2m-3} \frac{c(m)}{\alpha p_i^2/2n^{1/2} - i\beta w^{1/2}/n^{1/2}}. \tag{4.5.16}$$

Since $\sqrt{-w/n}$ is $-iw^{1/2}/n^{1/2}$ on the top branch of our contour, and $+iw^{1/2}/n^{1/2}$ on the bottom, we get $2i\,\mathcal{I}m(I(w,\vec{p}))$ for our integrand by taking into account the orientation of the integration along the top and bottom of the branch cut.

Thus, (4.5.15) reduces to

$$\lim_{n\to\infty}\lim_{\delta\searrow 0}\frac{1}{\pi}\int_0^\infty \mathcal{I}m\left\{\frac{n^{-(m-5/2)}c(m)\Omega z_c\sqrt{1+b}}{n^{-(2m-3)/2}\prod_{i=1}^{2m-3}(\alpha p_i^2/2 - i\beta w^{1/2})}\right\}\frac{dw}{(1+\frac{w}{z_c n})^{n+1} n z_c} \tag{4.5.17}$$

The factor of $n^{m-5/2}\Omega z_c\sqrt{1+b}$ comes from $t_n^{(m)}$ and the factor of $n^{-(m-3/2)}$ was pulled from the product in the square root in the denominator (recall the change of variables $z_c - z = -\frac{w}{n}$). The combination of the factors cancels the factor of $1/n$ in the measure.

Now, by using the identity

$$\frac{1}{p^2/2 - iw^{1/2}} = (2/z_c)^{1/2}\int_0^\infty dl e^{-l(2/z_c)^{1/2}(p^2/2 - iw^{1/2})} \text{ (since } p_i^2 > 0) \tag{4.5.18}$$

in (4.5.17) we obtain

$$\lim_{n\to\infty}\lim_{\delta\searrow 0}\frac{\tilde{c}(m)}{\pi}\int_0^\infty \frac{dw}{(1+\frac{w}{z_c n})^{n+1} z_c} I(w,\vec{p}) \tag{4.5.19}$$

where

$$I(w,\vec{p}) = \int_0^\infty \cdots \int_0^\infty dl_1 \ldots dl_{2m-3} e^{-\sum_{j=1}^{2m-3}\alpha l_j p_j^2/2\sqrt{1+b}}\,\mathcal{I}m\, e^{iw^{1/2}(2/z_c)^{1/2}L} \tag{4.5.20}$$

with $\tilde{c}(m) = c(m)\Omega z_c\sqrt{1+b}\frac{1}{z_c\pi}\left(\frac{2}{z_c\beta^2}\right)^{(2m-3)/2} = \frac{1}{z_c}(2/\pi)^{1/2}$ and $L = \sum_{j=1}^{2m-3} l_j$. By interchanging the order of integration (which is permissible for any $\delta > 0$) and then letting $\delta \searrow 0$ we obtain:

$$\lim_{n\to\infty}\int_0^\infty \cdots \int_0^\infty dl_1 \ldots dl_{2m-3} e^{-\sum_{j=1}^{2m-3} \alpha l_j \tilde{k}_j^2/2\sqrt{1+b}} (2/\pi)^{1/2} F_n(L) \quad (4.5.21)$$

where

$$F_n(L) = \int_0^\infty \frac{dw \ \sin((2w)^{1/2}L)}{(1+\frac{w}{n})^{n+1}} = \int_0^\infty \frac{dt L\cos(tL)}{(1+t^2/2n)^n} \quad (4.5.22)$$

(as can be seen by letting $(2w)^{1/2} = t$ and then integrating by parts). Once again we must appeal to special functions [GR] (p.959); in particular we will use representations 8.432 (5) and (9) for the the function $k_\nu(xz)$. By letting $y^2 = t^2/2n$ we arrive at the integral

$$\sqrt{2n}L\int_0^\infty dy \frac{\cos(\sqrt{2n}Ly)}{(1+y^2)^n} \equiv F_n(L). \quad (4.5.23)$$

By the integral representations discussed in [GR] we get (for $L \geq 1$)

$$\begin{aligned} F_n(L) &= \frac{c\sqrt{2n}}{4^n\Gamma(n)^2} L \int_0^\infty \frac{dt \ t^{2n-1}}{(t^2+1)^{1/2}} e^{-\sqrt{t^2+1}\cdot\sqrt{2n}L} && (4.5.24) \\ &\leq \frac{c\sqrt{2n}}{4^n\Gamma(n)^2} L \int_0^\infty e^{-\sqrt{n}t-\sqrt{n}L} \ t^{2n-1} dt \\ &\leq \frac{c\sqrt{2n}}{4^n\Gamma(n)^2} e^{-\sqrt{n}L} L \int_0^\infty dt e^{-t} \ t^{2n-1} n^{-n+1/2} \\ &\leq \frac{c\sqrt{2n}}{4^n\Gamma(n)^2} L e^{-\sqrt{n}L}\Gamma(2n-1) n^{-n+1/2} \\ &\overset{n\to\infty}{\sim} c' n^{3/2} L e^{-\sqrt{n}L}. \end{aligned}$$

Thus, $F_n(L) \leq c \ Le^{-L}$ for some $L \geq L_0$ and $n \geq n_0$. We are now justified in using Lebesque's dominated convergence theorem. By the same arguments as for the 2-point function,

$$\lim_{n\to\infty} F_n(L) = (\pi/2)^{1/2} L e^{-L^2/2}. \quad (4.5.25)$$

Inserting this result back into (4.5.21), we arrive at

$$\int_0^\infty \ldots \int_0^\infty dl_1 \ldots dl_{2m-3} L e^{-(L)^2/2-\sum_{j=1}^{2m-1} \alpha l_j \hat{k}_j^2/2\sqrt{1+b}}. \tag{4.5.26}$$

...■

As mentioned in the introduction, the diffusion constant D of (1.1.10) obeys $(2/\pi)^{1/2}D^2/d = \alpha/\sqrt{1+b}$, so that had we taken $\bar{D}^2 \equiv D^2(2/\pi d)^{1/2}$ as our scaling constant in the transformation $k_i \to k_i/\bar{D}n^{1/4}$, we would have wound up with (1.1.24) as per the content of Theorem 1.2.

The above establishes the convergence of the finite dimensional distributions of continuum lattice trees to integrated super-Brownian excursion as will be discussed in Chapter 5.

Proof of Lemma 4.7 We have already taken care of the the errors for the 2-point function limit, so from here on in, $m \geq 3$.

