ETALE K-THEORY

AND

IWASAWA THEORY OF NUMBER FIELDS

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Abstract

Results of W.G.Dwyer and E.M.Friedlander on étale K-theory of the S-integers O_E^S in a number field E are used to express the higher étale tame and wild kernel in terms of arithmetical invariants in the cyclotomic $\mathbf{Z}_l$-extension of $F = E(\zeta_l)$. Furthermore, properties of these groups are discussed, such as higher rank formulas and Galois descent.

Contents

1 Introduction

Over more than two centuries the ideal class number of a number field has interested and clearly fascinated many mathematicians. The problems and applications relating to the class number are widespread, just to mention the class number problem for imaginary quadratic fields or the still unknown Gauß conjecture on totally real quadratic number fields as well as the Vandiver conjecture on the divisibility of the class number of $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$, where ζ_p denotes a primitive p-th root of unity.

The by far most famous result regarding the class number $h(E)$ of the number field E is the analytic class number formula: Let $R(E)$ the regulator, $D(E)$ the discriminante of E and $\mu(E)$ th group of roots of unity in E, then

$$\lim_{s\to 1}(s-1)\zeta_E(s) = \frac{2^{r_1(E)} \cdot (2\pi)^{r_2(E)} \cdot R(E) \cdot h(E)}{\#\mu(E) \cdot \sqrt{|D(E)|}},$$

where $\zeta_E(s)$ is the Dedekind ζ-function of E and $[E:\mathbb{Q}] = r_1(E) + 2r_2(E)$. This formula should not be seen as a way to calculate explicitly the class number $h(E)$, but rather as a local-global principle. Namely, the ζ-function $\zeta_E(s)$ possesses a convergent Euler product expression

$$\zeta_E(s) = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}, \ Re(s) > 1,$$

where the product runs over all finite primes in E, and the analytic class number formula now says that the local factors give results on the global arithmetic of the number field.

It is common to modify the ζ-function by ∞-factors in such a way that the new function $Z_E(s)$ satisfies the functional equation $Z_E(s) = Z_E(1-s)$. Moreover, the Euler product of $Z_E(s)$ then clearly runs over all primes in E.

As we do so often in mathematics we can ask for generalizations of the analytic class number formula. A first such approach is given by replacing the number field by curves or even projective varieties. For example, let E be an elliptic curve defined over $\mathbf{Q}$. Then the ζ-function—usually called the L-series of E— is constructed by local data and exists at least conjecturally. The corresponding formula is then known as the conjecture of Birch and Swinnerton-Dyer.

Another idea is to stick to the number field E, and evaluate, if possible, the ζ- function at negative integers. For the Riemann ζ-function $\zeta_{\mathbf{Q}}(s)$, we have

$$\zeta_{\mathbf{Q}}(1-n) = -B_n/n \ ,$$

where B_n denotes the n-th Bernoulli number, and the Dedekind ζ-function can be evaluated at negative integers by means of the generalized Bernoulli numbers. But this is not exactly what we are looking for. Instead we would like to evaluate the ζ-function at negative integers in terms of invariants closely related to the number field E—such as the class number and the regulator.

Motivated by an analogous result for global fields of characteristic $p > 0$, J.Birch and J.Tate formulated the following

Conjecture 0.1 (Birch-Tate) *Let E be a totally real number field with ring of integers O_E, then*

$$\zeta_E(-1) = \pm \frac{\# K_2(O_E)}{w^{(2)}(E)} ,$$

where $w^{(2)}(E) := max\{m : Gal(E(\zeta_m)/E)^2 = 1\}$ and $K_2(O_E)$ is the tame kernel of E.

Nowadays, this conjecture is almost a theorem, i.e., it is known to be true up to 2-torsion, and if E is an abelian (and totally real) number field, the 2-part is also valid. The proof of this conjecture is based on several, very deep results in algebraic number theory, which we will now present here.
But before we do so, let us add the following generalization of the Birch-Tate conjecture to higher K-theory and arbitrary number fields.

Conjecture 0.2 (Lichtenbaum) *Let E be a number field with ring of integers O_E and $i \geq 2$ an integer, then up to 2-torsion*

$$\zeta_E(1-i)^* = \pm R_{i-1} \cdot \frac{\# K_{2i-2}(O_E)}{\# tor\ K_{2i-1}(O_E)} ,$$

where R_{i-1} is the higher regulator defined on $K_{2i-1}(O_E)$ and $\zeta_E(s)^*$ stands for the leading coefficient in the Taylor expansion of $\zeta_E(s)$.

Note, that by the work of A.S.Merkurjev and A.A.Suslin on $K_3(O_E)$ we have $\# l\text{-}tor\ K_3(O_E) = w_l^{(2)}(E)$, and in the case of a totally real field we also know from A.Borel that $R_2 = 1$. Thus the Lichtenbaum conjecture indeed generalizes the Birch-Tate conjecture. The cases, for which the Lichtenbaum conjecture is valid, are precisely the same as for the Birch-Tate conjecture with the exception of a few imaginary quadratic number fields and $i = 2$.

One of the major ingredients in the proof of the Birch-Tate conjecture is Iwasawa theory of number fields. But let us first consider the function field case. Remember that this was the motivation for J.Birch and J.Tate. Let C be a curve of genus g over $\mathbb{F}_q$ and $F = \mathbb{F}_q(C)$ its function field. If $\zeta_C(s)$ denotes the ζ-function of C, then we know from A.Weil that

$$\zeta_C(s) = \frac{\prod_{i=1}^{2g}(1 - \alpha_i q^{-s})}{(1 - q^{-s})(1 - q^{1-s})} ,$$

where the α_i's are the eigenvalues of the Frobenius acting on the Jacobian $J = J(C)$ of C. As we know, this result has a far reaching generalization to smooth, projective varieties X over $\mathbb{F}_q$, namely the Weil conjecture. The proof of the Weil conjecture makes heavy use of étale cohomology and its properties, and countless other theories. For example, the Lefschetz Fixed Point Theorem in étale cohomology implies the expression of the ζ-function as a product of determinants, and hence in particular the above formula, cf. chapter 3.

Note, that for the Dedekind ζ-function—also called the continuous case—the recent work of C.Deninger gives the local factors (including the ∞-factors) in a uniform way and leads to conjectures, which are analogous to the Weil conjecture.

Now Iwasawa's contribution was to construct an l-adic analogue for number fields. Let E be a totally real number field, l an odd prime and $F_\infty = E(W_l)$, where $W_l = \mathbb{Q}_l/\mathbb{Z}_l(1)$ is the group of all l-th power roots of unity. Further, let $A_\infty := l\text{-}tor\ Pic(O_\infty)$, O_∞ the ring of integers in F_∞. Then for every integer j one can associate to the eigenspace $\varepsilon_j A_\infty$ in a natural way a characteristic polynomial

$$g_j(T) = char\ Hom_{\mathbb{Z}_l}(\varepsilon_j A_\infty, \mathbb{Q}_l/\mathbb{Z}_l)$$

which plays the analogous role to $\prod_{i=1}^{2g}(1-\alpha_i T)$ in the above formula. On the other hand, for every integer j and every Dirichlet character χ one can construct a power series $H(T,\chi)$ using Stickelberger elements and the arithmetic in cyclotomic fields. This construction is due to K.Iwasawa for $E = \mathbb{Q}$ and due to P.Deligne and K.Ribet for an arbitrary totally real number field. Moreover, we set $\Gamma := Gal(F_\infty/F)$, $F = E(\zeta_l)$, and $\Delta := Gal(F/E)$. If $\kappa : \Gamma \to U_l^{(1)}$ resp. $\omega : \Delta \to \mu_{l-1}$ denote the cyclotomic resp. the Teichmüller character, then one can show that for $u = \kappa(\gamma_0)$, $\gamma_0 \in \Gamma$ a topological generator, and $s \in \mathbb{Z}_l - \{1\}$

$$H(u^s - 1, \omega^j) = \begin{cases} L_l(\omega^{1-j}, s) & \text{if } j \not\equiv 1 \; mod \; \#\Delta \\ (u^s - u) \cdot L_l(id, s) & \text{if } j \equiv 1 \; mod \; \#\Delta \end{cases},$$

where $L_l(\chi, s)$ denotes the l-adic L-function associated to χ. The Main Conjecture in Iwasawa theory, which has been proven by A.Wiles, relates now the characteristic polynomial $g_j(T)$ and the power series $H(T, \omega^j)$, namely

Theorem 0.3 *Let E be a totally real number field and $G_j(T) := g_j((1+T)^{-1} - 1)$. Then for $j \equiv 1 \; mod \; 2$,*

$$(G_j(T)) = (H(T, \omega^j)) \; in \; \Lambda = \mathbb{Z}_l[[T]] .$$

The remaining step concerning the proof of the Birch-Tate conjecture is an Iwasawa theoretical description of the tame kernel, which is due to J.Coates.

Theorem 0.4 *Let E be a totaly real number field and l an odd prime. Then*

$$l\text{-}tor \, K_2(O_E) \simeq \varepsilon_{-1} A_\infty(1)^\Gamma .$$

The proof is based on the following deep results.

(i) For $j = 1, 2$, there are isomorphisms

$$K_{2 \cdot 2 - j}(O_E) \otimes \mathbb{Z}_l \xrightarrow{\sim} H^1_{\acute{e}t}(O_E^S, \mathbb{Z}_l(2)) ,$$

where $O_E^S = O_E[\frac{1}{l}]$.

(ii) The tame kernel $K_2(O_E)$ is a finite group.

The first assertion was proven by J.Tate and later by A.S.Merkurjev and A.A.Suslin. The finiteness of the tame kernel was shown by J.Garland, and generalized to higher K-theory by A.Borel. That the theorems 0.3 and 0.4 now imply the Birch-Tate conjecture follows from standard arguments in Iwasawa theory, cf. chapter 4 and 5.

Let us recap what we have done so far. First we considered an appropriate characteristic polynomial $G_j(T)$ in Λ and then we constructed a power series $H(T,\chi)$, which is closely related to the arithmetic in cyclotomic fields. Next the Main Conjecture says that these two elements are equal up to a unit in Λ. Since the power series $H(T,\chi)$ is essentially the l-adic L-function $L_l(\chi,s)$, we obtain a relation betweeen $G_j(T)$ and the Dedekind ζ-function $\zeta_E(s)$ at negative integers. Finally we used the Iwasawa theoretical description of the tame kernel to prove the Birch-Tate conjecture.
Clearly we would like to take the same approach for the Lichtenbaum conjecture. But unfortunately this would lead to nowhere, since the l-adic L-function is simply trivial in all cases other than E totally real and $j \equiv 1 \; mod 2$. Moreover, the construction of the power series $H(T,\chi)$ depends on the fact, that the base field E is totally real. Nevertheless, there is a slight hope that a generalization of Coates's result to higher K-theory could indicate the

right construction of $H(T,\chi)$ as well as the correct relation to the Dedekind ζ-function.

To find such a generalization was the starting-point of this thesis. Simplifying Coates's proof we will see that his arguments give rather a statement on the étale (l-adic) cohomology of O_E^S, and thus the theorem 0.4 is then a simple consequence of (i) above. More generally there is the following

Conjecture 0.5 (Quillen) *Let E be a number field, l an odd prime and $O_E^S = O_E[\frac{1}{l}]$. Then for $j = 1,2$ and $i \geq 2$, the Chern classes induce isomorphisms*

$$K_{2i-j}(O_E) \otimes \mathbf{Z}_l \xrightarrow{\sim} H^j_{et}(O_E^S, \mathbf{Z}_l(i)) .$$

By introducing étale K-theory, which can be computed by the (continuous) étale cohomology via a fourth quadrant spectral sequence, W.G.Dwyer and E.M.Friedlander proved the s rjectivity of the Chern class map. In chapter 5, cf. 5.9, we prove the étale-theoretical generalization of 0.4, namely

Theorem 0.6 *Let E be a number field, l an odd prime and S the set of infinite and l-adic primes. Then for $i \geq 2$ and $G_\infty := Gal(E(W_l/E)$, there are exact sequences*

$$0 \to (U_\infty^{(S)} \otimes \mathbf{Q}_l/\mathbf{Z}_l(i-1))^{G_\infty}/max.\,div. \to K^{et}_{2i-2}(O_E^S) \to A_\infty^{(S)}(i-1)^{G_\infty} \to 0 ,$$

where $max.\,div.(U_\infty^{(S)} \otimes \mathbf{Q}_l/\mathbf{Z}_l(i-1))^{G_\infty} \simeq K^{et}_{2i-1}(O_E^S) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l$.

Using Iwasawa theory we can compute the structure and order of the left-hand group in the above sequence, cf. chapter 4. Furthermore if O_n^S denotes the ring of S-integers in $F_n = E(\zeta_n)$, then for every $i \geq 2$, we obtain a non-degenerate pairing

$$tor_\Lambda \mathcal{X}(-i) \times \varinjlim K_{2i-2}^{\acute{e}t}(O_n^S) \rightarrow \mathbf{Q}_l/\mathbf{Z}_l ,$$

where $\mathcal{X} = Gal(M_\infty/F_\infty)$ is the Galois group of the maximal abelian l-ramified, pro-l-extension of $F_\infty = E(W_l)$, cf. 5.14. From this pairing we deduce Galois descent for $K_{2i-2}^{\acute{e}t}(O_n^S)$ in the cyclotomic $\mathbf{Z}_l$-extension F_∞/F as well as conditions for the triviality of $K_{2i-2}^{\acute{e}t}(O_n^S)$, cf. 5.15–5.17. Another consequence of the pairing is that we can calculate the order of $K_{2i-2}^{\acute{e}t}(O_E^S)$ by evaluating the characteristic polynomial of $\varepsilon_{-i} tor_\Lambda \mathcal{X}$ at $u^{i-1} - 1$, cf. 5.18. Another classical invariant of a number field E is the wild kernel $WK_2(E)$. Following G.Banaszak, M.Kolster and T.Nguyen Quang Do we can define a higher wild kernel, in the algebraic as well as étale case, cf. 2.32. If we replace the units by the so-called Gross kernel $ker\, g_\infty \subseteq U_\infty^S \otimes \mathbf{Z}_l$ and the ideal class group by a certain idelé class group C_∞, cf. chapter 6, we obtain after solving a few technical problems the following analogue of 0.6, cf. 6.16

Theorem 0.7 *Let the notations be as in the previous theorem. Then for $i \geq 2$, there is exact sequence*

$$0 \rightarrow (ker\, g_\infty \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l(i-1))^{G_\infty}/max.\, div. \rightarrow WK_{2i-2}^{\acute{e}t}(E) \rightarrow l\text{-}tor\, C_\infty(i-1)^{G_\infty} \rightarrow 0 ,$$

where $max.\,div.(ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} \simeq K^{\acute{e}t}_{2i-1}(O^S_E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$.

As for the higher étale tame kernel $K^{\acute{e}t}_{2i-2}(O^S_E)$ we can construct a non-degenerate pairing

$$tor_\Lambda \tilde{\mathcal{X}}(-i) \times \varinjlim W K^{\acute{e}t}_{2i-2}(F_n) \to \mathbb{Q}_l/\mathbb{Z}_l\ ,$$

where $\tilde{\mathcal{X}}$ is a certain Λ-quotient module of $\mathcal{X}$, cf. chapter 6. The same machinery as above then applies to the higher wild kernel, and analogous results follow.

A crucial argument in the proof of the last two theorems is the triviality of $H^2_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i))$ for $i \geq 2$ or equivalently the finiteness of $A^S_\infty(i-1)^{G_\infty}$ resp. $l\text{-}tor\, C_\infty(i-1)^{G_\infty}$, cf. 5.5 and 6.15. This is supposed to be true in a wider sense, namely

Conjecture 0.8 (Schneider) *Let E be a number field, l an odd prime and $O^S_E = O_E[\frac{1}{l}]$. Then for $i \neq 1$,*

$$H^2_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0\ .$$

As mentioned above the Schneider conjecture holds for $i \geq 2$, and for $i = 0$ the assertion is equivalent to the well-known Leopoldt conjecture. If $i = 1$, the cohomology group $H^2_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(1))$ does not vanish unless the number field E has a unique l-adic prime, but the group of Galois invariants of the

ideal class group ${A_\infty^S}^{G_\infty}$ is still finite. This was conjectured by B.-H.Gross and just recently proven by T.Nguyen Quang Do, cf. 7.5.
We consider his proof and consequences towards the Schneider conjecture in chapter 7. We also explain there, why one can assume without loss of generality that the Iwasawa μ-invariant is trivial. Concerning the Schneider conjecture we use cup-product arguments to obtain a reformulation in terms of $K_{2i-2}^{\acute{e}t}(O_E^S)$ and $WK_{2i-2}^{\acute{e}t}(E)$, cf. 7.11. This gives us at least under ceratin conditions the validity of the Schneider conjecture. It should be pointed out here, that this conjecture is not even known to be true for the rational numbers $\mathbf{Q}$. At the end of chapter 7 we calculate higher rank formulas for $K_{2i-2}^{\acute{e}t}(O_E^S)$ and $WK_{2i-2}^{\acute{e}t}(E)$, which imply divisibility criteria, cf. 7.14.

In the last chapter we consider (l,i)-regular fields, which generalizes the earlier concepts of l-regular resp. l-rational fields, introduced by G.Gras and T.Nguyen Quang Do. Since the property (l,i)-regular is periodic $mod[E(\zeta_l):E]$, it is enough to consider positive values of i, which then translates into the triviality of $K_{2i-2}^{\acute{e}t}(O_E^S)$. G.Gras, J.-F.Jaulent and T.Nguyem Quang Do studied the problem of a going-up lemma for l-regular resp. l-rational fields. For (l,i)-regular fields this means that we have to deal with the question of Galois descent for étale K-theory, cf. 8.12. The advantage of our approach lies in the fact, that we get very explicit results and can deduce quite easily examples, which satisfy the imposed conditions, cf. 8.15 and 8.16.

Notations: For every abelian group A, the kernel of multiplication by m is denoted by ${}_mA = ker(A \xrightarrow{\times m} A)$. If $n \geq 1$, ζ_n stands for a primitive n-th root of unity, and the cardinality of a set M is denoted by $\#M$. Any other relevant notation will be introduced and frequently repeated in the text.

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1 Cohomology

This paragraph provides the basic definitions of the various cohomology theories of which we will make frequent use in what follows. We start with the cohomology of a profinite group G with coefficients in a G-module M. For this, let G be a fixed profinite group, e.g., the absolute Galois group $G_E = Gal(E^{sep}/E)$ of a field E with separable algebraic closure E^{sep}. Let M be a discrete G-module, and $j \geq 0$ an integer. The group of j-th cochains is defined by $C^j := \{\phi : G^j \to M\}$; here $G^0 := \{*\}$ is understood as the one-point set. In the usual way we have differentations $d^j : C^j(G,M) \to C^{j+1}(G,M)$, and so we obtain the cochain complex $\mathcal{C}(G,M) := (C^j(G,M); d^j)$.

Definition 1.1 *The cohomology of G with coefficients in M is the homology of the cochain complex $\mathcal{C}(G,M) := (C^j(G,M); d^j)$, i.e., for $j \geq 0$, the j-th cohomology group $H^j(G,M)$ of G with coefficients in M is defined by*

$$H^j(G,M) := ker\, d^j / im\, d^{j-1} .$$

A somewhat more functorial approach is given by the following. We fix again the profinite group G, and consider the category $M(G)$ of discrete G-modules. It is well-known, that $M(G)$ is an abelian category with enough injectives, to be precise, $M(G)$ is an abelian category satisfying **AB5** and **AB3***, and having generators, cf. [78]. So we can define the right derivatives $R^j\Gamma(G, \cdot)$

of the functor $\Gamma(G,\cdot) : M(G) \to Ab$, $\Gamma(G,M) := M^G$. Here and in the following, Ab denotes the category of abelian groups. Then

$$H^j(G,M) = R^j\Gamma(G,M) ,$$

namely $H^j(G,\cdot)$ and $R^j\Gamma(G,\cdot)$ are both universal δ-functors in the sense of Grothendieck, cf. [32], and they coincide in dimension zero. In particular, $H^j(G,\cdot)$ are functors from $M(G)$ to Ab, which in fact are additive. At this point we are not going to discuss the basic properties of these cohomology functors such as the existence of restriction and corestriction (transfer) maps, compatibility with direct sums and direct limits, cup-products and cohomological dimension etc. . The reader who is unfamilar with these concepts should consult any standard text-book on homology and cohomology of groups, such as [8] or [83].

If E is a field with absolute Galois group $G_E = Gal(E^{sep}/E)$, we denote as it is usual in the literature, the Galois cohomology groups by $H^j(E,M)$ instead of $H^j(G_E,M)$.

Example 1.2 *Let E be a field with separable algebraic closure E^{sep}, then $H^0(E,E^{sep}) = E^\star$, $H^1(E,E^{sep}) = 0$ by Hilbert '90', and $H^2(E,E^{sep}) = Br(E)$, the Brauer group of E. So if $n \geq 1$ is an integer, which is relatively* *prime to char(E), then the Kummer sequence of Galois modules*

$$0 \longrightarrow \mu_n \longrightarrow (E^{sep})^* \xrightarrow{\times n} (E^{sep})^* \longrightarrow 0$$

yields

$$0 \longrightarrow \mu_n(E) \longrightarrow E^* \xrightarrow{\times n} E^* \longrightarrow H^1(E, \mu_n) \longrightarrow 0$$

and

$$0 \longrightarrow H^2(E, \mu_n) \longrightarrow Br(E) \xrightarrow{\times n} Br(E)\,.$$

A quite important concept is the notion of Tate twists of a Galois module M. For this, let E be a field and l be a prime different from the characteristic $char(E)$. For $\nu \geq 1$ and $i \in \mathbb{Z}$, we set

$$\mathbb{Z}/l^\nu\mathbb{Z}(i) = \begin{cases} \mu_{l^\nu} \otimes_{\mathbb{Z}/l^\nu\mathbb{Z}} \ldots \otimes \mu_{l^\nu} = \mu_{l^\nu}^{\otimes i} & \text{if } i > 0 \\ \mathbb{Z}/l^\nu\mathbb{Z} & \text{if } i = 0 \\ Hom(\mathbb{Z}/l^\nu\mathbb{Z}(-i), \mathbb{Z}/l^\nu\mathbb{Z}) & \text{if } i < 0 \end{cases}.$$

By acting diagonally on the tensor product and canonically on the Hom-group, $\mathbb{Z}/l^\nu\mathbb{Z}(i)$ becomes a G_E-module, which as an abelian group is isomorphic to $\mathbb{Z}/l^\nu\mathbb{Z}$. Furthermore, let $\mathbb{Z}_l(1) := \varprojlim \mu_{l^\nu}$, then for $i \in \mathbb{Z}$, we set

$$\mathbb{Z}_l(i) = \begin{cases} \mathbb{Z}_l(1) \otimes_{\mathbb{Z}_l} \ldots \otimes \mathbb{Z}_l(1) = \mathbb{Z}_l^{\otimes i} & \text{if } i > 0 \\ \mathbb{Z}_l & \text{if } i = 0 \\ Hom(\mathbb{Z}_l(-i), \mathbb{Z}_l) & \text{if } i < 0 \end{cases}.$$

This definition is compatible with the above, since for example

$$\varprojlim(\mu_{l^\nu} \otimes \mu_{l^\nu}) \simeq \varprojlim \mu_{l^\nu} \otimes_{\mathbb{Z}_l} \varprojlim \mu_{l^\nu}\,.$$

Now let M be a G_E-module, and suppose that M is also a $\mathbb{Z}/l^\nu\mathbb{Z}$-module resp. $\mathbb{Z}_l$-module, then for $i \in \mathbb{Z}$, the i-th Tate twist $M(i)$ of M is defined by

$$M(i) = M \otimes_{\mathbb{Z}/l^\nu\mathbb{Z}} \mathbb{Z}/l^\nu\mathbb{Z}(i)$$

resp.

$$M(i) = M \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(i)$$

Again as abelian groups M and $M(i)$ are isomorphic, but in general not as G_E-modules.

A different approach to Tate twists is given by the following. Let $\chi : G_E \to \mathbb{Z}_l^*$ be the cyclotomic character given by the action of G_E on $W_l = \varinjlim \mu_{l^\nu}$. If M is a G_E-module, which is also a $\mathbb{Z}_l$-module, we define a new module $M[i]$ as follows: The underlying set of $M[i]$ is M and the Galois action is given by $g \circ m := \chi(g)^i \cdot g(m)$, where $g \in G_E$ and $m \in M$. Then this new definition of the Tate twist is actually compatible with the one given above, namely for all $i \in Z$, there are canonical isomorphisms $M(i) \simeq M[i]$ of G_E-modules.

The compatibility of cup-products in cohomology with Galois action gives the following trivial result, cf. [85].

Lemma 1.3 *Let E be a field, l a prime different from $char(E)$ and $d_\nu := [E(\zeta_{l^\nu}) : E]$. Then for $i \equiv j$ mod d_ν, the cup-product with a generator of $H^0(E, \mathbb{Z}/l^\nu\mathbb{Z}(j-i))$ yields canonical isomorphisms*

$$H^*(E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq H^*(E, \mathbb{Z}/l^\nu\mathbb{Z}(j)) .$$

If E contains the group of all l-th power roots of unity W_l, we obtain by passing to direct limits in 1.3

$$H^j(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq H^j(E, \mathbb{Q}_l/\mathbb{Z}_l)(i) .$$

If E does not contain W_l, we still have the following

Proposition 1.4 *Let E be a local or global field, l an odd prime different from $char(E)$, and $i \in \mathbb{Z}$. Then*

(i)

$$H^0(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) = \begin{cases} \mathbb{Q}_l/\mathbb{Z}_l & \text{if } i = 0 \\ \mathbb{Z}/w_l^{(i)}(E)\mathbb{Z} & \text{if } i \neq 0 \end{cases} ,$$

where $w_l^{(i)}(E) := max\ \{l^n : [E(\zeta_{l^n}) : E]$ divides $i\}$.

(ii)

$$H^2(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) = \begin{cases} l\text{-}tor\ Br(E) & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

(iii)

$$H^j(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0 \text{ for } j \geq 3$$

Proof: (i): trivial. (ii): For $i = 1$ this follows immediately from 1.2 by passing to direct limits. Suppose now that $i \neq 1$, let $F := E(\zeta_l)$ and F_∞/F the cyclotomic $\mathbb{Z}_l$-extension with Galois group $\Gamma = Gal(F_\infty/F)$. Set $G_\infty := Gal(F_\infty/E)$. For the l-cohomological dimension of G_∞ and

$Gal(E^{sep}/F_\infty)$, we have $cd_l G_\infty \leq 1$ and $cd_l Gal(E^{sep}/F_\infty) \leq 1$, cf. [S3], and thus the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_\infty, H^q(F_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i))) \Longrightarrow H^{p+q}(E, \mathbb{Q}_l/\mathbb{Z}_l(i))$$

degenerates, in particular

$$\begin{aligned} H^2(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) &\simeq H^1(G_\infty, H^1(F_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i))) \\ &\simeq H^1(G_\infty, H^1(F_\infty, \mathbb{Q}_l/\mathbb{Z}_l(1))(i-1)) \text{ since } W_l \subset F_\infty^* \\ &\simeq H^1(G_\infty, (F_\infty^* \otimes \mathbb{Q}_l/\mathbb{Z}_l)(i-1)) \text{ by 1.2} \end{aligned}$$

Let $\Delta = Gal(F/E)$. Now l does not divide $[F:E]$, and therefore

$$H^1(G_\infty, (F_\infty^* \otimes \mathbb{Q}_l/\mathbb{Z}_l)(i-1)) \simeq H^1(\Gamma, (F_\infty^* \otimes \mathbb{Q}_l/\mathbb{Z}_l)(i-1))^\Delta .$$

Since $i-1 \neq 0$, we have $H^1(\Gamma, (F_\infty^* \otimes \mathbb{Q}_l/\mathbb{Z}_l)(i-1)) = 0$ from the lemma below. (iii): This is clear from $cd_l(E) \leq 2$, cf. [83]. □

Lemma 1.5 *Let F be a local or global field containing a primitive l-th root of unity ζ_l, $l \neq char(F)$ a prime, and F_∞/F the cyclotomic $\mathbb{Z}_l$-extension with $\Gamma = Gal(F_\infty/F)$. Then for a discrete Γ-module M and $i \neq 0$, we have*

$$H^1(\Gamma, (M \otimes \mathbb{Q}_l/\mathbb{Z}_l)(i)) = 0 .$$

Proof: The case $i = 1$ is proven in [92], and the general case follows along the same lines. □

Remark 1.6 *The groups $H^1(E, \mathbb{Q}_l/\mathbb{Z}_l(i))$ are extensively discussed in [82], but see also below.*

Recall that cohomology is compatible with direct limits, e.g., $H^j(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) = \varinjlim H^j(E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$, and we might ask ourselves what happens, if we pass to the projective limit. One problem is immediately clear, namely the projective limit is not an exact functor. Nevertheless, we define the so-called l-adic Galois cohomology groups by

$$H^j(E, \mathbb{Z}_l(i)) := \varprojlim H^j(E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$$

Even if these groups are of certain interest, they lack functorial properties such as the existence of Hochschild-Serre spectral sequences etc. . One way out of this dilemma is to consider $\mathbb{Z}_l$ in its natural topology, which is the subject of continuous cohomology introduced by J.Tate, cf. [93].

For this, let G be a profinite group and M a topological G-module, i.e., a topological, abelian group, on which G acts continuously. For $j \geq 0$, we set $C^j_{cont}(G, M) := \{\phi : G^j \to M \text{ continuous }\}$, and in the same way as above, we obtain the continuous cochain complex $C_{cont}(G, M) := (C^j_{cont}(G, M); d^j_{cont})$.

Definition 1.7 *The continuous cohomology of G with coefficients in M is the homology of the complex $C_{cont}(G, M)$, i.e., for $j \geq 0$, the j-th continuous cohomology group $H^j_{cont}(G, M)$ of G with coefficients in M is defined by*

$$H^j_{cont}(G, M) := ker\, d^j_{cont}/im\, d^{j-1}_{cont}\ .$$

Jannsen's definition of continuous cohomology is analogous to the above, i.e., deriving a certain functor. For this, let $M(G)^{\mathbb{N}}$ be the category of inverse systems over $\mathbb{N}$ in $M(G)$. It is obvious that $M(G)^{\mathbb{N}}$ is an abelian category. In addition, $M(G)^{\mathbb{N}}$ has enough injectives, cf. [43], and hence we can define for $(M_n, \phi_n) \in M(G)^{\mathbb{N}}$,

$$H^j(G, (M_n, \phi_n)) := R^j\Gamma^{\mathbb{N}}(G, (M_n, \phi_n)) ,$$

where $\Gamma^{\mathbb{N}}(G, \cdot)$ is induced by $\Gamma(G, \cdot)$. The connection to continuous cohomology is given by the following result of Jannsen, cf. [43].

Theorem 1.8 *Let $(M_n, \phi_n) \in M(G)^{\mathbb{N}}$, and suppose that (M_n, ϕ_n) satisfies the Mittag-Leffler condition—M-L condition for short. Consider $M := \varprojlim M_n$ as a topological G-module, then for $j \geq 0$, there are functorial isomorphisms*

$$H^j_{cont}(G, M) \simeq H^j(G, (M_n, \phi_n)) .$$

Since *Ab* satisfies **AB4***, i.e., the second derivative of *lim* is trivial, we have functorial sequences for $(M_n, \phi_n) \in M(G)^{\mathbb{N}}$,

$$0 \longrightarrow \varprojlim{}^1 H^j(G, M_n) \longrightarrow H^j(G, (M_n, \phi_n)) \longrightarrow \varprojlim H^j(G, M_n) \longrightarrow 0$$

arising from the Grothendieck spectral sequence of composed functors

$$E_2^{p,q} = \varprojlim{}^p R^q\Gamma(G, M_n) \Longrightarrow R^{p+q}\Gamma^{\mathbb{N}}(G, (M_n, \phi_n)) ,$$

where $\varprojlim^p$ stands for the right derivatives $R^p\varprojlim$.

A generalization of Galois cohomology is, as we will see, étale cohomology. But the reader should keep in mind that étale cohomology was not defined in search for a generalization of Galois cohomology, but rather in order to find a suitable 'Weil-cohomology', cf. [18].

Let X be a scheme, and $X_{ét}$ the (small) étale site on X. A presheaf P of abelian groups on $X_{ét}$ is a contravariant functor $P : X_{ét} \to Ab$, and the presheaves form an abelian category $P(X_{ét})$ in an obvious way. Furthermore, $P(X_{ét})$ satisfies **AB5** and **AB3***, and posseses generators, cf. [32], thus $P(X_{ét})$ has enough injectives. Now the sheaves, which are roughly speaking presheaves whose sections are determined by local data, form a full subcategory $S(X_{ét})$ of $P(X_{ét})$.

Example 1.9 *(i) Let A be an abelian group, and for $Y \to X$ in $X_{ét}$, let c_Y be the number of connected components of Y, then $A_X(Y \to X) := A^{c_Y} = \prod A$ is called the constant sheaf corresponding to A.*

(ii) For $Y \to X$ in $X_{ét}$, $G_{m,X}(Y \to X) := \Gamma(Y, O_Y)^*$ is a sheaf on $X_{ét}$, where $\Gamma(Y, O_Y) := O_Y(Y)$ are the sections of the structure sheaf O_Y. If n is invertible in $\Gamma(X, O_X)$, in other words, X is a scheme over $\mathbb{Z}[\frac{1}{n}]$, we have the exact Kummer sequence of étale sheaves

$$0 \longrightarrow \mu_{n,X} \longrightarrow G_{m,X} \xrightarrow{\times n} G_{m,X} \longrightarrow 0 ,$$

where $\mu_{n,X}$ is just the kernel of multiplication by n.

If the reference to the scheme X is clear, we omit the subscript X in A_X, $G_{m,X}$, etc. .

A fundamental result of A. Grothendieck states that the natural functor $\iota : S(X_{\acute{e}t}) \to P(X_{\acute{e}t})$ has a left adjoint $\alpha : P(X_{\acute{e}t}) \to S(X_{\acute{e}t})$, cf. [1]. From this we see that $S(X_{\acute{e}t})$ is an abelian category with generators. Another consequence is, that $S(X_{\acute{e}t})$ satisfies **AB5** and **AB3***, thus $S(X_{\acute{e}t})$ has enough injectives. Hence the following definition makes sense.

Definition 1.10 *Let X be a scheme, and $S \in S(X_{\acute{e}t})$. Then for $j \geq 0$,*

$$H^j_{\acute{e}t}(X,S) := R^j\Gamma(X,S)$$

is called the j-th étale cohomology group of $X_{\acute{e}t}$ with coefficients in S, where $\Gamma(X,\cdot) : S(X_{\acute{e}t}) \to Ab$ is the section functor.

Remark 1.11 *There is also a cohomology theory defined by cochains, cocycles, etc., namely the Čech-cohomology $\check{H}^\bullet(X,S)$. But in contrast to cohomology of groups, the Čech-cohomology and the just defined étale cohomology do not agree in general. They are related to each other by a spectral sequence, cf. [84].*

Let X be a scheme, and G_m the étale sheaf defined in 1.9 , then $H^1_{\acute{e}t}(X,G_m) \simeq Pic(X)$, where $Pic(X)$ is the group of isomorphism classes of locally-free, rank one O_X-modules, cf. [37] or [65]. If in addition, l is invertible in X, then the Kummer sequence in 1.9 yields by passing to direct limits

$$0 \longrightarrow \Gamma(X, O_X)^* \otimes \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow H^1_{ét}(X, W_l) \longrightarrow l\text{-}tor\ Pic(X) \longrightarrow 0 .$$

Specializing to number fields, to be more precise to rings of integers in a number field, gives the following. Let E be a number field with ring of integers O_E, and $O_E^S = O_E[\frac{1}{l}]$ the ring of S-integers in E with S-units U_E^S. Then for $X = Spec(O_E^S)$, we certainly have $\Gamma(X, O_X)^* = U_E^S$ as well as $l\text{-}tor\ Pic(X) \simeq A_E^S$, where A_E^S is the Sylow-l-subgroup of the S-ideal class group $Cl(O_E^S)$ of E, cf. [37]. Therefore, the above sequence becomes in this particular case

$$0 \longrightarrow U_E^S \otimes \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow H^1_{ét}(Spec(O_E^S, W_l) \longrightarrow A_E^S \longrightarrow 0 ,$$

and thus it is obvious just from this sequence that certain étale cohomology groups of $X = Spec(O_E^S)$ play an important role in the arithmetic of a number field E. Furthermore, we will later give an explicit description of the group $H^1_{ét}(Spec(O_E^S), W_l)$.

As we said earlier, étale cohomology generalizes Galois cohomology. This is done in the following way. For a field E, let $X = Spec(E)$ and fix a separable algebraic closure E^{sep} with Galois group $G_E = Gal(E^{sep}/E)$. In other words, we choose a geometric point $\bar{x} \to \bar{X} = Spec(E^{sep})$ and set $G_E = \pi_1(\bar{X}, \bar{x})$. For $P \in P(X_{ét})$, we define the stalk at $\bar{x}$ by

$$P_{\bar{x}}\,(= M_P) := \varinjlim P(Spec(F)) ,$$

where the limit runs over all $[F:E]<\infty$. Then $P_{\bar{x}}$ becomes a discrete G_E-module via $\sigma^*: Spec(F^\sigma) \to Spec(F)$, $\sigma \in G_E$. On the other hand, given a discrete G_E-module M, we define

$$\begin{array}{rccl} S_M: & X_{\acute{e}t} & \longrightarrow & Ab \\ & (Y \to X) & \longmapsto & Hom_{G_E}(H(Y), M) \end{array}$$

where $H : Fin\, Et/X \to G_E\text{-}sets$ is the functor defined by $H(Y) := Hom_X(\bar{x}, Y)$ and $Fin\, Et/X$ is the category of étale schemes of finite type over X. Then S_M is, in fact, a sheaf, cf. [65], and the just defined correspondence induces an equivalence of categories $S(X_{\acute{e}t}) \leftrightarrow M(G_E)$, cf. [65]. Since $\Gamma(X,S) = S_{\bar{x}}^{G_E}$, we get canonical isomorphisms, $H^j_{\acute{e}t}(X,S) \simeq H^j(E, S_{\bar{x}})$ for $j \geq 0$.

As for G_E-modules, we can define the notion of the i-th Tate twist of sheaves of $\mathbb{Z}/l^\nu\mathbb{Z}$-modules resp. $\mathbb{Z}_l$-modules. This is done analogously to the case of G_E-modules on page 15 using the sheaf μ_{l^ν} defined in 1.9. Suppose that l is invertible in X and $j \geq 0$, then

$$H^j_{\acute{e}t}(X, \mathbb{Z}_l(i)) := \varprojlim H^j_{\acute{e}t}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i))$$

are called the l-adic cohomology groups of X.

Remark 1.12 *Let X_0 be a smooth, projective variety over the finite field $\mathbb{F}_q$, $char(\mathbb{F}_q) = p \neq l$, and $X := X_0 \times_{\mathbb{F}_q} \mathbb{F}_q^{sep}$ the extension to the separable closure $\mathbb{F}_q^{sep}$ of $\mathbb{F}_q$. Then*

$$H^j_{\acute{e}t}(X, \mathbb{Q}_l(i)) := H^j_{\acute{e}t}(X, \mathbb{Z}_l(i)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

is a Weil-cohomology for X, cf. [14] and [34].

At this point we face the same problem with the l-adic cohomology groups of X, as we did with l-adic Galois cohomology, i.e., the lack of spectral sequences etc. . Again we have to make our cohomology theory 'continuous'. For a scheme X, $S(X_{\acute{e}t})$ is an abelian category with enough injectives, cf. above, and hence the same is true for the category of inverse systems $S(X_{\acute{e}t})^{\mathbb{N}}$, cf. [43]. Thus we can make the following

Definition 1.13 *Let X be a scheme, and $(S_n, \phi_n) \in S(X_{\acute{e}t})^{\mathbb{N}}$. Then for $j \geq 0$, we set*

$$H^j(X, (S_n, \phi_n)) := R^j\Gamma^{\mathbb{N}}(X, (S_n, \phi_n)) ,$$

where $\Gamma^{\mathbb{N}}(X, \cdot)$ is induced by $\Gamma(X, \cdot)$. If l is invertible in X, and $S = (S_{l^\nu})$ is an l-adic sheaf, then for $j \geq 0$,

$$H^j_{\mathrm{cont}}(X, S) := H^j(X, (S_{l^\nu}))$$

is called the j-th continuous (étale) cohomology group of X with coefficients in S.

As for continuous cohomology of groups, we have functorial sequences in X and $(S_n, \phi_n) \in S(X_{\acute{e}t})^{\mathbb{N}}$

$$0 \longrightarrow \varprojlim{}^1 H^{j-1}_{\acute{e}t}(X, S_n) \longrightarrow H^j(X, (S_n, \phi_n)) \longrightarrow \varprojlim H^j_{\acute{e}t}(X, S_n) \longrightarrow 0 .$$

We have seen that étale cohomology generalizes Galois cohomology, cf. above, and it is not surprising that the same is true for continuous cohomology. Namely for a field E with absolute Galois group $G_E = Gal(E^{sep}/E)$, let $X = Spec(E)$ and $\bar{x} \to Spec(E^{sep})$ be a geometric point. We know that $S(X_{ét}) \to M(G_E)$ given by $S \mapsto S_{\bar{x}}$ induces an equivalence of categories and $\Gamma(X,S) = S_{\bar{x}}^{G_E}$. Let $S_n \in S(X_{ét})^{\mathbb{N}}$ satisfying the M-L condition and $S_{\bar{x}} := \varprojlim(S_n)_{\bar{x}}$, then $S_{\bar{x}}$ is a topological G_E-module, and we deduce from 1.8, there are functorial isomorphisms

$$H^j(X,(S_n)) \simeq H^j_{cont}(E, S_{\bar{x}}) \ , \ j \geq 0 \ .$$

Next we state without a proof a theorem of Jannsen on $\mathbb{Q}_l$-cohomology, which generalizes an earlier result by J. Tate, cf. [93].

