FREE VIBRATIONS OF SKIRT SUPPORTED PRESSURE VESSELS

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A Thesis
Submitted to the School of Graduate Studies in
Partial Fulfilment of the Requirements
For the Degree
Master of Engineering.

McMaster University,
August, 1977.

MASTER OF ENGINEERING (1977)
(Civil Engineering)

McMaster University
Hamilton, Ontario.

TITLE: Free Vibrations of Skirt Supported Pressure Vessels.

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SUPERVISOR: Dr. M. Levinson

NUMBER OF PAGES: viii and 95

ABSTRACT

The free vibrations of skirt supported pressure vessels are studied in this thesis; both cantilevered and fixed-pinned systems are considered. A hierarchy of models, ranging from a rigid mass (vessel) supported by a massless Euler-Bernoulli beam (skirt) to a model in which both components are represented by Timoshenko beams, is subjected to analysis. Several typical numerical examples are considered for both sets of boundary conditions. The results of these calculations indicate that whereas the cantilevered system may be modeled with fair accuracy, compared to the most sophisticated model considered, by a rigid mass supported by a massless beam capable of undergoing shear deformation, it is necessary to model all components of the fixed-pinned system by Timoshenko beams, i.e. the most sophisticated model considered. The first two mode shapes for all models of a typical case of each configuration are shown. Finally, some comments on the modeling and analysis of specific realistic systems are made.

ACKNOWLEDGEMENTS.

My sincere gratitude and thanks are due to Dr. M. Levinson for his guidance, inspiration, and encouragement during every stage of the research work.

I gratefully acknowledge the financial assistance from the National Research Council of Canada and McMaster University.

Also I would like to thank Mrs. P. Johnstone for making this report presentable.

And finally, and personally most important, for her understanding and encouragement, this thesis is dedicated to my wife.

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CHAPTER 1

INTRODUCTION

1.1 General.

In recent years considerable effort has been devoted to developing sophisticated methods of analysis to aid in the design of structures which will operate under conditions requiring a high level of confidence in their structural integrity. These methods include mathematical modeling, ones based on materials science, those based on rationally evolved design criteria, etc. Pressure vessel systems are among such structures. Some such systems consist of a comparatively heavy pressure vessel, possibly containing heat exchange elements or other innards as well as a working fluid, which is supported by an almost cylindrical support skirt, Fig. (1.1). The vessel is connected by relatively soft externals such as piping. These systems occur in the process and power generation industries, as well as aboard nuclear powered ships and in other industrial settings dealing with the generation, conversion or storage of energy.

1.2 Purpose of Research.

It is the purpose of this report to study the free vibrations of such skirt-vessel combinations, in order to determine their dynamic characteristics and to develop, if possible, simple formulae, based on detailed analysis of skirt-vessel systems, which may be applied to estimate the natural frequencies of such systems.

On occasion, one may wish to provide additional support or constraint to the system in order to raise its natural frequencies of vibration when the frequencies of expected external excitations are close to the predicted natural frequencies of the cantilevered system shown in Fig.(1.1.a). One simple way to raise the frequencies is to pin the top of the vessel to a support attached to a ceiling or deck above the pressure vessel. This arrangement is shown in Fig. (1.1.b).

The present work concerns itself with the study of a number of cases of each of the configurations shown in Fig.(1.1) using a hierarchy of six models of increasing complexity. The most simple model considers the skirt as a massless spring having the characteristics of an Euler-Bernoulli beam and the pressure vessel, including its contents, as a rigid body [1]* while the most sophisticated model considers the system as an assemblage of Timoshenko beams.

In the following chapter the hierarchy of model systems is described while in subsequent chapters the analyses of the models are performed and frequency and modal shape results for typical cases are presented. Finally comments and conclusions based on these results are given.

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* Numbers in square brackets indicate references listed at the end of this work. In [1] the model is analyzed without reference to any particular application.

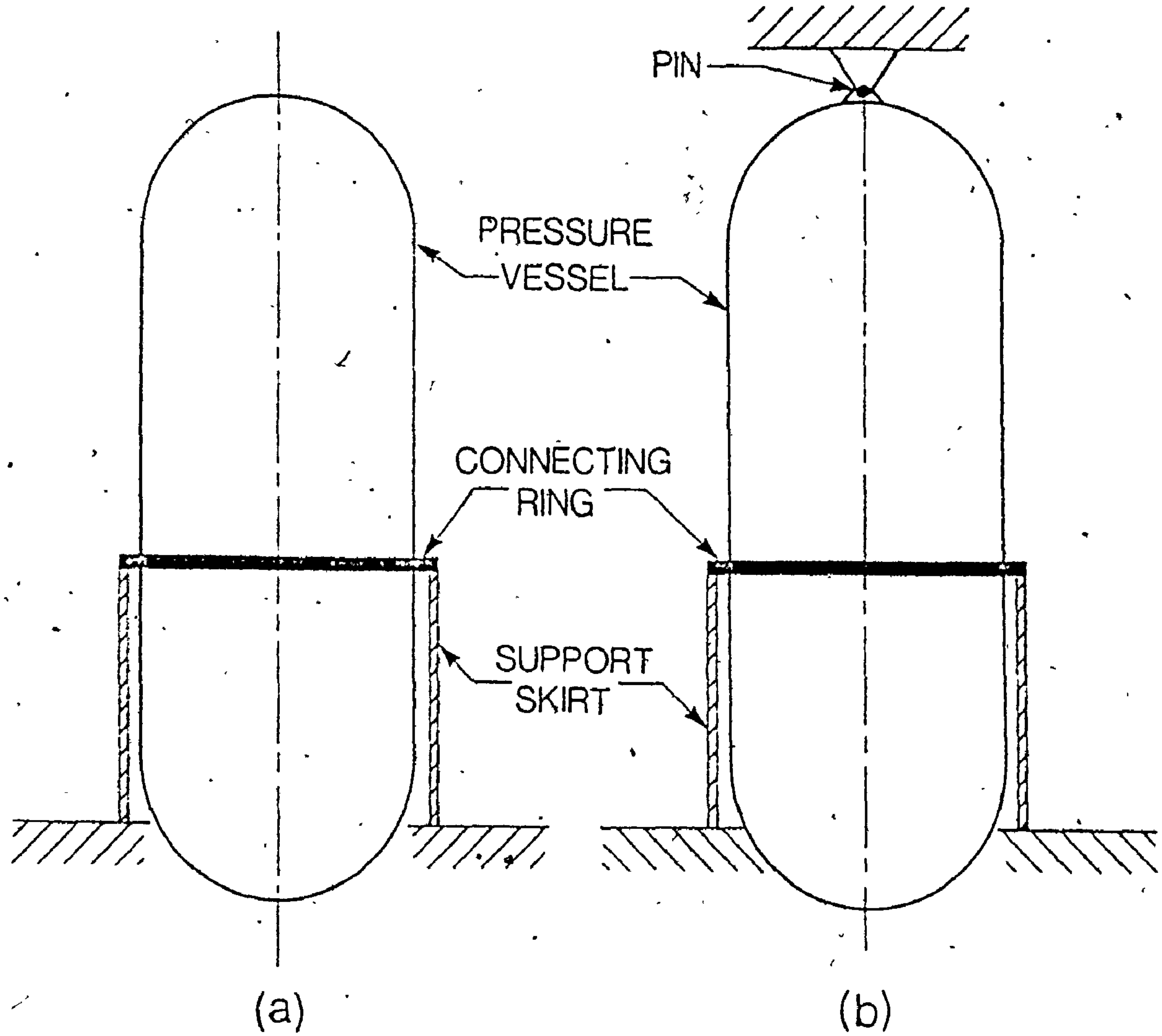

FIGURE 1.1: SKIRT SUPPORTED PRESSURE VESSELS.

CHAPTER 2

THE HIERARCHY OF MODELS.

2.1 Basic Assumptions.

In performing the dynamic study of the skirt-pressure vessel system, six mathematical models, as mentioned before, are considered. It was taken into account that the hierarchy of models used in this study should be able to show the effect of each parameter on the dynamic characteristics of the pressure vessel-skirt structure. The same hierarchy of models is used to analyze both of the configurations shown in Fig. (1.1). Although ultimately it will be seen that different degrees of sophistication are necessary to provide comparable estimates of the natural frequencies of the configuration shown in Fig. (1.1.a) and the configuration shown in Fig. (1.1.b). In all cases the interaction of the contents of the vessel with the rest of the system is neglected other than to include their mass with the mass of the vessel. This assumption is probably reasonable for a full, pressurized vessel; its worth, more generally, is not clear but to make any other assumption would make the problem intractable or almost so. In any event, the lessons to be learned from the present study may be applied to more sophisticated analyses which include the dynamics of the vessel's contents. The axial load in the skirt as well as that in the pressure vessel also are neglected in this study because they are at most a percent or two of the shell buckling loads, conservatively estimated [2], for these elements and an imperceptible portion of the column buckling loads. It is reasonable, for typical systems of the sort considered here, to assume that shell vibration frequencies and modes need not be studied since the excitations to be expected will usually have frequencies below 10^2 Hz and the lowest shell frequencies will likely be several times this value; this will be made clear by perusal of [3]. Further, to simplify the analysis, it is assumed that the support skirt, which is frequently slightly tapered, may be represented adequately by a hollow cylindrical beam and that the ring usually connecting the skirt and vessel is replaced by a massless rigid connector. A final simplification made is to neglect the typically small stiffening of the skirt-vessel combination by externals such as piping. These assumptions are common to all models of the hierarchy. Further assumptions may be made for each model individually.

2.2 Description of the Hierarchy of Models.

The hierarchy of models which is used in carrying out this study may be characterized in accordance with the assumptions used in the analysis of the skirt-vessel combination. Six different models are used in this study, Fig. (2.1), and may be described as follows:

Model (I): We assume in this model that the skirt may be represented by a massless Euler-Bernoulli beam and that the pressure vessel and its contents behave as a rigid body. The rationale for this model is that the pressure vessel is stiffer than the skirt, sometimes by an order of magnitude, and that the mass of the skirt is rarely as much as ten percent of the total mass of the system and usually less than that.

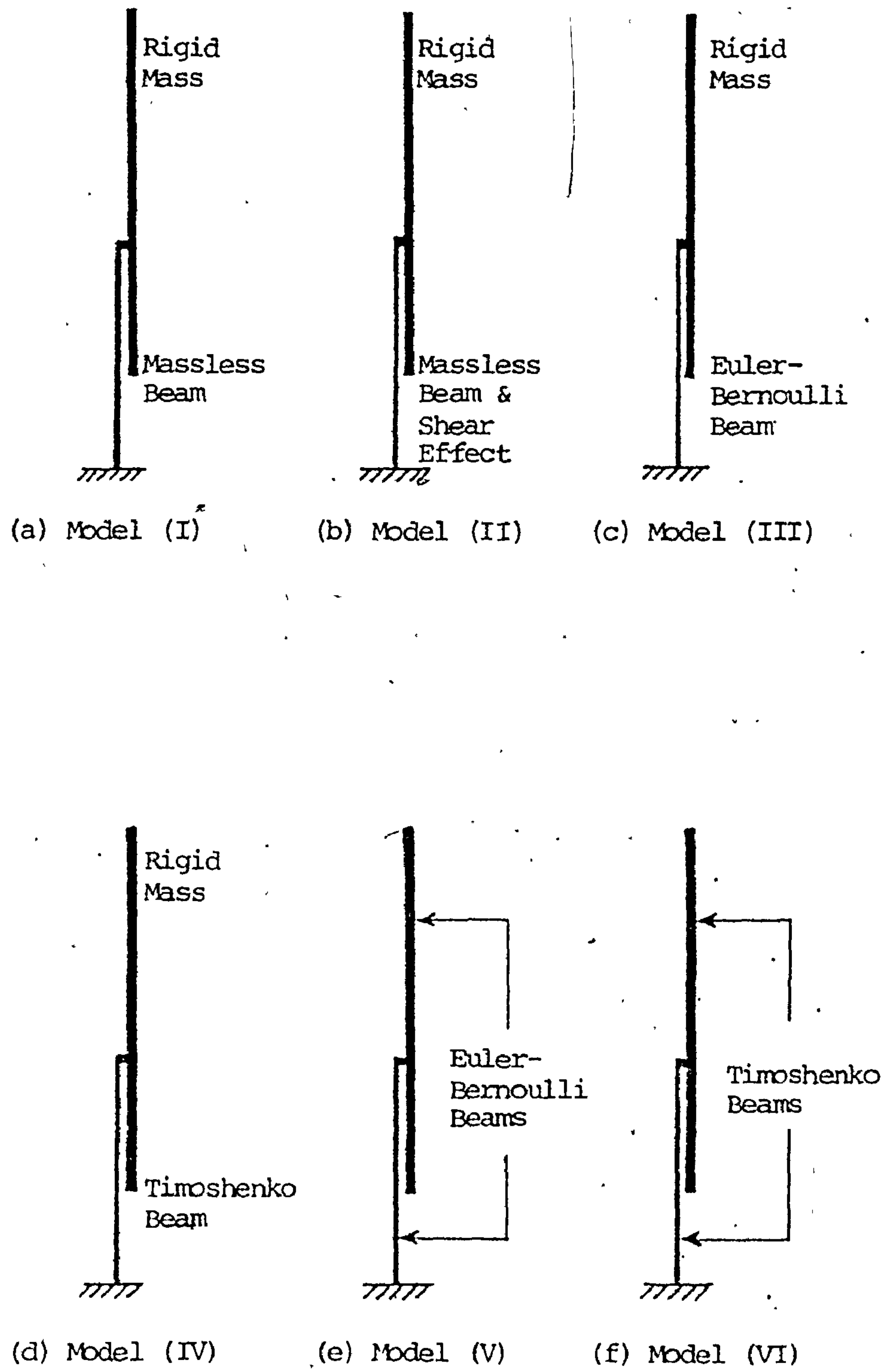

FIGURE 2.1: THE HIERARCHY OF MODELS

Model (II): In this model it is assumed that the support skirt deforms in shear as well as flexure. Otherwise this model is the same as Model (I). This additional degree-of-freedom in the behavior of the skirt is of great importance since the support skirt is usually a rather short beam.

Model (III): Here, the mass of the skirt is considered in the analysis, as well as its stiffness, and so the skirt will be analyzed as an Euler-Bernoulli beam with mass. Otherwise it is the same as Model (I). This change in modeling does more, however, than merely take into account the small additional mass of the skirt; it changes the model from one with a finite number of degrees-of-freedom to a continuous system with an infinite spectrum of natural frequencies. It should be noted that in case "a" Model (I) has two degrees-of-freedom while in case "b" it has only one degree-of-freedom.

Model (IV): Here the skirt is modeled as a Timoshenko beam* rather than as an Euler-Bernoulli beam with mass, i.e. shear deformations and rotatory inertia are accounted for as well as flexural deflections and transverse translational inertia. This model may be considered as an extension of either Model (II) or Model (III).

Model (V): Now both the skirt and the pressure vessel are modeled as Euler-Bernoulli beams. This is one of the simplest models which allows one to assess the influence of the flexibility of the vessel on the natural frequencies of the system.

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* The reader unfamiliar with the theory of the Timoshenko beam will find a lucid and detailed account in [4].

Model (VI): This ultimate model in the hierarchy used to study skirt-pressure vessel systems is the one in which both elements are taken to be Timoshenko beams and it is, finally, the yardstick against which all of the other models are to be measured. If a simpler model provides frequency estimates comparable to those provided by Model (VI) then the simpler model will be considered adequate to describe the system.

It is clear that various other simplified models to describe the skirt-vessel system could be postulated, e.g., the skirt might be represented by a massless shear beam and the vessel by a Timoshenko beam. However, it is believed that no further didactic purpose would be served by so extending the present work. Such models, especially suitable for particular situations, will be devised and analyzed without difficulty by the interested reader after a perusal of this work.

CHAPTER 3

DERIVATION OF FREQUENCY EQUATIONS.

3.1 The Notations and Sign Conventions:

Before presenting the analysis of the various models for both configurations of the skirt-vessel system, most of the notations and the sign conventions to be used in this study will be given; we note that not all variables and parameters appear in the analysis of each model. The notations and conventions conform to those used in [4]. It seems simplest to present this material graphically and this is done in Figures (3.1) - (3.2); also see Appendix-A. Figure (3.1.a) shows the nonstandard sign convention of [4] for bending moments, which makes eminent sense in the study of Timoshenko beams, as well as the usual, modern convention for transverse shear forces and distributed loads. In Fig. (3.1.b) the transverse displacement of the neutral surface and the rotation of a cross section of the beam are shown. Finally, Fig. (3.2) shows the coordinate systems to be used as well as the diagrammatic representation of the skirt and pressure vessel as uniform beams. It has proven easier to work with the vessel considered to consist of two parts, which join at the vessel's junction with the support skirt, rather than as one body; this approach also provides the formulation for a three component system which is a natural generalization of the system studied here. This generalization is of some practical importance because the innards of a vessel may indeed be nonuniformly distributed over its length and it is even possible to imagine that the vessel itself differs above

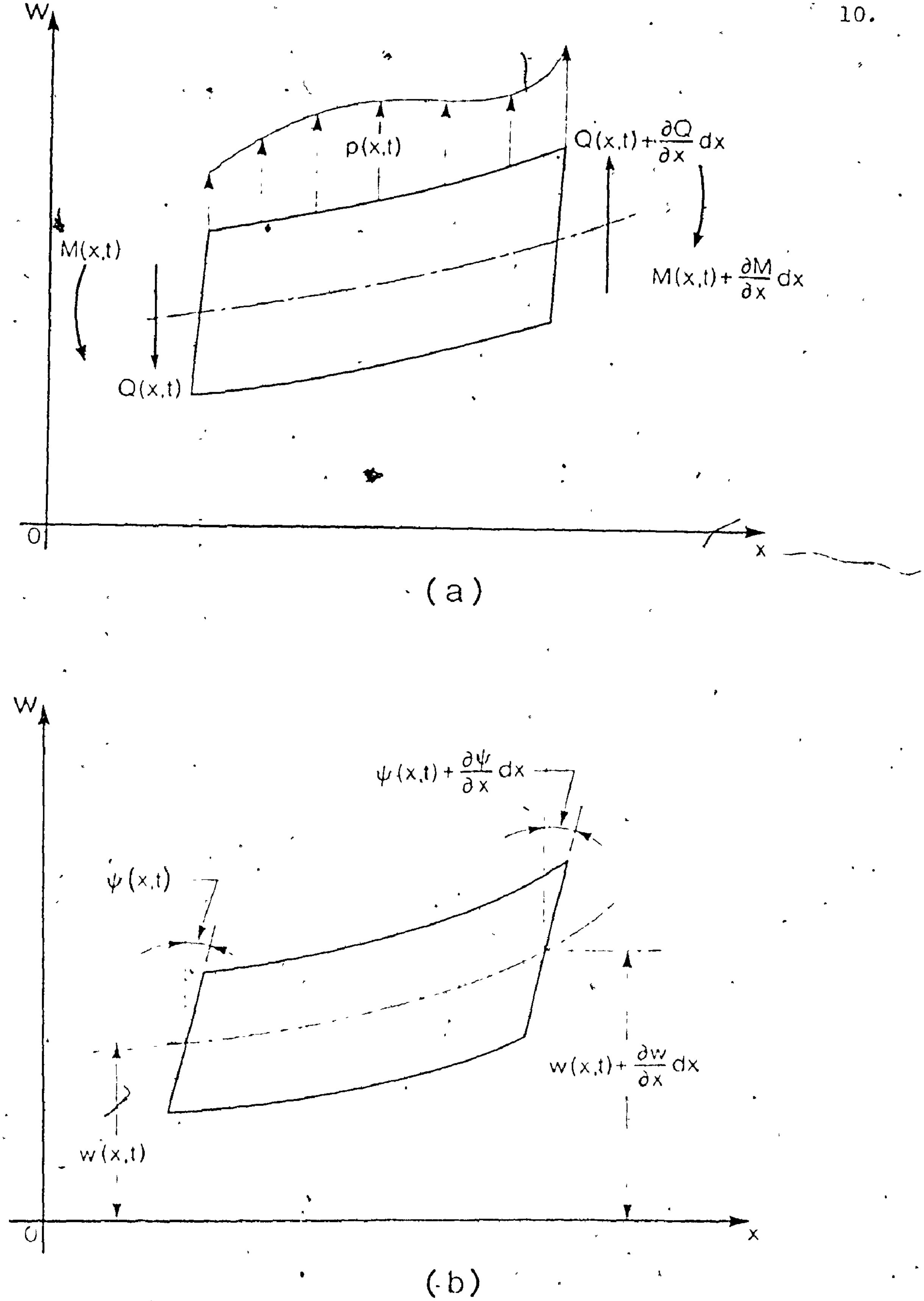


FIGURE 3.1: a) DEFINITION OF POSITIVE MOMENTS AND FORCES

b) DEFINITION OF POSITIVE DISPLACEMENTS

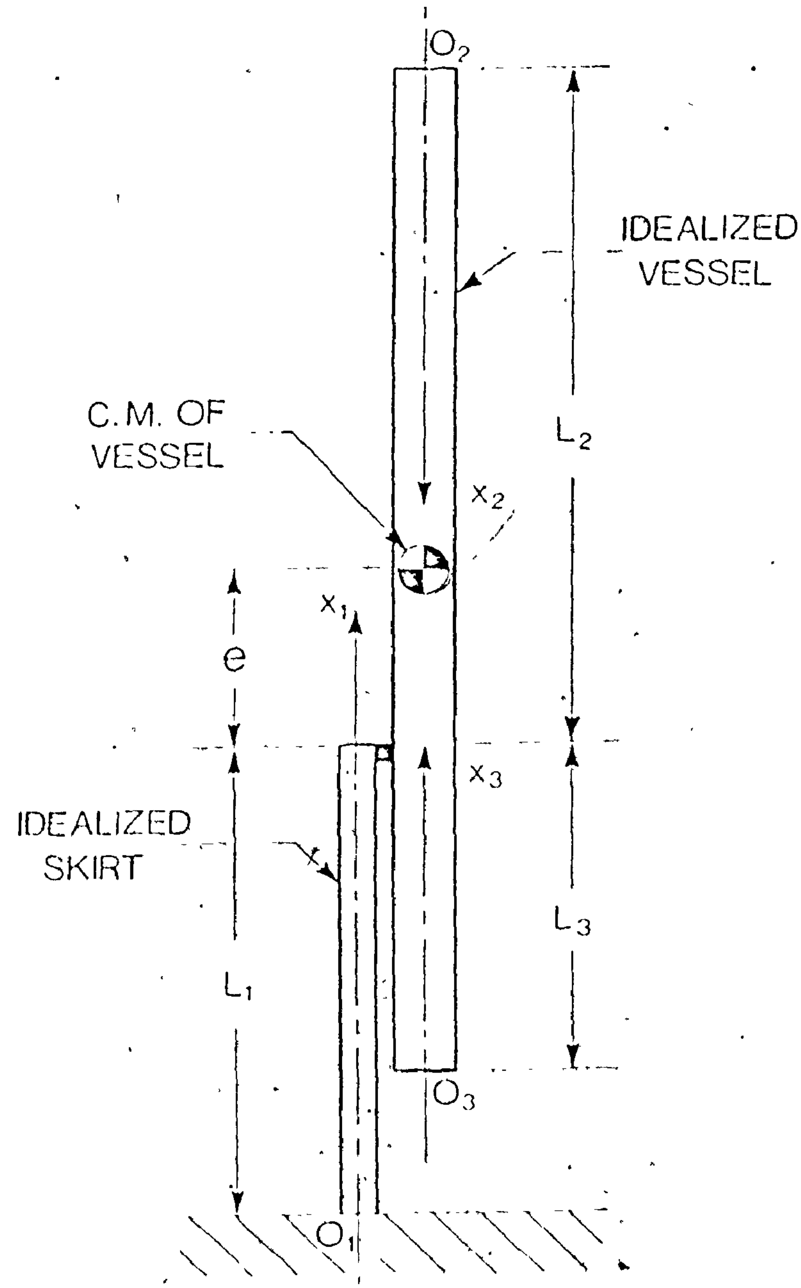


FIGURE 3.2: DEFINITION OF COORDINATE SYSTEMS AND RELATED SYSTEM PARAMETERS.