We will consider 2 cases. The first deals with terms in (4.5.9) for which $p_2 + p_3 \geq 1$. The last case assumes $p_2 = p_3 = 0$ but $q_2 \geq 1$ in (4.5.10). The following estimates will also simplify the analysis:

By Hölder's inequality (using counting measure),

$$(\sum_{i=1}^m x_i)^n \leq m^{n-1}(\sum_{i=1}^m x_i^n) \tag{4.5.27}$$

for any $x_i > 0$. Using the above and Lemmas 4.3, 4.4 and 4.2, we can write

$$\begin{aligned} |E_1(z,\kappa_i)|^{p_2} &\leq a_m|z_c - z|^{-p_2/2}\Big\{|z_c - z|^{+p_2\epsilon} \\ &+ n^{-p_2(1/2+\epsilon/4)}|z_c - z|^{-p_2/2} + n^{-p_2/2}|z_c - z|^{p_2(\epsilon-1/2)}\Big\}, \end{aligned} \tag{4.5.28}$$

and

$$|E_2(z,\kappa_i|^{p_3} \leq b_m|z_c - z|^{-p_3/2}(n^{-p_3\epsilon/4} + |z_c - z|^{+p_3\epsilon}). \tag{4.5.29}$$

Case (*i*) If $m \geq 3$ with $p_2 + p_3 \geq 1$ in (4.5.9), we can bound the second product in (4.5.6) on the disc $|z| \leq z_c$ by a constant, and then bound the first product by

$$\frac{cA^{p_2}C^{p_3}}{|z_c - z|^{p_1/2+p_2/2+p_3/2}}, \tag{4.5.30}$$

with $p_1 + p_2 + p_3 = 2m - 3$, and where by (4.5.28, and (4.5.29),

$$\begin{aligned} A^{p_2} &\leq a_m(|z_c - z|^{+p_2\epsilon} + n^{-p_2(1/2+\epsilon/4)}|z_c - z|^{-p_2/2}) \\ &+ a_m n^{-p_2/2}|z_c - z|^{p_2(\epsilon-1/2)}, \end{aligned} \tag{4.5.31}$$

and

$$C^{p_3} \leq b_m(n^{-p_3\epsilon/4} + |z_c - z|^{+p_3\epsilon}). \tag{4.5.32}$$

Recall that $p_2+p_3 \geq 1$. If $p_2 = 0$ the analysis is easy so we will focus on $p_2 \geq 1$. In the latter case, since (4.5.32) is bounded for $|z| \leq z_c$, we can deal solely with terms coming from (4.5.31). The denominator of (4.5.30) is $|z_c - z|^{-m+3/2}$, so applying Lemma 4.1 to (4.5.30) with the bounds in (4.5.31), we find that the coefficients of the terms under consideration are $O(n^\gamma z_c^{-n})$ with
$\gamma = (-1 + m - 3/2 - p_2\epsilon)$ or $(-1 + m - 3/2 - p_2(1/2 + \epsilon/4) + p_2/2)$ or $(-1 + m - 3/2 - p_2/2 - p_2(\epsilon - 1/2))$. Since $t_n^{(m)} \overset{n\to\infty}{\sim} cn^{m-5/2}z_c^{-n}$, the corresponding term in (4.5.12) will decay at least as fast as $O(n^{-\epsilon/4})$ for any $\epsilon < \min(1/2, (d-8)/4)$.

Case (*ii*) Suppose now that $p_1 = 2m - 3$ in (4.5.9) and $q_2 \geq 1$. Since $E_3(z, \vec{\kappa})$ is bounded on the disc $|z| \leq z_c$, it suffices to consider the case $q_2 = 1$. Such a term can be bounded by

$$c|z_c - z|^{-m+3/2}(|z_c - z|^\epsilon + n^{-\epsilon/4}), \tag{4.5.33}$$

for $|z| < z_c$. By Lemma 4.1 (*i*), the n^{th} coefficient of such a term is no worse than

$$c(n^{-1+-3/2+m-\epsilon} + n^{-1-\epsilon/4-3/2+m})z_c^{-n}.$$

Once again, upon dividing by $t_n^{(m)}$, the corresponding term in (4.5.12), will be $O(n^{-\epsilon/4})$ for any $\epsilon < \min(1/2, (d-8)/4)$. Since the error term in (4.5.2) is bounded by

$$c|z_c - z|^{-3/2} \cdot |z_c - |z||^{-m+3+\epsilon},$$

Lemma 4.1 once again shows its coefficients are no worse than $O(z_c^{-n} n^{m-5/2-\epsilon})$ for any $\epsilon < \min(1/2, (d-8)/4)$.

The end result is that all error terms in (4.5.12) are $O(n^{-\epsilon/4})$ for any $\epsilon < \min(1/2, (d-8)/4)$. ...■

Chapter 5

Discussions and conclusions

5.1 Discussions

Before moving to the conclusions, we will interpret our results.

With the notation from (1.1.24) (i.e. with the momenta rescaled by $\bar{D}$ to avoid scaling factors), we had:

$$\mathcal{P}(k_1, .., k_{2m+1}) = \int_0^\infty ... \int_0^\infty dl_1 ... dl_{2m+1} (\sum_{i=1}^{2m+1} l_i) e^{-(\sum_i^{2m+1} l_i)^2/2 - \sum_j^{2m+1} l_j k_j^2/2}. \tag{5.1.1}$$

Upon Fourier transforming back to x space, we recover

$$\begin{aligned} & \int_0^\infty ... \int_0^\infty dl_1 ... dl_{2m+1} \frac{(\sum_{i=1}^{2m+1} l_i)}{(\prod_{i=1}^{2m+1} 2\pi l_i)^{d/2}} e^{-(\sum_i^{2m+1} l_i)^2/2 - \sum_j^{2m+1} \Delta y_j / l_j} \\ \equiv \; & P^{(m)}(\Delta y_1, .., \Delta y_{2m+1}), \end{aligned} \tag{5.1.2}$$

where the Δy_i s are inter-nodal distances. By letting

$$p(\Delta y_i; l_i) = \frac{e^{-\Delta y_i^2/2l_i}}{(2\pi l_i)^{d/2}}, \tag{5.1.3}$$

we can recast (5.1.2) as

$$P^{(m)}(\cdot) = \int_0^\infty ... \int_0^\infty dl_1 ... dl_{2m+1} (\sum_{i=1}^{2m+1} l_i) e^{-(\sum_i^{2m+1} l_i)^2/2} \prod_{i=1}^{2m+1} p(\Delta y_i; l_i). \tag{5.1.4}$$

The above has the interpretation that the $p(\Delta y_i; l_i)$ s are the transition kernels for Brownian motion in $\mathbb{R}^d$ for particles moving from the 'left end-point' of Δy_i to the 'right end-point' in time l_i. The factor

$$(\sum_{i=1}^{2m+1} l_i)e^{-(\sum_i^{2m+1} l_i)^2/2} \tag{5.1.5}$$

is responsible for determining how these times are 'shared' among the different nodes.