Theorem 1.14 *Let X be a scheme, l invertible in X and $S = (S_n)$ a torsion-free l-adic sheaf on $X_{ét}$. We set $S \otimes \mathbb{Q}_l/\mathbb{Z}_l := \varinjlim(S \otimes \mathbb{Z}/l^\nu\mathbb{Z}) \in S(X_{ét})$, and for $j \geq 0$, $H^j_{cont}(X, S\otimes\mathbb{Q}_l) := H^j_{cont}(X,S) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Then there is a long exact sequence*

$$\cdots \to H^j_{cont}(X,S) \to H^j_{cont}(X,S\otimes\mathbb{Q}_l) \xrightarrow{\pi^j} H^j_{ét}(X,S\otimes\mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\delta^j} H^{j+1}_{cont}(X,S) \to \cdots$$

functorial in S and X with the properties

$$\begin{aligned} (a)\quad im\,\delta^j &= tor\text{-}H^{j+1}_{cont}(X,S) \\ (b)\quad ker\,\delta^j &= max.\ div.\, H^j(X, S\otimes\mathbb{Q}_l/\mathbb{Z}_l) \end{aligned}$$

In the following, we also need a slight generalization of continuous cohomology. Let Δ be the category of ordered finite sets $\underline{m} := \{0, 1, \ldots, m\}$ and monoton maps as morphisms. A simplicial scheme sX is a contravariant functor $sX : \Delta \to Sch$, where Sch is the category of schemes. Then in a natural way, there is an étale site $sX_{\acute{e}t}$ on sX, and a sheaf S is given for each $\underline{m} \in \Delta$, by sheaves $S_{\underline{m}}$ on $X_{\underline{m}}$ and compatible morphisms, cf. [25]. The sheaves on $sX_{\acute{e}t}$ form an abelian category $S(sX_{\acute{e}t})$ with enough injectives, and for $S \in S(sX_{\acute{e}t})$ and $j \geq 0$, we define the cohomology group of the simplicial scheme sX with coefficients in S by

$$H^j_{\acute{e}t}(sX, S) := R^j(\varprojlim H^0_{\acute{e}t}(X_{\underline{m}}, S_{\underline{m}}) .$$

Furthermore, if $(S_n) \in S(sX_{\acute{e}t})^{\mathbb{N}}$ is an inverse system on $sX_{\acute{e}t}$ and $j \geq 0$, we define

$$H^j(sX, (S_n)) := R^j(\varprojlim \varprojlim H^0_{\acute{e}t}(X_{\underline{m}}, S_{\underline{m},\underline{n}}) .$$

A different approach is given in [20], and U.Jannsen showed that they agree, cf. [43]. Note that we can associate to a scheme X the constant simplicial scheme sX, and obviously for $j \geq 0$,

$$H^j(X, (S_n)) = H^j(sX, (S_n)) .$$

At the end of this paragraph we want to consider the cohomology of number fields, and in particular of rings of integers. Let E be a number field, l an odd prime, S a finite set of primes containing all infinite and l-adic primes.

Further let O_E^S the ring of S-integers in E with S-units $U_E^S = (O_E^S)^*$. For any prime $\mathfrak{p}$ of E, the completion of E at $\mathfrak{p}$ is denoted by $E_\mathfrak{p}$, and if $\mathfrak{p}$ is a finite prime, the residue field of $E_\mathfrak{p}$ by $k_\mathfrak{p}$.

For simplicity, we write $H^j_{ét}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(i))$ instead of $H^j_{ét}(Spec(O_E^S), \mathbb{Z}/l^\nu\mathbb{Z}(i))$, and so on . We already know that

$$H^1(E, \mathbb{Q}_l/\mathbb{Z}_l(1)) \simeq E^* \otimes \mathbb{Q}_l/\mathbb{Z}_l \ ,$$

which is called the 'universal Kummer radical of E ', as well as

$$0 \longrightarrow U_E^S \otimes \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow H^1_{ét}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(1)) \xrightarrow{\phi_E} A_E^S \longrightarrow 0$$

is an exact sequence, where A_E^S is the Sylow-l-subgroup of the S-ideal class group of E. The cohomology group $H^1(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(1))$ and the map ϕ_E can be described explicitly via Kummer theory. Namely let

$$\Delta_E^{(\nu)} := \{\frac{1}{l^\nu} mod\, \mathbb{Z} \otimes z \in \frac{1}{l^\nu}\mathbb{Z}/\mathbb{Z} \otimes E^* : v_\mathfrak{p}(z) \equiv 0\, mod\, l^\nu\, \forall\, \mathfrak{p} \notin S\} \ ,$$

where $v_\mathfrak{p}$ is the (normalized) discrete valuation corresponding to the finite prime $\mathfrak{p}$. Then we get a commutative diagram

$$\begin{array}{ccc} \Delta_E^{(\nu)} & \longrightarrow & H^1_{ét}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(1)) \\ \downarrow & & \downarrow \\ \Delta_E^{(\nu+1)} & \longrightarrow & H^1_{ét}(O_E^S, \mathbb{Z}/l^{\nu+1}\mathbb{Z}(1)) \end{array}$$

and compatible morphisms

$$\begin{array}{rccc} \psi_E^\nu : & \Delta_E^{(\nu)} & \longrightarrow & {}_{l^\nu}A_E^S \\ & \frac{1}{l^\nu} \otimes z & \longmapsto & \prod [\mathfrak{p}]^{\frac{1}{l^\nu} v_\mathfrak{p}(z)} \end{array}$$

so that $\Delta_E^{(\infty)} := \varinjlim \Delta_E^{(\nu)} \simeq H^1_{\acute{e}t}(O_E^S, \mathbf{Q}_l/\mathbf{Z}_l(1))$ and $\psi_E^{(\infty)} := \varinjlim \psi_E^{(l^\nu)} \simeq \phi_E$.

Remark 1.15 *From the above sequence and Dirichlet's unit theorem, we get*

$$co\text{-}rk\, H^1_{\acute{e}t}(O_E^S, \mathbf{Q}_l/\mathbf{Z}_l(1)) = |S| - 1\ .$$

We will see later, that for $i \geq 2$, co-rk $H^1_{\acute{e}t}(O_E^S, \mathbf{Q}_l/\mathbf{Z}_l(i))$ depends on the parity of i and not on S. Conjecturally, this is even true for $i \leq 0$, cf. 1.23.

For $i \neq 1$, $H^1_{\acute{e}t}(O_E^S, \mathbf{Q}_l/\mathbf{Z}_l(i))$ does not arise from the sequence of sheaves $0 \to \mu_{l^\nu} \to G_m \overset{\times l^\nu}{\to} G_m \to 0$, since G_m is not an l-adic sheaf and so we can not take Tate twists. The right way to look at $H^1_{\acute{e}t}(O_E^S, \mathbf{Q}_l/\mathbf{Z}_l(i))$ — even for $i = 1$ — is to consider the sequence of inverse systems of sheaves

$$0 \longrightarrow \mathbf{Z}_l(i) \overset{\times l^\nu}{\longrightarrow} \mathbf{Z}_l(i) \longrightarrow \mathbf{Z}/l^\nu\mathbf{Z}(i) \longrightarrow 0\ .$$

Applying continuous cohomology and passing to direct limits yields

$$0 \to H^j_{cont}(O_E^S, \mathbf{Z}_l(i)) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l \to H^j_{\acute{e}t}(O_E^S, \mathbf{Q}_l/\mathbf{Z}_l(i)) \to l\text{-}tor\, H^{j+1}_{cont}(O_E^S, \mathbf{Z}_l(i)) \to 0\ .$$

Proposition 1.16 *Let E be a number field, l an odd prime and S a finite set of primes containing all infinite and l-adic primes, and O_E^S the ring of S-integers in E. Then*

$$\begin{array}{lrcl} (i) & H^*_{cont}(O_E^S, \mathbf{Z}_l(i)) & \simeq & H^*_{\acute{e}t}(O_E^S, \mathbf{Z}_l(i))\ . \\ (ii) & rk_{\mathbf{Z}_l} H^*_{cont}(O_E^S, \mathbf{Z}_l(i)) & = & co\text{-}rk\, H^*_{\acute{e}t}(O_E^S, \mathbf{Q}_l/\mathbf{Z}_l(i))\ . \\ (iii) & H^1_{cont}(O_E^S, \mathbf{Z}_l(1)) & \simeq & U_E^S \otimes \mathbf{Z}_l\ . \end{array}$$

Proof: (i): Since $H^*_{ét}(O^S_E, \mathbf{Z}/l^\nu\mathbf{Z}(i))$ is finite for all $\nu \geq 1$, cf. [90], we have $\varprojlim^1 H^*_{ét}(O^S_E, \mathbf{Z}/l^\nu\mathbf{Z}(i)) = 0$ is trivial and so $H^*_{cont}(O^S_E, \mathbf{Z}_l(i)) \simeq H^*_{ét}(O^S_E, \mathbf{Z}_l(i))$ is a finitely generated $\mathbf{Z}_l$-module. (ii): This is clear by (i) and the exact sequence above. (iii): Since A^S_E is finite and U^S_E is finitely generated, we get

$$\begin{aligned} H^1_{cont}(O^S_E, \mathbf{Z}_l(1)) &\simeq H^1_{ét}(O^S_E, \mathbf{Z}_l(1)) \\ &= \varprojlim H^1_{ét}(O^S_E, \mathbf{Z}/l^\nu\mathbf{Z}(1)) \\ &\simeq \varprojlim (U^S_E \otimes \mathbf{Z}/l^\nu\mathbf{Z}) \\ &\simeq U^S_E \otimes \mathbf{Z}_l \,. \end{aligned}$$

□

We can also compare the $\mathbf{Z}_l$-rank of $H^1_{ét}(O^S_E, \mathbf{Z}_l(i))$ with $H^2_{ét}(O^S_E, \mathbf{Z}_l(i))$, but at first we need the following

Lemma 1.17 *Let A be a Dedekind domain and l an odd prime, which is invertible in A. Assume that A does not contain all l-th powers of roots of unity. Then for $i \neq 0$,*

$$H^0_{ét}(A, \mathbf{Z}_l(i)) = 0 \,.$$

Proof: Without loss of generality we can assume that A contains ζ_l. Let $K = Quot(A)$ be the quotient field and $\kappa : G_K = \Gamma \to \mathbf{Z}_l^*$ the cyclotomic character. Then $H^0_{ét}(A, \mathbf{Z}_l(i))$ is the kernel of $\mathbf{Z}_l \xrightarrow{1-\kappa(\gamma)^i} \mathbf{Z}_l$, where $\gamma \in \Gamma$ is a topological generator, and by our assumption this map is injective. □

Proposition 1.18 *Let E be a number field, l an odd prime and S a finite set of primes containing all infinite and l-adic primes. Then*

$$rk_{\mathbb{Z}_l} H^1_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i)) = rk_{\mathbb{Z}_l} H^2_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i)) + \begin{cases} r_2(E)+1 & \text{if } i=0 \\ r_2(E) & \text{if } i \neq 0,\ i \equiv 0 \bmod 2 \\ r_1(E)+r_2(E) & \text{if } i \equiv 1 \bmod 2 \end{cases} .$$

In other words,

$$\text{co-}rk\, H^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) = \text{co-}rk\, H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) + \begin{cases} r_2(E)+1 & \text{if } i=0 \\ r_2(E) & \text{if } i \neq 0,\ i \equiv 0 \bmod 2 \\ r_1(E)+r_2(E) & \text{if } i \equiv 1 \bmod 2 \end{cases}$$

Proof: Considering the short exact sequence of inverse systems

$$0 \longrightarrow \mathbb{Z}_l(i) \xrightarrow{\times l} \mathbb{Z}_l(i) \longrightarrow \mathbb{Z}/l\mathbb{Z}(i) \longrightarrow 0$$

gives the long exact sequence

$$\begin{array}{rlllll} 0 & \to & H^0_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i)) & \xrightarrow{\times l} & H^0_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i)) & \to & H^0_{\acute{e}t}(O_E^S, \mathbb{Z}/l\mathbb{Z}(i)) \\ & \to & H^1_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i)) & \xrightarrow{\times l} & H^1_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i)) & \to & H^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l\mathbb{Z}(i)) \\ & \to & H^2_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i)) & \xrightarrow{\times l} & H^2_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i)) & \to & H^2_{\acute{e}t}(O_E^S, \mathbb{Z}/l\mathbb{Z}(i)) \to 0\,, \end{array}$$

and the assertion follows from

$$\sum_{j=0}^{2} (-1)^{j+1} \# H^j_{\acute{e}t}(O_E^S, \mathbb{Z}/l\mathbb{Z}(i)) = \begin{cases} r_2(E) & \text{if } i \equiv 0 \bmod 2 \\ r_1(E)+r_2(E) & \text{if } i \equiv 1 \bmod 2 \end{cases} ,$$

cf. [90]. □

We can also relate the cohomology of rings of integers to the cohomology of number fields, namely the so-called localization sequence in étale cohomology, cf. [85].

Theorem 1.19 *Let E be a number field, l an odd prime and O_E^S the ring of S-integers, where S is a finite set of primes containing the infinite and l-adic primes. Further let $\iota : Spec(E) \to Spec(O_E^S)$ be the canonical morphism. Then*

$$(\mathbb{Z}/l^\nu\mathbb{Z}(i))_{Spec(O_E^S)} \longrightarrow \iota_*(\mathbb{Z}/l^\nu\mathbb{Z}(i))_{Spec(E)}$$

is an isomorphism of sheaves on $Spec(O_E^S)$, and there is a long exact sequence

$$\begin{aligned} 0 &\to H^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to H^1(E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to \oplus_{\mathfrak{p}\notin S} H^0(k_\mathfrak{p}, \mathbb{Z}/l^\nu\mathbb{Z}(i-1)) \\ &\to H^2_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to H^2(E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to \oplus_{\mathfrak{p}\notin S} H^1(k_\mathfrak{p}, \mathbb{Z}/l^\nu\mathbb{Z}(i-1)) \\ &\to 0\,. \end{aligned}$$

Proof: For the first assertion, it is enough to show that it is an isomorphism at every stalk, and this follows at once, cf. [17]. Next consider the Leray spectral sequence

$$E_2^{p,q} = H^p_{\acute{e}t}(O_E^S, R^q\iota_*\mathbb{Z}/l^\nu\mathbb{Z}(i)) \Longrightarrow H^{p+q}(E, \mathbb{Z}/l^\nu\mathbb{Z}(i))\,.$$

Let $\mathcal{F} := R^q\iota_*\mathbb{Z}/l^\nu\mathbb{Z}(i)$, and for $\mathfrak{p} \notin S$, $\iota_\mathfrak{p} : Spec(k_\mathfrak{p}) \to Spec(O_E^S)$, then for $q \geq 1$,

$$\mathcal{F} \longrightarrow \bigoplus_{\mathfrak{p}\notin S} \iota_{\mathfrak{p}*}\iota_\mathfrak{p}^*\mathcal{F}$$

is an isomorphism of sheaves on $Spec(O_E^S)$, since it is an isomorphism at every stalk. Furthermore, $\iota_\mathfrak{p}$ is a finite morphism, and hence there are isomorphisms

$$\begin{aligned} H^p_{\acute{e}t}(O_E^S, \iota_{\mathfrak{p}*}\iota_\mathfrak{p}^*\mathcal{F}) &\simeq H^p(k_\mathfrak{p}, \iota_\mathfrak{p}^*\mathcal{F}) \\ &\simeq H^p(k_\mathfrak{p}, (\iota_\mathfrak{p}^*\mathcal{F})_{\bar{x}_\mathfrak{p}}) \\ &\simeq H^p(k_\mathfrak{p}, (\iota_\mathfrak{p}^* R^q \iota_* \mathbb{Z}/l^\nu\mathbb{Z}(i))_{\bar{x}_\mathfrak{p}}) \,, \end{aligned}$$

cf. [35], where $\bar{x}_\mathfrak{p} : Spec(k_\mathfrak{p}^{sep}) \to Spec(k_\mathfrak{p})$ is a geometric point. Since étale cohomology commutes with direct limits and $cd_l k_\mathfrak{p} = 1$ as well as $cd_l O_E^S \leq 2$, we get $E_2^{p,q} = 0$ for $q \neq 0, 1$ or $p \geq 3$ and thus we obtain the exact sequence

$$\begin{aligned} 0 &\to H^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to H^1(E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to \oplus_{\mathfrak{p}\notin S} H^0(k_\mathfrak{p}, (\iota_\mathfrak{p}^* R^1\iota_* \mathbb{Z}/l^\nu\mathbb{Z}(i))_{\bar{x}_\mathfrak{p}}) \\ &\to H^2_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to H^2(E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to \oplus_{\mathfrak{p}\notin S} H^1(k_\mathfrak{p}, (\iota_\mathfrak{p}^* R^1\iota_* \mathbb{Z}/l^\nu\mathbb{Z}(i))_{\bar{x}_\mathfrak{p}}) \\ &\to 0 \,. \end{aligned}$$

Let$\tilde{E}_\mathfrak{p}$ be the maximal unramified extension of $E_\mathfrak{p}$, then

$$\begin{aligned} (\iota_\mathfrak{p}^* R^1\iota_* \mathbb{Z}/l^\nu\mathbb{Z}(i))_{\bar{x}_\mathfrak{p}} &\simeq H^1(\tilde{E}_\mathfrak{p}, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \\ &\simeq H^1(\tilde{E}_\mathfrak{p}, \mathbb{Z}/l^\nu\mathbb{Z}(1))(i-1) \quad \text{, since } \zeta_{l^\nu} \in \tilde{E}_\mathfrak{p}^*, \\ &\simeq \mathbb{Z}/l^\nu\mathbb{Z}(i-1) \quad \text{, since } \mathfrak{p} \nmid l, \end{aligned}$$

cf. [83], and the theorem is proven. □

Corollary 1.20 *Let the notation be as in 1.19, and suppose that $i \neq 1$. Then there is a commutative diagram*

$$\begin{array}{ccc} H^1_{cont}(O_E^S, \mathbb{Z}_l(i)) & \xrightarrow{\sim} & H^1_{cont}(E, \mathbb{Z}_l(i)) \\ \downarrow \wr & & \downarrow \wr \\ H^1_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i)) & \xrightarrow{\sim} & H^1(E, \mathbb{Z}_l(i)) \end{array}$$

and an exact sequence

$$\begin{array}{llllll} 0 & \to & H^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) & \to & H^1(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) & \to \bigoplus_{\mathfrak{p}\notin S} H^0(k_{\mathfrak{p}}, \mathbb{Q}_l/\mathbb{Z}_l(i-1)) \\ & \to & H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) & \to 0 & & \end{array}$$

Proof: The vertical isomorphisms in the diagram follow from the finiteness of $H^0_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(i))$ resp. $H^0(E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$ for all $\nu \geq 0$. Since $H^0(k_{\mathfrak{p}}, \mathbb{Z}_l(i-1)) = 0$ by 1.17, the bottom horizontal isomorphism follows from passing to the projective limit in 1.19. The commutativity of the diagram is clear by functoriality, and so we deduce the upper horizontal isomorphism. By 1.4 $H^2(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0$ for $i \neq 1$, and passing to direct limits in 1.19 yields the above sequence. □

Next we state the duality theorems of Tate-Poitou, cf. [36] and [90], for Galois modules over local and global fields.

Proposition 1.21 *For $i \in \mathbb{Z}$ and $0 \leq j \leq 2$, there are canonical isomorphisms*

$$c_{\mathfrak{p}} = c^i_{\mathfrak{p}} : H^j(E_{\mathfrak{p}}, \mathbb{Z}_l(i)) \xrightarrow{\sim} H^{2-j}(E_{\mathfrak{p}}, \mathbb{Q}_l/\mathbb{Z}_l(1-i))^* .$$

Proof: Let $G_{\mathfrak{p}} = Gal(E_{\mathfrak{p}}^{sep}/E_{\mathfrak{p}})$ be the absolute Galois group of the local field $E_{\mathfrak{p}}$. For any finite $G_{\mathfrak{p}}$-module M, let $M^D := Hom(M, (E_{\mathfrak{p}}^{sep})^*)$ be its dual module, then the pairing $M \times M^D \to (E_{\mathfrak{p}}^{sep})^*$ induces the cup-product

$$H^j(E_{\mathfrak{p}}, M) \times H^{2-j}(E_{\mathfrak{p}}, M^D) \longrightarrow H^2(E_{\mathfrak{p}}, (E_{\mathfrak{p}}^{sep})^*) .$$

Composing with the invariant $inv_{\mathfrak{p}} : H^2(E_{\mathfrak{p}}, (E_{\mathfrak{p}}^{sep})^*) \xrightarrow{\sim} \mathbb{Q}_l/\mathbb{Z}_l$ gives isomorphisms

$$H^j(E_{\mathfrak{p}}, M) \xrightarrow{\sim} H^{2-j}(E_{\mathfrak{p}}, M^D)^* ,$$

cf. [66] or [83]. For $M = \mathbb{Z}/l^\nu\mathbb{Z}(i)$, we have $M^D = \mathbb{Z}/l^\nu\mathbb{Z}(1-i)$ and passing to the projective limit yields the assertion. □

For a proof of the global Tate-Poitou duality theorem, cf. [36] or [90].

Theorem 1.22 *Let $i \in \mathbb{Z}$, then.*

(i) The kernels of the natural morphisms

$$H^1_{ét}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to \oplus_{\mathfrak{p}|l} H^1(E_\mathfrak{p}, \mathbb{Q}_l/\mathbb{Z}_l(i))$$

and

$$H^2_{ét}(O_E^S, \mathbb{Z}_l(1-i)) \to \oplus_{\mathfrak{p}|l} H^2(E_\mathfrak{p}, \mathbb{Z}_l(1-i))$$

are canonically dual to each other.

(ii) There is a canonical exact sequence

$$\begin{array}{cclclcl} 0 & \to & H^0_{ét}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(1-i)) & \to \oplus_{\mathfrak{p}|l} H^2(E_\mathfrak{p}, \mathbb{Z}_l(i))^* & \to & H^2_{ét}(O_E^S, \mathbb{Z}_l(i))^* & \\ & \to & H^1_{ét}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(1-i)) & \to \oplus_{\mathfrak{p}|l} H^1(E_\mathfrak{p}, \mathbb{Z}_l(i))^* & \to & H^1_{ét}(O_E^S, \mathbb{Z}_l(i))^* & \\ & \to & H^2_{ét}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(1-i)) & \to \oplus_{\mathfrak{p}|l} H^0(E_\mathfrak{p}, \mathbb{Z}_l(i))^* & \to & H^0_{ét}(O_E^S, \mathbb{Z}_l(i))^* & \to 0 . \end{array}$$

In [82] P.Schneider proposed the following

Conjecture 1.23 (Schneider) *Let E be a number field, l an odd prime and S a finite of primes containing all infinite and l-adic primes. Then for $i \neq 1$,*

$$H^2_{ét}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0 .$$

Recall that for $i = 1$, we have

$$H^2_{ét}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(1)) \simeq (\mathbb{Q}_l/\mathbb{Z}_l)^{g_l(E)-1} ,$$

where $g_l(E) := \#\{\mathfrak{p} \subseteq E : \mathfrak{p}|l\}$ is the number of l-adic primes of E, cf. 1.16.

Putting the rank and co-rank formulas together, we get

Lemma 1.24 *For $i \neq 1$, let $R_{1-i}(E)$ be the kernel of the localization map*

$$H^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(1-i)) \longrightarrow \bigoplus_{\mathfrak{p}|l} H^1(E_\mathfrak{p}, \mathbb{Q}_l/\mathbb{Z}_l(1-i)) .$$

Then the following assertions are equivalent.

(i) $H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0$.

(ii) $\text{co-rk}\, H^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) = \begin{cases} r_2(E)+1 & \text{if } i=0 \\ r_2(E) & \text{if } i \neq 0,\ i \equiv 0 \bmod 2 \\ r_1(E)+r_2(E) & \text{if } i \equiv 1 \bmod 2 \end{cases}$.

(iii) $H^2_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i))$ *is finite* .

(iv) $rk_{\mathbb{Z}_l} H^1_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i)) = \begin{cases} r_2(E)+1 & \text{if } i=0 \\ r_2(E) & \text{if } i \neq 0,\ i \equiv 0 \bmod 2 \\ r_1(E)+r_2(E) & \text{if } i \equiv 1 \bmod 2 \end{cases}$.

(v) $R_{1-i}(E)$ *is finite* .

Proof: The equivalence of (i)-(iv) is clear by 1.16 and 1.18. On the other hand we obtain from 1.21 and 1.22 the exact sequence

$$\begin{aligned} 0 \longrightarrow\ & H^0_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(1-i)) \longrightarrow \oplus_{\mathfrak{p}|l} H^0(E_\mathfrak{p}, \mathbb{Q}_l/\mathbb{Z}_l(1-i)) \\ \longrightarrow\ & H^2_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i))^\star \longrightarrow R_{1-i}(E) \longrightarrow 0 , \end{aligned}$$

where $\star$ stands for the Pontryagin dual, cf. [36] and [82]. Since $i \neq 1$, the first two groups in this sequence are finite, cf. 1.4, and thus the equivalence of (iv) and (v). □

Using K-theory, in particular Borel's computation of the algebraic K-theory of rings of integers and Soulé's resp. Dwyer and Friedlander's results on the chern classes, we will see in chapter 2, cf. 2.28, that the Schneider-conjecture is true for $i \geq 2$. Another interesting case is $i = 0$, which is precisely the Leopoldt-conjecture, namely.

Conjecture 1.25 (Leopoldt) *For a number field E and a prime l, let $\tilde{E}$ the compositum of all $\mathbb{Z}_l$-extensions of E. Then*

$$Gal(\tilde{E}/E) \simeq \mathbb{Z}_l^{r_2(E)+1} .$$

Remark 1.26 *(i) The Leopoldt-conjecture is proven for abelian extensions over the rationals $\mathbb{Q}$ and over an imaginary quadratic field, cf. [9], as well as in several other specific cases, cf. [69]. (ii) We stated here an equivalent version of the original Leopoldt-conjecture on the non-vanishing of the l-adic regulator. For more on this subject, cf. [95]. (iii) It is also known that $r_2(E) + 1 \leq rk_{\mathbb{Z}_l} Gal(\tilde{E}/E) \leq [E : \mathbb{Q}]$, cf. [41], and we define the Leopoldt defect $\delta_E^{Leop} \geq 0$ by $rk_{\mathbb{Z}_l} Gal(\tilde{E}/E) = r_2(E) + 1 + \delta_E^{Leop}$. (iv) Since $H^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z})$ classifies the cyclic extensions of E of degree $\leq l^\nu$, which are unramified outside l, the Leopoldt conjecture is a special case of the Schneider conjecture.*

2 K-theory

As paragraph 1 this paragraph has an introductory character. Even if we define algebraic and étale K-theory in a quite general context, when it comes to examples - or say applications –, we will be mainly interested in the K-theory of number fields and rings of integers. Let us start, where algebraic K-theory has its origin, namely with the Grothendieck group of an exact category.

Definition 2.1 *Let $\mathcal{C}$ be an exact category, and assume that the isomorphism classes of objects in $\mathcal{C}$ form a set, e.g., if $\mathcal{C}$ is a small category. Then the Grothendieck group $K_0(\mathcal{C})$ of $\mathcal{C}$ is defined by*

$$K_0(\mathcal{C}) := \mathcal{F} / \mathcal{R} ,$$

where $\mathcal{F}$ is the free abelian group on the isomorphism classes of objects in $\mathcal{C}$, and $\mathcal{R}$ is the subcategory generated by classes $[M] - [M_1] - [M_2]$ for each exact sequence $0 \to M_1 \to M \to M_2 \to 0$ in $\mathcal{C}$.

When we consider rings and schemes, we assume that every ring is noetherian and contains 1, and that every scheme is locally noetherian. So if R is a ring, then $X = Spec(R)$ is a scheme. Now let R be a ring and $P(R)$ the category of finitely generated, projective R-modules, then we set

$$K_0(R) := K_0(P(R)) .$$

Example 2.2 *(i) Let R be a field or a local ring, then every projective module over R is free, and so $K_0(R) = \mathbb{Z}$. (ii) Let R be a Dedekind domain, then $K_0(R) \simeq \mathbb{Z} \oplus Cl(R)$, where $Cl(R)$ is the ideal class group of R, cf. [68].*

For any small category $\mathcal{C}$, let $B\mathcal{C} := |N\mathcal{C}|$ be the geometric realization of the nerve $N\mathcal{C}$ of $\mathcal{C}$. Then $B\mathcal{C}$ can be considered as a CW-complex, whose n-cells correspond to the non-degenerated n-simplices of $N\mathcal{C}$, i.e., the sequences $X_0 \to \cdots \to X_n$ in $\mathcal{C}$, which do not contain the identity. In particular, the 0-cells correspond to the objects in $\mathcal{C}$.

Example 2.3 *Let $\mathcal{C}$ be the category consisting of*

$$\begin{aligned} Ob(\mathcal{C}) &= \{0,1\} \\ Mor(\mathcal{C}) &= \{0 \xrightarrow{id} 0, 1 \xrightarrow{id} 1, 0 \to 1\} , \end{aligned}$$

then $B\mathcal{C} = [0,1] \subseteq \mathbb{R}$.

For any exact category $\mathcal{C}$, we denote the Quillen-category of $\mathcal{C}$ – also called the Q-construction – by $Q\mathcal{C}$, cf. [80]. Let $0 \in \mathcal{C}$ be a zero object in the exact category $\mathcal{C}$, then for any $M \in \mathcal{C}$, we have canonical maps

$$i_M : 0 \to M \text{ and } j_M : M \to 0 ,$$

which induce morphisms

$$i_{M!} : 0 \to M \text{ and } j_M^! : 0 \to M$$

in QC. Thus we obtain paths $\omega(i_{M!})$ and $\omega(j^!_M)$ from $\{0\}$ to $\{M\}$ in BQC, and thus a loop

$$s_M := \omega(j^!_M) \circ \omega(i_{M!})^{-1}$$

at $\{0\}$ in BQC.

Theorem 2.4 *Let C be a small, exact category and 0 a zero object in C. Then QC is a small category, and there is an isomorphism*

$$\begin{array}{cccc} \phi: & K_0(C) & \xrightarrow{\sim} & \pi_1(BQC, \{0\}), \\ & [M] & \mapsto & [s_M] \end{array}$$

where s_M is the loop defined above, and $[\cdot]$ stands for the isomorphism resp. homotopy class.

Proof: We just give here a sketch of the proof, for details cf. [80]. It is straightforward to show that ϕ is a well-defined homomorphism. Further, it is enough to show that the induced functor

$$\phi^* : \pi_1(BQC, \{0\})\text{-}Sets \longrightarrow K_0(C)\text{-}Sets$$

is an equivalence of categories. From the theory of covering spaces we know that

$$\pi_1(BQC, \{0\})\text{-}Sets \longleftrightarrow (Cov/BQC, *)$$

is an equivalence of categories, where $(Cov/BQC, *)$ denotes the category of covering spaces overBQC with base point $*$. Furthermore, it is not hard to see that

$$\begin{array}{ccc} (Cov/BQC,*) & \longleftrightarrow & \{\text{ invertible functors } QC \to Sets\} \\ (E,p) & \longmapsto & (F : x \mapsto p^{-1}\{x\}) \end{array}$$

is an equivalence as well. We define

$$\begin{array}{rcl} \psi : \{\text{ invertible functors } QC \to Sets\} & \longrightarrow & K_0(C)\text{-}Sets\ , \\ F & \longmapsto & F(0) \end{array}$$

then we obtain a commutative diagram

$$\begin{array}{ccc} \pi_1(BQC,\{0\})\text{-}Sets \xrightarrow{\phi^*} & & K_0(C) \\ \updownarrow & & \uparrow \psi \\ (Cov/BQC,*) & \longleftrightarrow & \{\text{ invertible functors } QC \to Sets\}\ . \end{array}$$

Using the universal property of QC, we deduce that ψ is an equivalence of categories, and the theorem is proven. □

The above theorem motivates

Definition 2.5 *Let C be a small, exact category. For $m \geq 0$, the m-th K-group of C is defined by*

$$K_m(C) := \pi_{m+1}(BQC,\{0\})\ ,$$

where BQC is the geometric realization of the Quillen-category QC, and $\{0\}$ is a zero object in C.

Remark 2.6 *$K_i(C)$ is well-defined, i.e., independent of the choice of the zero object 0. Given any other zero object $\tilde{0}$, the groups $\pi_{m+1}(BQC, \{0\})$ and $\pi_{m+1}(BQC, \{\tilde{0}\})$ are conjugate to each other.*

Next we define the K-theory of rings and schemes by

- *Let R be a ring, and $P(R)$ the category of finitely generated, projective, left R-modules, then for $m \geq 0$,*

$$K_m(R) := K_m(P(R)) .$$

- *Let X be a scheme, and $Sh(X)$ the category of locally-free O_X-modules of finite rank, then for $m \geq 0$,*

$$K_m(X) := K_m(Sh(X)) .$$

Let us observe a technical point here. $P(R)$ and $Sh(X)$ are of course not small categories, so actually we are considering a skeleton of $P(R)$ and $Sh(X)$. This is good enough, since the K-groups do not depend on the choice of the skeleton, cf. [80]. Note that the above definitions are compatible, since $P(R)$ and $Sh(X)$ are equivalent for $X = Spec(R)$.

Proposition 2.7 *For every $m \geq 0$, K_m gives rise to a covariant resp. contravariant functor*

$$\begin{aligned} K_m :&\ Rings \longrightarrow Ab \\ resp. \quad K_m :&\ Schemes \longrightarrow Ab . \end{aligned}$$

Proof: Since $P(R)$ and $Sh(X)$ are equivalent for $X = Spec(R)$, it is enough to show the functoriality of $K_m : Schemes \to Ab$. Let $f : X \to Y$ be a morphism of schemes, then the inverse image

$$\begin{array}{rcl} f^*: \ Sh(Y) & \longrightarrow & Sh(X) \\ \mathcal{G} & \longmapsto & f^{-1}\mathcal{G} \otimes_{f^{-1}O_Y} O_X \end{array}$$

is an exact functor. Using the universal mapping property of the Quillen-category gives a functor $Qf^* : QSh(Y) \to QSh(X)$, and thus a morphism $f_m : K_m(Y) \to K_m(X)$. □

Let X be a scheme and $\mathcal{F} \in Sh(X)$ a locally-free O_X-module of finite rank, then

$$\begin{array}{rcl} \mathcal{F}\otimes ?: \ Sh(X) & \longrightarrow & Sh(X) \\ \mathcal{G} & \longmapsto & \mathcal{F} \otimes_{O_X} \mathcal{G} \end{array}$$

is an exact functor, and so it induces a morphism

$$(\mathcal{F}\otimes ?)_* : K_*(X) \longrightarrow K_*(X) .$$

Furthermore, if $0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0$ is an exact sequence in $Sh(X)$, then $0 \longrightarrow \mathcal{F}_1\otimes ? \longrightarrow \mathcal{F}\otimes ? \longrightarrow \mathcal{F}_2\otimes ? \longrightarrow 0$ is an exact sequence of exact functors, and by [80] we obtain

$$(\mathcal{F}\otimes ?)_* = (\mathcal{F}_1\otimes ?)_* + (\mathcal{F}_2\otimes ?)_* : K_m(X) \to K_m(X) ,$$

i.e., the functor $(\mathcal{F}\otimes ?)_*$ depends only on the class of $\mathcal{F}$ in $K_0(X)$. Thus we can make $K_0(X)$ into a commutative ring resp. $K_m(X)$ into a $K_0(X)$-module by setting

$$\begin{array}{ccc} K_0(X) \times K_m(X) & \longrightarrow & K_m(X) . \\ ([\mathcal{F}], z) & \longmapsto & (\mathcal{F} \otimes ?)_m(z) \end{array}$$

Replacing X by a ring R gives the analogous result for the category of rings.

In [68] J. Milnor defined the nowadays called Milnor K-groups of a ring R. For this, let $GL(R)$ be the general linear group, $E(R)$ the group of elementary matrices and $St(R)$ the Steinberg group with generators $x_{ij}(\lambda)$, $\lambda \in R$. By $\varphi : St(R) \to E(R)$ we denote the canonical map, which sends a generator $x_{ij}(\lambda)$ to the elementary matrix $e_{ij}(\lambda) \in E(R)$. Then the Milnor K-groups are defined by

$$K_m^M(R) := \begin{cases} K_0(R) & \text{if } m = 0 \\ GL(R)/E(R) = GL(R)^{ab} & \text{if } m = 1 \\ ker\,\varphi & \text{if } m = 2 \end{cases} .$$

Since $0 \to K_2^M(R) \to St(R) \to E(R) \to 0$ is a universal central extension of $E(R)$, cf. [68], we get $K_2^M(R) \simeq H_2(E(R), \mathbf{Z})$, the Schur multiplier of $E(R)$, cf. [8].

Let us assume at this point, that R is commutative. Then the determinant $det : GL(R) \to R^*$ induces the split exact sequence

$$0 \longrightarrow SL(R) \longrightarrow GL(R) \xrightarrow{det} R^* \longrightarrow 0 ,$$

and so

$$K_1^M(R) \simeq SL(R)/E(R) \oplus R^\star .$$

Example 2.8 *Let R be a field or a principal ideal domain, then $K_1^M(R) \simeq R^\star$.*

By definition, the group $K_2^M(R)$ gives all the non-trivial relations in the group of elementary matrices, and for a field, we have the following result, cf. [62].

Theorem 2.9 *Let F be a field, then*

$$K_2^M(F) \simeq F^\star \otimes F^\star \,/\, \langle \{a \otimes 1 - a \,:\, a \in F \backslash \{0,1\}\} \rangle \,,$$

i.e., $K_2^M(F)$ is the universal symbol group of F.

This result as well as 2.2 and 2.8 motivate the following definition. Let F be a field and $m \geq 0$, we set

$$K_m^M(F) := (F^\star)^{\otimes m} \,/\, \langle \{a_1 \otimes \cdots \otimes a_m \,:\, a_j + a_k = 1 \text{ for } 1 \leq j < k \leq m\} \rangle \,,$$

where $(F^\star)^0 := \mathbb{Z}$ by definition. It is well-known, that the Milnor K-theory of a field does not agree with the algebraic K-theory; this happens already for $m = 3$. But nevertheless, Milnor K-theory is of interest in its own, e.g., the connection to quadratic forms or higher dimensional local class field theory, just to mention a few aspects. Even if $K_3^M(F) \neq K_3(F)$, the following question comes naturally to our mind

$$\text{Is } K_m^M(R) = K_m(R) \text{ for } m = 1, 2 \text{ ?}$$

It turns out, that this is indeed the case, cf. 2.12. We give now a different approach to K-theory of rings, which actually was Quillen's first definition of it, and which also allows us to be more precise for low dimensional K-groups. Again let $GL(R)$ be the general linear group of the ring R, considered as a discrete group. We denote by $BGL(R)$ the classifying space of $GL(R)$; this is an Eilenberg-MacLane space $K(GL(R), 1)$. Therefore, $BGL(R)$ is connected and

$$\pi_m(BGL(R)) = \begin{cases} GL(R) & \text{for } m = 1 \\ 0 & \text{for } m > 1 \end{cases} .$$

Remark 2.10 *That we used the notation BC for the geometric realization of a small category C, is no coincidence. Namely, let G be any discrete group, and define a category $\mathcal{G}$ by $Ob\{\mathcal{G}\} := \{*\}$, $Mor\{*, *\} := G$. Then $BG \simeq B\mathcal{G}$ as topological spaces, cf. [89].*

By attaching 2- and 3-cells to $BGL(R)$ we obtain a space $BGL(R)^+$ – unique up to homotopy – with the following property: There is an inclusion

$$i : BGL(R) \hookrightarrow BGL(R)^+$$

such that

1. $i_1 : \pi_1(BGL(R)) \to \pi_1(BGL(R)^+)$ is given by $GL(R) \to GL(R)/E(R) = GL(R)^{ab}$.

2. For any local coefficient system L on $BGL(R)^+$,

$$i_* : H_*(BGL(R), L) \longrightarrow H_*(BGL(R)^+, L)$$

is an isomorphism.

Here, $H_*(X, L)$ denotes the singular homology of the space X with coefficient in L. In the literature the attaching of 2- and 3-cells is known as the *plus*-construction, and it was Quillen who showed, that both approaches, the Q- and *plus*-construction, agree. Namely we have, cf. [28] or [89].