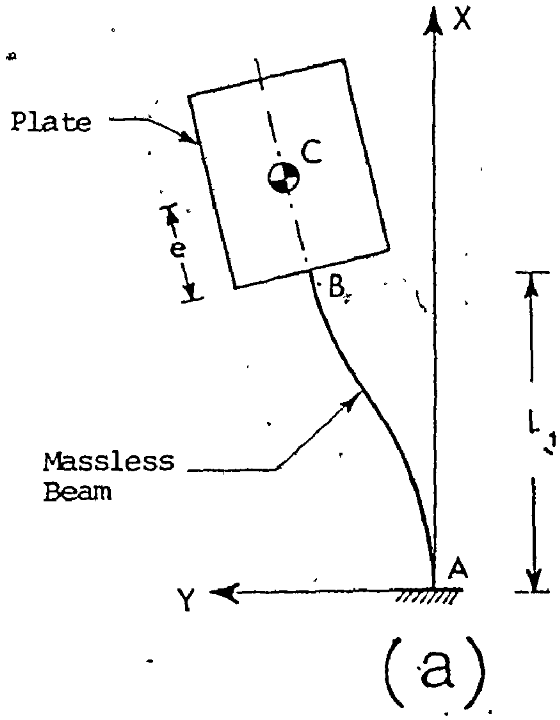
and below its junction with the skirt. The choice of coordinate systems is made for a practical reason, namely that this choice effectively reduces the most complicated frequency determinants considered in this study to ones of sixth order, whereas having all the x_i positive upward leads, if x_2 were measured from the skirt vessel junction, leads to determinants of eighth order; i.e., for the coordinate systems chosen, six of the twelve arbitrary constants of integration are easily eliminated when Models V and VI are studied.

In the following sections the labeling scheme will be that the Latin numeral indicates the corresponding model in the hierarchy given in chapter 2 and the letter indicates the corresponding configuration in Fig. (1.1).

3.2 Analysis of Model (I)-a.

As has been indicated previously, this case has been studied by Timoshenko [1] for the small vibration of a plate BC, attached to a prismatical bar AB, Fig. (3.3.a), assuming that the x-y plane is a principal plane of the bar and the center of mass of the plate, C, is on the prolongation of the axis of the bar. Proceeding with these assumptions, a quadratic frequency equation was obtained, the solution of which is

$$\omega_{1,2}^2 = \frac{6EI}{mL_1^3} \frac{1}{\left[1 + \frac{3e}{L_1} + \frac{3(e^2+i^2)}{L_1^2}\right] \pm \sqrt{\left[1 + \frac{3e}{L_1} + \frac{3(e^2+i^2)}{L_1^2}\right]^2 - \frac{3i^2}{L_1^2}}} \tag{3.2.1}$$



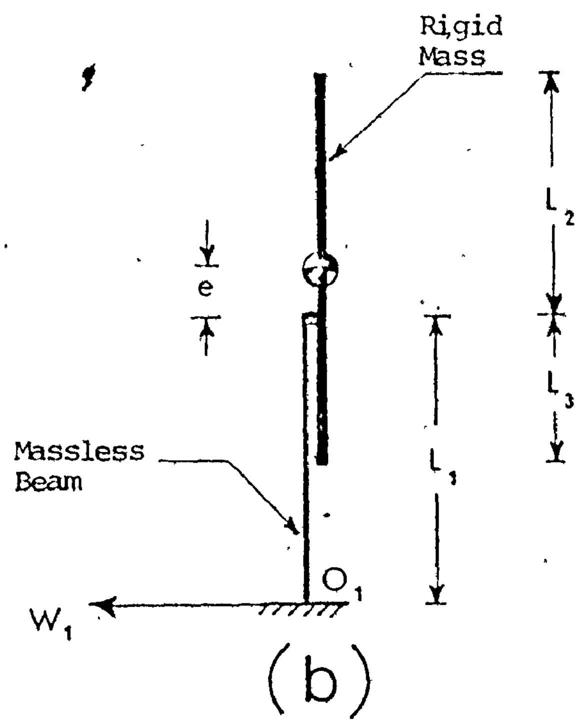


FIGURE 3.3: a) SMALL VIBRATIONS OF A PLATE ATTACHED TO A PRISMATICAL BAR.

b) MODEL (I)-a

where ω is the natural frequency in rad/sec, EI is the flexural rigidity of the bar (corresponding to the skirt), L_1 is the length of the bar, e is the distance between the plate (corresponding to the vessel) center of mass and end of the bar, i is the radius of gyration of the plate with respect to the axis normal to the plate through C and m is the total mass of the plate.

This equation given by Timoshenko can be used to obtain the natural frequencies of Model (I)-a, Fig.(3.3.b), by letting

$$\left.\begin{aligned} e &= \frac{L_2 - L_3}{2} \\ i &= \frac{(L_2 + L_3)^2}{12} \\ m &= \text{total mass of the vessel and its contents} \\ EI &= \text{flexural rigidity of the skirt} \end{aligned}\right\} \qquad (3.2.2)$$

3.3 Model (II)-a. Massless Timoshenko Beam Supporting a Rigid Mass

Consider now, the case of a massless Timoshenko Beam. The equations given in [4] become, when no distributed load acts,

$$EI \frac{d^2\psi}{dx^2} - \kappa^2 AG \left(\psi + \frac{dw}{dx}\right) = 0 \qquad (3.3.1)$$

$$\frac{d\psi}{dx} + \frac{d^2w}{dx^2} = 0 \qquad (3.3.2)$$

where A is cross-sectional area of the elastic beam, K^2 is the so-called shear coefficient and ψ is the angle between a cross-sectional plane and the horizontal plane passing through the y-axis; this implies that planes which are normal to the beam axis in the undeformed state generally do not remain normal to the beam axis in the deformed state.

The boundary conditions (b.c) for the case of Model (II)-a, Fig. (2.1.b), are

$$\text{at } x = 0 \qquad w = 0 \tag{3.3.3}$$

$$\text{and} \quad \psi = 0 \tag{3.3.4}$$

$$x = L_1 \qquad \overline{M} = EI \frac{d\psi}{dx} \tag{3.3.5}$$

$$\text{and} \quad \overline{Q} = \kappa^2 AG \left(\psi + \frac{dw}{dx}\right) \tag{3.3.6}$$

where $\overline{M}$ and $\overline{Q}$ are the moment and shear force exerted by the rigid mass on the end of the beam.

Equation (3.3.2) provides

$$\left(\psi + \frac{dw}{dx}\right) = \text{const.} = D \quad \ldots\ldots \quad (a)$$

Substituting from (a) into b.c. (3.3.6)

$$D = \frac{\overline{Q}}{\kappa^2 AG}, \quad \ldots\ldots \quad (b)$$

Using eq. (b), equation (3.3.1) becomes

$$EI \frac{d^2\psi}{dx^2} - \overline{Q} = 0 \tag{3.3.7}$$

Integrating twice and using b.c. (3.3.4) results in

$$EI\psi = \bar{Q}\,\frac{x^2}{2} + C_1 x \qquad (3.3.8)$$

and substitution of (3.2.8) and (3.2.3) into (3.2.2) results in

$$EIw = -\bar{Q}\,\frac{x^3}{6} - C_1\,\frac{x^2}{2} + C_2\,x \qquad (3.3.9)$$

From (a), (b), (3.2.8) and (3.3.9), the constant C_2 can be obtained and (3.3.9) becomes

$$EIw = -\bar{Q}\,\frac{x^3}{6} - C_1\,\frac{x^2}{2} + \frac{EI\,\bar{Q}}{\kappa^2 AG}\,x \qquad (3.3.10)$$

With the aid of (3.3.5) and (3.3.8) the constant C_1 can be found and finally (3.3.8) and (3.3.9) become

$$\psi_{(L_1)} = \frac{1}{EI}\left[-\frac{\bar{Q}L_1^2}{2} + \bar{M}L_1\right] \qquad (3.3.11)$$

$$w_{(L_1)} = \frac{1}{EI}\left[\bar{Q}\left(\frac{L_1^3}{3} + \frac{EIL_1}{\kappa^2 AG}\right) + \frac{\bar{M}L_1^2}{2}\right] \qquad (3.3.12)$$

The kinetic energy of the system consists of energy of rotation of the mass about its center of mass C and of translatory energy of the mass center. Thus

$$T = \frac{m}{2}\left(\dot{w}_{(L_1)} - e\,\dot{\psi}_{(L_1)}\right)^2 + \frac{m}{2}\,i^2\dot{\psi}^2_{(L_1)} \qquad (3.3.13)$$

Substituting T in Lagrange's equations of motion [4], the following expressions for $\bar{M}$ and $\bar{Q}$ are obtained

$$\bar{Q} = -m\left(\ddot{w}_{(L_1)} - e\,\ddot{\psi}_{(L_1)}\right) \qquad (3.3.14)$$

$$\overline{M} = -m\,[-e\,\ddot{w}_{(L_1)} + (e^2 + i^2)\,\ddot{\psi}_{(L_1)}] \qquad (3.3.15)$$

by inverting (3.3.11) and (3.3.12), expressions for $\overline{Q}$ and $\overline{M}$ can be obtained in terms of $w_{(L_1)}$ and $\psi_{(L_1)}$

$$\overline{Q} = \frac{EI}{\beta}\left[\frac{L_1}{2}\,\psi_{(L_1)} + w_{(L_1)}\right] \qquad (3.3.16)$$

$$\overline{M} = \frac{EI}{\beta}\left[\left(\frac{L_1^2}{3} + \frac{EI}{\kappa^2 AG}\right)\psi_{(L_1)} + \frac{L_1}{2}\,w_{(L_1)}\right] \qquad (3.3.17)$$

where

$$\beta = \left(\frac{L_1^3}{12} + \frac{EIL_1}{\kappa^2 AG}\right)$$

With the aid of equations (3.3.16) and (3.3.17), $\overline{Q}$ and $\overline{M}$ can be eliminated from equations (3.3.14) and (3.3.15). Thus we have

$$m\,[\ddot{w}_{(L_1)} - e\,\ddot{\psi}_{(L_1)}] + \frac{EI}{\beta}\left[\frac{L_1}{2}\,\psi_{(L_1)} + w_{(L_1)}\right] = 0 \qquad (3.3.18)$$

$$m[-e\,\ddot{w}_{(L_1)} + (e^2 + i^2)\ddot{\psi}_{(L_1)}] + \frac{EI}{\beta}\left[\left(\frac{L_1^2}{3} + \frac{EI}{\kappa^2 AG}\right)\psi_{(L_1)} + \frac{L_1}{2}\,w_{(L_1)}\right] = 0 \qquad (3.3.19)$$

Assume $w_{(L_1)}$ and $\psi_{(L_1)}$ to be harmonic in time i.e:

$$\left.\begin{aligned} w_{(L_1)} &= R_1 \sin \omega t \\ \psi_{(L_1)} &= R_2 \sin \omega t \end{aligned}\right\} \qquad (3.3.20)$$

By substituting (3.3.20) into (3.3.18) and (3.3.19), we obtain:

$$\left[\frac{EI}{\beta} - m\omega^2\right]R_1 + \left[\frac{EIL_1}{2\beta} + m\omega^2 e\right] R_2 = 0 \qquad (3.3.21)$$

$$\left[\frac{EIL_1}{2\beta} + m\omega^2 e\right]R_1 + \left[\frac{EI}{\beta}\left(\frac{L_1^2}{3} + \frac{EI}{K^2AG}\right) - m\omega^2(e^2 + i^2)\right]R_2 = 0 \qquad (3.3.22)$$

In order to obtain a nontrivial solution of this system of simultaneous, linear algebraic equations, the determinant of the coefficient matrix of (3.3.21) and (3.3.22) must be set equal to zero, i.e.

$$\begin{vmatrix} \left[\frac{EI}{\beta} - m\omega^2\right] & \left[\frac{EIL_1}{2\beta} + m\omega^2 e\right] \\ \left[\frac{EIL_1}{2\beta} + m\omega^2 e\right] & \left[\frac{EI}{\beta}\left(\frac{L_1^2}{3} + \frac{EI}{\kappa^2AG}\right) - m\omega^2(e^2 + i^2)\right] \end{vmatrix} = 0 \qquad (3.3.23)$$

Expanding (3.3.23), the following frequency equation is obtained:

$$\omega_{1,2}^2 = \frac{EI}{2\beta m i^2}\left\{ \left[eL_1 + (e^2 + i^2) + \left(\frac{L_1^2}{3} + \gamma\right)\right] \pm \sqrt{\left[eL_1+(e^2 + i^2)+\left(\frac{L_1^2}{3} + \gamma\right)\right]^2 - 4i^2\left(\frac{L_1^2}{12} + \gamma\right)}\right\} \qquad (3.3.24)$$

where $\gamma = \frac{EI}{\kappa^2AG}$

and, recall, $\beta = \left(\frac{L_1^3}{12} + \gamma L_1\right).$

3.4 Model (III)-a and (V)-a. i) Euler-Bernoulli Beam Supporting a Rigid Mass, and ii) A System of Euler-Bernoulli Beams.

3.4.A – Derivation of Equations of Motion and Boundary Conditions.

Consider the whole system to be elastic, use the coordinate axes shown in Fig.(3.4), and divide the system into three segments of lengths L_1, L_2 and L_3 in order to obtain the equations of motion and the boundary conditions by means of Hamilton's principle [5]. Note that

$$\left.\begin{aligned} w_{2(x_2,t)} &= w_{1(L_1,t)} + (L_2 - x_2)\frac{\partial w_1}{\partial x_1}_{(L_1,t)} + \bar{w}_{2(x_2,t)} \\ w_{3(x_3,t)} &= w_{1(L_1,t)} - (L_3 - x_3)\frac{\partial w_1}{\partial x_1}_{(L_1,t)} + \bar{w}_{3(x_3,t)} \end{aligned}\right\} \quad (3.4.1)$$

where $\bar{w}_{2(x_2,t)}$ and $\bar{w}_{3(x_3,t)}$ are the elastic displacements of segments 2 and 3 respectively. In the limiting case of the rigid model of the vessel, $\bar{w}_2$ and $\bar{w}_3$ are set equal to zero. We have let $\bar{x}_2 = x_2$ and $\bar{x}_3 = x_3$ since this approximation is consistent with the small deflection, linear theory being used in this study.

Now, making use of equations (3.4.1), the kinetic energy of the system may be written as

$$T = \int_0^{L_1} \frac{\bar{m}_1}{2} (\dot{w}_1)^2 dx_1 + \int_0^{L_2} \frac{\bar{m}_2}{2} [\dot{w}_{1(L_1,t)} + (L_2 - x_2)\frac{\partial \dot{w}_1}{\partial x_1}_{(L_1,t)} + \dot{\bar{w}}_2]^2 dx_2$$

$$+ \int_0^{L_3} \frac{\bar{m}_3}{2}\left[\dot{w}_{1(L_1,t)} - (L_3 - x_3)\frac{\partial \dot{w}_1}{\partial x_1}_{(L_1,t)} + \dot{\bar{w}}_3\right]^2 dx_3 \quad (3.4.2)$$

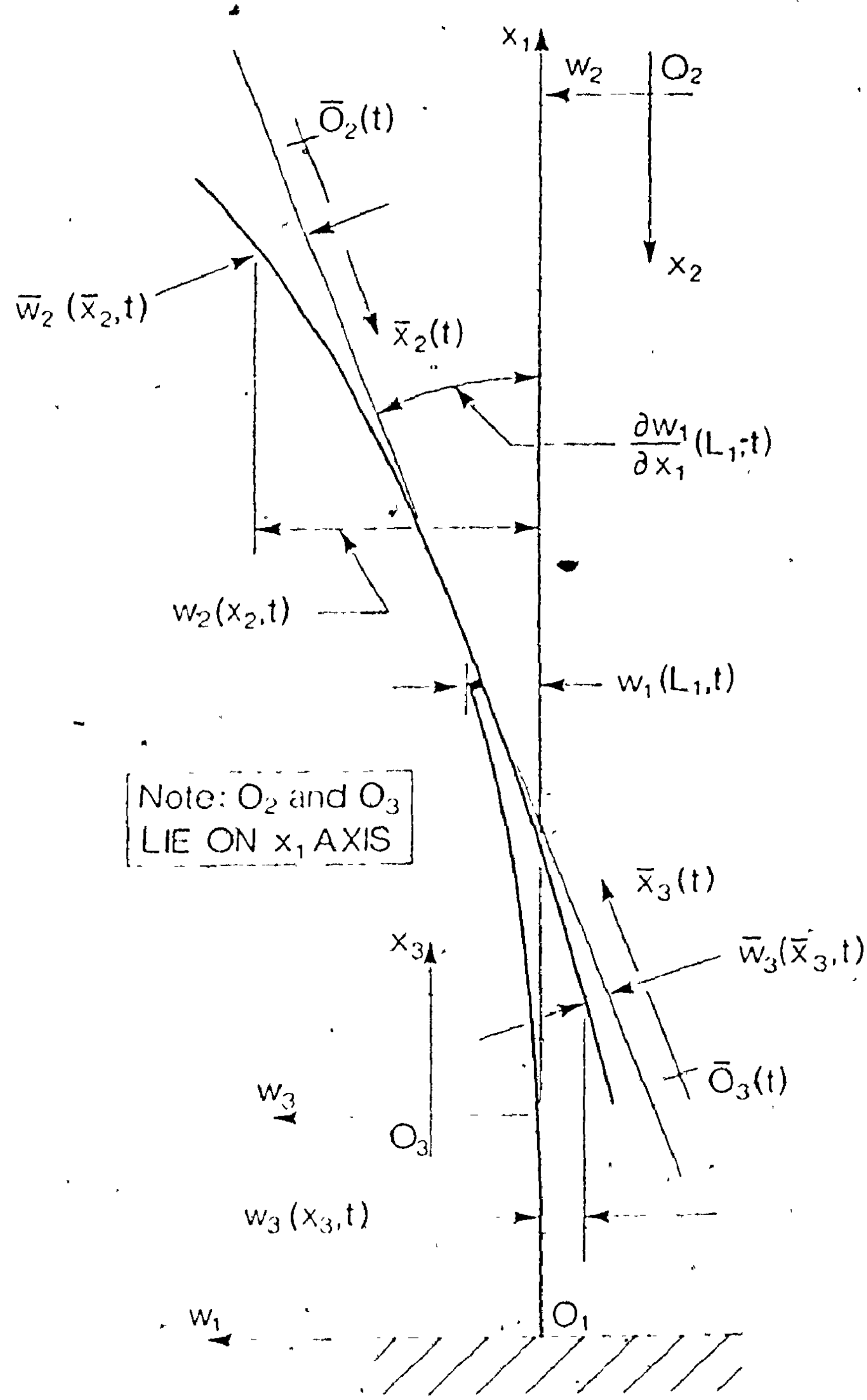


FIGURE 3.4: DECOMPOSITION OF W_2 AND W_3 IN MODELS (III) AND (V).

where the $\bar{m}_i$ are masses per unit length for the i-th segment.

The total strain energy is

$$U = \int_0^{L_1} \frac{EI_1}{2} (w_1'')^2 \, dx_1 + \int_0^{L_2} \frac{EI_2}{2} (\bar{w}_2'')^2 \, dx_2 + \int_0^{L_3} \frac{EI_2}{2} (\bar{w}_3'')^2 dx_3 \qquad (3.4.3)$$

Also, in general, one should consider the work, W, of external (generalized) forces.

$$W = \int_0^{L_i} (p_i \, w_i) \, dx_i + [\bar{M}_i \, (\frac{\partial w_i}{\partial x_i}) + \bar{Q}_i \, w_i \,] \Big|_0^{L_i} \qquad (a)$$

$$(i = 1,2,3)$$

where p_i is the distributed load and $\bar{M}_i$ and $\bar{Q}_i$ are the moments and shears acting upon the ends of each segment. When we reduce the system to our specific case, i.e. $P_i = \bar{Q}_i = \bar{M}_i = 0$, we note that

$$W = 0. \qquad (b)$$

By applying Hamilton's principle for elastic solids,

$$\int_{t_1}^{t_2} \delta (T - U + W) \, dt = 0, \qquad (3.4.4)$$

we may obtain the appropriate formulation of our problem.

Substituting eqs. (3.4.2) and (3.4.3) into (3.4.4) yields

$$\int_{t_1}^{t_2} \Big\{ - \int_0^{L_1} (EI_1 \, w_1^{iv} + \bar{m}_1 \ddot{w}_1) \, \delta \, w_1 \, dx_1 \; -$$

$$- \int_0^{L_2} [EI_2 \bar{w}_2^{iv} + \bar{m}_2 \ddot{\bar{w}}_2 + \bar{m}_2 (\ddot{w}_{1(L_1,t)} + (L_2-x_2) \frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)})] \delta\bar{w}_2 dx_2$$

$$- \int_0^{L_3} [EI_3 \bar{w}_3^{iv} + \bar{m}_3 \ddot{\bar{w}}_3 + \bar{m}_3 (\ddot{w}_{1(L_1,t)} - (L_3-x_3) \frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)})] \delta\bar{w}_3 dx_3$$

$$- [\int_0^{L_2} \bar{m}_2 (\ddot{w}_{1(L_1,t)} + (L_2-x_2) \frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)} + \ddot{\bar{w}}_2) dx_2$$

$$+ \int_0^{L_3} \bar{m}_3 (\ddot{w}_{1(L_1,t)} - (L_3-x_3) \frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)} + \ddot{\bar{w}}_3) dx_3 - EI_1 w'''_{1(L_1,t)}] \delta w_{1(L_1,t)}$$

$$- [\int_0^{L_2} \bar{m}_2 (\ddot{w}_{1(L_1,t)} + (L_2-x_2) \frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)} + \ddot{\bar{w}}_2) (L_2-x_2) dx_2$$

$$- \int_0^{L_3} \bar{m}_3 (\ddot{w}_{1(L_1,t)} - (L_3-x_3) \frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)} + \ddot{\bar{w}}_3) (L_3-x_3) dx_3$$

$$+ EI_1 w''_{1(L_1,t)}] \frac{\partial \delta w_1}{\partial x}_{1(L_1,t)}$$

$$- EI_1 w''_{1(0,t)} \frac{\partial \delta w_1}{\partial x}_{1(0,t)} + EI_1 w'''_{1(0,t)} \delta w_{1(0,t)}$$

$$- EI_2 \bar{w}''_{2(0,t)} \frac{\partial \delta \bar{w}_2}{\partial x}_{2(0,t)} - EI_2 \bar{w}''_{2(L_2,t)} \frac{\partial \delta \bar{w}_2}{\partial x}_{2(L_2,t)}$$

$$+ EI_2 \bar{w}'''_{2(0,t)} \delta\bar{w}_{2(0,t)} + EI_2 \bar{w}'''_{2(L_2,t)} \delta \bar{w}_{2(L_2,t)}$$

$$- EI_3 \bar{w}''_{3(0,t)} \frac{\partial \delta \bar{w}_3}{\partial x}_{3(0,t)} - EI_3 \bar{w}''_{3(L_3,t)} \frac{\partial \delta \bar{w}_3}{\partial x}_{3(L_3,t)}$$

$$+ \quad EI_3 \bar{w}'''_{3(0,t)} \delta\bar{w}_{3(0,t)} + EI_3 \bar{w}'''_{3(L_3,t)} \delta\bar{w}_{3(L_3,t)} \Big\} dt = 0 \qquad \ldots (3.4.5)$$

For any arbitrary time interval (t_1-t_2), the integrands of the double and single integrals (including the integration with respect to time) must vanish independently. Therefore the equations of motion are:

$$\left.\begin{aligned} & EI_1 w_1^{iv} + \bar{m}_1 \ddot{w}_1 = 0 \\ & EI_2 \bar{w}_2^{iv} + \bar{m}_2 \ddot{\bar{w}}_2 = -\bar{m}_2[\ddot{w}_{1(L_1,t)} + (L_2-x_2)\frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)}] \\ & EI_3 \bar{w}_3^{iv} + \bar{m}_3 \ddot{\bar{w}}_3 = -\bar{m}_3[w_{1(L_1,t)} - (L_3-x_3)\frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)}] \end{aligned}\right\} \qquad (3.4.6)$$

The boundary conditions for the system are:

$$\left.\begin{aligned} & \text{at } x_1 = 0 : \quad w_{1(0,t)} = 0, \quad w'_{1(0,t)} = 0 && \text{(a)} \\ & \text{at } x_2 = 0 : \quad EI_2 \bar{w}''_{2(0,t)} = 0, \quad EI_2 \bar{w}'''_{3(0,t)} = 0 && \text{(b)} \\ & \text{at } x_3 = 0 : \quad EI_3 \bar{w}''_{3(0,t)} = 0, \quad EI_3 \bar{w}'''_{(0,t)} = 0 && \text{(c)} \end{aligned}\right\} \qquad (3.4.7)$$

and at the junction of the three segments, i.e., $x_1 = L_1$, $x_2 = L_2$ and $x_3 = L_3$

$$EI_1 w'''_{1(L_1,t)} = \int_0^{L_2} \bar{m}_2[\ddot{w}_{1(L_1,t)} + (L_2-x_2)\frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)} + \ddot{\bar{w}}_2]dx_2$$

$$+ \int_0^{L_3} \bar{m}_3[\ddot{w}_{1(L_1,t)} - (L_3-x_3)\frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)} + \ddot{\bar{w}}_3]dx_3$$

and ... (3.4.8)

$$EI_1 w''_{1(L_1,t)} = -\int_0^{L_2} \bar{m}_2[\ddot{w}_{1(L_1,t)} + (L_2-x_2)\frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)} + \ddot{\bar{w}}_2](L_2-x_2)dx_2$$

$$+ \int_0^{L_3} \bar{m}_3[\ddot{w}_{1(L_1,t)} - (L_3-x_3)\frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)} + \ddot{\bar{w}}_3](L_3-x_3)dx_3$$

as well as $\bar{w}_2 = \bar{w}_3 = \bar{w}'_2 = \bar{w}'_3 = 0$ (3.4.9)

3.4.B Frequency Equation for the System when the Three Parts are Elastic, Model (V)-a.