We shall now discuss how, with the correct interpretation, the $\{P^{(m)}\}$ form a consistent family. Consider m external vertices $\{0, x_1, .., x_{m-1}\}$ along with $m-2$ internal vertices y_j. We can write $P^{(m)}$ in terms of these internal and external vertices, and then integrate out the external node x_{m-1} (for example) and the internal node y_{m-2} (for sake of argument) to which it was connected. After some tedious calculations, one can show that $P^{(m)}$ then reduces to $P^{(m-1)}$ as per the backbone determined by $\{0, .., x_{m-2}\}$ and $\{y_1, .., y_{m-3}\}$ (i.e. the backbone with the edge (y_{m-2}, x_{m-1}) removed).

Having given a physical explanation of (5.1.2), it is perhaps appropriate to briefly talk about d dimensional super-Brownian motion. Again, for the cognoscenti, this is special case of an (α, d, β)-Superprocess [DBS], with $\alpha = 2$ and $\beta = 1$; $\alpha = 2$ is the part responsible for the Brownian motion, and $\beta = 1$ will yield binary branching. As for the description, super-Brownian motion is a measure valued process $\{Z_t; t \geq 0\}$ which (roughly speaking) can be constructed in $\mathbb{R}^d$ by running a particle undergoing d dimensional Brownian motion and then at exponentially distributed times having it die, or split into two independent copies each with probability 1/2. If the particle splits, the process can then continue splitting until it dies out. The state space at time t is the number of particles in existence and their positions. ISE in turn, is a special case of super-Brownian motion.

Heuristically, ISE is a variant of superBrownian motion in which the time variable has been integrated out to yield a random measure. The process itself starts with infinitesimal mass at the origin and is then conditioned to have unit mass at extinction. The construction of such measures is very technical and complicated but for our purposes suffice it to say that Aldous' construction [A 2 -4] involved randomly embedding abstract trees into R^d. These abstract trees can, in turn, be constructed via a map from Brownian excursions to abstract trees (the 'extrema' of the former determining the branching structure of the latter). The probability density for these abstract trees was found to be (5.1.5).

Before moving to the conclusions, we should point out that there is another instance where the density (5.1.5) has been observed.

Via totally different methods, and via a different map $T_p(e, t_1, .., t_p) \rightarrow (\tau, x_1, .., x_p)$ from excursions e to 'trees', Le Gall [G], obtained the same density (5.1.5) in a paper that extended a result of Bismut [B]. His results essentially state that under the law of normalised excursion $n_{(1)}(de)$ (excursions conditioned on having duration one) the distribution of the tree $2T_p(e, t_1, ..t_p)$ under the probability measure

$$1_{[0,1]}(t_1)...1_{[0,1]}(t_p)n_{(1)}(de)dt_1...dt_p \tag{5.1.6}$$

has density

$$2^{-(p-1)}(\sum_{i=1}^{p} x_i)e^{-(\sum_{i=1}^{p} x_i)^2/2} \tag{5.1.7}$$

with respect to a measure $\Lambda(dT)$. This last measure on trees is the distribution of $T_p(e; t_1, .., t_p)$ under the measure

$$2^{-(p-1)}1_{[0,1]}(t_1)...1_{[0,1]}(t_p)n(de)dt_1...dt_p, \tag{5.1.8}$$

where $n(de)$ is Itô measure on positive excursions of linear Brownian motion with a special normalisation.

5.2 Conclusions

The main conclusion to be drawn from this work, is that the scaling limit of sufficiently spread-out lattice trees above eight dimensions (or of the n.n. model in sufficiently high dimension) is distributed as ISE.

More generally though, any model in statistical mechanics, for which the Fourier transform of its generating function (with radius of convergence z_0) obeys

$$\hat{G}_z(k) = \frac{c}{D^2k^2 + B(z_0 - z)^{1/2}} + E(z,k), \tag{5.2.9}$$

with $|\frac{d}{dz}E(z,k)| \leq c|z_0 - z|^{-3/2+\epsilon}$ for any $\epsilon > 0$, can be automatically known to have its scaling limit 2-point functions distributed as ISE.

Such considerations could be useful in the study of high dimensional percolation as conjectured by Hara and Slade in [DS], since the contour techniques in Theorem 4.8 reduce the scaling limit problem to analytic estimates similar to those of Lemma 4.7. However, one obvious hurdle that will require new methods to clear is proving the analogue of (1.1.9). This analogue has already been conjectured in [AN], and will likely be a major stumbling block, as will be the lack of a unique backbone structure.

Present work centers around the use of even more refined generating functions. By adding another variable ζ to our 2-point generation function, we can keep track of the number of bonds in the backbone of a tree connecting 2 points. Thus,

$$\hat{G}_{z,\zeta}(k) = \sum_{n=1}^{\infty}\sum_{m=1}^{n} \hat{t}_{m,n}(k)\zeta^m z^n \tag{5.2.10}$$

where $\hat{t}_{m,n}(k)$, is the Fourier transform for the number of n bond trees connecting 0 and x with exactly m bonds. This new generating function obeys

$$\hat{G}_{z,\zeta}(k) \sim \frac{c}{\alpha k^2/2 + \beta\sqrt{z_c - z} + 2\gamma^{-1}(1-\zeta)}, \tag{5.2.11}$$

where γ is a new constant governing the scaling of the backbone. This new work explains the rather artificial looking l_is in (5.1.2) with the following:

Theorem 5.1 *For sufficiently spread-out trees above $d = 8$ (or for $d \geq d_0$ for the n.n. model),*

$$\lim_{n\to\infty} (\gamma n^{1/2}) \frac{\hat{t}_{\lfloor l\gamma n^{1/2}\rfloor,n}(k/(\bar{D}n^{1/4}))}{\hat{t}_n} = le^{-l^2/2-lk^2/2}. \tag{5.2.12}$$

Similar statements hold for m point functions. The precise statements of the results of this work are contained in [DS].

The above result has the rather satisfying interpretation that if we let $l = m/(\gamma n^{1/2})$ in the right side of (5.2.12), we get a Riemann sum approximation to $\mathcal{P}(k)$ by dividing by $\gamma n^{1/2}$ and summing over all m. The main content of the above Theorem is that backbones of length $n^{1/2}$ are typical, and converge to Brownian motion sample paths in the scaling limit.

We now conclude with a comparison of the models discussed in this Chapter: Aldous' construction used probabilistic methods and led to an interpretation of (5.1.5) as involving inter-nodal edges of lengths l_i; Le Gall's results used very deep insights into excursion theory and also yielded (5.1.5) albeit with a slightly different notion of edge lengths; finally by analytical techniques (with generating functions) we also obtained families of consistent measures involving (5.1.5) from the scaling limit of a deterministic model, with the physical meaning of the l_i's as captured by (5.2.12).

