INEFFABILITY PROPERTIES OF $P_\kappa\lambda$

By

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ABSTRACT

We first use some results of Menas to prove that every normal filter on $P_\kappa\lambda$ extends the cub filter on $P_\kappa\lambda$ thereby settling a basic question in the structure theory of filters on $P_\kappa\lambda$.

Then we investigate ideal-theoretic and other aspects of ineffability properties of $P_\kappa\lambda$ with particular emphasis on those which can be viewed as $P_\kappa\lambda$ generalizations of weak compactness.

In the course of these studies, we came to view mild λ-ineffability as a $P_\kappa\lambda$ generalization of weak compactness in an ideal-theoretically weak sense, and sought a $P_\kappa\lambda$ generalization of weak compactness in an ideal-theoretically stronger sense.

To this end, we define the λ-Shelah property, a new ineffability property of $P_\kappa\lambda$ between mild λ-ineffability and almost λ-ineffability, and prove results which support the contention that this is the property we sought.

These results include characterizations of the λ-Shelah property in terms of a normal ideal on $P_\kappa\lambda$ and if $\lambda^{<\kappa} = \lambda$, in terms of a "normal" ultrafilter property. We also prove that κ is supercompact <u>iff</u> κ is λ-Shelah for every $\lambda \geq \kappa$ <u>iff</u> κ is almost λ-ineffable for every $\lambda \geq \kappa$ thereby improving a result of Magidor, and concluding that the λ-Shelah property and mild λ-ineffability are not provably equivalent for arbitrary $\lambda > \kappa$.

STUCTURE OF INEFFABILITY PROPERTIES OF $P_\kappa\lambda$ 1979 and 1981

All arrows represent direct implications; lower case letters and ⇒ are the 1979 picture; upper case letters and ➡ are provided here.

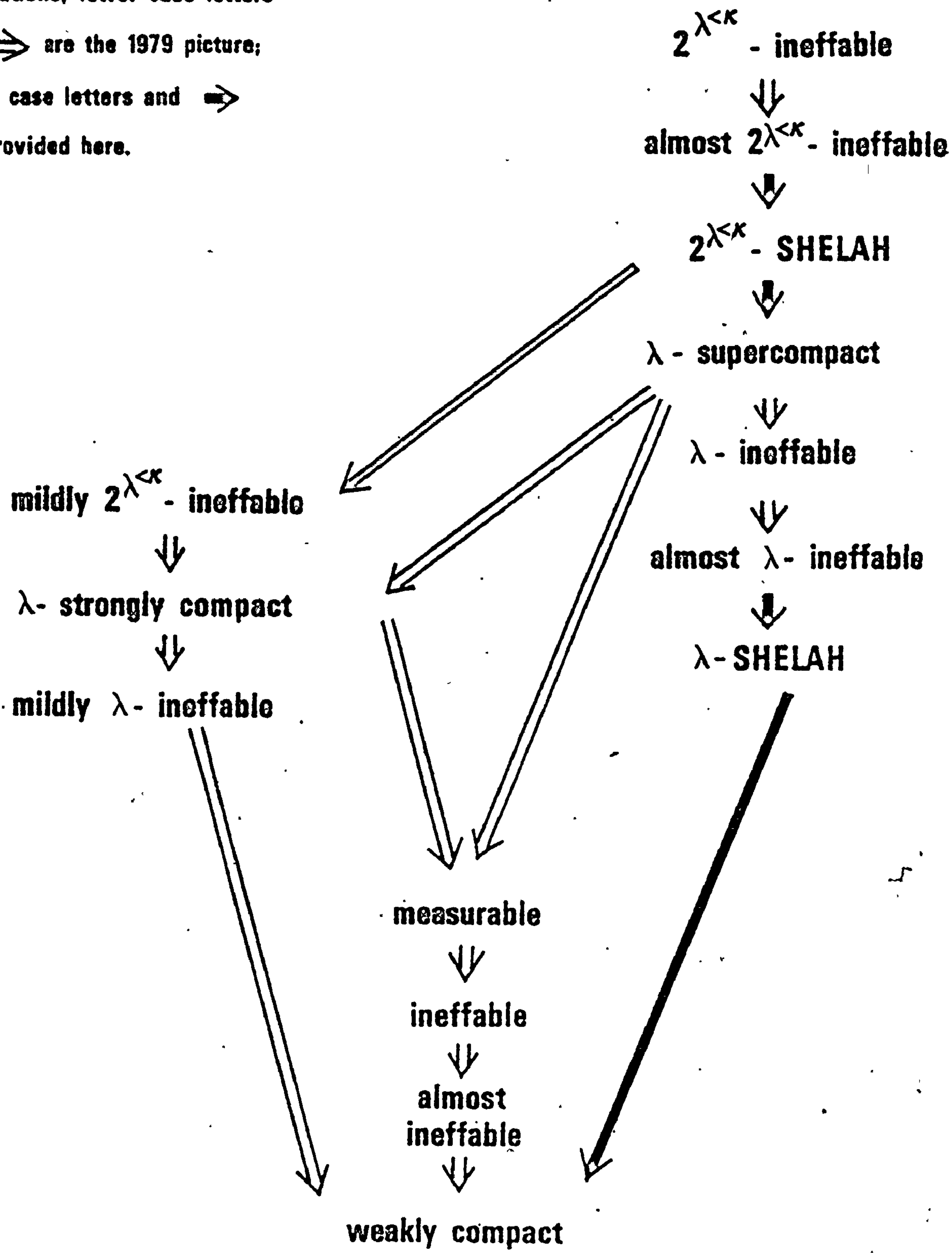

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TABLE OF CONTENTS

INTRODUCTION

It is well-known that the cub filter CF_κ on an uncountable regular cardinal κ is normal (Fodor [13]), that $CF_\kappa = \Delta FSF_\kappa$ (Neumer [28]) where FSF_κ is the filter generated on κ by $\{\{\beta<\kappa : \beta>\alpha\} : \alpha<\kappa\}$, and hence that every normal filter on κ extends CF_κ.

Jech [15] provided $P_\kappa\lambda$ generalizations of the basic concepts of combinatorial set theory, and extended some of the basic results of this theory to the context of $P_\kappa\lambda$. In particular, he provided suitable definitions of "closed", "unbounded", and "Δ" and proved that for every $\lambda \geq \kappa$, the cub filter $CF_{\kappa\lambda}$ on $P_\kappa\lambda$ is normal thus generalizing Fodor's result to the context of $P_\kappa\lambda$.

Menas [25] proved that $X \subset P_\kappa\lambda$ is non-stationary iff there is a function $f : X \to \lambda\times\lambda$ such that $(\forall x\in X)(f(x) \in x\times x)$ and $(\forall\alpha,\beta<\lambda)(f^{-1}(\{(\alpha,\beta)\})$ is not unbounded), and that the condition on f cannot be weakened to allow f to be a function from X into λ. These results of Menas are presented in the dual language of filters and diagonal intersections in the first two sections of chapter 1 below.

In the first section of chapter 1 we find a subset $SC_{\kappa\lambda}$ of the family of all cub subsets of $P_\kappa\lambda$ which generates a filter $SCF_{\kappa\lambda}$ on $P_\kappa\lambda$ with the property that $SCF_{\kappa\lambda} = \Delta FSF_{\kappa\lambda}$. We then use some work of Menas to show that $SCF_{\kappa\lambda} \subsetneq CF_{\kappa\lambda}$ if $\lambda > \kappa$, every normal filter on $P_\kappa\lambda$ extends $CF_{\kappa\lambda}$ and for every $\lambda > \kappa$, $SCF_{\kappa\lambda}$ is not normal.

In the final section of chapter 1 we prove that $\nabla\nabla\nabla I = \nabla\nabla I$ for any ideal I on $P_\kappa\lambda$ thus obtaining the $P_\kappa\lambda$ analogue of the fact that for any ideal I on κ, $\nabla\nabla I = \nabla I$ (see Baumgartner, Taylor, Wagon [5]).

The remaining chapters of this work deal with ideal-theoretic and other aspects of ineffability properties of $P_\kappa\lambda$ with particular emphasis on those which can be viewed as $P_\kappa\lambda$ generalizations of weak compactness.

Jech [15] defined λ-ineffability and almost λ-ineffability as natural generalizations of Jensen's notions (see [18]) of ineffability and almost ineffability. Magidor [23] subsequently proved that κ is supercompact iff κ is λ-ineffable for every $\lambda \geq \kappa$. In the final section of chapter 4 we improve Magidor's result by showing that κ is supercompact iff κ is almost λ-ineffable for every $\lambda \geq \kappa$.

DiPrisco and Zwicker [10] defined mild λ-ineffability, and proved that κ is strongly compact iff κ is mildly λ-ineffable for every $\lambda \geq \kappa$.

We study this notion of DiPrisco and Zwicker in chapter 2 of this work where we obtain characterizations of mild λ-ineffability which are $P_\kappa\lambda$ versions of well-known characterizations of weak compactness. In particular, we show that it is characterized by a $P_\kappa\lambda$ analogue of the tree property due to Jech [15] (section 1), and by a $P_\kappa\lambda$ analogue of the "filter extension property" (section 2) if $\lambda^{<\kappa} = \lambda$. We also show that a slight generalization of a $P_\kappa\lambda$ version of $\kappa \to (\kappa)^2_2$ due to Jech is a sufficient condition for mild λ-ineffability (section 2). Finally, we obtain an M-ultrafilter characterization of mild λ-ineffability in the spirit of Kunen's characterization [21] of weak compactness (section 4).

Inspired by Lévy's work in [22], Baumgartner [1], [2] showed that many small large cardinal properties can be characterized in terms of properties of certain normal ideals on κ. In [1] he showed that weak compactness, almost ineffability and ineffability (among others) can be so characterized. In [2] he observed that except for Π^1_1-indescribability, the usual characterizations of weak compactness are "ideal-theoretically weak", i.e. they do not yield the weakly compact ideal. He then "strengthened" these to obtain new characterizations of weak compactness which do yield this ideal.

In the first section of chapter 3 of this work, we obtain ideal-theoretic characterizations of λ-ineffability, almost λ-ineffability, and mild λ-ineffability. Whereas we find that the ideals characterizing λ-ineffability and almost λ-ineffability are normal, the one characterizing mild λ-ineffability is not normal. This together with our results of chapter 2 and the DiPrisco-Zwicker characterization of strong compactness led us to view mild λ-ineffability as a $P_\kappa\lambda$ generalization of weak compactness in the ideal-theoretically weak sense. Moreover, it prompted us to seek an ideal-theoretically stronger $P_\kappa\lambda$ generalization of weak compactness.

Motivated by some work of Shelah [29] together with a result of our own, we define such a notion in section 2 of chapter 3. There we define the "λ-Shelah property", a new ineffability property of $P_\kappa\lambda$ between mild λ-ineffability and almost λ-ineffability, and show that its associated ideal is normal.

The final chapter of this work is devoted to a detailed study of this new ineffability property. In section 1 we obtain a "normal" ultrafilter property which characterizes λ-Shelah cardinals if $\lambda^{<\kappa} = \lambda$. In section 2 we prove that if κ is $2^{\lambda^{<\kappa}}$-Shelah, then κ is λ-supercompact. From this we infer that κ is supercompact <u>iff</u> κ is λ-Shelah for every $\lambda \geq \kappa$ <u>iff</u> κ is almost λ-ineffable for every $\lambda \geq \kappa$. We also infer that mild λ-ineffability and the λ-Shelah property are not provably equivalent for arbitrary $\lambda > \kappa$.

CHAPTER 0. NOTATION AND BACKGROUND

1. Basic set-theoretic notation.

We work in ZFC, Zermelo-Fraenkel set theory with choice throughout; we refer the reader to [11] or [16] for statements of the axioms of this theory together with other basic set-theoretic background material assumed here.

For any set x, $\bigcup x$ and $\bigcap x$ denote the sets defined by $\bigcup x = \{z : (\exists y \in x)(z \in y)\}$ and $\bigcap x = \{z : (\forall y \in x)(z \in y)\}$. Further, $P(x)$ denotes the power set of x, i.e. the set of all subsets of x. Finally, for any model M of ZFC and any $x \in M$, $P^M(x) \in M$ denotes the power set of x in the sense of M.

For any function f, dom(f) denotes the domain of f, and ran(f) denotes its range. Further, for any $x \subset \mathrm{dom}(f)$, $f''(x)$ denotes the set $\{v \in \mathrm{ran}(f) : (\exists u \in x)(f(u) = v)\}$, and for any $y \subset \mathrm{ran}(f)$, $f^{-1}(y)$ denotes the set $\{u \in \mathrm{dom}(f) : f(u) \in y\}$. Finally, for any two sets x and y, ${}^x y$ denotes the set of all functions from x into y.

Small Greek letters are used to denote ordinals, and for any ordinal α, we write $\lim(\alpha)$ iff α is a limit ordinal. Cardinals are initial ordinals, and are denoted by κ, λ, γ. For any set x, ot(x) denotes the order type of x, and $|x|$ denotes its cardinality. Finally, for any set x and any cardinal $\kappa \le |x|$, $[x]^\kappa$ and $[x]^{<\kappa}$ are the sets defined by $[x]^\kappa = \{y \subset \kappa : |y| = \kappa\}$ and $[x]^{<\kappa} = \{y \subset x : |x| < \kappa\}$.

For any cardinal κ, κ^+ denotes the least cardinal $> \kappa$. κ is called a limit cardinal iff $\gamma^+ < \kappa$ for every cardinal $\gamma<\kappa$, and a limit cardinal κ is called a strong limit cardinal iff $(\forall\alpha<\kappa)(2^\alpha<\kappa)$. An uncountable cardinal κ is said to be regular iff it cannot be expressed as a sum of fewer than κ cardinals $< \kappa$. Finally, an uncountable cardinal κ is said to be inaccessible iff it is a regular strong limit cardinal.

An ideal over a set S is a subset $I \subset P(S)$ with the properties $(\forall X,Y \subset S)(X\subset Y \ \& \ Y\in I \Rightarrow X\in I)$ and $(\forall X,Y \in I)(X\cup Y\in I)$. Filters over S are defined dually. For any ideal I over S, $I^+ = P(S) - I$ is the set of all subsets of S of I-positive measure, and $I^* = \{X\subset S : S-X\in I\}$ is the filter dual to I. An ideal I over S is said to be proper iff $S \notin I$, to be non-principal iff $\bigcup I \notin I$, and to be prime iff it is proper and $(\forall X\subset S)(X\in I \text{ or } S-X\in I)$. The dual of a prime ideal is called an ultra-filter. Finally, for any cardinal κ, an ideal I is said to be κ-complete iff $(\forall X\in [I]^{<\kappa})(\bigcup X\in I)$.

In the sequel, κ always denotes an uncountable regular cardinal unless we specify otherwise, and λ is a cardinal $\geq \kappa$. For any such pair, $P_\kappa\lambda$ denotes the set $[\lambda]^{<\kappa} = \{x\subset\lambda : |x|<\kappa\}$, and $\lambda^{<\kappa}$ is the cardinality of this set. Finally, the phrase "filter over $P_\kappa\lambda$" always means a proper, non-principal, κ-complete filter over $P_\kappa\lambda$ extending the set $\{\{x\in P_\kappa\lambda : \alpha\in x\} : \alpha<\lambda\}$ unless we specify otherwise. The phrase "ideal over $P_\kappa\lambda$" is used dually.

2. Strong compactness and supercompactness.

Strong compactness and supercompactness are generalizations of weak compactness and measurability respectively.

For any uncountable regular cardinal κ, $L_{\kappa\omega}$ is the infinitary first-order language obtained by extending the finitary first-order language with equality to allow conjunctions and disjunctions of length $<\kappa$. These and more general infinitary languages are studied in depth in [9].

A set $\sum$ of $L_{\kappa\omega}$ sentences is <u>κ-satisfiable</u> iff every subset of $\sum$ of cardinality $< \kappa$ has a model. An uncountable regular cardinal κ is said to be <u>weakly compact</u> iff κ is inaccessible, and every κ-satisfiable set $\sum$ of $L_{\kappa\omega}$ sentences such that $|\sum| = \kappa$ has a model. A natural generalization of this was introduced by Keisler and Tarski [20]:

2.1 <u>Definition</u>. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, say that κ is <u>λ-compact</u> iff κ is inaccessible, and any κ-satisfiable set $\sum$ of $L_{\kappa\omega}$ sentences such that $|\sum| = \lambda$ has a model. Finally, say that κ is <u>strongly compact</u> iff κ is λ-compact for every $\lambda \geq \kappa$.

Standard arguments (eg. see [19], [30]) yield

2.2 <u>Theorem</u>. For any uncountable regular cardinal κ, t.f.a.e.

(1) κ is strongly compact.

(2) For every $\lambda \geq \kappa$, there is a non-principal κ-complete ultrafilter $\mathcal{U}$

over $P_\kappa\lambda$ such that $(\forall\alpha<\lambda)(\{x\in P_\kappa\lambda : \alpha\in x\} \in U)$.

(3) For every $\lambda \geq \kappa$, there is an inner model M and an elementary embedding $j : V \to M$ such that κ is the least ordinal moved, and $(\forall X\subset M)(|X| \leq \lambda \Rightarrow (\exists Y\in M)(X\subset Y \ \& \ M \models |Y| < j(\kappa)))$.

(4) For every $\lambda \geq \kappa$, any proper, $\leq \lambda$-generated, κ-complete filter over an arbitrary set S can be extended to a κ-complete ultrafilter over S. □

The properties given in (2), (3), (4) are often referred to as λ-strong compactness, i.e. κ is said to be <u>λ-strongly compact</u> iff there is a non-principal κ-complete ultrafilter over $P_\kappa\lambda$ extending $\{\{x\in P_\kappa\lambda : \alpha\in x\} : \alpha<\lambda\}$.

It is easy to see that if κ is strongly compact, then κ is measurable, i.e. there is a non-principal κ-complete ultrafilter over κ. In fact, κ is measurable iff κ is κ-strongly compact. However, measurability has certain features which are not generalized nicely by strong compactness. For instance, standard arguments (eg. see [30]) show that κ is measurable iff there is an inner model M and an elementary embedding $j : V \to M$ such that κ is the least ordinal moved and ${}^\kappa M \subset M$ but ${}^{\kappa^+}M \not\subset M$. 2.2(3) doesn't seem to be the nicest possible generalization of this. Solovay and Reinhardt (see [30]) provided a better one:

2.3 <u>Definition</u>. For any uncountable regular cardinal κ, and any cardinal $\lambda \geq \kappa$, say that κ is <u>λ-supercompact</u> iff there is an inner model M and an elementary embedding $j : V \to M$ such that

(1) $j \upharpoonright \kappa = \mathrm{id} \upharpoonright \kappa$ and $\lambda < j(\kappa)$, and

(2) ${}^{\lambda}M \subset M$.