In this section the entire system is considered to be elastic. Revert to total displacements, in order to put equations (3.4.6) in the simpler form

$$EI_i w_i^{iv} + \bar{m}_i \ddot{w}_i = 0, \qquad i = 1,2,3, \qquad (3.4.10)$$

Furthermore, boundary conditions (3.4.7) become

$$\begin{aligned} w_{1(0,t)} &= 0 \\ w'_1(0,t) &= 0 \\ w''_2(0,t) &= 0 \\ w'''_2(0,t) &= 0 \\ w''_3(0,t) &= 0 \\ w'''_3(0,t) &= 0 \end{aligned} \qquad (3.4.11)$$

and b·c·s (3.4.8) and (3.4.9) reduce to

$$\left.\begin{aligned} EI_1\, w'''_{1(L_1,t)} &= \int_0^{L_2} \bar{m}_2\, \ddot{w}_2\, dx_2 + \int_0^{L_3} \bar{m}_3\, \ddot{w}_3\, dx_3 \\ EI_1\, w''_{1(L_1,t)} &= -\int_0^{L_2} \bar{m}_2 \ddot{w}_2 (L_2-x_2)\,dx_2 + \int_0^{L_3} \bar{m}_3 \ddot{w}_3 (L_3-x_3)\,dx_3, \end{aligned}\right\} \quad (3.4.12)$$

and

$$\begin{aligned} w_{1(L_1,t)} &= w_{2(L_2,t)} = w_{3(L_3,t)} \\ w'_{1(L_1,t)} &= -w'_{(L_2,t)} = w'_{(L_3,t)} \end{aligned} \quad (3.4.13)$$

Using the method of separation of variables, the solution of equations (3.4.10) may be obtained [1] . Since we seek periodic solutions in time it is assumed that

$$w_i = W_i(x) \cdot \sin \omega t. \quad (3.4.14)$$

Substituting equation (3.4.14) into equation (3.4.10) yields the following ordinary differential equation

$$W_i^{iv} + a_i^4 W_i = 0 \quad (3.4.15)$$

the solution of which is

$$W_i(x) = B_i \sin a_i x_i + C_i \cos a_i x_i + D_i \sinh a_i x_i + F_i \cosh a_i x_i \quad \ldots\ldots \quad (3.4.16)$$

where $a_i^4 = \dfrac{\bar{m}_i \omega^2}{EI_i}$, $i = 1,2,3.$

If we now substitute equation (3.4.14) into (3.4.11) through (3.4.13), the following boundary conditions on the mode shapes are obtained:

$$W_{1(0)} = W'_{1(0)} = 0$$

$$W''_{2(0)} = W'''_{2(0)} = 0 \qquad (3.4.17)$$

$$W''_{3(0)} = W'''_{3(0)} = 0 ,$$

$$EI_1 W'''_{1(L_1)} = -\int_0^{L_2} \bar{m}_2\omega^2 W_2 dx_2 - \int_0^{L_3} \bar{m}_3\omega^2 W_3 dx_3 , \qquad (3.4.18)$$

$$EI_1 W''_{1(L_1)} = \int_0^{L_2} \bar{m}_2\omega^2 W_2 (L_2-x_2) dx_2 - \int_0^{L_3} \bar{m}_3\omega^2 W_3 (L_3-x_3) dx_3 , \qquad (3.4.19)$$

and

$$W_{1(L_1)} = W_{2(L_2)} = W_{3(L_3)}$$

$$W'_{1(L_1)} = -W'_{2(L_2)} = W'_{3(L_3)} \qquad (3.4.20)$$

equations (3.4.15) subject to b.c.s (3.4.16) leads to

$$W_1 = B_1(\sin a_1x_1 - \sinh a_1x_1) + C_1(\cos a_1x_1 - \cosh a_1x_1) \qquad (a)$$

$$W_2 = B_2(\sin a_2x_2 + \sinh a_2x_2) + C_2(\cos a_2x_2 + \cosh a_2x_2) \qquad (b)$$

$$W_3 = B_3(\sin a_3x_3 + \sinh a_3x_3) + C_3(\cos a_3x_3 + \cosh a_3x_3) \qquad (c)$$

..... (3.4.21)

Substituting (3.4.21) into the boundary conditions (3.4.18) through (3.4.20) yields

$$B_1(\sin a_1L_1 + \sinh a_1L_1) + C_1(\cos a_1L_1 + \cosh a_1L_1)$$
$$- B_2 \cdot R_1(\sin a_2L_2 - \sinh a_2L_2) - C_2 \cdot R_1(\cos a_2L_2 - \cosh a_2L_2)$$
$$+ B_3R_2(\sin a_3L_3 - \sinh a_3L_3) + C_3R_2(\cos a_3L_3 - \cosh a_3L_3) = 0 \quad \ldots\ldots (3.4.22)$$

where $R_1 = (\frac{a_2}{a_1})^2 \frac{I_2}{I_1}$ and $R_2 = (\frac{a_3}{a_1})^2 \frac{I_3}{I_1}$,

$$B_1(\cos a_1L_1 + \cosh a_1L_1) - C_1(\sin a_1L_1 - \sinh a_1L_1)$$
$$+ B_2R_3(\cos a_2L_2 - \cosh a_2L_2) - C_2R_3(\sin a_2L_2 + \sinh a_2L_2)$$
$$+ B_3R_4(\cos a_3L_3 - \cosh a_3L_3) - C_3R_4(\sin a_3L_3 + \sinh a_3L_3) = 0 \quad \ldots\ldots (3.4.23)$$

where $R_3 = (\frac{a_2}{a_1})^2 \frac{I_2}{I_1}$ and $R_4 = (\frac{a_3}{a_1})^3 \frac{I_3}{I_1}$,

$$B_1(\sin a_1L_1 - \sinh a_1L_1) + C_1(\cos a_1L_1 - \cosh a_1L_1)$$
$$- B_2(\sin a_2L_2 + \sinh a_2L_2) - C_2(\cos a_2L_2 + \cosh a_2L_2) = 0, \quad \ldots\ldots (3.4.24)$$

$$B_1(\cos a_1L_1 - \cosh a_1L_1) - C_1(\sin a_1L_1 + \sinh a_1L_1)$$
$$+ B_2R_5(\cos a_2L_2 + \cosh a_2L_2) - C_2R_5(\sin a_2L_2 - \sinh a_2L_2) = 0, \quad \ldots\ldots (3.4,25)$$

$$B_2(\sin a_2L_2 + \sinh a_2L_2) + C_2(\cos a_2L_2 + \cosh a_2L_2)$$

$$- B_3(\sin a_3L_3 + \sinh a_3L_3) - C_3(\cos a_3L_3 + \cosh a_3L_3) = 0, \quad \ldots\ldots (3.4.26)$$

and

$$B_2(\cos a_2L_2 + \cosh a_2L_2) - C_2(\sin a_2L_2 - \sinh a_2L_2)$$

$$+ B_3R_6(\cos a_3L_3 + \cosh a_3L_3) - C_3R_6(\sin a_3L_3 - \sinh a_3L_3) = 0 \quad \ldots\ldots (3.4.27)$$

where $R_5 = \dfrac{a_2}{a_1}$ and $R_6 = \dfrac{a_3}{a_2}$.

Equations (3.4.22) through (3.4.27) form a system of six homogeneous linear algebraic equations in six unknowns, $B_1, C_1, \ldots, C_3$.

Again, for the system to have a nontrivial solution, the determinant of the coefficient matrix must vanish. This provides the desired frequency equation from which the frequencies and, ultimately, with the use of the simultaneous equations, the modes may be obtained. Therefore, we set

$$\begin{vmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \vdots \\ \vdots & & a_{66} \end{vmatrix} = 0 \quad (3.4.28)^*$$

Now for any numerical example, the natural frequencies may be obtained as the roots of the transcendental equation (3.4.28)

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* For the detailed frequency determinant see Appendix C.1.

3.4.C Frequency Equation for the System when Considering the Vessel to be Rigid (Model (III)-a).

In this limiting case, with the stiffness of the pressure vessel approaching infinity, i.e. considered to be rigid, the elastic deformations, $\bar{w}_2$ and $\bar{w}_3$, must be set equal to zero in the preceding case. This yields the equation of motion

$$EI_1 w_1^{iv} + \bar{m}_1 \ddot{w}_1 = 0 \qquad (3.4.29)$$

and the boundary conditions

at $x_1 = 0$

$$w_{1(0,t)} = w'_{1(0,t)} = 0 \qquad (3.4.30)$$

at $x_1 = L_1$

$$EI\, w'''_{1(L_1,t)} = \int_0^{L_2} \bar{m}_2[\ddot{w}_{1(L_1,t)} + (L_2-x_2)\frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)}]dx_2 + \int_0^{L_3} \bar{m}_3[\ddot{w}_{1(L_1,t)} - (L_3-x_3)\frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)}]dx_2 \qquad (3.4.31)$$

$$EI\, w''_{1(L_1,t)} = -\int_0^{L_2} \bar{m}_2[\ddot{w}_{1(L_1,t)} + (L_2-x_2)\frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)}](L_2-x_2)dx_2 + \int_0^{L_3} \bar{m}_3[\ddot{w}_{1(L_1,t)} - (L_3-x_3)\frac{\partial \ddot{w}_1}{\partial x}_{1(L_1,t)}](L_3-x_3)dx_3$$

.... (3.4.32)

The solution of (3.4.28) is given, using separation of variables, by (3.4.14) and (3.4.15) and proceeding by an entirely analogous analysis, the following may be obtained, (note $\bar{m}_2 = \bar{m}_3$)

$$B_1[(\cos a_1L_1+ \cosh a_1L_1)-R_1(\sin a_1L_1- \sinh a_1L_1)-R_2(\cos a_1L_1-\cosh a_1L_1)]$$
$$-C_1[(\sin a_1L_1-\sinh a_1L_1)+R_1(\cos a_1L_1-\cosh a_1L_1)-R_2(\sin a_1L_1+\sinh a_1L_1)]$$
$$= 0 \qquad (3.4.33)$$

and

$$B_1[(\sin a_1L_1+\sinh a_1L_1)+R_2(\sin a_1L_1-\sinh a_1L_1)+R_3(\cos a_1L_1-\cosh a_1L_1)]$$
$$+C_1[(\cos a_1L_1+\cosh a_1L_1)+R_2(\cos a_1L_1-\cosh a_1L_1)-R_3(\sin a_1L_1+\sinh a_1L_1)]$$
$$= 0 \qquad (3.4.34)$$

where

$$R_1 = a_1L_1\left(\frac{\bar{m}_2}{\bar{m}_1}\right)\left(\frac{L_2+L_3}{L_1}\right),$$

$$R_2 = \tfrac{1}{2}(a_1L_1)^2\left(\frac{\bar{m}_2}{\bar{m}_1}\right)\left(\frac{L_2^2-L_3^2}{L_1^2}\right),$$

$$R_3 = \tfrac{1}{3}(a_1L_1)^3\left(\frac{\bar{m}_2}{\bar{m}_1}\right)\left(\frac{L_2^3+L_3^3}{L_1^3}\right).$$

For the system to have nontrivial solutions, the determinant of the coefficient matrix of equations (3.4.33) and (3.4.34) must be equal to zero. Thus the frequency equation is of the simpler form

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \qquad (3.4.35)^*$$

* See Appendix C.2

3.5. Mathematical Analysis for the System Using Timoshenko Beam Theory. Models (IV and VI)-a.

3.5.A Derivation of Equations of Motion and Boundary Conditions.

The dynamic Euler-Bernoulli beam theory is the most commonly used beam theory for technical applications and it proceeds upon two assumptions, one kinematic and the other kinetic in nature. The first is that planes which are normal to the beam axis in the undeformed state remain plane and normal to the beam axis in the deformed state. This assumption is equivalent to assuming the beam to be rigid with respect to shear deformations, i.e. that all deformations of the beam are due to longitudinal fibre extension and compression alone. The second assumption is that the effect of rotatory inertia is assumed to be negligible compared to transverse translational inertia and, therefore, the term $(\rho\ I\ddot{\psi})$ is neglected.

In some cases, specially for short beams, the effect of shear deformations and rotatory inertia should be taken into account. In such cases, the need for the use of what is usually referred to as the Timoshenko beam theory arises. This theory takes into account both transverse translational and rotatory inertia, and deformations due to both the flexure and shear deformations of the beam.

Now, considering the system shown in Fig.(3.5), it is clear that

$$\left.\begin{aligned} w_2(x_2,t) &= w_1(L_1,t) - \psi_1(L_1,t)\,(L_2 - x_2) + \bar{w}_2(x_2,t) \\ \psi_2(x_2,t) &= -\psi_1(L_1,t) + \bar{\psi}_2(x_2,t) \\ w_3(x_3,t) &= w_1(L_1,t) + \psi_1(L_1,t)\,(L_3 - x_3) + \bar{w}_3(x_3,t) \end{aligned}\right\} \qquad (3.5.1)$$

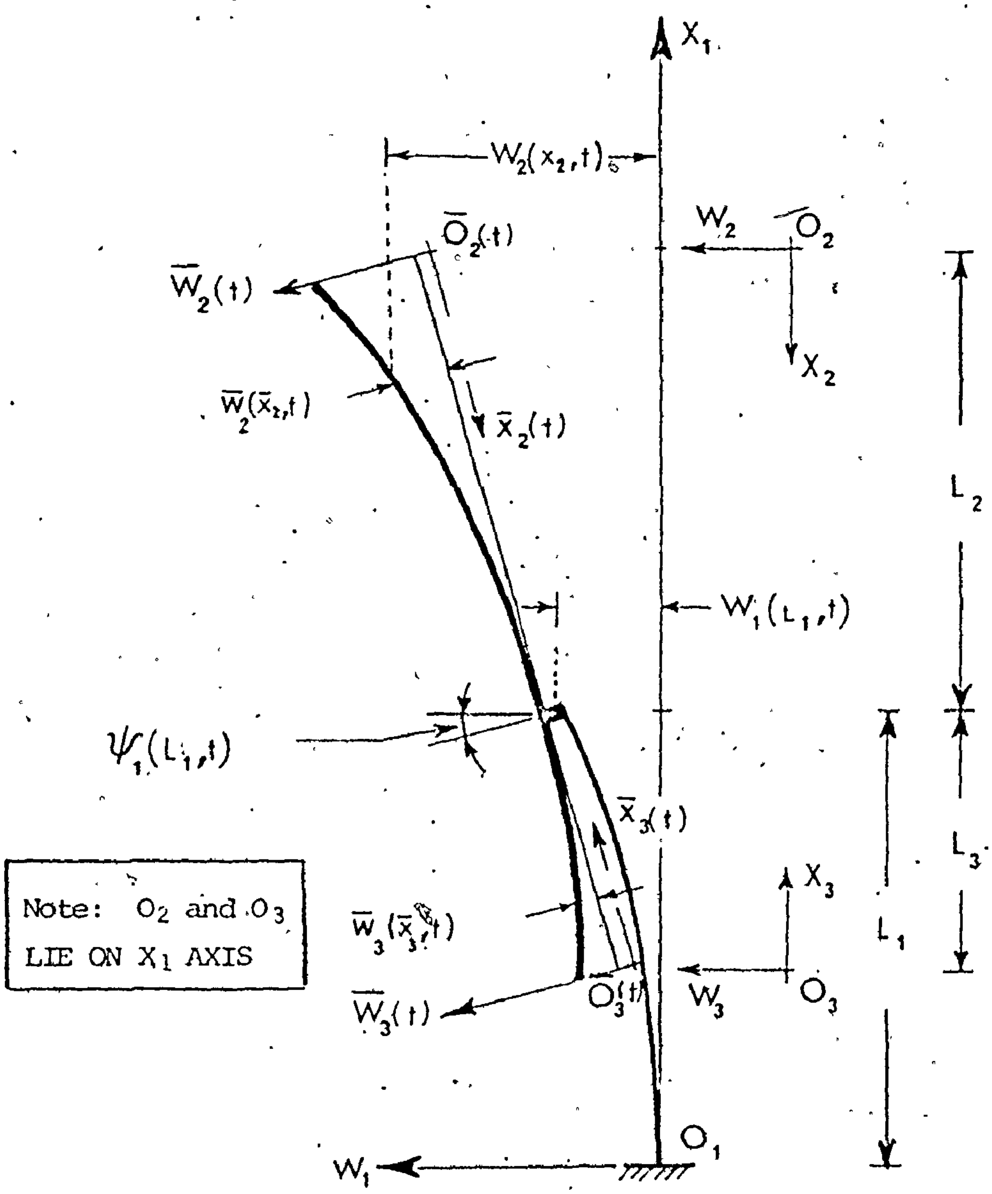


FIGURE 3.5: DECOMPOSITION OF W_2 AND W_3 IN MODELS (IV) AND (VI).

$$\psi_3(x_3,t) = \psi_1(L_1,t) - \bar{\psi}_3(x_3,t) \qquad (3.5.1)\ \text{cont.}$$

where $\bar{w}_2$, $\bar{w}_3$, $\bar{\psi}_2$ and $\bar{\psi}_3$ are elastic deformations.

The kinetic energy for the system is [4]

$$T = \frac{1}{2}\int_0^{L_i} \rho_i (I_i\, \dot{\psi}_i^2 + A_i\, \dot{w}_i^2)\, dx_i, \quad i = 1,2,3, \qquad (3.5.2)$$

and

$$U = \frac{1}{2}\int_0^{L_i} [M\frac{\partial\bar{\psi}_i}{\partial x_i} + Q(\bar{\psi}_i + \frac{\partial\bar{w}_i}{\partial x_i})]\, dx_i, \quad i = 1,2,3, \qquad (3.5.3)$$

where $\bar{w}_1 \equiv w_1$ and $\bar{\psi}_1 \equiv \psi_1$.

The constitutive equations (generalized stress-strain law) for the Timoshenko beam theory are:

$$M = EI\,\frac{\partial\psi}{\partial x}$$
$$Q = \kappa^2 AG\,(\psi + \frac{\partial w}{\partial x}) \qquad (3.5.4)$$

where A is the cross-sectional area, G is the shear modulus and K^2 is a numerical factor called Timoshenko's shear coefficient [6] which depends upon the shape of the cross section; in our case of a thin walled circular cylinder $K^2 = 0{,}53$. Substituting from (3.5.4) into (3.5.3) yields

$$U = \frac{1}{2}\int_0^{L_i} [EI_i(\frac{\partial\bar{\psi}_i}{\partial x_i})^2 + \kappa_i^2 A_i\, G(\bar{\psi}_i + \frac{\partial\bar{w}_i}{\partial x_i})^2]dx_i \qquad (3.5.5)$$

Again, for our case,

$$W = \int_0^{L_i} (p_i \cdot w_i) dx_i + \left| \bar{M}_i \cdot \psi_i + \bar{Q}_i w_i \right|_0^{L_i} = 0 \qquad (3.5.6)$$

Applying Hamilton's principle we have

$$\int_{t_1}^{t_2} \delta(T-U)\, dt = 0 \qquad (3.5.7)$$

$$= \int_{t_1}^{t_2} \{ -\int_0^{L_i} \rho_i (I_i \ddot{\psi}_i \cdot \delta\psi_i + A_i \ddot{w}_i \delta w_i) dx_i$$

$$+ \int_0^{L_i} [(EI_i \frac{\partial^2 \bar{\psi}_i}{\partial x_i^2} - \kappa_i^2 A_i G(\bar{\psi}_i + \frac{\partial \bar{w}_i}{\partial x_i})) \delta \bar{\psi}_i$$

$$+ \kappa_i^2 A_i G(\frac{\partial \bar{\psi}_i}{\partial x_i} + \frac{\partial^2 \bar{w}_i}{\partial x_i^2}) \delta \bar{w}_i] dx_i$$

$$- EI_i \frac{\partial \bar{\psi}_i}{\partial x_i} \delta \bar{\psi}_i \Big|_0^{L_i} - \kappa_i^2 A_i G(\bar{\psi}_i + \frac{\partial \bar{w}_i}{\partial x_i}) \delta \bar{w}_i \Big|_0^{L_i} \} dt = 0 \qquad (3.5.8)$$

where $i = 1, 2, 3,$ and $\bar{w}_1 \equiv w_1$ and $\bar{\psi}_1 \equiv \psi_1$

Now expanding (3.4.8), substituting from (3.4.1), and rearranging the terms yields

$$\int_{t_1}^{t_2} \{ \int_0^{L_1} [EI_1 \frac{\partial^2 \psi_1}{\partial x_1^2} - \kappa_1^2 A_1 G (\psi_1 + \frac{\partial w_1}{\partial x_1}) - \rho_1 I_1 \ddot{\psi}_1] \delta \psi_1 dx_1$$

$$+ \int_0^{L_1} [\kappa_1^2 A_1 G (\frac{\partial \psi_1}{\partial x_1} + \frac{\partial^2 w_1}{\partial x_1^2}) - \rho_1 A_1 \ddot{w}_1 - p_1] \delta w_1 dx_1$$

$$+ \int_0^{L_2} [EI_2 \frac{\partial^2 \bar{\psi}_2}{\partial x_2^2} - \kappa_2^2 A_2 G(\bar{\psi}_2 + \frac{\partial \bar{w}_2}{\partial x_2}) - \rho_2 I_2 (-\ddot{\psi}_{1(L_1,t)} + \ddot{\bar{\psi}}_2)] \delta \bar{\psi}_2 \, dx_2$$

$$+ \int_0^{L_2} [\kappa_2^2 A_2 G(\frac{\partial \bar{\psi}_2}{\partial x_2} + \frac{\partial^2 \bar{w}_2}{\partial x_2^2}) - \rho_2 A_2 (\ddot{w}_{1(L_1,t)} - (L_2 - x_2) \ddot{\psi}_{1(L_1,t)} + \ddot{\bar{w}}_2)] \delta \bar{w}_2 dx_2$$

$$+ \int_0^{L_3} [EI_3 \frac{\partial^2 \bar{\psi}_3}{\partial x_3^2} - \kappa_3^2 A_3 G(\bar{\psi}_3 + \frac{\partial \bar{w}_3}{\partial x_3}) - \rho_3 I_3 (\ddot{\psi}_{1(L_1,t)} + \ddot{\bar{\psi}}_2)] \delta \bar{\psi}_3 \, dx_3$$

$$+ \int_0^{L_3} [\kappa_3^2 A_3 G(\frac{\partial \bar{\psi}_3}{\partial x_3} + \frac{\partial^2 \bar{w}_3}{\partial x_3^2}) - \rho_3 A_3 (\ddot{w}_{1(L_1,t)} + (L_3 - x_3) \ddot{\psi}_{1(L_1,t)} + \ddot{\bar{w}}_3)] \delta \bar{w}_3 dx_3$$

$$+ \delta \psi_{1(L_1,t)} \Bigg[\int_0^{L_2} (\rho_2 I_2 (-\ddot{\psi}_{1(L_1,t)} + \ddot{\bar{\psi}}_2) + \rho_2 A_2 (\ddot{w}_{1(L_1,t)} - (L_2 - x_2)$$

$$\ddot{\psi}_{1(L_1,t)} + \ddot{\bar{w}}_2)(L_2 - x_2)) dx_2 - \int_0^{L_3} (\rho_3 I_3 (\ddot{\psi}_{1(L_1,t)} + \ddot{\bar{\psi}}_3) + \rho_3 A_3 (\ddot{w}_{1(L_1,t)}$$

$$+ (L_3 - x_3) \ddot{\psi}_{1(L_1,t)} + \ddot{\bar{w}}_3)(L_3 - x_3)) \, dx_3 - EI_1 \frac{\partial \psi_1}{\partial x_1}_{(L_1,t)} \Bigg]$$

$$+ \delta w_{1(L_1,t)} \Bigg[- \int_0^{L_2} \rho_2 A_2 (\ddot{w}_{1(L_1,t)} - (L_2 - x_2) \ddot{\psi}_{1(L_1,t)} + \ddot{\bar{w}}_2) \, dx_2$$

$$- \int_0^{L_3} \rho_3 A_3 (\ddot{w}_{1(L_1,t)} + (L_3 - x_3) \ddot{\psi}_{1(L_1,t)} + \ddot{\bar{w}}_3) dx_3 - \kappa_1^2 A_1 G(\psi_1 + \frac{\partial w_1}{\partial x_1})_{(L_1,t)} \Bigg]$$

$$- EI_1 \frac{\partial \psi_1}{\partial x_1} \delta \psi_{1(0,t)} - EI_2 \frac{\partial \bar{\psi}_2}{\partial x_2} \delta \bar{\psi}_2 \Big|_0^{L_2} - EI_3 \frac{\partial \bar{\psi}_3}{\partial x_3} \delta \bar{\psi}_3 \Big|_0^{L_3}$$

$$- \kappa_1^2 A_1 G(\psi_1 + \frac{\partial w_1}{\partial x_1}) \delta w_{1(0,t)} - \kappa_2^2 A_2 G(\bar{\psi}_2 + \frac{\partial \bar{w}_2}{\partial x_2}) \delta \bar{w}_2 \Big|_0^{L_2}$$

$$- \kappa_3^2 A_3 G(\bar{\psi}_3 + \frac{\partial \bar{w}_3}{\partial x_3}) \delta \bar{w}_3 \Big|_0^{L_3} \} \, dt = 0 \qquad (3.5.9)$$

For an arbitrary time interval $(t_1 - t_2)$, the integrands of the double and single integrals must vanish separately. In addition, by introducing the specific nature of the problem under consideration,

$$w_{1(0,t)} = \bar{w}_{2(L_2,t)} = \bar{w}_{3(L_3,t)} = 0$$

$$\psi_{1(0,t)} = \psi_{2(L_2,t)} = \psi_{3(L_3,t)} = 0 \qquad (3.5.10)$$

the following results are obtained.