Theorem 2.11 *For a topological space X, let ΩX be the loop space, then there is a homotopy equivalence*

$$BGL(R)^+ \longrightarrow (\Omega BQP(R))_o \, ,$$

where the subscript o *denotes the connected component of the trivial loop at $\{0\} \in BQP(R)$. In particular for $m \geq 1$,*

$$\pi_m(BGL(R)^+) \simeq \pi_{m+1}(BQP(R), \{0\}) = K_m(R) \, .$$

Corollary 2.12 *Let R be a ring, then $K_m^M(R) \simeq K_m(R)$ for $m = 1, 2$.*

Proof: For $m = 1$, this is property 1 of $i : BGL(R) \hookrightarrow BGL(R)^+$. For simplicity, we set $X := BGL(R)$ and $X^+ := BGL(R)^+$. Let $F(i)$ be the homotopy fibre of $X \times_{X^+} X^{+I} \to X^+$, where X^{+I} denotes the path space of X^+. Recall that X^+ is path-connected, and thus $F(i)$ is well-defined. Since X^+ is a CW-complex, the universal covering space $\tilde{X}^+$ exists, and the pull-back

$$\begin{array}{ccc} X \times_{X^+} \tilde{X}^+ & \xrightarrow{\tilde{i}} & \tilde{X}^+ \\ \downarrow & & \downarrow \\ X & \xrightarrow{i} & X^+ \end{array}$$

has a homeomorphic fibre $F(\tilde{i})$. Furthermore, we know from the theory of covering spaces that $X \times_{X^+} \tilde{X}^+$ is the covering space of X corresponding to $E(R) \subset GL(R) = \pi_1(BGL(R))$. In other words, $X \times_{X^+} \tilde{X}^+ \simeq BE(R) \simeq K(E(R), 1)$. The long exact sequence in homotopy theory gives

$$\begin{array}{ccccccccc} \pi_2(BE(R)) & \longrightarrow & \pi_2(\tilde{X}^+) & \longrightarrow & \pi_1(F(\tilde{i})) & \longrightarrow & \pi_1(BE(R)) & \longrightarrow & \pi_1(\tilde{X}^+)\,, \\ \| & & \| & & \| & & \| & & \| \\ 0 & & \pi_2(X^+) & & \pi_1(F(i)) & & E(R) & & 0 \end{array}$$

and $G := \pi_1(F(i))$ acts trivially on $ker(\pi_m(F(\tilde{i})) \to \pi_m(BE(R))$ for $m \geq 1$. If $m = 1$, this action is given by conjugation, and so the above sequence is a central extension of $E(R)$. On the other hand, it can be shown that $H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0$, cf. [88],and therefore, the extension is universal, i.e.,

$$K_2(R) \simeq \pi_2(BGL(R)^+) \simeq K_2^M(R)\,.$$

□

In [7] Browder considers algebraic K-theory with coefficients $\mathbb{Z}/k\mathbb{Z}$ via homotopy theory with coefficients. Let us recall the basic definitions here. For $m \geq 2$ and $k \geq 2$, let $Y_k^m := C_{\times k} \simeq S^{m-1} \cup_{\times k} e^m$ be the mapping cone of the cofibration

$$S^{m-1} \xrightarrow{\times k} S^{m-1} \xrightarrow{i} Y_k^m \,,$$

cf. [88]. Since the mapping cone C_i of the map i is homeomorphic to the suspension $\Sigma S^{m-1} \simeq S^m$, we obtain by continuing with the procedure the long co-exact Barratt-Puppe sequence

$$S^1 \xrightarrow{\times k} S^1 \to Y_k^2 \to S^2 \to \cdots \xrightarrow{\times k} S^{m-1} \to Y_k^m \,.$$

For any topological space X and $m \geq 2$, we set $\pi_m(X;\mathbb{Z}/k\mathbb{Z}) := [Y_k^m, X]$. Taking homotopy classes $[\cdot, X]$ in the Barratt-Puppe sequence yields

$$\pi_m(X;\mathbb{Z}/k\mathbb{Z}) \to \pi_{m-1}(X) \xrightarrow{\times k} \cdots \to \pi_2(X;\mathbb{Z}/k\mathbb{Z}) \to \pi_1(X) \to \pi_1(X) \,.$$

Definition 2.13 *Let C be a (small) exact category and BQC the geometric realization of the Quillen-category QC of C. Then the K-groups of C with coefficients in $\mathbb{Z}/k\mathbb{Z}$ are defined by*

$$K_m(C;\mathbb{Z}/k\mathbb{Z}) := \begin{cases} \pi_1(BQC)/k\pi_1(BQC) & \text{if } m = 0 \\ \pi_{m+1}(BQC;\mathbb{Z}/k\mathbb{Z}) & \text{if } m \geq 1 \end{cases} .$$

Remark 2.14 *Since BQC is an H-space, $K_m(C;\mathbb{Z}/k\mathbb{Z})$ is a group, cf. [98]. Even more is true, $K_m(C;\mathbb{Z}/k\mathbb{Z})$ is an abelian group and $k \cdot K_m(C;\mathbb{Z}/k\mathbb{Z}) = 0$*

resp. $(2k)\cdot K_m(C;\mathbb{Z}/k\mathbb{Z}) = 0$ *depending on* $k \not\equiv 0\,(2)$ *or* $k \equiv 0\,(2)$, *cf. [7] or [72].*

In all what follows, we are only interested in the K-theory of schemes, and even more important to us, of rings. So let X be a scheme, e.g., $X = Spec(R)$, R a ring.
With all the notations from above we get the long exact Bockstein sequence

$$K_m(X;\mathbb{Z}/k\mathbb{Z}) \to K_{m-1}(X) \overset{\times k}{\to} K_{m-1}(X) \to \cdots \overset{\times k}{\to} K_0(X) \to K_0(X;\mathbb{Z}/k\mathbb{Z}) \to 0\,,$$

which is functorial in X and the coefficients $\mathbb{Z}/k\mathbb{Z}$. Specializing to $k = l^\nu$, $\nu \geq 1$ and l a prime, we extract the short exact sequence

$$0 \to K_m(X)\otimes \mathbb{Z}/l^\nu\mathbb{Z} \to K_m(X;\mathbb{Z}/l^\nu\mathbb{Z}) \to {}_{l^\nu}K_{m-1}(X) \to 0\,.$$

Passing to the direct limit yields

$$0 \to K_m(X)\otimes \mathbb{Q}_l/\mathbb{Z}_l \to K_m(X;\mathbb{Q}_l/\mathbb{Z}_l) \to l\text{-}tor\,K_{m-1}(X) \to 0\,,$$

where $K_m(X;\mathbb{Q}_l/\mathbb{Z}_l) := \varinjlim K_m(X;\mathbb{Z}/l^\nu\mathbb{Z})$. As usual, passing to the projective limit causes more problems, but nevertheless, we make the following

Definition 2.15 *Let X be a scheme and l a prime. For $m \geq 0$, the l-adic K-groups of X are defined by*

$$K_m(X;\mathbb{Z}_l) := \varprojlim K_m(X;\mathbb{Z}/l^\nu\mathbb{Z})\,.$$

As in chapter 1 for cohomology, the l-adic K-groups do not behave as we want them to do. By comparing this to cohomology it seems to be clear what we have to do, if we want to consider K-groups with $\mathbb{Z}_l$-coefficients and certain properties. Namely, we have to take the projective limit 'inside', and this leads to continuous K-theory defined by Banaszak and Zelewski, cf. [3]. For $m \geq 2$, let $Y^m_{l^\infty} := \varinjlim Y^m_{l^\nu}$, then the continuous K-groups of X with coefficients in $\mathbb{Z}_l$ are defined by

$$K^{cont}_m(X;\mathbb{Z}_l) := [Y^m_{l^\infty}; BQSh(X)] \, .$$

In [3] the basic properties of continuous K-theory are proven such as a comparison theorem with l-adic K-theory, i.e., for $m \geq 1$, there is a short exact sequence

$$0 \to \varprojlim{}^1 K_{m+1}(X;\mathbb{Z}/l^\nu\mathbb{Z}) \to K^{cont}_m(X;\mathbb{Z}_l) \to K_m(X;\mathbb{Z}_l) \to 0 \, ,$$

and the existence of a long exact Bockstein sequence

$$\cdots \to K^{cont}_m(X;\mathbb{Z}_l) \xrightarrow{\times l^\nu} K^{cont}_m(X;\mathbb{Z}_l) \to K_m(X;\mathbb{Z}/l^\nu\mathbb{Z}) \to K^{cont}_{m-1}(X;\mathbb{Z}_l) \to \cdots \, .$$

Now we consider the algebraic K-theory of number fields and rings of integers. Let E be a number field, S a finite set of primes containing the infinite primes (and not necessarily all l-adic primes) and O^S_E the ring of S-integers with S-units U^S_E. For any prime $\mathfrak{p}$, the completion of E with respect to $\mathfrak{p}$ is denoted by $E_\mathfrak{p}$, and if $\mathfrak{p}$ is finite, the residue field by $k_\mathfrak{p}$. Quillen computed the K-theory of finite fields, cf. [79], and we have

$$K_m(k_\mathfrak{p}) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \\ 0 & \text{if } m = 2j,\ j \geq 1 \\ \mathbb{Z}/(q_\mathfrak{p}^j - 1)\mathbb{Z} & \text{if } m = 2j-1,\ j \geq 1 \text{ and } q_\mathfrak{p} = \# k_\mathfrak{p} \end{cases} .$$

From the work of Borel, cf. [5], we know that

$$K_m(O_E^S) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } m = 0 \\ \mathbb{Q}^{|S|-1} & \text{if } m = 1 \\ 0 & \text{if } m \equiv 0\,(2),\ m \geq 2 \\ \mathbb{Q}^{r_1(E)+r_2(E)} & \text{if } m \equiv 1\,(4),\ m \geq 5 \\ \mathbb{Q}^{r_2(E)} & \text{if } m \equiv 3\,(4) \end{cases} .$$

As in cohomology, cf. 1.19, the K-theory of the number field E is related to the K-theory of the ring of integers O_E^S by a localization sequence. Namely by Quillen, cf. [80], we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} K_m(O_E^S) & \rightarrow & K_m(E) & \xrightarrow{\partial^m} & \bigoplus_{\mathfrak{p} \notin S} K_{m-1}(k_\mathfrak{p}) & \rightarrow & K_{m-1}(O_E^S) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_m(O_E^S; \mathbb{Z}/k\mathbb{Z}) & \rightarrow & K_m(E; \mathbb{Z}/k\mathbb{Z}) & \xrightarrow{\partial_k^m} & \bigoplus_{\mathfrak{p} \notin S} K_{m-1}(k_\mathfrak{p}; \mathbb{Z}/k\mathbb{Z}) & \rightarrow & K_{m-1}(O_E^S; \mathbb{Z}/k\mathbb{Z}) \end{array},$$

where the vertical arrows are given by the corresponding Bockstein morphisms and the ∂^m's resp. ∂_k^m's are the so-called boundary maps.

For $m = 2$, ∂^2 is induced by the tame symbols $\tau_\mathfrak{p} : E^* \otimes E^* \rightarrow k_\mathfrak{p}^*$, and

Moore's reciprocity law, cf. [70], implies that ∂^2 is surjective. Obviously, $K_1(k_\mathfrak{p}) \to K_1(k_\mathfrak{p}; \mathbb{Z}/k\mathbb{Z})$ is surjective, and hence ∂_k^2 is surjective as well and we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 \to & K_2(O_E^S) & \to & K_2(E) & \to & \bigoplus_{\mathfrak{p}\notin S} K_1(k_\mathfrak{p}) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & K_2(O_E^S; \mathbb{Z}/k\mathbb{Z}) & \to & K_2(E; \mathbb{Z}/k\mathbb{Z}) & \to & \bigoplus_{\mathfrak{p}\notin S} K_1(k_\mathfrak{p}; \mathbb{Z}/k\mathbb{Z}) & \to 0 . \end{array}$$

Note, that the left arrow of the bottom row is not injective in general. For $S = S_\infty$, $K_2(O_E^S) = K_2(O_E)$ is called the tame kernel of E. One might hope that a diagram as above holds in a wider sense, and this is indeed the case by the next theorem, cf. [87].

Theorem 2.16 *Let E be a number field, S a finite set of primes containing all infinite primes and $m \geq 1$, $k \in \mathbb{Z}$. Then*

(i) There is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & K_{2m}(O_E^S) & \to & K_{2m}(E) & \to & \bigoplus_{\mathfrak{p}\notin S} K_{2m-1}(k_\mathfrak{p}) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & K_{2m}(O_E^S; \mathbb{Z}/k\mathbb{Z}) & \to & K_{2m}(E; \mathbb{Z}/k\mathbb{Z}) & \to & \bigoplus_{\mathfrak{p}\notin S} K_{2m-1}(k_\mathfrak{p}; \mathbb{Z}/k\mathbb{Z}) & \to 0 . \end{array}$$

(ii) There are isomorphisms

$$K_{2m+1}(O_E^S) \simeq K_{2m+1}(E) .$$

It was J.Tate who first proved a theorem relating K-theory of number fields with Galois cohomology, cf. [92] and [93]. Namely for a number field E and an odd prime l, there is an exact sequence

$$0 \to H^1_{cont}(E, \mathbb{Z}_l(2)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \to H^1(E, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to l\text{-}tor\, K_2(E) \to 0$$

and thus an isomorphism

$$l\text{-}tor\, H^2_{cont}(E, \mathbb{Z}_l(2)) \simeq l\text{-}tor\, K_2(E) .$$

Then A.S.Merkurjev and A.A.Suslin, cf. [63], extended Tate's result by determining the kernel and the cokernel of $K_m(E) \stackrel{\times l^\nu}{\to} K_m(E)$, $m = 2,3$. Namely

$$\begin{array}{rcccccc} 0 & \to & H^0(E, \mathbb{Z}/l^\nu\mathbb{Z}(2)) & \to & K_3(E) & \stackrel{\times l^\nu}{\to} & K_3(E) \\ & \to & H^1(E, \mathbb{Z}/l^\nu\mathbb{Z}(2)) & \to & K_2(E) & \stackrel{\times l^\nu}{\to} & K_2(E) \\ & \to & H^2(E, \mathbb{Z}/l^\nu\mathbb{Z}(2)) & \to & 0 & & \end{array}$$

is an exact sequence. This was partly (and independently) proven by M.Levine, cf. [59]. If S contains all l-adic primes, we might as well replace E by O_E^S and $H^*(E, \mathbb{Z}/l^\nu\mathbb{Z}(2))$ by $H^*_{et}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(2))$, and as a consequence we obtain for $j = 1,2$ and an odd prime l,

$$K_{2\cdot 2-j}(O_E^S) \otimes \mathbb{Z}_l \simeq H^j_{et}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(2)) .$$

This is a partly verification of the Quillen-conjecture, cf. [81].

Conjecture 2.17 (Quillen) *Let E be a number field, l an odd prime and S a finite set of primes containing all infinite and l-adic primes. Then for $i \geq 2$ and $j = 1,2$, there are isomorphisms*

$$ch_{i,j} : K_{2i-j}(O_E^S) \otimes \mathbb{Z}_l \simeq H^j_{et}(O_E^S, \mathbb{Z}_l(i)) ,$$

where $ch_{i,j}$ are induced by Chern classes, cf. below.

Remark 2.18 *Since the K-groups of rings of integers are finitely generated by Borel's work, we deduce from the localization sequence in K-theory and from Quillen's computation of the K-theory of finite fields, that for $i \geq 2$ and $j = 1, 2$,*

$$K_{2i-j}(O_E) \otimes \mathbb{Z}_l \simeq K_{2i-j}(O_E^S) \otimes \mathbb{Z}_l \, .$$

C.Soulé has constructed morphisms $ch_{i,j}$ via étale equivariant Chern classes due to A.Grothendieck, cf. [33], and has also proven surjectivity of $ch_{i,j}$ in certain cases, cf. [85], as well as $ch_{i,j} \otimes id : K_{2i-j}(O_E^S) \otimes \mathbb{Q}_l \to H^j_{ét}(O_E^S, \mathbb{Q}_l(i))$ is an isomorphism, cf. [86]. We recall very briefly the definition of the map $ch_{i,j}$, for details cf. [85].

Let $R = O_E^S$ be the ring of S-integers in a number field E, where S contains all infinite and l-adic primes. Suppose that P is a finitely generated, projective R-module of bounded rank over all residue fields $k_\mathfrak{p}$, and $\rho : G \to Aut(P)$ a representation of a discrete group G over P. For $i \geq 0$, $\nu \geq 1$, A.Grothendieck, cf. [33], defines Chern classes

$$ch_i(\rho) \in H^{2i}_{ét}(R, G; \mathbb{Z}/l^\nu\mathbb{Z}(i)) \, ,$$

where $H^*_{ét}(R, G; \cdot)$ denotes the étale equivariant cohomology with trivial G-action on R, cf. [8]. The Chern classes satisfy certain properties, just to mention

Functoriality: If $f : (R \to R',\ G \to G',\ P \to P')$ is a compatible system of morphisms, then

$$ch_i(f^*(\rho)) = f^*(ch_i(\rho))\ .$$

Additivity: Let $0 \to P_1 \to P \to P_2 \to 0$ be an exact sequence of $R[G]$-modules, projective over R. If ρ, ρ_1 and ρ_2 denote the corresponding representations, then

$$ch_i(\rho) = ch_i(\rho_1) \cup ch_i(\rho_2)\ .$$

The natural representation $id_n : GL_n(R) \to Aut(R^n)$, for every $n \geq 1$,induces $c_i(id_n) \in H^{2i}_{\acute{e}t}(R, GL_n(R); \mathbf{Z}/l^\nu\mathbf{Z}(i))$, and the above properties imply that $(c_i(id_n))_n$ form a compatible system under $i_n : GL_n(R) \hookrightarrow GL_{n+1}(R)$, i.e., $i_n^*(c_i(id_n)) = c_i(id_{n+1})$. After passing to the direct limit $GL(R) = \varinjlim GL_n(R)$ we obtain $c_i(id) \in H^{2i}_{\acute{e}t}(R, GL(R); \mathbf{Z}/l^\nu\mathbf{Z}(i))$. The Künneth formula induces a map

$$H^{2i}_{\acute{e}t}(R, GL(R); \mathbf{Z}/l^\nu\mathbf{Z}(i)) \longrightarrow \bigoplus_{j=0}^{2i} Hom(H_{2i}(GL(R), \mathbf{Z}/l^\nu\mathbf{Z}), H^j_{\acute{e}t}(R, \mathbf{Z}/l^\nu\mathbf{Z}))$$

and so for $0 \leq j \leq 2i$,

$$ch_{i,j}(id) : H_{2i-j}(GL(R), \mathbf{Z}/l^\nu\mathbf{Z}) \longrightarrow H^j_{\acute{e}t}(R, \mathbf{Z}/l^\nu\mathbf{Z}(i))\ .$$

Composing this map with the mod-l^ν Hurewicz map $K_{2i-j}(R; \mathbf{Z}/l^\nu\mathbf{Z}) \to H_{2i-j}(BGL(R)^+, \mathbf{Z}/l^\nu\mathbf{Z}) = H_{2i-j}(GL(R), \mathbf{Z}/l^\nu\mathbf{Z})$ yields a functorial morphism

$$\overline{ch}_{i,j} : K_{2i-j}(R; \mathbf{Z}/l^\nu\mathbf{Z}) \longrightarrow H^j_{\acute{e}t}(R, \mathbf{Z}/l^\nu\mathbf{Z}(i))$$

and passing to the projective limit, we finally obtain

$$ch_{i,j} : K_{2i-j}(R) \otimes \mathbf{Z}_l \longrightarrow H^j_{ét}(R, \mathbf{Z}_l(i)) ,$$

since the finite generation of $K_m(R)$ for all $m \geq 0$ implies $\varprojlim K_{2i-j}(R; \mathbf{Z}/l^\nu\mathbf{Z}) \simeq K_{2i-j}(R) \otimes \mathbf{Z}_l$.

In a series of papers, cf. [19] and [20], W.G.Dwyer and E.M.Friedlander studied étale K-theory of simplicial schemes X over $\mathbf{Z}[\frac{1}{l}]$, and proved the surjectivity of $ch_{i,j}$, $j = 1, 2$, in full generality, cf. 2.24. To define étale K-theory would require quite a lot of work in topology, and fortunately for us, the exact definition plays no role in our considerations and does not give us any further understanding. Since étale K-theory can be computed via a spectral sequence, whose E_2-terms are given by continuous cohomology, cf. 2.19, we omit the details of étale K-theory and rather list a few functorial properties. For any $m \geq 0$ and a prime l, étale K-theory gives rise to functors

$$K^{ét}_m(\cdot) : \text{ Simp Schemes}/\mathbf{Z}[\tfrac{1}{l}] \longrightarrow \mathbf{Z}_l\text{-modules}$$

$$K^{ét}_m(\cdot; \mathbf{Z}/l^\nu\mathbf{Z}) : \text{ Simp Schemes}/\mathbf{Z}[\tfrac{1}{l}] \longrightarrow \mathbf{Z}_l/l^\nu\mathbf{Z}_l\text{-modules} .$$

Here, Simp Schemes/$\mathbf{Z}[\frac{1}{l}]$ denotes the category of simplicial schemes over $\mathbf{Z}[\frac{1}{l}]$. As for algebraic K-theory, the groups $K^{ét}_m(X)$ and $K^{ét}_m(X; \mathbf{Z}/l^\nu\mathbf{Z})$ are related to each other by a long exact Bockstein sequence

$$\cdots \to K^{ét}_m(X) \xrightarrow{\times l^\nu} K^{ét}_m(X) \to K^{ét}_m(X; \mathbf{Z}/l^\nu\mathbf{Z}) \to K^{ét}_{m-1}(X) \to \cdots .$$

Let R be a ring and assume that $\frac{1}{l} \in R$, then for $m \geq 0$ the étale K-groups of R are defined by

$$\begin{aligned} K_m^{\acute{e}t}(R) &:= K_m^{\acute{e}t}(Spec(R)) \\ K_m^{\acute{e}t}(R; \mathbb{Z}/l^\nu\mathbb{Z}) &:= K_m^{\acute{e}t}(Spec(R); \mathbb{Z}/l^\nu\mathbb{Z}) , \end{aligned}$$

where $Spec(R)$ is considered as the constant simplicial scheme. In [20] W.G.Dwyer and E.M.Friedlander constructed highly non-trivial maps from algebraic to étale K-theory of rings, such that

$$\begin{array}{ccc} K_*(R) & \longrightarrow & K_*^{\acute{e}t}(R) \\ \downarrow & & \downarrow \\ K_*(R; \mathbb{Z}/l^\nu\mathbb{Z}) & \longrightarrow & K_*^{\acute{e}t}(R; \mathbb{Z}/l^\nu\mathbb{Z}) \end{array}$$

becomes a commutative diagram of rings – here, we have to assume that $l^\nu \neq 2$. For a proof of the next theorem, cf. [20].

Theorem 2.19 *Let X be a simplicial scheme over $\mathbb{Z}[\frac{1}{l}]$ with $cd_l X < \infty$. Then there are strongly convergent, fourth-quadrant spectral sequences*

$$\begin{aligned} (i)\quad E_2^{p,-q} &= H^p_{cont}(X, \mathbb{Z}_l(\tfrac{q}{2})) &\Longrightarrow\ & K^{\acute{e}t}_{q-p}(X) \\ (ii)\quad E_2^{p,-q} &= H^p_{cont}(X, \mathbb{Z}/l^\nu\mathbb{Z}(\tfrac{q}{2})) &\Longrightarrow\ & K^{\acute{e}t}_{q-p}(X; \mathbb{Z}/l^\nu\mathbb{Z}) , \end{aligned}$$

where by definition $\mathbb{Z}_l(\frac{q}{2}) = \mathbb{Z}/l^\nu\mathbb{Z}(\frac{q}{2}) = 0$ for $q \equiv 1\,(2)$.

Corollary 2.20 *Let E be number field, S a finite set of primes containing all infinite and l-adic primes, and O_E^S the ring of S-integers in E. Then for*

$X = Spec(E)$ resp. $X = Spec(O_E^S)$ and $i \geq 2$.

(i) $H^j_{cont}(X, \mathbb{Z}_l(i)) \quad \tilde{\to} \quad K^{\acute{e}t}_{2i-j}(X)$ *for* $j = 1, 2$.

(ii) $H^1_{\acute{e}t}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \quad \tilde{\to} \quad K^{\acute{e}t}_{2i-1}(X; \mathbb{Z}/l^\nu\mathbb{Z})$.

(iii) $0 \to H^2_{\acute{e}t}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to K^{\acute{e}t}_{2i-2}(X; \mathbb{Z}/l^\nu\mathbb{Z}) \to H^0_{\acute{e}t}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i-1)) \to 0$

is an exact sequence.

(iv) $H^2_{\acute{e}t}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \quad \tilde{\to} \quad K^{\acute{e}t}_{2i-2}(X) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l/l^\nu\mathbb{Z}_l$.

Proof: (i),(ii),(iii): All differentials $d_r^{p,q}$, $r \geq 2$ of the spectral sequence in 2.19 are trivaial, so in all three cases $E_2^{p,-q} = E_\infty^{p,-q}$. Since E does not contain all l-th power roots of unity, $H^0_{cont}(X, \mathbb{Z}_l(i)) = 0$ by 1.17, and the assertion follows immediately from 2.19. (iv): Again we have $H^0_{cont}(X, \mathbb{Z}_l(i)) = 0$, and thus $H^0_{\acute{e}t}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i-1)) \simeq {}_{l^\nu}H^1_{cont}(X, \mathbb{Z}_l(i-1))$. By (i),(ii),(iii) and the Bockstein sequence in étale K-theory we get the claim. □

Corollary 2.21 *Let the notation be as in 2.20. Set $E_\infty = E(W_l)$, and let O_∞^S be the ring of S-integers in E_∞. Then for $X = Spec(E_\infty)$ resp.*

$X = Spec(O_\infty^S)$ and $i \geq 2$.

(i) $H^1_{cont}(X, \mathbb{Z}_l(i)) \xrightarrow{\simeq} K^{et}_{2i-1}(X)$.

(ii) $0 \to H^2_{cont}(X, \mathbb{Z}_l(i)) \to K^{et}_{2i-2}(X; \mathbb{Z}/l^\nu\mathbb{Z}) \to H^0_{et}(X, \mathbb{Z}_l(i-1)) \to 0$

is an exact sequence.

(iii) $H^1_{et}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \xrightarrow{\simeq} K^{et}_{2i-1}(X; \mathbb{Z}/l^\nu\mathbb{Z})$.

(iv) $K^{et}_{2i-2}(X; \mathbb{Z}/l^\nu\mathbb{Z}) \xrightarrow{\simeq} H^0_{et}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i-1))$.

Proof: Analogous to the proof of 2.20. Just observe, that in this case $\mathbb{Z}_l(i-1) = H^0_{et}(X, \mathbb{Z}_l(i-1)) \neq 0$ and $H^2_{et}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i)) = 0$. □

Corollary 2.22 *Let $\mathbb{F}_q$ be a finite field with $char(\mathbb{F}_q) = p \neq l$. Then for $X = Spec(\mathbb{F}_q)$ and $i \geq 2$.*

(i) $H^1_{cont}(X, \mathbb{Z}_l(i)) \xrightarrow{\simeq} K^{et}_{2i-1}(X)$.

(ii) $K^{et}_{2i-2}(X) \xrightarrow{\simeq} H^0_{cont}(X, \mathbb{Z}_l(i-1))$.

(iii) $H^1_{et}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \xrightarrow{\simeq} K^{et}_{2i-1}(X; \mathbb{Z}/l^\nu\mathbb{Z})$.

(iv) $K^{et}_{2i-2}(X; \mathbb{Z}/l^\nu\mathbb{Z}) \xrightarrow{\simeq} H^0_{et}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i-1))$.

Proof: Since $H^1_{et}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i))$ is finite and $cd_l\mathbb{F}_q = 1$, we get $H^2_{cont}(X, \mathbb{Z}_l(i)) = H^2_{et}(X, \mathbb{Z}_l(i)) = 0$ as well as $H^2_{et}(X, \mathbb{Z}/l^\nu\mathbb{Z}(i)) = 0$. Therefore, the corollary is trivial by 2.19. □

Remark 2.23 *All morphisms occuring in 2.20 - 2.22 are edgemorphisms with respect to the corresponding spectral sequence 2.19. So they are functorial in X and the coefficients.*

As mentioned earlier W.G.Dwyer and E.M.Friedlander proved the surjectivity of the Chern class map for rings of integers, cf. [20], to be more precise.

Theorem 2.24 *Let l be an odd prime, and R a field or the ring of S-integers in a global field, where S is a finite set of primes containing all infinite and l-adic primes. Suppose that R satisfies*

1. $\frac{1}{l} \in R$
2. $cd_l R \leq 2$.

Then for $m \geq 1$,

$$K_m(R; \mathbb{Z}/l^\nu\mathbb{Z}) \longrightarrow K_m^{\acute{e}t}(R; \mathbb{Z}/l^\nu\mathbb{Z})$$

is surjective.

Remark 2.25 *For $l = 2$, we have to assume in addition that $\nu \geq 2$ and that R contains a primitive 4-th root of unity ζ_4.*

Corollary 2.26 *Let $\mathbb{F}_q$ be a finite field, $char(\mathbb{F}_q) = p \neq l$, l an odd prime. Then for $m \geq 1$.*

$$\text{(i)} \quad K_m(\mathbb{F}_q; \mathbb{Z}/l^\nu\mathbb{Z}) \xrightarrow{\sim} K_m^{\acute{e}t}(\mathbb{F}_q; \mathbb{Z}/l^\nu\mathbb{Z}) .$$

$$\text{(ii)} \quad K_m(\mathbb{F}_q) \otimes \mathbb{Z}_l \xrightarrow{\sim} K_m^{\acute{e}t}(\mathbb{F}_q) .$$

Proof: A finite field $\mathbb{F}_q$ certainly satisfies the condition in 2.24, and hence $K_m(\mathbb{F}_q;\mathbb{Z}/l^\nu\mathbb{Z}) \to K_m^{ét}(\mathbb{F}_q;\mathbb{Z}/l^\nu\mathbb{Z})$ is surjective. Since both groups are of the same finite order, cf. page 53 and 2.22, it is, in fact, an isomorphism. The second assertion follows from

$$K_m(\mathbb{F}_q)\otimes\mathbb{Z}_l \simeq \varprojlim K_m(\mathbb{F}_q;\mathbb{Z}/l^\nu\mathbb{Z}) \simeq \varprojlim K_m^{ét}(\mathbb{F}_q;\mathbb{Z}/l^\nu\mathbb{Z}) \simeq K_m^{ét}(\mathbb{F}_q)\,.$$

□

Corollary 2.27 *Let E be a number field, l an odd prime and S a finite set of primes containing all infinite and l-adic primes. Then for $m \geq 2$,*

$$K_m(O_E^S)\otimes\mathbb{Z}_l \longrightarrow K_m^{ét}(O_E^S)$$

is surjective.

Proof: The conditions in 2.24 are satisfied for O_E^S, and thus we obtain as in the proof of 2.26 the chain of morphisms

$$\begin{aligned} K_m(O_E^S)\otimes\mathbb{Z}_l &\xrightarrow{\simeq} \varprojlim K_m(O_E^S;\mathbb{Z}/l^\nu\mathbb{Z}) \\ &\to \varprojlim K_m^{ét}(O_E^S;\mathbb{Z}/l^\nu\mathbb{Z}) \\ &\xrightarrow{\simeq} K_m^{ét}(O_E^S)\,, \end{aligned}$$

where the middle morphism is surjective. The isomorphisms follow from the finite generation of $K_m(O_E^S)$ and $K_m^{ét}(O_E^S)$. □

By the work of Borel, cf. page 53, for $i \geq 2$, $K_{2i-2}(O_E^S)$ is finite, and so the same holds for $K_{2i-2}^{\acute{e}t}(O_E^S)$. Since the later group is isomorphic to $H^2_{cont}(O_E^S, \mathbb{Z}_l(i))$, we have proven, cf. 1.24,

Theorem 2.28 *Let E be a number field, l an odd prime and S a finite set of primes containing all infinite and l-adic primes. Then for $i \geq 2$,*

$$H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0 ,$$

i.e., the Schneider conjecture is true for $i \geq 2$.

If we want to obtain analogous results to 2.27 for the number field E, it is clear that we have to modify our arguments. First of all, everything works equally well for the odd-dimensional K-theory of E, since $K_{2i-1}(O_E^S) \simeq K_{2i-1}(E)$ by 2.16 and $K_{2i-1}^{\acute{e}t}(O_E^S) \simeq K_{2i-1}^{\acute{e}t}(E)$ by 1.20 and 2.20. For the even-dimensional K-theory of E, we still have the following

Corollary 2.29 *Let E be a number field, l an odd prime. Then for $i \geq 2$,*

$$l\text{-}tor\ K_{2i-2}(E) \longrightarrow l\text{-}tor\ K_{2i-2}^{\acute{e}t}(E)$$

is surjective.

Proof: From the long exact Bockstein sequence we get the commutative diagram

$$\begin{array}{ccccc} K_{2i-1}(E;\mathbb{Z}/l^\nu\mathbb{Z}) & \longrightarrow & {}_{l^\nu}K_{2i-2}(E) & \longrightarrow 0 \\ \downarrow & & \downarrow & \\ K_{2i-1}^{\acute{e}t}(E;\mathbb{Z}/l^\nu\mathbb{Z}) & \longrightarrow & {}_{l^\nu}K_{2i-2}^{\acute{e}t}(E) & \longrightarrow 0 , \end{array}$$

where the left vertical arrow is surjective. Passing to the direct limit gives the assertion. □

Note that the above proof certainly carries over to the ring of S-integers O_E^S in the following sense. For $i \geq 2$, $K_{2i-2}(O_E^S) \otimes \mathbb{Z}_l \to l\text{-}tor\, K_{2i-2}^{\acute{e}t}(O_E^S)$ is surjective. But as we see, this does not give the finiteness of $K_{2i-2}^{\acute{e}t}(O_E^S) \simeq H^2_{cont}(O_E^S, \mathbb{Z}_l(i))$, and so we could not deduce the theorem 2.28.

If we want to compare étale K-theory of number fields and rings of integers, then immediately the word localization sequence comes to our mind. But there is not ad hoc a localization sequence in étale K-theory. We can go around this problem by considering the cohomological one and using the above results, relating étale K-theory to cohomology. Recall our standard notation for number fields on page 52. For $i \geq 2$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & H^2_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(i)) & \to & K_{2i-2}^{\acute{e}t}(O_E^S; \mathbb{Z}/l^\nu\mathbb{Z}) & \to & H^0_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(i-1)) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \wr & & \\ 0 & \to & H^2(E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) & \to & K_{2i-2}^{\acute{e}t}(E; \mathbb{Z}/l^\nu\mathbb{Z}) & \to & H^0(E, \mathbb{Z}/l^\nu\mathbb{Z}(i-1)) & \to & 0\,. \end{array}$$

Using the localization sequence in étale cohomology, cf. 1.19, and appealing once more to 2.20 as well as 2.22 we deduce the long exact sequence for $i \geq 2$,

$$\begin{array}{ccccccc} 0 & \to & K_{2i-1}^{\acute{e}t}(O_E^S; \mathbb{Z}/l^\nu\mathbb{Z}) & \to & K_{2i-1}^{\acute{e}t}(E; \mathbb{Z}/l^\nu\mathbb{Z}) & \to & \bigoplus_{\mathfrak{p} \notin S} K_{2i-2}^{\acute{e}t}(k_\mathfrak{p}; \mathbb{Z}/l^\nu\mathbb{Z}) \\ & \to & K_{2i-2}^{\acute{e}t}(O_E^S; \mathbb{Z}/l^\nu\mathbb{Z}) & \to & K_{2i-2}^{\acute{e}t}(E; \mathbb{Z}/l^\nu\mathbb{Z}) & \to & \bigoplus_{\mathfrak{p} \notin S} K_{2i-3}^{\acute{e}t}(k_\mathfrak{p}; \mathbb{Z}/l^\nu\mathbb{Z}) \to 0\,. \end{array}$$

For the odd-dimensional étale K-theory, we already know that

$$K_{2i-1}^{\acute{e}t}(O_E^S) \simeq K_{2i-1}^{\acute{e}t}(E)\ , i \geq 2,$$

and for the even-dimensional étale K-theory, we proceed as follows. By passing to the direct limit in 1.19 and applying 2.28, we get

$$0 \to H_{\acute{e}t}^1(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to H^1(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to \bigoplus_{\mathfrak{p} \notin S} H^0(k_{\mathfrak{p}}, \mathbb{Q}_l/\mathbb{Z}_l(i-1)) \to 0\ , i \geq 2,$$

and after dividing the maximal divisible subgroup $H_{cont}^1(O_E^S, \mathbb{Z}_l(i))$ resp., we finally obtain

Lemma 2.30 *Let $i \geq 2$, then with the above notations there is an exact sequence*

$$0 \to K_{2i-2}^{\acute{e}t}(O_E^S) \to l\text{-}tor\, K_{2i-2}^{\acute{e}t}(E) \to \bigoplus_{\mathfrak{p} \notin S} K_{2i-3}^{\acute{e}t}(k_{\mathfrak{p}}) \to 0\ .$$

On the other hand, we could pass to the projective limit in the long exact sequence above, and then

$$0 \to \varprojlim \ker j_{l^\nu} \to K_{2i-2}^{\acute{e}t}(O_E^S) \to \varprojlim K_{2i-2}^{\acute{e}t}(E; \mathbb{Z}/l^\nu\mathbb{Z}) \to \bigoplus_{\mathfrak{p} \notin S} K_{2i-3}^{\acute{e}t}(k_{\mathfrak{p}}) \to 0\ ,$$

is exact, where $j_{l^\nu} : K_{2i-2}^{\acute{e}t}(O_E^S; \mathbb{Z}/l^\nu\mathbb{Z}) \to K_{2i-2}^{\acute{e}t}(E; \mathbb{Z}/l^\nu\mathbb{Z})$.

Lemma 2.31 *Let $i \geq 2$, then we have*

$$\begin{aligned} \varprojlim \ker j_{l^\nu} &\simeq l\text{-}tor\left(\varprojlim{}^1{}_{l^\nu} K_{2i-2}^{\acute{e}t}(E)\right) \\ &\simeq l\text{-}tor\left(l\text{-}div\, K_{2i-2}^{\acute{e}t}(E)\right)\ . \end{aligned}$$

Proof: By passing to the projective limit in the commutative diagram

$$\begin{array}{ccc} H^1(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) & \longrightarrow & \oplus_{\mathfrak{p}\notin S} H^0(k_\mathfrak{p}, \mathbb{Q}_l/\mathbb{Z}_l(i-1)) \\ \downarrow & & \downarrow \\ H^2(E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) & \longrightarrow & \oplus_{\mathfrak{p}\notin S} H^1(k_\mathfrak{p}, \mathbb{Z}/l^\nu\mathbb{Z}(i-1)) \end{array}$$

we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & H^2_{cont}(O_E^S, \mathbb{Z}_l(i)) & \to & l\text{-}tor\, H^2_{cont}(E, \mathbb{Z}_l(i)) & \to & \oplus_{\mathfrak{p}\notin S} H^1_{cont}(k_\mathfrak{p}, \mathbb{Z}_l(i-1)) & \to 0 \\ & \downarrow \wr & & \downarrow & & \downarrow \wr & \\ & H^2_{\acute{e}t}(O_E^S, \mathbb{Z}_l(i)) & \to & H^2(E, \mathbb{Z}_l(i)) & \to & \oplus_{\mathfrak{p}\notin S} H^1_{\acute{e}t}(k_\mathfrak{p}, \mathbb{Z}_l(i-1)) & \to 0\,, \end{array}$$

where the kernel of the middle vertical map is equal to $l\text{-}tor\, \varprojlim^1 H^1(E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$. Now identifying the cohomology groups with the corresponding K-groups, e.g., $H^2(E, \mathbb{Z}_l(i)) \simeq \varprojlim K^{\acute{e}t}_{2i-2}(E; \mathbb{Z}/l^\nu\mathbb{Z})$, we deduce

$$\varprojlim ker\, j_{l^\nu} \simeq l\text{-}tor\, \left(\varprojlim^1 K^{\acute{e}t}_{2i-1}(E; \mathbb{Z}/l^\nu\mathbb{Z})\right) .$$

Since $K^{\acute{e}t}_{2i-1}(E)\otimes_{\mathbb{Z}_l}\mathbb{Z}_l/l^\nu\mathbb{Z}_l \simeq K^{\acute{e}t}_{2i-1}(O_E^S)\otimes_{\mathbb{Z}_l}\mathbb{Z}_l/l^\nu\mathbb{Z}_l$ satisfies the M-L condition, we get

$$\begin{aligned} l\text{-}tor\, \left(\varprojlim^1 K^{\acute{e}t}_{2i-1}(E; \mathbb{Z}/l^\nu\mathbb{Z})\right) &\simeq l\text{-}tor\, \left(\varprojlim^1{}_{l^\nu} K^{\acute{e}t}_{2i-2}(E)\right) \\ &\simeq l\text{-}tor\, \left(l\text{-}div.\, K^{\acute{e}t}_{2i-2}(E)\right) . \end{aligned}$$

The second isomorphism is obvious, since $l\text{-}tor\, K^{\acute{e}t}_{2i-2}(E)$ does not contain any divisible subgroup. □

There is a more natural setting for the group of l-divisible elements in

$K^{et}_{2i-2}(E)$. At this point we follow ideas of G.Banaszak, M.Kolster and T.Nguyen Quang Do, cf.[2], [53] and [76], who defined higher algebraic and étale wild kernels of a number field E. Let us first recall the definition of the classical wild kernel $WK_2(E)$. For this, let $\lambda = (\lambda_\mathfrak{p})_\mathfrak{p} : K_2(E) \to \bigoplus'_\mathfrak{p} \mu(E_\mathfrak{p})$ be the morphism induced by the Hilbert symbol $(\cdot,\cdot)_\mathfrak{p}$. Here, and in the following $\bigoplus'_\mathfrak{p}$ stands for the sum over all non-complex primes $\mathfrak{p}$ of E. Then

$$WK_2(E) := ker\,\lambda\ ,$$

and by uniqueness of reciprocity, cf. [70], there is an exact sequence

$$0 \longrightarrow WK_2(E) \longrightarrow K_2(E) \xrightarrow{\lambda} \bigoplus_\mathfrak{p}{}'\mu(E_\mathfrak{p}) \xrightarrow{\pi} \mu(E) \longrightarrow 0\ ,$$

where $\pi((\zeta_\mathfrak{p})_\mathfrak{p}) := \prod \zeta_\mathfrak{p}^{\frac{m_\mathfrak{p}}{m}}$ with $m_\mathfrak{p} := \#\mu(E_\mathfrak{p})$ and $m := \mu(E)$.