Chapter A

Appendix

A Estimating integrals

We seek upper bounds on the volume cutoff of integrals of the form:

$$\int_{\lambda>|k_1|,\ldots,|k_m|} \frac{d^d k_1 \ldots d^d k_m}{\prod_{i=1}^{m}(k_i^2+\mu^2)^{n_i} \prod_{1\le i<j\le m}((k_i+k_j)^2+\mu^2)^{n_{ij}}}, \tag{A.13}$$

where n_i and n_{ij} are natural numbers. For our application, $m \le 4$ and $\mu = 1$; for $m = 4$ we will have $n_{ij} = 0$ for those loops not sharing momenta k_i, k_j. The following is a self-contained version of the method developed by Thomas Riesz[1]; the argument amounts to showing that (A.13) can be dominated by a sum of integrals in which the mass $\mu \equiv 0$ but where the domain of integration excludes the resulting poles in the integrand (cf. Theorem A.1). Once this is achieved, one can estimate the cutoff behavior in terms of power counting in all sub-graphs via induction on m (the number of loops). In essence, the overall behavior is determined by the worst sub-graph (the example in section 'B' explains these steps much more explicitly). We start by re-casting (A.13) with

[1]which itself is based on the work of Hahn & Zimmerman

the following notation: Let $N = m + m(m-1)/2$ and define $l_i = \sum_{j=1}^{m} C_{ij}k_j$ for $1 \le i \le N$, where $\text{rank}(C_{ij}) = m$ and any row of C_{ij} has at most 2 non-zero entries. Then (A.13) can be expressed as

$$\int_{\lambda > |k_1|,\dots,|k_m|} \frac{d^dk_1 \dots d^dk_m}{\prod_{i=1}^{N}(l_i^2+1)^{p_i}}, \tag{A.14}$$

where the p_i correspond to the appropriate n_i or n_{ij}.

Define $D^\lambda = \{(k_1,\dots,k_m) \in \mathbb{R}^{dm} : |l_i| < \lambda,\ 1 \le i \le N\}$. Since D^λ contains a convex neighbourhood of the origin, there is some $c' > 0$ for which $D^{c'\lambda} \supset [-\lambda,\lambda]^{dm}$; by scaling out this constant through the measure, (A.13) can be dominated by

$$C\int_{D^\lambda} \frac{d^dk_1 \dots d^dk_m}{\prod_{i=1}^{N}(l_i^2+1)^{p_i}} \tag{A.15}$$

Now let $D^{\lambda,1} = \{(k_1,\dots,k_m) \in \mathbb{R}^{dm} : 1 \le |l_i| \le \lambda\ , 1 \le i \le N\}$. We will show:

Theorem A.1 $\exists\ C, c > 1$ *independent of* λ *(for* λ *sufficiently large) such that for* $\bar{\lambda} = c\lambda$*:*

$$\int_{\lambda > |k_1|,\dots,|k_m|} \frac{d^dk_1 \cdots d^dk_m}{\prod_{i=1}^{N}(l_i^2+1)^{p_i}} \le C\int_{D^{\bar{\lambda},1}} \frac{d^dk_1 \dots d^dk_m}{\prod_{i=1}^{N}(l_i^2+1)^{p_i}} \tag{A.16}$$

Note that in $D^{\bar{\lambda},1}$, the l_i 's are bounded away from zero! This will later allow us to let the mass vanish in the propagators and obtain a homogeneous denominator.

The proof of (A.15) hinges on the observation that if $P = \{1,\dots,N\}$ and $S \subseteq P$ we can chop-up D^λ into what we might call *'Riesz's pieces'*:

$$X_S^\lambda = \left\{ (k_1,\dots,k_m) \in \mathbb{R}^{dm} : \begin{array}{ll} 1 \le |l_j| \le \lambda & \text{if } j \in S \\ |l_j| \le 1 & \text{if } j \in P \setminus S \end{array} \right\}, \tag{A.17}$$

where $\bigcup_{S \subseteq P} X_S^\lambda = D^\lambda$. If

$$I(X_s^\lambda) = \int_{X_S^\lambda} \frac{d^dk_1 \dots d^dk_m}{\prod_{i=1}^{N}(l_i^2+1)^{p_i}}, \tag{A.18}$$

then it suffices to show that $\exists\ C(S) > 0$ (again independent of λ) such that: $I(X_S^\lambda) \le C(S) I(X_P^{\bar\lambda})\ \forall\ S \subseteq P$, for then

$$\int_{D^\lambda} \frac{d^d k_1 \dots d^d k_m}{\prod_{i=1}^N (l_i^2+1)^{p_i}} \le \sum_{S \subseteq P} I(X_S^{\bar\lambda}) \le \tilde{C} I(X_P^{\bar\lambda}). \tag{A.19}$$

Since $X_P^{\bar\lambda}$ is by definition $D^{\bar\lambda,1}$, the theorem will be established.

Proof of Theorem A.1: Let $S \subseteq P$ be chosen (if $S = P$ there is nothing to show; if $S = \emptyset$ see below). By re-numbering we can arrange so that: $(a = |P| - |S|)$

$$X_S^\lambda = \left\{ \begin{array}{ll} |l_j| \le 1 & \text{for }, 1 \le j \le a \\ 1 \le |l_j| \le \lambda & \text{for }, a+1 \le j \le N \end{array} \right\} \tag{A.20}$$

By (possibly) re-numbering again, we can also have:

1) $\dot{l} = \{l_1, ..., l_b\}$ is a basis of $\{l_1, ..., l_a\}$ $(b \le a)$. In particular, if $a > b$ then for $b+1 \le j \le a$, $l_j = \sum_{i=1}^b a_{ji} l_i$.

2) $\ddot{l} = \{l_{c+1}, ..., l_N\}$ $(a \le c)$ completes $\dot{l}$ to a basis of $\{l_1, .., l_N\}$ ($\ddot{l}$ need not however be a basis for $\{l_{a+1}, ..., l_N\}$).

3) for some e $\le c$, $\{l_{e+1}, ..., l_N\}$ are all linear combinations from $\ddot{l}$ (e is chosen as small as possible).

It then follows that $l_j = l_j(\dot{l}, \ddot{l})$ for $a+1 \le j \le e$, where to summarise, $1 < b \le a < e \le c < N$.