Say that κ is <u>supercompact</u> iff κ is λ-supercompact for every $\lambda \geq \kappa$.

Standard arguments (eg. see [19], [30]) yield the following well known theorem.

2.4 <u>Theorem</u>. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, κ is λ-supercompact iff there is a non-principal ultrafilter $\mathcal{U}$ over $P_\kappa\lambda$ such that

(1) $(\forall\alpha<\lambda)(\{x\in P_\kappa\lambda : \alpha\in x\} \in \mathcal{U})$, and

(2) $\mathcal{U}$ is <u>normal</u>, i.e. for any function $f : P_\kappa\lambda \to \lambda$ such that $\{x\in P_\kappa\lambda : f(x)\in x\} \in \mathcal{U}$, $(\exists\alpha<\lambda)(f^{-1}(\{\alpha\})\in\mathcal{U})$. □

It is clear that if κ is supercompact, then κ is strongly compact. However, studies of Menas [25] show that the converse of this isn't provably true. He showed that if there is a measurable cardinal κ which is the limit of strongly compact cardinals, then κ is strongly compact, but the least such κ isn't even 2^κ-supercompact. This result has since been subsumed by Magidor's work [24].

Inspired by Kunen's work in [21], Jech [15] provided a notion of λ-supercompactness relative to a transitive model M of ZFC:

2.5 <u>Definition</u>. For any transitive model M of ZFC and any cardinals $\kappa \leq \lambda$ in M, say that κ is <u>λ-supercompact with respect to M</u> iff there is an elementary embedding $j: M \to N$ such that

(1) $j\restriction\kappa = id\restriction\kappa$

(2) $j(\kappa) > \lambda$

(3) $j''(\lambda) \in N$, and

(4) M and N admit the same λ-sequences.

We will discuss this notion further in section 4 of chapter 2.

3. Combinatorial aspects of $P_\kappa\lambda$.

In [15] Jech formulated $P_\kappa\lambda$ versions of the basic concepts of combinatorial set theory, and extended some of the basic results of this theory to the context of $P_\kappa\lambda$. We begin with a brief sketch of a portion of the theory he sought to generalize. The reader is referred to [15], [16] and references cited therein for a more detailed description of this theory.

For any uncountable regular cardinal κ, $X \subset \kappa$ is said to be unbounded or cofinal in κ iff $(\forall\alpha<\kappa)(\exists\beta\in X)(\alpha\leq\beta)$. X is said to be closed in κ iff it is closed in the order topology on κ, i.e. iff $(\forall\alpha<\kappa)(\bigcup(X\cap\alpha)\in X)$, equivalently iff $(\forall Y\in[X]^{<\kappa})(\bigcup Y\in X)$. Finally, $X \subset \kappa$ is called a cub iff it is both closed and unbounded in κ.

It is well-known that the set of all cub subsets of κ generates a proper, non-principal, κ-complete filter CF_κ on κ. It is also well-known that CF_κ is normal (Fodor [13]), i.e. that $(\forall\langle X_\alpha : \alpha<\kappa\rangle\in{}^\kappa CF_\kappa)(\Delta\{X_\alpha : \alpha<\kappa\} = \{\beta<\kappa : (\forall\alpha<\beta)(\beta\in X_\alpha)\}\in CF_\kappa)$. Finally, it is also well-known that $CF_\kappa = \Delta FSF_\kappa$ (Neumer [28]) where FSF_κ is the filter generated on κ by $\{\{\beta<\kappa : \beta>\alpha\} : \alpha<\kappa\}$, and ΔFSF_κ is the filter $\{X\subset\kappa : (\exists\langle X_\alpha : \alpha<\kappa\rangle\in{}^\kappa FSF_\kappa)(X=\Delta\{X_\alpha : \alpha<\kappa\})\}$. An immediate consequence of these facts is that every normal filter on κ extends CF_κ.

We now outline the particular of Jech's work.

3.1 For each $x \in P_\kappa\lambda$, let $\hat{x}$ denote the set $\{y\in P_\kappa\lambda : x\subset y\}$. It is easy to see that the family $\{\hat{x} : x\in P_\kappa\lambda\}$ generates a proper non-principal, κ-complete filter over $P_\kappa\lambda$. We denote this filter by $FSF_{\kappa\lambda}$ (the "final segment filter" on $P_\kappa\lambda$), and its dual by $I_{\kappa\lambda}$.

By a filter on $P_\kappa\lambda$ we mean a proper, non-principal, κ-complete filter on $P_\kappa\lambda$ extending $FSF_{\kappa\lambda}$. Dually, an ideal on $P_\kappa\lambda$ is a proper, non-principal, κ-complete ideal on $P_\kappa\lambda$ extending $I_{\kappa\lambda}$.

3.2 As in Jech [15], we call $X \subset P_\kappa\lambda$ unbounded iff $(\forall y\in P_\kappa\lambda)(X\cap\hat{y}\neq 0)$. Thus $I_{\kappa\lambda}$ is the ideal of "not unbounded" subsets of $P_\kappa\lambda$.

$C \subset P_\kappa\lambda$ is said to be closed iff $(\forall X\subset C)(|X|<\kappa \;\&\; X \text{ is directed} \Rightarrow \bigcup X\in C)$. Note that by a result of Solovay (eg. see [23]), $C \subset P_\kappa\lambda$ is closed iff $(\forall\delta<\kappa)(\forall\{x_\nu : \nu<\delta\}\subset C)(x_0\subset..\subset x_\nu\subset..(\nu<\delta) \Rightarrow \bigcup\{x_\nu : \nu<\delta\}\in C)$. Finally $C \subset P_\kappa\lambda$ is called a cub iff it is both closed and unbounded.

We denote the family of all cub subsets of $P_\kappa\lambda$ by $C_{\kappa\lambda}$, and say that $S \subset P_\kappa\lambda$ is stationary iff $(\forall C\in C_{\kappa\lambda})(S\cap C\neq 0)$.

$C_{\kappa\lambda}$ is easily seen to generate a filter on $P_\kappa\lambda$ (eg. see Jech [15]). We denote this filter by $CF_{\kappa\lambda}$ and call it the cub filter on $P_\kappa\lambda$. Its dual $NS_{\kappa\lambda}$ is the non-stationary ideal on $P_\kappa\lambda$.

3.3 The diagonal intersection $\Delta\{X_\alpha : \alpha<\lambda\}$ and the diagonal union $\nabla\{X_\alpha : \alpha<\lambda\}$ of any $(X_\alpha : \alpha<\lambda) \in {}^\lambda\mathcal{P}(P_\kappa\lambda)$ are defined by $\Delta\{X_\alpha : \alpha\in\lambda\} = \{x\in P_\kappa\lambda : (\forall\alpha\in x)(x\in X_\alpha)\}$ and $\nabla\{X_\alpha : \alpha<\lambda\} = \{x\in P_\kappa\lambda : (\exists\alpha\in x)(x\in X_\alpha)\}$.

A filter F (an ideal I) on $P_\kappa\lambda$ is said to be <u>normal</u> iff F (I) is closed under diagonal intersections (diagonal unions).

For any filter F on $P_\kappa\lambda$, ΔF denotes the set $\{X \in P_\kappa\lambda : (\exists \langle X_\alpha : \alpha<\lambda\rangle \in {}^\lambda F)(X = \Delta\{X_\alpha : \alpha<\lambda\})\}$. ∇I is defined dually for an ideal I on $P_\kappa\lambda$. Clearly F (I) is normal iff $F = \Delta F$ ($I = \nabla I$).

It is easy to see that ΔF is a (possibly improper) filter extending F and dually that ∇I is a (possibly improper) ideal extending I. Also, it is easy to see that if F and I are dual to one another, then so are ΔF and ∇I.

Jech [15] proved that $CF_{\kappa\lambda}$ is normal for every $\lambda \geq \kappa$ thus generalizing Fodor's result to the context of $P_\kappa\lambda$. He also showed that for every $\lambda \geq \kappa$, if κ is λ-supercompact, then every normal ultrafilter $\mathcal{U}$ over $P_\kappa\lambda$ extends $CF_{\kappa\lambda}$ using the properties of the ultrapower $Ult_{\mathcal{U}}(V)$.

Menas proved in [25] that for every $\lambda \geq \kappa$ and any $X \subset P_\kappa\lambda$, X is non-stationary iff there is an $f : X \to \lambda\times\lambda$ such that $(\forall x\in X)(f(x)\in x\times x)$ and $(\forall\alpha,\beta<\lambda)(f^{-1}(\{(\alpha,\beta)\})\in I_{\kappa\lambda})$, and if $\lambda > \kappa$, the condition on f cannot be weakened to allow f to be a function from X into λ. The first two sections of chapter 1 below include a presentation of these results of Menas in the dual language of filters and diagonal intersections. This yields the $P_\kappa\lambda$ analogue of Neumer's theorem together with the minimality of $CF_{\kappa\lambda}$. We conclude this section with a sketch of the particulars of those notions and results of Menas which are needed in chapter 1.

3.4 Definition. For any finite $n \geq 1$ and any $w : \lambda^n \to P_\kappa\lambda$, let $C(\{w\})$ be the set defined by $C(\{w\}) = \{x \in P_\kappa\lambda : (\forall \bar{\alpha} \in x^n)(w(\bar{\alpha}) \subset x)\}$.

A routine argument yields the following useful fact:

3.5 Lemma. For any finite $n \geq 1$ and any $w : \lambda^n \to P_\kappa\lambda$, $C(\{w\})$ is cub in $P_\kappa\lambda$. Moreover, $C = C(\{w\})$ has the property $(\forall X \subset C)(X \neq 0 \to \bigcap X \in C)$. □

More work is required to obtain Menas' representation theorem for cubs:

3.6 Theorem. For any cub subset C of $P_\kappa\lambda$, there is a $w : \lambda^2 \to P_\kappa\lambda$ such that $C(\{w\}) \subset C$. □

Remark. Menas called closed subsets of $P_\kappa\lambda$ with the property given in the last sentence of 3.5 "strongly closed". However, we wish to reserve this term for a stronger notion (definition 1.1.1 below). We therefore call those closed subsets of $P_\kappa\lambda$ with the above property "Menas closed":

3.7 Definition. $C \subset P_\kappa\lambda$ is said to be Menas closed iff it is closed and has the property $(\forall X \subset C)(X \neq 0 \to \bigcap X \in C)$. Thus $C \subset P_\kappa\lambda$ is called a Menas cub iff it is unbounded and Menas closed.

It is easy to see that the intersection of any $<\kappa$-sequence of Menas cubs is a Menas cub, and that the diagonal intersection of any λ-sequence of Menas cubs is a Menas cub (Menas [25]). Thus the family of all Menas cubs generates a normal filter $MCF_{\kappa\lambda}$ on $P_\kappa\lambda$. An immediate consequence of 3.6 is that $MCF_{\kappa\lambda} = CF_{\kappa\lambda}$ (Menas [25]).

4. Ineffability properties.

Jensen [18] defined an uncountable regular cardinal κ to be ineffable iff for any $(A_\alpha : \alpha<\kappa)$ such that $(\forall\alpha<\kappa)(A_\alpha\subset\alpha)$, $(\exists A\subset\kappa)(H=\{\alpha<\kappa : A_\alpha=A\cap\alpha\}\in NS_\kappa^+)$ where NS_κ is the ideal of non-stationary subsets of κ. Further, he defined κ to be almost ineffable iff for any $(A_\alpha : \alpha<\kappa)$ such that $(\forall\alpha<\kappa)(A_\alpha\subset\alpha)$, $(\exists A\subset\kappa)(H=\{\alpha<\kappa : A_\alpha=A\cap\alpha\}\in I_\kappa^+)$ where I_κ is the ideal $[\kappa]^{<\kappa}$ of bounded subsets of κ.

Inspired by Lévy's work in [22], Baumgartner [1], [2] showed that these and other small large cardinal properties can be viewed as properties of normal ideals on κ. For instance, he showed in [1] that κ is almost ineffable iff there is a normal ideal $NAIn_\kappa$ on κ such that for every $X\subset\kappa$, $X\in NAIn_\kappa^+$ iff for any $(A_\alpha : \alpha\in X)$ such that $(\forall\alpha\in X)(A_\alpha\subset\alpha)$, $(\exists A\subset\kappa)(H=\{\alpha\in X : A_\alpha=A\cap\alpha\}\in I_\kappa^+)$.

Jech [15] provided $P_\kappa\lambda$ generalizations of ineffability and almost ineffability:

4.1 Definition. For any uncountable regular cardinal κ and any cardinal $\lambda\geq\kappa$, κ is said to be

(1) λ-ineffable iff for every $(A_x : x\in P_\kappa\lambda)$ such that $(\forall x\in P_\kappa\lambda)(A_x\subset x)$, $(\exists A\subset\lambda)(H=\{x\in P_\kappa\lambda : A_x=A\cap x\}\in NS_{\kappa\lambda}^+)$, and

(2) almost λ-ineffable iff for every $(A_x : x\in P_\kappa\lambda)$ such that $(\forall x\in P_\kappa\lambda)(A_x\subset x)$, $(\exists A\subset\lambda)(H=\{x\in P_\kappa\lambda : A_x=A\cap x\}\in I_{\kappa\lambda}^+)$.

Magidor [23] showed that if κ is λ-supercompact then κ is λ-ineffable, and if κ is $2^{\lambda^{<\kappa}}$-ineffable then κ is λ-supercompact. Thus λ is supercompact iff κ is λ-ineffable for every $\lambda \geq \kappa$.

DiPrisco and Zwicker [10] defined mild λ-ineffability, and showed that it characterizes strong compactness in the same way; i.e. that κ is strongly compact iff κ is mildly λ-ineffable for every $\lambda \geq \kappa$. The definition we give in 4.2 below is easily seen to be equivalent to the one given in [10].

4.2 Definition. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, κ is said to be mildly λ-ineffable iff for any $(A_x : x \in P_\kappa\lambda)$ such that $(\forall x \in P_\kappa\lambda)(A_x \subset x)$, $(\exists A \subset \lambda)(\forall x \in P_\kappa\lambda)(H_x = \{y \in \hat{x} : A_y \cap x = A \cap x\} \neq 0)$.

Chapter 2 of this work is a detailed study of this notion of DiPrisco and Zwicker.

5. Aspects of weak compactness.

Recall that an uncountable cardinal κ is said to be weakly compact iff κ is inaccessible, and any κ-satisfiable set Σ of $L_{\kappa\omega}$ sentences such that $|\Sigma| = \kappa$ has a model. It is well known that this notion has several other characterizations. Some of these are given in the following theorem whose proof may be found in many standard references - eg. [9], [11], [16].

5.1 Theorem. For any inaccessible cardinal κ, the following are equivalent:

(1) κ is weakly compact

(2) κ is Π^1_1-indescribable, i.e. for any $R \subset V_\kappa$ and any Π^1_1-sentence φ, if $(V_\kappa, \in, R) \models \varphi$, then $(V_\alpha, \in, R \cap V_\alpha) \models \varphi$ for some $\alpha < \kappa$.

(3) $\kappa \to (\kappa)^2_2$, i.e. for any $f : [\kappa]^2 \to 2$, $(\exists H \in I_\kappa{}^+)(|f''([H]^2)| = 1)$.

(4) κ has the tree property, i.e. through any tree T of height κ such that every element of T has fewer than κ immediate successors there is a path of length κ.

(5) $(UP)_\kappa$ holds, i.e. there is a non-principal κ-complete ultrafilter in any κ-complete field B of subsets of κ such that $\{\{\alpha\} : \alpha < \kappa\} \subset B$ and $|B| = \kappa$. □

5.2 Remark. A tree T of height κ such that every element of T has fewer than γ immediate successors is often called a (κ,γ) tree. The proof that (3) $\Leftrightarrow$ (4) in 5.1 really shows that 5.1(4) is equivalent with the assertion that through every $(\kappa,3)$ tree on κ there is a path of length κ.

It is easy to see that each of (2) - (5) in 5.1 above has an ideal-theoretic equivalent.

eg. 5.1(2) holds iff the set $N\Pi^1_1$ consisting just of those $X \subset \kappa$ for which there is an $R \subset V_\kappa$ and a Π^1_1 sentence φ such that $(V_\kappa, \epsilon, R) \models \varphi$ but $(\forall\alpha\in X)((V_\alpha, \epsilon, R\cap V_\alpha) \not\models \varphi)$ is a proper, non-principal, κ-complete ideal on κ.

Lévy [22] and Baumgartner [1] showed that κ is Π^1_1-indescribable iff $N\Pi^1_1$ is a normal ideal on κ. Baumgartner calls this the <u>weakly compact ideal</u>.

Baumgartner remarked in [2] that 5.1(3) - (5) do not yield the weakly compact ideal. In fact, they just yield I_κ. He then proceeded in [2] to show that each of 5.1(3) - (5) can be "strengthened" to obtain properties which do yield the weakly compact ideal. His proof of this yields the following theorem.

5.3 <u>Theorem</u>. For any inaccessible cardinal κ, the following are equivalent:

(1) κ is Π^1_1-indescribable.

(2) $\kappa \to (\kappa, \text{stationary set})^2$, i.e. for any $f : [\kappa]^2 \to 2$, <u>either</u> $(\exists H\in I_\kappa^+)(f''([H]^2) = \{0\})$ <u>or else</u> $(\exists H\in NS_\kappa^+)(f''([H]^2) = \{1\})$.

(3) Through every tree $(T, \leq_T)$ on κ with the properties that $\alpha \leq_T \beta \Rightarrow \alpha < \beta$ and each element of T has non-stationarily many immediate successors there is a path of length κ.

(4) $(NUP)_\kappa$ holds, i.e. for any κ-complete field B of subsets of κ and any set G of regressive functions on κ such that $|B| = |G| = \kappa$ and $(\forall g\in G)(\forall\alpha<\kappa)(g^{-1}(\{\alpha\})\in B)$, there is a non-principal κ-complete ultrafilter $\mathcal{U}$ in B such that every function in G is constant on a set in $\mathcal{U}$. □

We call the characterizations of weak compactness expressed by 5.1(3) - (5) <u>ideal-theoretically weak</u> characterizations. Those expressed by 5.3(1) - (4) are called <u>ideal-theoretically strong</u>.

Another ideal-theoretically strong characterization of weak compactness was given by Shelah in [29]. A perusal of Shelah's paper suggests the following definition together with the formulation of his result which succeeds it.

5.4 <u>Definition</u>. For any uncountable regular cardinal κ, $X \subset \kappa$ is said to have the <u>Shelah property</u> iff for any $(f_\alpha : \alpha\in X)$ such that $(\forall\alpha\in X)(f_\alpha : \alpha\to\alpha)$, $(\exists f : \kappa\to\kappa)(\forall\alpha<\kappa)(\{\beta\in[\alpha,\kappa)\cap X : f_\beta\restriction\alpha = f\restriction\alpha\} \neq 0)$ where $[\alpha,\kappa) = \{\beta<\kappa : \alpha\leq\beta\}$. Further, define the set NSh_κ by $NSh_\kappa = \{X\subset\kappa : X$ does <u>not</u> have the Shelah property$\}$.