Following Schneider, cf. [82], let

$$D_i(E) := ker\left(H^1(E,\mathbb{Q}_l/\mathbb{Z}_l(i))/max.\,div. \to \bigoplus_{\mathfrak{p}\nmid\infty} H^1(E_\mathfrak{p},\mathbb{Q}_l/\mathbb{Z}_l(i))/max.\,div.\right), i \neq 1$$

where $\cdot/max.\,div.$ stands for dividing by the maximal divisible subgroup resp. Then $D_i(E)$ equals the group of l-divisible elements in l-$tor\,H^2_{cont}(E,\mathbb{Z}_l(i))$, i.e., for $i \geq 2$, $D_i(E) \simeq \varprojlim ker\,j_{l^\nu}$. Since for $i \neq 1$, $H^2_{cont}(E_\mathfrak{p},\mathbb{Z}_l(i)) = H^2(E_\mathfrak{p},\mathbb{Z}_l(i))$ is finitely generated and $H^2(E_\mathfrak{p},\mathbb{Q}_l/\mathbb{Z}_l(i)) = 0$ by 1.4, we have

$$H^1(E_\mathfrak{p},\mathbb{Q}_l/\mathbb{Z}_l(i))/max.\,div. \simeq H^2_{cont}(E_\mathfrak{p},\mathbb{Z}_l(i))\ .$$

Furthermore, we know that the Schneider conjecture is true for $i \geq 2$, cf. 2.28, and so in that case

$$\begin{array}{ccccc} 0 \to D_i(E) \to & H^1_{\acute{e}t}(O^S_E, \mathbf{Q}_l/\mathbf{Z}_l(i))/max.div. & \to & \bigoplus_{\mathfrak{p}|l} H^1(E_{\mathfrak{p}}, \mathbf{Q}_l/\mathbf{Z}_l(i))/max.div. \\ & \downarrow \wr & & \downarrow \wr \\ & H^2_{cont}(O^S_E, \mathbf{Z}_l(i)) & \to & \bigoplus_{\mathfrak{p}|l} H^2_{cont}(E_{\mathfrak{p}}, \mathbf{Z}_l(i)) \end{array}$$

is a commutative diagram, where by 1.22 the cokernel of the (lower) horizontal map is equal to $H^0_{\acute{e}t}(O^S_E, \mathbf{Q}_l/\mathbf{Z}_l(1-i))^* = W_l^{(1-i)}(E)^*$. Since $H^2_{cont}(E_{\mathfrak{p}}, \mathbf{Z}_l(i)) \simeq H^0(E_{\mathfrak{p}}, \mathbf{Q}_l/\mathbf{Z}_l(1-i))^*$ by 1.21, we obtain for $i \geq 2$,

$$\begin{array}{l} l\text{-}tor\, K_{2i-2}(E) \\ \quad\downarrow \\ l\text{-}tor\, K^{\acute{e}t}_{2i-2}(E) \to \bigoplus_{\mathfrak{p}\nmid\infty} H^0(E_{\mathfrak{p}}, \mathbf{Q}_l/\mathbf{Z}_l(1-i))^* \to H^0(E, \mathbf{Q}_l/\mathbf{Z}_l(1-i))^* \to 0\,. \\ \quad\downarrow \\ \quad 0 \end{array}$$

It is clear what we are up to now, but let us first consider the case $i = 2$. Let $\delta_E : E^\star \to H^1_{\acute{e}t}(E, \mathbf{Z}_l(1))$ be the connecting morphism induced by the exact sequence of Galois modules

$$0 \longrightarrow \mathbf{Z}_l(1) \longrightarrow \varprojlim(E^{sep})^\star \longrightarrow (E^{sep})^\star \longrightarrow 0\,,$$

and $g_E : K_2(E) \to H^2_{cont}(E, \mathbf{Z}_l(2))$ the unique morphism —the so-called Galois symbol—such that $g_E(\{a, b\}) = \delta_E a \cup \delta_E b$, cf. [93]. For the local Galois symbol, we simply write $g_{\mathfrak{p}} = g_{E_{\mathfrak{p}}}$. With all these notations we get the diagram

$$\begin{array}{ccccc} l\text{-}tor\, K_2(E) & \to & \bigoplus_{\mathfrak{p}\nmid\infty} l\text{-}tor\, K_2(E_{\mathfrak{p}}) & \xrightarrow{\oplus\lambda_{\mathfrak{p}}} & \bigoplus_{\mathfrak{p}\nmid\infty} l\text{-}tor\, \mu(E_{\mathfrak{p}}) \\ \downarrow g_E & & \downarrow \oplus g_{\mathfrak{p}} & & \uparrow \oplus eval_{\mathfrak{p}}^{(1)} \\ l\text{-}tor\, H^2_{cont}(E,\mathbb{Z}_l(2)) & \to & \bigoplus_{\mathfrak{p}\nmid\infty} H^2_{cont}(E_{\mathfrak{p}},\mathbb{Z}_l(2)) & \xrightarrow{\oplus c_{\mathfrak{p}}} & \bigoplus_{\mathfrak{p}\nmid\infty} H^0(E_{\mathfrak{p}},\mathbb{Q}_l/\mathbb{Z}_l(-1))^* \end{array},$$

where $eval_{\mathfrak{p}}^{(1)}$ has still to be defined. Nevertheless, the left square is commutative, and we want to define $eval_{\mathfrak{p}}^{(1)}$ such that the right square becomes commutative as well. Thus we are left with the local case. Let $l^{\nu} := \# l\text{-}tor\,\mu(E_{\mathfrak{p}})$ and consider the diagram

$$\begin{array}{ccc} l\text{-}tor\, K_2(E_{\mathfrak{p}}) & \xrightarrow{\lambda_{\mathfrak{p}}} & \mathbb{Z}/l^{\nu}\mathbb{Z}(1) \\ \downarrow g_{\mathfrak{p}} & & \uparrow eval_{\mathfrak{p}}^{(1)} \\ H^2(E_{\mathfrak{p}},\mathbb{Z}/l^{\nu}\mathbb{Z}(2)) & \xrightarrow{c_{\mathfrak{p}}} & H^0(E_{\mathfrak{p}},\mathbb{Z}/l^{\nu}\mathbb{Z}(-1))^* \end{array}.$$

The map $\lambda_{\mathfrak{p}}$ is given by Kummer theory and the local norm residue symbol, which is induced by the cup-product with $inv^{-1}\{\frac{1}{l^{\nu}} \bmod \mathbb{Z}\}$. If we set $eval_{\mathfrak{p}}^{(1)} : (\mathbb{Z}/l^{\nu}\mathbb{Z}(-1))^* \to \mathbb{Z}/l^{\nu}\mathbb{Z}(1)$ by $eval_{\mathfrak{p}}^{(1)}(\phi) := \phi(1+l^{\nu}\mathbb{Z})$, then a straightforward, but tedious calculation shows the commutativity of the diagram, and therefore also of the diagram above. Now for $i \geq 2$, we define $eval_{\mathfrak{p}}^{(i-1)} : W_l^{(1-i)}(E_{\mathfrak{p}})^* \xrightarrow{\sim} W_l^{(i-1)}(E_{\mathfrak{p}})$ resp. $eval^{(i-1)} : W_l^{(1-i)}(E)^* \xrightarrow{\sim} W_l^{(i-1)}(E)$ in the same manner as above and we obtain for $i \geq 2$,

$$\begin{array}{lcl} l\text{-}tor\, K_{2i-2}(E) & & \\ \downarrow & \searrow^{\lambda_{i-1}} & \\ l\text{-}tor\, K^{\acute{e}t}_{2i-2}(E) & \overset{\lambda^{\acute{e}t}_{i-1}}{\rightarrow} & \bigoplus_{\mathfrak{p}\nmid\infty} W_l^{(i-1)}(E_{\mathfrak{p}}) \rightarrow W_l^{(i-1)}(E) \rightarrow 0\,. \end{array}$$

Definition 2.32 *Let E be a number field, l an odd prime and $i \geq 2$.*

(i) The Sylow-l-subgroup of the higher wild kernel is defined by

$$l\text{-}tor\, WK_{2i-2}(E) := ker\, \lambda_{i-1}\,.$$

(ii) The higher étale wild kernel is defined by

$$WK^{\acute{e}t}_{2i-2}(E) := ker\, \lambda^{\acute{e}t}_{i-1}\,.$$

Remark 2.33 *(i) As we have shown, this notion of the higher wild kernel is consistent with the one for the classical wild kernel. Unfortunately, we do not know of any appropriate definition for $l = 2$, neither in the algebraic nor in the étale case. (ii) In [2] Banaszak showed that*

$$l\text{-}tor\, WK_{2i-2}(E) \rightarrow WK^{\acute{e}t}_{2i-2}(E) \rightarrow 0$$

is canonically split onto the group of l-divisible elements in l-tor $K_{2i-2}(E)$.

We see in chapter 6, that just like the relation between $H^{*}_{cont}(O^S_E, \mathbb{Z}_l(i))$ and $K^{ét}_{2i-*}(O^S_E)$, there is a linkage between certain subgroups of $H^2_{cont}(O^S_E, \mathbb{Z}_l(i))$ resp. $H^1_{ét}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i))$ and $WK^{ét}_{2i-2}(E)$. But let us close this chapter with the following

Proposition 2.34 *Let E be a number field, l an odd prime and $i \geq 2$. Then*

$$\frac{\#l\text{-}tor\, K_{2i-2}(O^S_E)}{\#l\text{-}tor\, WK_{2i-2}(E)} = \frac{\#K^{ét}_{2i-2}(O^S_E)}{\#WK^{ét}_{2i-2}(E)} = \frac{\prod_{\mathfrak{p}|l} \#W_l^{(i-1)}(E_\mathfrak{p})}{\#W_l^{(i-1)}(E)}.$$

Proof: With all the notations we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & l\text{-}tor\, WK_{2i-2}(E) & \to & l\text{-}tor\, K_{2i-2}(E) & \to & \bigoplus_{\mathfrak{p}\nmid\infty} W_l^{(i-1)}(E_\mathfrak{p}) & \to W_l^{(i-1)}(E) \to 0 \\ & \downarrow & & \downarrow id & & \downarrow & \\ 0 \to & l\text{-}tor\, K_{2i-2}(O^S_E) & \to & l\text{-}tor\, K_{2i-2}(E) & \to & \bigoplus_{\mathfrak{p}\notin S} l\text{-}tor\, K_{2i-3}(k_\mathfrak{p}) & \to 0\,. \end{array}$$

For $\mathfrak{p} \notin S$, $W_l^{(i-1)}(E_\mathfrak{p}) \simeq H^1(E_\mathfrak{p}, \mathbb{Z}_l(i-1)) \simeq H^1(k_\mathfrak{p}, \mathbb{Z}_l(i-1))$, thus the formula

$$\frac{\#l\text{-}tor\, K_{2i-2}(O^S_E)}{\#l\text{-}tor\, WK_{2i-2}(E)} = \frac{\prod_{\mathfrak{p}|l} \#W_l^{(i-1)}(E_\mathfrak{p})}{\#W_l^{(i-1)}(E)}$$

follows at once from the diagram, cf. also [2]. The second formula is evident from $K_{2i-3}(k_\mathfrak{p}) \simeq K^{ét}_{2i-3}(k_\mathfrak{p})$, cf. 2.26. □

3 Preliminaries on Iwasawa theory

Much of Iwasawa's work on cyclotomic fields was inspired by Weil's solution of conjectures on curves over finite fields, cf. [96] and [97]. We recall briefly a few of these aspects, which are crucial for understanding Iwasawa's ideas. Let X be a smooth, projective variety over a finite field $\mathbb{F}_q$ of $char(\mathbb{F}_q) = p$, and $X_\infty := X \times_{\mathbb{F}_q} \mathbb{F}_{q^\infty}$ the variety over a separable closure $\mathbb{F}_{q^\infty}$ of $\mathbb{F}_q$ obtained by extending the scalars. For every $r \geq 1$, let N_r be the number of points of X_∞, which are rational over $\mathbb{F}_{q^r}$, i.e., whose coordinates lie in $\mathbb{F}_{q^r}$. Certainly, these numbers N_r are of great interest from a number-theoretical point of view; just to mention N_r of the Fermat curve over the finite field $\mathbb{F}_q$.
We define the Z-function of X by

$$Z_X(T) := \exp\left(\sum_{r=1}^{\infty} N_r \frac{T^r}{r}\right) ,$$

which we consider as a formal power series in $\mathbb{Q}[[T]]$. The analogue to the Riemann ζ-function—more generally, Dedekind ζ-function—is given by

Lemma 3.1 *Let $X = C$ be a curve over $\mathbb{F}_q$, and $F := \mathbb{F}_q(C)$ its function field. Then*

$$Z_X(q^{-s}) = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1} ,$$

where the product runs over all primes $\mathfrak{p}$ of F and $N(\mathfrak{p})$ denotes the absolute norm of a prime $\mathfrak{p}$.

Proof: If $deg(\mathfrak{p}) := [k(\mathfrak{p}) : \mathbb{F}_q]$ denotes the degree of $\mathfrak{p}$, where $k(\mathfrak{p})$ stands for the residue field of $\mathfrak{p}$, then $N(\mathfrak{p}) = q^{deg(\mathfrak{p})}$. Since there is a one-to-one correspondence between closed points $x \in X$ and primes $\mathfrak{p} \subseteq F$, we get $N_r = \sum_{d|r} d \cdot a_d$, where a_d denotes the number of primes resp. closed points of degree d. So the right hand side in 3.1 becomes for $u := q^{-s}$,

$$\prod_{d=1}^{\infty}(\frac{1}{1-u^d})^{a_d} .$$

Taking the logarithmic derivative and expanding it into a geometric series gives

$$\frac{1}{u}\sum_{r=1}^{\infty}\left(\sum_{d|r} d \cdot a_d\right) \cdot u^r = \sum_{r=1}^{\infty} N_r \cdot u^{r-1} .$$

After integrating and taking the exponential we get the formula in 3.1. □

In 1949 A. Weil formulated conjectures on the Z-function $Z_X(T)$ of a smooth variety over $\mathbb{F}_q$, cf. [97].

Conjecture 3.2 (Weil) *Let X be a smooth, projective variety of dimension d over $\mathbb{F}_q$. Then*

1.

$$Z_X(T) = \frac{P_{1,X}(T) \cdot P_{3,X}(T) \cdot \ldots \cdot P_{2d-1,X}(T)}{P_{0,X}(T) \cdot \ldots \cdot P_{2d,X}(T)}$$

and each $P_{j,X}(T)$ is a polynomial with coefficients in a field of characteristic zero.

2. *$P_{j,X}(T) = \prod_{i=1}^{b_j}(1 - \alpha_{ji}T)$, where α_{ji} are algebraic integers with $|\alpha_{ji}| = q^{j/2}$.*

3. *$Z_X(T)$ satisfies a functional equation*

$$Z_X(\frac{1}{q^d T}) = \pm q^{d \cdot E/2} T^E Z_X(T) ,$$

where E is the self-intersection number of the diagonal Δ of $X \times X$.

4. *If X is obtained by reduction mod q of a smooth, projective variety Y over a number field, then*

$$b_j := deg\, P_{j,X}(T) = rk_{\mathbf{Z}} H_j(Y(\mathbf{C})) ,$$

where $Y(\mathbf{C})$ is the complex manifold defined by Y, cf. [37].

Weil also proposed the idea of a suitable cohomology theory for X_∞, so that one could count the fixed points of a morphism $f : X_\infty \to X_\infty$ as for singular cohomology via Lefschetz's Fixed Point Theorem. Clearly, this requires a cohomology theory with coefficients in a field of characteristic zero, and as Serre showed, there can not be a cohomology theory over the rational or real numbers, nor over $\mathbf{Q}_p$. Then A.Grothendieck in collaboration with J.-L.Verdier and M.Artin developed étale topology and cohomology. In particular, they defined cohomology groups with coefficients in $\mathbb{Q}_l$, cf. 1.12. Namely,

$$H^j_{\acute{e}t}(X_\infty, \mathbb{Q}_l) := H^j_{\acute{e}t}(X_\infty, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l ,$$

l a prime different from $char(\mathbb{F}_q) = p$. Along several theorems on l-adic cohomology they proved a Lefschetz's Fixed Point Theorem. If $f : X_\infty \to X_\infty$ has isolated fixed points, then the Lefschetz number

$$L(f) := \sum(-1)^j Trace\,(f^j : H^j_{\acute{e}t}(X_\infty, \mathbb{Q}_l) \to H^j_{\acute{e}t}(X_\infty, \mathbb{Q}_l))$$

equals the number of fixed points counted according to the multiplicities. Now the Frobenius automorphism of $\mathbb{F}_{q^\infty}$ induces a morphism $\varphi : X_\infty \to X_\infty$ by acting on the coordinates of a point $x \in X_\infty$. It is plain from the definition that N_r equals the number of fixed points of φ^r, and hence

$$N_r = L(\varphi^r) .$$

Substituting back into the definition of the Z-function we get

$$\begin{aligned} Z_X(T) &= \exp\left(\textstyle\sum_{r=1}^{\infty} L(\varphi^r)\frac{T^r}{r}\right) \\ &= \exp\left(\textstyle\sum_{r=1}^{\infty}\sum_{j=0}^{2d} Trace\,(\varphi^{r*} : H^j_{\acute{e}t}(X_\infty, \mathbb{Q}_l) \to H^j_{\acute{e}t}(X_\infty, \mathbb{Q}_l))\frac{T^r}{r}\right) \\ &= \textstyle\prod_{j=0}^{2d} det\,(1 - T\varphi^* : H^j_{\acute{e}t}(X_\infty, \mathbb{Q}_l) \to H^j_{\acute{e}t}(X_\infty, \mathbb{Q}_l))^{(-1)^{j+1}} \\ &= \frac{P_{1,X}(T)\cdot\ldots\cdot P_{2d-1,X}(T)}{P_{0,X}(T)\cdot\ldots\cdot P_{2d,X}(T)} , \end{aligned}$$

where $P_{j,X}(T) := det\,(1 - T\varphi^* : H^j_{\acute{e}t}(X_\infty, \mathbb{Q}_l) \to H^j_{\acute{e}t}(X_\infty, \mathbb{Q}_l))$, $0 \le j \le 2d$. Thus the first part of the Weil conjectures is thereby proven. Furthermore, Poincaré duality in l-adic cohomology implies the functional equation with $E = \sum(-1)^j dim_{\mathbb{Q}_l} H^j_{ét}(X_\infty, \mathbb{Q}_l)$. The last statement follows from the comparison theorem between étale and singular cohomology of smooth, projective varieties, cf. [65]. Note, that the polynomials $P_{j,X}(T)$ are not ad hoc independent of l, so to be correct, we should have denoted $P_{j,X}(T)$ by $P^{(l)}_{j,X}(T)$ instead. But P. Deligne, cf. [14] and [15], finally succeeded in proving the remaining and hardest part of the Weil conjectures, in particular $P_{j,X}(T)$ has integer coefficients. Next we specialize to the case $X = C$ a curve of genus $g = g(C)$. In order to calculate the Z-function $Z_X(T)$ of X, we have to determine the polynomials $P_{j,X}(T)$, and thus the maps

$$(\varphi^*)^j : H^j_{ét}(X_\infty, \mathbb{Q}_l) \to H^j_{ét}(X_\infty, \mathbb{Q}_l) ,$$

which are induced by the Frobenius $\varphi : X_\infty \to X_\infty$. Clearly, $(\varphi^*)^0 = id$, and so $P_{0,X}(T) = 1 - T$. Furthermore, since $\mathbb{F}_{q^\infty}$ contains all l-th roots of unity, we have

$$H^j_{ét}(X_\infty, \mathbb{Q}_l(1)) \simeq H^j_{ét}(X_\infty, \mathbb{Q}_l)(1) ,$$

and so,

$$P_{j,X}(T) = det\,(1 - T\varphi^* : H^j_{ét}(X_\infty, \mathbb{Q}_l(1)) \to H^j_{ét}(X_\infty, \mathbb{Q}_l(1)) .$$

Note, that φ^* is induced by the Frobenius acting on X_∞ and not on the l-adic sheaf $\mathbb{Z}_l$. From the exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z}/l^\nu\mathbb{Z}(1) \longrightarrow \mathcal{O}^*_{X_\infty} \xrightarrow{\times l^\nu} \mathcal{O}^*_{X_\infty} \longrightarrow 0 ,$$

we obtain in conjunction with $H^1_{\acute{e}t}(X_\infty, \mathcal{O}^*_{X_\infty}) = Pic(X_\infty)$, cf. chapter 1, and $H^2_{\acute{e}t}(X_\infty, \mathcal{O}^*_{X_\infty}) = 0$, which is a consequence of Tsen's theorem, cf. [66],

$$\begin{aligned} H^1_{\acute{e}t}(X_\infty, \mathbb{Q}_l(1)) &\simeq \left(\varprojlim ker\,(Pic(X_\infty) \stackrel{\times l^\nu}{\to} Pic(X_\infty)\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \\ H^2_{\acute{e}t}(X_\infty, \mathbb{Q}_l(1)) &\simeq \left(\varprojlim coker\,(Pic(X_\infty) \stackrel{\times l^\nu}{\to} Pic(X_\infty)\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\,. \end{aligned}$$

Now consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Pic^0(X_\infty) & \longrightarrow & Pic(X_\infty) & \stackrel{deg}{\longrightarrow} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow \times l^\nu & & \downarrow \times l^\nu & & \downarrow \times l^\nu & & \\ 0 & \longrightarrow & Pic^0(X_\infty) & \longrightarrow & Pic(X_\infty) & \stackrel{deg}{\longrightarrow} & \mathbb{Z} & \longrightarrow & 0\,, \end{array}$$

where $deg : Pic(X_\infty) \to \mathbb{Z}$ is the degree map. The group $Pic^0(X_\infty)$ of divisor classes of degree zero is isomorphic to the Jacobian variety of X_∞, which is an abelian variety of dimension g, cf. [56] and [67]. Hence the left vertical arrow in the above diagram is surjective and ${}_{l^\nu}Pic^0(X_\infty) \simeq (\mathbb{Z}/l^\nu\mathbb{Z})^{2g}$, cf. [56]. Finally the commutative diagram

$$\begin{array}{ccc} Pic(X_\infty) & \stackrel{deg}{\longrightarrow} & \mathbb{Z} \\ \downarrow \varphi^* & & \downarrow \times(deg\,\varphi^*)=\times q \\ Pic(X_\infty) & \stackrel{deg}{\longrightarrow} & \mathbb{Z} \end{array}$$

and the considerations above show that

$$\begin{aligned} H^1_{\acute{e}t}(X_\infty, \mathbb{Q}_l(1)) &\simeq (\varprojlim {}_{l^\nu}Pic^0(X_\infty)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \simeq \mathbb{Q}_l^{2g} \\ H^2_{\acute{e}t}(X_\infty, \mathbb{Q}_l(1)) &\simeq (\varprojlim {}_{l^\nu}\mathbb{Z}/l^\nu\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \simeq \mathbb{Q}_l \end{aligned}$$

as well as $P_{2,X}(T) = 1 - q \cdot T$. Putting all things together we obtain

$$Z_X(T) = \frac{\prod_{j=1}^{2g}(1-\alpha_i T)}{(1-T)(1-qT)},$$

where α_i, $1 \le i \le 2g$, are the eigenvalues of φ acting on $(\varprojlim_{l^\nu} Pic^0(X_\infty)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \simeq Hom_{\mathbb{Z}_l}(l\text{-}tor\ Pic^0(X_\infty), \mathbb{Q}_l/\mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$.

What is of interest for us at this point, is, that on the one hand we have the analytically defined zeta-function $Z_X(T)$ and on the other hand the algebraically defined characteristic polynomial of the Frobenius acting on a certain vector space. The connection between them is then given by the Lefschetz's Fixed Point Theorem, which implies the description of the zeta-function as a product of determinants, cf. above.

One of Iwasawa's contributions was to find an l-adic analogue for a number field E. For the zeta-function, we take the l-adic L-function attached to a certain character. At least this makes sense, if E is a totally real number field. Furthermore, for the Frobenius, we choose a suitable generator of $\Gamma := Gal(E(W_l)/E(\zeta_l))$, and the Jacobian is replaced by $A_\infty := \varinjlim A_n$ resp. $A_\infty^S := \varinjlim A_n^S$, where A_n resp. A_n^S is the Sylow-l-subgroup of the ideal resp. S-ideal class group of $E(\zeta_{l^n})$. But before we go into the details, we give a short introduction to $\Lambda := \mathbb{Z}_l[[\Gamma]]$-modules and their basic properties.

Remark 3.3 *Just recently C.Deninger, cf. [73], proved some astonishing theorems on the Riemann ζ-function $\zeta_{\mathbb{Q}}(s)$, which led him to a list of conjectures—*

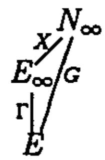
the so-called Deninger conjecture—, i.e., there should exist a global cohomology theory for $X = Spec(\mathbb{Z} \cup \{\infty\})$ *consisting of complex vector spaces* $H^j(X/\mathcal{L})$, $j = 0,1,2$ *of countable dimension together with an operator* θ *satisfying certain properties, including a Lefschetz's Fixed Point Theorem. As for étale cohomology, this would imply a description of* $\zeta_{\mathbb{Q}}(s)$ *as a product of determinants, where the determinant of an operator is defined by the regulized product over its eigenvalues. The significance of the Deninger conjecture can be seen just from the fact that the Riemann Hypothesis is a trivial consequence of it.*

Let l be a prime, and Γ a compact, abelian group isomorphic to the additive group $\mathbb{Z}_l$. For $n \geq 0$, let $\Gamma_n \subseteq \Gamma$ such that $\Gamma/\Gamma_n \simeq \mathbb{Z}/l^n\mathbb{Z}$, so $\Gamma_0 = \Gamma$. Then for a fixed topological generator $\gamma \in \Gamma$,

$$\begin{array}{rcl} \mathbb{Z}_l[[\Gamma]] := \varprojlim \mathbb{Z}_l[\Gamma/\Gamma_n] & \longrightarrow & \mathbb{Z}_l[[T]] \\ \gamma & \longmapsto & 1+T \end{array}$$

is an isomorphism, and $\Lambda := \mathbb{Z}_l[[T]]$ is a noetherian, regular, local domain of Krull dimension 2 with maximal ideal $(l,T) \subseteq \Lambda$. In the following we identify $\mathbb{Z}_l[[\Gamma]]$ with Λ via the above isomorphism. If $\mathfrak{p} \subseteq \Lambda$ is a prime ideal of height 1, then $\mathfrak{p} = (l)$ or $\mathfrak{p} = (P(T))$, where $P(T)$ is a distinguished, irreducible polynomial, cf. [95]. A typical example how Λ-modules arise is given by the following construction. Let E be a field and E_∞/E a $\mathbb{Z}_l$-extension, i.e., $Gal(E_\infty/E) \simeq \mathbb{Z}_l$, and N_∞/E_∞ an abelian pro-l-extension with Galois group

$X = Gal(N_\infty/E_\infty)$. Suppose that N_∞/E is a Galois extension with Galois group $G = Gal(N_\infty/E)$,



Then X admits a structure of a Λ-module by defining for $\gamma \in \Gamma$, $x \in X$,

$$\gamma(x) := \tilde{\gamma} \cdot x \cdot \tilde{\gamma}^{-1} ,$$

where $\tilde{\gamma} \in G$ is a lift of γ to N_∞. For the rest of this chapter, we make the following

Assumption : *All Λ-modules in question are finitely generated as Λ-modules.*

Proposition 3.4 *For a Λ-module X and a prime $\mathfrak{p}$ of height 1, let $X_\mathfrak{p} := X \otimes_\Lambda \Lambda_\mathfrak{p}$ be the localization of X with respect to $\mathfrak{p}$. Then.*

(i) *Let $x \in X$, $\beta \in \Lambda$, then $x \otimes 1/\beta = 0$ in $X_\mathfrak{p}$ if and only if there exists $\alpha \in \Lambda \backslash \mathfrak{p}$ such that $\alpha \cdot x = 0$ in X.*

(ii) *If X is not Λ-torsion, then $X_\mathfrak{p} \neq 0$ for all $\mathfrak{p}$.*

(iii) *If X is Λ-torsion, then $X_\mathfrak{p} = 0$ for almost all $\mathfrak{p}$.*

(iv) *X is finite if and only if $X_\mathfrak{p} = 0$ for all $\mathfrak{p}$.*

Proof: (i): cf. [6]. (ii): Trivial consequence of (i). (iii): Let $\{x_1, \ldots, x_n\} \subseteq X$ be a set of generators and $\mu_j \in \Lambda$, $1 \le j \le n$, such that $\mu_j \cdot x_j = 0$. Set $\mu := \prod \mu_j$, then $\mu \cdot X = 0$ and μ is contained in finitely many prime ideals. Thus by (i), $X_{\mathfrak{p}} = 0$ for almost all $\mathfrak{p}$. (iv): Let $\{x_1, \ldots, x_n\} \subseteq X$ be a minimal set of generators. If $n = 1$, then $X \simeq \Lambda/\mathcal{A}$, where $\mathcal{A} \subseteq \Lambda$ is the annihilator of $x_1 \in X$. If X is finite, then we must have $\mathcal{A} = (l^k, P(T)^m)$ for suitable $k, m \ge 1$, and so $X_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ by (i). The converse is trivial. Suppose now that $n > 1$, then

$$0 \longrightarrow < x_1 > \longrightarrow X \longrightarrow X/ < x_1 > \longrightarrow 0$$

is an exact sequence of Λ-modules. By induction and exactness of localization we are done. □

Definition 3.5 *Let $\varphi : X \to Y$ be a morphism of Λ-modules. Then φ is called a pseudo-isomorphism if $\varphi_{\mathfrak{p}} : X_{\mathfrak{p}} \to Y_{\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p}$. We write $X \sim Y$, if there exists a pseudo-isomorphism $\varphi : X \to Y$.*

Remark 3.6 *(i) By 3.4 $\varphi : X \to Y$ is a pseudo-isomorphism if and only if $\ker \varphi$ and $\operatorname{coker} \varphi$ are finite.*
(ii) $\sim$ is not an equivalence relation, e.g., $(l, T) \sim \Lambda$, but $\Lambda \not\sim (l, T)$. It can be shown, that $\sim$ is an equivalence relation on the category of Λ-torsion modules, cf. [38].

Λ-modules behave almost as nicely as modules over principal ideal domains, namely we have the following structural result, cf. [38].

Theorem 3.7 *Let X be a Λ-module. Then there exists a unique Λ-module $E(X) := \Lambda \oplus \bigoplus_{j=1}^{m} \Lambda/\mathfrak{p}_j^{e_j}$, where $e_j \geq 0$ for $0 \leq j \leq m$, and $\mathfrak{p}_j$ are prime ideals of height 1 in Λ, such that*

$$X \sim E(X) .$$

Modules of the form $E(X)$ are called elementary Λ-modules. For the most part, the pseudo-isomorphism $X \sim E(X)$ still gives us enough information on X, when we consider the elementary module $E(X)$ instead. For a classification of Λ-modules up-to isomorphism, cf. [44]. We denote the Λ-torsion module of X by $\Lambda\text{-}tor\, X = t_\Lambda X$ and the torsion free quotient module by $X/\Lambda\text{-}tor\, X = f_\Lambda X$. For X and $E(X)$ as in 3.7, we define the divisor of X by

$$div(X) := \sum_{j=1}^{m} e_j \cdot \mathfrak{p}_j ,$$

in particular, $div(X) = div(t_\Lambda X)$. Next we introduce two invariants associated to X, namely the so-called λ- and μ-invariants, which measure the complexity of X—rather $t_\Lambda X$. For a prime ideal $\mathfrak{p}$ of height 1 in Λ, we set

$$\lambda(\mathfrak{p}) := \begin{cases} 0 & \text{if } \mathfrak{p} = (l) \\ deg\, P(T) & \text{if } \mathfrak{p} = (P(T)) \end{cases}$$

$$\mu(\mathfrak{p}) := \begin{cases} 1 & \text{if } \mathfrak{p} = (l) \\ 0 & \text{if } \mathfrak{p} = (P(T)) \end{cases} ,$$

and extend this definition by linearity to the divisor group of Λ. Finally for a Λ-module X, we define

$$\lambda(X) := \lambda(div(X)) \text{ and } \mu(X) := \mu(div(X)) .$$

Furthermore, the characteristic ideal (polynomial) of X is defined by

$$char(X) := \prod_{j=1}^{m} \mathfrak{p}_j^{e_j} ,$$

where $\mathfrak{p}_j$ and e_j, $1 \leq j \leq m$ are given by 3.7. Certainly, $\lambda(X)$, $\mu(X)$ and $char(X)$ just depend on $t_\Lambda X$. For convenience (and simplicity), we make the following notation

$$\omega_n := \gamma^{l^n} - 1 = (1+T)^{l^n} - 1 \qquad \text{for } n \geq 0$$

$$\xi_0 := \omega_0 = T \text{ and } \xi_n := \omega_n/\omega_{n-1} \qquad \text{for } n \geq 1$$

$$\nu_{n,m} := \omega_m/\omega_n = \xi_{n+1} \cdot \ldots \cdot \xi_m \qquad \text{for } m \geq n \geq 0 .$$

Lemma 3.8 *Let X be a Λ-module and for $n \geq 0$, let $X_n := X/\omega_n X$. Then*

(i) $dim_{\mathbb{F}_l}\left(X_n/lX_n\right) < \infty$ *for all* $n \geq 0$.

(ii) *X is a Λ-torsion module if and only if $dim_{\mathbb{Q}_l}(X \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) < \infty$ if and only if $dim_{\mathbb{Q}_l}(X_n \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$ is bounded independently of $n \geq 0$.*

(iii) *X is a Λ-torsion module with $\mu(X) = 0$ if and only if $dim_{\mathbb{F}_l} X/lX < \infty$ if and only if $dim_{\mathbb{F}_l} X_n/lX_n$ is bounded independently of $n \geq 0$.*

Proof: Let $\varphi : X \to E(X)$ be the pseudo-isomorphism from 3.7, then the kernel and cokernel of $\varphi_n : X/\omega_n X \to E(X)/\omega_n E(X)$ are bounded independently of $n \geq 0$. Hence it is enough to prove the lemma for an elementary module, and then again we are reduced to the cases $E(X) = \Lambda$ and $E(X) = \Lambda/\mathfrak{p}^e$ with $\mathfrak{p} = (l)$ or $\mathfrak{p} = (P(T))$. The lemma follows now easily. □

Proposition 3.9 *Let X be a Λ-module, ana suppose that for all $n \geq 0$, $X/\omega_n X$ is finite. Then there exist integers n_0 and $\nu(X)$ such that for all $n \geq n_0$,*

$$\#X/\omega_n X = \mu(X) \cdot l^n + \lambda(X) \cdot n + \nu(X) .$$

Proof: Since $X/\omega_n X$ is finite for all $n \geq 0$, X is a Λ-torsion module by 3.8, and so there exists an elementary Λ-torsion module $E = E(X)$ such that $E \sim X$, cf. 3.6. Since an elementary Λ-module does not contain any non-trivial finite Λ-submodule, we get the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & E & \to & X & \to & A & \to 0 \\ & \downarrow \omega_n & & \downarrow \omega_n & & \downarrow \omega_n & \\ 0 \to & E & \to & X & \to & A & \to 0 , \end{array}$$

where A is a finite Λ-module. The finiteness of A implies $ker\, \omega_{n,A} = A$ and $coker\, \omega_{n,A} = A$ for $n \geq 0$ large enough. Hence

$$\#X/\omega_n X = l^c \cdot \#E/\omega_n E \; , n \gg 0,$$

with some constant $c \geq 0$. Thus it is enough to prove the formula for an elementary Λ-module E. This can be done directly, for details cf. [57] or [95]. □

Lemma 3.10 *Let X be a Λ-module, then*

$$ker\left(X/\omega_n X \xrightarrow{\times\nu_{n,m}} X/\omega_m X\right)$$

is bounded independently of $m \geq n \geq 0$, and therefore,

$$\varinjlim ker\,(X/\omega_n X \longrightarrow X/\omega_m X)$$

is finite.

Proof: Let $\varphi : X \to E = E(X)$ be the pseudo-isomorphism of 3.7, then we obtain a commutative diagram

$$\begin{array}{ccc} X/\omega_n X & \xrightarrow{\times\nu_{n,m}} & X/\omega_m X \\ \downarrow & & \downarrow \\ E/\omega_n E & \xrightarrow{\times\nu_{n,m}} & E/\omega_m E\,, \end{array}$$

and there is an integer $N \geq 0$, such that for all $n \geq N$,

$$E/\omega_n E \xrightarrow{\times\nu_{n,m}} E/\omega_m E$$

is injective. Thus for $m \geq n \geq N$,

$$ker(X/\omega_n X \to X/\omega_m X) \subseteq ker(X/\omega_n X \to E/\omega_n E)\,.$$

But $\#ker(X/\omega_n \to E/\omega_n E) \leq max(\#ker\,\varphi, \#coker\,\varphi)^2$. □

Remark 3.11 *We might as well replace* ω_n *by* $\nu_{e,n}$ *in 3.9 and 3.10, where* $e \geq 0$ *is a fixed integer.*

Now suppose that X is a Λ-torsion module, then by 3.4, $X_{\mathfrak{p}} \neq 0$ if and only if $\mathfrak{p}$ divides $div(X)$, and we define

$$\begin{aligned} \psi_X : X &\longrightarrow \bigoplus_{\mathfrak{p}} X_{\mathfrak{p}} \,. \\ x &\longmapsto (x \otimes 1)_{\mathfrak{p}} \end{aligned}$$

Lemma 3.12 *Let* X *be a* Λ*-torsion module, then* $ker\,\psi_X$ *is the maximal finite* Λ*-submodule of* X.

Proof: Let Y be a Λ-submodule of X, and $\iota : Y \hookrightarrow X$ the canonical embedding, then

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & X \\ \downarrow \psi_Y & & \downarrow \psi_X \\ \bigoplus_{\mathfrak{p}} Y_{\mathfrak{p}} & \xrightarrow{(\iota_{\mathfrak{p}})} & \bigoplus_{\mathfrak{p}} X_{\mathfrak{p}} \end{array}$$

is a commutative diagram. Thus, $Y \subseteq ker\,\psi_X$ if and only if $\psi_X \circ \iota$ is trivial if and only if $(\iota_{\mathfrak{p}}) \circ \psi_Y$ is trivial. If Y is finite, then by 3.4, $Y_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$, thus $Y \subseteq ker\,\psi_X$. Conversely, if Y is not finite, then $Y_{\mathfrak{p}} \neq 0$ for at least one $\mathfrak{p}$. Therefore, ψ_Y is not trivial, and by exactness of localization $\iota_{\mathfrak{p}} : Y_{\mathfrak{p}} \to X_{\mathfrak{p}}$ is not trivial either, so $Y \not\subseteq ker\,\psi_X$. □

Definition 3.13 *Let X be a Λ-torsion module, then*

$$\beta(X) := coker\,\psi_X$$

is called the co-adjoint and

$$\alpha(X) := Hom_{\mathbb{Z}_l}(\beta(X), \mathbb{Q}_l/\mathbb{Z}_l)$$

the adjoint of X. They are Λ-modules in a natural way.

Remark 3.14 *(i) If X is finite, then $\beta(X) = \alpha(X) = 0$. (ii) β is a right and pseudo-left exact functor on the category of Λ-torsion modules, and obviously dual statements for α.*

We can give a more precise description of the co-adjoint and adjoint of a Λ-torsion module X. We call a sequence of elements $\{\sigma_n\} \subseteq \Lambda$ admissible, if

1. $\sigma_0 \in (l, T)\Lambda$.
3. $\sigma_n \neq 0$ for all $n \geq 0$.
2. $\sigma_{n+1} \in \sigma_n \cdot (l, T)\Lambda$.

For example, $\{\omega_n\}$, $\{\xi_{n+1}\}$ and $\{\nu_{n,m}\}$ are admissible sequences. If X is a Λ-module, then $\{\sigma_n\} \subseteq \Lambda$ is called X-admissible, if $\{\sigma_n\}$ is admissible and for all $n \geq 0$, σ_n and $div(X)$ are relatively prime. It is plain from the definition and 3.7, that given a Λ-module X, there always exists an X-admissible sequence $\{\sigma_n\}$.

Theorem 3.15 *Let X be a Λ-torsion module, and $\{\sigma_n\}$ an X-admissible sequence. Then there are non-canonical isomorphisms*

$$\beta(X) \simeq \varinjlim X/\sigma_n X \text{ and } \alpha(X) \simeq Hom_{\mathbb{Z}_l}(\varinjlim X/\sigma_n X, \mathbb{Q}_l/\mathbb{Z}_l)\,.$$

Proof: Let

$$\begin{array}{rccc} \phi_X: & X \otimes_\Lambda \varinjlim \frac{1}{\sigma_n}\Lambda & \longrightarrow & \bigoplus_{\mathfrak{p}} X_{\mathfrak{p}} \\ & x \otimes \frac{1}{\sigma_n} & \longmapsto & (x \otimes \frac{1}{\sigma_n}) \end{array}$$

be the diagonal embedding. Then following Federer, cf. [22], ϕ_X is a Λ-module isomorphism. Consider

$$\begin{array}{ccc} X \otimes_\Lambda \varinjlim \frac{1}{\sigma_n}\Lambda & \longrightarrow & X \otimes_\Lambda \varinjlim \frac{1}{\sigma_n}\Lambda/\Lambda \\ \downarrow\wr & & \downarrow\wr \\ \bigoplus_{\mathfrak{p}} X_{\mathfrak{p}} & \longrightarrow & \beta(X)\,, \end{array}$$

where the vertical arrows are given by ϕ_X. The assertion follows now from the isomorphism $\varinjlim X/\sigma_n X \simeq X \otimes_\Lambda \varinjlim \frac{1}{\sigma_n}\Lambda/\Lambda$. □

Remark 3.16 *Let $f : X \to Y$ be a morphism of Λ-torsion modules, and suppose that $\{\sigma_n\}$ is an X-admissible as well as Y-admissible sequence. Then*

$$\begin{array}{ccc} \varinjlim X/\sigma_n X & \to & \varinjlim Y/\sigma_n Y \\ \downarrow\wr & & \downarrow\wr \\ \beta(X) & \to & \beta(Y) \end{array}$$

is commutative.

For a Λ-module X, we define a new module X^{-1} by $X^{-1} = X$ (equality as sets) and inverted Γ-action. The next theorem is due to Iwasawa, for a proof cf. [22].

Theorem 3.17 *Let E be an elementary Λ-torsion module, then*

$$\alpha(E) \simeq E^{-1} .$$

Corollary 3.18 *Let X be a Λ-torsion module.*

(i) *$\alpha(X)$ does not contain any non-trivial finite Λ-submodule.*

(ii) $\alpha(X) \sim X^{-1}$.

Proof: By assumption there are an elementary Λ-torsion module $E = E(X)$ and a finite Λ-module A, such that

$$0 \longrightarrow E \longrightarrow X \longrightarrow A \longrightarrow 0$$

is an exact sequence of Λ-modules. Taking adjoints and applying 3.14 yields

$$0 \longrightarrow \alpha(X) \longrightarrow \alpha(E)$$

with finite cokernel. By 3.15 and the fact that an elementary Λ-module does not contain any non-trivial finite Λ-submodule, we are done. □

So far it was essential that the Λ-module X is a Λ-torsion module in order to get relevant information on X resp. $X/\omega_n X$. In certain cases we can do even better. Namely, let X be a Λ-module and $n, a \geq 0$ integers, then we define $i(n, a; X) \geq 0$ by

$$\# X/(\omega_n X + l^a X) = l^{i(n,a;X)} .$$

The invariant $i(n, a; X)$ is certainly additive in X, and if $X \sim Y$, then $i(n, a; X) - i(n, a; Y)$ is bounded independently of $n, a \geq 0$. In particular, $i(n, a; X)$ is finite, since this is true for an elementary Λ-module. For a proof of the next proposition, cf. [39].

Proposition 3.19 *Let X be a Λ-module, and $E(X) = \Lambda^{e_0} \oplus \bigoplus_{j=1}^{m} \Lambda/\mathfrak{p}_j^{e_j}$ be the elementary Λ-module associated to X, cf. ref3.7. We set $d := \sum_j deg\, \xi_{n_j}$, where the summation runs over all those j, such that $\mathfrak{p}_j = (\xi_{n_j})$ for some integer $n_j \geq 0$. Then there exist an integer n_0 and an integer-valued function $a(n)$, such that*

$$i(n, a; X) - ((e_0 l^n + d)a + \mu l^n + (\lambda - d)n)$$

is bounded for $n \geq n_0$ and $a \geq a(n)$. Here, $\lambda = \lambda(X)$ and $\mu = \mu(X)$ are the λ- and μ-invariants of X.