At this point it will be appropriate to invoke:

Proposition A.1 : *Having chosen S, and keeping the re-arrangement in (A.20) and the corresponding constants a, b, c, e and bases $\dot{l}, \ddot{l}$ as before, define:*

$$Y_S^\lambda(\ddot{l}) = \left\{ (\dot{l}) \in \mathbb{R}^{bd} : \begin{array}{ll} |l_j| \leq 1 & for\ 1 \leq b \\ |l_j(\dot{l})| \leq 1 & if\ b+1 \leq j \leq a \\ 1 \leq |l_j(\dot{l},\ddot{l})| \leq \lambda & for\ a+1 \leq j \leq e \end{array} \right\} \tag{A.21}$$

(If $S = \emptyset$, then $a = P$ and the theorem follows directly from this lemma). Then $\exists\ C, c > 1$ (indep. of λ for λ sufficiently large) such that for $\bar{\lambda} = c\lambda$

$$\begin{aligned} J(Y^\lambda(\ddot{l})) &= \int_{Y^\lambda(\ddot{l})} \frac{d\dot{l}}{\prod_{j=1}^{e}(l_j^2+1)^{p_j}} \\ &\leq c' \int_{W^\lambda(\ddot{l})} \frac{d\dot{l}}{\prod_{j=1}^{e}(l_j^2+1)^{p_j}} = c' J(W^{\bar{\lambda}}(\ddot{l})), \end{aligned} \tag{A.22}$$

where $W^{\bar{\lambda}}(\ddot{l}) = \{(\dot{l}) \in R^{bd} : 1 \leq |l_j(\dot{l},\ddot{l})| \leq \bar{\lambda}\ , 1 \leq j \leq e\}$, and

$$l_j = \sum_{i=1}^{b} b_{ji} l_i + \sum_{i=c+1}^{N} d_{ji} l_i.$$

(Proof of proposition following Thm A.1)

Back to the proof of Thm A.1:

Define $Z_S^\lambda = \{\ddot{l} = (l_{c+1}, \ldots, l_N) \in \mathbb{R}^{cd} : 1 \leq |l_j| \leq \lambda,\ e+1 \leq j \leq N\}$. Now use $(\dot{l},\ddot{l})$ as integration variables, and let d_S be the Jacobian of $(k_1, \ldots, k_m)$ with respect to the $(\dot{l},\ddot{l})$ variables. Let $E_1(\ddot{l}) = \prod_{i \in N_1}(l_i(\ddot{l})^2+1)^{p_i}$ where $N_1 = \{e+1, .., N\}$, and $E_2(\dot{l},\ddot{l}) = \prod_{i \in N_2}(l_i(\dot{l},\ddot{l})^2+1)^{p_i}$ where $N_2 = \{1, \ldots, e\}$. Then by the proposition A.1,

$$\begin{aligned} I(X_S^\lambda) = &\ d_S \int_{Z_S^\lambda} \frac{d\ddot{l}}{E_1(\ddot{l})} \int_{Y_S^\lambda} \frac{d\dot{l}}{E_2(\dot{l},\ddot{l})} \\ \leq C_S &\ d_S \int_{Z_S^\lambda} \frac{d\ddot{l}}{E_1(\ddot{l})} \int_{W^\lambda(\ddot{l})} \frac{d\dot{l}}{E_2(\dot{l},\ddot{l})} = C_S I(X_P^{\bar{\lambda}}) \end{aligned} \tag{A.23}$$

This concludes the proof of Theorem A.1. ...■

Before proving proposition A.1 we need

Lemma A.2 *There exist a finite number of compact sets σ_1 and $\{\sigma_{2,j}\}_{j=1}^{n_0}$ in $\mathbb{R}^{bd}$, of positive measure, with the following properties:*
(1) $Y^{\lambda}(\ddot{l}) \subset \sigma_1$
(2) For any $(\ddot{l}) \in Z_S^{\lambda}$, $\exists\ j$ such that $\sigma_{2,j} \subset W^{\tilde{\lambda}}(\ddot{l})$, where $\tilde{\lambda} = c\lambda$ for some $c > 1$.
(3) The sets can be chosen independent of λ for λ sufficiently large.

Proof of proposition A.1 using Lemma A.2 Let $\sigma = \sigma_1 \cup (\cup_{j=1}^{n_0} \sigma_{2,j})$. Having constructed the above sets, the idea is to obtain upper and lower bounds for the integrands in (A.22) for all $\dot{l}$ in $\sigma \equiv \sigma_1 \cup (\cup_{j=1}^{n_0} \sigma_{2,j})$. From (A.21) we have $l_j(\dot{l}, \ddot{l}) = \sum_{i=1}^{b} b_{ji} l_i + q_j(\ddot{l})$ for $(1 \leq j \leq e)$ where

$$q_j(\ddot{l}) \equiv \sum_{i=c+1}^{N} d_{ji} l_i \tag{A.24}$$

(of course b_{ji} and d_{ji} will have blocks of zero and identity matrices embedded within). Now, pick $M > 0$ such that $|\sum_{i=1}^{b} b_{ji} l_i| \leq M\ \forall\ \dot{l} \in \sigma$ $(1 \leq j \leq e)$. We then have

$$\begin{aligned} g(\ddot{l}) &\equiv \frac{1}{\prod_{j=1}^{e}(2M^2 + 2q_j^2 + 1)^{p_j}} \qquad \text{(A.25)} \\ &\leq \frac{1}{\prod_{j=1}^{e}((\sum_{i=1}^{b} b_{ji} l_i + q_j(\ddot{l}))^2 + 1)^{p_j}} \equiv \mathcal{I}(\dot{l}, \ddot{l}) \end{aligned}$$

To obtain a similar upper bound for $\mathcal{I}(\dot{l}, \ddot{l})$, note that for $1 \leq j \leq e$ and for $\dot{l} \in \sigma_1$,

$$\begin{aligned} \frac{2q_j^2(\ddot{l})^2 + 2M^2 + 1}{(\sum_{i=1}^{b} b_{ji} l_i + q_j(\ddot{l}))^2 + 1} &\leq \frac{2((q_j + \sum_{i=1}^{b} b_{ji} l_i) - \sum_{i=1}^{b} b_{ji} l_i)^2 + 2M^2 + 4}{(\sum_{i=1}^{b} b_{ji} l_i + q_j)^2 + 1} \\ &\leq 4 + 6M^2 \end{aligned} \tag{A.26}$$

where, as in (A.25), we used the inequality $(x+y)^2 \leq 2(x^2+y^2)$. Thus, by (A.25) and (A.26) (with some appropriate $\tilde{c} > 0$), it follows that

$$\begin{aligned} &g(\ddot{l}) \leq \mathcal{I}(\dot{l}, \ddot{l}) \quad \text{for } \dot{l} \in \sigma \\ &\mathcal{I}(\dot{l}, \ddot{l}) \leq \tilde{c} g(\ddot{l}) \quad \text{for } \dot{l} \in \sigma_1 \end{aligned}, \tag{A.27}$$

respectively. Then using (A.27), we can prove (A.22) of Lemma A.1 since:

$$J(Y^\lambda(\ddot{l})) \leq \int_{\sigma_1} d\dot{l}\; \mathcal{I}(\dot{l}, \ddot{l}) \leq \text{vol}(\sigma_1)\tilde{c} g(\ddot{l}) \tag{A.28}$$

and

$$g(\ddot{l})\text{vol}(\sigma_{2j}) \leq \int_{\sigma_{2j}} d\dot{l}\; \mathcal{I}(\dot{l}, \ddot{l}) \leq J(W^\lambda(\ddot{l})), \tag{A.29}$$

can be combined, so that

$$J(Y^\lambda(\ddot{l})) \leq \tilde{c} \left(\min_{1 \leq j \leq n_0} \left(\frac{\text{vol}(\sigma_1)}{\text{vol}(\sigma_{2j})}\right) \right) J(W^\lambda(\ddot{l})). \tag{A.30}$$