Shelah proved that κ is weakly compact iff κ has the Shelah property, and that if κ has the Shelah property, then NSh_κ is a normal ideal on κ.

We conclude this section by noting that it is easy to see that κ has the Shelah property iff κ satisfies "Baumgartner's principle" where the latter is defined as in 5.5 below. This definition is easily seen to be equivalent to the version appearing in Erdös et al. [12].

5.5 Definition. An uncountable regular cardinal κ is said to satisfy Baumgartner's principle iff for any κ-sequence $(g_\alpha : \alpha<\kappa)$ of regressive functions on κ, $(\exists f : \kappa\to\kappa)(\forall\alpha<\kappa)(\{\beta\in[\alpha,\kappa) : (\forall\xi<\alpha)(g_\xi(\beta) = f(\xi))\}\neq 0)$.

CHAPTER 1. THE MINIMAL NORMAL FILTER ON $P_\kappa\lambda$.

1. The strong cub filter and its dual.

1.1 Definition. $C \subset P_\kappa\lambda$ is said to be strongly closed iff $(\forall X \in [C]^{<\kappa})(\bigcup X \in C)$. Thus $C \subset P_\kappa\lambda$ is called a strong cub iff it is unbounded and strongly closed.

Remark. In [25], Menas used the term "strongly closed" for a different but related concept. See 0.3.7 above for the particulars of his concept.

It is easy to see that the intersection of any $<\kappa$ sequence of strong cubs is a strong cub and hence that the family $SC_{\kappa\lambda}$ of strong cub subsets of $P_\kappa\lambda$ generates a κ-complete filter on $P_\kappa\lambda$. We call this the strong cub filter and denote it by $SCF_{\kappa\lambda}$. $SCF_{\kappa\lambda}$ is easily obtained from $FSF_{\kappa\lambda}$ as we now show.

1.2 Theorem. $(\forall\lambda \geq \kappa)(SCF_{\kappa\lambda} = \Delta FSF_{\kappa\lambda})$.

Proof. First, pick $(x_\alpha : \alpha<\lambda) \in {}^{\lambda}P_\kappa\lambda$ and set $C = \Delta\{\hat{x}_\alpha : \alpha<\lambda\} = \{x \in P_\kappa\lambda : (\forall\alpha\in x)(x_\alpha \subset x)\}$. Clearly C is cub. Let $X \in [C]^{<\kappa}$. Clearly $\bigcup X \in P_\kappa\lambda$. Now let $\alpha \in \bigcup X$ and pick $x \in X \subset C$ such that $\alpha \in x$. Then $x_\alpha \subset x \subset \bigcup X$, so $X \in C$.

Conversely, let $C \subset P_\kappa\lambda$ be a strong cub. For each $\alpha < \lambda$ pick $x_\alpha \in C$ such that $\alpha \in x_\alpha$. We show that $\Delta\{\hat{x}_\alpha : a<\lambda\} \subset C$. Pick $x \in \Delta\{\hat{x}_\alpha : \alpha<\lambda\}$. Since $(\forall\alpha\in x)(x_\alpha \subset x)$ and since $x \subset \bigcup\{x_\alpha : \alpha\in x\}$ it is clear that $x = \bigcup\{x_\alpha : \alpha \in x\}$. Then since C is strongly closed, it follows that $x \in C$. □

Note that the proof of 1.2 yields the following useful fact:

1.3 $\Delta\{\widehat{x_\alpha} : \alpha<\lambda\}$ is a strong cub for any $(x_\alpha : \alpha<\lambda) \in {}^{\lambda}P_\kappa\lambda$. □

An easy consequence of theorem 1.2 is the minimality of $CF_{\kappa\kappa}$:

1.4 <u>Corollary</u>. $CF_{\kappa\kappa} = SCF_{\kappa\kappa}$ is the smallest normal filter on $P_\kappa\kappa$.

<u>proof</u>. It is clear that $SCF_{\kappa\kappa} \subset CF_{\kappa\kappa}$. To obtain the reverse inclusion, first recall that κ is cub in $P_\kappa\kappa$ and hence that for any cub $C \subset P_\kappa\kappa$, $C \cap \kappa$ is cub in $P_\kappa\kappa$. Now notice that for any cub $C \subset P_\kappa\kappa$, $C \cap \kappa$ is strongly closed in $P_\kappa\kappa$. Thus $CF_{\kappa\kappa} = SCF_{\kappa\kappa}$.

Since $\Delta FSF_{\kappa\kappa} = CF_{\kappa\kappa}$ is normal and since every normal filter on $P_\kappa\kappa$ must extend $\Delta FSF_{\kappa\kappa}$ the minimality of $CF_{\kappa\kappa}$ is clear. □

1.5 <u>Remark</u>. There is another way to obtain the result of corollary 1.4 as we now outline.

It is easy to see that for any <u>normal</u> filter F on $P_\kappa\kappa$, $F\restriction\kappa = \{X\cap\kappa : X\in F\}$ is a normal filter on κ, and that $F = \{X\subset P_\kappa\kappa : X\cap\kappa\in F\restriction\kappa\}$. Moreover, it is easy to see that $CF_\kappa = CF_{\kappa\kappa}\restriction\kappa$ and hence that $CF_{\kappa\kappa} = \{X\subset P_\kappa\kappa : X\cap\kappa\in CF_\kappa\}$. Thus the minimality of CF_κ implies the minimality of $CF_{\kappa\kappa}$.

It is clear that $(\forall\lambda\geq\kappa)(SCF_{\kappa\lambda}\subset CF_{\kappa\lambda})$, and as shown in the proof of 1.4, this inclusion reverses if $\lambda = \kappa$. If $\lambda > \kappa$ however, then as a careful examination of Menas' proof of 1.7 in [25] reveals, $SCF_{\kappa\lambda} \subsetneq CF_{\kappa\lambda}$. For the sake of completeness, we will give all of the particulars here (1.7 and 1.8 below). This requires the following simple preliminary.

1.6 Lemma. The following are Menas cubs:

(1) $\Delta\{\hat{x}_\alpha : \alpha<\lambda\}$ for any $(x_\alpha : \alpha<\lambda) \in {}^\lambda P_\kappa\lambda$, and

(2) $C_f = \{x \in P_\kappa\lambda : f''(x\times x) \subset x\}$ for any $f : \lambda\times\lambda \to \lambda$.

proof. Recall that for any finite $n \geq 1$ and any $w : \lambda^n \to P_\kappa\lambda$, $C(\{w\}) = \{x \in P_\kappa\lambda : (\forall\bar{\alpha}\in x^n)(w(\bar{\alpha}) \subset x)\}$ is a Menas cub (see 0.3.5 above).

(1) Pick $(x_\alpha : \alpha<\lambda) \in {}^\lambda P_\kappa\lambda$ and define $w : \lambda \to P_\kappa\lambda$ by $w(\alpha) = x_\alpha$. It is easy to see that $\Delta\{\hat{x}_\alpha : \alpha<\lambda\} = C(\{w\})$.

(2) Pick $f : \lambda\times\lambda \to \lambda$ and define $w : \lambda\times\lambda \to P_\kappa\lambda$ by $w(\alpha,\beta) = \{f(\alpha,\beta)\}$. It is clear that $C_f = C(\{w\})$. □

1.7 Lemma. For any regular cardinal $\lambda > \kappa$ and any bijection $f : \lambda\times\lambda \leftrightarrow \lambda$, $C_f \in CF_{\kappa\lambda} - SCF_{\kappa\lambda}$.

proof. It follows from 1.6 that $C_f \in CF_{\kappa\lambda}$. Suppose by way of contradiction that $C_f \in SCF_{\kappa\lambda} = \Delta FSF_{\kappa\lambda}$ and let $(z_\alpha : \alpha<\lambda) \in {}^\lambda P_\kappa\lambda$ be such that $C = \Delta\{\hat{z}_\alpha : \alpha<\lambda\} \subset C_f$.

We construct a regressive function $g : \lambda - \kappa \to \lambda$ and then use this to derive the required contradiction. This will require a few preliminaries.

For each $\alpha,\beta < \lambda$ define $x_\alpha = \bigcap\{x\in C : \alpha\in x\}$, $x_\beta = \bigcap\{x\in C : \beta\in x\}$, $x_{\alpha\beta} = \bigcap\{x\in C : \{\alpha,\beta\}\subset x\}$. By 1.6, C is Menas closed so x_α, x_β, $x_{\alpha\beta}$ are all in $C \subset C_f$. And by 1.3 above, C is also strongly closed so

$x_\alpha \cup x_\beta \in C$. Thus $x_{\alpha\beta} = x_\alpha \cup x_\beta$.

Now pick $\alpha \in \lambda - \kappa$. Note that since f is one-one, $|\{f(\alpha,\beta) : \beta<\alpha\}| \geq \kappa$. But $|x_\alpha| < \kappa$ so $(\exists\beta<\alpha)(f(\alpha,\beta) \notin x_\alpha)$. We are now ready to define g. For each $\alpha \in \lambda - \kappa$ pick $\beta_\alpha < \alpha$ such that $f(\alpha,\beta_\alpha) \notin x_\alpha$ and then set $g(\alpha) = \beta_\alpha$. Clearly g is regressive.

We can now derive the required contradiction. Pick $\beta < \lambda$ such that $X = g^{-1}(\{\beta\}) \in NS_\lambda^{\,+}$. The definition of g guarantees that $(\forall\alpha\in X)(f(\alpha,\beta) \notin x_\alpha)$. But $(\forall\alpha\in X)(f(\alpha,\beta) \in x_{\alpha\beta} = x_\alpha \cup x_\beta)$ since $\{\alpha,\beta\} \subset x_{\alpha\beta} = x_\alpha \cup x_\beta$ and $C \subset C_f$. This means that $(\forall\alpha\in X)(f(\alpha,\beta) \in x_\beta)$ thus contradicting the one-one-ness of f since $|x_\beta| < \kappa < \lambda = |X|$. □

1.8 Theorem. For every $\lambda > \kappa$, $SCF_{\kappa\lambda} \subsetneq CF_{\kappa\lambda}$.

proof. Pick $\lambda > \kappa$, let $p : \lambda\times\lambda \leftrightarrow \lambda$ be the canonical bijection, and set $C_p = \{y \in P_\kappa\lambda : p''(y\times y) \subset y\}$. Further, set $q = p \restriction \kappa^+\times\kappa^+$ and notice that q is the canonical bijection on $\kappa^+\times\kappa^+$. We will show that $C_p \notin SCF_{\kappa\lambda}$.

Suppose by way of contradiction that $C_p \in SCF_{\kappa\lambda} = \Delta FSF_{\kappa\lambda}$; let $(z_\alpha : \alpha<\lambda) \in {}^\lambda P_\kappa\lambda$ be such that $C = \Delta\{\hat{z}_\alpha : \alpha<\lambda\} \subset C_p$. We will construct a sequence $(z_\alpha^* : \alpha<\kappa^+) \in {}^{\kappa^+}P_\kappa\kappa^+$ such that $C^* = \Delta\{\widehat{z_\alpha^*} : \alpha<\kappa^+\} \subset C_q$ thus contradicting the preceeding lemma.

For each $\alpha < \kappa^+$ define $z_\alpha^* = (\bigcap\{y\in C : \alpha\in y\}) \cap \kappa^+$; write $z_\alpha^* = y_\alpha \cap \kappa^+$ where $y_\alpha = \bigcap\{y\in C : \alpha\in y\}$.

Now pick $x \in C^* = \Delta\{\widehat{z_\alpha^*} : \alpha<\kappa^+\}$. We show that $x \in C_q$. So pick $\alpha,\beta \in x \subset \kappa^+$ and set $y = y_\alpha \cup y_\beta$. Note that $\{\alpha,\beta\} \subset y \in C \subset C_p$ so $p(\alpha,\beta) \in y$. Then since $\alpha,\beta < \kappa^+$ we have $p(\alpha,\beta) = q(\alpha,\beta) < \kappa^+$. Thus

$q(\alpha,\beta) \in y \cap \kappa^+ = (y_\alpha \cup y_\beta) \cap \kappa^+ = (y_\alpha \cap \kappa^+) \cup (y_\beta \cap \kappa^+) = z_\alpha^* \cup z_\beta^* \subset x$ so $q(\alpha,\beta) \in x$. Thus $C^* \subset C_q$. □

In light of theorems 1.2 and 1.8 above, we see that Neumer's theorem $CF_\kappa = \dot{\Delta}FSF_\kappa$ does not generalize to $P_\kappa\lambda$ if $\lambda > \kappa$. We obtain the $P_\kappa\lambda$ analogue of Neumer's theorem in section 2 below.

We conclude this section with some brief remarks on the ideal I dual to $SCF_{\kappa\lambda}$. Notice that for any $X \subset P_\kappa\lambda$, $X \in I$ iff $P_\kappa\lambda - X$ contains a strong cub. Thus we make the following definition:

1.9 <u>Definition</u>. We call the ideal dual to $SCF_{\kappa\lambda}$ the ideal of <u>strongly non-stationary</u> subsets of $P_\kappa\lambda$, and denote it by $SNS_{\kappa\lambda}$.

Now notice that an easy argument yields the following useful fact.

1.10 <u>Proposition</u>. For any ideal I on $P_\kappa\lambda$ and any $X \subset P_\kappa\lambda$, $X \in \nabla I$ iff there is an <u>I-small regressive function</u> on X, ie. a function $f : X \to \lambda$ with the properties:

(1) $(\forall x \in X)(f(x) \in x)$, and

(2) $(\forall \alpha < \lambda)(f^{-1}(\{\alpha\}) \in I)$.

As an easy consequence of this together with theorem 1.2 we obtain

1.11 <u>For any $X \subset P_\kappa\lambda$, $X \in SNS_{\kappa\lambda}$ iff there is an $I_{\kappa\lambda}$-small regressive function on X.</u> □

2. The minimality of $CF_{\kappa\lambda}$.

In this section we use a result of Menas (2.3 below) to obtain the $P_\kappa\lambda$ analogue of Neumer's theorem (2.4 below) from which we obtain the minimality of $CF_{\kappa\lambda}$ together with the non-normality of $SCF_{\kappa\lambda}$ $(\lambda < \kappa)$.

2.1 Definition. For any finite $n \geq 1$ and any $\{X_{\bar\alpha} : \bar\alpha \in \lambda^n\} \subset P(P_\kappa\lambda)$ define $\Delta^{(n)}\{X_{\bar\alpha} : \bar\alpha \in \lambda^n\}$ by induction over $k < n$:

$\Delta^{(1)}\{X_\alpha : \alpha<\lambda\} = \Delta\{X_\alpha : \alpha<\lambda\}$,

$\Delta^{(k+1)}\{X_{\bar\alpha} : \bar\alpha = (\alpha_0,..,\alpha_{k-1},\alpha_k) \in \lambda^{k+1}\} = \Delta\{\Delta^{(k)}\{X_{\bar\alpha} : (\alpha_0,..,\alpha_{k-1}) \in \lambda^k$ &

$(\bar\alpha)_k = \alpha_k\} : \alpha_k < \lambda\}$.

For any finite $n \geq 1$ and any $w : \lambda^n \to P_\kappa\lambda$ let $\Delta(w)$ denote the set $\Delta^{(n)}\{\widehat{w(\bar\alpha)} : \bar\alpha \in \lambda^n\}$.

Recall that for any finite $n \geq 1$ and any $w : \lambda^n \to P_\kappa\lambda$, $C(\{w\})$ denotes the set $\{x \in P_\kappa\lambda : (\forall\bar\alpha \in x^n)(w(\bar\alpha) \subset x)\}$, and that this is a Menas cub (see 0.3.5 above).

2.2 Lemma. For any finite $n \geq 1$ and any $w : \lambda^n \to P_\kappa\lambda$,
$\Delta(w) = \Delta^{(n)}\{\widehat{w(\bar\alpha)} : \bar\alpha \in \lambda^n\} = \{x \in P_\kappa\lambda : (\forall\bar\alpha \in x^n)(w(\bar\alpha) \subset x)\} = C(\{w\})$.

proof. It is clear that for any $w : \lambda \to P_\kappa\lambda$, $\Delta(w) = \Delta\{\widehat{w(\alpha)} : \alpha<\lambda\}$ $= \{x \in P_\kappa\lambda : (\forall\alpha \in x)(w(\alpha) \subset x)\}$. Thus pick $k \geq 1$, assume that the lemma holds for every $w : \lambda^k \to P_\kappa\lambda$ and let $w : \lambda^{k+1} \to P_\kappa\lambda$. For each $\beta < \lambda$ define $w_\beta : \lambda^k \to P_\kappa\lambda$ by $w_\beta(\bar\alpha) = w(\bar\alpha\beta)$ where $\bar\alpha\beta$ denotes the concatenation $(\alpha_0,..,\alpha_{k-1},\beta)$ of $\bar\alpha$ and β. Then for each $\beta < \lambda$ we have

$\Delta^{(k)}\{\widehat{w(\bar{\alpha}\beta)} : \bar{\alpha}\in\lambda^{\kappa}\} = \Delta(w_{\beta})$; thus $\Delta(w) = \Delta\{\Delta(w_{\beta}) : \beta<\lambda\}$.

It now follows by our induction hypothesis that for any $x\in P_{\kappa}\lambda$, $x\in\Delta(w)$ iff $(\forall\beta\in x)(x\in\Delta(w_{\beta}))$ iff $(\forall\beta\in x)(\forall\bar{\alpha}\in x^{k})(w_{\beta}(\bar{\alpha})\subset x)$ iff $(\forall\bar{\alpha}\in x^{k})(\forall\beta\in x)(w(\bar{\alpha}\beta)\subset x)$ iff $(\forall\bar{\alpha}\in x^{k+1})(w(\bar{\alpha})\subset x)$. □

2.3 Lemma. (Menas) For any cub subset C of $P_{\kappa}\lambda$ there is a $w : \lambda^2 \to P_{\kappa}\lambda$ such that $C(\{w\}) \subset C$.

proof. Menas [25].

We can now obtain the $P_{\kappa}\lambda$ analogue of Neumer's theorem:

2.4 Theorem. For every $\lambda \geq \kappa$, $CF_{\kappa\lambda} = \Delta\Delta FSF_{\kappa\lambda}$.

proof. Since $FSF_{\kappa\lambda} \subset CF_{\kappa\lambda}$ and since $CF_{\kappa\lambda}$ is normal, it is clear that $\Delta\Delta FSF_{\kappa\lambda} \subset CF_{\kappa\lambda}$. Thus it remains to prove the reverse inclusion.