The equations of motion are:

$$EI_1 \frac{\partial^2 \psi_1}{\partial x_1^2} - \kappa_1^2 A_1 G(\psi_1 + \frac{\partial w_1}{\partial x_1}) - \rho_1 I_1 \ddot{\psi}_1 = 0 \qquad (a)$$

$$(3.5.11)$$

$$\kappa_1^2 A_1 G (\frac{\partial \psi_1}{\partial x_1} + \frac{\partial^2 w_1}{\partial x_1^2}) - \rho_1 A_1 \ddot{w}_1 = 0 \qquad (b)$$

$$EI_2 \frac{\partial^2 \bar{\psi}_2}{\partial x_2^2} - \kappa_2^2 A_2 G(\bar{\psi}_2 + \frac{\partial \bar{w}_2}{\partial x_2}) - \rho_2 I_2 \ddot{\bar{\psi}}_2 = -\rho_2 I_2 \ddot{\psi}_{1(L_1,t)} \qquad (a)$$

$$\ldots\ldots \qquad (3.5.12)$$

$$\kappa_2^2 A_2 G(\frac{\partial \bar{\psi}_2}{\partial x_2} + \frac{\partial^2 \bar{w}_2}{\partial x_2^2}) - \rho_2 A_2 \ddot{\bar{w}}_2 = \rho_2 A_2 (\ddot{w}_{1(L_1,t)} - (L_2 - x_2)\ddot{\psi}_{1(L_1,t)}) \qquad (b)$$

and

$$EI_3 \frac{\partial^2 \bar{\psi}_3}{\partial x_3^2} - \kappa_3^2 A_3 G(\bar{\psi}_3 + \frac{\partial \bar{w}_3}{\partial x_3}) - \rho_3 I_3 \ddot{\bar{\psi}}_3 = \rho_3 I_3 \ddot{\psi}_{1(L_1,t)} \qquad (a)$$

$$\ldots\ldots \qquad (3.5.13)$$

$$\kappa_3^2 A_3 G(\frac{\partial \psi_3}{\partial x_3} + \frac{\partial^2 \bar{w}_3}{\partial x_3^2}) - \rho_3 A_3 \ddot{\bar{w}}_3 = \rho_3 A_3 (\ddot{w}_{1(L_1,t)} + (L_3 - x_3)\ddot{\psi}_{1(L_1,t)}) \qquad (b)$$

The boundary conditions for the system are:

$$
\left.
\begin{aligned}
&\text{at } x_1 = 0, \\
&\quad w_{1(0,t)} = \psi_{1(0,t)} = 0 \\
&\text{at } x_2 = 0, \\
&\quad \kappa_2^2 A_2 G \left(\bar{\psi}_2 + \frac{\partial \bar{w}_2}{\partial x_2}\right)_{(0,t)} = EI_2 \frac{\partial \bar{\psi}_2}{\partial x_2}_{(0,t)} = 0 \\
&\text{at } x_3 = 0, \\
&\quad \kappa_3^2 A_3 G \left(\bar{\psi}_3 + \frac{\partial \bar{w}_3}{\partial x_3}\right)_{(0,t)} = EI_3 \frac{\partial \bar{\psi}_3}{\partial x_{3(0,t)}} = 0
\end{aligned}
\right\} \quad (3.5.14)
$$

and the junction conditions are:

$$
\begin{aligned}
EI_1 \frac{\partial \psi_1}{\partial x_{1(L_1,t)}} = \int_0^{L_2} &[\rho_2 I_2(-\ddot{\psi}_{1(L_1,t)} + \ddot{\bar{\psi}}_2) + \rho_2 A_2(\ddot{w}_{1(L_1,t)} - (L_2 - x_2) \\
&\ddot{\psi}_{1(L_1,t)} + \ddot{\bar{w}}_2)(L_2 - x_2)]dx_2 - \int_0^{L_3} [\rho_3 I_3(\ddot{\psi}_{1(L_1,t)} + \ddot{\bar{\psi}}_3) \\
&+ \rho_3 A_3(\ddot{w}_{1(L_1,t)} + (L_3 - x_3)\ddot{\psi}_{1(L_1,t)} + \ddot{\bar{w}}_3)(L_3 - x_3)]dx_3,
\end{aligned}
$$

.... (3.5.15)

$$
\begin{aligned}
\kappa_1^2 A_1 G\left(\psi_1 + \frac{\partial w_1}{\partial x_1}\right)_{(L_1,t)} = -\int_0^{L_2} &\rho_2 A_2(\dddot{w}_{1(L_1,t)} - (L_2 - x_2)\ddot{\psi}_{1(L_1,t)} + \ddot{\bar{w}}_2)dx_2 \\
&- \int_0^{L_3} \rho_3 A_3(w_{1(L_1,t)} + (L_3 - x_3)\ddot{\psi}_{1(L_1,t)} + \ddot{\bar{w}}_3)dx_3,
\end{aligned}
$$

and

$$
\bar{w}_{2(L_2,t)} = \bar{\psi}_{2(L_2,t)} = \bar{w}_{3(L_3,t)} = \bar{\psi}_{3(L_3,t)} = 0 \quad (3.5.16)
$$

3.5.B Frequency Equation for the System Considered as a Collection of Timoshenko Beams (Model (VI)-a).

With the aid of (3.5.1), which defines the total displacements of the system, the equations of motion may be put into the following form:

$$EI_i\, \psi_i'' - \kappa_i^2 A_i G(\psi_i + w_i') - \rho_i I_i \ddot{\psi}_i = 0$$

$$\kappa_i^2 A_i G(\psi_i' + w_i'') - \rho_i A_i \ddot{w}_i = 0 \quad ; \qquad i = 1,2,3 \tag{3.5.17}$$

The boundary conditions become

$$\left.\begin{aligned} w_{1(0,t)} &= \psi_{1(0,t)} = 0 \\ \left(\psi_2 + \frac{\partial w_2}{\partial x_2}\right)_{(0,t)} &= \frac{\partial \psi_2}{\partial x_2}_{(0,t)} = 0 \\ \left(\psi_3 + \frac{\partial w_3}{\partial x_3}\right)_{(0,t)} &= \frac{\partial \psi_3}{\partial x_3}_{(0,t)} = 0 \end{aligned}\right\} \tag{3.5.18}$$

while the junction conditions are

$$EI_1\, \psi'_{1(L_1,t)} = \int_0^{L_2} [\rho_2 I_2 \ddot{\psi}_2 + \rho_2 A_2 \ddot{w}_2 (L_2 - x_2)]\, dx_2 - \int_0^{L_3} [\rho_3 I_3 \ddot{\psi}_3 + \rho_3 A_3 \ddot{w}_3 (L_3 - x_3)]\, dx_3 \,, \tag{3.5.19}$$

$$\kappa_1^2 A_1 G\, (\psi_1 + w_1')_{(L_1,t)} = -\int_0^{L_2} \rho_2 A_2 \ddot{w}_2 dx_2 - \int_0^{L_3} \rho_3 A_3 \ddot{w}_3 dx_3 \,, \tag{3.5.20}$$

and

$$
\begin{aligned}
w_{1(L_1,t)} &= w_{2(L_2,t)} = w_{3(L_3,t)} \\
\psi_{1(L_1,t)} &= -\psi_{2(L_2,t)} = \psi_{3(L_3,t)}
\end{aligned}
\tag{3.5.21}
$$

Equations (3.5.17) are the appropriate Timoshenko beam equations. The solution of these equations again may be obtained by using the method of separation of variables, see Appendix B. The solution is

$$
\begin{aligned}
w_{i(x,t)} &= W_i(x) \cdot \sin \omega t \\
\psi_{i(x,t)} &= \Psi_i(x) \cdot \sin \omega t
\end{aligned}
\tag{3.5.22}
$$

$$
W_{i(x)} = B_i \sinh \lambda_{1i} x_i + C_i \cosh \lambda_{1i} x_i + D_i \sin \lambda_{2i} x_i + F_i \cos \lambda_{2i} x_i \quad \ldots\ldots \tag{3.5.23}
$$

$$
\begin{aligned}
\Psi_{i(x)} = &-\frac{\alpha_i}{\lambda_{1i}} (B_i \cosh \lambda_{1i} x_i + C_i \sinh \lambda_{1i} x_i) \\
&+ \frac{\beta_i}{\lambda_{2i}} (D_i \cos \lambda_{2i} x_i - F_i \sin \lambda_{2i} x_i)
\end{aligned}
\tag{3.5.24}
$$

where

$$
\left.
\begin{aligned}
\lambda_{1i} &= \left[\left(\frac{b_i^4 \omega^4}{4} + a_i \omega^2 \right)^{\frac{1}{2}} - \left(\frac{b_i^2 \omega^2}{2} \right) \right]^{\frac{1}{2}} , \\
\lambda_{2i} &= \left[\left(\frac{b_i^4 \omega^4}{4} + a_i \omega^2 \right)^{\frac{1}{2}} + \left(\frac{b_i^2 \omega^2}{2} \right) \right]^{\frac{1}{2}} ; \\
b_i^2 &= \frac{\rho_i}{E} \left(1 + \frac{E}{\kappa_i^2 G} \right) ,
\end{aligned}
\right|
\tag{3.5.25}
$$

$$a_i = \frac{\rho_i}{E}\left(\frac{A_i}{I_i} - \frac{\rho_i \omega^2}{\kappa_i^2 G}\right),$$

$$\alpha_i = \left(\frac{\rho_i \omega^2}{\kappa_i^2 G} + \lambda_{1i}^2\right),$$

and

$$\beta_i = \left(\frac{\rho_i \omega^2}{\kappa_i^2 G} - \lambda_{2i}^2\right). \qquad (3.5.25)\ \text{(cont)}$$

$$(i = 1,2,3)$$

Now substituting equations (3.5.22) into equations (3.5.18) through (3.5.21) yields the following form for the boundary conditions and junction conditions on the mode functions.

$$W_{1(0)} = 0, \quad \Psi_{1(0)} = 0 \qquad (3.5.26)$$

$$\Psi'_{2(0)} = 0, \quad \Psi_{2(0)} + W'_{2(0)} = 0 \qquad (3.5.27)$$

$$\Psi'_{3(0)} = 0, \quad \Psi_{3(0)} + W'_{3(0)} = 0 \qquad (3.5.28)$$

$$W_{1(L_1)} = W_{2(L_2)} = W_{3(L_3)} \qquad (3.5.29)$$

$$\Psi_{1(L_1)} = -\Psi_{2(L_2)} = \Psi_{3(L_3)} \qquad (3.5.30)$$

$$EI_1\Psi'_{1(L_1)} + \rho_2\omega^2\int_0^{L_2}[\ I_2\Psi_2 + A_2\ W_2(L_2 - x_2)]\ dx_2$$

$$- \rho_3\omega^2\int_0^{L_3}[\ I_3\Psi_3 + A_3\ W_3(L_3 - x_3)]\ dx_3 = 0 \qquad (3.5.31)$$

$$\kappa_1^2 A_1 G(\Psi_1 + W_1')_{(L_1)} - \int_0^{L_2}\rho_2 A_2\omega^2 W_2 dx_2 - \int_0^{L_3}\rho_3 A_3\omega^2 W_3 dx_3 = 0 \qquad (3.5.32)$$

Proceeding as in section 3.4.B, we find that six of the twelve constants in equations (3.5.23) and (3.5.24) may be eliminated with the aid of equations (3.5.26) through (3.5.28). Using the conditions at the junction, (3.5.29) through (3.5.32), the following six equations are obtained.

$$B_1[\sinh\ \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}})(\frac{\alpha_1}{\beta_1})\ \sin\ \lambda_{21}L_1] + C_1[\cosh\ \lambda_{11}L_1 - \cos\ \lambda_{21}L_1]$$
$$- B_2[\sinh\ \lambda_{12}L_2 + (\frac{\lambda_{22}}{\lambda_{12}})\sin\ \lambda_{22}L_2] - C_2[\cosh\ \lambda_{12}L_2 - (\frac{\alpha_2}{\beta_2})\cos\ \lambda_{22}L_2] = 0,$$
$$\ldots\ldots \qquad (3.5.33)$$

$$B_2[\sinh\ \lambda_{12}L_2 + (\frac{\lambda_{22}}{\lambda_{12}})\ \sin\ \lambda_{22}L_2] + C_2[\cosh\ \lambda_{12}L_2 - (\frac{\alpha_2}{\beta_2})\ \cos\ \lambda_{22}L_2]$$
$$- B_3[\sinh\ \lambda_{13}L_3 + (\frac{\lambda_{23}}{\lambda_{13}})\sin\ \lambda_{23}L_3] - C_3[\cosh\ \lambda_{13}L_3 - (\frac{\alpha_3}{\beta_3})\cos\ \lambda_{23}L_3] = 0,$$
$$\ldots\ldots \qquad (3.5.34)$$

$$B_1[(\frac{\alpha_1}{\lambda_{11}})(\cosh\lambda_{11}L_1-\cos\lambda_{21}L_1)] + C_1[(\frac{\alpha_1}{\lambda_{11}})(\sinh\lambda_{11}L_1-(\frac{\lambda_{11}}{\lambda_{21}})(\frac{\beta_1}{\alpha_1})\sin\lambda_{21}L_1)]$$

$$+B_2[(\frac{\alpha_2}{\lambda_{12}})(\cosh\lambda_{12}L_2-(\frac{\beta_2}{\alpha_2})\cos\lambda_{22}L_2)]+C_2[(\frac{\alpha_2}{\lambda_{12}})(\sinh\lambda_{12}L_2-(\frac{\lambda_{12}}{\lambda_{22}})\sin\lambda_{22}L_2)] = 0, \quad (3.5.35)$$

$$B_2[(\frac{\alpha_2}{\lambda_{12}})(\cosh\lambda_{12}L_2-(\frac{\beta_2}{\alpha_2})\cos\lambda_{22}L_2)]+C_2[(\frac{\alpha_2}{\lambda_{12}})(\sinh\lambda_{12}L_2-(\frac{\lambda_{12}}{\lambda_{22}})\sin\lambda_{22}L_2)]$$

$$+B_3[(\frac{\alpha_3}{\lambda_{13}})(\cosh\lambda_{13}L_3-(\frac{\beta_3}{\alpha_3})\cos\lambda_{23}L_3)] + C_3[(\frac{\alpha_3}{\lambda_{13}})(\sinh\lambda_{13}L_3 - (\frac{\lambda_{13}}{\lambda_{23}})\sin\lambda_{23}L_3)] = 0, \quad (3.5.36)$$

$$B_1[EI_1\alpha_1(\sinh\lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}})\sin\lambda_{21}L_1)] + C_1[EI_1\alpha_1(\cosh\lambda_{11}L_1-(\frac{\beta_1}{\alpha_1})\cos\lambda_{21}L_1)]$$

$$+ B_2[\frac{\rho_2 I_2\omega^2\alpha_2}{\lambda_{12}^2}(\sinh\lambda_{12}L_2 - (\frac{\beta_2}{\alpha_2})(\frac{\lambda_{12}}{\lambda_{22}})\sin\lambda_{22}L_2) - \frac{\rho_2 A_2\omega^2}{\lambda_{12}^2}(\sinh\lambda_{12}L_2 - (\frac{\lambda_{12}}{\lambda_{22}})\sin\lambda_{22}L_2)]$$

$$+ C_2\{\frac{\rho_2 I_2\omega^2\alpha_2}{\lambda_{12}^2}[(\cosh\lambda_{12}L_2 + (\frac{\lambda_{12}}{\lambda_{22}})^2\cos\lambda_{22}L_2) - (1 + (\frac{\lambda_{12}}{\lambda_{22}})^2)] - \frac{\rho_2 A_2\omega^2}{\lambda_{12}^2}[(\cosh\lambda_{12}L_2+(\frac{\alpha_2}{\beta_2})(\frac{\lambda_{12}}{\lambda_{22}})^2\cos\lambda_{22}L_2)-(1 + \frac{\alpha_2}{\beta_2}(\frac{\lambda_{12}}{\lambda_{22}})^2)]\}$$

$$- B_3[\frac{\rho_3 I_3\omega^2\alpha_3}{\lambda_{13}^2}(\sinh\lambda_{13}L_3 - (\frac{\beta_3}{\alpha_3})(\frac{\lambda_{13}}{\lambda_{23}})\sin\lambda_{23}L_3) - \frac{\rho_3 A_3\omega^2}{\lambda_{13}^2}(\sinh\lambda_{13}L_3 - (\frac{\lambda_{13}}{\lambda_{23}})\sin\lambda_{23}L_3)]$$

$$- C_3\{\frac{\rho_3 I_3\omega^2\alpha_3}{\lambda_{13}^2}[(\cosh\lambda_{13}L_3 + (\frac{\lambda_{13}}{\lambda_{23}})^2\cos\lambda_{23}L_3) - (1 + (\frac{\lambda_{13}}{\lambda_{23}})^2)]$$

$$- \frac{\rho_3 A_3 \omega^2}{\lambda_{13}^2} \left[(\cosh \lambda_{13} L_3 + (\frac{\alpha_3}{\beta_3}) (\frac{\lambda_{13}}{\lambda_{23}})^2 \cos \lambda_{23} L_3) - (1 + \frac{\alpha_3}{\beta_3} (\frac{\lambda_{13}}{\lambda_{23}})^2) \right] \Big\} = 0, \qquad (3.5.37)$$

and finally,

$$B_1 \{\kappa_1^2 A_1 G[\cosh \lambda_{11} L_1 (\lambda_{11} - \frac{\alpha_1}{\lambda_{11}}) + \cos \lambda_{21} L_1 (\frac{\alpha_1}{\lambda_{11}}) (1 + \frac{\lambda_{21}^2}{\beta_1})]\}$$

$$+ \quad C_1 \{\kappa_1^2 A_1 G[\sinh \lambda_{11} L_1 (\lambda_{11} - \frac{\alpha_1}{\lambda_{11}}) + \sin \lambda_{21} L_1 (\lambda_{21} + \frac{\beta_1}{\lambda_{21}})]\}$$

$$- \quad B_2 \left[\frac{\rho_2 A_2 \omega^2}{\lambda_{12}} (\cosh \lambda_{12} L_2 - \cos \lambda_{22} L_2)\right]$$

$$- \quad C_2 \left[\frac{\rho_2 A_2 \omega^2}{\lambda_{12}} (\sinh \lambda_{12} L_2 - (\frac{\alpha_2}{\beta_2}) (\frac{\lambda_{12}}{\lambda_{22}}) \sin \lambda_{22} L_2)\right]$$

$$- \quad B_3 \left[\frac{\rho_3 A_3 \omega^2}{\lambda_{13}} (\cosh \lambda_{13} L_3 - \cos \lambda_{23} L_3)\right]$$

$$- \quad C_3 \left[\frac{\rho_3 A_3 \omega^2}{\lambda_{13}} (\sinh \lambda_{13} L_3 - (\frac{\alpha_3}{\beta_3}) (\frac{\lambda_{13}}{\lambda_{23}}) \sin \lambda_{23} L_3)\right] = 0 \qquad (3.5.38)$$

For the system to have nontrivial solutions, the determinant of the coefficient matrix must be equal to zero. From this the frequency equation and the frequencies may be obtained as before. Again, the frequency equation is of the form

$$\begin{vmatrix} a_{11} & a_{12} - - - & a_{16} \\ a_{21} & a_{22} & \\ a_{61} & & a_{66} \end{vmatrix} = 0 \qquad (3.5.39)^*$$

---

* See Appendix C.3

3.5.C Frequency Equation for the System when the Vessel is Considered to be Rigid (Model (IV)-a).

Now consider the pressure vessel to be a rigid body, i.e., the elastic deformations are set equal to zero. Since

$$\bar{w}_2 = \bar{w}_3 = \bar{\psi}_2 = \bar{\psi}_3 = 0, \tag{3.5.40}$$

we have as the equations of motion

$$EI_1 \frac{\partial^2 \psi_1}{\partial x_1^2} - \kappa_1^2 A_1 G(\psi_1 + \frac{\partial w_1}{\partial x_1}) - \rho_1 I_1 \ddot{\psi}_1 = 0$$

$$\kappa_1^2 A_1 G(\frac{\partial \psi_1}{\partial x_1} + \frac{\partial^2 w_1}{\partial x_1^2}) - \rho_1 A_1 \ddot{w}_1 = 0 \tag{3.5.41}$$

The boundary conditions are now

at $x_1 = 0$

$$w_{1(0,t)} = \psi_{1(0,t)} = 0 \tag{3.5.42}$$

and at $x_1 = L_1$

$$EI_1 \frac{\partial \psi_1}{\partial x}_{1(L_1,t)} = -\int_0^{L_2} [\rho_2 I_2 \ddot{\psi}_{1(L_1,t)} - \rho_2 A_2 (\ddot{w}_{1(L_1,t)} - (L_2-x_2)\ddot{\psi}_{1(L_1,t)})$$

$$(L_2-x_2)]\, dx_2$$

$$-\int_0^{L_3} [\rho_3 I_3 \ddot{\psi}_{1(L_1,t)} - \rho_3 A_3 (\ddot{w}_{1(L_1,t)} + (L_3-x_3)\ddot{\psi}_{1(L_1,t)})$$

$$(L_3-x_3)]\, dx_3 \tag{3.5.43}$$

$$\kappa_1^2 A_1 G \left(\psi_1 + \frac{\partial w_1}{\partial x_1}\right)_{(L_1,t)} = -\int_0^{L_2} \rho_2 A_2 (\ddot{w}_{1(L_1,t)} - (L_2-x_2)\ddot{\psi}_{1(L_1,t)})dx_2$$

$$-\int_0^{L_3} \rho_3 A_3 (\ddot{w}_{1(L_1,t)} + (L_3-x_3)\ddot{\psi}_{1(L_1,t)})dx_3 \qquad \ldots. \quad (3.5.44)$$

The solutions to equations (3.5.41) have been presented in the preceding section by equations (3.5.22) through (3.5.25), for i = 1, and hence

$$W_{1(x)} = B_1 \sinh \lambda_{11} x_1 + C_1 \cosh \lambda_{11} x_1 + D_1 \sin \lambda_{21} x_1 + F_2 \cos \lambda_{21} x_1$$

$$\Psi_{1(x)} = -\frac{\alpha_1}{\lambda_{11}}(B_1 \cosh \lambda_{11} x_1 + C_1 \sinh \lambda_{11} x_1) + \frac{\beta_1}{\lambda_{21}}(D_1 \cos \lambda_{21} x_1 - F_1 \sin \lambda_{21} x_1) \quad (3.5.45)$$

Substituting from equation (3.5.22) into equation (3.5.42) through (3.5.44) yields the following boundary conditions on the modal functions

$$W_{1(0)} = 0$$
$$\Psi_{1(0)} = 0 \quad (3.5.46)$$

and

$$EI_1 \Psi'_{1(L_1)} - R_1 \Psi_{1(L_1)} + R_2 W_{1(L_1)} = 0$$
$$R_3 W_{1(L_1)} - \kappa_1 A_1 G_1 W'_{(L_1)} - R_4 \Psi_{1(L_1)} = 0 \quad (3.5.47)$$

where $R_1 = \rho_2 A_2 \omega^2 [(\frac{L_2^3 + L_3^3}{3}) + \frac{I_2}{A_2} (L_2 + L_3)]$,

$$R_2 = \rho_2 A_2 \omega^2 (\frac{L_2^2 - L_3^2}{2}),$$

$$R_3 = \rho_2 A_2 \omega^2 (L_2 + L_3), \qquad (3.5.48)$$

and $R_4 = \kappa_1^2 A_1 G + \rho_2 A_2 \omega^2 (\frac{L_2^2 - L_3^2}{2})$.