At the first glance the above proposition does not seem to give any valuable results on X. But if we can calculate $i(n, a; X)$ by a different method—and that is what we will actually do for certain Λ-modules, cf. chapter 4,—then we get the information on e_0, μ and λ back. Let us add a simple, but quite useful lemma at the end of this chapter.

Lemma 3.20 *Let X be a Λ-module, and $n \geq 0$.*

(i) If $X_{\Gamma_n} = X/\omega_n X$ is finite, then X^{Γ_n} is finite.

(ii) If X is in addition a Λ-torsion module, then X_{Γ_n} is finite if and only if X^{Γ_n} is finite.

Proof: Let $\varphi : X \to E = E(X)$ be the pseudo-isomorphism from 3.7, then

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & ker\,\varphi & \longrightarrow & X & \longrightarrow & E & \longrightarrow & coker\,\varphi & \longrightarrow & 0 \\ & & \downarrow \omega_n & & \downarrow \omega_n & & \downarrow \omega_n & & \downarrow \omega_n & & \\ 0 & \longrightarrow & ker\,\varphi & \longrightarrow & X & \longrightarrow & E & \longrightarrow & coker\,\varphi & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows. Since $ker\,\varphi$ and $coker\,\varphi$ are finite, it is enough to show the assertion for an elementary Λ-torsion module $E = E(X)$, and hence we can assume that $E = \Lambda/(l)$ or $E = \Lambda/(P(T))$, for which the lemma is obvious. $\square$

4 Iwasawa theory of number fields

In this chapter we study Λ-modules over number fields, and by using Kummer theory and basic class field theory, we can make precise statements on certain Λ-modules. Even if most results are still valid for the prime $l = 2$, we assume for simplicity, that l is an odd prime.

Let E be a number field, l an odd prime and E_∞/E a $\mathbf{Z}_l$-extension with intermediate fields E_n, $n \geq 0$, and $E_0 = E$. We set $\Gamma := Gal(E_\infty/E) \simeq \mathbf{Z}_l$, and for $n \geq 0$, $\Gamma_n := Gal(E_\infty/E_n)$, so $\Gamma_0 = \Gamma$. For example, let E_∞/E be the cyclotomic $\mathbf{Z}_l$-extension, i.e., E_∞ is the unique subfield of $E(W_l)$ such that $Gal(E_\infty/E) \simeq \mathbf{Z}_l$. After choosing a topological generator $\gamma \in \Gamma$, we identify $\mathbf{Z}_l[[\Gamma]]$ with $\mathbf{Z}_l[[T]]$ via

$$\begin{array}{ccc} \mathbf{Z}_l[[\Gamma]] & \longrightarrow & \mathbf{Z}_l[[T]] \,, \\ \gamma & \longmapsto & 1+T \end{array}$$

cf. page 80. If X is a pro-l-group which admits a Γ-module structure, then X becomes a Λ-module via the above isomorphism. Let N_∞/E_∞ be an abelian pro-l-extension, and assume that N_∞/E is a Galois extension, then $X := Gal(N_\infty/E_\infty)$ is a Λ-module as defined on page 81. For $n \geq 0$, let N_n be the maximal abelian extension of E_n contained in N_∞, then $X/\omega_n X = Gal(N_n/E_\infty)$. By Nakayama's lemma, X is a finitely generated Λ-module if and only if $X/\omega_0 X$ is a finitely generated $\mathbf{Z}_l$-module, cf. [13].

Proposition 4.1 *Let E_∞/E be a $\mathbb{Z}_l$-extension of the number field E, and M_∞/E_∞ the maximal abelian pro-l-extension of E_∞, which is unramified outside l. Then $\mathcal{X} = Gal(M_\infty/E_\infty)$ is a finitely generated Λ-module, and hence by 3.7*

$$\mathcal{X} \sim \Lambda^{e_0} \oplus \bigoplus_{j=1}^{m} \Lambda/\mathfrak{p}_j^{e_j} ,$$

where $\mathfrak{p}_j \subseteq \Lambda$ are prime ideals of height 1 and e_j are integers.

Proof: From the maximality of M_∞, it follows at once that M_∞/E is a Galois extension, and so $\mathcal{X}$ is a Λ-module. If M_0 is the maximal abelian extension of $E_0 = E$ contained in M_∞, we have to show that $\mathcal{X}/\omega_0\mathcal{X} = Gal(M_0/E_\infty)$ is a finitely generated $\mathbb{Z}_l$-module. Let K be the compositum of all $\mathbb{Z}_l$-extensions of E, then by class field theory $[M_0 : K] < \infty$ and $Gal(K/E) \simeq \mathbb{Z}_l^{1+r_2(E)+\delta_E^{Leop}}$ with $1 + r_2(E) + \delta_E^{Leop} \leq [E : \mathbb{Q}]$, cf. 1.26. □

For a $\mathbb{Z}_l$-extension E_∞/E, we fix a couple of notations. Let $n_0 = n_0(E_\infty/E)$ be the smallest integer n such that all primes, which ramify in the $\mathbb{Z}_l$-extension, are totally ramified in E_∞/E_n, and $r = r(E_\infty/E)$ the number of ramified primes. Clearly, if $\mathfrak{p}$ ramifies in E_∞/E, then $\mathfrak{p}$ is an l-adic prime and there is at least one ramified prime. It is not necessarily true that all l-adic primes ramify, but this is the case for the cyclotomic $\mathbb{Z}_l$-extension E_∞/E. Let O_E, U_E and A_E be the ring of integers, the units and the Sylow-l-subgroup of the ideal class group $Cl(E)$ of E. For the fields E_n, we simply write O_n, U_n and

A_n instead of O_{E_n}, U_{E_n} and A_{E_n}. Furthermore, if S is a finite set of primes containing all infinite and l-adic primes, then the corresponding objects are written with a superscript, so O_n^S denotes the ring of S-integers in E_n and so on.

Proposition 4.2 *Let E_∞/E be a $\mathbb{Z}_l$-extension, and L_∞/E_∞ the maximal abelian, unramified, pro-l-extension of E_∞. Then $X_\infty := Gal(L_\infty/E_\infty)$ is a Λ-torsion module.*

Proof: Since X_∞ is a quotient module of $\mathcal{X}$, it is a finitely generated Λ-module. Let L_n be the maximal abelian extension of E_n contained in L_∞, so $Gal(L_n/E_\infty) = X_\infty/\omega_n X_\infty$ and we have to show that $dim_{\mathbb{Q}_l}(X_\infty/\omega_n X_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$ is bounded independently of $n \geq 0$, cf. 3.8. We denote the subfield of the Hilbert class field of E_n belonging to A_n by H_n, i.e., $Gal(H_n/E_n) \simeq A_n$ under the reciprocity map. Then for $n \geq n_0$, $Gal(L_n/H_n) \simeq I_{\mathfrak{p}_1} \cdot \ldots \cdot I_{\mathfrak{p}_r}$, where $I_{\mathfrak{p}_j} \simeq \mathbb{Z}_l$ is the inertia group of a ramified prime $\mathfrak{p}_j$. Since $[H_n : E_n] = \# A_n < \infty$, we get

$$dim_{\mathbb{Q}_l}(X_\infty/\omega_n X_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \leq r - 1 \ .$$

□

Remark 4.3 *Let L_∞^S be the maximal abelian, unramified pro-l-extension of E_∞, in which all l-adic primes of E_∞ are completely decomposed. Then along the same lines as 4.2, $X_\infty^S := Gal(L_\infty^S/E_\infty)$ is a Λ-torsion module.*

Corollary 4.4 *Suppose that E contains a primitive l-th root of unity ζ_l, and let E_∞/E be the cyclotomic $\mathbb{Z}_l$-extension of E. Then for the constant e_0 in 4.1, we have $e_0 = r_2(E)$.*

Proof: Let δ_n^{Leop} be the Leopoldt defect for the field E_n. With the notation of 4.1 we have

$$Gal(M_n/E_\infty) = \mathcal{X}/\omega_n\mathcal{X} \simeq \mathbb{Z}_l^{r_2(E)\cdot l^n+\delta_n^{Leop}} \oplus B_n ,$$

where B_n is a finite group for $n \geq 0$. Since X is a Λ-torsion module by 4.2, one can show using Kummer theory that $\delta_n^{Leop} \leq \lambda(X)$, cf. [29], i.e., δ_n^{Leop} is bounded independently of n, and the corollary follows immediately. □

We reprove 4.4 below and give also a description of the Λ-torsion submodule of a certain quotient module of $\mathcal{X}$, cf. 4.14. One of Iwasawa's first results is a formula on the growth of the class number in $\mathbb{Z}_l$-extension, cf. [38].

Theorem 4.5 *Let E_∞/E be a $\mathbb{Z}_l$-extension, and A_n the Sylow-l-subgroup of the ideal class group of E_n. Then there exist integers $\lambda = \lambda(E) \geq 0$, $\mu = \mu(E) \geq 0$, $\nu = \nu(E) \geq 0$ and $N \geq 0$ such that for all $n \geq N$,*

$$\#A_n = l^{\mu\cdot l^n+\lambda\cdot n+\nu} .$$

Proof: Recall the notation in the proof of 4.2, and let $Y_\infty := Gal(L_\infty/E_\infty H_{n_0})$. Then for $n \geq n_0$,

$$A_n \simeq Gal(H_n/E_n) \simeq Gal(E_\infty H_n/E_\infty) \simeq X_\infty/\nu_{n_0,n}Y_\infty ,$$

and since $\#X_\infty/\nu_{n_0,n}Y_\infty = \#X_\infty/Y_\infty \cdot \#Y_\infty/\nu_{n_0,n}Y_\infty$, the formula follows from 3.9. □

Remark 4.6 *(i) Of course, there is a similar result, if we consider S-ideal class groups. (ii) If E_∞/E is the cyclotomic $\mathbb{Z}_l$-extension, then it is conjectured that $\mu = 0$, cf. [42]. This is known to be true, if E is an abelian number field, cf. [24]. Iwasawa has also shown that the conjecture is not valid in an arbitrary $\mathbb{Z}_l$-extension E_∞/E, cf. [42] or [57]. (iii) If E_∞/E is the cyclotomic $\mathbb{Z}_l$-extension of a totally real number field E, then not a single example of a field E is known such that $\lambda > 0$. Following Greenberg, cf. [29], it is conjectured that $\lambda = 0$. Unfortunately, not much is known in general about this conjecture. It has been shown for just a few specific examples. (iv) If E_∞/E is the cyclotomic $\mathbb{Z}_l$-extension, then for $p \neq l$, the order of the Sylow-p-subgroup of the ideal class group $Cl(E_n)$ of E_n is bounded independently of $n \geq 0$, cf. [94].*

Proposition 4.7 *Let E_∞/E be a $\mathbb{Z}_l$-extension, and set $A_\infty := \varinjlim A_n$ resp. $A_\infty^S := \varinjlim A_n^S$. Then the orders of the groups*

$$ker(A_n \to A_m) \quad and \quad ker(A_n \to A_\infty)$$

resp.

$$ker(A_n^S \to A_m^S) \quad and \quad ker(A_n^S \to A_\infty^S)$$

are bounded independently of $m \geq n \geq 0$.

Proof: It is enough to show that for $m \geq n \geq n_0 = n_0(E_\infty/E)$, the order of $ker(A_n \to A_m)$ is bounded independently of m and n. The proof for the S-ideal class groups follows then along the same lines. With the notation of 4.5 we have for $m \geq n \geq n_0$

$$\begin{aligned} \#ker(A_n \to A_m) &= \#ker(X_\infty/\nu_{n_0,n}Y_\infty \xrightarrow{\nu_{n,m}} X_\infty/\nu_{n_0,m}Y_\infty) \\ &\leq \#X_\infty/Y_\infty \cdot \#ker(Y_\infty/\nu_{n_0,n}Y_\infty \xrightarrow{\nu_{n,m}} Y_\infty/\nu_{n_0,m}Y_\infty)\,, \end{aligned}$$

and we deduce the proposition from 3.10. □

Corollary 4.8 *Let* E_∞/E *be a* $\mathbb{Z}_l$*-extension, and* U_n^S *resp.* U_∞^S *the group of* S*-units in* E_n *resp.* E_∞*. Then the orders of the groups*

$$H^1(\Gamma_n/\Gamma_m, U_m^S) \text{ and } H^1(\Gamma_n, U_\infty^S)$$

are bounded independently of $m \geq n \geq 0$.

Proof: This is immediate from 4.7 and the well-known isomorphisms $ker(A_n^S \to A_m^S) \simeq H^1(\Gamma_n/\Gamma_m, U_m^S)$ and $ker(A_n^S \to A_\infty^S) \simeq H^1(\Gamma_n, U_\infty^S)$. □

It is natural for us to consider $Hom_{\mathbb{Z}_l}(A_\infty, \mathbb{Q}_l/\mathbb{Z}_l)$ resp. $Hom_{\mathbb{Z}_l}(A_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l)$—remember what we said on page 79—and we can make now the following statement

Proposition 4.9 *Let E_∞/E be a $\mathbb{Z}_l$-extension, and X_∞ resp. Y_∞ as in the proof of 4.2. Then*

$$Hom_{\mathbb{Z}_l}(A_\infty, \mathbb{Q}_l/\mathbb{Z}_l) \simeq \alpha(Y_\infty) \sim X_\infty{}^{-1} ,$$

and with similar notations for A_∞^S, cf. 4.3,

$$Hom_{\mathbb{Z}_l}(A_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l) \simeq \alpha(Y_\infty^S) \sim X_\infty^{S\,-1} .$$

In particular, $Hom_{\mathbb{Z}_l}(A_\infty, \mathbb{Q}_l/\mathbb{Z}_l)$ and $Hom_{\mathbb{Z}_l}(A_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l)$ do not contain any non-trivial finite Λ-submodules.

Proof: As usual we just prove the assertion for $Hom_{\mathbb{Z}_l}(A_\infty, \mathbb{Q}_l/\mathbb{Z}_l)$. With the notation from above

$$A_\infty \simeq \varinjlim X_\infty/\nu_{n_0,n} Y_\infty .$$

The finiteness of X_∞/Y_∞ implies $\varinjlim X_\infty/Y_\infty = 0$, and hence

$$A_\infty \simeq \varinjlim Y_\infty/\nu_{n_0,n} Y_\infty .$$

We set $\sigma_n := \nu_{n_0,n_0+n}$. Since $Y_\infty/\sigma_n Y_\infty$ is finite, $\{\sigma_n\}_{n\geq 0}$ is disjoint from $div(Y_\infty)$, and hence $\{\sigma_n\}_{n\geq 0}$ is an Y_∞-admissible sequence. By 3.15 we get

$$Hom_{\mathbb{Z}_l}(A_\infty, \mathbb{Q}_l/\mathbb{Z}_l) \simeq \alpha(Y_\infty) ,$$

and again since X_∞/Y_∞ is finite, we obtain

$$\alpha(Y_\infty) \sim X_\infty{}^{-1} .$$

The last statement is obvious, since the adjoint $\alpha(Y_\infty)$ does not contain any non-trivial finite Λ-submodule, cf. 3.18. □

So far we considered arbitrary $\mathbb{Z}_l$-extension, and therefore the results are of a quite general type. One might hope that the situation gets better, if we consider the cyclotomic $\mathbb{Z}_l$-extension E_∞/E, which indeed is the case. Furthermore, we want to apply Kummer theory, which means that we have to have enough roots of unity contained in our base field. Hence we just adjoin them and take suitable eigenspaces. To be precise, let $F := E(\zeta_l)$ and F_∞/F the cyclotomic $\mathbb{Z}_l$-extension of F with intermediate fields F_n. We set $G_\infty := Gal(F_\infty/E) \simeq \Gamma \times \Delta$, where $\Gamma := Gal(F_\infty/F)$ and $\Delta := Gal(F/E)$ is cyclic of order $d := \#\Delta$ dividing $l-1$. Further we define $G_n := Gal(F_n/E)$ and $\Gamma_n := Gal(F_\infty/F_n)$. The action of G_∞ on $W_l = \mathbb{Q}_l/\mathbb{Z}_l(1)$ induces the so-called cyclotomic character and its restriction to Γ and Δ,

$$\begin{array}{llll} \chi : & G_\infty & \longrightarrow & \mathbb{Z}_l^* \\ \kappa : & \Gamma & \longrightarrow & U_l^{(1)} \subseteq \mathbb{Z}_l^* \\ \omega : & \Delta & \longrightarrow & \mu_{l-1} \subseteq \mathbb{Z}_l^* . \end{array}$$

Next we fix a topological generator of Γ in the following way. Let $l^k = \#W_l(F)$, then set

$$u := exp(l^k) = \sum_{n\geq 0} \frac{(l^k)^n}{n!} \in U_l^{(1)} ,$$

and let $\gamma_0 \in \Gamma$ be the preimage of u under κ. Whenever we make a Γ-module, which is also a $\mathbb{Z}_l$-module, into Λ-module, we do it by means of this fixed generator $\gamma_0 \in \Gamma$. It should be pointed out here, that even if many results on Γ- resp. Λ-modules depend on the choice of a topological generator $\gamma \in \Gamma$, any other generator than γ_0 would work equally well.

Now the orthogonal idempotents in $\mathbb{Z}_l[\Delta]$ are defined by

$$\varepsilon_j := \frac{1}{d} \sum_{\tau \in \Delta} \omega^j(\tau^{-1})\tau$$

and satisfy

1. $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}\varepsilon_j$.
2. $\sum_{j=1}^{d} \varepsilon_j = 1$.
3. $\varepsilon_j \cdot \rho = \omega^j(\rho)\varepsilon_j \, , \rho \in \Delta$.

The following lemma is obvious from the definition and the properties (1)-(3).

Lemma 4.10 *Let M be a $\mathbb{Z}_l$-module with continuous G_∞-action. For $j \in \mathbb{Z}$, let $M^{[j]} := M_{\omega^j} := \varepsilon_j M$. Then*

(i) M posseses a decomposition into eigenspaces with eigenvalues $\omega^j(\rho)$ corresponding to the action of $\rho \in \Delta$, i.e., $M = \bigoplus_{j=1}^{d} M^{[j]}$.

(ii) The decomposition of M into eigenspaces is compatible with the Γ-action, i.e., for $\gamma \in \Gamma$, $\gamma(M^{[j]}) = M^{[j]}$.

(iii) There are canonical G_∞-module isomorphisms $M(i)^\Delta \simeq M(i)_\Delta \simeq (\varepsilon_{-i}M)(i)$ and $\varepsilon_j M(i) \simeq (\varepsilon_{j-i}M)(i)$.

Since F_∞ contains all l-th power roots of unity, Kummer theory gives a pairing

$$<\cdot,\cdot>: Gal(K_\infty/F_\infty) \times F_\infty^\star \otimes \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow W_l = \mathbb{Q}_l/\mathbb{Z}_l(1)\ ,$$

where K_∞ is the maximal abelian, pro-l-extension of F_∞. By pairing we always understand a non-degenerate pairing. For $g \in G_\infty$,

$$< g(x), g(\alpha) >= g(< x, \alpha >) =< x, \alpha >^{\chi(g)}\ ,$$

and thus

$$Gal(K_\infty/F_\infty) \xrightarrow{\sim} Hom_{\mathbb{Z}_l}(F_\infty^\star \otimes \mathbb{Q}_l/\mathbb{Z}_l, W_l)$$

is a G_∞-module isomorphism. Regarding the decomposition into eigenspaces, we obtain for $i + j \equiv 1\ mod\ d$ the pairing

$$<\cdot,\cdot>: Gal(K_\infty/F_\infty)^{[i]} \times (F_\infty^\star \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{[j]} \longrightarrow W_l = \mathbb{Q}_l/\mathbb{Z}_l(1)\ .$$

Let M_∞ be the maximal abelian, pro-l-extension of F_∞, which is unramified outside l, and $\mathcal{X} := Gal(M_\infty/F_\infty)$. Then the orthogonal complement $\mathcal{M} := Gal(K_\infty/M_\infty)^\perp$ is given by

$$\mathcal{M} = \Delta_\infty^{(\infty)} = \varinjlim \varinjlim \Delta_n^{(\nu)}\ ,$$

cf. [41] or [60]. Since étale cohomology commutes with direct limits, we obtain from Kummer theory, cf. chapter 1, $\mathcal{M} \simeq H^1_{\acute{e}t}(O_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l(1))$. Furthermore,

$$<\cdot,\cdot>: \mathcal{X} \times \mathcal{M} \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l(1)$$

is again a pairing of G_∞-modules. Twisting with -1 induces the pairing

$$<\cdot,\cdot>: \mathcal{X}(-1) \times \mathcal{M} \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l$$

with the property $< g(x), g(\alpha) >= g(< x, \alpha >) =< x, \alpha >$, $g \in G_\infty$. Thus $\mathcal{X}(-1)$ and $\mathcal{M}$ are dual to each other in the sense of Pontryagin, so e.g., $\mathcal{X}(-1) \simeq Hom_{\mathbb{Z}_l}(\mathcal{M}, \mathbb{Q}_l/\mathbb{Z}_l)$ is an isomorphism of G_∞-modules.

Remark 4.11 *Instead of the above isomorphism we often find in the literature the •-construction and a corresponding pairing*

$$\ll \cdot,\cdot \gg: \dot{\mathcal{X}} \times \mathcal{M} \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l\,.$$

The Γ*-module structure of* $\dot{\mathcal{X}} = \mathcal{X}$ *(equality as sets) is defined by* $\gamma \cdot x := \kappa(\gamma)\gamma^{-1}(x)$, $\gamma \in \Gamma$, $x \in \mathcal{X}$. *So* $\dot{\mathcal{X}}$ *is the* (-1)*-twist* $\mathcal{X}(-1)$ *with inverse* Γ*-action, and in particular* $\ddot{\mathcal{X}} = \mathcal{X}$ *as* Γ*-modules. After fixing an isomorphism* $W_l \simeq \mathbb{Q}_l/\mathbb{Z}_l$*, the above pairing satisfies* $\ll \gamma \cdot x, \alpha \gg = \ll x, \gamma(\alpha) \gg$. *Contrary to the usual* Γ*-module structure on* $Hom_{\mathbb{Z}_l}(\cdot, \mathbb{Q}_l/\mathbb{Z}_l)$*, we define for* $f \in \dot{Hom}_{\mathbb{Z}_l}(\mathcal{M}, \mathbb{Q}_l/\mathbb{Z}_l) = Hom_{\mathbb{Z}_l}(\mathcal{M}, \mathbb{Q}_l/\mathbb{Z}_l)$ *(again equality as sets) and* $\gamma \in \Gamma$, $(\gamma \cdot f)(\alpha) := f(\gamma(\alpha))$*, which is just the canonical* Γ*-module structure on* $Hom_{\mathbb{Z}_l}(\mathcal{M}, \mathbb{Q}_l/\mathbb{Z}_l)$ *inverted. With these notations we get a* Γ*-module isomorphism*

$$\dot{\mathcal{X}} \simeq Hom_{\mathbb{Z}_l}(\mathcal{M}, \mathbb{Q}_l/\mathbb{Z}_l)\,.$$

In terms of Λ-modules, this leads to the following construction. Let $\dot{T} \in (l, T) \subseteq \Lambda$ be the power series such that

$$(1+T)(1+\dot{T}) = u = \kappa(\gamma_0)\ ,$$

where γ_0 is the fixed topological generator of Γ. Then $\bullet : \Lambda \to \Lambda$ defined by $\xi = \xi(T) \mapsto \dot{\xi} = \xi(\dot{T})$ is an involution of Λ over $\mathbb{Z}_l$, and we define for $\xi \in \Lambda$, $x \in \mathcal{X}$, $\xi \cdot x := \dot{\xi}x$. One sees immediately that both constructions are compatible with our fixed isomorphism $\mathbb{Z}_l[[\Gamma]] \xrightarrow{\sim} \Lambda$.

As metioned above we reprove 4.4 and give a more precise statement. Let $N_\infty^S := F_\infty(\sqrt[l^\infty]{U_\infty^S})$ be the field generated over F_∞ by all l^ν-th roots of S-units U_∞^S, then $N_\infty^S \subseteq M_\infty$ and $Gal(M_\infty/N_\infty^S)$ is the orthogonal complement of $(U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l)$. Thus, if $\mathcal{Y}^S := Gal(N_\infty^S/F_\infty)$, then we obtain the pairing

$$< \cdot, \cdot >: \mathcal{Y}^S(-1) \times U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l\ .$$

For simplicity, we set $Z := \mathcal{Y}^S(-1)$, and for $n \geq 0$, let $Z_n := (U_n^S \otimes \mathbb{Q}_l/\mathbb{Z}_l)^\perp$ be the orthogonal complement of $U_n^S \otimes \mathbb{Q}_l/\mathbb{Z}_l$. Since $U_n^S \otimes \mathbb{Q}_l/\mathbb{Z}_l \subseteq (U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{\Gamma_n}$, we have $\omega_n Z \subseteq Z_n \subseteq Z$ and hence

$$\begin{aligned} Z/Z_n &\simeq Hom_{\mathbb{Z}_l}(U_n^S \otimes \mathbb{Q}_l/\mathbb{Z}_l, \mathbb{Q}_l/\mathbb{Z}_l) \\ Z_n/\omega_n Z &\simeq Hom_{\mathbb{Z}_l}((U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{\Gamma_n}/U_n^S \otimes \mathbb{Q}_l/\mathbb{Z}_l, \mathbb{Q}_l/\mathbb{Z}_l)\ . \end{aligned}$$

Let s_n be the number of l-adic primes in F_n. Since F_∞ is the cyclotomic $\mathbb{Z}_l$-extension of F, all l-adic primes eventually ramify, i.e., $s_n = s_{n_0}$ for $n \geq n_0$, and we denote the number of l-adic primes in F_∞ by $s = s(F_\infty/F)$. For $n \geq n_0$, we deduce from Dirichlet's unit theorem and the above

$$Z/Z_n \simeq \mathbf{Z}_l^{r_2(F)\cdot l^n+s-1} .$$

Since $H^1(\Gamma_n, \mathbf{Q}_l/\mathbf{Z}_l(1)) = 0$, cf. 1.5, 4.8 implies

$$Z_n/\omega_n Z \simeq Hom_{\mathbf{Z}_l}(H^1(\Gamma_n, U_\infty^S), \mathbf{Q}_l/\mathbf{Z}_l)$$

is of bounded order independently of n. Therefore, $i(n, a; Z) - (r_2(F)l^n + s - 1)\cdot a$ is bounded for $n \geq n_0$ and $a \geq 0$, and thus by 3.19, there is a pseudo-isomorphism

$$\varphi_Z : Z \longrightarrow E(Z) := \Lambda^{r_2(F)} \oplus \bigoplus_{j=1}^{t} \Lambda/(\xi_{n_j})$$

with $\sum_{j=1}^{t} deg\, \xi_{n_j} = \lambda(Z) = s-1$. Since Z is a torsion-free $\mathbf{Z}_l$-module, φ_Z is, in fact, injective. The following lemma is trivial, but worth mentioning.

Lemma 4.12 *Let $\gamma_0 \in \Gamma$ be our fixed topological generator and $u = \kappa(\gamma_0) \in U_l^{(1)}$. Then for $i \in \mathbf{Z}$, there are Λ-module isomorphisms*

$$\Lambda(i) \to \Lambda \text{ and } \Lambda/(g(T))(i) \to \Lambda/(g(u^{-i}(1+T)-1)) \cdot$$

Putting all things together we have proven the following, cf. [41],

Theorem 4.13 *Let N_∞^S be the field generated over F_∞ by all l^n-th roots of S-units U_∞^S, and set $\mathcal{Y}^S := Gal(N_\infty^S/F_\infty)$. Then $\mathcal{Y}^S$ is a torsion-free $\mathbf{Z}_l$-module,* *and there is an injective* Λ*-module morphism*

$$\mathcal{Y}^S \longrightarrow \Lambda^{r_2(F)} \oplus \bigoplus_{j=1}^{t} \Lambda/(\xi_{n_j}(u^{-1}(1+T)-1))$$

with $\sum_{j=1}^{t} deg\, \xi_{n_j} = s-1$ *and finite cokernel.*

Corollary 4.14 *Let* M_∞ *be the maximal abelian, pro-l-extension of* F_∞*, which is unramified outside* l*, and* $\mathcal{X} := Gal(M_\infty/F_\infty)$*. Then* $\mathcal{X}$ *does not contain any non-trivial finite* Λ*-submodule and there is an injective* Λ*-module morphism*

$$\mathcal{X} \longrightarrow \Lambda^{r_2(F)} \oplus \bigoplus_{j=1}^{m} \Lambda/\mathfrak{p}_j^{e_j}$$

with finite cokernel.

Proof: Let $\mathcal{M} = H^1_{\acute{e}t}(O^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l(1))$ as above, then

$$0 \longrightarrow U^S_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow \mathcal{M} \longrightarrow A^S_\infty \longrightarrow 0$$

is an exact sequence. For N^S_∞ as in 4.13, this implies

$$Gal(M_\infty/N^S_\infty)(-1) \simeq Hom_{\mathbb{Z}_l}(A^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l)\,.$$

By 4.9 the later group is isomorphic to the adjoint of a certain Λ-torsion module Y^S_∞, and the corollary follows now directly from 4.13. □

Corollary 4.15 *Let N_∞ be the field generated over F_∞ by all l^n-th roots of units U_∞ of F_∞, and $\mathcal{Y} = Gal(N_\infty/F_\infty)$, then there exists an integer s_0, $0 \le s_0 \le s$, such that*

$$\mathcal{Y} \longrightarrow \Lambda^{r_2(F)} \oplus \bigoplus_{j=1}^{t_0} \Lambda/(\xi_{n_j}(u^{-1}(1+T)-1))$$

is an injective Λ-module morphism with $\sum_{j=1}^{t_0} deg\, \xi_{n_j} = s_0 - 1$ and finite cokernel.

Proof: Since $\mathcal{Y}(-1)/\omega_n\mathcal{Y}(-1)$ is dual to $(U_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{\Gamma_n}$, which contains $U_n \otimes \mathbb{Q}_l/\mathbb{Z}_l$, Dirichlet's unit theorem implies that $rk_{\mathbb{Z}_l}\mathcal{Y}(-1)/\omega_n\mathcal{Y}(-1) \ge r_2(F)l^n - 1$. On the other hand, the surjective morphism

$$\mathcal{Y}^S(-1)/\omega_n\mathcal{Y}^S(-1) \longrightarrow \mathcal{Y}(-1)/\omega_n\mathcal{Y}(-1) \longrightarrow 0$$

and the calculations above show that there exists an integer s_0, $0 \le s_0 \le s$, such that for $n \ge 0$ large enough, $rk_{\mathbb{Z}_l}\mathcal{Y}(-1)/\omega_n\mathcal{Y}(-1) = r_2(F)l^n + s_0 - 1$. The assertion now follows again from 3.19. □

We have seen that $f_\Lambda(\mathcal{Y}^S(-1)) = \mathcal{Y}^S(-1)/tor_\Lambda\mathcal{Y}^S(-1)$ is contained in $\Lambda^{r_2(F)}$ with finite cokernel, say H. As usual when finite groups are involved, one is led to the question what else other than the finiteness is known about this group, e.g., order, exponent etc. . Even if we give a precise statement on H, cf. 4.16, this result will be of a rather theoretical than pratical use.

Again let $Z = \mathcal{Y}^S(-1)$ and $E(Z) = \Lambda^{r_2(F)} \oplus \bigoplus_{j=1}^{t} \Lambda/(\xi_{n_j})$ as above, then for

$i \in \mathbb{Z}$, we get the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & tor_\Lambda Z(i) & \to & Z(i) & \to & f_\Lambda Z(i) & \to 0 & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & tor_\Lambda E(Z)(i) & \to & E(Z)(i) & \to & \Lambda^{r_2(F)}(i) & \to 0\,, & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & A(i) & \to & B(i) & \to & H(i) & \to 0 & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

diagram 1

where A, B and H are finite Λ-modules.

Case (a): $i = 0$

Let $m := max\,\{n_j : 1 \leq j \leq t\}$ and assume that $n \geq m$, then $\omega_n : \Lambda/(\xi_{n_j}) \to \Lambda/(\xi_{n_j})$ is the trivial map, i.e.,

$$\begin{array}{rclclcrcl} E(Z)^{\Gamma_n} & = & (tor_\Lambda E(Z))^{\Gamma_n} & = & tor_\Lambda E(Z) & \text{and} & \omega_n \cdot tor_\Lambda E(Z) & = & 0 \\ Z^{\Gamma_n} & = & (tor_\Lambda Z)^{\Gamma_n} & = & tor_\Lambda Z & \text{and} & \omega_n \cdot tor_\Lambda Z & = & 0\,. \end{array}$$

Hence we obtain from the middle horizontal map in diagram 1 the split exact sequence

$$0 \to tor_\Lambda E(Z) \to E(Z)/\omega_n E(Z) \to (\Lambda/\omega_n\Lambda)^{r_2(F)} \to 0 ,$$

and in particular, $E(Z)/\omega_n E(Z)$ is $\mathbf{Z}_l$-torsion-free. From this and diagram 1 we deduce

$$\begin{aligned} tor_{\mathbf{Z}_l} Z/\omega_n Z &= ker(Z/\omega_n Z \to E(Z)/\omega_n E(Z)) \\ &= im(B^{\Gamma_n} \to Z/\omega_n Z) \\ &\simeq im(B^{\Gamma_n} \to H^{\Gamma_n}) , \end{aligned}$$

and on the other hand we have

$$\begin{aligned} tor_{\mathbf{Z}_l} Z/\omega_n Z &= Z_n/\omega_n Z \\ &\simeq Hom_{\mathbf{Z}_l}(H^1(\Gamma_n, U_\infty^S), \mathbf{Q}_l/\mathbf{Z}_l) . \end{aligned}$$

Since B and H are finite Λ-modules, $B^{\Gamma_n} = B$ and $H^{\Gamma_n} = H$ for $n \geq 0$ large enough, and thus the following

Proposition 4.16 *Let H be the cokernel of $\mathcal{X}(-1)/tor_\Lambda \mathcal{X}(-1) \to \Lambda^{r_2(F)}$, cf. 4.14. Then for $n \geq 0$ large enough,*

$$\begin{aligned} H &\simeq Hom_{\mathbf{Z}_l}(H^1(\Gamma_n, U_\infty^S), \mathbf{Q}_l/\mathbf{Z}_l) \\ &\simeq Hom_{\mathbf{Z}_l}(ker(A_n^S \to A_\infty^S), \mathbf{Q}_l/\mathbf{Z}_l) . \end{aligned}$$

Case (b): $i \neq 0$

Recall the isomorphism of 4.12

$$\Lambda(i) \simeq \Lambda \text{ and } \Lambda/(\xi_{n_j})(i) \simeq \Lambda/(\xi_{n_j}(u^{-i}(1+T)-1)) ,$$

where $u = \kappa(\gamma_0) \in U_l^c \backslash U_l^{c+1}$ and $l^c = \# W_l(F)$. Since ξ_{n_j} is irreducible and $\omega_n(u^i \zeta_{l^{n_j}} - 1) \neq 0$ for all $n \geq 0$, $i \neq 0$, ω_n and $\xi_{n_j}(u^{-i}(1+T)-1)$ are relatively prime. Hence

$$\omega_n : \Lambda/(\xi_{n_j})(i) \longrightarrow \Lambda/(\xi_{n_j})(i)$$

is an injective morphism with finite cokernel isomorphic as $\mathbf{Z}_l$-module to $\mathbf{Z}_l[\alpha_{n_j}]/\omega_n(\alpha_{n_j})\mathbf{Z}_l[\alpha_{n_j}]$, where $\alpha_{n_j} := u^i \zeta_{l^{n_j}} - 1$ is a root of $\xi_{n_j}(u^{-i}(1+T)-1)$. Thus we obtain the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & A(i)^{\Gamma_n} & \to & B(i)^{\Gamma_n} & \to & H(i)^{\Gamma_n} & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & t_\Lambda Z(i)/\omega_n t_\Lambda Z(i) & \to & Z(i)/\omega_n Z(i) & \to & f_\Lambda Z(i)/\omega_n f_\Lambda Z(i) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & t_\Lambda E(Z)(i)/\omega_n t_\Lambda E(Z)(i) & \to & E(Z)(i)/\omega_n E(Z)(i) & \to & (\Lambda/\omega_n\Lambda)^{r_2(F)} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & A(i)/\omega_n A(i) & \to & B(i)/\omega_n B(i) & \to & H(i)/\omega_n H(i) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}
$$

diagram 2

From the diagram we conclude

$$rk_{\mathbf{Z}_l} Z(i)/\omega_n Z(i) = r_2(F) \cdot l^n$$

and

$$0 \to tor_\Lambda Z(i)/\omega_n tor_\Lambda Z(i) \to tor_{\mathbf{Z}_l} Z(i)/\omega_n Z(i) \to H(i)^{\Gamma_n} \to 0$$

is an exact sequence of finite $\mathbf{Z}_l$-modules. In certain cases we can calculate the structure and order of $tor_\Lambda E(Z)(i)/\omega_n tor_\Lambda E(Z)(i)$, and appealing to the left vertical column in diagram 2 this gives us at least the order of $tor_\Lambda Z(i)/\omega_n tor_\Lambda Z(i)$. Namely we have the following

Lemma 4.17 *Let $Z := \mathcal{Y}^S(-1)$ and $E(Z) := \Lambda^{r_2(F)} \oplus \bigoplus_{j=1}^t \Lambda/(\xi_{n_j})$ as in 4.13. Then for $i \neq 0$.*

(i)

$$tor_\Lambda E(Z)(i)/\omega_0 tor_\Lambda E(Z)(i) \simeq \bigoplus_{j:n_j=0} \mathbb{Z}/l^{e+\nu_l(i)}\mathbb{Z} \oplus \bigoplus_{j:n_j\neq 0} \mathbb{Z}/l\mathbb{Z}\,.$$

(ii) If $\xi_{n_j} = T$, i.e., $n_j = 0$ for all j, or if $n \geq 0$ is large enough, then

$$tor_\Lambda E(Z)(i)/\omega_n tor_\Lambda E(Z)(i) \simeq (\mathbb{Z}/l^{e+n+\nu_l(i)}\mathbb{Z})^{s-1}\ ,$$

where $s = s(F_\infty/F)$ is the number of l-adic primes in F_∞.

Proof: (i) Since $\omega_0 = T$ is relatively prime to $\xi_{n_j}(u^{-i}(1+T)-1)$, we get

$$coker\left(\omega_0 : \Lambda/(\xi_{n_j})(i) \to \Lambda/(\xi_{n_j})(i)\right) \simeq \mathbb{Z}_l/l^{\nu_l(\xi_{n_j}(u^{-i}-1))}\mathbb{Z}_l\,.$$

For $n_j = 0$,we have by definition of $u = \kappa(\gamma_0)$, $\nu_l(u^{-i}-1) = e + \nu_l(i)$, and for $n_j \neq 0$, $\nu_l(\xi_{n_j}(u^{-i}-1)) = \nu_l(u^{-il^{n_j}}-1) - \nu_l(u^{-il^{n_j-1}}-1) = 1$, and thus by additivity the claim.

(ii) For $\alpha_{n_j} = u^i\zeta_{l^{n_j}} - 1$ and $n \geq n_j$, we have $\omega_n(\alpha_{n_j}) = u^{i\cdot l^n} - 1$, and hence

$$\begin{aligned} coker\left(\omega_n : \Lambda/(\xi_{n_j})(i) \to \Lambda/(\xi_{n_j})(i)\right) &\simeq \mathbb{Z}_l[\alpha_{n_j}]/\omega_n(\alpha_{n_j})\mathbb{Z}_l[\alpha_{n_j}] \\ &\simeq (\mathbb{Z}_l/l^{e+n+\nu_l(i)}\mathbb{Z}_l)^{deg\,\xi_{n_j}}\,. \end{aligned}$$

Since $\sum_{j=1}^t deg\,\xi_{n_j} = s-1$, the assertion follows again by additivity. □

The next corollary is obvious from the considerations above.

Corollary 4.18 *Let the notations be as in the previous lemma. We define* $j_0 := \#\{j : n_j = 0\}$ *and* $j_1 := \#\{j : n_j \neq 0\}$, *then for* $i \neq 0$.

(i)

$$\# tor_\Lambda Z(i)/\omega_0 tor_\Lambda Z(i) = l^{j_0(e+\nu_l(i))+j_1} ,$$

and if $n_j = 0$ *for all* j *or if* $n \geq 0$ *is large enough, then*

$$\# tor_\Lambda Z(i)/\omega_n tor_\Lambda Z(i) = (l^{e+n+\nu_l(i)})^{s-1} .$$

(ii)

$$\# tor_{\mathbb{Z}_l} Z(i)/\omega_0 Z(i) = l^{j_0(e+\nu_l(i))+j_1} \cdot \# H(i)^\Gamma ,$$

and if $n_j = 0$ *for all* j *or if* $n \geq 0$ *is large enough, then*

$$\# tor_{\mathbb{Z}_l} Z(i)/\omega_n Z(i) = (l^{e+n+\nu_l(i)})^{s-1} \cdot \# H(i)^{\Gamma_n} .$$

Remark 4.19 *If we consider* $Z_0 := \mathcal{Y}(-1)$ *and* $E(Z_0) := \Lambda^{r_2(F)} \oplus \bigoplus_{j=1}^{t_0} \Lambda/(\xi_{n_j})$ *with* $\sum_{j=1}^{t_0} deg\, \xi_{n_j} = s_0 - 1$ *as in 4.15, then the above calculations show, that as long as* $i \neq 0$ *we might replace* Z *by* Z_0, H *by* H_0 *etc. . But it should be pointed out, that 4.16 does not have an analogous formulation.*

5 The higher étale tame kernel

Using earlier results of Tate on K_2 and Galois cohomology, cf. [92], J. Coates proved in [11] among other things an Iwasawa theoretical description of the Sylow-l-subgroup of the tame kernel $K_2(O_E)$ for a totally real number field E. Namely, let l be an odd prime, $F = E(\zeta_l)$ and F_∞/F the cyclotomic $\mathbb{Z}_l$-extension of F. For the intermediate field F_n, let A_n be the Sylow-l-subgroup of the ideal class group and $A_\infty^- = \varinjlim A_n^-$ the minus part of A_∞, then Coates's result reads

$$l\text{-}tor\, K_2(O_E) \simeq (A_\infty^-(1))^{G_\infty} ,$$

where $G_\infty = Gal(F_\infty/E)$. In a series of papers, cf. [47],[48],[51],[52], M. Kolster generalized this to arbitrary number fields as well to the prime $l = 2$. Simplifying Coates's and Kolster's arguments it turns out that the above isomorphism and its generalization are rather statements on $K_2^{\acute{e}t}(O_E^S)$ than on $K_2(O_E)$. Keeping this in mind generalizations to arbitary number fields and higher étale K-groups follow quite easily.