Let $C \subset P_{\kappa}\lambda$ be a cub and let $w : \lambda^2 \to P_{\kappa}\lambda$ be such that $C(\{w\}) \subset C$. Then by lemma 2.2 above $\Delta(w) = \Delta^2\{\widehat{w(\alpha,\beta)} : (\alpha,\beta)\in\lambda^2\} = \Delta\{\Delta\{\widehat{w(\alpha,\beta)} : \alpha < \lambda\} : \beta < \lambda\} \in \Delta\Delta FSF_{\kappa\lambda}$. □

2.5 Corollary. For every $\lambda \geq \kappa$, $CF_{\kappa\lambda}$ is the smallest normal filter on $P_{\kappa}\lambda$.

proof. Immediate from 2.4 since every normal filter on $P_{\kappa}\lambda$ must extend $\Delta\Delta FSF_{\kappa\lambda}$, and since $\Delta\Delta FSF_{\kappa\lambda}$ is normal. □

2.6 Corollary. For every $\lambda > \kappa$, $SCF_{\kappa\lambda}$ is not normal.

proof. Immediate by theorems 1.8 and 2.4 for if $\lambda > \kappa$, then $\Delta FSF_{\kappa\lambda} = SCF_{\kappa\lambda} \subsetneq CF_{\kappa\lambda} = \Delta\Delta FSF_{\kappa\lambda}$. □

2.7 Remark. An immediate consequence of 2.6 is that the family of strong cub subsets of $P_\kappa\lambda(\lambda>\kappa)$ is not closed under diagonal intersections. An argument due to Jech [17] yields a direct proof of this fact:

For each $\alpha < \lambda$ define $C_\alpha \subset P_\kappa\lambda$ by $C_\alpha = \{x\in P_\kappa\lambda$: either $\max(x)$ does not exist or else $(\exists\gamma<\lambda)(\max(x) = \gamma+\alpha)\}$. It is easy to see that $(\forall\alpha<\lambda)(C_\alpha$ is a strong cub & $\{\alpha\}\in C_\alpha)$. Set $C = \Delta\{C_\alpha : \alpha<\lambda\}$. It is clear that $(\forall\alpha<\lambda)(\{\alpha\}\in C)$. Now let $\alpha,\beta < \lambda$ be such that $\alpha < \beta$ and β is indecomposible. Then $\bigcup\{\{\alpha\},\{\beta\}\} = \{\alpha,\beta\} \notin C$ since $\{\alpha,\beta\} \notin C_\alpha$.

We conclude this section with some remarks on the ideal-theoretic version of theorem 2.4 above. Menas used the result expressed in 2.3 above to prove that an unbounded subset of X of $P_\kappa\lambda$ is non-stationary iff there is a function $f : X \to \lambda\times\lambda$ with the properties $(\forall x\in X)(f(x)\in x\times x)$, and $(\forall\alpha,\beta<\lambda)(f^{-1}(\{(\alpha,\beta)\})\in I_{\kappa\lambda})$.

The above result can also be obtained from the "dual" $NS_{\kappa\lambda} = \nabla\nabla I_{\kappa\lambda}$ of theorem 2.4 together with the following easily proved fact which is of some interest in its own right.

2.8 Proposition. For any ideal I on $P_{\kappa\lambda}$ and any $X \subset P_\kappa\lambda$, $X \in \nabla\nabla I$ iff there is a function $f: X \to \lambda\times\lambda$ satisfying

(1) $(\forall x\in X)(f(x)\in x\times x)$, and

(2) $(\forall\alpha,\beta<\lambda)(f^{-1}(\{(\alpha,\beta)\}) \in I)$.

proof. First, pick $X \in \nabla\nabla I$ and $\{X_{\alpha\beta} : (\alpha,\beta)\in\lambda^2\} \subset I$ such that $X = \nabla\{\nabla\{X_{\alpha\beta} : \alpha<\lambda\} : \beta<\lambda\}$. For each $x \in X$, pick $(\alpha_x,\beta_x) \in x\times x$ such that

$x \in X_{\alpha_x \beta_x}$. Then define $f : X \to \lambda \times \lambda$ by $f(x) = (\alpha_x, \beta_x)$. It is easy to see see that this f has the required properties.

Conversely, pick $X \subset P_\kappa \lambda$ and $f : X \to \lambda \times \lambda$ satisfying (1) and (2). It is easy to see that $X = \nabla\{\nabla\{f^{-1}(\{(\alpha,\beta)\}) : \alpha < \lambda\} : \beta < \lambda\}$. □

3. "$\nabla\nabla\nabla I = \nabla\nabla I$"

Baumgartner, Taylor and Wagon [5] used Neumer's theorem to prove that for any κ-complete ideal I on κ extending I_κ, $\nabla\nabla I = \nabla I$ and hence concluded that the smallest normal ideal extending I is ∇I. In view of the fact that the $P_\kappa\lambda$ analogue of Neumer's theorem is $NS_{\kappa\lambda} = \nabla\nabla I_{\kappa\lambda}$ (theorem 2.4 above) and the fact that $\nabla I_{\kappa\lambda} \subsetneqq NS_{\kappa\lambda}$ ($\lambda > \kappa$) (theorems 1.2 and 1.7), it is reasonable to expect that the $P_\kappa\lambda$ analogue of $\nabla\nabla I = \nabla I$ is $\nabla\nabla\nabla I = \nabla\nabla I$. This is true as we now show.

3.1 Theorem. For any $\lambda \geq \kappa$ and any ideal I on $P_\kappa\lambda$, $\nabla\nabla\nabla I = \nabla\nabla I$.

proof. Clearly $\nabla\nabla I \subset \nabla\nabla\nabla I$ so it remains to show that the reverse inclusion holds.

Pick $X \in \nabla\nabla\nabla I$ and let $f : X \to \lambda$ be a $\nabla\nabla I$-small regressive function on X. For each $\alpha < \lambda$ set $X_\alpha = f^{-1}(\{\alpha\})$ and recall that $(\forall\alpha<\lambda)(X_\alpha \in \nabla\nabla I)$. Thus for each $\alpha < \lambda$ let $f_\alpha : X_\alpha \to \lambda$ be a ∇I-small regressive function on X_α.

Now let $p : \lambda\times\lambda \leftrightarrow \lambda$ be any bijection and set $C = \{x \in P_\kappa\lambda : p''(x\times x) \subset x\}$. Since $X - C \in NS_{\kappa\lambda} = \nabla\nabla I_{\kappa\lambda} \subset \nabla\nabla I$, we can complete the proof by showing that $X \cap C \in \nabla\nabla I$.

Define $g: X \cap C \to \lambda$ by $g(x) = p(f(x), f_{f(x)}(x))$. We show that g is ∇I - small and regressive. It is clear that g is regressive on $X \cap C$ since f is regressive on x, since $f_{f(x)}$ is regressive on $X_{f(x)} \subset X$ and since $X \cap C \subset C$. Now pick $\beta < \lambda$ and let $\beta_0, \beta_1 < \lambda$ be such that $\beta = p(\beta_0, \beta_1)$. Then $x \in g^{-1}(\{\beta\}) \Rightarrow f(x) = \beta_0$ & $f_{f(x)}(x) = \beta_1 \Rightarrow x \in f_{\beta_0}^{-1}(\{\beta_1\})$. Thus $g^{-1}(\{\beta\}) \subset f_{\beta_0}^{-1}(\{\beta_1\})$; thus $g^{-1}(\{\beta\}) \in \nabla I$. □

An even simpler argument than the one used to prove the proceeding theorem yields the following useful fact.

3.2 <u>For any $\lambda \geq \kappa$ and any ideal I on $P_\kappa\lambda$, if I extends $SNS_{\kappa\lambda}$, then $\nabla\nabla I = \nabla I$.</u> □

The main results of this chapter were first reported in [7] and were presented at the CMS Summer Research Workshop in Set Theory and Set-theoretic Topology, Erindale College, University of Toronto, July-August 1980.

CHAPTER 2. ASPECTS OF MILD λ-INEFFABILITY

1. Mild λ-ineffability and the solvability of binary (κ,λ) messes.

In [15] Jech provided a $P_\kappa\lambda$ analogue of the tree property (see definition 1.3 below). We will show that this characterizes mild λ-ineffability (theorem 1.5 below). But first we give some elementary facts which will be useful in the sequel.

Recall that an unaccountable regular cardinal κ is said to be mildly λ-ineffable iff for every $(A_x : x \in P_\kappa\lambda)$ such that $(\forall x \in P_\kappa\lambda)(A_x \subset x)$, $(\exists A \subset \lambda)(\forall x \in P_\kappa\lambda)(H_x = \{y \in x : A_y \cap x = A \cap x\} \neq 0)$.

Notice that $(\forall x \in P_\kappa\lambda)(H_x \neq 0)$ can be replaced by $(\forall x \in P_\kappa\lambda)(H_x \in I_{\kappa\lambda}{}^+)$ in this definition.

1.1 Proposition. If κ is mildly λ-ineffable, then κ is mildly γ-ineffable for every $\gamma \in [\kappa,\lambda]$,

proof. Pick $\gamma \in [\kappa,\lambda]$ and let $(A_x : x \in P_\kappa\gamma)$ be such that $(\forall x \in P_\kappa\gamma)(A_x \subset x)$. For each $y \in P_\kappa\lambda$ define $B_y \subset y$ by $B_y = A_{y\cap\gamma}$. Now let $A \subset \lambda$ be such that $(\forall y \in P_\kappa\lambda)(\exists z \in P_\kappa\lambda)(y \subset z \;\&\; B_z \cap y = A \cap y)$. We show that $A \subset \gamma$ and that $(\forall x \in P_\kappa\lambda)(\exists y \in P_\kappa\lambda)(x \subset y \;\&\; A_y \cap x = A \cap x)$.

Suppose by way of contradiction that $A - \gamma \neq 0$ and let $\alpha \in A - \gamma$. Now let $z \in P_\kappa\lambda$ be such that $\alpha \in z$. Then $B_z = A_{z\cap\gamma} \subset z \cap \gamma$ so $\alpha \notin B_z$ so $A \cap \{\alpha\} \neq B_z \cap \{\alpha\}$ thus contradicting the mild λ-ineffability of κ. Thus

$A \subset \gamma$. Now pick $x \in P_\kappa \gamma \subset P_\kappa \lambda$ and $z \in P_\kappa \lambda$ such that $x \subset z$ and $B_z \cap x = A \cap x$. Then $A \cap x = A_{z\cap\gamma} \cap x$. □

1.2 Proposition. If κ is mildly λ-ineffable for some $\lambda \geq \kappa$, then κ is weakly compact.

proof. In view of 1.1 above, we know that κ is mildly κ-ineffable. We will show that this implies that for any $(A_\alpha : \alpha<\kappa)$ such that $(\forall\alpha<\kappa)(A_\alpha\subset\alpha)$, $(\exists A\subset\kappa)(\forall\alpha<\kappa)(\exists\beta\in[\alpha,\kappa))(A_\beta\cap\alpha = A\cap\alpha)$ and hence that $(\exists A\subset\kappa)(\forall\alpha<\kappa)(X_\alpha = \{\beta\in[\alpha,\kappa) : A_\beta\cap\alpha = A\cap\alpha\} \in I_\kappa^+)$. Then we use this fact to prove that κ is weakly compact.

Let $(A_\alpha : \alpha<\kappa)$ be such that $(\forall\alpha<\kappa)(A_\alpha\subset\alpha)$. For each $x \in P_\kappa\kappa$ pick $\gamma_x < \kappa$ such that $x \subset \gamma_x$ and define $B_x = A_{\gamma_x} \cap x$. Now let $A \subset \kappa$ be such that $(\forall x\in P_\kappa\kappa)(\exists y\in\hat{x})(B_y\cap x = A\cap x)$. Pick $\alpha < \kappa \subset P_\kappa\kappa$ and $y \in \hat{\alpha}$ such that $B_y \cap \alpha = A \cap \alpha$. Then $\alpha \leq \gamma_y$ and $A_{\gamma_y} \cap \alpha = A \cap \alpha$.

We complete the proof by showing that κ is inaccessible and has the tree property. Suppose by way of contradiction that κ is not inaccessible and let $\delta < \kappa$ be such that $2^\delta \geq \kappa$. Let $(A_\alpha : \alpha\in[\delta,\kappa))$ be a family of distinct subsets of δ. Notice that $(\forall\alpha\in[\delta,\kappa))(\forall\beta,\gamma\in[\alpha\ \kappa))$ $(A_\beta\cap\alpha = A_\gamma\cap\alpha \leftrightarrow A_\beta = A_\gamma)$. This means that no $A \subset \kappa$ can satisfy the conclusion of the preceeding paragraph. Thus κ is inaccessible.

Let $\mathcal{T} = (T,\leq_T)$ be a (κ,κ) tree; assume w.l.o.g. that $T = \kappa$. For each $\alpha < \kappa$ define $A_\alpha \subset \alpha$ by $A_\alpha = \{\xi<\alpha : \xi\leq_T\alpha\}$. Now let $A \subset \kappa$ be such that $(\forall\alpha<\kappa)(\exists\beta\in[\alpha,\kappa))(A_\beta\cap\alpha = A\cap\alpha)$, and for each $\alpha < \kappa$ note that $X_\alpha = \{\beta\in[\alpha,\kappa) : A_\beta\cap\alpha = A\cap\alpha\} \in I_\kappa^+$. We will show that A is a path of length κ through T.

Suppose by way of contradiction that $(\exists\delta<\kappa)(T_\delta \cap A = 0)$. Then $(\forall\xi\in T_\delta)(\xi\notin A)$, so $(\forall\xi\in T_\delta)(\forall\alpha>\xi)(\forall\beta\in X_\alpha)(\xi\notin A_\beta)$, so $(\forall\xi\in T_\delta)(\forall\alpha>\xi)$ $(\forall\beta\in X_\alpha)(\xi\not\leq_T\beta)$. Thus if we pick $\alpha > \sup\{\xi : \xi\in T_\delta\}$ we have that $(\forall\beta\in X_\alpha)(\forall\xi\in T_\delta)(\xi\not\leq_T\beta)$. From this it follows that $X_\alpha \subset \bigcup\{T_\nu : \nu<\delta\}$ for if $\beta \notin \bigcup\{T_\nu : \nu<\delta\}$, $(\exists\xi\in T_\delta)(\xi\leq_T\beta)$, so $\beta \notin X_\alpha$.

Now pick $\delta < \kappa$ and suppose by way of contradiction that $|T_\delta \cap A| > 1$. Let $\xi,\xi' \in T_\delta \cap A$ and pick $\alpha > \xi,\xi'$. Then $(\forall\beta\in X_\alpha)(\xi,\xi'\in A_\beta)$, so $(\forall\beta\in X_\alpha)(\xi\leq_T\beta \;\&\; \xi'\leq_T\beta)$. But this means that $X_\alpha = 0$ since $\{\beta<\kappa : \xi\leq_T\beta\} \cap \{\beta<\kappa : \xi'\leq_T\beta\} = 0$. □

In [15] Jech provided a $P_\kappa\lambda$ generalization of the tree property:

1.3 <u>Definition</u>. Let $S_{\kappa\lambda}$ denote the set of all functions u with $\text{dom}(u) \in P_\kappa\lambda$ and $\text{ran}(u) \subset 2$. As in Jech [15] we call $M \subset S_{\kappa\lambda}$ a <u>binary (κ,λ) mess</u> iff

(1) $(\forall x\in P_\kappa\lambda)(\exists u\in M)(\text{dom}(u) = x)$, and

(2) $(\forall u,v\in S_{\kappa\lambda})(u\subset v \;\&\; v\in M \Rightarrow u\in M)$.

A binary (κ,λ) mess M is said to be <u>solvable</u> iff $(\exists f : \lambda \to 2)$ $(\forall x\in P_\kappa\lambda)(f\restriction x\in M)$; any such f is called a <u>solution</u> of M.

It is easy to see that if κ is inaccessible, then to every binary (κ,κ) mess M there corresponds a $(\kappa,3)$ tree T_M, and that any path of length κ through T_M is a solution of M. Conversely, to every $(\kappa,3)$ tree T on κ there corresponds a binary (κ,κ) mess M, and any solution of M is a path of length κ through T. Thus (in ZFC), an inaccessible cardinal κ is weakly compact iff every binary (κ,κ) mess is solvable.

Jech's proof of 4.2 in [15] shows that if κ is almost λ-ineffable, then every binary (κ,λ) mess is solvable. We observed that the assumption "κ is mildly λ-ineffable" is enough to make his argument work. The converse of this is true too, as we now show.

1.4 Theorem. κ is mildly λ-ineffable iff every binary (κ,λ) mess is solvable.[1]

proof. First, suppose that κ is mildly λ-ineffable and let M be a binary (κ,λ) mess. We argue as in Jech's proof of 4.2 in [15]. For each $x \in P_\kappa\lambda$ pick $u_x \in M$ such that $\mathrm{dom}(u_x) = x$ and set $A_x = u_x^{-1}(\{1\}) \subset x$. Now let $A \subset \lambda$ be such that $(\forall x \in P_\kappa\lambda)(\exists y \in \hat{x})(A_y \cap x = A \cap x)$. Then the characteristic function χ_A of A is easily seen to be a solution of M.

Conversely, suppose that every binary (κ,λ) mess is solvable, and let $\langle A_x : x \in P_\kappa\lambda\rangle$ be such that $(\forall x \in P_\kappa\lambda)(A_x \subset x)$. For each $x \in P_\kappa\lambda$ let u_x be the characteristic function of A_x. Clearly $M = \{u_y \restriction x : x,y \in P_\kappa\lambda \ \& \ y \in \hat{x}\}$ is a binary (κ,λ) mess. Let f be a solution of M. It is easy to see that $A = f^{-1}(\{1\})$ has the required property. $\square$

Notice that in our proof of proposition 1.2 we showed that "κ is mildly λ-ineffable" implies that "for any $\langle A_\alpha : \alpha < \kappa\rangle$ such that

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1 After we had obtained this result we learned that Hickin and Plotkin [14] had defined a property equivalent to mild λ-ineffability and remarked that their property holds iff every binary (κ,λ) mess is solvable.

$(\forall\alpha<\kappa)(A_\alpha\subset\alpha)$, $(\exists A\subset\kappa)(\forall\alpha<\kappa)(\exists\beta\in[\alpha,\kappa))(A_\beta\cap\alpha = A\cap\alpha)$" and that this implies that κ is weakly compact. In fact, this characterizes weak compactness:

1.5 Corollary. For any uncountable regular cardinal κ, the following are equivalent:

(1) κ is weakly compact,

(2) κ is mildly κ-ineffable, and

(3) for any $(A_\alpha : \alpha<\kappa)$ such that $(\forall\alpha<\kappa)(A_\alpha\subset\alpha)$, $(\exists A\subset\kappa)(\forall\alpha<\kappa)(\exists\beta\in[\alpha,\kappa))$ $(A_\beta\cap\alpha = A\cap\alpha)$.

proof. Immediate by proposition 1.2 together with theorem 1.4 and the remark following definition 1.3. Of course it is also possible to give easy direct proofs of (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). □

The result expressed in theorem 1.4 above was first reported in [6] and was described together with some of the other results of this chapter at the 774th meeting of the American Mathematical Society in Boulder, Colorado, 27-29 March 1980.