Note that segments 2 and 3 are assumed to have the same cross sectional area and the same mass density.

Now in equations (3.5.45), subject to boundary conditions (3.5.46), two of the constants can be eliminated and then, with the aid of boundary conditions (3.5.47), the following two equations are obtained.

$$B_1[R_1(\frac{\alpha_1}{\lambda_{11}})(\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1) + R_2(\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}})(\frac{\alpha_1}{\beta_1})\sin \lambda_{21}L_1)$$
$$- EI_1\alpha_1(\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}}) \sin \lambda_{21}L_1)]$$
$$+ C_1[R_1(\frac{\alpha_1}{\lambda_{11}})(\sinh \lambda_{11}L_1 - (\frac{\lambda_{11}}{\lambda_{21}})(\frac{\beta_1}{\alpha_1})\sin \lambda_{21}L_1) + R_2(\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1)$$
$$- EI_1\alpha_1(\cosh \lambda_{11}L_1 - \frac{\beta_1}{\alpha_1} \cos \lambda_{21}L_1)] = 0 \qquad (3.5.49)$$

$$B_1[R_3(\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}})(\frac{\alpha_1}{\beta_1})\sin \lambda_{21}L_1) + R_4(\frac{\alpha_1}{\lambda_{11}})(\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1)$$
$$- R_5(\cosh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}})^2 (\frac{\alpha_1}{\beta_1}) \cos \lambda_{21}L_1)]$$

$$+ C_1[R_3(\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1) + R_4(\frac{\alpha_1}{\lambda_{11}})(\sinh \lambda_{11}L_1 - (\frac{\lambda_{11}}{\lambda_{21}})(\frac{\beta_1}{\alpha_1})\sin \lambda_{21}L_1)$$

$$- R_5(\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}}) \sin \lambda_{21}L_1)] = 0 \qquad (3.5.50)$$

where $R_5 = \kappa_1^2 A_1 G \lambda_{11}$

Again, for the system to have a nontrivial solution, the determinant of the coefficient matrix in equations (3.5.49) and (3.5.50) must vanish.

Hence the frequency equation is of the simpler form

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \qquad (3.5.51)^*$$

3.6 Analysis of Model (I)-b.

We now turn our attention to the case when the pressure vessel is pinned at the top as shown in Fig. (1.1.b). In this case the system is reduced to a single degree-of-freedom, namely the angle of the rotation of the rigid vessel about an axis through the pin. This angle, (ϕ) Fig. (3.6.a), is given by

$$\phi = -\frac{dw_1}{dx_1}_{(L_1)} = \frac{w_1(L_1)}{L_2} \qquad (3.6.1)$$

---

* See Appendix C.4.

Elementary rigid body mechanics shows that, for the system shown in Fig. (3.6.b)

$$\frac{m}{3}(L_2 + L_3)^2 \ddot{\phi} = -[\bar{M} + \bar{Q} \cdot L_2] \quad (3.6.2)$$

The formulae of elementary beam theory provide

$$w_{1(L_1)} = \frac{\bar{Q}L_1^3}{3EI} - \frac{\bar{M}L_1^2}{2EI} \quad (3.6.3)$$

and
$$\frac{dw_1}{dx_1}\bigg|_{(L_1)} = \frac{\bar{Q}L_1^2}{2EI} - \frac{\bar{M}L_1}{EI} \quad (3.6.4)$$

With the aid of equations (3.6.1), (3.6.3), and (3.6.4) the following expressions for $\bar{M}$ and $\bar{Q}$ may be obtained

$$\bar{M} = \frac{2EI}{L_1^2}(2L_1 + 3L_2)\,\phi \quad (3.6.5)$$

and

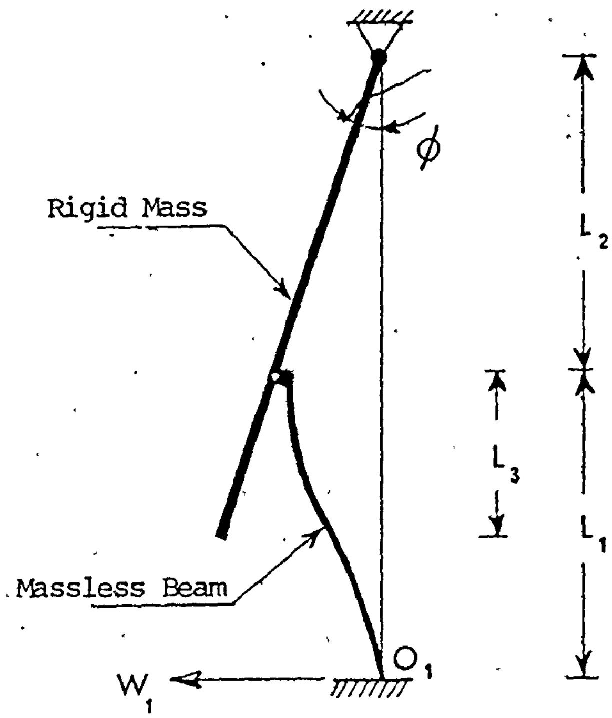
$$\bar{Q} = \frac{6EI}{L_1^3}(L_1 + 2L_2)\,\phi \quad (3.6.6)$$

Now using equations (3.6.5) and (3.6.6) the equation of motion (3.6.2) becomes

$$\ddot{\phi} + \frac{12EI}{mL_1^3}\,\frac{(L_1^2 + 3L_1L_2 + 3L_2^2)}{(L_2 + L_3)^2}\,\phi = 0 \quad (3.6.7)$$

so that the natural frequency of Model (I)-b is

$$\omega = \left[\frac{12EI}{mL_1^3} \cdot \frac{(L_1^2 + 3L_1L_2 + 3L_2^2)}{(L_2 + L_3)^2}\right]^{1/2} \quad (3.6.8)$$



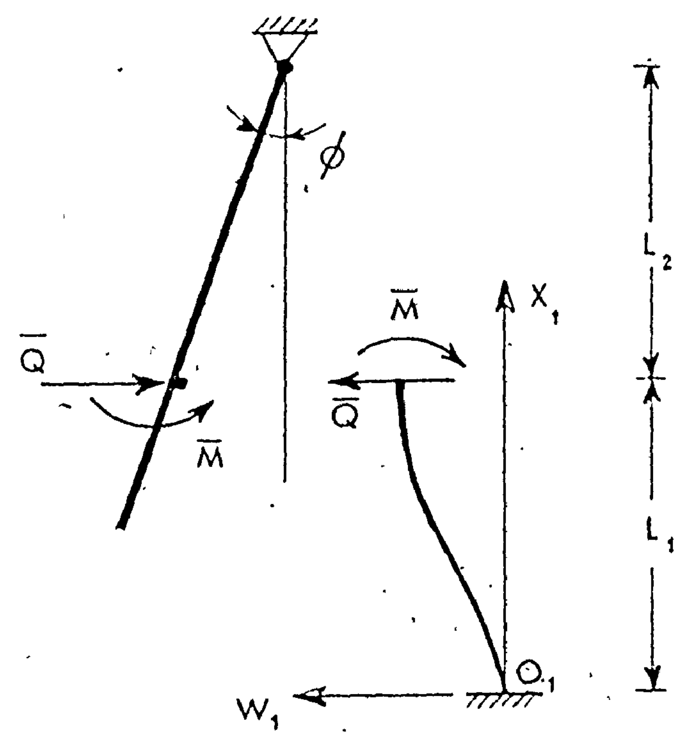


FIGURE 3.6: SINGLE DEGREE-OF-FREEDOM SYSTEM, MODELS (I & II)-b.

3.7 Analysis of Model (II)-b.

Again a rigid pressure vessel, pinned at its top, is considered, but now it is supported by a skirt modeled as a massless Timoshenko beam. As in the case of Model (I)-b, for this single degree-of-freedom system,

$$\phi = \psi_1(L_1) = \frac{w_1(L_1)}{L_2} \qquad (3.7.1)$$

Now it is possible, as in Model (I)-b, to write expressions for $\bar{M}$ and $\bar{Q}$.

$$\bar{M} = \frac{6EI}{L_1} \left[\frac{L_1L_2 + 2(L_1^3/3 + \alpha)}{12\alpha + L_1^2} \right] \phi \qquad (3.7.2)$$

and

$$\bar{Q} = \frac{6EI}{L_1} \left[\frac{L_1 + 2L_2}{12\alpha + L_1^2} \right] \phi \qquad (3.7.3)$$

where $\alpha = EI/\kappa^2AG$

The equation of motion is

$$\frac{m}{3}(L_2 + L_3)^2 \ddot{\phi} = -[\bar{M} + \bar{Q} \cdot L_2] \qquad (3.7.4)$$

or

$$\ddot{\phi} + \frac{36EI}{mL_1(L_2+L_3)^2} \left[\frac{L_1L_2 + L_2^2 + L_1^2/3 + \alpha}{12\alpha + L_1^2} \right] \phi = 0 \qquad (3.7.5)$$

so that the natural frequency is

$$\omega = \left[\frac{36EI}{mL_1} \frac{L_1L_2 + L_2^2 + L_1^2/3 + \alpha}{(12\alpha + L_1^2)(L_2 + L_3)^2} \right]^{1/2} \qquad (3.7.6)$$

Equation (3.7.6) reduces to equation (3.6.8) if $\kappa^2 AG \to \infty$, i.e., $\alpha \to \infty$.

3.8 Analysis of Models (III)-b and (IV)-b.

In these two cases a rigid pressure vessel, pinned at its top, is supported by a flexible skirt with mass. Since the role of these two models is, again, to assess the validity of the assumption made in Models (I)-a through (II)-b that the mass of the skirt may be neglected for typical real systems and since that assumption can be assessed by Models (III)-a and (IV)-a, no separate detailed analysis for these two models will be given and the interested reader may obtain the frequency determinants for these two models by following the same procedure described in sections 3.4 and 3.5 and imposing the following boundary conditions. For

a) Model (III)-b

$$\frac{\partial w_1}{\partial x_{1\,(L_1)}} = - \frac{w_{1\,(L_1)}}{L_2}$$

and for

b) Model (IV)-b

$$\psi_{1\,(L_1)} = \frac{w_{1\,(L_1)}}{L_2}$$

3.9 Analysis of Model (V)-b.

This case differs from the case of Model (V)-a in only one particular, namely the boundary conditions at the top of the pressure vessel, i.e. at $x_2 = 0$, equations (3.4.17)(b) are replaced by

$$\begin{aligned} W_{2(0)} &= 0 \\ W''_{2(0)} &= 0 \end{aligned} \qquad (3.9.1)$$

since a free end is now replaced by a pinned end. Consequently the only change in the modal functions is that $W_{2(x)}$ is given now by

$$W_{2(x_2)} = B_2 \sin a_2 x_2 + D_2 \sinh a_2 x_2 \qquad (3.9.2)$$

instead of equation (3.4.21)(b).

Once again, substituting equations (3.4.21)(a) and (c), and (3.9.2) into the boundary conditions at the junction, equations (3.4.18), (3.4.19) and (3.4.20), yields the following six equations.

$$B_1(\sin a_1L_1 - \sinh a_1L_1) + C_1(\cos a_1L_1 - \cosh a_1L_1) - B_2 \sin a_2L_2 - D_2 \sinh a_2L_2 = 0, \qquad (3.9.3)$$

$$B_1(\cos a_1L_1 - \cosh a_1L_1) - C_1(\sin a_1L_1 + \sinh a_1L_1) + B_2(\frac{a_2}{a_1} \cdot \cos a_2L_2) + D_2(\frac{a_2}{a_1} \cdot \cosh a_2L_2) = 0, \qquad (3.9.4)$$

$$B_2 \sin a_2L_2 + D_2 \sinh a_2L_2 - B_3(\sin a_3L_3 + \sinh a_3L_3) - C_3(\cos a_3L_3 + \cosh a_3L_3) = 0, \qquad (3.9.5)$$

$$B_2 \cos a_2L_2 + D_2 \cosh a_2L_2 + B_3(\cos a_3L_3 + \cosh a_3L_3)$$
$$- C_3(\sin a_3L_3 - \sinh a_3L_3) = 0, \qquad (3.9.6)$$

$$B_1(\sin a_1L_1 + \sinh a_1L_1) + C_1(\cos a_1L_1 + \cosh a_1L_1)$$
$$- B_2(\sin a_2L_2 - a_2L_2) R_1 + D_2(\sinh a_2L_2 - a_2L_2)R_1$$
$$+ B_3(\sin a_3L_3 - \sinh a_3L_3)R_2 + C_3(\cos a_3L_3 - \cosh a_3L_3)R_2$$
$$= 0, \qquad (3.9.7)$$

and

$$B_1(\cos a_1L_1 + \cosh a_1L_1) - C_1(\sin a_1L_1 - \sinh a_1L_1)$$
$$+ B_2(\cos a_2L_2 - 1)R_3 - D_2(\cosh a_2L_2 - 1)R_3$$
$$+ B_3(\cos a_3L_3 - \cosh a_3L_3)R_4 - C_3(\sin a_3L_3 + \sinh a_3L_3)R_4$$
$$= 0. \qquad (3.9.8)$$

where R_1 through R_4 have been defined in section (3.4).

As before, the frequency equation may be obtained by setting determinant of the coefficient matrix for equations (3.9.3) through (3.9.8) equal to zero*.

As previously mentioned, Model (III)-b is obtained from Model (V)-b by again letting $\bar{w}_2 = \bar{w}_3 = 0$ at the appropriate point in the analysis.

3.10 Analysis of Model (VI)-b.

Finally, as in the case of Model (V)-b, only the boundary conditions at $x_2 = 0$ need be changed in the analysis of Model (VI)-a.

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* See Appendix C.5

The appropriate boundary conditions are

$$w_{2(0,t)} = \psi'_{2(0,t)} = 0$$

which lead to the following six equations for the constants of integration.

$$B_1[\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}})(\frac{\alpha_1}{\beta_1}) \sin \lambda_{21}L_1] + C_1[\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1]$$
$$- B_2 \sinh \lambda_{12}L_2 - D_2 \sin \lambda_{22}L_2 = 0, \quad (3.10.1)$$

$$B_2 \sinh \lambda_{12}L_2 + D_2 \sin \lambda_{22}L_2 - B_3[\sinh \lambda_{13}L_3 + (\frac{\lambda_{23}}{\lambda_{13}}) \sin \lambda_{23}L_3]$$
$$- C_3[\cosh \lambda_{13}L_3 - (\frac{\alpha_3}{\beta_3}) \cos \lambda_{23}L_3] = 0, \quad (3.10.2)$$

$$B_1[(\frac{\alpha_1}{\lambda_{11}})(\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1)] + C_1[(\frac{\alpha_1}{\lambda_{11}})(\sinh \lambda_{11}L_1 - (\frac{\lambda_{11}}{\lambda_{21}})$$
$$(\frac{\beta_1}{\alpha_1}) \sin \lambda_{21}L_1)] + B_2[(\frac{\alpha_2}{\lambda_{12}}) \cosh \lambda_{12}L_2] - D_2[(\frac{\beta_2}{\lambda_{22}}) \cos \lambda_{22}L_2] = 0,$$
$$\ldots (3.10.3)$$

$$B_2[(\frac{\alpha_2}{\lambda_{12}}) \cosh \lambda_{12}L_2] - D_2[(\frac{\beta_2}{\lambda_{22}}) \cos \lambda_{22}L_2]$$
$$+ B_3[(\frac{\alpha_3}{\lambda_{13}})(\cosh \lambda_{13}L_3 - (\frac{\beta_3}{\alpha_3}) \cos \lambda_{23}L_3)]$$
$$+ C_3[(\frac{\alpha_3}{\lambda_{13}})(\sinh \lambda_{13}L_3 - (\frac{\lambda_{13}}{\lambda_{23}}) \sin \lambda_{23}L_3)] = 0, \quad (3.10.4)$$

$$B_1[EI_1\alpha_1(\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}}) \sin \lambda_{21}L_1)]$$
$$+ C_1[EI_1\alpha_1(\cosh \lambda_{11}L_1 - (\frac{\beta_1}{\alpha_1}) \cos \lambda_{21}L_1)]$$

$$+ B_2[\frac{\rho_2 I_2 \omega^2 \alpha_2}{\lambda_{12}^2} \sinh \lambda_{12}L_2 - \frac{\rho_2 A_2 \omega^2}{\lambda_{12}^2} (\sinh \lambda_{12}L_2 - \lambda_{12}L_2)]$$

$$- D_2[\frac{\rho_2 I_2 \omega^2 \alpha_2}{\lambda_{12}^2} (\frac{\lambda_{12}}{\lambda_{22}})^2 (\frac{\beta_2}{\alpha_2}) \sin \lambda_{22}L_2 - \frac{\rho_2 A_2 \omega^2}{\lambda_{12}^2} (\frac{\lambda_{12}}{\lambda_{22}})^2 (\sin \lambda_{22}L_2 - \lambda_{22}L_2)]$$

$$- B_3[\frac{\rho_3 I_3 \omega^3 \alpha_3}{\lambda_{13}^2} (\sinh \lambda_{13}L_3 - (\frac{\beta_3}{\alpha_3})(\frac{\lambda_{13}}{\lambda_{23}}) \sin \lambda_{23}L_3)$$

$$- \frac{\rho_3 A_3 \omega^2}{\lambda_{13}^2} (\sinh \lambda_{13}L_3 - (\frac{\lambda_{13}}{\lambda_{23}}) \sin \lambda_{23}L_3)] .$$

$$- C_3\left[\frac{\rho_3 I_3 \omega^2 \alpha_3}{\lambda_{13}^2}\left(\left(\cosh \lambda_{13}L_3 + (\frac{\lambda_{13}}{\lambda_{23}})^2 \cos \lambda_{23}L_3\right) - \left(1 + (\frac{\lambda_{13}}{\lambda_{23}})^2\right)\right)\right.$$

$$- \frac{\rho_3 A_3 \omega^2}{\lambda_{13}^2}\left(\left(\cosh \lambda_{13}L_3 + (\frac{\alpha_3}{\beta_3})(\frac{\lambda_{13}}{\lambda_{23}})^2 \cos \lambda_{23}L_3\right)\right.$$

$$\left.\left. - \left(1 + \frac{\alpha_3}{\beta_3} (\frac{\lambda_{13}}{\lambda_{23}})^2\right)\right)\right] = 0, \tag{3.10.5}$$

and, finally,

$$B_1[\kappa_1^2 A_1 G\left(\cosh \lambda_{11}L_1 (\lambda_{11} - \frac{\alpha_1}{\lambda_{11}}) + \cos \lambda_{21}L_1 (\frac{\alpha_1}{\lambda_{11}})(1 + \frac{\lambda_{21}^2}{\beta_1})\right)]$$

$$+ C_1[\kappa_1^2 A_1 G\left(\sinh \lambda_{11}L_1 (\lambda_{11} - \frac{\alpha_1}{\lambda_{11}}) + \sin \lambda_{21}L_1 (\lambda_{21} + \frac{\beta_1}{\lambda_{21}})\right)]$$

$$- B_2[\frac{\rho_2 A_2 \omega^2}{\lambda_{12}} (\cosh \lambda_{12}L_2 - 1)]$$

$$+ D_2[\frac{\rho_2 A_2 \omega^2}{\lambda_{12}} (\frac{\lambda_{12}}{\lambda_{22}}) (\cos \lambda_{22}L_2 - 1)]$$

$$- B_3[\frac{\rho_3 A_3 \omega^2}{\lambda_{13}} (\cosh \lambda_{13} L_3 - \cos \lambda_{23} L_3)]$$

$$- C_3[\frac{\rho_3 A_3 \omega^2}{\lambda_{13}} (\sinh \lambda_{13} L_3 - (\frac{\alpha_3}{\beta_3})(\frac{\lambda_{13}}{\lambda_{23}}) \sin \lambda_{23} L_3)] = 0. \quad (3.10.6)$$

Again equations (3.10.1) through (3.10.6) provide the frequency equation* for Model (VI)-b. Also Model (IV)-b is obtained from Model (VI)-b by letting $\bar{w}_2 = \bar{w}_3 = \bar{\psi}_2 = \bar{\psi}_3 = 0$ in equations (3.5.1) and proceeding in the analysis by the same procedure as before.

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* See Appendix C.6.

CHAPTER 4

NUMERICAL EXAMPLES AND RESULTS.

We now present numerical results for four typical skirt-vessel external geometries and for each of these four different vessel wall thicknesses are considered, i.e. a total of sixteen geometries are examined. Each of these is considered for both sets of boundary conditions. Only the first two natural frequencies, model permitting, are calculated for each case and model, since the higher frequencies usually are not of interest. If required, the third and higher frequencies may be calculated easily using the same procedure which is used to calculate the first two frequencies. The computer program used was based on a standard Fortran IV subroutine for evaluating determinants and required no great programming effort since it merely involved a marching routine to determine values of the frequency for which the determinant vanished, i.e. at which it changed sign, see Appendix D.

In all cases the vessel and skirt were assumed to be made of steel with a specific weight of 490 lbs/ft^3, a Young's modulus of 29 × 10^6 psi., and a Poisson's ratio of 0.3*. The specific weight of the vessel's contents was taken to be 105 lbs/ft^3., i.e. 90% water by volume and 10% steel. The first skirt-vessel combination considered was

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* Poisson's ratio is needed to find κ^2[6]

A) $L_1 = 6$ ft., $L_2 = 6$ ft., $L_3 = 4$ ft.,and $R = 2$ ft.

where R is the outer radius of the vessel and the mean radius of the skirt. The other cases were

B) $L_1 = 6$ ft., $L_2 = 10$ ft., $L_3 = 5$ ft., and $R = 2$ ft.

C) $L_1 = 6$ ft., $L_2 = 4$ ft., $L_3 = 6$ ft., and $R = 2$ ft.

and

D) $L_1 = 6$ ft., $L_2 = 7$ ft., $L_3 = 8$ ft., and $R = 2$ ft.

The skirt thickness is always taken to be 0.5 in. while the vessel thickness is considered to be 1 in., 2 in., 4 in., or 6 in. The first thickness representing, perhaps, a boiler and the last one a nuclear reactor.

The results are presented in eight tables below, each of which gives the first two natural frequencies for each model of a given case for the four vessel thicknesses considered. We find it convenient, for ease of presentation, to limit each table to either the cantilevered system or to the fixed-pinned system. The upper number in each entry is the fundamental natural frequency, in Hz., of the example considered and the lower number is the second frequency.