Let E be a number field, l an odd prime and $F = E(\zeta_l)$. Moreover, let F_∞/F be the cyclotomic $\mathbb{Z}_l$-extension of F with intermediate fields F_n, $F_0 = F$, and $G_\infty = Gal(F_\infty/E) \simeq \Gamma \times \Delta$. All other notations will be standard, cf. page 94. Recall the exact sequence

$$0 \to H^1_{cont}(O_E^S, \mathbb{Z}_l(i)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \to H^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to l\text{-}tor\, H^2_{cont}(O_E^S, \mathbb{Z}_l(i)) \to 0 ,$$

and for $j = 1, 2$, $i \geq 2$, the isomorphisms, cf. 2.20,

$$H^j_{\mathrm{cont}}(O_E^S, \mathbf{Z}_l(i)) \simeq K^{\acute{e}t}_{2i-j}(O_E^S) .$$

The finiteness of $K^{\acute{e}t}_{2i-2}(O_E^S)$ implies the exactness of

$$0 \longrightarrow K^{\acute{e}t}_{2i-1}(O_E^S) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l \longrightarrow H^1_{\acute{e}t}(O_E^S, \mathbf{Q}_l/\mathbf{Z}_l(i)) \longrightarrow K^{\acute{e}t}_{2i-2}(O_E^S) \longrightarrow 0 .$$

On the other hand we have for $\mathcal{M} = H^1_{\acute{e}t}(O_\infty^S, \mathbf{Q}_l/\mathbf{Z}_l(1))$ the exact sequence

$$0 \longrightarrow U_\infty^S \otimes \mathbf{Q}_l/\mathbf{Z}_l(i-1) \longrightarrow \mathcal{M}(i-1) \longrightarrow A_\infty^S(i-1) \longrightarrow 0 .$$

The idea now is to study Galois descent for $\mathcal{M}(i-1) \simeq H^1_{\acute{e}t}(O_\infty^S, \mathbf{Q}_l/\mathbf{Z}_l(i))$, i.e., to consider the restriction map $H^1_{\acute{e}t}(O_E^S, \mathbf{Q}_l/\mathbf{Z}_l(i)) \to H^1_{\acute{e}t}(O_\infty^S, \mathbf{Q}_l/\mathbf{Z}_l(i))^{G_\infty}$, and further to find a morphism $K^{\acute{e}t}_{2i-1}(O_E^S) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l \to (U_\infty^S \otimes \mathbf{Q}_l/\mathbf{Z}_l(i-1))^{G_\infty}$ such that

$$\begin{array}{ccc} K^{\acute{e}t}_{2i-1}(O_E^S) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l & \longrightarrow & H^1_{\acute{e}t}(O_E^S, \mathbf{Q}_l/\mathbf{Z}_l(i)) \\ \downarrow & & \downarrow \\ (U_\infty^S \otimes \mathbf{Q}_l/\mathbf{Z}_l(i-1))^{G_\infty} & \longrightarrow & H^1_{\acute{e}t}(O_\infty^S, \mathbf{Q}_l/\mathbf{Z}_l(i))^{G_\infty} \end{array}$$

is commutative.

Remark 5.1 *Using the explicit description of $\mathcal{M} = \varinjlim\varinjlim \Delta_n^{(\nu)}$, cf. page 28, we might as well give a direct proof for the exactness of the above sequence, which then also shows that*

$$0 \longrightarrow U_\infty \otimes \mathbf{Q}_l/\mathbf{Z}_l(i-1) \longrightarrow \mathcal{M}(i-1) \longrightarrow A_\infty(i-1) \longrightarrow 0$$

is still exact. It is worth mentioning that there is no such sequence on finite levels.

For the Galois descent, we have the following

Lemma 5.2

(*i*) *If* $i \neq 0$*, then*

$$H^1_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq H^1_{\acute{e}t}(O^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i))^{G_\infty}$$

and

$$H^2_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq H^1(G_\infty, H^1_{\acute{e}t}(O^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i)) .$$

(*ii*) *If* $i = 0$*, then*

$$0 \rightarrow H^1(G_\infty, \mathbb{Q}_l/\mathbb{Z}_l) \simeq \mathbb{Q}_l/\mathbb{Z}_l \rightarrow H^1_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow H^1_{\acute{e}t}(O^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l)^{G_\infty} \rightarrow 0$$

is an exact sequence and

$$H^2_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l) \simeq H^1(G_\infty, H^1_{\acute{e}t}(O^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l)) .$$

Proof: Consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_\infty, H^q_{\acute{e}t}(O^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i))) \Longrightarrow H^{p+q}_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) .$$

Since $cd_l G_\infty = 1$ and $cd_l O^S_\infty = 1$, we get $E_2^{p,q} = 0$ for $p \geq 2$ or $q \geq 2$. Thus

$$H^2_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq H^1(G_\infty, H^1_{\acute{e}t}(O^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i)))$$

and

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \longrightarrow E_2^{0,1} \longrightarrow 0$$

is an exact sequence. Furthermore, $E_2^{1,0} \simeq H^1(\Gamma, \mathbb{Q}_l/\mathbb{Z}_l(i))^{\Delta}$. Let $\gamma \in \Gamma$ be a topological generator, then $H^1(\Gamma, \mathbb{Q}_l/\mathbb{Z}_l(i))$ is the cokernel of

$$1 - \gamma : \mathbb{Q}_l/\mathbb{Z}_l(i) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l(i) ,$$

which is multiplication by $1 - \kappa(\gamma)^i$ on $\mathbb{Q}_l/\mathbb{Z}_l$, and so $E_2^{1,0} = 0$ for $i \neq 0$. □

Remark 5.3 *The above proof also shows that* $F_\infty^\star \otimes \mathbb{Q}_l/\mathbb{Z}_l \simeq H^1(F_\infty, \mathbb{Q}_l/\mathbb{Z}_l(1))$ *satisfies Galois descent. For a 'direct', but somewhat complicated proof cf. [46].*

On the finite level we still have the following

Corollary 5.4 *Let $\Gamma_n := Gal(F_\infty/F_n)$ and $G_n := Gal(F_n/E)$ with the convention $\Gamma_0 = \Gamma$ and $G_0 = \Delta$.*

(*i*) *If $i \neq 0$, then*

$$H^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))^{G_n}$$

and

$$\begin{aligned} 0 \;\rightarrow\; & H^1(G_n, H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))) \;\rightarrow\; H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \;\rightarrow\; H^2_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))^{G_n} \\ \rightarrow\; & H^2(G_n, H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))) \;\rightarrow 0 \end{aligned}$$

is an exact sequence as well for $j = 1, 2$

$$H^j(G_n, H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))) \simeq H^j(G_n, H^2_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))) \,.$$

(*ii*) *If $i = 0$, then*

$$0 \rightarrow H^1(G_n, \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow H^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l)^{G_n} \rightarrow 0$$

and

$$\begin{aligned} 0 \;\rightarrow\; & H^1(G_n, H^1_{\acute{e}t}(O_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l)^{\Gamma_n}) \;\rightarrow\; H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l) \;\rightarrow\; H^2_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l)^{G_n} \\ \rightarrow\; & H^2(G_n, H^1_{\acute{e}t}(O_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l)^{\Gamma_n}) \;\rightarrow 0 \end{aligned}$$

are exact sequences.

Proof: (i) Again we consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_n, H^q_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))) \Longrightarrow H^{p+q}_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \,.$$

From 5.2 we obtain $E_2^{1,0} \hookrightarrow H^1(G_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0$, which by the finiteness of $H^0_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))$ implies $E_2^{k,0} = 0$ for $k \geq 1$, i.e., $E_2^{p,q} = 0$ for $p \neq 0$ and

$q \neq 1, 2$. A simple spectral sequence argument gives the assertion. (ii) This follows immediately from the commutativity of the diagrams

$$\begin{array}{ccc} H^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l) & \to & H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l)^{G_n} \\ \downarrow & & \downarrow \\ H^1_{\acute{e}t}(O_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l)^{G_\infty} & \to & H^1_{\acute{e}t}(O_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l)^{G_\infty} \end{array}$$

with surjective vertical arrows and

$$\begin{array}{ccc} H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l) & \to & H^2_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l)^{G_n} \\ \downarrow \wr & & \downarrow \wr \\ H^1(G_\infty, H^1_{\acute{e}t}(O_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l)) & \to & H^1(\Gamma_n, H^1_{\acute{e}t}(O_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l))^{G_n} . \end{array}$$

□

Next we consider the exact sequences relating $\mathcal{M}(i-1)$ with ideal class groups and units, cf. above. We are interested in their Galois cohomological behaviour. Since $H^1(G_\infty, U_\infty^{(S)} \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))$ is dual— in the sense of Pontryagin—to $\mathcal{Y}_\infty^{(S)}(-i)^{G_\infty}$, cf. chapter 4, we get by diagram 2 in chapter 4 and 4.19, $H^1(G_\infty, U_\infty^{(S)} \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1)) = 0$ for $i \neq 1$. Hence, if $i \neq 1$, then

$$0 \to (U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} \to \mathcal{M}(i-1)^{G_\infty} \to A_\infty^S(i-1)^{G_\infty} \to 0$$

and

$$0 \to (U_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} \to \mathcal{M}(i-1)^{G_\infty} \to A_\infty(i-1)^{G_\infty} \to 0$$

are exact sequences. Even more is true, namely

Proposition 5.5 *Let E be a number field, l an odd prime, $F = E(\zeta_l)$ and F_∞/F the cyclotomic $\mathbb{Z}_l$-extension. For the intermediate field F_n, let A_n resp. A_n^S be the Sylow-l-subgroup of its ideal resp. S-ideal class group. Set $A_\infty := \varinjlim A_n$ resp. $A_\infty^S := \varinjlim A_n^S$. Then for $i \neq 1$, the following are equivalent.*

(i) $H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0$, *i.e., the Schneider conjecture is valid for E and l.*

(ii) $A_\infty^S(i-1)^{G_\infty}$ *is finite.*

(iii) $A_\infty^S(i-1)_{G_\infty} = 0$.

(iv) $A_\infty(i-1)^{G_\infty}$ *is finite.*

(v) $A_\infty(i-1)_{G_\infty} = 0$.

Proof: By 4.9 we know that

$$Hom_{\mathbb{Z}_l}(A_\infty, \mathbb{Q}_l/\mathbb{Z}_l) \simeq \alpha(Y_\infty) \text{ and } Hom_{\mathbb{Z}_l}(A_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l) \simeq \alpha(Y_\infty^S)\,,$$

where Y_∞ and Y_∞^S are certain Λ-torsion modules. The above consideration gives the chain of isomorphisms, $i \neq 1$,

$$H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq H^1(G_\infty, \mathcal{M}(i-1)) \simeq H^1(G_\infty, A_\infty(i-1)) \simeq \left(\alpha(Y_\infty)(1-i)^{G_\infty}\right)^*$$

as well as

$$H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq H^1(G_\infty, \mathcal{M}(i-1)) \simeq H^1(G_\infty, A_\infty^S(i-1)) \simeq \left(\alpha(Y_\infty^S)(1-i)^{G_\infty}\right)^*\,.$$

Since $\alpha(Y_\infty)$ and $\alpha(Y_\infty^S)$ are Λ-torsion modules, which do not contain any non-trivial finite Λ-submodule, the equivalence of the assertions is now obvious. □

Remark 5.6 *For $i = 1$, the situation is quite different. Namely in that case we have co-rk $(U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{G_\infty} = |S| - 1$ and co-rk $H^1(G_\infty, U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l) = g_l(E) - 1$, where $g_l(E)$ denotes the number of l-adic primes in E, cf. chapter 4. But on the other hand, we know that co-rk $\mathcal{M}^{G_\infty}$ = co-rk $H^1_{ét}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(1)) = |S|-1$ and co-rk $H^1(G_\infty, \mathcal{M})$ = co-rk $H^2_{ét}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(2)) = g_l(E)-1$, cf. 1.18. Thus considering the exact sequence*

$$(\star) \qquad 0 \to U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l \to \mathcal{M} \to A_\infty^S \to 0$$

one might suggest that ${A_\infty^S}^{G_\infty}$ is finite or equivalently $H^1(G_\infty, A_\infty^S) = (A_\infty^S)_{G_\infty} = 0$. This would follow immediately, if $(\star)$ splits as a sequence of Λ-modules—certainly, it is split exact considered as a sequence of abelian groups. In [77] T.Nguyen Quang Do showed that $(\star)$ admits a pseudo-splitting. As always for Λ-modules, this is already enough to obtain the finiteness of ${A_\infty^S}^{G_\infty}$, but cf. also 7.5.

Corollary 5.7 *For a given field E and an odd prime l,*

$$H^2_{ét}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0$$

for almost all $i \in \mathbb{Z}$.

Proof: Since the characteristic polynomial $char(X_\infty^S)$ of X_∞^S has only finitely many roots, the corollary follows from 4.9 and 5.5. □

Corollary 5.8 *Suppose that $F = E(\zeta_l)$ is a CM-field with maximal real subfield F^+. Furthermore, assume that the μ- and λ-invariants of F_∞^+/F^+ are trivial, cf. 4.6. Then for all $i-1 \equiv 0 \mod 2$,*

$$H^2_{\acute{e}t}(O^S_{F^+}, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0 \ .$$

Proof: For $i-1 \equiv 0 \mod 2$, we have

$$A_\infty^S(i-1)^{G_\infty} \simeq (\varepsilon_{1-i}A_\infty^S)(i-1)^\Gamma \subseteq A^S_{F^+_\infty}(i-1)^\Gamma \ ,$$

and by assumption $A^S_{F^+_\infty} = 0$. □

At this point we are already far enough to prove the Iwasawa theoretical description of $K^{\acute{e}t}_{2i-2}(O_E^S)$ mentioned in the beginning of this chapter.

Theorem 5.9 *Let E be a number field, l an odd prime, $F = E(\zeta_l)$ and F_∞/F the cyclotomic $\mathbb{Z}_l$-extension with Galois group $G_\infty = Gal(F_\infty/E)$. Then for $i \geq 2$,*

$$0 \to (U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty}/K^{\acute{e}t}_{2i-1}(O_E^S) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \to K^{\acute{e}t}_{2i-2}(O_E^S) \to A_\infty^S(i-1)^{G_\infty} \to 0$$

and

$$0 \to (U_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty}/K^{\acute{e}t}_{2i-1}(O_E^S) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \to K^{\acute{e}t}_{2i-2}(O_E^S) \to A_\infty(i-1)^{G_\infty} \to 0$$

are exact sequences, and in particular

$$K_{2i-1}^{\acute{e}t}(O_E^S)\otimes_{\mathbb{Z}_l}\mathbb{Q}_l/\mathbb{Z}_l \simeq max.\,div.\,(U_\infty^S\otimes\mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} \simeq max.\,div.\,(U_\infty\otimes\mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty}\ .$$

Proof: Since the Schneider conjecture is valid for $i \geq 2$, cf. 2.28, we obtain the finiteness of $A_\infty^S(i-1)^{G_\infty}$ and $A_\infty(i-1)^{G_\infty}$ from 5.5, and thus

$$max.\,div.\,\mathcal{M}(i-1)^{G_\infty} \subseteq (U_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} \subseteq (U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty}\ .$$

On the other hand

$$max.\,div.\,H_{\acute{e}t}^1(O_E^S,\mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq K_{2i-1}^{\acute{e}t}(O_E^S)\otimes_{\mathbb{Z}_l}\mathbb{Q}_l/\mathbb{Z}_l\ ,$$

so that we get the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & K_{2i-1}^{\acute{e}t}(O_E^S)\otimes_{\mathbb{Z}_l}\mathbb{Q}_l/\mathbb{Z}_l & \to & H_{\acute{e}t}^1(O_E^S,\mathbb{Q}_l/\mathbb{Z}_l(i)) & \to & K_{2i-2}^{\acute{e}t}(O_E^S) & \to 0 \\ & \downarrow & & \downarrow\wr & & \downarrow & \\ 0 \to & (U_\infty^{(S)}\otimes\mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} & \to & \mathcal{M}(i-1)^{G_\infty} & \to & A_\infty^{(S)}(i-1)^{G_\infty} & \to 0\ , \end{array}$$

and the theorem is proven. □

Remark 5.10 *Let E be a totally real number field, l an odd prime and $i-1 \equiv 1 \bmod 2$. Then $(U_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} = 0$, and thus*

$$K_{2i-2}^{\acute{e}t}(O_E^S) \simeq A_\infty(i-1)^{G_\infty}\ .$$

In particular, for $i = 2$, we get

$$l\text{-}tor\ K_2(O_E) \simeq A_\infty(1)^{G_\infty}\ ,$$

which is precisely Coates's result mentioned in the beginning of this chapter.

Passing to the direct limits over the sequences in 5.9 we get on the infinite level the following

Corollary 5.11 *For $i \geq 2$, there are exact sequences*

$$0 \to U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1)/\varinjlim K_{2i-1}^{ét}(O_n^S) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \to \varinjlim K_{2i-2}^{ét}(O_n^S) \to A_\infty^S(i-1) \to 0$$

and

$$0 \to U_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1)/\varinjlim K_{2i-1}^{ét}(O_n^S) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \to \varinjlim K_{2i-2}^{ét}(O_n^S) \to A_\infty(i-1) \to 0\,.$$

Remark 5.12 *Since O_∞^S contains all l-th power roots of unity, $H^1_{\mathrm{cont}}(O_\infty^S, \mathbb{Z}_l(i)) \simeq K_{2i-1}^{ét}(O_\infty^S)$ is torsion-free, which implies $\varinjlim K_{2i-1}^{ét}(O_n^S) \neq K_{2i-1}^{ét}(O_\infty^S)$. But*

$$K_{2i-1}^{ét}(O_n^S) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \subseteq (K_{2i-1}^{ét}(O_\infty^S) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l)^{\Gamma_n}\,,$$

for all $n \geq 0$ and passing to the direct limit yields

$$\begin{array}{ccccccccc} 0 \to & \varinjlim K_{2i-1}^{ét}(O_n^S) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l & \to & \mathcal{M}(i-1) & \to & \varinjlim K_{2i-2}^{ét}(O_n^S) & \to 0 \\ & \downarrow & & \downarrow \mathrm{id} & & \downarrow & \\ 0 \to & K_{2i-1}^{ét}(O_\infty^S) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l & \to & \mathcal{M}(i-1) & \to & l\text{-}tor\, K_{2i-2}^{ét}(O_\infty^S) & \to 0\,. \end{array}$$

It is not clear to us, whether the left or equivalently right arrow is an isomorphism or not.

Since $ker\,(A_n^S \to A_\infty^S)$ is finite independently of n, say of exponent l^{n_1}, a similar sequence to 5.9 resp. 5.11 already exists on finite level, namely

Corollary 5.13 *For $i \geq 2$ and $1 \leq \nu \leq n-n_1$, there is a split exact sequence*

$$0 \to U_n^S \otimes \tfrac{1}{l^\nu}\mathbf{Z}/\mathbf{Z}(i-1)/K_{2i-1}^{et}(O_n^S) \otimes_{\mathbf{Z}_l} \tfrac{1}{l^\nu}\mathbf{Z}_l/\mathbf{Z}_l \to {}_{l^\nu}K_{2i-2}^{et}(O_n^S) \to {}_{l^\nu}A_n^S(i-1) \to 0\,,$$

and

$$U_n^S \otimes \tfrac{1}{l^\nu}\mathbf{Z}/\mathbf{Z}(i-1)/K_{2i-1}^{et}(O_n^S) \otimes_{\mathbf{Z}_l} \tfrac{1}{l^\nu}\mathbf{Z}_l/\mathbf{Z}_l \simeq (\mathbf{Z}/l^\nu\mathbf{Z})^{g_l(F_n)-1}\,,$$

where $g_l(F_n)$ denotes the number of l-adic primes in F_n.

Proof: From the diagram

$$\begin{array}{ccccccccc}
0 \to & K_{2i-1}^{et}(O_n^S) \otimes_{\mathbf{Z}_l} \frac{1}{l^n}\mathbf{Z}_l/\mathbf{Z}_l & \to & H_{et}^1(O_n^S, \mathbf{Z}/l^n\mathbf{Z}(i)) & \to & {}_{l^n}K_{2i-2}^{et}(O_n^S) & \to 0 \\
 & & & \downarrow \wr & & & \\
0 \to & U_n^S \otimes \frac{1}{l^n}\mathbf{Z}/\mathbf{Z}(i-1) & \to & H_{et}^1(O_n^S, \mathbf{Z}/l^n\mathbf{Z}(1))(i-1) & \to & {}_{l^n}A_n^S(i-1) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & U_\infty^S \otimes \mathbf{Q}_l/\mathbf{Z}_l(i-1) & \to & \mathcal{M}(i-1) & \to & A_\infty^S(i-1) & \to 0
\end{array}$$

we get after certain identifications, e.g., $H_{et}^1(O_n^S, \mathbf{Z}/l^n\mathbf{Z}(i)) = H_{et}^1(O_n^S, \mathbf{Z}/l^n\mathbf{Z}(1))(i-1)$, for $z \in K_{2i-1}^{et}(O_n^S) \otimes_{\mathbf{Z}_l} \frac{1}{l^n}\mathbf{Z}_l/\mathbf{Z}_l \subseteq H_{et}^1(O_n^S, \mathbf{Z}/l^n\mathbf{Z}(1))(i-1)$,

$$z^{l^{n_1}} \in U_n^S \otimes \frac{1}{l^{n-n_1}}\mathbf{Z}/\mathbf{Z}(i-1)\,.$$

Therefore,

$$K_{2i-1}^{et}(O_n^S) \otimes_{\mathbf{Z}_l} \tfrac{1}{l^\nu}\mathbf{Z}_l/\mathbf{Z}_l \subseteq U_n^S \otimes \tfrac{1}{l^\nu}\mathbf{Z}/\mathbf{Z}(i-1)\,,$$

and the rest of the corollary is then obvious. □

Next we describe the image of $\varinjlim K_{2i-1}^{et}(O_n^S) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l \to U_\infty^S \otimes \mathbf{Q}_l/\mathbf{Z}_l(i-1)$ via

Iwasawa theory. At first we recall a few notations from chapter 4. Let M_∞ be the maximal abelian, pro-l-extension of F_∞, which is unramified outside S, and N_∞^S the field generated over F_∞ by all l-th power roots of S-units U_∞^S of F_∞. We set $\mathcal{X} := Gal(M_\infty/F_\infty)$ and $\mathcal{Y}^S := Gal(N_\infty^S/F_\infty)$. Then Kummer theory induces pairings of Λ-modules

$$\mathcal{X}(-1) \times \mathcal{M} \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l$$

and

$$\mathcal{Y}^S(-1) \times U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l\,.$$

Let $\mathcal{N} := (tor_\Lambda \mathcal{X}(-1))^\perp$ be the orthogonal complement of the Λ-torsion submodule of $\mathcal{X}(-1)$. Then by 4.14 $\mathcal{N} \subseteq U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l$—we even have $\mathcal{N} \subseteq U_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l$, cf. 4.15,—and for all $i \in \mathbb{Z}$, we get the pairing

$$\mathcal{Y}^S(-i)/tor_\Lambda \mathcal{Y}^S(-i) \times \mathcal{N}(i-1) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l$$

resp.

$$tor_\Lambda \mathcal{Y}^S(-i) \times U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1)/\mathcal{N}(i-1) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l\,.$$

Since $(\mathcal{Y}^S(-i)/tor_\Lambda \mathcal{Y}^S(-i))^{\Gamma_n} = 0$ for all $n \geq 0$, we get $\mathcal{N}(i-1)/\omega_n\mathcal{N}(i-1) = 0$ by duality, which implies

$$\begin{aligned}(U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{\Gamma_n}/\mathcal{N}(i-1)^{\Gamma_n} &\simeq \left(U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1)/\mathcal{N}(i-1)\right)^{\Gamma_n} \\ &\simeq Hom_{\mathbb{Z}_l}(t_\Lambda \mathcal{Y}^S(-i)/\omega_n t_\Lambda \mathcal{Y}^S(-i), \mathbb{Q}_l/\mathbb{Z}_l)\,.\end{aligned}$$

By diagram 2 in chapter 4 $t_\Lambda\mathcal{Y}^S(-i)/\omega_n t_\Lambda\mathcal{Y}^S(-i)$ is finite for $i \neq 1$, and so

$$max.\,div.\,(U^S_\infty \otimes \mathbf{Q}_l/\mathbf{Z}_l(i-1))^{\Gamma_n} = max.\,div.\,\mathcal{N}(i-1)^{\Gamma_n}\,.$$

But for $i \geq 2$, we have by 5.9

$$max.\,div.\,(U^S_\infty \otimes \mathbf{Q}_l/\mathbf{Z}_l(i-1))^{\Gamma_n} = K^{\acute{e}t}_{2i-1}(O^S_n) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l\,,$$

and hence for $i \geq 2$, we obtain from diagram 2 in chapter 4

$$\begin{aligned}\mathcal{N}(i-1)^{\Gamma_n}/K^{\acute{e}t}_{2i-1}(O^S_n) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l &\simeq Hom_{\mathbf{Z}_l}(\mathbf{Z}_l\text{-}tor\,(f_\Lambda\mathcal{Y}^S(-i))_{\Gamma_n}, \mathbf{Q}_l/\mathbf{Z}_l) \\ &\simeq Hom_{\mathbf{Z}_l}(H(1-i)^{\Gamma_n}, \mathbf{Q}_l/\mathbf{Z}_l)\,,\end{aligned}$$

where H is the finite Λ-module defined by diagram 1 in chapter 4. The finiteness of H implies

$$\begin{aligned}\varinjlim K^{\acute{e}t}_{2i-1}(O^S_n) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l &= \varinjlim max.\,div.\,\mathcal{N}(i-1)^{\Gamma_n} \\ &= \varinjlim \mathcal{N}(i-1)^{\Gamma_n} \\ &= \mathcal{N}(i-1)\,,\end{aligned}$$

and we have proven

Theorem 5.14 *Let $\mathcal{N} := (tor_\Lambda\mathcal{X}(-1))^\perp$ be the orthogonal complement of the Λ-torsion module of $\mathcal{X}(-1)$ in the pairing $\mathcal{X}(-1) \times \mathcal{M} \to \mathbf{Q}_l/\mathbf{Z}_l$. Then for $i \geq 2$,*

$$\varinjlim K^{\acute{e}t}_{2i-1}(O^S_n) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l = \mathcal{N}(i-1)\,,$$

and there is a pairing

$$tor_\Lambda\mathcal{X}(-i) \times \varinjlim K^{\acute{e}t}_{2i-2}(O^S_n) \longrightarrow \mathbf{Q}_l/\mathbf{Z}_l\,.$$

Since $\mathcal{N}(i-1)/\omega_n\mathcal{N}(i-1) = 0$, cf. above, we obtain the exact sequence

$$0 \to \mathcal{N}(i-1)^{\Gamma_n} \to \mathcal{M}(i-1)^{\Gamma_n} \to (\varinjlim K^{\acute{e}t}_{2i-2}(O^S_m))^{\Gamma_n} \to 0 ,$$

and dividing by the maximal divisible subgroup $K^{\acute{e}t}_{2i-1}(O^S_n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$ yields the exact sequence

$$0 \to (H(1-i)^{\Gamma_n})^* \to K^{\acute{e}t}_{2i-2}(O^S_n) \to (\varinjlim K^{\acute{e}t}_{2i-2}(O^S_m))^{\Gamma_n} \to 0 ,$$

where $*$ stands for the Pontryagin dual. Let l^h be the exponent of the finite group H, then for $i_1 \equiv i_2 \; mod \, l^{h-1}$,

$$H(i_1) \simeq H(i_2) \text{ as } \Gamma_n\text{-modules} ,$$

and so we can find an integer $n_h \geq 0$ independent of $i \in \mathbb{Z}$, such that for all $n \geq n_h$

$$H(1-i)^{\Gamma_n} = H(1-i) \simeq H .$$

Thus we obtain

Corollary 5.15 *Let H be the finite Λ-module defined by diagram 1 in chapter 4, then for all $n \geq 0$,*

$$0 \to (H(1-i)^{\Gamma_n})^* \to K^{\acute{e}t}_{2i-2}(O^S_n) \to (\varinjlim K^{\acute{e}t}_{2i-2}(O^S_m))^{\Gamma_n} \to 0$$

is an exact sequence, and there is an integer $n_h \geq 0$ independent of $i \geq 2$, such that for all $n \geq n_h$,

$$0 \to H^* \to K^{\acute{e}t}_{2i-2}(O^S_n) \to (\varinjlim K^{\acute{e}t}_{2i-2}(O^S_m))^{\Gamma_n} \to 0$$

is exact as well.

Corollary 5.16 *If $K^{\acute{e}t}_{2i-2}(O^S_n) = 0$ for some $n \geq 0$ and some $i \geq 2$, then $K^{\acute{e}t}_{2i-2}(O^S_n) = 0$ for all $n \geq 0$ and all $i \geq 2$.*

Proof: Suppose $K^{\acute{e}t}_{2i-2}(O^S_n) = 0$ for some $n \geq 0$ and $i \geq 2$, then by 5.14 and 5.15,

$$H(1-i)^{\Gamma_n} = 0 \text{ and } tor_\Lambda \mathcal{X}(-i)/\omega_n tor_\Lambda \mathcal{X}(-i) = 0 \ ,$$

which already implies $H = 0$ and $tor_\Lambda \mathcal{X} = 0$. □

Unfortunately, the condition of the above corollary is hardly ever satisfied, namely we have

Proposition 5.17 *Let $s = s(F_\infty/F)$ be the number of l-adic primes in F_∞, then $K^{\acute{e}t}_{2i-2}(O^S_n) = 0$ for some $n \geq 0$ and $i \geq 2$ if and only if $s = 1$ and $A^S_n = 0$ for all $n \geq 0$.*

Proof: We may assume that $n \geq 0$ is large enough, and so by duality we obtain from 4.18

$$\begin{aligned} \#(U^S_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{\Gamma_n}/K^{\acute{e}t}_{2i-1}(O^S_n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l &= \#\mathbb{Z}_l\text{-}tor\, \mathcal{Y}^S(-i)/\omega_n \mathcal{Y}^S(-i) \\ &= (l^{c+n+\nu_l(1-i)})^{s-1} \cdot \#H(1-i)^{\Gamma_n} \ . \end{aligned}$$

Hence 5.9 implies

$$K^{\acute{e}t}_{2i-2}(O^S_n) = 0 \text{ if and only if } s = 1,\ H = 0 \text{ and } A^S_\infty(i-1)^{\Gamma_n} = 0 \ .$$

Since $Hom_{\mathbb{Z}_l}(A_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l) \simeq Gal(M_\infty/N_\infty^S)(-1)$, the triviality of $A_\infty^S(i-1)^{\Gamma_n}$ already implies $A_\infty^S = 0$. Using $H = 0$, we deduce that $A_n^S = 0$ for all $n \geq n_1$. But since $s = 1$, the norm map $A_n^S \to A_{n-1}^S$ is surjective for all $n \geq 0$, and thus $A_n^S = 0$ for all $n \geq 0$. The converse is trivial. □

From what we showed above, we can now calculate the order of the even-dimensional étale K-groups of a number field E by evaluating the characteristic polynomial of a certain submodule of $tor_\Lambda \mathcal{X}$ at negative integers. For a totally real number field E, this automatically leads to the Main Conjecture in Iwasawa theory—now proven by A. Wiles—, which has to be seen as the l-adic (number theoretical) analogue to Weil's description of the ζ-function of a curve over a finite field, cf. chapter 3. We stick here to our standard notations, so E is an arbitrary number field and so on. By 5.14 $\varinjlim K_{2i-2}^{\acute{e}t}(O_m^S) \simeq \mathcal{M}/\mathcal{N}(i-1)$, and so for $j + (i-1) \equiv 0 \; mod \, \#\Delta$, we get

$$(\varinjlim K_{2i-2}^{\acute{e}t}(O_m^S))^{G_\infty} \simeq (\varepsilon_j \mathcal{M}/\mathcal{N})(i-1)^\Gamma ,$$

cf. 4.10. We know that $(\varepsilon_j \mathcal{M}/\mathcal{N})(i-1)^\Gamma$ is dual to $(\varepsilon_{1-j} tor_\Lambda \mathcal{X})(-i)_\Gamma$, and the order of the later (finite) group can be calculated via the characteristic polynomial of $\varepsilon_{1-j} tor_\Lambda \mathcal{X}$. Namely, since $tor_\Lambda \mathcal{X}$ does not contain any non-trivial finite Λ-submodule, the same is true for the eigenspace $\varepsilon_k tor_\Lambda \mathcal{X}$, $k = 1 - j$. Hence there is an exact sequence of Λ-modules

$$0 \to \varepsilon_k tor_\Lambda \mathcal{X} \to \bigoplus_{r=1}^{r_k} \Lambda/(f_{rr_k}(T)) \to D_k \to 0 ,$$

where D_k is a finite Λ-module. Twisting the above sequence and noting that $(\bigoplus_{r=1}^{r_k}\Lambda/(f_{rr_k}(T))\,(-i))^{\Gamma}=0$, yields the exact sequence

$$0\to D_k(-i)^{\Gamma}\to(\varepsilon_k tor_{\Lambda}\mathcal{X})(-i)_{\Gamma}\to\left(\bigoplus_{r=1}^{r_k}\Lambda/(f_{rr_k}(u^i(1+T)-1))\right)_{\Gamma}\to D_k(-i)_{\Gamma}\to 0\,.$$

Thus for $g_j(T):=\prod_{r=1}^{r_k}f_{rr_k}(u(1+T)-1)$, we get

$$\begin{aligned}\#(\varinjlim K_{2i-2}^{\acute{e}t}(O_m^S))^{G_\infty} &= \#(\varepsilon_k tor_{\Lambda}\mathcal{X})(-i)_{\Gamma}\\ &= \#\left(\bigoplus_{r=1}^{r_k}\Lambda/(f_{rr_k}(u^i(1+T)-1))\right)_{\Gamma}\\ &= l^{\nu_l(g_j(u^{i-1}-1))}\,.\end{aligned}$$

Combining this formula with 5.15 gives

Proposition 5.18 *Let $i\geq 2$ and $j+(i-1)\equiv 0\ mod\,\#\Delta$, then*

$$\#K_{2i-2}^{\acute{e}t}(O_E^S)=l^{\nu_l(g_j(u^{i-1}-1))}\cdot\#(\varepsilon_{i-1}H)(1-i)^{\Gamma}\,.$$

Now suppose that E is a totally real number field, and let $\chi=\omega^j$ be the Teichmüller character. Following Iwasawa's ideas, cf. [40], P. Deligne and K. Ribet showed using Stickelberger ideals that one can associate to a Dirichlet character χ a power series $G(T,\chi^{-1}\omega)\in Quot(\Lambda)$, such that for $u=\kappa(\gamma_0)$ and $s\in\mathbf{Z}_l$-$\{1\}$,

$$G(u^s-1,\chi^{-1}\omega)=L_l(\chi,s)\,,$$

cf. [16]. Here $L_l(\chi,s)$ denotes the l-adic L-function of E and χ, i.e., the continuous function $L_l(\chi,\cdot):\mathbf{Z}_l$-$\{1\}\to\mathbf{C}_l$ uniquely determined by the values on negative integers

$$L_l(\chi, 1-n) = L(\chi\omega^{-n}, 1-n) \cdot \prod_{\mathfrak{p}|l}(1 - \chi\omega^{-n}(\mathfrak{p})N(\mathfrak{p})^{n-1}) \text{ for all } n \geq 1 ,$$

where $L(\psi, s)$ is the classical L-function attached to ψ. We set

$$G_j(T) := g_j((1+T)^{-1} - 1) ,$$

where $g_j(T)$ is as above, and

$$H(T,\omega^j) := \begin{cases} G(T,\omega^j) & \text{if } j \not\equiv 1 \; mod \, \#\Delta \\ (1+T-u)\cdot G(T,\omega) & \text{if } j \equiv 1 \; mod \, \#\Delta , \end{cases}$$

then $H(T,\omega^j) \in \Lambda$, cf. [13]. Now the Main Conjecture proven by A. Wiles, cf. [99], states that these two elements of Λ generate the same ideal, i.e.,

Theorem 5.19 *Let E be a totally real number field, and ω the Teichmüller character on $\Delta = Gal(F/E)$, $F = E(\zeta_l)$. Then for $j \equiv 1 \; mod \, 2$,*

$$(G_j(T)) = (H(T,\omega^j)) .$$

Remark 5.20 *For the prime $l = 2$, we have to modify the assertion slightly, cf. [21], and then it remains valid at least for an abelian (totally real) number field E, cf. [99].*

Since the units U_n^+ of the maximal real subfield F_n^+ of F_n have index 1 or 2 in U_n, $\varepsilon_j(U_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l) \subseteq (U_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l)^- = 0$ for $j \equiv 1 \; mod \, 2$, and hence for $j \equiv 1 \; mod \, 2$,

$$\varepsilon_j \mathcal{N} = 0 \text{ as well as } (\varepsilon_j U_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l)(i-1)/K_{2i-1}^{et}(O_n^S) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l = 0 .$$

Considering the exact sequence

$$0 \to (\varepsilon_k Gal(M_\infty/N_\infty))(-1) \to (\varepsilon_k tor_\Lambda \mathcal{X})(-1) \to (\varepsilon_k tor_\Lambda \mathcal{Y})(-1) \to 0$$

we get for $k = 1-j$, $j \equiv 1 \; mod\, 2$,

$$(\varepsilon_k Gal(M_\infty/N_\infty))(-1) \simeq (\varepsilon_k tor_\Lambda \mathcal{X})(-1) ,$$

and hence

$$\begin{aligned} g_j(T) &:= char((\varepsilon_k tor_\Lambda \mathcal{X})(-1)) \\ &= char(Hom_{\mathbb{Z}_l}(\varepsilon_j A_\infty, \mathbb{Q}_l/\mathbb{Z}_l)) , \end{aligned}$$

so that the analogy with the function field case becomes now evident, cf. chapter 3. We can also give a more precise statement of 5.18 in the case E totally real and $i-1 \equiv 1 \; mod\, 2$. At first we get for $j + (i-1) \equiv 0 \; mod\, \#\Delta$,

$$\begin{aligned} K^{\acute{e}t}_{2i-2}(O^S_E) &\simeq (\varinjlim K^{\acute{e}t}_{2i-2}(O^S_m))^{G_\infty} \\ &\simeq A_\infty(i-1)^{G_\infty} , \end{aligned}$$

and so by the Main Conjecture

$$\begin{aligned} \#K^{\acute{e}t}_{2i-2}(O^S_E) &= |H(u^{-(i-1)} - 1, \omega^j)|_l^{-1} \\ &= \begin{cases} |G(u^{-(i-1)} - 1, \omega^j)|_l^{-1} & \text{if } j \not\equiv 1 \; mod\, \#\Delta \\ |u^{-(i-1)} - u|_l^{-1} \cdot |G(u^{-(i-1)} - 1, \omega^j)|_l^{-1} & \text{if } j \equiv 1 \; mod\, \#\Delta . \end{cases} \end{aligned}$$

In order to evaluate the right hand side further, we need the following

Lemma 5.21 *Let K be an arbitrary number field, l an odd prime and for $i \neq 0$, $w_l^{(i)}(K) := \#H^0(K, \mathbb{Q}_l/\mathbb{Z}_l(i))$. Further let K_∞/K_0 be the cyclotomic*

$\mathbb{Z}_l$-extension of $K_0 = K(\zeta_l)$, and $u = \kappa(\gamma)$, where $\gamma \in \Gamma$ is a topological generator and $\kappa : \Gamma \longrightarrow U_l^1$ the cyclotomic character. Then

$$w_l^{(i)}(K) = \begin{cases} 1 & \text{if } i \not\equiv 0 \; mod\,[K_0 : K] \\ |u^{-(i-1)} - u|_l^{-1} & \text{if } i \equiv 0 \; mod\,[K_0 : K]\,. \end{cases}$$

Proof: If $i \not\equiv 0 \; mod\,[K_0 : K]$, then clearly $w_l^{(i)}(K) = 1$. Otherwise, $l^r | w_l^{(i)}(K)$ if and only if $\sigma^i = 1$ for all $\sigma \in Gal(K(\zeta_{l^r})/K_0)$ if and only if $\gamma^i(\zeta_{l^r}) = \zeta_{l^r}$ if and only if $l^r | (u^i - 1)$. □

We know that $H^0_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq tor_{\mathbb{Z}_l} H^1_{cont}(O_E^S, \mathbb{Z}_l(i))$ in general, but in the case we are considering we already have

$$w_l^{(i)}(E) = \# H^1_{cont}(O_E^S, \mathbb{Z}_l(i)) = \# K^{\acute{e}t}_{2i-1}(O_E^S)\,.$$

Furthermore, it follows quite easily, cf. [13], that

$$|G(u^{-(i-1)} - 1, \omega^j)|_l^{-1} = |\zeta_E(1-i)|_l^{-1}\,.$$

Putting all things together we finally obtain

Theorem 5.22 *Let E be a totally real number field and $i - 1 \equiv 1 \; mod\, 2$, then*

$$\frac{\# K^{\acute{e}t}_{2i-2}(O_E^S)}{\# K^{\acute{e}t}_{2i-1}(O_E^S)} = |\zeta_E(1-i)|_l^{-1}\,.$$

Since the Quillen conjecture is valid for K_2 and K_3, cf. 2.17, we get

$$\# l\text{-}tor\, K_2(O_E^S) = w_l^{(2)}(E) \cdot |\zeta_E(-1)|_l^{-1}\,.$$

The tame kernel over $\mathfrak{p}|l$ vanishes on elements of l-th power order, which means l-*tor* $K_2(O_E) = l$-*tor* $K_2(O_E^S)$, and we have proven the so-called Birch-Tate conjecture up to 2-torsion, i.e.,

Conjecture 5.23 (Birch-Tate) *Let E be a totally real number field and $w^{(2)}(E) = \#H^0(E, \mathbb{Q}/\mathbb{Z}(2))$, then*

$$\#K_2(O_E) = w^{(2)}(E) \cdot |\zeta_E(-1)| .$$

Remark 5.24 *(i) Motivated by an analogous result for curves over finite fields, J. Birch and J. Tate formulated this conjecture at the International Congress of Mathematics, Nice,1970, cf. [91], and we have seen that it is nowadays a theorem up to 2-torsion. (ii) Using his results on the structure of the Sylow-2-subgroup of $K_2(O_E)$, i.e., the Iwasawa theoretical description of it, cf. [48], M. Kolster then showed that the Main Conjecture for the prime $l = 2$, cf. [21], implies the 2-part of the Birch-Tate conjecture, cf. [49]. Thus, for an abelian (totally real) number field E, the Birch-Tate conjecture is true in full generality.*

Certainly we would like to generalize 5.22 to arbitrary number fields and $i - 1 \equiv 0 \ mod\, 2$, $i \geq 2$, at least conjecturally. In order to do so, let us firstly take a look at the analytic class number formula, which we state in the following equivalent form

$$\lim_{s \to 0} s^{-(r_1(E)+r_2(E)-1)} \cdot \zeta_E(s) = \pm R(E) \cdot \frac{h(E)}{w^{(1)}(E)} ,$$

where $R(E) = R_0(E)$ is the usual regulator of E and $w^{(1)}(E) := \#H^0(E, \mathbb{Q}/\mathbb{Z}(1))$ is the number of roots of unity in E. This formula tells us two things, firstly, up to a non-zero rational number the leading coefficient of the Taylor expansion of the ζ-function at $s = 0$ is given by the regulator of E, and secondly, this rational number is precisely—up to signs—the quotient of the order of $\tilde{K}_0(O_E) := K_0(O_E)/\mathbb{Z} \simeq Cl(E)$ and the torsion part of $K_1(O_E) = U_E$. So any generalization depends now first of all on the existence of higher regulators defined on $K_{2i-1}(O_E)$. For this, let $\mathbb{R}(i-1) := (2\pi\zeta_4)^{i-1}\mathbb{R}$, then the complex conjugation ρ acts diagonally on $\mathbb{Z}^{Hom(E,\mathbb{C})} \otimes \mathbb{R}(i-1)$, and we denote the fixed space by $(\mathbb{Z}^{Hom(E,\mathbb{C})} \otimes \mathbb{R}(i-1))^+$. Obviously,

$$dim_{\mathbb{R}}(\mathbb{Z}^{Hom(E,\mathbb{C})} \otimes \mathbb{R}(i-1))^+ = \begin{cases} r_2(E) & \text{if } i-1 \equiv 1 \; mod\, 2 \\ r_1(E) + r_2(E) & \text{if } i-1 \equiv 0 \; mod\, 2\,. \end{cases}$$

For $i \geq 2$, there are highly non-trivial morphisms

$$r_{i-1} : K_{2i-1}(O_E) \longrightarrow (\mathbb{Z}^{Hom(E,\mathbb{C})} \otimes \mathbb{R}(i-1))^+$$

extending the usual regulator map, and A. Borel has shown, cf. chapter 2,

1. $ker\, r_{i-1}$ is finite.