This proves proposition A.1 assuming Lemma A.2. ...■

Proof of Lemma A.2. We now set about the construction of the sets in σ. First it is clear that

$$Y^\lambda(\ddot{l}) \subset \otimes_{i=1}^{b}[-1,1]^d \equiv \sigma_1. \tag{A.31}$$

The trouble lies in the construction of $\{\sigma_{2,j}\}_{j=1}^{n_0}$, for one cannot construct a single set $\tilde{\sigma}$ such that $\tilde{\sigma} \subset \cup_{\ddot{l} \in Z_S^\lambda} W_S^\lambda(\ddot{l})$ since the l_j s in $\ddot{l}$ can vary [cf (A.22]; their magnitude, however, is bounded by $\lambda + 2$.

Choose $a_0 = 0 < a_1 < \ldots < a_{N-e} \in R$ such that the set of $(l_1, ..., l_b) \in \mathbb{R}^{bd}$ satisfying

$$\begin{aligned} &|l_i| \geq 1 && \text{for } 1 \leq i \leq b \text{ and} \\ &2 + a_{j-1} \leq |\textstyle\sum_{j=1}^{b} b_{ij} l_j| \leq a_j && \text{for } 1 \leq i \leq e \end{aligned} \tag{A.32}$$

contains a compact set of positive measure for $1 \leq j \leq N-e$ (these compact sets can be chosen since $\sum_{j=1}^{b} b_{ij} l_j$ defines a hyperplane in $\mathbb{R}^{db}$). Call these sets $\sigma_{2,j}$. The task is to show that they possess property *(2)* of Lemma A.2 (property *(3)* is automatic).

For $1 \leq i \leq N-e$ and some $\tau > 2$, let

$$Q_j = \{(\bar{q}_{c+1}, ..., \bar{q}_N) \in \mathbb{R}^{(N-c)d} : |\bar{q}_i| \leq a_{j-1}+1 \text{ or } \lambda+\tau \geq |\bar{q}_i| \geq a_j+1\} \quad \text{(A.33)}$$

for $e+1 \leq i \leq N$ where $\bar{q}_i(\ddot{l}) = \sum_{j=c+1}^{N} f_{ij} l_j$ by the re-numbering below (A.20). Also let

$$Q_{N-e+1} = \{(\bar{q}_{c+1}, ..., \bar{q}_N) \in \mathbb{R}^{(N-c)d} : |\bar{q}_i| \leq a_{N-e} + 1 \text{ for } e+1 \leq i \leq N\}. \quad \text{(A.34)}$$

Note that $\{(\bar{q}_{c+1}, ..., \bar{q}_N) : |\bar{q}_i| \leq \lambda + 2,\ e+1 \leq i \leq N\} \subset \cup_{i=1}^{N-e+1} Q_i$. Thus, $Z_S^\lambda \subset \cup_{i=1}^{N-e+1} Q_i$ for some sufficiently large τ.

Now, if $(\bar{q}_{c+1}, ..., \bar{q}_N) \in Q_j$, then by (A.32) either

$$\begin{array}{ll} 2 + a_{j-1} - (1 + a_{j-1}) \leq |\sum_{j=1}^{b} b_{ij} l_j| - |\bar{q}_i| \leq |l_i| & \text{for } e+1 \leq i \leq N, \text{ or} \\ (a_j + 1) - a_j \leq |\bar{q}_i| - |\sum_{j=1}^{b} b_{ij} l_j| \leq |l_i| & \text{for } e+1 \leq i \leq N \end{array} \quad \text{(A.35)}$$

for $(\ddot{l}) \in \sigma_{2,j}$. However, no matter the value of j,
$|l_i| \leq |\bar{q}_i| + |\sum_{j=1}^{b} b_{ij} l_j| \leq c\lambda$ for some $c > 1$. Thus, for $(\bar{q}_{c+1}, ..., \bar{q}_N) \in Q_j$ we have $\sigma_{2,j} \in W_S^{\bar{\lambda}}(\ddot{l})$, with $\bar{\lambda} = c\lambda$. This concludes the proof of Lemma A.2.

...■

We now set about estimating (A.14). Before proceeding, we need to define the maximum degree of divergence of an integral of the form (A.14) as follows:

Definition A.1 *Let* $P = \{Q \subset \{1, ..., N\} : |Q| \leq m-1\}$, *where* $|Q|$ *denotes the cardinality of* Q. *Now let*

$$\omega(S) = d(m - |S|) - 2(\sum_{i=1}^{n} n_i - \sum_{j\in S} n_j). \quad \text{(A.36)}$$

Then, $\omega(H)$ is defined as follows:

$$\omega(H) \equiv \max_{SCP}(\omega(S)) \tag{A.37}$$

By the remarks before (A.15), it suffices to obtain estimates for (A.18). Thus, with the notation for (A.18) with X_S^λ replaced by D^λ, we have:

Theorem A.3

$$I(D^\lambda) \le C^* \begin{Bmatrix} 1 & if\ \omega(H) < 0 \\ (\log\lambda)^m \lambda^{\omega(H)} & if\ \omega(H) \ge 0 \end{Bmatrix} \tag{A.38}$$

Proof. For $1 \le \xi \le N$, let

$$D_\xi^{\lambda,1} = \{(k_1, .., k_m) \in \mathbb{R}^{dm} : \lambda \ge |l_i| \ge |l_\xi| \ge 1\ , 1 \le i \le N\}. \tag{A.39}$$

Then $D^{\lambda,1} \subset \cup_{\xi=1}^N D_\xi^\lambda$. By Theorem A.1, $\exists\ c > 1$ so that as per the notation in (A.18):

$$I([-\lambda,\lambda]^{dm}) \le c' I(D^{c\lambda,1}) \le c'' \sum_{\xi=1}^N I(D_\xi^{c\lambda}).$$

By changing C^* in (A.38), we may assume w.l.o.g. that $c = 1$ for the factor scaling λ. Fix ξ, $1 \le \xi \le N$ and choose a non-singular $m \times m$ matrix $(A(\xi))_{ij}$ such that $t_i = \sum_{j=1}^m A(\xi)_{ij} k_j$ where $t_j \in \mathbb{R}^d$ with $\det A(\xi) = 1$ and $t_1 = l_\xi(\vec{k})$. Thus, $\vec{k}(\vec{t}) = A^{-1}(\xi)\vec{t}$. Then,

$$I(D_\xi^\lambda) \le c \int_{V(\lambda)} d^d t_1 \int_{U(\lambda)} \frac{d^d t_2 ... d^d t_m}{\prod_{i=1}^N (l_i^2 + 0)^{\mu_i}}, \tag{A.40}$$

where $V(\lambda) = \{t_1 \in \mathbb{R}^d : 1 \le |t_1| \le \lambda\}$ and $U(\lambda) = \{(t_2, .., t_m) \in \mathbb{R}^{d(m-1)} : \lambda \ge |l_i(A_\xi^{-1}(\vec{t})| \ge t_1 \ge 1\}$. Note that by construction, the domain of integration in the above integral is bounded away from the zeroes of the integrand. This allows us to let the mass vanish and thereby obtain a homogeneous denominator.