2. A partition - theoretic result.

In [15] Jech provided $P_\kappa\lambda$ generalizations of $\kappa \to (\kappa)^2_2$ and $\kappa \to (\text{stationary})^2_2$. Define $D_\kappa\lambda = \{\{x,y\}\in[P_\kappa\lambda]^2 : x\subsetneq y \text{ or } y\subsetneq x\}$ and say that Part (κ,λ) holds iff for any $f : D_\kappa\lambda \to 2$, $(\exists H\in I^+_{\kappa\lambda})(|f''(D_\kappa\lambda\cap[H]^2)| = 1)$; say that Part* (κ,λ) holds iff for any $f : D_\kappa\lambda \to 2$, $(\exists H\in NS^+_{\kappa\lambda})(|f''(D_\kappa\lambda\cap[H]^2)| = 1)$.

Magidor [23] showed that if Part $*(\kappa,\lambda)$ holds then κ is λ-ineffable, and Menas [26] showed that if κ is supercompact then Part*(κ,λ) holds for every $\lambda > \kappa$. But the strength of Part (κ,λ) is not so clear.

Jech proved that if Part (κ,λ) holds for some $\lambda \geq \kappa$, then κ is weakly compact. In 2.1 we give a generalization of Jech's notation, and then prove (theorem 2.2) that if Part $(\kappa,\lambda)^3$ holds, then κ is mildly λ-ineffable. From this it follows, by DiPrisco and Zwicker [10] that κ is strongly compact if Part $(\kappa,\lambda)^3$ holds for every $\lambda \geq \kappa$. Results similar to ours have been obtained independently by Baumgartner [3].

2.1 Definition. For each finite $n \geq 2$ let $D^n_\kappa\lambda$ denote the set $\{\{x_0,\ldots,x_{n-1}\}\in[P_\kappa\lambda]^n : x_0\subsetneq\ldots\subsetneq x_{n-1}\}$. We will sometimes write $\{x_0,\ldots,x_{n-1}\}_{\subsetneq}$ to mean $x_0\subsetneq\ldots\subsetneq x_{n-1}$. For each $n \geq 2$ say that Part $(\kappa,\lambda)^n$ holds iff every partition $f : D^n_\kappa\lambda \to 2$ has an unbounded homogeneous set, i.e. iff for any $f : D^n_\kappa\lambda \to 2$, $(\exists H\in I^+_{\kappa\lambda})(|f''(D^n_\kappa\lambda\cap[H]^n)| = 1)$.

In accordance with Jech's notation we write Part (κ,λ) in place of Part $(\kappa,\lambda)^2$.

It is easy to see that if Part $(\kappa,\lambda)^n$ holds, then Part $(\kappa,\gamma)^n$ holds for every $\gamma \in [\kappa,\lambda]$. It is also easy to see that $(\forall n\geq 2)(\text{Part }(\kappa,\lambda)^{n+1} \Rightarrow \text{Part }(\kappa,\lambda)^n)$. Thus it follows by Jech's work that if Part $(\kappa,\lambda)^n$ holds for some $n \geq 2$, then κ is weakly compact.

2.2 Theorem. If Part $(\kappa,\lambda)^3$ holds, then κ is mildly λ-ineffable.

proof. First, note that Part $(\kappa,\lambda)^3$ implies that κ is inaccessible. Let $(A_x : x\in P_\kappa\lambda)$ be such that $(\forall x\in P_\kappa\lambda)(A_x\subset x)$. Define $f : D^3_\kappa\lambda \to 2$ by $f(\{x,y,z\}_{\subsetneq}) = \begin{cases} 0 & \text{if } A_y\cap x = A_z\cap x \\ 1 & \text{otherwise} \end{cases}$
Now let $H \in I^+_{\kappa\lambda}$ be homogeneous for f. We will show that $f''(D^3_\kappa\lambda\cap[H]^3) = \{0\}$ and then use this fact to define the required $A \subset \lambda$.

Pick $x \in P_\kappa\lambda$ and set $\hat{\hat{x}} = \hat{x} - \{x\}$. Then pick $w \subset x$ such that $H_w = \{y\in H\cap\hat{\hat{x}} : A_y\cap x = w\} \in I^+_{\kappa\lambda}$. Note that such a w must exist since the inaccessibility of κ guarantees that $|P(x)| < \kappa$, and since $I_{\kappa\lambda}$ is κ-complete. Now observe that $(\forall y,z\in H_w)(A_y\cap x = A_z\cap x)$. The homogeneity of of H now guarantees that $f''(D^3_\kappa\lambda\cap[H]^3) = \{0\}$.

We can now define A. For each $\alpha < \lambda$ pick $x_\alpha \in H \cap \widehat{\{\alpha\}}$ and note that $(\forall y,z\ H\cap\hat{\hat{x}}_\alpha)(\alpha\in A_y \Leftrightarrow \alpha\in A_z)$. Then define $A = \{\alpha<\lambda : (\forall y\in H\cap\hat{\hat{x}}_\alpha)(\alpha\in A_y)\}$. It is easy to see that this A has the required property. □

2.3 Corollary. If Part $(\kappa,\lambda)^3$ holds for every $\lambda \geq \kappa$, then κ is strongly compact.

proof. Immediate by 2.2 together with the fact that for every $\lambda \geq \kappa$, if κ is mildly $2^{\lambda^{<\kappa}}$-ineffable, then κ is λ-strongly compact by [10]. □

In view of the fact that κ is weakly compact <u>iff</u> $\kappa \to (\kappa)^2_2$ <u>iff</u> $\kappa \to (\kappa)^n_r$ for every finite $n, r \geq 1$, natural questions that arise at this stage include

Does $\text{Part}(\kappa,\lambda)^2$ imply that κ is mildly λ-ineffable?

Does mild λ-ineffability imply $\text{Part}(\kappa,\lambda)^n$ for some $n \geq 2$?

Both of these questions are open. The arguments used to obtain the above "κ results" do not appear to have $P_\kappa\lambda$ analogues which would answer these questions. In fact, attempts to generalize these arguments suggest that $\text{Part}(\kappa,\lambda)$ might not be the best possible $P_\kappa\lambda$ generalization of $\kappa \to (\kappa)^2_2$. Baumgartner [3] has proposed another notion which might be more suitable.

He defined the relation $Q(\kappa,\lambda)^n$ to hold iff for every partition $f : D^n_\kappa\lambda \to 2$ there is an $h : P_\kappa\lambda \to P_\kappa\lambda$ such that

(i) $(\forall x \in P_\kappa\lambda)(h(x) \subset x)$, and

(ii) $(\exists i \in 2)(\forall \{x_0,\ldots,x_{n-1}\}_{\subsetneq} \in D^n_\kappa\lambda)(|x_0| < \ldots < |x_{n-1}| \Rightarrow h(x_0) \subsetneq \ldots \subsetneq h(x_{n-1})$

$\& \ f(\{h(x_0),\ldots,h(x_{n-1})\}) = i)$.

His studies yield the following facts:

(1) $\text{Part}(\kappa,\lambda)^n \Rightarrow Q(\kappa,\lambda)^n$ for every finite $n \geq 2$.

(2) $Q(\kappa,\lambda)^3 \Rightarrow \kappa$ is mildly λ-ineffable.

(3) If $\lambda^{<\kappa} = \lambda$ and if κ is mildly λ-ineffable, then $Q(\kappa,\lambda)^n$ holds for every finite $n \geq 2$.

3. A filter extension property characterizing mild λ-ineffability.

In this section we will prove a result (theorem 3.2) which is needed in the sequel and which is interesting in its own right.

3.1 Definition. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, let $(\text{FEP})_{\kappa\lambda}$ denote the following filter extension property:

> Every proper κ-complete filter in a κ-complete field B of sets such that $|B| = \lambda$ can be extended to a κ-complete ultrafilter in B.

In theorem 3.2 below we prove that for any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$ satisfying $\lambda^{<\kappa} = \lambda$, κ is mildly λ-ineffable iff $(\text{FEP})_{\kappa\lambda}$ holds. A result similar to this has also been obtained independently by Baumgartner [3]. The proof of the forward implication is via λ-compactness (see 0.2.1) and that of the reverse implication is essentially due to DiPrisco and Zwicker [10].

Jech proved that if κ is inaccessible and if $\lambda^{<\kappa} = \lambda$, then κ is λ-compact iff every binary (κ,λ) mess is solvable (theorem 2.2 in [15]). Our proof of (1) $\Rightarrow$ (2) in theorem 3.2 below is a reformulation and expansion of his proof of the reverse implication of his theorem.

3.2 Theorem. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$ satisfying $\lambda^{<\kappa} = \lambda$, the following are equivalent:

(1) κ is mildly λ-ineffable,

(2) κ is λ-compact,

(3) $(FEP)_{\kappa\lambda}$ holds.

proof. (1) $\Rightarrow$ (2). First note that by 1.2 above, κ is inaccessible. Now let $\sum$ be a κ-satisfiable set of $L_{\kappa\omega}$ sentences such that $|\sum| = \lambda$. Since $\lambda^{<\kappa} = \lambda$ we may assume w.l.o.g. that $\sum$ contains all the Skolem sentences in the $L_{\kappa\omega}$ language L of $\sum$.

Now let $\{\phi_\alpha : \alpha<\lambda\}$ be an enumeration of all L sentences. For each $x \in P_\kappa\lambda$ let $\mathcal{G}_x$ be a model of $\sum \cap \{\phi_\alpha : \alpha \in x\}$, and define $A_x \subset x$ by $A_x = \{\alpha \in x : \mathcal{G}_x \models \phi_\alpha\}$. Now let $A \subset \lambda$ be such that $(\forall x \in P_\kappa\lambda)(\exists y \in \hat{x})$ $(A_y \cap x = A \cap x)$ and set $\sum^* = \{\phi_\alpha : \alpha \in A\}$. It is easy to see that $\sum \subset \sum^*$, and that for every L sentence ϕ, precisely one of ϕ or $\neg\phi$ is in $\sum^*$. We can now construct a model of $\sum^*$ and hence of $\sum$ as in Henkin's proof of the completeness theorem for $L_{\omega\omega}$.

(2) $\Rightarrow$ (3). Let B be a κ-complete field of subsets of a set S such that $|B| = \lambda$, and let F be a proper κ-complete filter in B. Further, let c be an individual constant symbol, and for each $X \in B$, let $\underline{X}$ be a unary predicate symbol. Finally, let μ be the similarity type $(\{\epsilon\} \cup \{c\} \cup \{\underline{X} : X \in B\})$, and let $\sum$ be the set of $L_{\kappa\omega}(\mu)$ sentences consisting of

(i) the complete $L_{\kappa\omega}(\mu-\{c\})$ theory of $(S,X)_{X\in B}$ under the natural interpretations of the symbols in $\mu-\{c\}$, and

(ii) the sentences in $\{c\in \underline{X} : X\in F\}$.

Clearly $|\Sigma| = \lambda$. Moreover, it is clear that Σ is κ-satisfiable; $(S,X)_{X\in B}$ is a model of each $\Gamma \in [\Sigma]^{<\kappa}$ if we interpret each $\underline{X}$ appearing in Γ by X itself, and if we interpret c by an element of $\bigcap\{X\in F : "c\in X"\in\Gamma\}$.

Now let $\mathfrak{A}$ by any model of Σ. Standard arguments show that the set $\mathcal{U} \subset P(S)$ defined by "$X \in \mathcal{U}$ iff $X \in B$ and $\mathfrak{A} \models c \in X$" is a κ-complete ultrafilter in B extending F.

(3) $\Rightarrow$ (1). Let $(A_x : x\in P_\kappa\lambda)$ be such that $(\forall x\in P_\kappa\lambda)(A_x\subset x)$. For each $\alpha < \lambda$ set $Y_\alpha = \{x\in P_\kappa\lambda : \alpha\in A_x\}$. Further, let B be the smallest κ-complete field of subsets of $P_\kappa\lambda$ containing $\{\widehat{\{\alpha\}} : \alpha<\lambda\} \cup \{Y_\alpha : \alpha<\lambda\}$. The assumption $\lambda^{<\kappa} = \lambda$ guarantees that $|B| = \lambda$. Now let $\mathcal{U}$ be any κ-complete ultrafilter in B which extends the proper κ-complete filter generated in B by $\{\widehat{\{\alpha\}} : \alpha<\lambda\}$, and set $A = \{\alpha<\lambda : Y_\alpha\in\mathcal{U}\}$. We show that A has the required property.

Pick $x \in P_\kappa\lambda$. Notice that $(\forall\alpha\in x\cap A)(Y_\alpha\in\mathcal{U})$ and $(\forall\alpha\in x-A)(P_\kappa\lambda-Y_\alpha\in\mathcal{U})$. Thus $X = (\bigcap\{Y_\alpha : \alpha\in x\cap A\}) \cap (\bigcap\{P_\kappa\lambda-Y_\alpha : \alpha\in x-A\}) \in \mathcal{U}$. It is easy to see that $(\forall y\in X\cap\hat{x})(A_y\cap x = A\cap x)$. □

We make use of this result in section 4 below. We can also use it and its proof to obtain a characterization of mild λ-ineffability in the spirit of $(UP)_\kappa$ (see item (5) in 0.5.1).

3.3 <u>Definition</u>. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, let $(UP)_{\kappa\lambda}$ denote the following ultrafilter property:

> There is a non-principal κ-complete ultrafilter containing $\{\widehat{\{\alpha\}} : \alpha<\lambda\}$ in any κ-complete field B of subsets of $P_\kappa\lambda$ such that $\{\widehat{\{\alpha\}} : \alpha<\lambda\} \subset B$ and $|B| = \lambda$.

3.4 <u>Corollary</u>. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$ satisfying $\lambda^{<\kappa} = \lambda$, κ is mildly λ-ineffable iff $(UP)_{\kappa\lambda}$ holds. $\square$

4. M-ultrafilters over $P_\kappa^M\lambda$; generalization of a result of Kunen.

The following definition is a natural $P_\kappa\lambda$ generalization of Kunen's notion [21] of an M-ultrafilter over κ.

4.1 Definition. Let M be a transitive model of ZFC and let $\kappa \le \lambda$ be cardinals in M. Call $\mathcal{U} \subset P^M(P_\kappa^M\lambda)$ an M-ultrafilter over $P_\kappa^M\lambda$ iff

(1) $0 \notin \mathcal{U}$ and $(\forall x \in P_\kappa^M\lambda)(\{x\} \notin \mathcal{U})$,

(2) $(\forall\alpha<\lambda)(\widehat{\{\alpha\}} \cap M \in \mathcal{U})$,

(3) $(\forall X \in P^M(P_\kappa^M\lambda))(X \in \mathcal{U}$ or $P_\kappa^M\lambda - X \in \mathcal{U})$,

(4) for any $\delta < \kappa$ and any $(X_\alpha : \alpha<\delta) \in M$ such that $(\forall\alpha<\delta)(X_\alpha \in \mathcal{U})$, $\bigcap\{X_\alpha : \alpha<\delta\} \in \mathcal{U}$, and

(5) for any $(X_\alpha : \alpha<\lambda) \in M$, $\{\alpha<\lambda : X_\alpha \in \mathcal{U}\} \in M$.

$\mathcal{U} \subset P^M(P_\kappa^M\lambda)$ is called a normal M-ultrafilter on $P_\kappa^M\lambda$ iff it satisfies 4.1(1) - (5) together with

(6) for any $(X_\alpha : \alpha<\lambda) \in M$ such that $(\forall\alpha<\lambda)(X_\alpha \in \mathcal{U})$, $\Delta_M\{X_\alpha : \alpha<\lambda\} = \{x \in P_\kappa^M\lambda : (\forall\alpha \in x)(x \in X_\alpha)\} \in \mathcal{U}$; equivalently,

(6)' for any $f \in M$ such that $f : P_\kappa^M\lambda \to \lambda$ and $\{x \in P_\kappa\lambda : f(x) \in x\} \in \mathcal{U}$, $(\exists\alpha<\lambda)(f^{-1}(\{\alpha\}) \in \mathcal{U})$.

Notice that we do not require $\mathcal{U}$ to be an element of M. However, we can form the ultrapower $\mathrm{Ult}_\mathcal{U}(M)$ of M mod $\mathcal{U}$ using only these functions from $P_\kappa^M\lambda$ into M which are elements of M. The fundamental theorem of ultrapowers goes through for this situation by the usual arguments. Call an M-ultrafilter $\mathcal{U}$ over $P^M_\kappa\lambda$ <u>well-founded</u> iff $\mathrm{Ult}_{\mathcal{U}}(M)$ is well-founded.

Standard arguments (e.g. see [19], [30]) show that if κ is λ-supercompact w.r.t. M in the sense of Jech's definition (also see 0.2.5 above), then there is a well-founded normal M-ultrafilter over $P^M_\kappa\lambda$. We concern ourselves here with "not necessarily well-founded, not necessarily normal" M-ultrafilters over $P^M_\kappa\lambda$.

In [15] Jech showed that if κ is λ-supercompact w.r.t. M, then $M \models$ "κ is λ-ineffable". We use arguments essentially due to DiPrisco and Zwicker [10] and to Magidor [23] to strengthen this as follows.

4.2 <u>Theorem</u>. Let M be a transitive model of ZFC, let $\kappa \leq \lambda$ be cardinals in M, and let $\mathcal{U}$ be an M-ultrafilter over $P^M_\kappa\lambda$. Then
(1) $M \models$ "κ is mildly λ-ineffable", and
(2) if $\mathcal{U}$ is M-normal, then $M \models$ "κ is λ-ineffable".

<u>proof</u>. Let $(A_x : x \in P^M_\kappa\lambda) \in M$ be such that $M \models (\forall x \in P_\kappa\lambda)(A_x \subset x)$. For each $\alpha < \lambda$ set $Y_\alpha = \{x \in P^M_\kappa\lambda : \alpha \in A_x\} \in M$. Then $(Y_\alpha : \alpha < \lambda) \in M$ so by 4.1(5), $A = \{\alpha < \lambda : Y_\alpha \in \mathcal{U}\} \in M$.

(1) Pick $x \in P^M_\kappa\lambda$. Note that $(\forall \alpha \in x \cap A)(Y_\alpha \in \mathcal{U})$ & $(\forall \alpha \in x - A)(P^M_\kappa\lambda - Y_\alpha \in \mathcal{U})$. Thus $X = (\bigcap\{Y_\alpha : \alpha \in x \cap A\}) \cap (\bigcap\{P^M_\kappa\lambda - Y_\alpha : \alpha \in x - A\}) \in \mathcal{U}$. Now let y be any element of $X \cap \hat{x}$. Then $(\forall \alpha \in x \cap A)(\alpha \in A_y)$ & $(\forall \alpha \in x - A)(\alpha \notin A_y)$, so $A_y \cap x = A \cap x$.