2. $im\, r_{i-1}$ is a lattice in $(\mathbb{Z}^{Hom(E,\mathbb{C})} \otimes \mathbb{R}(i-1))^+$.

3. If R_{i-1} denotes the volume of the lattice in 2., then $\zeta_E(1-i)^* \sim_{\mathbb{Q}^*} R_{i-1}$.

Here, $\zeta_E(1-i)^*$ denotes the leading coefficient in the Taylor expansion of $\zeta_E(s)$ at $s = 1 - i$, and $_{\mathbb{Q}^*}$ means equality up to a non-zero rational number. The R_{i-1}'s are called higher regulators. In analogy with the Birch-Tate conjecture and the analytic class number formula, S. Lichtenbaum proposed the following beautiful conjecture, cf. [61],

Conjecture 5.25 (Lichtenbaum) *Let E be a number field, $i \geq 2$ an integer, $R_{i-1} = R_{i-1}(E)$ the higher regulator of E and $w^{(i)}(E) := \#H^0(E, \mathbb{Q}/\mathbb{Z}(i))$. Then*

$$\zeta_E(1-i)^* = \pm R_{i-1} \cdot \frac{\#K_{2i-2}(O_E)}{w^{(i)}(E)} .$$

Remark 5.26 *We changed here the original conjecture by replacing the order of the torsion-part of $K_{2i-1}(O_E)$ with $w^{(i)}(E)$, since otherwise it would already fail for $E = \mathbb{Q}$ and $i = 2$; namely, $\#K_2(\mathbb{Z}) = 2$ and $\#K_3(\mathbb{Z}) = 48$, but $\zeta_{\mathbb{Q}}(-1) = -\frac{1}{12}$, cf. [58].*

6 The higher étale wild kernel

Our purpose in this chapter is to give an Iwasawa theoretical description of $WK^{\acute{e}t}_{2i-2}(E)$ analogous to the one for $K^{\acute{e}t}_{2i-2}(O^S_E)$, cf. 5.9. As we mentioned at the end of chapter 2, there is a close relation between the higher étale wild kernel $WK^{\acute{e}t}_{2i-2}(E)$, $i \geq 2$, of the number field E and certain subgroups of $H^1_{\acute{e}t}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$. By investigating these groups very closely we will see that not just the result will be similar to 5.9, but also the way to derive it will be quite the same.

Again we use our standard notation, e.g., $F = E(\zeta_l)$ and F_∞/F denotes the cyclotomic $\mathbb{Z}_l$-extension with intermediate fields F_n, $n \geq 0$.

First of all we have to deal with certain subgroups of $H^1_{\acute{e}t}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$. For $i \in \mathbb{Z}$, consider the localization map

$$H^1_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \xrightarrow{loc(i)} \bigoplus_{\mathfrak{p}|l} H^2(E_\mathfrak{p}, \mathbb{Z}_l(i)) \simeq \bigoplus_{\mathfrak{p}|l} H^0(E_\mathfrak{p}, \mathbb{Q}_l/\mathbb{Z}_l(1-i))^* \,,$$

and let

$$\pi_l^{(\nu)}(i) : H^1_{\acute{e}t}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \longrightarrow H^1_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \xrightarrow{loc(i)} \bigoplus_{\mathfrak{p}|l} H^2_{cont}(E_\mathfrak{p}, \mathbb{Z}_l(i)) \,.$$

Then for $i \neq 1$ and $\nu \geq 1$, we set

$$\tilde{H}^1_{\acute{e}t}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) := ker\, \pi_l^{(\nu)}(i)$$

and

$$\tilde{H}^1_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) := \varinjlim ker\, \pi_l^{(\nu)}(i) = ker\, loc(i)$$

Then from chapter 2 we obtain the exact sequences

$$0 \to K^{et}_{2i-1}(O^S_E) \otimes_{\mathbb{Z}_l} \tfrac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l \to \tilde{H}^1_{et}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to {}_{l^\nu}WK^{et}_{2i-2}(E) \to 0$$

and

$$0 \to K^{et}_{2i-1}(O^S_E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \to \tilde{H}^1_{et}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to WK^{et}_{2i-2}(E) \to 0 .$$

Now the first step is already done, namely the linkage between $WK^{et}_{2i-2}(E)$ and 'cohomology'. Next we have to find a subgroup $D^{(\nu)}_E \subseteq \Delta^{(\nu)}_E$ and a sequence, which are analogous to

$$\Delta^{(\infty)}_E := \varinjlim \Delta^{(\nu)}_E \simeq \varinjlim H^1_{et}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(1)) = H^1_{et}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(1))$$

and

$$0 \to U^S_E \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^1_{et}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(1)) \to A^S_E \to 0 ,$$

so that we can relate $\tilde{H}^1_{et}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$ resp. $\tilde{H}^1_{et}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i))$ (and hence $WK^{et}_{2i-2}(E)$) to rather classical objects in number theory such as units and (ideal) class groups. As in chapter 2 let us first consider the case $i = 2$. For $1 \le \nu \le n$, the cup-product induces an isomorphism $\varphi^{(\nu)}_n(1) : \Delta^{(\nu)}_n(1) \xrightarrow{\sim} H^1_{et}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(2))$, and we define

$$D^{(\nu)}_n(1) := (\varphi^{(\nu)}_n(1))^{-1}\{\tilde{H}^1_{et}(O^S_n, \mathbb{Z}/l^\nu\mathbb{Z}(2))\} .$$

We deduce the commutative diagram

$$\begin{array}{ccccccc}
 & & D_n^{(\nu)}(1) & \rightarrow & \Delta_n^{(\nu)}(1) & & \\
 & & \downarrow\wr & & \downarrow\wr & & \\
0 \rightarrow & & \tilde{H}^1_{\acute{e}t}(O_n^S, \mathbb{Z}/l^\nu\mathbb{Z}(2)) & \rightarrow & H^1_{\acute{e}t}(O_n^S, \mathbb{Z}/l^\nu\mathbb{Z}(2)) & \rightarrow & \bigoplus_{\mathfrak{p}_n | l}\, {}_{l^\nu}H^2(F_{n,\mathfrak{p}_n}, \mathbb{Z}_l(2)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \rightarrow & & {}_{l^\nu}WK_2^{\acute{e}t}(F_n) & \rightarrow & {}_{l^\nu}K_2^{\acute{e}t}(O_n^S) & \rightarrow & \bigoplus_{\mathfrak{p}_n | l}\, {}_{l^\nu}W_l(F_{n,\mathfrak{p}_n}) = \bigoplus_{\mathfrak{p}_n | l} \mathbb{Z}/l^\nu\mathbb{Z}(1)\,.
\end{array}$$

Since $K_2^{\acute{e}t}(O_n^S) \rightarrow \bigoplus_{\mathfrak{p}_n | l} \mathbb{Z}/l^\nu\mathbb{Z}(1)$ is induced by the Hilbert symbols $(\cdot,\cdot)_{\mathfrak{p}_n}$, cf. chapter 2, we immediately get

$$D_n^{(\nu)} = \{\frac{1}{l^\nu} \, mod\, \mathbb{Z} \otimes z \in \tfrac{1}{l^\nu}\mathbb{Z}/\mathbb{Z} \otimes F_n^* : (\zeta_{l^\nu}, z)_{\mathfrak{p}} = 1 \text{ for all } \mathfrak{p}_n \not| \infty\}.$$

On the other hand for the arbitrary number field E and $\nu \geq 1$, we define

$$D_E^{(\nu)} := \{\frac{1}{l^\nu} mod \mathbb{Z} \otimes z \in \tfrac{1}{l^\nu}\mathbb{Z}/\mathbb{Z} \otimes E^* : z = a_{\mathfrak{p}} \cdot b_{\mathfrak{p}}^{l^\nu} \text{ with } a_{\mathfrak{p}} \in N_{\mathfrak{p}}, b_{\mathfrak{p}} \in E_{\mathfrak{p}}^* \text{ for all } \mathfrak{p} \not| \infty\}\,,$$

where $N_{\mathfrak{p}} := \bigcap_{n \geq 0} N(E_{\mathfrak{p}}(\zeta_{l^n})^*)$ is the local cyclotomic norm group of $E_{\mathfrak{p}}$, and

$$D_E^{(\infty)} := \varinjlim D_E^{(\nu)}\,.$$

That our definitions of $D_E^{(\nu)}$ are compatible is a consequence of the following

Lemma 6.1 *Let K be a local field, $[K : \mathbb{Q}_p] < \infty$ and $e \geq 1$ maximal with $\zeta_{l^e} \in K^*$, l an odd prime. Then for all $n \geq 0$,*

$$N(K(\zeta_{l^{e+n}})^*) = N_K \cdot K^{*l^n}\,,$$

where N_K is the cyclotomic norm group of K, i.e., $N_K = \bigcap_{m \geq 0} N(K(\zeta_{l^m})^*)$.

Proof: For all $m \geq n$, $K(\zeta_{l^{e+n}}) = K(\sqrt[l^n]{K^*}) \cap K(\zeta_{l^{e+m}})$, and so by local class field theory

$$N(K(\zeta_{l^{e+n}})^*) = \bigcap_{m \geq n} K^{*l^n} N(K(\zeta_{l^{e+m}})^*) .$$

On the other hand,

$$K^{*l^n} N_K = \bigcap_{m \geq n} K^{*l^n} N(K(\zeta_{l^{e+m}})^*) .$$

Namely, $\subseteq$ is obvious and $\supseteq$ at least if $l \neq p$, i.e., if $K(\zeta_{l^{e+n}})/K$ is unramified. Now suppose that K_∞/K is ramified, then for $m_0 \geq 0$ large enough, there are prime elements $\pi_m \in K_m^*$, such that $N(\pi_m) \in N_K$ for all $m \geq m_0$. Now let $z \in \bigcap_{m \geq n} K^{*l^n} N(K(\zeta_{l^{e+m}})^*)$, then

$$z = z_m^{l^n} N(u_m \pi_m^{t_m}) \text{ with } z_m \in K^*,\ u_m \in U_m,\ t_m \in \mathbb{Z} ,$$

and so, $N(u_m) = z z_m N(\pi_m)^{-t_m} \in z K^{*l^n} N_K$, i.e.,

$$z K^{*l^n} N_K \cap N(U_m) \neq \emptyset \text{ for all } m \geq m_0 .$$

Since $zK^{*l^n} N_K$ is closed and $N(U_m)$ compact, we get

$$z K^{*l^n} N_K \cap \bigcap_{m \geq m_0} N(U_m) \neq \emptyset ,$$

say, $u \in zK^{*l^n} N_K$ and $u \in \bigcap_{m \geq m_0} N(U_m) \subseteq N_K$, and thus

$$z \in K^{*l^n} N_K .$$

□

For the number field E, let $N_{\mathfrak{p}} := E_{\mathfrak{p}}^{*}$, if $\mathfrak{p}|\infty$ and $X_E := \prod_{\mathfrak{p}} N_{\mathfrak{p}} \subseteq J_E$, J_E the idelé group of E, then by Hasse's norm theorem

$$0 \to N_E \otimes \tfrac{1}{l^\nu}\mathbb{Z}/\mathbb{Z} \to D_E^{(\nu)} \xrightarrow{h_E^{(\nu)}} {}_{l^\nu}(J_E/E^*X_E) \to 0$$

is an exact sequence, where $h_E^{(\nu)}(\frac{1}{l^\nu} \, mod \, \mathbb{Z} \otimes z) := (b_{\mathfrak{p}})_{\mathfrak{p}} mod\ E^*X_E$ with $z = a_{\mathfrak{p}} \cdot b_{\mathfrak{p}}^{l^\nu}$ and $N_E := \bigcap_{n\geq 0} N(E_n^*)$ is the cyclotomic norm group of E, i.e., E_n is the n-th layer in the cyclotomic $\mathbb{Z}_l$-extension E_∞/E. Since $E_{\mathfrak{p}}^*/N_{\mathfrak{p}}$ is torsion-free, the above sequence is, in fact, split exact. Passing to direct limits yields the split exact sequence

$$0 \to N_E \otimes \mathbb{Q}_l/\mathbb{Z}_l \to D_E^{(\infty)} \xrightarrow{h_E^{(\infty)}} l\text{-}tor\,(J_E/E^*X_E) \to 0\,.$$

Obviously $N_E \subseteq U_E^S$ and $\mu(E) \subseteq N_E$, but the $\mathbb{Z}$-rank of N_E seems to be inaccessible in general, just a few examples are known, cf. [4]. Another disadvantage is, that E^*X_E is not necessarily closed in the idelé topology and hence (J_E/E^*X_E) does not correspond to an abelian extension of E via global class field theory. One way out of this dilemma is to consider the closure. Let $\overline{E^*X_E}$ be the closure of E^*X_E in the idelé group J_E, then $\overline{E^*X_E}/E^*X_E$ is l-divisible, cf. [50] or [54], and hence for $C_E := J_E/\overline{E^*X_E}$, we obtain the split exact sequence

$$0 \to ker\,\overline{h}_E^{(\nu)} \to D_E^{(\nu)} \xrightarrow{\overline{h}_E^{(\nu)}} {}_{l^\nu}C_E \to 0\,,$$

and passing to the direct limit yields

$$0 \to ker\,\overline{h}_E^{(\infty)} \to D_E^{(\infty)} \overset{\overline{h}_E^{(\infty)}}{\to} l\text{-}tor\,C_E \to 0\ .$$

Following Sinnott, cf [23], we have

Proposition 6.2 *Let E be a number field, l an odd prime and $C_E = J_E/\overline{E^\star X_E}$ with $X_E = \prod_\mathfrak{p} N_\mathfrak{p}$, where $N_\mathfrak{p}$ is the local cyclotomic norm group of $E_\mathfrak{p}$, if $\mathfrak{p} \nmid \infty$, and $N_\mathfrak{p} = E_\mathfrak{p}^\star$, if $\mathfrak{p}|\infty$. Then the class field belonging to C_E is the maximal abelian extension $\tilde{L}_E^S$ of E, which is unramified outside l and contains the cyclotomic $\mathbb{Z}_l$-extension E_∞ of E such that all l-adic primes of E_∞ split completely in $\tilde{L}_E^S$. Thus*

$$Gal(\tilde{L}_E^S/E) \simeq C_E\ .$$

Furthermore, if L_E^S denotes the maximal pro-l-extension of E contained in $\tilde{L}_E^S$, then $[\tilde{L}_E^S : L_E^S] < \infty$ is prime to l.

Our next goal is to give an interpretation of $ker\,\overline{h}_E^{(\nu)}$, $\nu \geq 1$, and $ker\,\overline{h}_E^{(\infty)}$, which are determined by the so-called Gross kernel of E, cf. 6.5. Let log_l be the l-adic logarithm normalized via $log_l(l) = 0$, and define

$$\begin{array}{rccl} \lambda_E: & E^\star & \longrightarrow & \bigoplus_{\mathfrak{p}|l} \mathbb{Z}_l \cdot \mathfrak{p} \\ & z & \longmapsto & \sum_{\mathfrak{p}|l} log_l(N_{E_\mathfrak{p}/\mathbb{Q}_l}(z)) \cdot \mathfrak{p}\ . \end{array}$$

Let $\overline{im\,\lambda_E}$ be the closure of $im\,\lambda_E$, then $\overline{im\,\lambda_E} = \bigoplus_{\mathfrak{p}|l} log_l(N_{E_\mathfrak{p}/\mathbb{Q}_l}(E_\mathfrak{p}^\star)) \cdot \mathfrak{p} \simeq \bigoplus_{\mathfrak{p}|l} E_\mathfrak{p}^\star/N_\mathfrak{p}$, and by restricting λ_E to U_E^S and then extending it to $U_E^S \otimes \mathbb{Z}_l$ by linearity, we obtain the exact sequence, cf. [23],

$$0 \to ker\, g_E \to U_E^S \otimes \mathbb{Z}_l \xrightarrow{g_E} \overline{im\, \lambda_E} \xrightarrow{k_E} C_E \to Cl^S(E) \to 0$$

with $k_E(\sum_{\mathfrak{p}|l} b_\mathfrak{p}\, mod\, N_\mathfrak{p}) := (b_\mathfrak{p})\, mod\, \overline{E^\star X_E}$. The kernel of g_E is called the Gross kernel of the number field E. By reciprocity $rk_{\mathbb{Z}_l} im\, g_E \leq g_l(E) - 1$, where $g_l(E)$ denotes the number of l-adic primes in E. We define the Gross defect $\delta_E^{Gro} \geq 0$ by

$$rk_{\mathbb{Z}_l} ker\, g_E = r_1(E) + r_2(E) + \delta_E^{Gro} \ ,$$

where $r_1(E)$ and $r_2(E)$ have the usual meaning. The Gross conjecture reads

Conjecture 6.3 (Gross) *Let E be a number field, l an odd prime and δ_E^{Gro} the Gross defect. Then*

$$\delta_E^{Gro} = 0 \ .$$

The above version of the Gross conjecture is due to Jaulent, cf. [45]. It was known to be true for abelian number fields, cf. [31] or [45], until recently T.Nguyen Quang Do proved another equivalent version, cf. 7.7.

Theorem 6.4 *Let E be a number field, l an odd prime and E_∞ the cyclotomic $\mathbb{Z}_l$-extension of E with intermediate fields E_n and Galois groups $\Gamma_n := Gal(E_\infty/E_n)$. Further let $A_{E_n}^S$ be the Sylow-l-subgroup of the S-ideal class group of E_n and $A_{E_\infty}^S := \varinjlim A_{E_n}^S$. Then the following assertions are all* *equivalent.*

(i) $\delta_E^{Gro} = 0$.

(ii) $(A^S_{E_\infty})^\Gamma$ *is finite.*

(iii) $(A^S_{E_\infty})_\Gamma$ *is finite—in fact* $(A^S_{E_\infty})_\Gamma = 0$.

(iv) $(A^S_{E_n})^{G_n}$ *has bounded order independent of* $n \geq 0$, $G_n :=$ $Gal(E_n/E)$.

Proof: Since $ker\,(A^S_{E_n} \to A^S_{E_\infty})$ has bounded order independent of $n \geq 0$, cf. 4.7, the equivalence of (ii) and (iv) is obvious. Let L^S_∞ be the maximal abelian, pro-l-extension of E_∞, which is unramified outside l and all l-adic primes of E_∞ split completely in L^S_∞. Then $X^S_\infty := Gal(L^S_\infty/E_\infty)$ is a Λ-torsion module and there is a pseudo-isomorphism, cf. 4.9,

$$Hom_{\mathbb{Z}_l}(A^S_{E_\infty}, \mathbb{Q}_l/\mathbb{Z}_l) \simeq \alpha(Y^S_\infty) \sim X^{S\,-1}_\infty ,$$

where $Y^S_\infty := Gal(L^S_\infty/E_\infty H^S_{n_0}) \subseteq X^S_\infty$ and $H^S_{n_0}$ is the Hilbert class field corresponding to $A^S_{E_{n_0}}$. Here, $n_0 \geq 0$ is the smallest integer such that all l-adic primes are totally ramified in E_∞/E_n for all $n \geq n_0$. By 6.2 we get $\delta_E^{Gro} = 0$ if and only if $[L^S_E : E_\infty] < \infty$ if and only if $X^S_{\infty\Gamma}$ is finite, which by the pseudo-isomorphism is the case, if and only if $(A^S_{E_\infty})^\Gamma$ is finite. Since X^S_∞ is a Λ-torsion module, the finiteness of $X^S_{\infty\Gamma}$ is equivalent to the one of $X^{S\,\Gamma}_\infty$, cf. 3.20. Again by the pseudo-isomorphism this is equivalent to $(A^S_{E_\infty})_\Gamma$ is finite. But since $Hom_{\mathbb{Z}_l}(A^S_{E_\infty}, \mathbb{Q}_l/\mathbb{Z}_l) \simeq \alpha(Y^S_\infty)$ does not contain any non-trivial finite submodule, cf. 3.18, this already implies $(A^S_{E_\infty})_\Gamma = 0$.

Hence the equivalence. □

The relation between the Gross kernel $ker\, g_E$ and $ker\,\overline{h}_E^{(\nu)}$ resp. $ker\,\overline{h}_E^{(\infty)}$ is given by the following result, cf. [50].

Theorem 6.5 *Let E be a number field, l an odd prime and $ker\, g_E$ the Gross kernel of E. Then*

$$ker\,\overline{h}_E^{(\nu)} = ker\, g_E \otimes_{\mathbb{Z}_l} \tfrac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l$$

resp.

$$ker\,\overline{h}_E^{(\infty)} = ker\, g_E \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l\,,$$

and so in particular, there are exact sequences

$$0 \to ker\, g_E \otimes_{\mathbb{Z}_l} \tfrac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l \to D_E^{(\nu)} \to {}_{l^\nu}C_E \to 0$$

resp.

$$0 \to ker\, g_E \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \to D_E^{(\infty)} \to l\text{-}tor\, C_E \to 0\,.$$

The next step is to describe the Gross kernel via Iwasawa theory. Let F_∞ be the cyclotomic $\mathbb{Z}_l$-extension of $F = E(\zeta_l)$ with intermediate fields F_n and Galois groups $\Gamma_n := Gal(F_\infty/F_n)$. Any other relevant object in the extension F_∞/F is simply indexed by n, i.e., we write $D_n^{(\nu)}$, $ker\, g_n$, C_n etc. instead of $D_{F_n}^{(\nu)}$, $ker\, g_{F_n}$, C_{F_n} etc. .

Lemma 6.6 *Let $D_\infty^{(\infty)} := \varinjlim D_n^{(\infty)}$, then $D_\infty^{(\infty)}$ is a Γ_n-module in a natural way, and*

$$D_\infty^{(\infty)\Gamma_n} \simeq D_n^{(\infty)}\,.$$

Proof: That Γ_n acts naturally on $D_\infty^{(\infty)}$ is obvious. Further we know that $\Delta_\infty^{(\infty)\Gamma_n} \simeq \Delta_n^{(\infty)}$, cf. 5.2, and clearly $\Delta_n^{(\infty)} \cap D_\infty^{(\infty)} = D_n^{(\infty)}$, which immediately gives the assertion. □

Passing to direct limits in 6.5 we obtain the exact sequence of Γ_n-modules

$$0 \to ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \to D_\infty^{(\infty)} \to l\text{-}tor\, C_\infty \to 0\ ,$$

where $ker\, g_\infty := \varinjlim ker\, g_n$ and $C_\infty := \varinjlim C_n$. Recall the Kummer pairing

$$\mathcal{X}(-1) \times \mathcal{M} \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l\ ,$$

where $\mathcal{M} = \Delta_\infty^{(\infty)}$. Let $\mathcal{N} := (tor_\Lambda \mathcal{X}(-1))^\perp$ be the orthogonal complement of the Λ-torsion module of $\mathcal{X}(-1)$, and $K_\infty := F_\infty(\sqrt[l^\infty]{ker\, g_\infty})$ the field generated over F_∞ by adjoining all l-th roots of $ker\, g_\infty$ with Galois group $\mathcal{Z}^S := Gal(K_\infty/F_\infty)$, then

$$\mathcal{Z}^S(-1) \times ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l$$

is again a pairing, and we have the following

Theorem 6.7 *Let E be number field, l an odd prime and $F = E(\zeta_l)$. Then with the notations as above there is an injective Λ-module morphism*

$$\mathcal{Z}^S \longrightarrow \Lambda^{r_2(F)} \oplus \bigoplus_{j=1}^{t_1} \Lambda/(\xi_{n_j})(1)$$

with finite cokernel, where $\sum_{j=1}^{t_1} deg\, \xi_{n_j} = \delta_\infty^{Gro} = max\, \delta_n^{Gro}$. Thus

$$\mathcal{N} \subseteq ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$$

with equality if and only if $\delta_n^{Gro} = 0$ *for all* $n \geq 0$.

Proof: We set $Z := \mathcal{Z}^S(-1)$, and let $Z_n := (ker\, g_n \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l)^\perp$ be the orthogonal complement of $(ker\, g_n \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l)$ in the above pairing, then $\omega Z \subseteq Z_n \subseteq Z$ and

$$\begin{aligned} Z/Z_n &\simeq Hom_{\mathbb{Z}_l}(ker\, g_n \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l, \mathbb{Q}_l/\mathbb{Z}_l) \\ &\simeq \mathbb{Z}_l^{r_2(F)l^n + \delta_n^{Gro}} \end{aligned}$$

resp.

$$\begin{aligned} Z_n/\omega_n Z &\simeq Hom_{\mathbb{Z}_l}((ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l)^{\Gamma_n}/ker\, g_n \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l, \mathbb{Q}_l/\mathbb{Z}_l) \\ &\simeq Hom_{\mathbb{Z}_l}(ker(l\text{-}tor\, C_n \to l\text{-}tor\, C_\infty), \mathbb{Q}_l/\mathbb{Z}_l) \text{ by 6.6.} \end{aligned}$$

Since $l\text{-}tor\, C_n \subseteq X_\infty^S/\omega_n X_\infty^S$, we get

$$ker(l\text{-}tor\, C_n \to l\text{-}tor\, C_m) \subseteq ker(X_\infty^S/\omega_n X_\infty^S \to X_\infty^S/\omega_m X_\infty^S)\ ,$$

and the order of the later group is bounded independently of $m \geq n$, cf. 3.10. Therefore, $Z_n/\omega_n Z$ is finite independently of n, and so we obtain as in chapter 4 for $\mathcal{Y}^S$,

$$Z \sim \Lambda^{r_2(F)} \oplus \bigoplus_{j=1}^{t_1} \Lambda/(\xi_{n_j})$$

with $\sum_{j=1}^{t_1} deg\, \xi_{n_j} = \delta_\infty^{Gro}$. The assertion of the theorem is now evident. □

Replacing $U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l$ by $ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$ resp. $\mathcal{Y}^S$ by $\mathcal{Z}^S$ at the end of chapter 4 we obtain from 6.6

Lemma 6.8 *Let H be the cokernel of $\mathcal{X}(-1)/tor_\Lambda \mathcal{X}(-1) \to \Lambda^{r_2(F)}$, cf. 4.14. Then for $n \geq 0$ large enough*

$$\begin{aligned} H &\simeq Hom_{\mathbb{Z}_l}(H^1(\Gamma_n, ker\, g_\infty), \mathbb{Q}_l/\mathbb{Z}_l) \\ &\simeq Hom_{\mathbb{Z}_l}(ker(l\text{-}tor\, C_n \to l\text{-}tor\, C_\infty), \mathbb{Q}_l/\mathbb{Z}_l) \,. \end{aligned}$$

The above theorem also gives the following characterization of the Gross conjecture

Corollary 6.9 *Let $F = E(\zeta_l)$, then the following are equivalent.*

(i) $\delta_n^{Gro} = 0$ *for all* $n \geq 0$.

(ii) $H^1(\Gamma_n, D_\infty^{(\infty)}) = D_\infty^{(\infty)}{}_{\Gamma_n}$ *is finite for all* $n \geq 0$.

Proof: Let $\tilde{M}_\infty$ be the field corresponding to the group $D_\infty^{(\infty)}$ via Kummer theory, then by the above theorem $Gal(\tilde{M}_\infty/K_\infty)(-1) \simeq Hom_{\mathbb{Z}_l}(l\text{-}tor\, C_\infty, \mathbb{Q}_l/\mathbb{Z}_l)$ is a Λ-torsion module. Thus $l\text{-}tor\, C_\infty^{\Gamma_n}$ is finite if and only if $l\text{-}tor\, C_{\infty\Gamma_n}$ is finite. Assume that $\delta_n^{Gro} = 0$ for all $n \geq 0$. Then by the above theorem

$$0 \to \mathcal{N}^{\Gamma_n} \to D_\infty^{(\infty)\Gamma_n} \to l\text{-}tor\, C_\infty^{\Gamma_n} \to \mathcal{N}_{\Gamma_n} \to D_\infty^{(\infty)}{}_{\Gamma_n} \to l\text{-}tor\, C_{\infty\Gamma_n} \to 0$$

is exact. But

$$\mathcal{N}_{\Gamma_n} \simeq Hom_{\mathbb{Z}_l}(f_\Lambda \mathcal{X}(-1)^{\Gamma_n}, \mathbb{Q}_l/\mathbb{Z}_l) = 0 \text{ and } \mathcal{N}^{\Gamma_n} \simeq (\mathbb{Q}_l/\mathbb{Z}_l)^{r_2(F)l^n} \oplus (\text{finite})$$

as well as

$$D_\infty^{(\infty)\Gamma_n} \simeq D_n^{(\infty)} \simeq (\mathbb{Q}_l/\mathbb{Z}_l)^{r_2(F)l^n} \oplus (\text{finite}) \,,$$

cf. 6.5 and 6.6. Therefore, $l\text{-}tor\, C_\infty^{\Gamma_n}$ is finite, and so is $l\text{-}tor\, C_{\infty\Gamma_n}$. Appealing once more to the sequence gives, $H^1(\Gamma_n, D_\infty^{(\infty)}) = D_\infty^{(\infty)}{}_{\Gamma_n}$ is finite. Conversely, assume that $D_\infty^{(\infty)}{}_{\Gamma_n}$ is finite and consider the sequence

$$\begin{array}{llllll} 0 & \to & (ker\, g_\infty \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l)^{\Gamma_n} & \to & D_\infty^{(\infty)\Gamma_n} & \to \ l\text{-}tor\, C_\infty^{\Gamma_n} \\ & \to & (ker\, g_\infty \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l)_{\Gamma_n} & \to & D_\infty^{(\infty)}{}_{\Gamma_n} & \to \ l\text{-}tor\, C_{\infty\Gamma_n} \to 0\,. \end{array}$$

With the notation from the above theorem, we get for $n \geq 0$ large enough,

$$\begin{aligned} (ker\, g_\infty \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l)_{\Gamma_n} &\simeq Hom_{\mathbf{Z}_l}(Z^{\Gamma_n}, \mathbf{Q}_l/\mathbf{Z}_l) \\ &\simeq Hom_{\mathbf{Z}_l}((tor_\Lambda Z)^{\Gamma_n}, \mathbf{Q}_l/\mathbf{Z}_l) \\ &\simeq Hom_{\mathbf{Z}_l}(tor_\Lambda Z, \mathbf{Q}_l/\mathbf{Z}_l) \\ &\simeq (\mathbf{Q}_l/\mathbf{Z}_l)^{\delta_\infty^{Gro}} \oplus (\text{finite})\,, \end{aligned}$$

and from the sequence we deduce $\delta_\infty^{Gro} = 0$. □

Note that the corresponding result for $\Delta_\infty^{(\infty)}$ is not satisfied, i.e., $\Delta_\infty^{(\infty)}{}_{G_\infty} \simeq H^2_{\acute{e}t}(O_E^S, \mathbf{Q}_l/\mathbf{Z}_l(1)) \neq 0$, cf. 5.6. There is another trivial consequence of the theorem 6.7, namely.

Corollary 6.10 *Let E be a number field, l an odd prime and $F_\infty = E(W_l)$ with Galois group $G_\infty := Gal(F_\infty/E)$. Then for $i \neq 1$,*

$$0 \to (ker\, g_\infty \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l(i-1))^{G_\infty} \to D_\infty^{(\infty)}(i-1)^{G_\infty} \to l\text{-}tor\, C_\infty(i-1)^{G_\infty} \to 0$$

is an exact sequence, which remains exact in the case $i = 1$ as long as we assume $\delta_n^{Gro} = 0$ for all $n \geq 0$.

Proof: By duality

$$(ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l(i-1))_{G_\infty} \simeq Hom_{\mathbb{Z}_l}(\mathcal{Z}^S(-i)^{G_\infty}, \mathbb{Q}_l/\mathbb{Z}_l)\ ,$$

and the triviality of $\mathcal{Z}^S(-i)^{G_\infty}$ follows along the same lines as for $\mathcal{Y}^S(-i)^{G_\infty}$, cf. diagram 2 in chapter 4. The second statement is obvious form 6.9 and its proof. □

The sequence in corollary 6.10 is one of two which are of interest to us. The other one is

$$0 \to K^{\acute{e}t}_{2i-1}(O^S_E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \to \tilde{H}^1_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to WK^{\acute{e}t}_{2i-2}(E) \to 0\ ,$$

and we want to find the relation between $D^{(\infty)}_\infty(i-1)$ and $\tilde{H}^1_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i))$. First we need the following

Lemma 6.11 *Let E be a number field, l an odd prime and $F_\nu = E(\zeta_{l^\nu})$.*

(*i*) *Let $d_\nu := [F_\nu : E]$ and $i, j \in \mathbb{Z}$, $i, j \neq 1$, then for $i \equiv j \bmod d_\nu$,*

$$\tilde{H}^1_{\acute{e}t}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq \tilde{H}^1_{\acute{e}t}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(j))(i-j).$$

(*ii*) *For $i \neq 1$, the cup-product induces an isomorphism*

$$\tilde{H}^1_{\acute{e}t}(O^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq D^{(\infty)}_\infty(i-1).$$

Proof: (i) This follows from the commutative diagram

$$\begin{array}{ccc} H^1_{\acute{e}t}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) & \to & \bigoplus_{\mathfrak{p}|l}\ {}_{l^\nu}H^0(E_\mathfrak{p}, \mathbb{Q}_l/\mathbb{Z}_l(1-i))^* \\ \downarrow\wr & & \downarrow\wr \\ H^1_{\acute{e}t}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(j))(i-j) & \to & \bigoplus_{\mathfrak{p}|l}\ {}_{l^\nu}H^0(E_\mathfrak{p}, \mathbb{Q}_l/\mathbb{Z}_l(1-j))^*(i-j)\ . \end{array}$$

(ii) For $m \geq n \geq \nu$,

$$\begin{array}{ccc} \tilde{H}^1_{\acute{e}t}(O^S_n, \mathbf{Z}/l^\nu\mathbf{Z}(i)) & \rightarrow & \tilde{H}^1_{\acute{e}t}(O^S_m, \mathbf{Z}/l^\nu\mathbf{Z}(i)) \\ \downarrow\wr & & \downarrow\wr \\ \tilde{H}^1_{\acute{e}t}(O^S_n, \mathbf{Z}/l^\nu\mathbf{Z}(1))(i-1) & \rightarrow & \tilde{H}^1_{\acute{e}t}(O^S_m, \mathbf{Z}/l^\nu\mathbf{Z}(1))(i-1) \end{array}$$

is commutative, and so cup-product and direct limit are compatible. Hence we get

$$\begin{aligned} \tilde{H}^1_{\acute{e}t}(O^S_\infty, \mathbf{Q}_l/\mathbf{Z}_l(i)) &\simeq \varinjlim \varinjlim \tilde{H}^1_{\acute{e}t}(O^S_n, \mathbf{Z}/l^\nu\mathbf{Z}(i)) \\ &\simeq \varinjlim \varinjlim D^{(\nu)}_n(i-1) \\ &= D^{(\infty)}_\infty(i-1)\,. \end{aligned}$$

□

The remaining step is to show Galois descent for $\tilde{H}^1_{\acute{e}t}(O^S_n, \mathbf{Q}_l/\mathbf{Z}_l(i))$ resp. $\tilde{H}^1_{\acute{e}t}(O^S_\infty, \mathbf{Q}_l/\mathbf{Z}_l(i)) := \varinjlim \tilde{H}^1_{\acute{e}t}(O^S_n, \mathbf{Q}_l/\mathbf{Z}_l(i))$, $i \geq 2$. From the definition of $\tilde{H}^1_{\acute{e}t}(\cdot, \mathbf{Q}_l/\mathbf{Z}_l(i))$ it is clear that we have to consider the exact sequence

$$\begin{aligned} 0 \rightarrow \; & \tilde{H}^1_{\acute{e}t}(O^S_n, \mathbf{Q}_l/\mathbf{Z}_l(i)) \;\rightarrow\; H^1_{\acute{e}t}(O^S_n, \mathbf{Q}_l/\mathbf{Z}_l(i)) \rightarrow \oplus_{\mathfrak{p}|l} \left(\oplus_{\mathfrak{p}_n|\mathfrak{p}} H^2(F_{n,\mathfrak{p}_n}, \mathbf{Z}_l(i))\right) \\ \rightarrow \; & H^0(F_n, \mathbf{Q}_l/\mathbf{Z}_l(i))^* \;\rightarrow 0 \quad , \end{aligned}$$

cf. also chapter 2. Here, the first sum $\oplus_{\mathfrak{p}|l}$ runs over all l-adic primes of the field E. Let $G_n := Gal(F_n/E)$, then the G_n-module structure on $\oplus_{\mathfrak{p}_n|\mathfrak{p}} H^2(F_{n,\mathfrak{p}_n}, \mathbf{Z}_l(i))$ is defined in the following way. If $\gamma \in G_n$, then $\gamma : F_{n,\mathfrak{p}_n} \rightarrow F_{n,\gamma\mathfrak{p}_n}$ is an $E_\mathfrak{p}$-isomorphism, and therefore we obtain a morphism

$\gamma^* : H^2(F_{n,\mathfrak{p}_n}, \mathbb{Z}_l(i)) \to H^2(F_{n,\gamma\mathfrak{p}_n}, \mathbb{Z}_l(i))$, which induces

$$\begin{array}{rccc} \gamma^* : & \bigoplus_{\mathfrak{p}_n|\mathfrak{p}} H^2(F_{n,\mathfrak{p}_n}, \mathbb{Z}_l(i)) & \longrightarrow & \bigoplus_{\mathfrak{p}_n|\mathfrak{p}} H^2(F_{n,\mathfrak{p}_n}, \mathbb{Z}_l(i)) \, . \\ & x = (x_\mathfrak{p}) & \longmapsto & \gamma^*(x) = (\gamma^*(x_\mathfrak{p})) \end{array}$$

We let G_n act on $\bigoplus_{\mathfrak{p}|l} \left(\bigoplus_{\mathfrak{p}_n|\mathfrak{p}} H^2(F_{n,\mathfrak{p}_n}, \mathbb{Z}_l(i)) \right)$ componentwise.

Lemma 6.12 *With the just defined G_n-module structure on $\bigoplus_{\mathfrak{p}_n|l} H^2(F_{n,\mathfrak{p}_n}, \mathbb{Z}_l(i))$, the map*

$$loc_n = loc_n(i) : H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \longrightarrow \bigoplus_{\mathfrak{p}|l} \left(\bigoplus_{\mathfrak{p}_n|\mathfrak{p}} H^2(F_{n,\mathfrak{p}_n}, \mathbb{Z}_l(i)) \right)$$

is a G_n-module morphism.