Now, if $\omega_0 = dm - 2\sum_{i=1}^N p_i$, we let $(t_2, ..., t_m) = |t_1|(t'_2, ..., t'_m)$, with $t'_1 = t_1/|t_1|$, so as to exploit the homogeneity of the denominator by using

$$|l_i(A_\xi^{-1}(\vec{t'}))|_{t'_1 \equiv 0} \leq 2|l_i(A_\xi^{-1}(\vec{t'}))|$$

to write

$$\begin{aligned} I(D_\xi^\lambda) &\leq c\int_{V(\lambda)} \frac{dt_1}{t_1} t_1^{\omega_0} \int_{\tilde{U}(\tilde{\lambda})} \frac{d^d t'_2 ... d^d t'_m}{\prod_{i\neq\xi}^N l_i(A_\xi^{-1}(t'))^{2p_i}} && \text{(A.41)} \\ &\leq c' \int_{V(\lambda)} \frac{dt_1}{t_1} t_1^{\omega_0} \int_{\tilde{U}(\tilde{\lambda})} \frac{d^d t'_2 ... d^d t'_m}{\prod_{i\neq\xi}^N l_i(A_\xi^{-1}(\vec{t'}))^{2p_i}|_{t'_1\equiv 0}}, && \text{(A.42)} \end{aligned}$$

where $\tilde{\lambda} = \lambda/|t_1|$ and

$$\tilde{U}(\tilde{\lambda}) = \{(t'_2, ..., t'_m) \in \mathbb{R}^{d(m-1)} : 1 \leq |l_i(A_\xi^{-1}(t'))|_{t'\equiv 0} \leq \tilde{\lambda}\}.$$

We now have:

$$I(D_\xi^\lambda) \leq c\int_{V(\lambda)} \frac{dt_1}{t_1} t_1^{\omega_0} \int_{\tilde{U}(\tilde{\lambda})} \frac{d^d t'_2 ... d^d t'_m}{\prod_{i\neq\xi}^N l_i(A_\xi^{-1}(t'))^{2p_i}|_{t'_1\equiv 0}} \qquad \text{(A.43)}$$

Remark A.1 *The inner integral now corresponds to an $m-1$ loop diagram related to its 'parent' m loop diagram by 'erasing' the line corresponding to the momenta flowing through l_ξ.*

Before we perform the induction on the number of loops we state:

Lemma A.4 *Choose $a_i \in R$, $1 \leq i \leq m$ and let $c = \max_{1\leq i\leq m}(a_i)$, then*

$$\int_{\lambda > t_1 > 1} \frac{dt_1}{t_1} t_1^{a_1} \int_{\lambda/t_1 > t_2 > 1} \frac{dt_2}{t_2} t_2^{a_2} \cdots \int_{\lambda/(t_1 \cdots t_{m-1}) > t_m > 1} \frac{dt_m}{t_m} t_m^{a_m}$$
$$\leq C \left\{ \begin{array}{cc} 1 & if\ c < 0 \\ (\log\lambda)^m \lambda^c & if\ c \geq 0 \end{array} \right\} \qquad \text{(A.44)}$$

Proof.

(A.44) is true for $m = 1$; let us assume it is true for $k = m - 1$, then if $b = \max_{1 \le i \le m-1}(a_i)$ and $b \geq 0$: (If $b < 0$ we may assume $c = a_1$)

$$\begin{aligned}
&\int_{\lambda > t_1 > 1} \frac{dt_1}{t_1} t_1^{a_1} \int_{\lambda/t_1 > t_2 > 1} \frac{dt_2}{t_2} t_2^{a_2} \ldots\ldots \int_{\lambda/(t_1 \ldots t_{m-1}) > t_m > 1} \frac{dt_m}{t_m} t_m^{a_m} \\
&\leq c \int_{\lambda > t_1 > 1} \frac{dt_1}{t_1} t_1^{a_1} (\lambda/t_1)^b \log^{m-1}(\lambda/t_1) \\
&\leq c \int_{\lambda > t_1 > 1} \frac{dt_1}{t_1} t_1^{a_1} (\lambda/t_1)^b \log^{m-1}(\lambda) \\
&\leq c \left\{ \begin{array}{cc} 1 & if\ c < 0 \\ \lambda^c (\log \lambda)^m & if\ c \geq 0 \end{array} \right\}
\end{aligned} \tag{A.45}$$

This proves (A.44). ...■

Theorem A.3 follows by nesting the procedure leading up to (A.43) on the remaining variables $t_2', \ldots, t_m'$. This will generate a sum of integrals of the type described in Lemma A.4; in every such integral, the a_is will correspond to a sequence $\{\omega_0, \ldots, \omega_{m-1}\}$ which in turn depends on the order in which the t_i's are integrated out. As per remark A.1, the ω_1 for the inner integral in (A.43) will be $d(m-1) - 2(n - n_\xi)$ (since the line corresponding to l_ξ was absorbed in the inner integral). By applying Lemma A.4 to all possible nestings, we deduce Theorem A.3. ...■

B The 2 loop case

We start by assuming Theorem A.1. Consider:

$$\int_{\lambda > |k_1|, |k_2|} \frac{d^d k_1 d^d k_2}{\prod_{i=1}^{3} (l_i^2 + 1)^{n_i}} \tag{B.1}$$

where $l_1 = k_1$, $l_2 = k_1 + k_2$ and $l_3 = k_2$ ($n_i \geq 1$ $i = 1, 2, 3$). As a consequence of Theorem A.1 and the comments preceding (A.15), (B.1) can be dominated by:

$$C \sum_{j=1}^{3} \int_{D_j^{c\lambda}} \frac{d^d k_1 d^d k_2}{\prod_{i=1}^{3} (l_i^2 + 1)^{n_i}} \tag{B.2}$$

where $D_j^{c\lambda} = \{(k_1, k_2) \in \mathbb{R}^{2d} : c\lambda \geq |l_i| \geq |l_j| \geq 1\ , 1 \leq i \leq 3\}$, and the constants $C, c > 1$ are independent of λ for λ sufficiently large.