(2) Now assume that $\mathcal{U}$ is M-normal. For each $\alpha \in A$ set $X_\alpha = Y_\alpha$, and for each $\alpha \in \lambda - A$ set $X_\alpha = P^M_\kappa\lambda - Y_\alpha$. Then $(\forall \alpha < \lambda)(X_\alpha \in \mathcal{U})$ so by

M-normality, $\Delta_M\{X_\alpha : \alpha<\lambda\} = \{x\in P^M_\kappa\lambda : (\forall\alpha\in x)(x\in X_\alpha)\} \in \mathcal{U}$. It is easy to see that $\Delta_M\{X_\alpha : \alpha<\lambda\} \subset \{x\in P^M_\kappa\lambda : A_x = A\cap x\}$. □

Part (1) of the preceeding theorem is reminiscent of Kunen's result [21] that if there is an M-ultrafilter over κ, then $M \models$ "κ is weakly compact". Recall that Kunen also proved that if $|P^M(\kappa)| = \aleph_0$ and $M \models$ "κ is weakly compact", then there is an M-ultrafilter over κ. It is natural to wonder whether or not the obvious $P_\kappa\lambda$ analogue of this is also true. We show that it is.

4.3 <u>Theorem</u>. Let M be a transitive model of ZFC, and let $\kappa \le \lambda$ be cardinals in M. If $|P^M(P^M_\kappa\lambda)| = \aleph_0$ and if $M \models$ "$\lambda^{<\kappa} = \lambda$ & κ is mildly λ-ineffable", then there is an M-ultrafilter over $P^M_\kappa\lambda$.

<u>proof</u>. Note that since $|P^M(P^M_\kappa\lambda)| = \aleph_0$, $|({}^\lambda P^M(P^M_\kappa\lambda))^M| = \aleph_0$ also. Thus let $(f_n : n\in\omega)$ enumerate $({}^\lambda P^M(P^M_\kappa\lambda))^M$. For each $n \in \omega$ let $B_n \in M$ be such that $M \models$ "B_n is the κ-complete field of subsets of $P_\kappa\lambda$ which is generated by $\bigcup\{ran(f_m) : m \le n\}$". We inductively construct a sequence $(\mathcal{U}_n : n\in\omega)$ such that for each $n \in \omega$, $\mathcal{U}_n \in M$, $\mathcal{U}_n \subset \mathcal{U}_{n+1}$ and $M \models$ "$\mathcal{U}_n$ is a non-principal κ-complete ultrafilter in B_n which includes $\{\widehat{\{\alpha\}} : \alpha<\lambda\} \cap B_n$".

Pick $n \in \omega$ and suppose that we have constructed $\mathcal{U}_0 \subset, \ldots \subset \mathcal{U}_{n-1}$. Since $B_{n-1} \subset B_n$ and since $M \models$ "$\lambda^{<\kappa} = \lambda$ & κ is mildly λ-ineffable", it followed by 3.2 above that there is a $\mathcal{U}_n \in M$ such that $M \models$ "$\mathcal{U}_n$ is a non-principal κ-complete ultrafilter in B_n which extends the proper, non-principal, κ-complete filter generated in B_n by $\mathcal{U}_{n-1} \cup (\{\widehat{\{\alpha\}} : \alpha<\lambda\}\cap B_n)$".

We claim that $\mathcal{U} = \bigcup\{\mathcal{U}_n : n\in\omega\}$ is an M-ultrafilter over $P^M_\kappa\lambda$. Clearly $\mathcal{U}$ satisfies 4.1(1) - (3). Pick $\delta < \kappa$ and $(X_\alpha : \alpha<\delta) \in M$ s.t. $(\forall\alpha<\delta)(X_\alpha\in\mathcal{U})$. Now let $n \in \omega$ be such that $(X_\alpha : \alpha<\delta) = f_n \restriction \delta$. Then $\{X_\alpha : \alpha<\delta\} \subset \mathcal{U}_n$ so $\bigcap\{X_\alpha : \alpha<\delta\} \in \mathcal{U}_n \subset \mathcal{U}$. Thus $\mathcal{U}$ satisfies 4.1(4). Now pick $(X_\alpha : \alpha<\lambda) \in M$ and let $n \in \omega$ be such that $(X_\alpha : \alpha<\lambda) = f_n$. Then $\{\alpha<\lambda : X_\alpha\in\mathcal{U}\} = \{\alpha<\lambda : X_\alpha\in\mathcal{U}_n\}$, so $\{\alpha<\lambda : X_\alpha\in\mathcal{U}\} \in M$ since $\mathcal{U}_n \in M$. Thus $\mathcal{U}$ satisfies 4.1(5). □

We conclude this section with a consideration of the question: "is there a 'nice' transitive model M of ZFC such that there is an M-ultrafilter over $P^M_\kappa\lambda$?" We know by Kunen's work in [21] that there is a well-founded normal L-ultrafilter over $P^L_\kappa\kappa$ iff $0^\#$ exists. But what if $\lambda > \kappa$?

Jech proved in [15] that if κ is $(\kappa^+)^M$-supercompact w.r.t. M, then $M \neq L$ and $M \neq L[\mathcal{U}]$ for any measure ultrafilter $\mathcal{U}$ on κ. We offer the following strengthening of this result.

4.4 <u>Theorem</u>. Let M be a transitive model of ZFC and let κ be a cardinal in M.

(1) If there is an M-ultrafilter over $P^M_\kappa(\kappa^+)^M$ then $M \neq L$.

(2) If there is an M-ultrafilter over $P^M_\kappa(\kappa^{++})^M$, then $M \neq L[\mathcal{F}]$ for any ultrafilter sequence $\mathcal{F}$.

<u>proof</u>. First, note that by 4.2 above together with DiPrisco and Zwicker [10] and Magidor [23], if there is an M-ultrafilter $\mathcal{U}$ over $P^M_\kappa\gamma$ where $\gamma \geq 2^{\lambda^{<\kappa}}$, then $M \models$ "κ is λ-strongly compact", and if $\mathcal{U}$ is M-normal, then $M \models$ "κ is λ-supercompact". Next recall that $L \models$ GCH and that for

any ultrafilter sequence F, $L[F] \models$ GCH (Mitchell [27]). Thus $L \models$ "$2^{\kappa^{<\kappa}} = \kappa^{+}$" and $L[F] \models$ "$2^{(\kappa^{+})^{<\kappa}} = \kappa^{++}$". The theorem follows from these facts together with the facts that there are no measurable cardinals in L (Scott; e.g. see [19]) and there are no κ^{+}-strongly compact cardinals in any $L[F]$ (Mitchell, [27]). $\square$

We do not know if there is an ultrafilter sequence F with respect to which there can be an $L[F]$ ultrafilter over $P^{L[F]}(\kappa^{+})^{L[F]}$.

CHAPTER 3. IDEALS ARISING FROM INEFFABILITY PROPERTIES OF $P_\kappa\lambda$

1. The ideals $NMIn_{\kappa\lambda}$, $NAIn_{\kappa\lambda}$ and $NIn_{\kappa\lambda}$.

In this section we will characterize mild λ-ineffability, almost λ-ineffability and λ-ineffability as properties of certain ideals on $P_\kappa\lambda$.

1.1 Definition. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, say that $X \subset P_\kappa\lambda$ is

(1) mildly λ-ineffable iff for any $(A_x : x \in X)$ such that $(\forall x \in X)(A_x \subset x)$, $(\exists A \subset \lambda)(\forall x \in P_\kappa\lambda)(H_x = \{y \in X \cap \hat{x} : A_y \cap x = A \cap x\} \neq 0)$,

(2) almost λ-ineffable iff for any $(A_x : x \in X)$ such that $(\forall x \in X)(A_x \subset x)$, $(\exists A \subset \lambda)(H = \{x \in X : A_x = A \cap x\} \in I^+_{\kappa\lambda})$,

(3) λ-ineffable iff for any $(A_x : x \in X)$ such that $(\forall x \in X)(A_x \subset x)$, $(\exists A \subset \lambda)(H = \{x \in X : A_x = A \cap x\} \in NS^+_{\kappa\lambda})$.

Notice that the condition given in the conclusion of (2) can be replaced by $(\exists H \in P(X) \cap I^+_{\kappa\lambda})(\forall x,y \in H)(x \subset y \Rightarrow A_x = A_y \cap x)$. Similarly, the condition in the conclusion of (3) can be replaced by $(\exists H \in P(X) \cap NS^+_{\kappa\lambda})$ $(\forall x,y \in H)(x \subset y \Rightarrow A_x = A_y \cap x)$.

Now define the sets $NMIn_{\kappa\lambda}$, $NAIn_{\kappa\lambda}$ and $NIn_{\kappa\lambda}$ by

$NMIn_{\kappa\lambda} = \{X \subset P_\kappa\lambda : X$ is not mildly λ-ineffable$\}$,

$NAIn_{\kappa\lambda} = \{X \subset P_\kappa\lambda : X$ is not almost λ-ineffable$\}$, and

$NIn_{\kappa\lambda} = \{X \subset P_\kappa\lambda : X$ is not λ-ineffable$\}$. It is clear that $I_{\kappa\lambda} \subset NMIn_{\kappa\lambda} \subset NAIn_{\kappa\lambda} \subset NIn_{\kappa\lambda}$. Moreover, it is clear that κ is mildly λ-

ineffable iff $P_\kappa\lambda \notin NMIn_{\kappa\lambda}$, that κ is almost λ-ineffable iff $P_\kappa\lambda \notin NAIn_{\kappa\lambda}$, and that κ is λ-ineffable iff $P_\kappa\lambda \notin NIn_{\kappa\lambda}$.

1.2 Proposition. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, κ is mildly λ-ineffable iff $NMIn_{\kappa\lambda} = I_{\kappa\lambda}$.

proof. In view of the preceeding remarks, it is clear that if $NMIn_{\kappa\lambda} = I_{\kappa\lambda}$, then κ is mildly λ-ineffable, and if κ is mildly λ-ineffable then $I_{\kappa\lambda} \subset NMIn_{\kappa\lambda}$. We complete the proof by showing that if κ is mildly λ-ineffable, then $I_{\kappa\lambda}{}^+ \subset NMIn_{\kappa\lambda}{}^+$. Pick $X \in I_{\kappa\lambda}{}^+$ and let $(A_x : x \in X)$ be such that $(\forall x \in X)(A_x \subset x)$. For each $z \in P_\kappa\lambda - X$ pick $x_z \in X \cap \hat{z}$, and for each $z \in X$ set $x_z = z$. Define $A_z' = A_{x_z} \cap z$ and let $A \subset \lambda$ be s.t. $(\forall z \in P_\kappa\lambda)(\exists y \in \hat{z})(A_y' \cap z = A \cap z)$. Then $(\forall z \in P_\kappa\lambda)(\exists y \in \hat{z})(A_{x_y} \cap y \cap z = A \cap z)$, so $(\forall z \in P_\kappa\lambda)(\exists x \in X \cap \hat{z})(A_x \cap z = A \cap z)$. □

The ideals $NAIn_{\kappa\lambda}$ and $NIn_{\kappa\lambda}$ are considerably more interesting:

1.3 Theorem. For any uncountable regular cardinal κ, and any cardinal $\lambda \geq \kappa$,

(1) κ is almost λ-ineffable iff $NAIn_{\kappa\lambda}$ is a normal ideal on $P_\kappa\lambda$, and

(2) κ is λ-ineffable iff $NIn_{\kappa\lambda}$ is a normal ideal on $P_\kappa\lambda$.

proof. We will just prove (1); (2) follows by a similar but even simpler argument. The reverse implication of (1) is clear, so it remains to prove the forward one.

The assumption that κ is almost λ-ineffable guarantees that $NAIn_{\kappa\lambda}$ is proper. Also, it is easy to see that $I_{\kappa\lambda} \subset NAIn_{\kappa\lambda}$ and that $(\forall X, Y \subset P_\kappa\lambda)(X \subset Y \ \& \ Y \in NAIn_{\kappa\lambda} \rightarrow X \in NAIn_{\kappa\lambda})$.

Now pick $\delta < \kappa$ and $\{X_\nu : \nu<\delta\} \subset NAIn_{\kappa\lambda}$. For each $\nu < \delta$, let $(A_x^\nu : x \in X_\nu)$ witness $X_\nu \in NAIn_{\kappa\lambda}$. Set $X = \bigcup\{X_\nu : \nu<\delta\}$ and suppose by way of contradiction that $X \notin NAIn_{\kappa\lambda}$. For each $x \in X$, let $\nu(x)$ be the least $\nu < \delta$ such that $x \in X_\nu$, and set $A_x = A_x^{\nu(x)}$. Now let $A \subset \lambda$ be such that $H = \{x \in X : A_x = A \cap x\} \in I_{\kappa\lambda}^+$. The κ-completeness of $I_{\kappa\lambda}$ guarantees that $(\exists\nu<\delta)(H \cap X_\nu \in I_{\kappa\lambda}^+)$. Let γ be the least ordinal $< \delta$ such that $H \cap X_\gamma \in I_{\kappa\lambda}^+$. Note that $(\forall x \in H \cap X_\gamma)(\nu(x) \leq \gamma)$. Then the minimality of γ together with the minimality of the $\nu(x)$'s imply that $\{x \in X_\gamma : \nu(x) = \gamma\} \in I_{\kappa\lambda}^+$ thus contradicting $X_\gamma \in NAIn_{\kappa\lambda}$.

We next show that $NS_{\kappa\lambda} \subset NAIn_{\kappa\lambda}$ and then use this fact to prove that $NAIn_{\kappa\lambda}$ is normal. Pick $X \in NS_{\kappa\lambda} = \nabla\nabla I_{\kappa\lambda}$, and let $f : P_\kappa\lambda \to \lambda \times \lambda$ be such that $(\forall x \in X)(f(x) \in x \times x)$ and $(\forall \alpha,\beta<\lambda)(f^{-1}(\{(\alpha,\beta)\}) \in I_{\kappa\lambda})$. For each $x \in X$ set $f(x) = (\alpha_x,\beta_x) \in x \times x$, and define $A_x \subset x$ by $A_x = \bigcup f(x) = \{\alpha_x,\beta_x\}$. Now notice that $(\forall x \in X)(\forall y \in X \cap \hat{x})(A_x = A_y \cap x \Rightarrow y \in f^{-1}(\{(\alpha_x,\beta_x)\}) \cup f^{-1}(\{(\beta_x,\alpha_x)\}))$. Thus we have that $(\forall x \in X)(\{y \in X \cap \hat{x} : A_x = A_y \cap x\} \in I_{\kappa\lambda})$, so $X \in NAIn_{\kappa\lambda}$.

Now pick $X \in NAIn_{\kappa\lambda}^+$, let $p : \lambda \times \lambda \rightarrowtail \lambda$, and set $C_p = \{x \in P_\kappa\lambda : p''(x \times x) \subset x\}$. Since C_p is cub in $P_\kappa\lambda$, it follows by the previous paragraph that $X \cap C_p \in NAIn_{\kappa\lambda}^+$. We will show that $X \cap C_p \notin \nabla NAIn_{\kappa\lambda}$; it will then follow that $X \notin \nabla NAIn_{\kappa\lambda}$. Suppose by way of contradiction that $X \cap C_p \in \nabla NAIn_{\kappa\lambda}$, and let $f : X \cap C_p \to \lambda$ be $NAIn_{\kappa\lambda}$-small and regressive on $X \cap C_p$. For each $\alpha < \lambda$, let $(A_x^\alpha : x \in f^{-1}(\{\alpha\}))$ witness $f^{-1}(\{\alpha\}) \in NAIn_{\kappa\lambda}$. Then define $(A_x : x \in C_p \cap X)$ by $A_x = \{p(\xi,f(x)) : \xi \in A_x^{f(x)}\}$, and let $H \subset X \cap C_p$ be such that $H \in I_{\kappa\lambda}^+$ and $(\forall x,y \in H)(x \subset y \Rightarrow A_x = A_y \cap x)$. Notice that $A_x = A_y \cap x \Rightarrow f(x) = f(y)$. This shows that $(\exists\alpha<\lambda)(\forall x\in H)(f(x)=\alpha)$ and hence that $(\exists\alpha<\lambda)(H\subset f^{-1}(\{\alpha\}))$ thus contradicting $f^{-1}(\{\alpha\}) \in NAIn_{\kappa\lambda}$. □

As an easy consequence of the preceeding theorem we obtain the following useful and suggestive characterizations of almost λ-ineffable and λ-ineffable subsets of $P_\kappa\lambda$:

1.4 <u>Corollary</u>. $X \subset P_\kappa\lambda$ is almost λ-ineffable (λ-ineffable) iff for any $(f_x : x\in X)$ such that $(\forall x\in X)(f_x : x\to x)$,
$(\exists f : \lambda\to\lambda)[H=\{x\in X : f_x = f\restriction x\}\in I_{\kappa\lambda}{}^+(NS_{\kappa\lambda}{}^+)]$.

<u>proof</u>. The reverse implication is clear, so it remains to prove the forward one. Pick $X \in NAIn_{\kappa\lambda}{}^+$ and let $(f_x : x\in X)$ be such that $(\forall x\in X)(f_x : x\to x)$. Further, let $p : \lambda\times\lambda \twoheadrightarrow \lambda$ and set
$C_p = \{x\in P_\kappa\lambda : p''(x\times x)\subset x\}$. The normality of $NAIn_{\kappa\lambda}$ together with the minimality of $NS_{\kappa\lambda}$ (1.2.5 above) guarantees that $X \cap C_p \in NAIn_{\kappa\lambda}{}^+$.

For each $x \in X\cap C_p$, define $A_x \subset x$ by $A_x = \{p(\xi,f_x(\xi)) : \xi\in x\}$, and then let $H \subset X \cap C_p$ be such that $H \in I_{\kappa\lambda}{}^+$ and $(\forall x,y\in H)(x\subset y \Rightarrow A_x = A_y\cap x)$. Notice that $A_x = A_y\cap x$ means that $\{p(\xi,f_x(\xi)) : \xi\in x\} = \{p(\xi,f_y(\xi)) : \xi\in y\} \cap x$ and hence that $f_x = f_y\restriction x$. Thus define $f : \lambda\to\lambda$ by $f(\alpha) = f_x(\alpha)$ where x is any element of $H\cap\widehat{\{\alpha\}}$. □

2. A new ineffability property of $P_\kappa\lambda$.

Motivated by Shelah's work in [29] together with our 1.4 above, we define a new ineffability property of $P_\kappa\lambda$ between mild λ-ineffability and almost λ-ineffability which can be characterized as a property of a normal ideal on $P_\kappa\lambda$.