Figures (4.1) through (4.8) show the first two mode shapes for the various models of case A for both the cantilevered and fixed-pinned systems when thickness is 1 in. These are normalized with respect to W_{max} in each case. For those models where shear deformations are taken into account this provides the relative magnitude of Ψ as well. The reader should note the dramatic differences in mode shapes for the various models of the fixed-pinned system as opposed to the comparatively minor differences in the mode shapes for the various models of the cantilevered system. A cautionary note. While the mode shapes for W do indeed provide the shape of the deflected neutral surfaces of the system, there is no comparably simple way of visualizing the mode shapes for Ψ which is, recall, the angle of rotation of a cross-section of the skirt or vessel.

TABLE 1

Model / Thickness	I	II	III	IV	V	VI
1 in.	41.5 218.9	36.1 117.2	41.3 214.0	35.1 105.7	40.6 193.0	34.2 95.5
2 in.	36.1 190.3	31.4 101.8	35.9 187.1	30.6 92.6	35.6 176.5	30.2 87.5
4 in.	30.0 158.0	26.1 84.6	29.9 156.0	25.5 77.7	29.5 149.6	25.4 75.2
6 in.	26.5 139.9	23.1 74.9	26.5 138.6	22.6 69.2	26.4 135.0	22.5 67.6

Natural Frequencies for Cantilevered System

Case-A

TABLE 2

Model / Thickness	I	II	III	IV	V	VI
1 in.	195.3 –	96.6 –	190.1 2472	91.7 613.3	63.4 221.5	51.0 118.7
2 in.	169.8 –	83.9 –	166.2 2464	80.3 611.0	55.4 202.7	44.7 109.7
4 in.	141.0 –	69.7 –	139.0 2450	67.2 608.8	45.7 171.9	38.0 95.0
6 in.	124.8 –	61.7 –	123.4 2446	59.8 607.7	40.9 155.1	33.2 85.8

Natural Frequencies for Fixed-Pinned System.

Case-A.

TABLE 3

Model / Thickness	I	II	III	IV	V	VI
1 in.	24.4 165.5	22.8 82.5	24.4 162.2	22.4 78.4	23.3 137.8	21.1 70.2
2 in.	21.2 143.9	19.8 71.7	21.2 141.7	19.5 68.5	20.7 129.3	18.9 64.4
4 in.	17.6 119.5	16.4 59.6	17.6 118.3	16.2 57.3	17.2 111.1	15.9 55.3
6 in.	15.6 105.8	14.6 52.8	15.6 105.0	14.4 50.9	15.4 100.7	14.2 49.6

Natural Frequencies for Cantilevered System.
Case-B

TABLE 4

Model / Thickness	I	II	III	IV	V	VI
1 in.	152.1 -	73.0 -	149.1 2455	70.8 610.8	42.0 158.1	36.8 83.9
2 in.	132.3 -	63.5 -	130.4 2450	61.9 609.1	36.6 146.4	32.3 77.4
4 in.	109.8 -	52.7 -	108.5 2445	51.6 607.4	30.2 124.7	27.0 66.6
6 in.	97.2 -	46.7 -	96.4 2442	45.9 606.6	27.1 112.9	23.9 59.9

Natural Frequencies for Fixed-Pinned System.
Case-B.

TABLE 5

Model / Thickness	I	II	III	IV	V	VI
1 in.	58.0 156.8	47.0 90.0	57.5 154.9	45.2 82.5	57.1 126.1	44.5 71.3
2 in.	50.5 136.0	40.9 78.2	50.1 135.0	39.5 72.1	49.9 119.9	39.2 66.3
4 in.	41.9 113.0	33.9 65.0	41.7 112.3	33.0 60.2	41.2 103.8	32.8 57.5
6 in.	37.1 100.0	30.0 57.5	37.0 99.6	29.3 53.6	36.9 94.3	29.2 51.7

Natural Frequencies for Cantilevered System.

Case-C.

TABLE 6

Model / Thickness	I	II	III	IV	V	VI
1 in.	153.7 –	78.7 –	151.5 2450	75.5 608.8	79.7 126.3	56.4 81.5
2 in.	133.6 –	68.4 –	134.2 2446	65.9 607.6	69.7 120.4	50.1 76.0
4 in.	110.9 –	56.8 –	109.9 2442	55.0 606.4	57.6 104.5	42.1 66.1
6 in.	98.2 –	50.3 –	97.7 2440	49.2 605.8	51.6 95.1	37.5 59.8

Natural Frequencies for Fixed-Pinned System.

Case-C.

TABLE 7

Model / Thickness	I	II	III	IV	V	VI
1 in.	34.2 117.9	31.6 59.4	34.1 116.7	30.7 57.5	33.4 85.6	29.7 48.8
2 in.	29.8 102.5	27.5 51.7	29.7 101.7	26.8 50.2	29.3 84.7	26.3 45.7
4 in.	24.7 85.1	22.8 42.9	24.7 84.7	22.3 41.8	24.3 75.2	22.1 39.6
6 in.	21.9 75.4	20.2 38.0	21.9 75.0	19.8 37.1	20.7 69.0	19.6 35.7

Natural Frequencies for Cantilerered System.
Case-D.

TABLE 8

Model / Thickness	I	II	III	IV	V	VI
1 in.	117.7 –	57.6 –	116.5 2445	56.4 607.6	50.8 87.8	45.9 50.1
2 in.	102.3 –	50.1 –	101.5 2442	49.1 606.6	44.7 85.5	40.5 47.2
4 in.	85.0 –	41.6 –	84.5 2440	40.9 605.7	37.1 75.4	33.8 41.2
6 in.	75.2 –	36.8 –	74.9 2438	36.3 605.3	33.2 69.1	30.0 37.1

Natural Frequencies for Fixed-Pinned System.
Case-D.

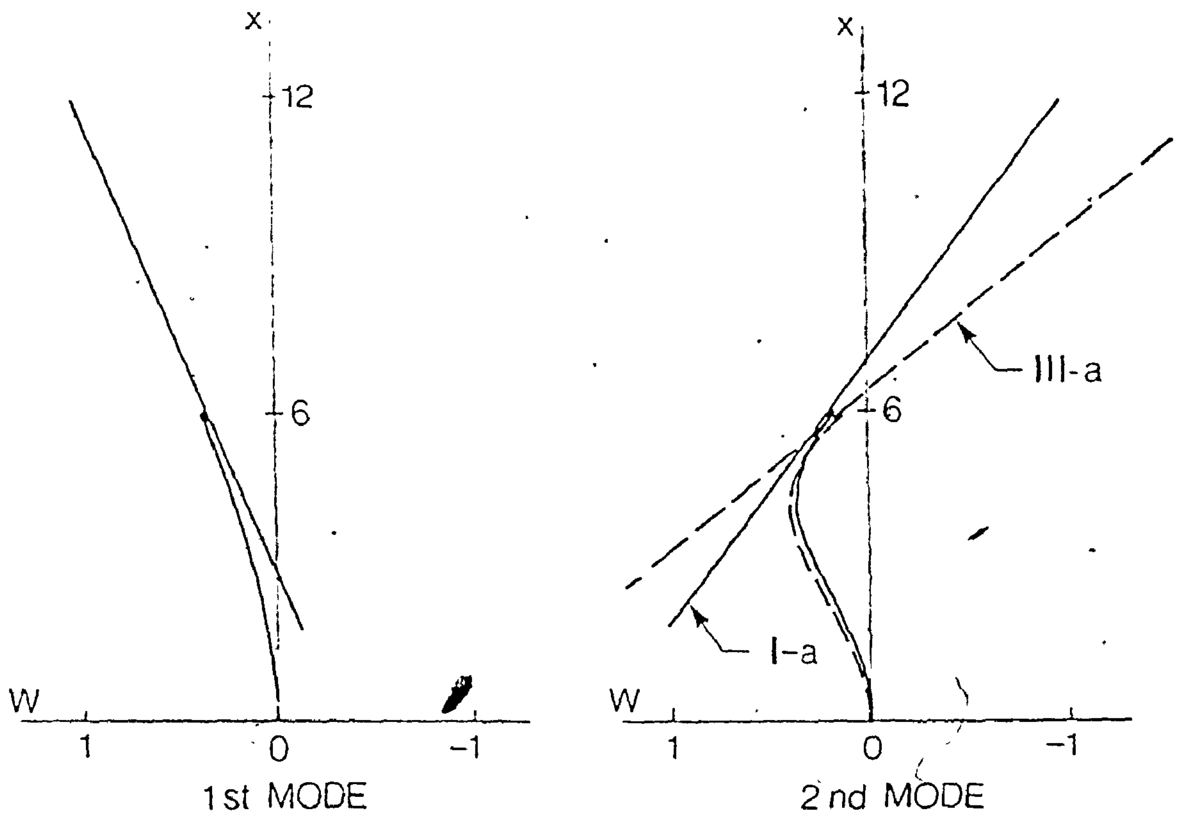

FIGURE 4.1: MODE SHAPES FOR MODELS (I)-a AND (III)-a, CASE-A.

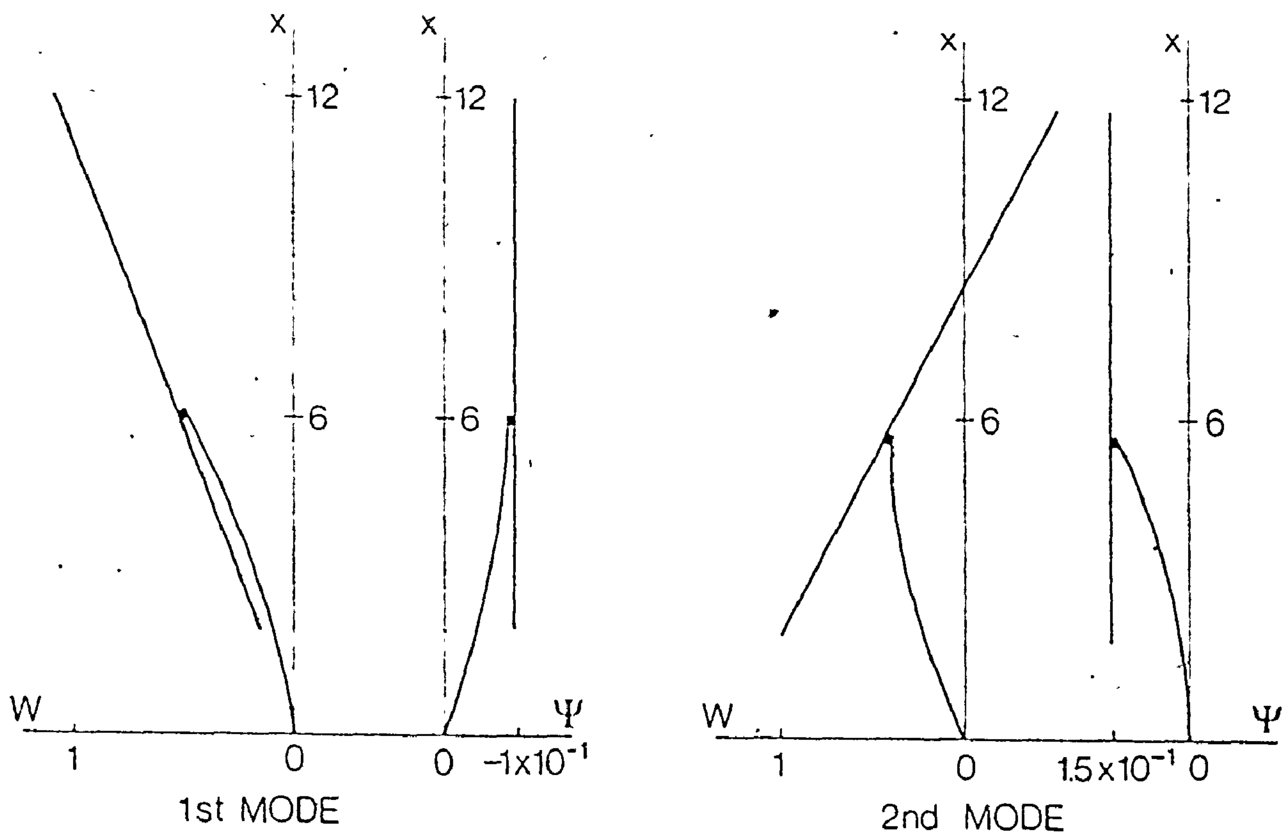

FIGURE 4.2: MODE SHAPES FOR MODELS (II)-a AND (IV)-a , CASE-A.

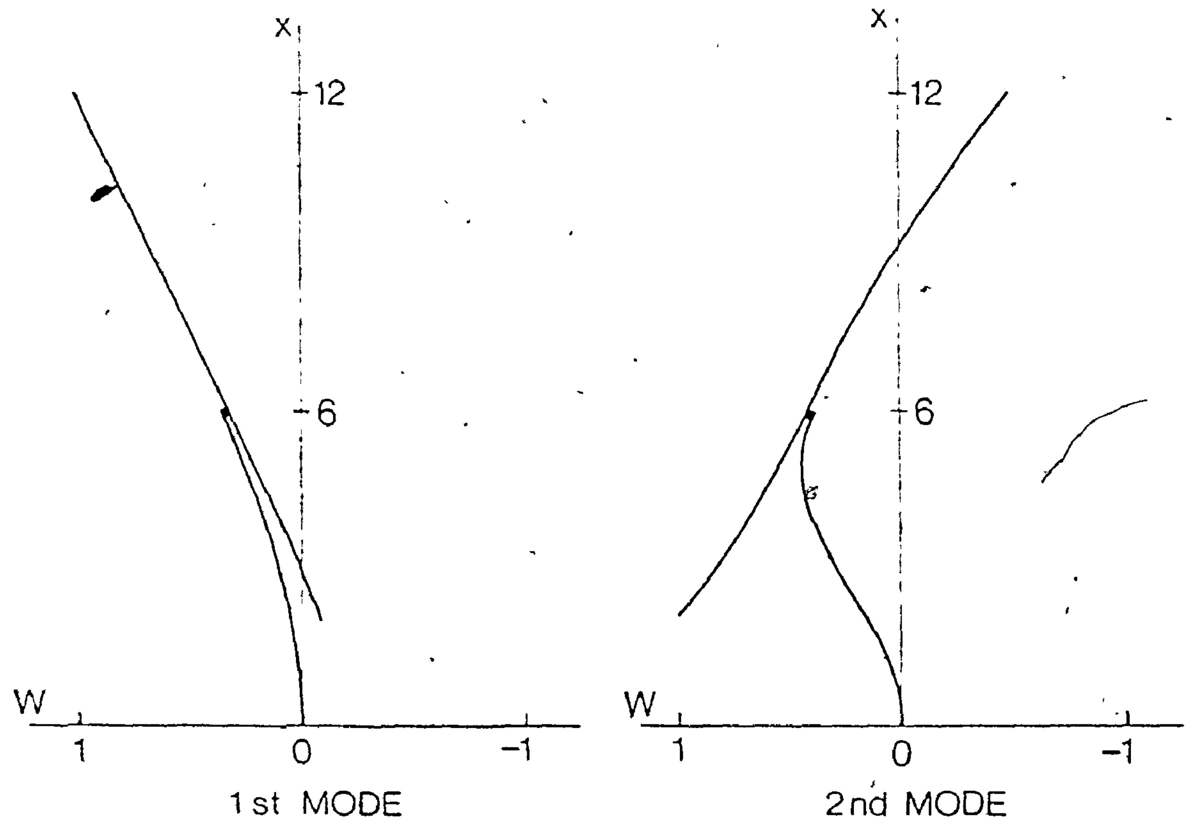

FIGURE 4.3: MODE SHAPES FOR MODEL (V)-a, CASE-A.

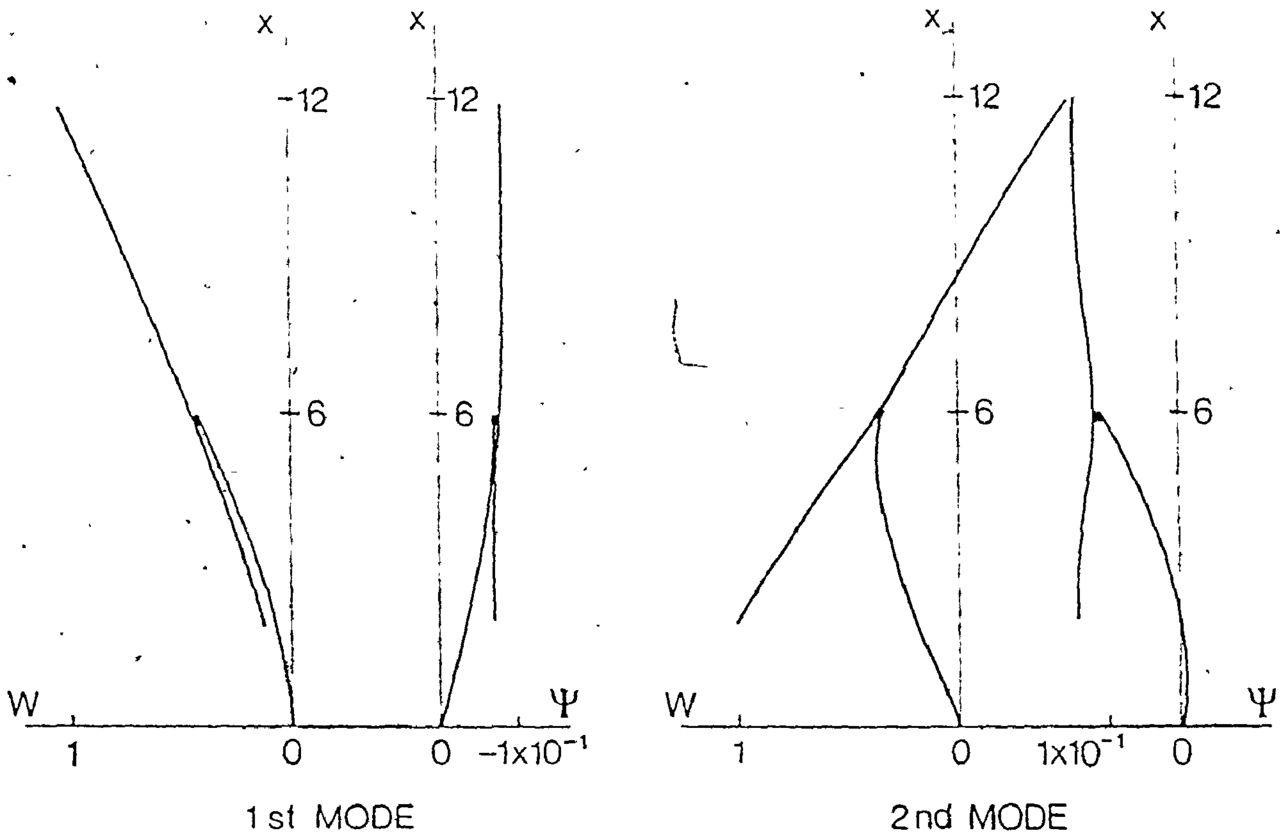

FIGURE 4.4: MODE SHAPES FOR MODEL (VI)-a, CASE-A.

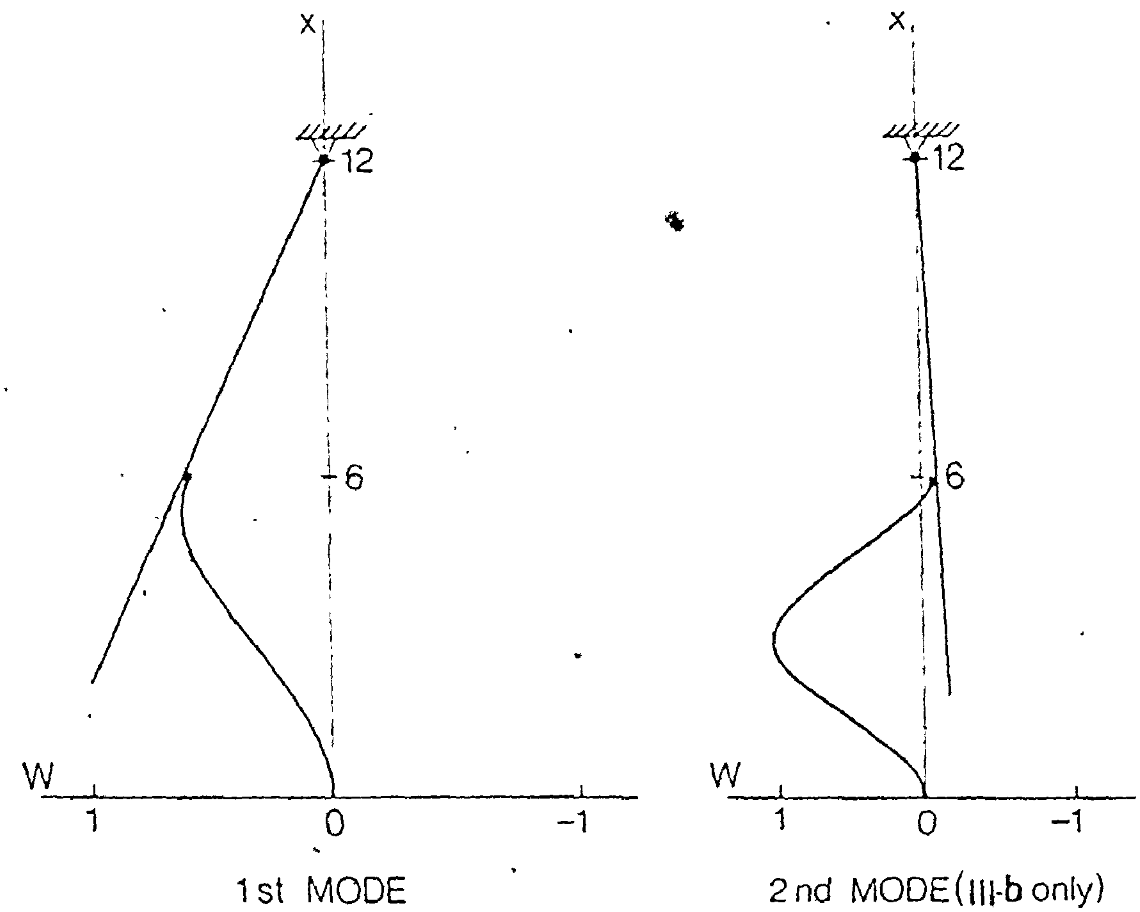

FIGURE 4.5: MODE SHAPES FOR MODELS (I)-b AND (III)-b, CASE-A.

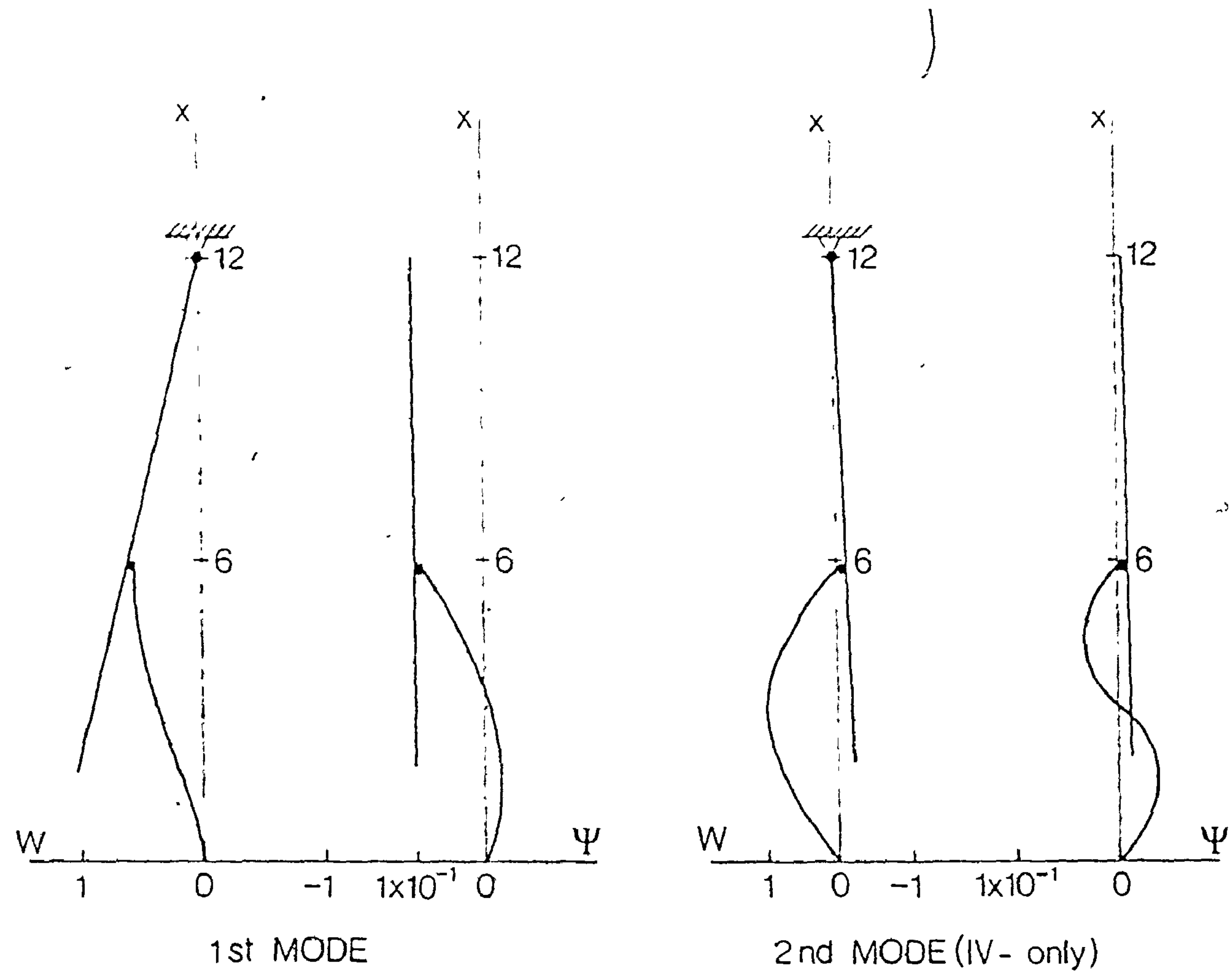

FIGURE 4.6: MODE SHAPES FOR MODELS (II)-b AND (IV)-b, CASE-A.