Let us now focus our attention on $j = 2$:

$$\begin{aligned} I(D_2^{c\lambda}) &= \int_{D_2^{c\lambda}} \frac{d^d k_1 d^d k_2}{\prod_{i=1}^{3} (l_i^2 + 1)^{n_i}} = \int_{\tilde{D}_j^{c\lambda}} \frac{d^d l_1 d^d l_2}{(l_1^2 + 1)^{n_1} (l_2^2 + 1)^{n_2} ((l_2 - l_1)^2 + 1)^{n_3}} \\ &\leq \int_{c\lambda > |l_2| > 1} \frac{d^d l_2}{l_2^{2n_2}} \int_{c\lambda > |l_2 - l_1|, |l_1| > 1} \frac{d^d l_1}{l_1^{2n_1} (l_2 - l_1)^{2n_3}} \end{aligned} \tag{B.3}$$

where $\tilde{D}_2^{c\lambda} = \{(k_1, k_2) \in \mathbb{R}^{2d} : c\lambda \geq |l_2 - l_1|, |l_1| \geq |l_2| \geq 1\}$.

Now let $l_2' = \frac{l_2}{|l_2|}$ and $l_1' = \frac{l_1}{|l_2|}$. Then,

$$\begin{aligned} I(D_2^{c\lambda}) &\leq c' \int_{c\lambda > |l_2| > 1} \frac{d^d l_2 |l_2|^d}{l_2^{2(n_1+n_2+n_3)}} \int_{c\lambda/|l_2| > |l_2' - l_1'|, |l_1'| > 1} \frac{d^d l_1'}{(l_1')^{2n_1} (l_2' - l_1')^{2n_3}} \\ &\leq c'' \int_{c\lambda > l_2 > 1} \frac{dl_2}{l_2} l_2^{2d - 2(n_1+n_2+n_3)} \int_{c\lambda/|l_2| > |l_1'| > 1} \frac{d^d l_1'}{|l_1'|^{2(n_1+n_3)}} \end{aligned} \tag{B.4}$$

since $|l_1'| \leq 2|l_1' - l_2'|$. We therefore have: ($n \equiv \sum_{i=1}^{3} n_i$)

$$\begin{aligned} I(D_2^{c\lambda}) &\leq c' 2^d \int_{c\lambda > l_2 > 1} \frac{dl_2}{l_2} l_2^{2d-2n} \int_{c\lambda/l_2 > l_2 > 1} \frac{dl_1}{l_1} l_1^{d - 2(n_1+n_3)} \\ &\leq c'' \left\{ \begin{array}{cc} 1 & if\ m < 0 \\ (\log \lambda)^2 \lambda^m & if\ m \geq 0 \end{array} \right\}, \end{aligned} \tag{B.5}$$

where $m = \max\{2d - 2n, d - 2(n_1 + n_3)\}$ (cf. Lemma A.4). Had we chosen $D_3^{c\lambda}$, we would have arrived at the same estimate with

$m = \max\{2d-2n, d-2(n_1+n_2)\}$; for $D_1^{c,\lambda}$, $m = \max\{2d-2n, d-2(n_2+n_3)\}$.

The above are the 3 sub-diagrams corresponding to the 2 loop graph.

C Fractional derivatives

We state without proof the results in [MS] regarding fractional derivatives:

Definition C.1 *Given $\epsilon > 0$ and a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, let the fractional derivative of $f(z)$ be defined as*

$$\delta_z^{\epsilon} f(z) \equiv \sum_{n=0}^{\infty} n^{\epsilon} a_n z^n.$$

We can estimate the above fractional derivative via Lemma 6.3.1 in [MS].

Lemma A.5 *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R. Then for any z with $|z| < R$, and for any $\epsilon \in (0,1)$,*

$$\delta_z^{\epsilon} f(z) = c(1-\epsilon) z \int_0^{\infty} f'(ze^{-\lambda^{1/(1-\epsilon)}}) e^{-\lambda^{1/(1-\epsilon)}} d\lambda,$$

where $c(1-\epsilon) = ((1-\epsilon)\Gamma(1-\epsilon))^{-1}$. *Similarly,*

$$\delta_z^{-\alpha} f(z) = c(\alpha) \int_0^{\infty} [f(ze^{-\lambda^{1/(1-\epsilon)}}) - f(0)] d\lambda.$$

(both the above also holds for $z = R$ if $a_n \geq 0$). For a proof see p.188 [MS].

...■

The usefulness of the above result will come to light after we state Lemma 6.3.2 of [MS]:

Lemma A.6 *Consider $\epsilon \in (0,1)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Suppose that $A_\epsilon \equiv \sum_{n=0}^{\infty} n^\epsilon |a_n| R^{n-\epsilon} < \infty$ for some $R > 0$ (so that in particular, $f(z)$ converges for $|z| \leq R$. Then for any $|z| \leq R$,*

$$|f(z) - f(R)| \leq 2^{1-\epsilon} A_\epsilon |R - \epsilon|^\epsilon. \tag{C.1}$$

For a proof see p.189 [MS]. ...■

When combining the above results with Lemma 4.1 of chapter 4 (which is Lemma 6.3.3 of [MS]), we can obtain information about the asymptotics of the coefficients a_n of $f(z)$. For convenience, we will compute the fractional derivative of a function under the following hypotheses: suppose that

$$\left|\frac{d}{dz} f(z)\right| \leq c |z_c - |z||^{-p} / |z| \tag{C.2}$$

for $|z| < z_c$. Then with $z_\lambda = z_c e^{-\lambda^{1/(1-\epsilon)}}$,

$$|f'(z_\lambda)/z_\lambda| \leq c |1 - e^{-\lambda^{1/(1-\epsilon)}}|^{-p} / z_\lambda \leq c \lambda^{-p/(1-\epsilon)} / z_\lambda \tag{C.3}$$

Thus,

$$\begin{aligned} |\delta_z^\epsilon f(z_c)| &\leq C_{1-\epsilon} \int_0^\infty d\lambda e^{-\lambda^{1/(1-\epsilon)}} |f(z_\lambda)'| \\ &\leq c \int_0^\infty d\lambda \, \lambda^{-p/(1-\epsilon)} < \infty, \end{aligned} \tag{C.4}$$

if $p < 1 - \epsilon$.

Now, in particular, since $\frac{d}{dz} f = (\mathcal{D} f - f)/z$, and since for $|z| < z_c$, [by Lemma 3.3 combined with remark 3.5.1, and (2.2.1) respectively]

$$|\mathcal{D}\hat{\Psi}_z(k)| \leq |\mathcal{D}\hat{\Psi}_z(0)| \leq c |z_c - |z||^{-(1-\epsilon)},$$

and

$$|\mathcal{D}\hat{\Pi}_z(k)| \leq |\mathcal{D}\hat{\Pi}_z(0)| \leq |z_c - z|^{-(1-\epsilon)},$$

it follows that the ϵ fractional derivatives of $\hat{\Psi}_z(k)$ and $\hat{\Pi}_z(k)$ are finite.

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