2.1 Definition. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, say that $X \subset P_\kappa\lambda$ has the <u>λ-Shelah property</u> iff for any $(f_x : x\in X)$ such that $(\forall x\in X)(f_x : x\to x)$, $(\exists f : \lambda\to\lambda)(\forall x\in P_\kappa\lambda)(H_x = \{y\in X\cap\hat{x} : f_y\restriction x = f\restriction x\} \neq 0)$.

Further, define the set $NSh_{\kappa\lambda}$ by $NSh_{\kappa\lambda} = \{X\subset P_\kappa\lambda$: X does <u>not</u> have the λ-Shelah property$\}$, and say that <u>κ is λ-Shelah</u> iff $P_\kappa\lambda \notin NSh_{\kappa\lambda}$.

It is clear that $I_{\kappa\lambda} \subset NMIn_{\kappa\lambda} \subset NSh_{\kappa\lambda}$, and in view of 1.4 above, it is also clear that $NSh_{\kappa\lambda} \subset NAIn_{\kappa\lambda}$.

The main result of this section is that κ is λ-Shelah iff $NSh_{\kappa\lambda}$ is a normal ideal on $P_\kappa\lambda$ (theorem 2.3 below). We start with the following simple preliminary which was inspired by a result of Baumgartner and Laver [4].

2.2 Lemma. Suppose that $I \subsetneq P(P_\kappa\lambda)$ is such that

(1) $(\forall X,Y\subset P_\kappa\lambda)(X\subset Y$ & $Y\in I \Rightarrow X\in I)$,

(2) $(\forall X\in I)(\forall Y\in I_{\kappa\lambda})(X\cup Y\in I)$, and

(3) $\nabla I \subset I$.

Then I is a κ-complete normal ideal on $P_\kappa\lambda$.

proof. It suffices to prove that I is κ-complete. In fact, we will show that $I^* = \{X \in P_\kappa\lambda : P_\kappa\lambda - X \in I\}$ is κ-complete.

Pick $\delta < \kappa$ and let $(X_\alpha : \alpha<\delta) \in {}^{\delta}I^*$. For each $\beta < \lambda$ write $\beta = \delta\gamma + \alpha$ where $\gamma \geq 0$ and $\alpha < \delta$, and set $Y_\beta = X_\alpha$. By (3), $Y = \Delta\{Y_\beta : \beta<\lambda\} = \{y \in P_\kappa\lambda : (\forall\beta\in y)(y \in Y_\beta)\} \in I^*$, so it follows by (2) that $Y \cap \hat{\delta} \in I^*$. Now note that $(\forall y \in Y\cap\hat{\delta})(\delta \subset y \;\&\; (\forall\beta\in y)(y\in Y_\beta))$. Thus $Y \cap \hat{\delta} \subset \bigcap\{X_\alpha : \alpha<\delta\}$. It now follows by (1) that $\bigcap\{X_\alpha : \alpha<\delta\} \in I^*$. □

2.3 Theorem. For any uncountable regular cardinal κ and any cardinal $\kappa \geq \lambda$, κ is λ-Shelah iff $NSh_{\kappa\lambda}$ is a normal ideal on $P_\kappa\lambda$.

proof. The reverse implication is clear. Moreover, it is clear that if κ is λ-Shelah, then $NSh_{\kappa\lambda}$ is proper, and that if $X \subset P_\kappa\lambda$ is λ-Shelah, then every $Y \supset X$ is λ-Shelah and $X \in I_{\kappa\lambda}{}^+$. Thus $NSh_{\kappa\lambda}$ satisfies (1) and (2) of the preceeding lemma, so we complete the proof by showing that it also satisfies (3).

Let $(X_\nu : \nu<\lambda) \in {}^{\lambda}NSh_{\kappa\lambda}$, and for each $\nu < \lambda$ let $(f^\nu_x : x \in X_\nu)$ witness $X_\nu \in NSh_{\kappa\lambda}$. Set $X = \nabla\{X_\nu : \nu<\lambda\}$. Suppose that $X \notin NSh_{\kappa\lambda}$; we derive the required contradiction as follows.

For each $x \in X$ let $(\alpha^x_0, .., \alpha^x_\nu, .., (\nu<ot(x)))$ enumerate x in increasing order. Notice that in view of the fact that $I_{\kappa\lambda} \subset NSh_{\kappa\lambda}$, we may assume w.l.o.g. that $X \subset \widehat{\{0\}}$ and hence that $(\forall x\in X)(\alpha^x_0 = 0)$.

For each $x \in X$ pick $\gamma(x) \in x$ so that $x \in X_{\gamma(x)}$ and define $g_x : x \to x$ by

$$g_x(\xi) = \begin{cases} \gamma(x) & \text{if } \xi = \alpha^x_\nu \text{ where } \nu = 0 \text{ or } \lim(\nu). \\ f_x^{\gamma(x)}(\alpha^x_\mu) & \text{if } \xi = \alpha^x_\nu \text{ where } \nu = \mu + 1. \end{cases}$$

Now let $g : \lambda \to \lambda$ be such that $(\forall x \in P_\kappa\lambda)(\exists y \in X \cap \hat{x})(g_y \restriction x = g \restriction x)$, and set $\gamma = g(0)$. Finally, define $f : \lambda \to \lambda$ by $f(\xi) = g(\xi+1)$. We show that $(\forall x \in P_\kappa\lambda)(\exists y \in X_\gamma \cap \hat{x})(f^\gamma_y \restriction x = f \restriction x)$ thus contradicting $X_\gamma \in NSh_{\kappa\lambda}$.

Pick $x \in P_\kappa\lambda$ and set $x' = x \cup \{0\} \cup \{\xi+1 : \xi \in x\}$. Now pick $y \in X \cap \hat{x}'$ such that $g_y \restriction x' = g \restriction x'$. Notice that since $0 \in x' \subset y$, $g_y(0) = \gamma$, so $\gamma(y) = \gamma$. Thus observe that for each $\xi \in x$ we have $f(\xi) = g(\xi+1) = g_y(\xi+1) = f_y(\xi)$ since $\{\xi, \xi+1\} \subset x' \subset y$. □

3. Projection of ideals from $P_\kappa\kappa$ to κ.

We conclude this chapter with a brief study of the "projections" of the ideals defined in sections 1 and 2 from $P_\kappa\kappa$ to κ. We start with the following simple preliminary which is interesting in its own right.

3.1 Lemma. For any normal ideal I on $P_\kappa\kappa$, $I\restriction\kappa = \{Y\cap\kappa : Y\in I\}$ is a normal ideal on κ. Moreover, $(I\restriction\kappa)^+ = \{Y\cap\kappa : Y\in I^+\}$ and $I = \{Y\subset P_\kappa\kappa : Y\cap\kappa\in I\restriction\kappa\}$.

proof. The first assertion follows from the easily verified facts that $I\restriction\kappa \subset I$ and $(\forall X\subset\kappa)(X\in I\restriction\kappa \leftrightarrow X\in I)$ together with the normality of I.

Clearly $(I\restriction\kappa)^+ \subset \{Y\cap\kappa : Y\in I^+\}$. Since the normality of I guarantees that κ has I-measure one, it follows that this inclusion reverses: $Y \in I^+ \Rightarrow Y \cap \kappa \in I^+ \Rightarrow Y \cap \kappa \in (I\restriction\kappa)^+$ since $Y \cap \kappa \subset \kappa$.

Clearly $I \subset \{Y\subset P_\kappa\lambda : Y\cap\kappa\in I\restriction\kappa\}$. The fact proved in the preceeding paragraph guarantees that this inclusion reverses. □

3.2 Remark. It is well known that the conclusions of the above lemma follow from the hypotheses that I is a prime ideal on $P_\kappa\kappa$ (eg. see Kanamori and Magidor [19]). And it is easy to see that for any ideal I on $P_\kappa\kappa$, $I\restriction\kappa = \{Y\cap\kappa : Y\in I\}$ is a (possibly improper) ideal on κ. Even if it is proper, we could have $(I\restriction\kappa)^+ \subsetneq \{Y\cap\kappa : Y\in I^+\}$ if I is neither prime nor normal. For instance, it is easy to see that $I_\kappa = I_{\kappa\kappa}\restriction\kappa$. However,

$I_\kappa^+ \subsetneq \{Y\cap\kappa : Y\in I^+\}$ since $\{x\in P_\kappa\kappa : x\notin\kappa\} \notin I_{\kappa\kappa}^+$ but $\{x\in P_\kappa\kappa : x\notin\kappa\} \cap \kappa = 0 \in I_\kappa$.

We conclude with the following theorem, an immediate consequence of which is that κ has the Shelah property (see 0.5.4 above) iff κ is κ-Shelah together with the well known facts that κ is almost ineffable (ineffable) iff κ is almost κ-ineffable (κ-ineffable).

3.3 <u>Theorem</u>.

For any uncountable regular cardinal κ, $\begin{pmatrix} NSh_\kappa \\ NAIn_\kappa \\ NIn_\kappa \end{pmatrix} = \begin{pmatrix} NSh_{\kappa\kappa}\restriction\kappa \\ NAIn_{\kappa\kappa}\restriction\kappa \\ NIn_{\kappa\kappa}\restriction\kappa \end{pmatrix}$

<u>proof</u>. We will just prove that $NSh_\kappa = NSh_{\kappa\kappa}\restriction\kappa$; the others are proved in a similar manner. To this end, note that by 2.2 and 3.1 above, it will suffice to prove that $(\forall Y\subset P_\kappa\kappa)(Y\in NSh_{\kappa\kappa}^+ \Leftrightarrow Y\cap\kappa\in NSh_\kappa^+)$. First pick $Y \subset P_\kappa\kappa$ such that $Y \in NSh_{\kappa\kappa}^+$. Then since κ is cub in $P_\kappa\kappa$, it follows by 2.3 that $Y \cap \kappa \in NSh_{\kappa\kappa}^+$ and hence by 2.1 that $Y \cap \kappa \in NSh_\kappa^+$.

Conversely, let $Y \subset P_\kappa\kappa$ be such that $X = Y \cap \kappa \in NSh_\kappa^+$. We show that $X \in NSh_{\kappa\kappa}^+$; then since $X \subset Y$ it will follow that $Y \in NSh_{\kappa\kappa}^+$. Let $(f_\alpha : \alpha\in X)$ be such that $(\forall\alpha\in X)(f_\alpha : \alpha\to\alpha)$ and let $f : \kappa\to\kappa$ be such that $(\forall\alpha<\kappa)(\exists\beta\in X\cap[\alpha,\kappa))(f_\beta\restriction\alpha = f\restriction\alpha)$. Pick $x \in P_\kappa\kappa$ and then pick $\alpha_x < \kappa$ such that $x \subset \alpha_x$. Finally, pick $\beta_x \in X \cap [\alpha_x,\kappa)$ such that $f_{\beta_x}\restriction\alpha_x = f\restriction\alpha_x$. Then $f_{\beta_x}\restriction x = f\restriction x$. □

CHAPTER 4. THE λ-SHELAH PROPERTY

1. λ-Shelah cardinals.

Recall that for any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, κ is said to be λ-Shelah iff for any $(f_x : x \in P_\kappa \lambda)$ such that $(\forall x \in P_\kappa \lambda)(f_x : x \to x)$, $(\exists f : \lambda \to \lambda)(\forall x \in P_\kappa \lambda)(H_x = \{y \in x : f_y \restriction x = f \restriction x\} \neq 0)$.

It is easy to see that if κ is λ-Shelah for some $\lambda \geq \kappa$, then κ is mildly λ-ineffable and hence weakly compact. Also, an argument similar to the one used to prove proposition 2.1.1 above shows that if κ is λ-Shelah, then κ is γ-Shelah for every $\gamma \in [\kappa, \lambda]$. Moreover, it is possible to give a characterization of λ-Shelah κ's much like the one given for mildly λ-ineffable κ's in theorem 2.1.4 above.

Let $S_{\kappa\lambda}$ denote the set of all functions u such that $\mathrm{dom}(u) \in P_\kappa \lambda$ and $\mathrm{ran}(u) \subset \lambda$. Call $M \subset S_{\kappa\lambda}$ a <u>(κ,λ) mess</u> iff

(i) $(\forall x \in P_\kappa \lambda)(M \cap {}^x x \neq 0)$, and

(ii) $(\forall u, v \in S_{\kappa\lambda})(u \subset v \ \&\ v \in M \Rightarrow u \in M)$.

Further, say that M is <u>solvable</u> iff $(\exists f : \lambda \to \lambda)(\forall x \in P_\kappa \lambda)(f \restriction x \in M)$. An argument similar to the proof of theorem 2.1.4 yields:

1.1 <u>κ is λ-Shelah iff every (κ,λ) mess is solvable</u>. □

In 2.3.4 above we gave an ultrafilter property $(\mathrm{UP})_{\kappa\lambda}$ which characterizes mild λ-ineffability if $\lambda^{<\kappa} = \lambda$. In 1.4 below we define a "normal" ultrafilter property $(\mathrm{NUP})_{\kappa\lambda}$ which is a $P_\kappa\lambda$ generalization of the property given in 0.5.3(4) which characterizes weak compactness. Then we show (in theorem 1.5 below) that if $\lambda^{<\kappa} = \lambda$, then $(\mathrm{NUP})_{\kappa\lambda}$ holds iff κ is λ-Shelah.

We start with the following simple preliminary which shows that the λ-Shelah property can be viewed as a $P_\kappa\lambda$ generalization of Baumgartner's principle (see 0.5.5 above). We will also make use of this fact in the succeeding sections of this chapter.

1.2 <u>Proposition</u>. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, κ is λ-Shelah iff for any λ-sequence $(g_\alpha : \alpha<\lambda)$ of regressive functions on $P_\kappa\lambda$, $(\exists f : \lambda\to\lambda)(\forall x\in P_\kappa\lambda)(E_x = \{y\in\hat{x} : (\forall\alpha\in x)(g_\alpha(y) = f(\alpha))\} \neq 0)$.

<u>proof</u>. First, suppose that κ is λ-Shelah and let $(g_\alpha : \alpha<\lambda)$ be a λ-sequence of regressive functions on $P_\kappa\lambda$. For each $x\in P_\kappa\lambda$, define $f_x : x\to x$ by $f(\alpha) = g_\alpha(x)$. It is clear that for each $f : \lambda\to\lambda$ and each $x\in P_\kappa\lambda$, $E_x = \{y\in\hat{x} : (\forall\alpha\in x)(g_\alpha(y) = f(\alpha))\} = \{y\in\hat{x} : f_y\restriction x = f\restriction x\} = H_x$.

Conversely, suppose that the stated condition holds, and let $(f_x : x\in P_\kappa\lambda)$ be such that $(\forall x\in P_\kappa\lambda)(f_x : x\to x)$. For each $x \in P_\kappa\lambda$ fix an element τ_x of x and then for each $\alpha < \lambda$ define $g_\alpha : P_\kappa\lambda \to \lambda$ by

$$g_\alpha(x) = \begin{cases} f_x(\alpha) & \text{if } \alpha \in x. \\ \tau_x & \text{otherwise.} \end{cases}$$

It is clear that each g_α is regressive. Moreover, it is clear that for each $f : \lambda\to\lambda$ and each $x \in P_\kappa\lambda$, $H_x = \{y\in\hat{x} : f_y\restriction x = f\restriction x\} = \{y\in\hat{x} : (\forall\alpha\in x)(g_\alpha(y) = f(\alpha))\} = E_x$. $\square$

1.3 Remark. The condition $(\exists f : \lambda\to\lambda)(\forall x\in P_\kappa\lambda)(E_x\neq 0)$ in proposition 1.2 is clearly equivalent to $(\exists f : \lambda\to\lambda)(\forall x\in P_\kappa\lambda)(E_x\in I_{\kappa\lambda}{}^+)$. In section 3 we show that if $\lambda^{<\kappa} = \lambda$, then it is actually equivalent to $(\exists f : \lambda\to\lambda)(\forall x\in P_\kappa\lambda)(E_x\in NS_{\kappa\lambda}{}^+)$.

We are now ready to define $(NUP)_{\kappa\lambda}$.

1.4 Definition. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, let $(NUP)_{\kappa\lambda}$ denote the following "normal" ultrafilter property:

> For any κ-complete field B of subsets of $P_\kappa\lambda$ such that $|B| = \lambda$ and $(\forall\alpha<\lambda)(\widehat{\{\alpha\}}\in B)$, and any λ-sequence $G = (g_\alpha : \alpha<\lambda)$ of regressive functions on $P_\kappa\lambda$ such that $(\forall\alpha<\lambda)(\forall\beta<\lambda)(g_\alpha{}^{-1}(\{\beta\})\in B)$, there is κ-complete ultrafilter U in B such that $(\forall\alpha<\lambda)(\widehat{\{\alpha\}}\in U)$ and every function in G is constant on a set in U.

1.5 Theorem. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$ satisfying $\lambda^{<\kappa} = \lambda$, κ is λ-Shelah iff $(NUP)_{\kappa\lambda}$ holds.

proof. First, assume that κ is λ-Shelah and let B,G be as in the hypothesis of $(NUP)_{\kappa\lambda}$. Further, let $f : \lambda\to\lambda$ be such that $(\forall x\in P_\kappa\lambda)(E_x = \{y\in\hat{x} : (\forall\alpha\in x)(g_\alpha(y) = f(\alpha))\}\in I_{\kappa\lambda}{}^+)$. It is easy to see that for each $\delta < \kappa$ and each $\{x_\nu : \nu<\delta\} \subset P_\kappa\lambda$, $\bigcap\{E_{x_\nu} : \nu<\delta\} = E_x$ where $x = \bigcup\{x_\nu : \nu<\delta\}$. Also, it is clear that $(\forall\alpha<\lambda)(E_{\{\alpha\}} \subset \widehat{\{\alpha\}})$, and that our assumptions on B,G guarantee that $(\forall\alpha<\lambda)(E_{\{\alpha\}}\in B)$. Thus $\{E_{\{\alpha\}} : \alpha<\lambda\}$ generates a proper κ-complete filter F in B containing $\{\widehat{\{\alpha\}} : \alpha<\lambda\}$. Then since κ is λ-Shelah and hence mildly λ-ineffable, and since $\lambda^{<\kappa} = \lambda$, it follows by $(FEP)_{\kappa\lambda}$ that there is a κ-complete ultrafilter U in B extending F. It is clear that $(\forall\alpha<\lambda)(\widehat{\{\alpha\}}\in U)$, and that every function in G is constant on a set in $\mathcal{U}$.