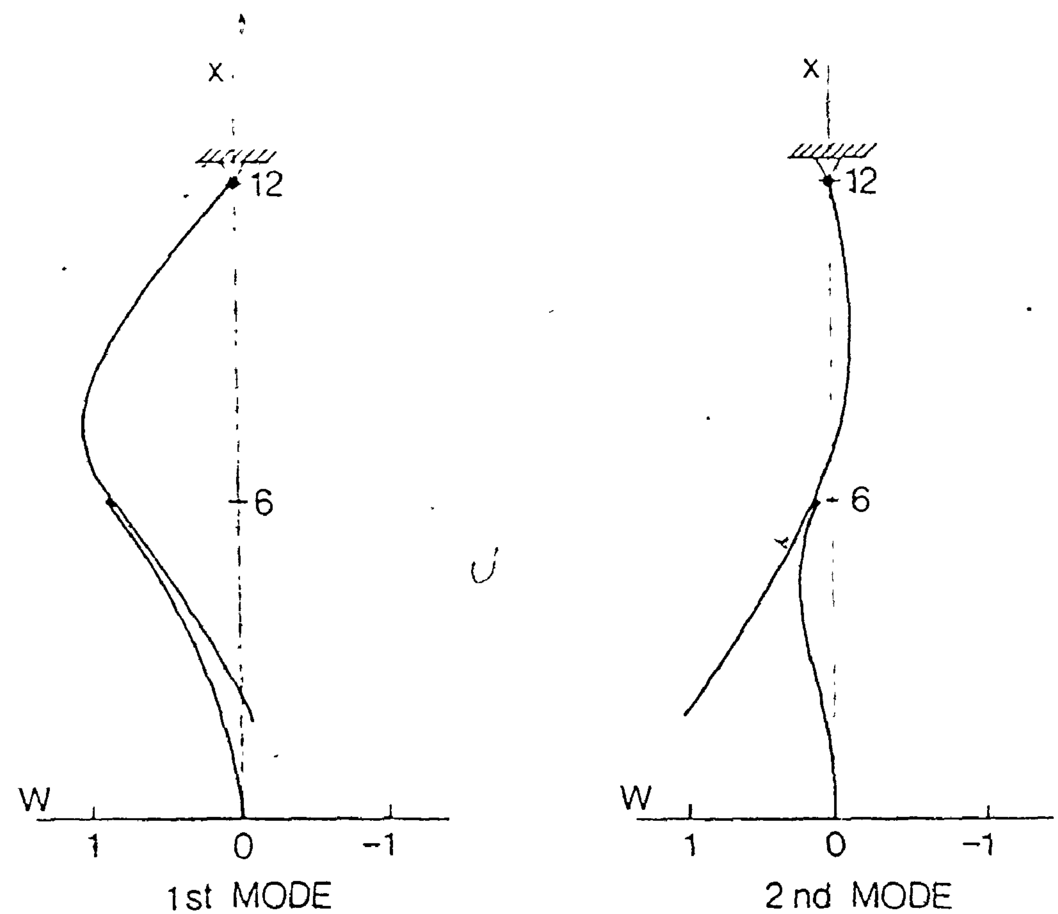

FIGURE 4.7: MODE SHAPES FOR MODEL (V)-b, CASE-A.

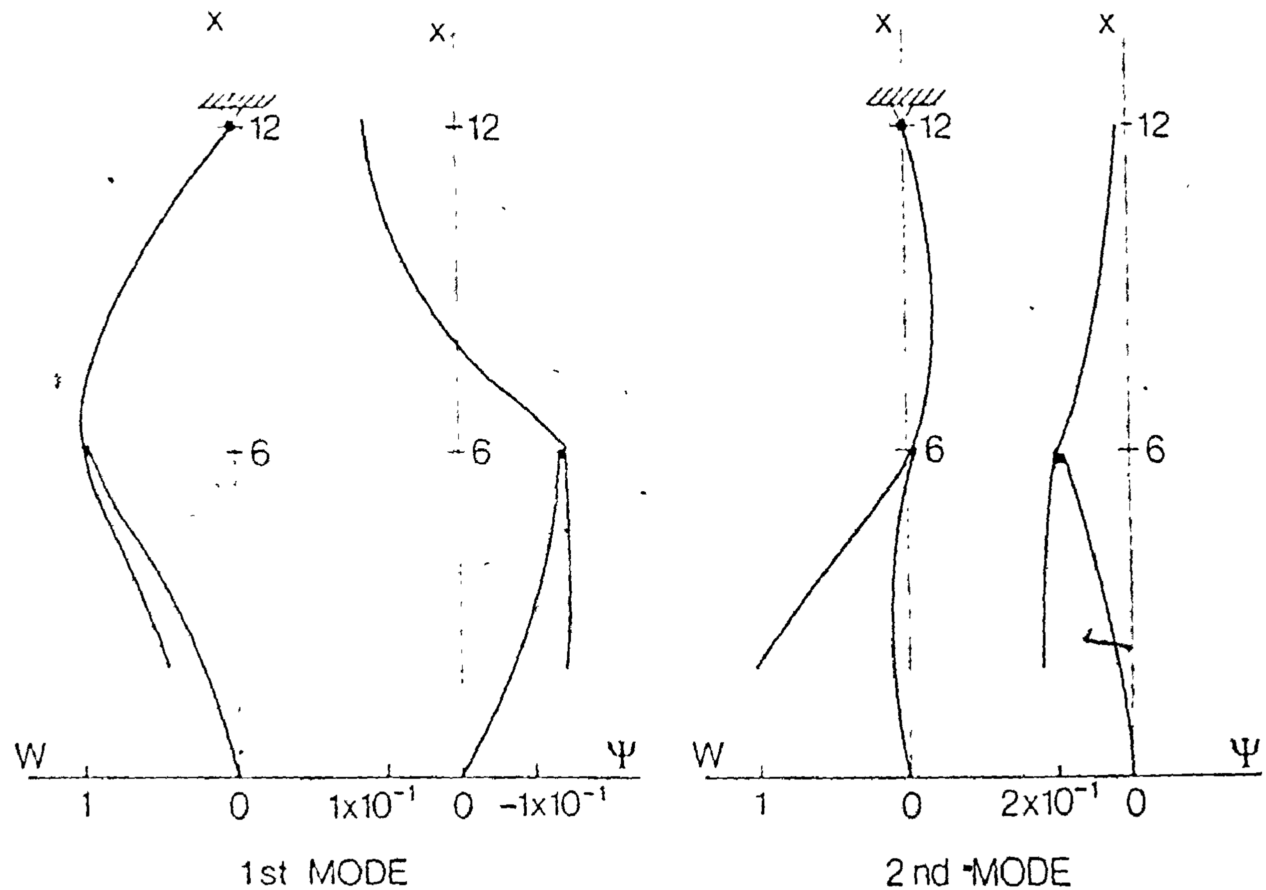

FIGURE 4.8: MODE SHAPES FOR MODEL (VI)-b, CASE-A.

CHAPTER 5

CONCLUSIONS

It is immediately evident from examination of Tables 1, 3, 5, and 7 that for the cantilevered systems Model (II)-a, a relatively simple two degree-of-freedom model, provides reasonable estimates of the first two natural frequencies when compared to the most sophisticated model considered, Model (VI)-a. However, it is equally clear from Tables 2, 4, 6, and 8 that no simpler model provides a reasonable approximation to the frequencies calculated from Model (VI)-b for the fixed-pinned systems. This difference in behavior of the differently considered skirt-vessel systems is not difficult to fathom since the pin constraint at the top of the pressure vessel induces significant flexure and shearing of the portion of the vessel above the connection of the vessel to the skirt; this is clearly evident in Figure (4.8). Tables 2, 4, 6, and 8 would seem to indicate that this conclusion is independent of vessel thickness. The comparative absence of these effects when the vessel is free at the top is equally evident in Figure (4.4), which differs little from Figure (4.2).

Our study leads us to the conclusion that the use of Timoshenko beams to model skirt-vessel systems is feasible and that this model should be used for other than cantilevered skirt-vessel systems of normal proportions when a beam theory model is used to model the system. If the situation warrants, it is also the way to model cantilevered systems since Model (II)-a, which provides an excellent approximation to the fundamental frequency, provides only a fair estimate of the second natural frequency of the system. If one requires higher frequencies then of course, one has no other choice than to use Model (VI) for reasonable results (at least within the hierarchy of models considered here) for any set of boundary conditions.

We note, now, that we have not exhausted reasonable models for skirt-vessel systems since it is certainly reasonable, for example, to model the skirt by a massless beam capable of undergoing shear deformation and the vessel by a Timoshenko beam. This is suggested by the results for Models (I) and (III) and Models (II) and (IV). These examples show the small effect of the skirt mass on the natural frequencies of skirt-vessel systems. This is easily understood since for our examples the skirt weighed about 1,500 lbs. and the vessels, including contents, ranged in weight from about 15,000 to 56,000 lbs. However, little computational advantage would be obtained from use of this model, as compared to the use of Model (VI).

Finally, we note that somewhat more realistic models of skirt-vessel systems may be studied without undue difficulty. For example, the ends of the vessel and appropriate portions of the innards, such as tube sheets, may be modeled as a rigid mass attached to cylindrical portion of the vessel. Such studies probably are performed most easily using finite element techniques, with Timoshenko beam elements, rather than by the analytical methods used in this thesis where analytical methods were appropriate since the major purpose of the present work was to study the dynamic modeling of skirt-pressure vessel systems rather than to provide precise design data for particular configurations.

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APPENDIX A

NOMENCLATURE

L_1	Length of the skirt.
L_2	Length of the portion of the vessel above the junction.
L_3	Length of the portion of the vessel below the junction.
x_i & y_i	Coordinate axes for the i-th segment.
E	Modulus of Elasticity.
I_i	Cross-sectional movements of inertia for the i-th segment.
$\bar{m}_i$	Masses per unit length for the i-th segment.
w_i	Total displacements for the i-th segment.
$\bar{w}_i$ & $\bar{\psi}_i$	Elastic displacements for the i-th segment.
ψ_i	Angle between a cross-sectional plane and the horizontal plane passing through the y-axis.
A_i	Cross-sectional area of the i-th segment.
ρ_i	Mass density of the i-th segment.
G	Shear modulus.
κ_i^2	Timoshenko's shear coefficient of the i-th segment.
$(\dot{\ })$	Differentiation with respect to time, $\frac{d}{dt}$.
$(\ )'$	Differentiation with respect to x, $\frac{d}{dx}$.
ω_i	Natural frequencies of the system.
B_i, C_i, D_i, and F_i	Constants of integration.
p_i	Distributed load.
m	Total mass of the pressure vessel and its contents.

APPENDIX B

SOLUTION FOR TIMOSHENKO BEAM'S EQUATIONS.

The Timoshenko beam's equations of motion are

$$EI\,\psi'' - \kappa^2 AG(\psi + w') - \rho I \ddot{\psi} = 0 \qquad (1)$$

$$\kappa^2 AG(\psi' + w'') - \rho A \ddot{w} = 0 \qquad (2)$$

By eliminating ψ from equations (1) and (2) we have

$$EI\cdot w^{iv} + A\,\ddot{w} - (\rho I + \frac{EI\rho}{\kappa^2 G})\,\ddot{w}'' + \frac{\rho^2 I}{\kappa^2 G}\,\ddddot{w} = 0 \qquad (3)$$

To solve equation (3) the method of separation of variables is employed and since we seek periodic solutions in time it is assumed that

$$\left.\begin{aligned} w(x,t) &= W(x)\cdot \sin\omega t \qquad &(a)\\ \psi(x,t) &= \Psi(x)\cdot \sin\omega t \qquad &(b)\end{aligned}\right\} \qquad (4)$$

Substituting from equation (4)-a into equation (3) yields ,

$$W^{iv} + b^2\omega^2\,W'' - a\omega^2\,W = 0 \qquad (5)$$

where

$$\left.\begin{aligned} b^2 &= \frac{\rho}{E}\,(1 + \frac{E}{\kappa^2 G})\\ a &= \frac{\rho}{E}\,(\frac{A}{I} - \frac{\rho\omega^2}{\kappa^2 G})\end{aligned}\right\} \qquad (6)$$

We note that usually $\frac{A}{I} > \frac{\rho\omega^2}{\kappa^2 G}$

Equation (5) is an ordinary differential equation of the fourth order, the solution of which is

$$W_{(x)} = B \sinh \lambda_1 x + C \cosh \lambda_1 x + D \sin \lambda_2 x + F \cos \lambda_2 x \qquad (7)$$

where B, C, D, and F are the constants of integration and

$$\lambda_1 = \left[\left(\frac{b^4\omega^4}{4} + a\omega^2\right)^{\frac{1}{2}} - \left(\frac{b^2\omega^2}{2}\right) \right]^{\frac{1}{2}}$$

$$\lambda_2 = \left[\left(\frac{b^4\omega^4}{4} + a\omega^2\right)^{\frac{1}{2}} + \left(\frac{b^2\omega^2}{2}\right) \right]^{\frac{1}{2}} \qquad (8)$$

Now, from equation (2) we have

$$\psi' = \frac{\rho}{\kappa^2 G} \ddot{W} - W'' \qquad (9)$$

Substituting from equation (4) into equation (9) yields

$$\Psi'_{(x)} = - \left[\frac{\rho\omega^2}{\kappa^2 G} W_{(x)} + W'' \right] \qquad (10)$$

Substituting from equation (7) into equation (10) and integrating once results in

$$\Psi_{(x)} = - \left(\frac{\alpha}{\lambda_1}\right) (B \cosh \lambda_1 x + C \sinh \lambda_1 x) + \left(\frac{\beta}{\lambda_2}\right) (D \cos \lambda_2 x - F \sin \lambda_2 x) + H \qquad (11)$$

where $\alpha = \left(\frac{\rho\omega^2}{\kappa^2 G} + \lambda_1^2\right)$, $\beta = \left(\frac{\rho\omega^2}{\kappa^2 G} - \lambda_2^2\right)$,

and we note that equation (1) requires that the constant of integration (H) in equation (11) should be equal to zero.

APPENDIX C

THE FREQUENCY DETERMINANTS

C.1 Model (V)-a

The frequency determinant of Model (V)-a is of the sixth order and its elements are:

$$a_{11} = \sin a_1L_1 - \sinh a_1L_1$$

$$a_{12} = \cos a_1L_1 - \cosh a_1L$$

$$a_{13} = -(\sin a_2L_2 + \sinh a_2L_2)$$

$$a_{14} = -(\cos a_2L_2 + \cosh a_2L_2)$$

$$a_{15} = 0$$

$$a_{16} = 0$$

$$a_{21} = \cos a_1L_1 - \cosh a_1L_1$$

$$a_{22} = -(\sin a_1L_1 + \sinh a_1L_1)$$

$$a_{23} = R_5(\cos a_2L_2 + \cosh a_2L_2)$$

$$a_{24} = -R_5(\sin a_2L_2 - \sinh a_2L_2)$$

$$a_{25} = 0$$

$$a_{26} = 0$$

$$a_{31} = \sin a_1L_1 + \sinh a_1L_1$$

$$a_{32} = \cos a_1L_1 + \cosh a_1L_1$$

$$a_{33} = -R_1(\sin a_2L_2 - \sinh a_2L_2)$$

$$a_{34} = -R_1(\cos a_2L_2 - \cosh a_2L_2)$$

$$a_{35} = R_2(\sin a_3L_3 - \sinh a_3L_3)$$

$$a_{36} = R_2(\cos a_3L_3 - \cosh a_3L_3)$$

$$a_{41} = \cos a_1L_1 + \cosh a_1L_1$$

$$a_{42} = -(\sin a_1L_1 - \sinh a_1L_1)$$

$$a_{43} = R_3(\cos a_2L_2 - \cosh a_2L_2)$$

$$a_{44} = -R_3(\sin a_2L_2 + \sinh a_2L_2)$$

$$a_{45} = R_4(\cos a_3L_3 - \cosh a_3L_3)$$

$$a_{46} = -R_4(\sin a_3L_3 + \sinh a_3L_3)$$

$$a_{51} = 0$$

$$a_{52} = 0$$

$$a_{53} = \sin a_2L_2 + \sinh a_2L_2$$

$$a_{54} = \cos a_2L_2 + \cosh a_2L_2$$

$$a_{55} = -(\sin a_3L_3 + \sinh a_3L_3)$$

$$a_{56} = -(\cos a_3L_3 + \cosh a_3L_3)$$

$$a_{61} = 0$$

$$a_{62} = 0$$

$$a_{63} = \cos a_2L_2 + \cosh a_2L_2$$

$$a_{64} = -(\sin a_2L_2 - \sinh a_2L_2)$$

$$a_{65} = R_6(\cos a_3L_3 + \cosh a_3L_3)$$

$$a_{66} = -R_6(\sin a_3L_3 - \sin a_3L_3)$$

where $a_i = \dfrac{\bar{m}_i\omega^2}{EI_i}$, $i = 1,2,3,$

$$a_3 = a_2$$

$$R_1 = (\frac{a_2}{a_1})^2 \frac{I_2}{I_1} ,$$

$$R_2 = (\frac{a_3}{a_1})^2 \frac{I_3}{I_1} ,$$

$$R_3 = (\frac{a_2}{a_1})^3 \frac{I_2}{I_1} ,$$

$$R_4 = (\frac{a_3}{a_1})^3 \frac{I_3}{I_1} ,$$

$$R_5 = \frac{a_2}{a_1} ,$$

and $R_6 = \dfrac{a_3}{a_1}$.

C.2 Model (III)-a

Here, the frequency determinant is of the second order and its elements are:

$$a_{11} = (\cos a_1L_1 + \cosh a_1L) - R_1(\sin a_1L_1 - \sinh a_1L_1) - R_2(\cos a_1L_1 - \cosh a_1L_1) \; .$$

$$a_{12} = -\left[(\sin a_1L_1 - \sinh a_1L_1) + R_1(\cos a_1L_1 - \cosh a_1L_1) - R_2(\sin a_1L_1 + \sinh a_1L_1) \right]$$

$$a_{21} = (\sin a_1L_1 + \sinh a_1L_1) + R_2(\sin a_1L_1 - \sinh a_1L_1) + R_3(\cos a_1L_1 - \cosh a_1L_1)$$

$$a_{22} = (\cos a_1L_1 + \cosh a_1L_1) + R_2(\cos a_1L_1 - \cosh a_1L_1) - R_3(\sin a_1L_1 + \sinh a_1L_1) \, ,$$

where $$R_1 = a_1L_1 \left(\frac{\bar{m}_2}{m_1}\right) \left(\frac{L_2 + L_3}{L_1}\right) \, ,$$

$$R_2 = \frac{1}{2} (a_1L_1)^2 \left(\frac{\bar{m}_2}{\bar{m}_1}\right) \left(\frac{L_2^2 - L_3^2}{L_1^2}\right) \, ,$$

and $$R_3 = \frac{1}{3} (a_1L_1)^3 \left(\frac{\bar{m}_2}{\bar{m}_1}\right) \left(\frac{L_2^3 + L_3^3}{L_1^3}\right) \, .$$

C.3 Model (VI)-a

Although this model of the skirt-vessel system is the most sophisticated model considered, it also leads to a sixth order frequency determinant so that the algebraic structure of Model (VI)-a is the same as that for Model (V)-a. The complexity of individual elements in the frequency determinant is, of course, greater in the present model. The elements of this determinant are:

$$a_{11} = \sin \lambda_{11}L_1 + \left(\frac{\lambda_{21}}{\lambda_{11}}\right)\left(\frac{\alpha_1}{\beta_1}\right) \sin \lambda_{21}L_1$$

$$a_{12} = \cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1$$

$$a_{13} = -\left[\sinh \lambda_{12}L_2 + \left(\frac{\lambda_{22}}{\lambda_{12}}\right) \sin \lambda_{22}L_2\right]$$

$$a_{14} = -\left[\cosh \lambda_{12}L_2 - \left(\frac{\alpha_2}{\beta_2}\right) \cos \lambda_{22}L_2\right]$$

$$a_{15} = 0$$

$$a_{16} = 0$$

$$a_{21} = \frac{\alpha_1}{\lambda_{11}} (\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1)$$

$$a_{22} = \frac{\alpha_1}{\lambda_{11}} \left(\sinh \lambda_{11}L_1 - \left(\frac{\lambda_{11}}{\lambda_{21}}\right)\left(\frac{\beta_1}{\alpha_1}\right) \sin \lambda_{21}L_1\right)$$

$$a_{23} = \frac{\alpha_2}{\lambda_{12}} \left(\cosh \lambda_{12}L_2 - \left(\frac{\beta_2}{\alpha_2}\right) \cos \lambda_{22}L_2\right)$$

$$a_{24} = \frac{\alpha_2}{\lambda_{12}} (\sinh \lambda_{12}L_2 - (\frac{\lambda_{12}}{\lambda_{22}}) \sin \lambda_{22}L_2)$$

$$a_{25} = 0$$

$$a_{26} = 0$$

$$a_{31} = EI_1\ \alpha_1(\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}}) \sin \lambda_{21}L_1)$$

$$a_{32} = EI_1\ \alpha_1(\cosh \lambda_{11}L_1 - (\frac{\beta_1}{\alpha_1}) \cos \lambda_{21}L_1)$$

$$a_{33} = \frac{\rho_2 I_2\omega^2\alpha_2}{\lambda_{12}^2} (\sin \lambda_{12}L_2 - (\frac{\beta_2}{\alpha_2})(\frac{\lambda_{12}}{\lambda_{22}}) \sin \lambda_{22}L_2) - \frac{\rho_2 A_2\omega^2}{\lambda_{12}^2} (\sinh \lambda_{12}L_2 - (\frac{\lambda_{12}}{\lambda_{22}}) \sin \lambda_{22}L_2)$$

$$a_{34} = \frac{\rho_2 I_2\omega^2\alpha_2}{\lambda_{12}^2} \left[\left(\cosh \lambda_{12}L_2 + (\frac{\lambda_{12}}{\lambda_{22}})^2 \cos \lambda_{22}L_2\right) - \left(1 + (\frac{\lambda_{12}}{\lambda_{22}})^2\right)\right] - \frac{\rho_2 A_2\omega^2}{\lambda_{12}^2}\left[\left(\cosh \lambda_{12}L_2 + (\frac{\alpha_2}{\beta_2})(\frac{\lambda_{12}}{\lambda_{22}})^2 \cos \lambda_{22}L_2\right) - \left(1 + \frac{\alpha_2}{\beta_2}(\frac{\lambda_{12}}{\lambda_{22}})^2\right)\right]$$

$$a_{35} = -\left[\frac{\rho_3 I_3\omega^2\alpha_3}{\lambda_{13}^2}\left(\sinh \lambda_{13}L_3 - (\frac{\beta_3}{\alpha_3})(\frac{\lambda_{13}}{\lambda_{23}}) \sin \lambda_{23}L_3\right) - \frac{\rho_3 A_3\omega^2}{\lambda_{13}^2}\left(\sinh \lambda_{13}L_3 - (\frac{\lambda_{13}}{\lambda_{23}}) \sin \lambda_{23}L_3\right)\right]$$

$$a_{36} = -\frac{\rho_3 I_3\omega^2\alpha_3}{\lambda_{13}^2}\left[\left(\cosh \lambda_{13}L_3 + \left(\frac{\lambda_{13}}{\lambda_{23}}\right)^2 \cos \lambda_{23}L_3\right) - \left(1 + \left(\frac{\lambda_{13}}{\lambda_{23}}\right)^2\right)\right] + \frac{\rho_3 A_3\omega^2}{\lambda_{13}^2}\left[\left(\cosh \lambda_{13}L_3 + (\frac{\alpha_3}{\beta_3})(\frac{\lambda_{13}}{\lambda_{23}})^2 \cos \lambda_{23}L_3\right) - \left(1 + \frac{\alpha_3}{\beta_3}\frac{\lambda_{13}}{\lambda_{23}}\right)\right]$$

$$a_{41} = \kappa_1^2 A_1 G[\cosh \lambda_{11}L_1(\lambda_{11} - \frac{\alpha_1}{\lambda_{11}}) + \cos \lambda_{21}L_1(\frac{\alpha_1}{\lambda_{11}})(1 + \frac{\lambda_{21}^2}{\beta_1})]$$

$$a_{42} = \kappa_1^2 A_1 G[\sinh \lambda_{11}L_1(\lambda_{11} - \frac{\alpha_1}{\lambda_{11}}) + \sin \lambda_{21}L_1(\lambda_{21} + \frac{\beta_1}{\lambda_{21}})]$$

$$a_{43} = -[\frac{\rho_2 A_2 \omega^2}{\lambda_{12}} (\cosh \lambda_{12}L_2 - \cos \lambda_{22}L_2)]$$

$$a_{44} = -[\frac{\rho_2 A_2 \omega^2}{\lambda_{12}} (\sinh \lambda_{12}L_2 - (\frac{\alpha_2}{\beta_2})(\frac{\lambda_{12}}{\lambda_{22}}) \sin \lambda_{22}L_2)]$$

$$a_{45} = -[\frac{\rho_3 A_3 \omega^2}{\lambda_{13}} (\cosh \lambda_{13}L_3 - \cos \lambda_{23}L_3)]$$

$$a_{46} = -[\frac{\rho_3 A_3 \omega^2}{\lambda_{13}} (\sinh \lambda_{13}L_3 - (\frac{\alpha_3}{\beta_3})(\frac{\lambda_{13}}{\lambda_{23}}) \sin \lambda_{23}L_3)]$$

$$a_{51} = 0$$

$$a_{52} = 0$$

$$a_{53} = \sinh \lambda_{12}L_2 + (\frac{\lambda_{22}}{\lambda_{12}}) \sin \lambda_{22}L_2$$

$$a_{54} = \cosh \lambda_{12}L_2 - (\frac{\alpha_2}{\beta_2}) \cos \lambda_{22}L_2$$

$$a_{55} = -[\sinh \lambda_{13}L_3 + (\frac{\lambda_{23}}{\lambda_{13}}) \sin \lambda_{23}L_3]$$

$$a_{56} = -[\cosh \lambda_{13}L_3 - (\frac{\alpha_3}{\beta_3}) \cos \lambda_{23}L_3]$$

$$a_{61} = 0$$

$$a_{62} = 0$$

$$a_{63} = \frac{\alpha_2}{\lambda_{12}} (\cosh \lambda_{12}L_2 - (\frac{\beta_2}{\alpha_2}) \cos \lambda_{22}L_2)$$

$$a_{64} = \frac{\alpha_2}{\lambda_{12}} (\sinh \lambda_{12}L_2 - (\frac{\lambda_{12}}{\lambda_{22}}) \sin \lambda_{22}L_2)$$

$$a_{65} = \frac{\alpha_3}{\lambda_{13}} (\cosh \lambda_{13}L_3 - (\frac{\beta_3}{\alpha_3}) \cos \lambda_{23}L_3)$$

$$a_{66} = \frac{\alpha_3}{\lambda_{13}} (\sinh \lambda_{13}L_3 - (\frac{\lambda_{13}}{\lambda_{23}}) \sin \lambda_{23}L_3)$$

where λ_{1i}, λ_{2i}, α_i, and β_i $(i = 1,2,3)$ are defined in sec.(3.5).

C.4 Model (IV)-a

Again, as in the case of Model (III)-a, the frequency determinant is of the second order and its elements are:

$$a_{11} = R_1 (\frac{\alpha_1}{\lambda_{11}}) (\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1) + R_2 (\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}}) (\frac{\alpha_1}{\beta_1}) \sin \lambda_{21}L_1) - EI_1\alpha_1 (\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}}) \sin \lambda_{21}L_1)$$

$$a_{12} = R_1 (\frac{\alpha_1}{\lambda_{11}}) (\sinh \lambda_{11}L_1 - (\frac{\lambda_{11}}{\lambda_{21}}) (\frac{\beta_1}{\alpha_1}) \sin \lambda_{21}L_1) + R_2 (\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1) - EI_1\alpha_1 (\cosh \lambda_{11}L_1 - \frac{\beta_1}{\alpha_1} \cos \lambda_{21}L_1)$$

$$a_{21} = R_3 (\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}}) (\frac{\alpha_1}{\beta_1}) \sin \lambda_{21}L_1) + R_4 (\frac{\alpha_1}{\lambda_{11}}) (\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1) - R_5 (\cosh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}})^2 (\frac{\alpha_1}{\beta_1}) \cos \lambda_{21}L_1)$$

$$a_{22} = R_3(\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1) + R_4 (\frac{\alpha_1}{\lambda_{11}})(\sinh \lambda_{11}L_1 - (\frac{\lambda_{11}}{\lambda_{21}})(\frac{\beta_1}{\alpha_1})$$

$$\sin \lambda_{21}L_1) - R_5(\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}}) \sin \lambda_{21}L_1)$$

where $R_1 = \rho_2 A_2 \omega^2 [(\frac{L_2^3 + L_3^3}{3}) + \frac{I_2}{A_2}(L_2 + L_3)]$,

$R_2 = \rho_2 A_2 \omega^2 (\frac{L_2^2 - L_3^2}{2})$,

$R_3 = \rho_2 A_2 \omega^2 (L_2 + L_3)$,

$R_4 = \kappa_1^2 A_1 G + \rho_2 A_2 \omega^2 (\frac{L_2^2 - L_3^2}{2})$,

and $R_5 = \kappa_1^2 A_1 G \lambda_{11}$

C.5 Model (V)-b.

The elements of the frequency determinant are:

$a_{11} = \sin a_1 L_1 - \sinh a_1 L_1$

$a_{12} = \cos a_1 L_1 - \cosh a_1 L_1$

$a_{13} = - \sin a_2 L_2$

$a_{14} = - \sinh a_2 L_2$

$a_{15} = 0$

$a_{16} = 0$

$$a_{21} = \cos a_1L_1 - \cosh a_1L_1$$

$$a_{22} = -(\sin a_1L_1 + \sinh a_1L_1)$$

$$a_{23} = (\frac{a_2}{a_1}) \cos a_2L_2$$

$$a_{24} = (\frac{a_2}{a_1}) \cosh a_2L_2$$

$$a_{25} = 0$$

$$a_{26} = 0$$

$$a_{31} = \sin a_1L_1 + \sinh a_1L_1$$

$$a_{32} = \cos a_1L_1 + \cosh a_1L_1$$

$$a_{33} = -R_1(\sin a_2L_2 - a_2L_2)$$

$$a_{34} = R_1(\sinh a_2L_2 - a_2L_2)$$

$$a_{35} = R_2(\sin a_3L_3 - \sinh a_3L_3)$$

$$a_{36} = R_2(\cos a_3L_3 - \cosh a_3L_3)$$

$$a_{41} = \cos a_1L_1 + \cosh a_1L_1$$

$$a_{42} = -(\sin a_1L_1 - \sinh a_1L_1)$$

$$a_{43} = R_3(\cos a_2L_2 - 1)$$

$$a_{44} = -R_3(\cosh a_2L_2 - 1)$$

$$a_{45} = R_4(\cos a_3L_3 - \cosh a_3L_3)$$

$$a_{46} = -R_4(\sin a_3L_3 + \sinh a_3L_3)$$

where R_1 through R_4 have been defined in section (C.2)

$$a_{51} = 0$$

$$a_{52} = 0$$

$$a_{53} = \sin a_2L_2$$

$$a_{54} = \sinh a_2L_2$$

$$a_{55} = -(\sin a_3L_3 + \sinh a_3L_3)$$

$$a_{56} = -(\cos a_3L_3 + \cosh a_3L_3)$$

$$a_{61} = 0$$

$$a_{62} = 0$$

$$a_{63} = \cos a_2L_2$$

$$a_{64} = \cosh a_2L_2$$

$$a_{65} = \cos a_3L_3 + \cosh a_3L_3$$

$$a_{66} = -(\sin a_3L_3 - \sinh a_3L_3)$$

C.6 Model (VI)-b

The elements of the frequency determinant are:

$$a_{11} = \sinh \lambda_{11}L_1 + \left(\frac{\lambda_{21}}{\lambda_{11}}\right)\left(\frac{\alpha_1}{\beta_1}\right) \sin \lambda_{21}L_1$$

$$a_{12} = \cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1$$

$$a_{13} = -\sinh \lambda_{12}L_2$$

$$a_{14} = -\sin \lambda_{22}L_2$$

$$a_{15} = 0$$

$$a_{16} = 0$$

$$a_{21} = \frac{\alpha_1}{\lambda_{11}}(\cosh \lambda_{11}L_1 - \cos \lambda_{21}L_1)$$

$$a_{22} = \frac{\alpha_1}{\lambda_{11}}(\sinh \lambda_{11}L_1 - (\frac{\lambda_{11}}{\lambda_{21}})(\frac{\beta_1}{\alpha_1}) \sin \lambda_{21}L_1)$$

$$a_{23} = \frac{\alpha_2}{\lambda_{12}} \cdot \cosh \lambda_{12}L_2$$

$$a_{24} = -\frac{\beta_2}{\lambda_{22}} \cdot \cos \lambda_{22}L_2$$

$$a_{25} = 0$$

$$a_{25} = 0$$

$$a_{31} = EI_1\alpha_1(\sinh \lambda_{11}L_1 + (\frac{\lambda_{21}}{\lambda_{11}}) \sin \lambda_{21}L_1)$$

$$a_{32} = EI_1\alpha_1 (\cosh \lambda_{11}L_1 - (\frac{\beta_1}{\alpha_1}) \cos \lambda_{21}L_1)$$

$$a_{33} = \frac{\rho_2 I_2\omega^2\alpha_2}{\lambda_{12}^2} \sinh \lambda_{12}L_2 - \frac{\rho_2 A_2\omega^2}{\lambda_{12}^2} (\sinh \lambda_{12}L_2 - \lambda_{12}L_2)$$

$$a_{34} = -[\frac{\rho_2 I_2\omega^2\alpha_2}{\lambda_{12}^2} (\frac{\lambda_{12}}{\lambda_{22}}) (\frac{\beta_2}{\alpha_2}) \sin \lambda_{22}L_2$$

$$- \frac{\rho_2 A_2\omega^2}{\lambda_{12}^2} (\frac{\lambda_{12}}{\lambda_{22}})^2 (\sin \lambda_{22}L_2 - \lambda_{22}L_2)]$$

$$a_{35} = -\left[\frac{\rho_3 I_3 \omega^2 \alpha_3}{\lambda_{13}^2} \left(\sinh \lambda_{13}L_3 - \frac{\beta_3}{\alpha_3} \left(\frac{\lambda_{13}}{\lambda_{23}}\right) \sin \lambda_{23}L_3\right) - \frac{\rho_3 A_3 \omega^2}{\lambda_{13}^2} \left(\sinh \lambda_{13}L_3 - \left(\frac{\lambda_{13}}{\lambda_{23}}\right) \sin \lambda_{23}L_3\right) \right]$$

$$a_{36} = - \frac{\rho_3 I_3 \omega^2 \alpha_3}{\lambda_{13}^2} \left[\left(\cosh \lambda_{13}L_3 + \left(\frac{\lambda_{13}}{\lambda_{23}}\right)^2 \cos \lambda_{23}L_3\right) - \left(1 + \left(\frac{\lambda_{13}}{\lambda_{23}}\right)^2\right)\right] + \frac{\rho_3 A_3 \omega^2}{\lambda_{13}^2} \left[\left(\cosh \lambda_{13}L_3 + \frac{\alpha_3}{\beta_3}\left(\frac{\lambda_{13}}{\lambda_{23}}\right)^2 \cos \lambda_{23}L_3\right) - \left(1 + \frac{\alpha_3}{\beta_3} \left(\frac{\lambda_{13}}{\lambda_{23}}\right)^2\right)\right]$$

$$a_{41} = \kappa_1^2 A_1 G \left[\cosh \lambda_{11}L_1 \left(\lambda_{11} - \frac{\alpha_1}{\lambda_{11}}\right) + \cos \lambda_{21}L_1 \left(\frac{\alpha_1}{\lambda_{11}}\right) \left(1 + \frac{\lambda_{21}^2}{\beta_1}\right)\right]$$

$$a_{42} = \kappa_1^2 A_1 G \left[\sinh \lambda_{11}L_1 \left(\lambda_{11} - \frac{\alpha_1}{\lambda_{11}}\right) + \sin \lambda_{21}L_1 \left(\lambda_{21} + \frac{\beta_1}{\lambda_{21}}\right)\right]$$

$$a_{43} = -\left[\frac{\rho_2 A_2 \omega^2}{\lambda_{12}} (\cosh \lambda_{12}L_2 - 1)\right]$$

$$a_{44} = \left[\frac{\rho_2 A_2 \omega^2}{\lambda_{12}} \left(\frac{\lambda_{12}}{\lambda_{22}}\right) (\cos \lambda_{22}L_2 - 1)\right]$$

$$a_{45} = -\left[\frac{\rho_3 A_3 \omega^2}{\lambda_{13}} (\cosh \lambda_{13}L_3 - \cos \lambda_{23}L_3)\right]$$

$$a_{46} = -\left[\frac{\rho_3 A_3 \omega^2}{\lambda_{13}} \left(\sinh \lambda_{13}L_3 - \frac{\alpha_3}{\beta_3} \left(\frac{\lambda_{13}}{\lambda_{23}}\right) \sin \lambda_{23}L_3\right)\right]$$

$$a_{51} = 0$$

$$a_{52} = 0$$

$$a_{53} = \sinh \lambda_{12}L_2$$

$$a_{54} = \sin \lambda_{22}L_2$$

$$a_{55} = -\left[\sinh \lambda_{13}L_3 + \left(\frac{\lambda_{23}}{\lambda_{13}}\right) \sin \lambda_{23}L_3 \right]$$

$$a_{56} = -\left[\cosh \lambda_{13}L_3 - \left(\frac{\alpha_3}{\beta_3}\right) \cos \lambda_{23}L_3 \right]$$

$$a_{61} = 0$$

$$a_{62} = 0$$

$$a_{63} = \frac{\alpha_2}{\lambda_{12}} \cosh \lambda_{12}L_2$$

$$a_{64} = -\frac{\beta_2}{\lambda_{22}} \cos \lambda_{22}L_2$$

$$a_{65} = \frac{\alpha_3}{\lambda_{13}} \left(\cosh \lambda_{13}L_3 - \left(\frac{\beta_3}{\alpha_3}\right) \cos \lambda_{23}L_3\right)$$

$$a_{66} = \frac{\alpha_3}{\lambda_{13}} \left(\sinh \lambda_{13}L_3 - \left(\frac{\lambda_{13}}{\lambda_{23}}\right) \sin \lambda_{23}L_3\right)$$

where λ_{1i}, λ_{2i}, α_i, and β_i $(i = 1,2,3)$ have been defined in section (3.5).

APPENDIX D

COMPUTER PROGRAM

The computer program listed in this appendix was written to solve the transcendental frequency equations for the models of the hierarchy used in this study using an IMS. Library routine, SUBROUTINE ZREAL 1, which calculates n real zeros of a real function F where the initial guesses are not known to be good. The listed program is for the case of Model (VI)-a, the most sophisticated model, and the only difference between this case and any other model in the programming is in the function F, which is the frequency determinant.

List of Symbols Used in the Program:

X	Frequency in rad/sec.
W	Frequency in Hz.
N	Size of the frequency determinant.
U	Poisson's ratio.
XLI	Length of i-th segment, (I = 1,2,3).
AI	Cross-sectional area of i-th segment, (I = 1,2,3).
RO1	Mass density of the skirt.
RO2	Mass density of the vessel and its contents.
XI1	Moment of inertia of the cross-section of the skirt.
XI2	Moment of inertia of the cross-section of the vessel.
E	Modulus of elasticity.
G	Shear modulus.
XK	Timoshenko shear coefficient.

```
      C*******************************************************************
      C   PROGRAM FOR OBTAINING THE FREQENCIES OF MODEL ( VI )-A.        *
      C*******************************************************************
ATTACH,IMSLIB.
FTN(R=3,OPT=0)
LDSET,LIB=IMSLIB.
LGO.
≠
          6400 END OF RECORD
       PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
       DIMENSION X(3)  , W(3)
       EXTERNAL F
       READ (5,*)          (X(I),I=1,3)
       WRITE (6,11)(X(I),I=1,3)
    11 FORMAT (3F10.4)
       DO 1   J=2,151
       X1 =FLOAT (J-1)*20.0
       Y = F(X1)
       WRITE (6,12) X1 ,Y
     1 CONTINUE
       CALL ZREAL1(F,1.E-05,1.E-06,1.E-02,2,3,X,500,IER)
       WRITE (6,12) X
    12 FORMAT (5X,6E13.6/)
       PI = 4.0*ATAN(1.0)
       CC = 2.0*PI
       DO 2 I = 1,3
       W(I) = X(I)/CC
     2 CONTINUE
       WRITE (6,13) W
    13 FORMAT (1H1,3F15.4/)
       STOP
       END
       FUNCTION F(X)
       DIMENSION A(6,6)
       N = 6
       U = 0.3
       E = 29.0*144.0*1000000.0
       G = E/(2.0*(1.0+U))
       RO1 = 15.217391
       XK  = 2.0*(1.0+U)/(4.0+3.0*U)
       XI1 = 1.219492
       EI1 = E*XI1
       XL1 = 6.0
       XL2 = 7.0
       XL3 = 8.0
       A1 = 0.5508695
       XI2 = 1.966202
       A2 = 1.025206
       RO2 = 46.169829
       B1 = (RO1/E)*(1.0+E/(XK*G))
       B2 = (RO2/E)*(1.0+E/(XK*G))
       B3 = B2
       S1 = (RO1/E)*(A1/XI1-RO1*X**2/(XK*G))
       S2 = (RO2/E)*(A2/XI2-RO2*X**2/(XK*G))
       S3 = S2
       D11= ((B1**2*X**4/4.0  +S1*X**2)**0.5-(B1*X**2/2.0))**0.5
       D21= ((B1**2*X**4/4.0  +S1*X**2)**0.5+(B1*X**2/2.0))**0.5
       D12= ((B2**2*X**4/4.0  +S2*X**2)**0.5-(B2*X**2/2.0))**0.5
       D22= ((B2**2*X**4/4.0  +S2*X**2)**0.5+(B2*X**2/2.0))**0.5
       D13 = D21/D11
       D23 = D22/D12
```

```
D13 = D12
D23 = D22
D3B = D2B
C11 = (RO1*X**2/(XK*G)+C11**2)
C21 = (RO1*X**2/(XK*G)-C21**2)
C12 = (RO2*X**2/(YK*G)+C12**2)
C22 = (RO2*X**2/(XK*G)-C22**2)
C13 = C12
C23 = C22
C1B = C11/C21
C2B = C12 /C22
C3B = C2B
X1  = D11*XL1
X2  = D21*XL1
Y1  = D12*XL2
Y2  = D22*XL2
Z1  = D13*XL3
Z2  = D23*XL3
R1  = EI1*C11
R2  = RO2*XI2*X**2*C12/(C12**2)
R3  = RO2*A2*X**2/(D12**2)
R4  = SINH(Y1)-(1.0/(D2B*C2B))*SIN(Y2)
R5  = SINH(Y1)-(1.0/D2B)*SIN(Y2)
R6  = COSH(Y1)+(1.0/D2B**2)*COS(Y2)-(1.0+(1.0/D2B**2))
R7  = COSH(Y1)+(C2B/D2B**2)*COS(Y2)-(1.0+(C2B/D2B**2))
R8  = SINH(Z1)-(1.0/(C3B*D3B))*SIN(Z2)
R9  = SINH(Z1)-(1.0/D3B)*SIN(Z2)
R10= COSH(Z1)+(1.0/D3B**2)*COS(Z2)-(1.0+(1.0/D3B**2))
R11= COSH(Z1)+(C3B/D3B**2)*COS(Z2)-(1.0+(C3B/D3B**2))
F1  = XK*A1*G
F2  = RO2*A2*Y**2/C12
F3  = D11-C11/D11
F4  = (C11/D11)*(1.0+D21**2/C21)
F5  = D21+C21/D21
A(1,1)  =  SINH(X1)+D1B*C1B*SIN(X2)
A(1,2)  =  COSH(X1)-COS(X2)
A(1,3)  =  -(SINH(Y1)+C2B*SIN(Y2))
A(1,4)  =  -(COSH(Y1)-C2B*COS(Y2))
A(1,5)  =  0.0
A(1,6)  =  0.0
A(2,1)  =  (C11/D11)*(COSH(X1)-COS(X2))
A(2,2)  =  (C11/D11)*(SINH(X1)-(1.0/(D1B*C1B))*SIN(X2))
A(2,3)  =  (C12/D12)*(COSH(Y1)-(1.0/C2B)*COS(Y2))
A(2,4)  =  (C12/D12)*(SINH(Y1)-(1.0/D2B)*SIN(Y2))
A(2,5)  =  0.0
A(2,6)  =  0.0
A(3,1)  =  -F1*(SINH(X1)+C1B*SIN(X2))
A(3,2)  =  -F1*(COSH(X1)-(1.0/C1B)*COS(X2))
A(3,3)  =  -R2*R4+R3*R5
A(3,4)  =  -R2*R6+R3*R7
A(3,5)  =  R2*R8-R3*R9
A(3,6)  =  R2*R10-R3*R11
A(4,1)  =  F1*(F3*COSH(X1)+F4*COS(X2))
A(4,2)  =  F1*(F3*SINH(X1)+F5*SIN(X2))
A(4,3)  =  -F2*(COSH(Y1)-COS(Y2))
A(4,4)  =  -F2*(SINH(Y1)-(C2B/D2B)*SIN(Y2))
A(4,5)  =  -F2*(COSH(Z1)-COS(Z2))
A(4,6)  =  -F2*(SINH(Z1)-(C3B/D3B)*SIN(Z2))
A(5,1)  =  0.0
A(5,2)  =  0.0
```

```
      A(5,3)  = SINH(Y1)+D28*SIN(Y2)
      A(5,4)  = COSH(Y1)-C28*COS(Y2)
      A(5,5)  = -(SINH(Z1)+D38*SIN(Z2))
      A(5,6)  = -(COSH(Z1)-C38*COS(Z2))
      A(6,1)  = 0.0
      A(6,2)  = 0.0
      A(6,3)  = (C12/D12)*(COSH(Y1)-(1.0/C28)*COS(Y2))
      A(6,4)  = (C12/D12)*(SINH(Y1)-(1.0/D28)*SIN(Y2))
      A(6,5)  = (C13/D13)*(COSH(Z1)-(1.0/C38)*COS(Z2))
      A(6,6)  = (C13/D13)*(SINH(Z1)-(1.0/D38)*SIN(Z2))
      K=2
      L=1
    5 DO 10 I=K,N
      R = A(I,L) / A(L,L)
      DO 10 J = K,N
   10 A(I,J) = A(I,J) - A(L,J) *R
      IF (K-N) 15,20,20
   15 L=K
      K=K+1
      GO TO 5
   20 DT =1
      DO 25 L= 1,N
   25 DT = DT * A(L,L)
      F = DT
      RETURN
      END
≠        6400  END OF RECORD
END LISTING
≠        6400  END OF RECORD
≠        6400  END OF RECORD
≠        6400  END OF RECORD
```