Conversely, suppose that $(\mathrm{NUP})_{\kappa\lambda}$ holds and let $G = (g_\alpha : \alpha<\lambda)$ be a λ-sequence of regressive functions on $P_\kappa\lambda$. Let B be the κ-complete field of subsets of $P_\kappa\lambda$ generated by $\{g_\alpha^{-1}(\{\beta\}) : \alpha,\beta<\lambda\} \cup \{\widehat{\{\alpha\}} : \alpha<\lambda\}$. Since $\lambda^{<\kappa} = \lambda$, it is clear that $|B| = \lambda$. Now let $\mathcal{U}$ be any κ-complete ultrafilter in B such that $(\forall\alpha<\lambda)(\widehat{\{\alpha\}}\in\mathcal{U})$ and every function in G is constant on a set in $\mathcal{U}$. For each $\alpha < \lambda$, let $\tau_\alpha < \lambda$ be such that $g_\alpha^{-1}(\{\tau_\alpha\})\in\mathcal{U}$. Then define $f : \lambda \to \lambda$ by $f(\alpha) = \tau_\alpha$. It is easy to see that f has the required property. $\square$

2. λ-Shelah cardinals and supercompactness.

It is well known that if κ is λ-supercompact, then κ is λ-ineffable, and that if κ is $2^{\lambda^{<\kappa}}$-ineffable, then κ is λ-supercompact (Magidor [23]). Then since κ is λ-Shelah if κ is λ-ineffable, it is clear that if κ is λ-supercompact, then κ is λ-Shelah. The main result of this section is that if κ is $2^{\lambda^{<\kappa}}$-Shelah, then κ is λ-supercompact (theorem 2.1 below). From this together with our 3.1.4 above we obtain the following improvement of Magidor's characterization of supercompactness: κ is supercompact iff κ is λ-Shelah for every $\lambda \geq \kappa$ iff κ is almost λ-ineffable for every $\lambda \geq \kappa$ (corollary 2.2 below).

2.1 Theorem. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$, if κ is $2^{\lambda^{<\kappa}}$-Shelah, then κ is λ-supercompact.

proof. Set $\gamma = 2^{\lambda^{<\kappa}}$ and let $(f_\alpha : \alpha<\gamma)$ enumerate the set of all regressive functions on $P_\kappa\lambda$. For each $y \in P_\kappa\gamma$ fix an element τ_y of y, and then for each $\alpha < \gamma$ define $g_\alpha : P_\kappa\gamma \to \gamma$ by

$$g_\alpha(y) = \begin{cases} f_\alpha(y\cap\lambda) & \text{if } y\cap\lambda\neq 0 \\ \tau_y & \text{otherwise.} \end{cases}$$

It is clear that for each $\alpha<\gamma$, g_α is regressive and that $f_\alpha = g_\alpha \restriction P_\kappa\lambda$.

Now let $g : \gamma \to \gamma$ be such that

$(\forall y\in P_\kappa\gamma)(E_y = \{z\in P_\kappa\gamma : y\subset z \ \&\ (\forall\alpha\in y)(g_\alpha(z) = g(\alpha))\}\in I_{\kappa\lambda}{}^+)$. Notice that $g : \gamma \to \lambda$.

For each $y \in P_\kappa\gamma$, set $E'_y = \{z\cap\lambda : z\in E_y\}$. It is easy to see that:

(i) $(\forall y\in P_\kappa\gamma)(E'_y \in I_{\kappa\lambda}{}^+)$,

(ii) $(\forall y\in P_\kappa\gamma)(E'_y \subset \bigcap\{E'_{\{\alpha\}} : \alpha\in y\})$, and

(iii) $(\forall\alpha<\lambda)(E'_{\{\alpha\}} \subset \widehat{\{\alpha\}} \cap P_\kappa\lambda)$.

An immediate consequence of (i) - (iii) above is that $\{E'_{\{\alpha\}} : \alpha<\gamma\}$ generates a proper κ-complete filter F in $P(P_\kappa\lambda)$ extending $FSF_{\kappa\lambda}$. We claim that F is a <u>normal</u> <u>ultra</u>filter.

Let $f : P_\kappa\lambda \to \lambda$ be such that $X = \{x\in P_\kappa\lambda : f(x)\in x\} \in F$ and let β be any ordinal $< \gamma$ such that $f_\beta \restriction X = f \restriction X$. Then since $E'_{\{\beta\}} \in F$ and since $E'_{\{\beta\}} - \{0\} = \{x\in P_\kappa\lambda : f_\beta(x) = g(\beta)\}$, it follows that $f^{-1}(\{g(\beta)\}) \in F$.

Pick $X \subset P_\kappa\lambda$ and let $\chi_X : P_\kappa\lambda \to \{0,1\}$ be its characteristic function. Notice that χ_X is regressive on $\widehat{\{0,1\}} \cap P_\kappa\lambda \in F$. An argument similar to the one used in the preceeding paragraph shows that $(\exists\beta<\gamma)(g(\beta) \in \{0,1\}\ \&\ \chi_X{}^{-1}(\{g(\beta)\}) \in F)$. □

2.2 <u>Corollary</u>. κ is supercompact <u>iff</u> κ is λ-Shelah for every $\lambda \geq \kappa$ <u>iff</u> κ is almost λ-ineffable for every $\lambda \geq \kappa$.

<u>proof</u>. An argument similar to the one used to prove 2.4.2(2) above yields Magidor's result that if κ is λ-supercompact, then κ is λ-ineffable and hence almost λ-ineffable and λ-Shelah. The rest follows immediately from the preceeding theorem together with the fact that $(\forall\gamma\geq\kappa)(NSh_{\kappa\gamma} \subset NAIn_{\kappa\gamma})$ (3.1.4 above). □

We conclude this section by showing that the λ-Shelah property and mild λ-ineffability are not provably equivalent for arbitrary $\lambda > \kappa$. Baumgartner [3] obtained a result which amounts to the same thing.

2.3 Corollary. The λ-Shelah property and mild λ-ineffability are not provably equivalent for arbitrary $\lambda > \kappa$.

proof. By theorem 2.1 together with the DiPrisco-Zwicker characterization [10] of strong compactness, and Menas' result [25] that the least measurable cardinal which is a limit of strongly compact cardinals is itself strongly compact, but is not 2^{κ}-supercompact. □

3. A combinatorial result.

In view of the fact that every normal ultrafilter on $P_\kappa\lambda$ extends $CF_{\kappa\lambda}$, it is clear that each of the $E'_{\{\alpha\}}$'s ($\alpha<\gamma$) constructed in the proof of theorem 2.1 above is actually stationary in $P_\kappa\lambda$. It is natural to wonder if the condition $(\exists f : \lambda\to\lambda)(\forall x\in P_\kappa\lambda)(E_x\neq 0)$ in 1.2 above is equivalent with $(\exists f : \lambda\to\lambda)(\forall x\in P_\kappa\lambda)(E_x\in NS_{\kappa\lambda}{}^+)$. We have already observed that the former is equivalent with $(\exists f : \lambda\to\lambda)(\forall x\in P_\kappa\lambda)(E_x\in I_{\kappa\lambda}{}^+)$. We use this fact together with the facts that $(\forall\lambda\geq\kappa)(NS_{\kappa\lambda} = \nabla SNS_{\kappa\lambda} = \nabla\nabla I_{\kappa\lambda})$ (1.1.2 and 1.2.4 above) and some ideas suggested by Baumgartner [3] to prove that the above conditions are equivalent if $\lambda^{<\kappa} = \lambda$ (3.2 below). This will require the following preliminary.

3.1 Lemma. Let κ be an uncountable regular cardinal, let λ be a cardinal $\geq\kappa$ satisfying $\lambda^{<\kappa} = \lambda$, and let $F = \{f_\nu : \nu<\lambda\}$ be any family of regressive functions on $P_\kappa\lambda$. Then for any ideal I on $P_\kappa\lambda$, there is a family $G = \{g_\alpha : \alpha<\lambda\}$ of regressive functions on $P_\kappa\lambda$ such that

(1) $F \subseteq G$

(2) for any $x\in P_\kappa\lambda$ and any $h : x\to\lambda$ such that $E_{xh} = \{y\in P_\kappa\lambda : (\forall\alpha\in x)(g_\alpha(y) = h(\alpha))\}\in\nabla I$, $(\exists\gamma<\lambda)(\forall\alpha<\lambda)(E_{xh}\cap g_\gamma{}^{-1}(\{\alpha\})\in I)$, and

(3) for any ideal J on $P_\kappa\lambda$, if there is a $g : \lambda\to\lambda$ such that $(\forall x\in P_\kappa\lambda)(E'_{xg} = \{y\in P_\kappa\lambda : (\forall\alpha\in x)(g_\alpha(y) = g(\alpha))\}\in J^+)$, then there is an $f : \lambda\to\lambda$ such that $(\forall x\in P_\kappa\lambda)(E_{xf} = \{y\in P_\kappa\lambda : (\forall\nu\in x)(f_\nu(y) = f(\nu))\}\in J^+)$.

<u>proof</u>. We construct a sequence $F_0 \subset \ldots \subset F_\alpha \subset \ldots (\alpha<\lambda)$ of families of regressive functions on $P_\kappa\lambda$, each of cardinality $\leq \lambda$ inductively as follows.

First, set $F_0 = F$. Then pick $\alpha < \lambda$ and suppose that we have found $F_0 \subset \ldots \subset F_\xi \subset \ldots (\xi<\alpha)$. If $\lim(\alpha)$ set $F_\alpha = \bigcup\{F_\xi : \xi<\alpha\}$. Clearly $|F_\alpha| = \lambda$.

Now suppose that $\alpha = \beta + 1$ and let $(f^\beta_\nu : \nu < \lambda)$ be an enumeration of F_β without repetition. For each $x \in P_\kappa\lambda$ and for each $h : x \to \lambda$ such $E^\beta_{xh} \in \nabla I$, let g_{xh} be a regressive function on $P_\kappa\lambda$ with the property $(\forall\nu<\lambda)(E^\beta_{xh} \cap g^{-1}_{xh}(\{\nu\}) \in I)$. Now let F_α be the union of F_β and the set consisting of all of these g_{xh}'s. The assumption $\lambda^{<\kappa} = \lambda$ guarantees that $|F_\alpha| = \lambda$.

Finally, set $G = \bigcup\{F_\alpha : \alpha<\lambda\}$. It is clear that $|G| = \lambda$ and that $F \subset G$. It remains to prove (2) and (3). To this end, let $(g_\alpha : \alpha<\lambda)$ be an enumeration of G without repetitions.

Let $x \in P_\kappa\lambda$ and $h : x \to \lambda$ be such that $E_{xh} \in \nabla I$. Now let $\beta < \lambda$ be such that $\{g_\nu : \nu\in x\} \subset F_\beta$; notice that such a β exists since the assumption $\lambda^{<\kappa} = \lambda$ guarantees that $cf(\lambda) \geq \kappa$. For each $\nu \in x$ let $\alpha_\nu < \lambda$ be such that $g_\nu = f^\beta_{\alpha_\nu}$. Set $x' = \{\alpha_\nu : \nu\in x\}$ and define $h' : x' \to \lambda$ by $h'(\alpha_\nu) = h(\nu)$. Then for each $y \in P_\kappa\lambda$, $y \in E^\beta_{x'h'}$ iff $(\forall\nu\in x)(f^\beta_{\alpha_\nu}(y) = h'(\alpha_\nu))$ iff $(\forall\nu\in x)(g_\nu(y) = h(\nu))$ iff $y \in E_{xh}$. Thus $E^\beta_{x'h'} \in \nabla I$. Then since $g_{x'h'} \in F_{\beta+1} \subset G$ is I-small on $E^\beta_{x'h'} = E_{xh}$, it follows that (2) holds.

Now let J be any ideal on $P_\kappa\lambda$ and suppose that $g : \lambda \to \lambda$ is such that $(\forall x \in P_\kappa\lambda)(E_{xg} \in J^+)$. For each $\nu < \lambda$, let $\alpha_\nu < \lambda$ be such that $f_\nu = g_{\alpha_\nu}$, and define $f : \lambda \to \lambda$ by $f(\nu) = g(\alpha_\nu)$. For each $x \in P_\kappa\lambda$, set $x' = \{\alpha_\nu : \nu \in x\}$ and notice that $E_{xf} = E_{x'g}$. □

3.2 Theorem. For any uncountable regular cardinal κ and any cardinal $\lambda \geq \kappa$ satisfying $\lambda^{<\kappa} = \lambda$, the following are equivalent.

(1) κ is λ-Shelah.

(2) For any λ-sequence $(f_\nu : \nu<\lambda)$ of regressive functions on $P_\kappa\lambda$,

$$(\exists f : \lambda\to\lambda)(\forall x \in P_\kappa\lambda)(E_{xf} = \{y \in \hat{x} : (\forall \nu \in x)(f_\nu(y) = f(\nu))\} \in SNS_{\kappa\lambda}{}^+).$$

(3) For any λ-sequence $(f_\nu : \nu<\lambda)$ of regressive functions on $P_\kappa\lambda$,

$$(\exists f : \lambda\to\lambda)(\forall x \in P_\kappa\lambda)(E_{xf} \in NS_{\kappa\lambda}{}^+).$$

proof. It is clear that (3) ⇒ (2) ⇒ (1).

(1) ⇒ (2). Let $(f_\nu : \nu<\lambda)$ be a λ-sequence of regressive functions on $P_\kappa\lambda$, and let $(g_\alpha : \alpha<\lambda)$ be a sequence satisfying the preceeding lemma. Further, let $g : \lambda \to \lambda$ be such that $(\forall x \in P_\kappa\lambda)(E_{xg} = \{y \in x : (\forall \alpha \in x)(g_\alpha(y) = g(\alpha))\} \in I_{\kappa\lambda}{}^+)$. We show that $(\forall x \in P_\kappa\lambda)(E_{xg} \in SNS_{\kappa\lambda}{}^+)$; (2) will then follow by 3.1(3) above.

Suppose by way of contradiction that $(\exists x \in P_\kappa\lambda)(E_{xg} \in SNS_{\kappa\lambda} = \nabla I_{\kappa\lambda})$, and let $\gamma < \lambda$ be such that $(\forall \alpha<\lambda)(E_{xg} \cap g_\gamma^{-1}(\{\alpha\}) \in I_{\kappa\lambda})$. Set $z = x \cup \{\gamma\}$. Then for any $y \in \hat{z}$, $y \in E_{zg}$ iff $(\forall \alpha \in z)(g_\alpha(y) = g(\alpha))$ iff $(\forall_\alpha \in x)(g_\alpha(y) = g(\alpha))$ & $g_\gamma(y) = g(\gamma)$ iff $y \in E_{xg} \cap g_\gamma^{-1}(\{g(\gamma)\})$. Thus $E_{zg} \in I_{\kappa\lambda}$ thus contradicting (1).

(2) ⇒ (3) follows by an argument similar to the one used to prove (1) ⇒ (2) using the fact that $NS_{\kappa\lambda} = \nabla SNS_{\kappa\lambda}$. □

BIBLIOGRAPHY

1. Baumgartner, J.E. "Ineffability properties of cardinals I", in Infinite and Finite Sets, Colloquia Mathematica Societatis Janos Bolyai, 10 (dedicated to P. Erdös), North-Holland, Amsterdam, 1975, 109-130.

2. Baumgartner, J.E. "Ineffability properties of cardinals II", in Logic, Foundations of Mathematics and Computability Theory, D. Reidel Publishing Company, Dordrecht-Holland, 1977, 87-106.

3. Baumgartner, J.E. "Generalizing weak compactness to the (κ,λ)-context", handwritten notes.

4. Baumgartner, J.E. and R. Laver. "Iterated perfect-set forcing", Ann. Math Logic 17 (1979), 271-288.

5. Baumgartner, J.E., A.D. Taylor, S. Wagon, "Structural properties of ideals", to appear in Dissertationes Mathematicae.

6. Carr, D.M. "Mild λ-ineffability and the solvability of binary (κ,λ) messes", Amer. Math. Soc. Abstracts 1 (1980), abstract no. 774-E4, p. 199.

7. Carr, D.M. "The minimal normal ideal on $P_\kappa\lambda$", Amer. Math. Soc. Abstracts 1 (1980), abstract no. 80T-E51, p. 394.

8. Carr, D.M. "The minimal normal filter on $P_\kappa\lambda$", submitted

9. Dickmann, M.A. Large Infinitary Languages, North-Holland, Amsterdam, 1975.

10. DiPrisco, C.A. and W.S. Zwicker. "Flipping properties and supercompact cardinals", Fund. Math. 109 (1980), 31-36.

11. Drake, F.R. Set Theory, North-Holland, Amsterdam, 1974.

12. Erdös, P., A. Hajnal, A. Máté and R. Rado. Combinatorial Set Theory, manuscript.

13. Fodor, G. "Eine Bemerkung zur Theorie der regressiven Funktionen", Acta Sci. Math. (Széged) 17 (1956), 139-142.

14. Hickin, K.K. and J.M. Plotkin. "Some algebraic properties of weakly compact and compact cardinals", Fund. Math. 97 (1977), 177-185.

15. Jech, T.J. "Some combinatorial problems concerning uncountable cardinals", Ann. Math. Logic 5 (1973), 165-198.

16. Jech, T.J. Set Theory, Academic Press, New York, 1978.

17. Jech, T.J. handwritten note.

18. Jensen, R.B. and K. Kunen. "Some combinatorial properties of L and V", circulated notes.

19. Kanamori, A. and M. Magidor. "The evolution of large cardinal axioms in set theory", in Higher Set Theory, Lecture Notes in Math. 669, Springer-Verlag, Berlin, 1978, 99-275.

20. Keisler, H.J. and A. Tarski. "From accessible to inaccessible cardinals", Fund. Math. 53 (1964), 225-308.

21. Kunen, K. "Some applications of iterated ultrapowers in set theory", Ann. Math. Logic 1 (1970), 179-297.

22. Lévy, A. "The sizes of the indescribable cardinals" in D.S. Scott, ed., Axiomatic Set Theory, Proc. Symp. Pure Math. 13(1), Amer. Math. Soc., Providence R.I., 1971, 205-218.

23. Magidor, M. "Combinatorial characterization of supercompact cardinals", Proc. Amer. Math. Soc. 42 (1974) 279-285.

24. Magidor, M. "How large is the first strongly compact cardinal?", Ann. Math. Logic, 10 (1976), 33-57.

25. Menas, T.K. "On strong compactness and supercompactness", Ann. Math. Logic 7 (1974), 327-359.

26. Menas, T.K. "A combinatorial property of $P_\kappa\lambda$", J. Sym. Logic 44 (1979), 225-233.

27. Mitchell, W.J. "Hypermeasurable cardinals", in M. Boffa et al., eds., Logic Colloquium 78, North-Holland, Amsterdam, 1979.

28. Neumer, J. "Verallgemeinerung eines Staz von Alexandroff und Urysohn", Math. Zeitschrift 54 (1951), 254-261.

29. Shelah, S. "Weakly compact cardinals : a combinatorial proof", J. Sym. Logic 44 (1979), 559-562.

30. Solovay, R.M., W.N. Reinhardt, A. Kanamori. "Strong axioms of infinity and elementary embeddings", Ann. Math. Logic 13 (1978), 73-116.