Proof: From the commutativity of the diagram

$$\begin{array}{ccc} H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) & \xrightarrow{\gamma^*} & H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \\ \downarrow loc_{\mathfrak{p}_n} & & \downarrow loc_{\gamma\mathfrak{p}_n} \\ H^2(F_{n,\mathfrak{p}_n}, \mathbb{Z}_l(i)) & \xrightarrow{\gamma^*} & H^2(F_{n,\gamma\mathfrak{p}_n}, \mathbb{Z}_l(i)) \end{array}$$

we deduce

$$\begin{aligned} loc_n(\gamma^*(z)) &= \bigoplus_{\mathfrak{p}|l} \bigoplus_{\mathfrak{p}_n|\mathfrak{p}} loc_{\mathfrak{p}_n}(\gamma^*(z)) \\ &= \bigoplus_{\mathfrak{p}|l} \bigoplus_{\mathfrak{p}_n|\mathfrak{p}} loc_{\gamma\mathfrak{p}_n}(\gamma^*(z)) \\ &= \bigoplus_{\mathfrak{p}|l} \bigoplus_{\mathfrak{p}_n|\mathfrak{p}} \gamma^*(loc_{\mathfrak{p}_n}(z)) \\ &= \gamma^*(loc_n(z)) \, . \end{aligned}$$

□

Recall the definitions of $eval_n^{(i-1)}$ resp. $eval_{\mathfrak{p}_n}^{(i-1)}$ on page 70, e.g., let l^{m_n} be the order of the finite group $H^0(F_n, \mathbb{Q}_l/\mathbb{Z}_l(i-1))$, $i \neq 1$, then

$$\begin{array}{ccc} H^0(F_n, \mathbb{Q}_l/\mathbb{Z}_l(1-i))^* = \mathbb{Z}/l^{m_n}\mathbb{Z}(1-i)^* & \overset{eval_n^{(i-1)}}{\longrightarrow} & H^0(F_n, \mathbb{Q}_l/\mathbb{Z}_l(i-1)) = \mathbb{Z}/l^{m_n}\mathbb{Z}(i-1) \\ \varphi & \longmapsto & \varphi(1+l^{m_n}\mathbb{Z}) \end{array}$$

is a G_n-module isomorphism, namely for $\gamma \in G_n$ and $\chi : G_\infty \to \mathbb{Z}_l^*$ the cyclotomic character, we have

$$\begin{aligned} \gamma(eval_n^{(i-1)}(\varphi)) &= \gamma(\varphi(1+l^{m_n}\mathbb{Z})) \\ &= \chi(\gamma)^{i-1}\varphi(1+l^{m_n}\mathbb{Z}) \\ &= (\gamma(\varphi))(1+l^{m_n}\mathbb{Z}) \\ &= eval_n^{(i-1)}(\gamma(\varphi))\,. \end{aligned}$$

In the same manner $eval_{\mathfrak{p}_n}^{(i-1)}$ resp. $\bigoplus eval_{\mathfrak{p}_n}^{(i-1)}$ are Galois equivariant maps. By the naturality of the map $c_{\mathfrak{p}_n} : H^2(F_{n,\mathfrak{p}_n}, \mathbb{Z}_l(i)) \to H^0(F_{n,\mathfrak{p}_n}, \mathbb{Q}_l/\mathbb{Z}_l(1-i))^*$, cf. 1.21, we finally obtain the exact sequence of G_n-modules

$$0 \to \tilde{H}^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to \bigoplus_{\mathfrak{p}_n | l} W_l^{(i-1)}(F_{n,\mathfrak{p}_n}) \to W_l^{(i-1)}(F_n) \to 0\,.$$

For $i \neq 0, 1$, consider the commutative diagram, cf. 5.2,

$$\begin{array}{ccc} H^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) & \to & \bigoplus_{\mathfrak{p}|l} W_l^{(i-1)}(E_\mathfrak{p}) \\ \downarrow l & & \downarrow \oplus diag_\mathfrak{p} \\ H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))^{G_n} & \to & \bigoplus_{\mathfrak{p}|l} \left(\bigoplus_{\mathfrak{p}_n|\mathfrak{p}} W_l^{(i-1)}(F_{n,\mathfrak{p}_n})\right)^{G_n} \end{array},$$

where $diag_\mathfrak{p} : W_l^{(i-1)}(E_\mathfrak{p}) \to \left(\bigoplus_{\mathfrak{p}_n|\mathfrak{p}} W_l^{(i-1)}(F_{n,\mathfrak{p}_n})\right)^{G_n}$ is defined in the following way. Let $a_\mathfrak{p}$ be a generator of $W_l^{(i-1)}(E_\mathfrak{p})$ and $w_l^{(i-1)} := \#W_l^{(i-1)}(E_\mathfrak{p})$, then $diag_{\mathfrak{p}_n}(a_\mathfrak{p}) := a_\mathfrak{p}^{z_\mathfrak{p}} \in W_l^{(i-1)}(F_{n,\mathfrak{p}_n})$ with $z_\mathfrak{p} \in \{1, \dots, w_l^{(i-1)}\}$ and $diag_\mathfrak{p} = \oplus diag_{\mathfrak{p}_n}$.

Since $cd_l E_\mathfrak{p} \leq 2$ resp. $cd_l F_{n,\mathfrak{p}_n} \leq 2$, the transfer map $tr : H^2(F_{n,\mathfrak{p}_n}, \mathbb{Z}_l(i)) \to H^2(E_\mathfrak{p}, \mathbb{Z}_l(i))$ is surjective, and hence the same is true for $tr : W_l^{(i-1)}(F_{n,\mathfrak{p}_n}) \to W_l^{(i-1)}(E_\mathfrak{p})$. As for $i = 2$, it is easy to see that this already forces $diag_\mathfrak{p}$ to be injective, and since $\#W_l^{(i-1)}(E_\mathfrak{p}) = \#(\oplus_{\mathfrak{p}_n|\mathfrak{p}} W_n^{(i-1)}(F_{n,\mathfrak{p}_n}))^{G_n}$, we get the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \to & \tilde{H}^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) & \to & H^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) & \to & \oplus_{\mathfrak{p}|l} W_l^{(i-1)}(E_\mathfrak{p}) \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ 0 & \to & \tilde{H}^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))^{G_n} & \to & H^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))^{G_n} & \to & \oplus_{\mathfrak{p}|l} \left(\oplus_{\mathfrak{p}_n|\mathfrak{p}} W_l^{(i-1)}(F_{n,\mathfrak{p}_n})\right)^{G_n} \end{array}$$

$$\begin{array}{ccc} \to & W_l^{(i-1)}(E) & \to 0 \\ & \downarrow \wr & \\ \to & W_l^{(i-1)}(F_n)^{G_n} & \to 0\,. \end{array}$$

Proposition 6.13 *Let E be a number field, l an odd prime, $F = E(\zeta_l)$ and F_∞ the cyclotomic $\mathbb{Z}_l$-extension of F with intermediate fields F_n and Galois groups $G_n := Gal(F_n/E)$, $n \leq \infty$. Then for $i \neq 0, 1$,*

$$\tilde{H}^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq \tilde{H}^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l(i))^{G_n}\,.$$

Proof: For $n < \infty$, see above. We choose successively generators of $W_l^{(i-1)}(F_{n,\mathfrak{p}_n})$, $n \geq 0$, such that $\{eval_l^{(i-1)}\}_{n\geq 0}$ becomes a family of morphisms compatible with the limits. Then by passing to the direct limit the assertion follows for $n = \infty$ as well. □

Proposition 6.14 *With the notations as in 6.13,*

$$0 \to H^1(G_n, \mathbb{Q}_l/\mathbb{Z}_l) \to \tilde{H}^1_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l) \to \tilde{H}^1_{\acute{e}t}(O_n^S, \mathbb{Q}_l/\mathbb{Z}_l)^{G_n} \to 0$$

is an exact sequence for $n \leq \infty$.

Proof: This is obvious by 5.2 and 5.4. □

Using the above considerations we also obtain new formulations of the Schneider conjecture, cf. 5.5.

Theorem 6.15 *Let E be a number field, $F_\infty = E(W_l)$, l an odd prime and $G_\infty := Gal(F_\infty/E)$. Then for $i \neq 1$, the following are equivalent.*

(i) $H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0$, *i.e., the Schneider conjecture is valid for E and l.*

(ii) l*-tor* $C_\infty(i-1)^{G_\infty}$ *is finite.*

(iii) l*-tor* $C_\infty(i-1)_{G_\infty} = 0$.

Proof: Since $\bigoplus_{\mathfrak{p}_n|\mathfrak{p}} W_l^{(i-1)}(F_{n,\mathfrak{p}_n}) \to W_l^{(i-1)}(E_\mathfrak{p}) \to (\bigoplus_{\mathfrak{p}_n|\mathfrak{p}} W_l^{(i-1)}(F_{n,\mathfrak{p}_n}))^{G_n}$ is surjective, we get

$$\varinjlim H^1(G_n, \bigoplus_{\mathfrak{p}_n|\mathfrak{p}} W_l^{(i-1)}(F_{n,\mathfrak{p}_n})) = 0 \,,$$

and hence

$$\begin{aligned} H^1(G_\infty, \tilde{H}^1_{\acute{e}t}(O^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i))) &= H^1(G_\infty, H^1_{\acute{e}t}(O^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i))) \\ &\simeq H^2_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) . \end{aligned}$$

By 6.10 and 6.11 we have

$$\begin{aligned} H^1(G_\infty, \tilde{H}^1_{\acute{e}t}(O^S_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i))) &\simeq H^1(G_\infty, D^{(\infty)}_\infty(i-1)) \\ &\simeq H^1(G_\infty, l\text{-}tor\, C_\infty(i-1)) , \end{aligned}$$

and the theorem follows now in the same way as 5.5. □

We are now able to state and prove the Iwasawa theoretical description of the higher wild kernel mentioned in the beginning of this chapter, namely

Theorem 6.16 *Let E be a number field, l an odd prime and $F_\infty = E(W_l)$ with Galois group $G_\infty := Gal(F_\infty/E)$. Then for $i \geq 2$, there is an exact sequence*

$$0 \to (ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty}/max.\,div. \to WK^{\acute{e}t}_{2i-2}(E) \to l\text{-}tor\, C_\infty(i-1)^{G_\infty} \to 0 ,$$

where $max.\,div.\,(ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} \simeq K^{\acute{e}t}_{2i-1}(O^S_E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$.

Proof: For $i \geq 2$, $l\text{-}tor\, C_\infty(i-1)^{G_\infty}$ is finite and so

$$max.\,div.\, D^{(\infty)}_\infty(i-1)^{G_\infty} \subseteq (ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} .$$

Under the isomorphism $\tilde{H}^1_{\acute{e}t}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq D^{(\infty)}_\infty(i-1)^{G_\infty}$ we have

$$K^{\acute{e}t}_{2i-1}(O^S_E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \simeq max.\,div.\, D^{(\infty)}_\infty(i-1)^{G_\infty} ,$$

and thus we get the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & K^{et}_{2i-1}(O^S_E)\otimes_{\mathbb{Z}_l}\mathbb{Q}_l/\mathbb{Z}_l & \to & \tilde{H}^1_{et}(O^S_E,\mathbb{Q}_l/\mathbb{Z}_l(i)) & \to & WK^{et}_{2i-2}(E) & \to 0 \\ & \downarrow & & \downarrow\wr & & \downarrow & \\ 0 \to & (ker\, g_\infty \otimes_{\mathbb{Z}_l}\mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} & \to & D^{(\infty)}_\infty(i-1)^{G_\infty} & \to & l\text{-}tor\, C_\infty(i-1)^{G_\infty} & \to 0\,, \end{array}$$

form which the theorem follows. □

Remark 6.17 *(i) A 'different' argument for 6.16 is given by the following. Since* $\varinjlim K^{et}_{2i-1}(O^S_n) \simeq \mathcal{N}(i-1) \hookrightarrow ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l(i-1)$, *cf. 5.14 and 6.7, the above diagram is commutative and since* $WK^{et}_{2i-2}(E)$ *is finite, the same is true for* $l\text{-}tor\, C_\infty(i-1)^{G_\infty}$. *But we do not gain any new information from this. (ii) Let* $\mathcal{N} := (tor_\Lambda \mathcal{X}(-1))^\perp$ *be the orthogonal complement of* $tor_\Lambda \mathcal{X}(-1)$ *in the pairing* $\mathcal{X}(-1) \times \mathcal{M} \to \mathbb{Q}_l/\mathbb{Z}_l$, *and assume that the Gross conjecture holds. Then by 6.7* $ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l = \mathcal{N}$, *and thus*

$$(ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty}/K^{et}_{2i-1}(O^S_E)\otimes_{\mathbb{Z}_l}\mathbb{Q}_l/\mathbb{Z}_l \simeq (H(1-i)^{G_\infty})^*\,.$$

This follows from diagram 2 in chapter 4, namely

$$\begin{aligned} \mathcal{N}(i-1)^{G_\infty}/max.div. &\simeq Hom_{\mathbb{Z}_l}(l\text{-}tor\, f_\Lambda \mathcal{X}(-i)_{G_\infty}, \mathbb{Q}_l/\mathbb{Z}_l) \\ &\simeq Hom_{\mathbb{Z}_l}(H(1-i)^{G_\infty}, \mathbb{Q}_l/\mathbb{Z}_l)\,. \end{aligned}$$

Passing to the direct limit in 6.16 implies

Corollary 6.18 *Let $\delta_\infty^{Gro} = max\, \delta_n^{Gro}$, cf. 6.7. Then for $i \geq 2$, there is an exact sequence*

$$0 \to (\mathbf{Q}_l/\mathbf{Z}_l(i-1))^{\delta_\infty^{Gro}} \to \varinjlim WK_{2i-2}^{\acute{e}t}(F_n) \to l\text{-}tor\, C_\infty(i-1) \to 0\ .$$

Since $ker(l\text{-}tor\, C_n \to l\text{-}tor\, C_m)$ is finite independent of $m \geq n \geq 0$, say of exponent $l^{\tilde{n}_1}$, we still have on the finite level

Corollary 6.19 *For $i \geq 2$ and $1 \leq \nu \leq n - \tilde{n}_1$, there is a split exact sequence*

$$0 \to ker\, g_n \otimes_{\mathbf{Z}_l} \tfrac{1}{l^\nu}\mathbf{Z}_l/\mathbf{Z}_l(i-1)/K_{2i-1}^{\acute{e}t}(O_n^S) \otimes_{\mathbf{Z}_l} \tfrac{1}{l^\nu}\mathbf{Z}_l/\mathbf{Z}_l \to {}_{l^\nu}WK_{2i-2}^{\acute{e}t}(F_n) \to {}_{l^\nu}C_n(i-1) \to 0\ .$$

and

$$ker\, g_n \otimes_{\mathbf{Z}_l} \tfrac{1}{l^\nu}\mathbf{Z}_l/\mathbf{Z}_l(i-1)/K_{2i-1}^{\acute{e}t}(O_n^S) \otimes_{\mathbf{Z}_l} \tfrac{1}{l^\nu}\mathbf{Z}_l/\mathbf{Z}_l \simeq (\mathbf{Z}/l^\nu\mathbf{Z})^{\delta_n^{Gro}}\ ,$$

Proof: The proof of 5.13 carries over word by word. □

Let $\tilde{\mathcal{X}} := Gal(\tilde{M}_\infty/F_\infty)$, where $\tilde{M}_\infty$ corresponds via Kummer theory to $D_\infty^{(\infty)}$, i.e.,

$$\tilde{\mathcal{X}} \times D_\infty^{(\infty)} \longrightarrow \mathbf{Q}_l/\mathbf{Z}_l$$

is a pairing of Λ-modules, and since $\varinjlim K_{2i-1}^{\acute{e}t}(O_n^S) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l \simeq \mathcal{N}(i-1)$, for $i \geq 2$,

$$tor_\Lambda \tilde{\mathcal{X}} \times \varinjlim WK_{2i-2}^{\acute{e}t}(F_n) \longrightarrow \mathbf{Q}_l/\mathbf{Z}_l$$

is a pairing as well. By replacing $\mathcal{X}$ by $\tilde{\mathcal{X}}$ and $U_\infty^S \otimes \mathbf{Q}_l/\mathbf{Z}_l$ by $ker\, g_\infty \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l$ in the end of chapter 4, we obtain some more analogies of $WK_{2i-2}^{\acute{e}t}(\cdot)$ with $K_{2i-2}^{\acute{e}t}(\cdot)$.

Corollary 6.20 *Let H be the cokernel of $\mathcal{X}(-1)/tor_{\Lambda}\mathcal{X}(-1) \to \Lambda^{r_2(F)}$, cf. 4.14. Then for all $n \geq 0$,*

$$0 \to (H(1-i)^{\Gamma_n})^* \to WK^{ét}_{2i-2}(F_n) \to (\varprojlim WK^{ét}_{2i-2}(F_m))^{\Gamma_n} \to 0$$

is an exact sequence, and there is an integer $n_h \geq 0$ independent of $i \geq 2$, such that for all $n \geq n_h$,

$$0 \to H^* \to WK^{ét}_{2i-2}(F_n) \to (\varprojlim WK^{ét}_{2i-2}(F_m))^{\Gamma_n} \to 0$$

is exact as well.

Corollary 6.21 *If $WK^{ét}_{2i-2}(F_n) = 0$ for some $n \geq 0$ and $i \geq 2$, then $WK^{ét}_{2i-2}(F_n) = 0$ for all $n \geq 0$ and all $i \geq 2$.*

There is also an analogue to 5.17, but clearly we do not get any information on the number $s = s(F_\infty/F)$ of l-adic primes in F_∞.

Proposition 6.22 *$WK^{ét}_{2i-2}(F_n) = 0$ for some $n \geq 0$ and $i \geq 2$ if and only if l-tor $C_n = 0$ for all $n \geq 0$.*

Proof: By 6.16 we have $WK^{ét}_{2i-2}(F_n) = 0$ if and only if $H(1-i)^{\Gamma_n} = 0$ and $(l\text{-}tor\, C_\infty(i-1))^{\Gamma_n} = 0$, from which we deduce

$$H = 0 \text{ and } l\text{-}tor\, C_\infty = 0\,.$$

Using $H = 0$ and 6.8 we get l-*tor* $C_n = 0$ for all $n \geq 0$ large enough. But the norm(restriction) map l-*tor* $C_n \to l$-*tor* C_{n-1} is obviously surjective, thus

l-$tor\, C_n = 0$ for all $n \geq 0$. The converse is trivial. □

Combining 2.34 and 5.22 we obtain

Proposition 6.23 *Let E be a totally real number field and $i - 1 \equiv 1 \bmod 2$, then*

$$\#WK_{2i-2}^{et}(E) = \left| \frac{\#W_l^{(i)}(E) \cdot \zeta_E(1-i)}{\prod \#W_l^{(i-1)}(E_p)} \right|_l^{-1} .$$

Let us return to the case of an arbitrary number field.

Proposition 6.24 *Let E be a number field, l an odd prime and $F_\infty = E(W_l)$ with Galois group $G_\infty = Gal(F_\infty/E)$. Assume that the Gross conjecture holds, then for every $i \in \mathbb{Z}$,*

$$\#l\text{-}tor\, C_\infty(i-1)^{G_\infty} = \#A_\infty^S(i-1)^{G_\infty} .$$

Proof: Recall the notation from 4.9. Since the Gross conjecture holds, l-$tor\, C_n \simeq X_\infty^S/\omega_n X_\infty^S$, and hence l-$tor\, C_\infty \simeq \varinjlim X_\infty^S/\omega_n X_\infty^S$. From the finiteness of $X_\infty^S/\omega_n X_\infty^S$ we obtain by the theory of adjoints, cf. 3.15,

$$Hom_{\mathbb{Z}_l}(l\text{-}tor\, C_\infty, \mathbb{Q}_l/\mathbb{Z}_l) \simeq \alpha(X_\infty^S) .$$

On the other hand we know

$$Hom_{\mathbb{Z}_l}(A_\infty^S, \mathbb{Q}_l/\mathbb{Z}_l) \simeq \alpha(Y_\infty^S) .$$

Since X_∞^S/Y_∞^S is finite, we get

$$0 \to \alpha(X_\infty^S) \to \alpha(Y_\infty^S)$$

with finite cokernel, cf. 3.14. This implies

$$A_\infty^S \to l\text{-}tor\, C_\infty \to 0$$

with finite kernel, and we deduce the formula. □

Remark 6.25 *The previous proposition in conjunction with 5.5 also gives an alternative proof of 6.15 under the assumption of the Gross conjecture.*

The next corollary is an immediate consequence of 2.34.

Corollary 6.26 *Let the notation be as in the previous proposition. Then for $i \geq 2$*

$$\begin{aligned} \#tor_{\mathbb{Z}_l}\mathcal{Y}^S(-i)_{G_\infty} &= \#(U_\infty^S \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty}/K_{2i-1}^{et}(O_E^S) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \\ &= \#H(1-i)^{G_\infty} \cdot \frac{\prod_{\mathfrak{p}|l} \#W_l^{(i-1)}(E_\mathfrak{p})}{\#W_l^{(i-1)}(E)}, \end{aligned}$$

where $\mathcal{Y}^S := Gal(N_\infty^S/F_\infty)$ with $N_\infty^S := F_\infty(\sqrt[l^\infty]{U_\infty^S})$.

Another quite interesting application is the following

Corollary 6.27 *Let $F = E(\zeta_l)$, $l^e = W_l(F)$ and j_0 resp. j_1 as in 4.18. Assume that the Gross conjecture holds, then for $i \geq 2$,*

$$l^{(j_0+1)(e+\nu_l(1-i))+j_1} = \prod_{\mathfrak{p}|l} \#W_l^{(i-1)}(F_\mathfrak{p}).$$

Proof: Follows immediately from 4.18 and 6.26. □

For a given field F, this corollary enables us to calculate j_0 and j_1, e.g., let $i = 2$ and $F = \mathbb{Q}(\zeta_m)$ with $m = l^e \cdot r$, then

$$j_0 = g_l(F) - 1 \text{ and } j_1 = 0 \ ,$$

i.e., $\xi_{n_j} = T$ in 4.13 and hence

$$\mathcal{Y}^S \longrightarrow \Lambda^{r_2(F)} \oplus \left(\Lambda/((1+l^e)^{-1}(1+T)-1)\right)^{g_l(F)-1}$$

is an injective Λ-module morphism with finite cokernel.

7 The finiteness of $A_\infty^S(i-1)^{G_\infty}$

We begin this chapter by discussing the Gross conjecture, i.e., the finiteness of ${A_\infty^S}^{G_\infty}$, cf. 6.4, and to some extent also its proof, cf. [77]. We also explain, why we can assume without loss of generality, that the μ-invariant of the cyclotomic $\mathbb{Z}_l$-extension F_∞/F is trivial, cf. 7.5. In the second part we study the Schneider conjecture, i.e., the finiteness of $A_\infty^S(i-1)^{G_\infty}$ and $l\text{-}tor\, C_\infty(i-1)^{G_\infty}$ for $i \neq 1$, cf. 5.5 and 6.15 or 6.24, and its relation to étale K-theory. This leads then automatically to higher rank formulas for $K_{2i-2}^{et}(O_E^S)$ and $WK_{2i-2}^{et}(E)$.

Let E be a number field, l an odd prime, $F = E(\zeta_l)$ and F_∞ the cyclotomic $\mathbb{Z}_l$-extension of F with intermediate fields F_n and Galois groups $\Gamma_n = Gal(F_\infty/F_n)$, $n \geq 0$. As in chapter 4, let L_∞^S be the maximal abelian, pro-l-extension of F_∞, unramified outside S, and in which all l-adic primes are totally decomposed. Then $X_\infty^S := Gal(L_\infty^S/F_\infty)$ is a Λ-torsion module, cf. 4.3, and the Gross conjecture for F_n is equivalent to the finiteness of $X_\infty^S/\omega_n X_\infty^S$, cf. 6.4. If $i-1 \geq 1$, then we know that $X_\infty^S(i-1)/\omega_n X_\infty^S(i-1)$ is finite, cf. 5.5. Following P. Schneider, cf. [82], we can make an even better statement.

Proposition 7.1 *Let $i-1\geq 1$, then there is a canonical isomorphism*

$$h_n^{(i-1)}: X_\infty^S(i-1)/\omega_n X_\infty^S(i-1) \xrightarrow{\sim} WK_{2i-2}^{ét}(F_n)\ ,$$

which is compatible with direct limits, and thus we get

$$h_\infty^{(i-1)}: \varinjlim X_\infty^S(i-1)/\omega_n X_\infty^S(i-1) \xrightarrow{\sim} \varinjlim WK_{2i-2}^{ét}(F_n)\ .$$

Proof: For simplicity, we assume that $n\geq n_0 = n_0(F_\infty/F)$, i.e., all l-adic primes in F_n are totally ramified in F_∞, and thus $\Gamma_n \simeq Gal(F_{\infty,\mathfrak{p}_\infty}/F_{n,\mathfrak{p}_n})$. Since $1-i\neq 0$, we have $H^1(\Gamma_n, \mathbf{Q}_l/\mathbf{Z}_l(1-i))=0$, cf. 1.5 or the proof of 5.2. Hence we get the commutative diagram

$$\begin{array}{ccc} H^1_{ét}(O_n^S, \mathbf{Q}_l/\mathbf{Z}_l(1-i)) & \rightarrow & \bigoplus_{\mathfrak{p}_n|l} H^1(F_{n,\mathfrak{p}_n}, \mathbf{Q}_l/\mathbf{Z}_l(1-i)) \\ \downarrow\wr & & \downarrow\wr \\ H^1_{ét}(O_\infty^S, \mathbf{Q}_l/\mathbf{Z}_l(1-i))^{\Gamma_n} & \rightarrow & \bigoplus_{\mathfrak{p}_\infty|l} H^1(F_{\infty,\mathfrak{p}_\infty}, \mathbf{Q}_l/\mathbf{Z}_l(1-i))^{\Gamma_n}\ , \end{array}$$

and

$$\begin{aligned} & ker\left(H^1_{ét}(O_\infty^S, \mathbf{Q}_l/\mathbf{Z}_l(1-i))^{\Gamma_n} \rightarrow \bigoplus_{\mathfrak{p}_\infty|l} H^1(F_{\infty,\mathfrak{p}_\infty}, \mathbf{Q}_l/\mathbf{Z}_l(1-i))^{\Gamma_n}\right) \\ = \ & ker\left(H^1_{ét}(O_\infty^S, \mathbf{Q}_l/\mathbf{Z}_l(1)) \rightarrow \bigoplus_{\mathfrak{p}_\infty|l} H^1(F_{\infty,\mathfrak{p}_\infty}, \mathbf{Q}_l/\mathbf{Z}_l(1))\right)(-i)^{\Gamma_n} \\ \simeq \ & Hom_{\mathbf{Z}_l}(X_\infty^S, W_l)(-i)^{\Gamma_n} \\ \simeq \ & \left(X_\infty^S(i-1)/\omega_n X_\infty^S(i-1)\right)^*\ . \end{aligned}$$

By Tate-Poitou duality we have

$$\begin{aligned} & ker\left(H^1_{ét}(O_n^S, \mathbf{Q}_l/\mathbf{Z}_l(1-i)) \rightarrow \bigoplus_{\mathfrak{p}_n|l} H^1(F_{n,\mathfrak{p}_n}, \mathbf{Q}_l/\mathbf{Z}_l(1-i))\right) \\ \simeq \ & ker\left(H^2_{ét}(O_n^S, \mathbf{Z}_l(i)) \rightarrow \bigoplus_{\mathfrak{p}_n|l} H^2(F_{n,\mathfrak{p}_n}, \mathbf{Z}_l(i))\right)^* \\ = \ & WK_{2i-2}^{ét}(F_n)^*\ . \end{aligned}$$

The compatibility of $h_n^{(i-1)}$ with direct limits follows from the morphisms involved. □

Corollary 7.2 *Let* $i-1 \geq 1$, *then*

$$A_\infty^S(i-1) \sim \varinjlim WK_{2i-2}^{\acute{e}t}(F_n)\,.$$

Proof: For $Y_\infty^S := Gal(L_\infty^S/F_\infty H_{n_0}^S)$ as in 4.9, we have

$$\begin{aligned} A_\infty^S(i-1) &\simeq \left(\varinjlim Y_\infty^S/\nu_{n_0,n} Y_\infty^S\right)(i-1) \\ &\simeq \beta(Y_\infty^S)(i-1) \\ &\simeq \beta(Y_\infty^S(i-1))\,. \end{aligned}$$

Since X_∞^S/Y_∞^S is finite, $\beta(Y_\infty^S(i-1)) \to \beta(X_\infty^S(i-1))$ is surjective with finite kernel. By the previous proposition ω_n is an $X_\infty^S(i-1)$-admissible sequence, and thus $\varinjlim X_\infty^S(i-1)/\omega_n X_\infty^S(i-1) \simeq \beta(X_\infty^S(i-1))$. Putting all things together, we obtain a surjective Λ-morphism

$$A_\infty^S(i-1) \longrightarrow \varinjlim WK_{2i-2}^{\acute{e}t}(F_n)$$

with finite kernel. □

Consider

$$\begin{array}{ccc} \varinjlim WK_{2i-2}^{\acute{e}t}(F_n) & & \\ \downarrow & \searrow^{\pi_{i-1}} & \\ \mathcal{M}(i-1)/\mathcal{N}(i-1) & \to & A_\infty^S(i-1)\,, \end{array}$$

then the fundamental result in [77] is the following

Theorem 7.3 *Let $i-1 \geq 1$, and suppose that the μ-invariant of F_∞/F is trivial, cf. 4.6. Then*

$$\pi(i-1) : \varinjlim WK_{2i-2}^{\acute{e}t}(F_n) \longrightarrow A_\infty^S(i-1)$$

has finite kernel.

From 7.2 we obtain

Corollary 7.4 *With the notations as in 7.3,*

$$\pi(i-1) : \varinjlim WK_{2i-2}^{\acute{e}t}(F_n) \longrightarrow A_\infty^S(i-1)^{G_\infty}$$

is a pseudo-isomorphism.

Regarding the Gross conjecture we have

Theorem 7.5 *Let $i-1 \geq 1$, then*

$$\pi(i-1) : \varinjlim WK_{2i-2}^{\acute{e}t}(F_n) \longrightarrow A_\infty^S(i-1)^{G_\infty}$$

is a surjective Λ-module morphism with finite kernel, and the Gross conjecture is valid.

Proof: Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & \ker g_\infty \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l(i-1)/\mathcal{N}(i-1) & \to & \varinjlim WK_{2i-2}^{\acute{e}t}(F_n) & \to & l\text{-}tor\, C_\infty(i-1) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & U_\infty^S \otimes \mathbf{Q}_l/\mathbf{Z}_l(i-1)/\mathcal{N}(i-1) & \to & \varinjlim K_{2i-2}^{\acute{e}t}(F_n) & \to & A_\infty^S(i-1) & \to 0,
\end{array}$$

cf. 5.11 and 6.18. Then we deduce the exact sequence

$$0 \to \left(ker\, g_\infty \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l/\mathcal{N}\right)(i-1) \to ker\,\pi(i-1) \to \left(U_\infty^S \otimes \mathbf{Q}_l/\mathbf{Z}_l/ker\, g_\infty \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l\right)(i-1)\,.$$

Since the dual of the left resp. right group has trivial μ-invariant, cf. 4.13 and 6.7, we can assume without loss of generality that the μ-invariant of the cyclotomic $\mathbf{Z}_l$-extension F_∞/F is trivial. Hence we obtain from 7.4

$$\pi(i-1) : \varinjlim WK_{2i-2}^{ét}(F_n) \longrightarrow A_\infty^S(i-1)$$

is a pseudo-isomorphism. Since $\left(ker\, g_\infty \otimes_{\mathbf{Z}_l} \mathbf{Q}_l/\mathbf{Z}_l/\mathcal{N}\right)(i-1) \simeq (\mathbf{Q}_l/\mathbf{Z}_l(i-1))^{\delta_\infty^{Gro}}$, cf. 6.7 or 6.16, this already implies the validity of the Gross conjecture and an isomorphism

$$\varinjlim WK_{2i-2}^{ét}(F_n) \xrightarrow{\sim} l\text{-}tor\, C_\infty(i-1)\,.$$

Therefore, we are left with showing that the canonical map

$$\pi : l\text{-}tor\, C_\infty \longrightarrow A_\infty^S$$

is surjective. But this is obvious, since the Gross conjecture holds, and thus in particular

$$X_\infty^S/\omega_n X_\infty^S \simeq l\text{-}tor\, C_n\,.$$

□

Remark 7.6 *As mentioned earlier, cf. 5.6, the fact, that $\mathcal{M}/\mathcal{N} \to A^S_\infty$ admits a pseudo-splitting, already implies the Gross conjecture.*

Regarding the Galois invariants $A^S_\infty(i-1)^{G_\infty}$ and $l\text{-}tor\, C_\infty(i-1)^{G_\infty}$, we know the following

- *$A^S_\infty{}^{G_\infty}$ and $l\text{-}tor\, C^{G_\infty}_\infty$ are finite, since the Gross conjecture holds, cf. 6.4 and 6.24.*
- *$A^S_\infty(i-1)^{G_\infty}$ and $l\text{-}tor\, C_\infty(i-1)^{G_\infty}$ are finite for $i-1 \geq 1$, since the Schneider conjecture is valid for $i \geq 2$, cf. 5.5 and 6.24.*

When considering the Schneider conjecture it turns out, that the idelé class group $l\text{-}tor\, C_\infty$ and the higher wild kernel $WK^{\acute{e}t}_{2i-2}(E)$ seem to be the right objects. This does not come as a surprise; namely M. Kolster proved a reflection theorem for $WK^{\acute{e}t}_{2i-2}(E)$ generalizing the well-known 'Spiegelung' theorem of Leopoldt, cf. [53].

Lemma 7.7 *Let $i \in \mathbb{Z}$, and suppose that $\varepsilon_{i-1}H = 0$, where H is the finite Λ-module defined in diagram 1 chapter 4. Then*

$$0 \to \varepsilon_{i-1} ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l(i-1) \to \varepsilon_{i-1} D^{(\infty)}_\infty(i-1) \to \varepsilon_{i-1} l\text{-}tor\, C_\infty(i-1) \to 0$$

is a split exact sequence of Λ-modules, and thus

$$0 \to (ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} \to D^{(\infty)}_\infty(i-1)^{G_\infty} \to l\text{-}tor\, C_\infty(i-1)^{G_\infty} \to 0$$

is split exact as well.

Proof: We already know the exactness of both sequences, cf. 6.10. Since the Gross conjecture holds, we have $ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l = \mathcal{N}$, where $\mathcal{N} = (tor_\Lambda \mathcal{X}(-1))^\perp$ is the orthogonal complement of the Λ-torsion submodule of $\mathcal{X} = Gal(M_\infty/F_\infty)(-1)$. The split exactness of the first sequence—and thus of the second as well— follows now from duality and the assumption $\varepsilon_{i-1}H = 0$, cf. chapter 4. □

For all $\nu \geq 1$, the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_\infty, H^q_{ét}(O^S_\infty, \mathbb{Z}/l^\nu\mathbb{Z}(i))) \Rightarrow H^{p+q}_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$$

almost degenerates, i.e., there is an exact sequence

$$0 \to H^1(G_\infty, H^0_{ét}(O^S_\infty, \mathbb{Z}/l^\nu\mathbb{Z}(i))) \to H^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to H^1_{ét}(O^S_\infty, \mathbb{Z}/l^\nu\mathbb{Z}(i))^{G_\infty} \to 0$$

and an isomorphism

$$H^2_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq H^1(G_\infty, H^1_{ét}(O^S_\infty, \mathbb{Z}/l^\nu\mathbb{Z}(i))) .$$

All remaining terms are trivial. Clearly $H^0_{ét}(O^S_\infty, \mathbb{Z}/l^\nu\mathbb{Z}(i)) = \mathbb{Z}/l^\nu\mathbb{Z}(i)$, and for $i \in \mathbb{Z}$ and $\nu \geq 1$, we set

$$f\text{-}H^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) := H^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i))/H^1(G_\infty, \mathbb{Z}/l^\nu\mathbb{Z}(i)) ,$$

and similarly

$$f\text{-}\tilde{H}^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) := \tilde{H}^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i))/H^1(G_\infty, \mathbb{Z}/l^\nu\mathbb{Z}(i)) .$$

Then for $G_n := Gal(F_n/E)$, we have

$$f\text{-}H^1_{et}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq H^1_{et}(O^S_\infty, \mathbb{Z}/l^\nu\mathbb{Z}(i))^{G_\infty} \text{ and } f\text{-}H^1_{et}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq f\text{-}H^1_{et}(O^S_n, \mathbb{Z}/l^\nu\mathbb{Z}(i))^{G_n}$$

as well as

$$f\text{-}\tilde{H}^1_{et}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq \tilde{H}^1_{et}(O^S_\infty, \mathbb{Z}/l^\nu\mathbb{Z}(i))^{G_\infty} \text{ and } f\text{-}\tilde{H}^1_{et}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq \tilde{H}^1_{et}(O^S_n, \mathbb{Z}/l^\nu\mathbb{Z}(i))^{G_n}.$$

Let $l^e = \#W_l(F)$ and suppose that $\varepsilon_{i-1}H = 0$. Then for $1 \le \nu \le e+n$, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to & (ker\, g_n/\mu_{l^\nu} \otimes_{\mathbb{Z}_l} \frac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l(i-1))^{G_n} & \to & f\text{-}\tilde{H}^1_{et}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) & \overset{\varphi_3}{\to} & {}_{l^\nu}C_n(i-1)^{G_n} & \\
& \downarrow & & \downarrow \wr \, res & & \downarrow & \\
0 \to & {}_{l^\nu}(ker\, g_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} & \to & {}_{l^\nu}D^{(\infty)}_\infty(i-1)^{G_\infty} & \to & {}_{l^\nu}C_\infty(i-1)^{G_\infty} & \to 0\,.
\end{array}$$

Remark 7.8 *(i) Let $\nu = e+n$, and suppose that $i-1 \not\equiv 0 \bmod l$. Then $ker\, g_n/\mu_{l^\nu} \otimes_{\mathbb{Z}_l} \frac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l(i-1)$ is a cohomologically trivial G_n-module, cf. [64] or [83], and hence the upper right, horizontal arrow in the above diagram is surjective. (ii) Let $i \ge 2$, then l-tor $C_\infty(i-1)^{G_\infty}$ is finite, and thus for $\nu \ge 1$ large enough, the bottom row is exact without assuming $\varepsilon_{i-1}H = 0$. But the disadvantage is clear, namely $\nu \ge 1$ depends on $i \ge 2$ and we can not gain any information on the interesting case $i \le 0$. Hence the importance of 7.7 is now evident.*

Theorem 7.9 *Let $i-1 \ne 1$, and suppose that $\varepsilon_{i-1}H = 0$. Assume that there exist $n \ge 0$ and ν, $1 \le \nu \le e+n$, such that*

$$rk_{l^\nu} l\text{-}tor\, C_n(i-1)^{G_n} = 0\,,$$

then

$$H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0 \ .$$

If $i-1 \not\equiv 0 \bmod l$, *the converse is also satisfied for* $\nu = e+n$.

Proof: By 6.15 the triviality of $H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i))$ is equivalent to the finiteness of $l\text{-}tor\, C_\infty(i-1)^{G_\infty}$, and thus the first part is clear by the above diagram. The kernel of ${}_{l^\nu}l\text{-}tor\, C_n(i-1)^{G_n} \to {}_{l^\nu}l\text{-}tor\, C_\infty(i-1)^{G_\infty}$ has bounded order independently of $n \geq 0$. Moreover, if $s_{l^\nu} : {}_{l^\nu}l\text{-}tor\, C_\infty(i-1)^{G_\infty} \to {}_{l^\nu}D_\infty^{(\infty)}(i-1)^{G_\infty}$ denotes the morphism induced by the section $l\text{-}tor\, C_\infty(i-1) \to D_\infty^{(\infty)}(i-1)$, then $\varphi_n \circ res^{-1} \circ s_{l^\nu} : {}_{l^\nu}l\text{-}tor\, C_\infty(i-1)^{G_\infty} \to {}_{l^\nu}l\text{-}tor\, C_n(i-1)^{G_n}$ is a section of ${}_{l^\nu}l\text{-}tor\, C_n(i-1)^{G_n} \to {}_{l^\nu}l\text{-}tor\, C_\infty(i-1)^{G_\infty}$. Hence by 7.8 the second part follows as well. □

Corollary 7.10 *Suppose that there exist* $n \geq 0$ *and* ν, $1 \leq \nu \leq e+n$, *such that*

$$rk_{l^\nu}\, l\text{-}tor\, C_n = 0 \ ,$$

then for all $i \neq 1$ *with* $\varepsilon_{i-1}H = 0$,

$$H^2_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(i)) = 0 \ .$$

Let us now consider negative twists, i.e. the case $i \leq 0$, in particular. If we want to relate this to étale K-theory, which is rather the case $i \geq 2$, it becomes clear that we have to use cup-product arguments. To be precise, let $d_n := [E(\zeta_{l^n}) : E]$, then for $i \equiv k \; mod \, d_n$ and $1 \leq \nu \leq n$, the cup-product induces an isomorphism, cf. 1.3 and 6.11,

$$H^j_{\acute{e}t}(O^S_E, \mathbf{Z}/l^\nu\mathbf{Z}(i)) \simeq H^j_{\acute{e}t}(O^S_E, \mathbf{Z}/l^\nu\mathbf{Z}(k))(i-k)$$

resp.

$$\tilde{H}^1_{\acute{e}t}(O^S_E, \mathbf{Z}/l^\nu\mathbf{Z}(i)) \simeq \tilde{H}^1_{\acute{e}t}(O^S_E, \mathbf{Z}/l^\nu\mathbf{Z}(k))(i-k) \; ,$$

which is compatible with the cup-product in étale K-theory, cf. [20], e.g., for $i, k \geq 2$,

$$\begin{array}{ccc} H^2_{\acute{e}t}(O^S_E, \mathbf{Z}/l^\nu\mathbf{Z}(i)) & \stackrel{\sim}{\rightarrow} & H^2_{\acute{e}t}(O^S_E, \mathbf{Z}/l^\nu\mathbf{Z}(k))(i-k) \\ \downarrow \wr & & \downarrow \wr \\ K^{\acute{e}t}_{2i-2}(O^S_E) \otimes_{\mathbf{Z}_l} \mathbf{Z}_l/l^\nu\mathbf{Z}_l & \stackrel{\sim}{\rightarrow} & K^{\acute{e}t}_{2k-2}(O^S_E) \otimes_{\mathbf{Z}_l} \mathbf{Z}_l/l^\nu\mathbf{Z}_l(i-k) \end{array}$$

is a commutative diagram, cf. 2.20.

Theorem 7.11 *Let $i \neq 1$ be an integer and $d_n := [E(\zeta_{l^n}) : E]$. Then the following are equivalent.*

(i) $H^2_{ét}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i))$ *is finite, i.e., the Schneider conjecture is valid for* E *and* l.

(ii) *For some* $n \geq 1$ *and* $i \equiv k \bmod d_n$, $k \geq 2$,

$$rk_{l^n} K^{ét}_{2k-2}(O^S_E) = 0 \ .$$

(iii) *For some* $n \geq 1$ *and* $i \equiv k \bmod d_n$, $k \geq 2$,

$$rk_{l^n} WK^{ét}_{2k-2}(E) = 0 \ .$$

Proof: Consider the exact sequence

$$H^2_{cont}(O^S_E, \mathbb{Z}_l(i)) \stackrel{\times l^n}{\to} H^2_{cont}(O^S_E, \mathbb{Z}_l(i)) \to H^2_{ét}(O^S_E, \mathbb{Z}/l^n\mathbb{Z}(i)) \to 0$$

arising from $0 \to \mathbb{Z}_l(i) \stackrel{\times l^n}{\to} \mathbb{Z}_l(i) \to \mathbb{Z}/l^n\mathbb{Z}(i) \to 0$. If $H^2_{cont}(O^S_E, \mathbb{Z}_l(i))$ is finite, then $rk_{l^n} H^2_{ét}(O^S_E, \mathbb{Z}/l^n\mathbb{Z}(i)) = 0$ for some $n \geq 1$. The cup-product isomorphism implies $rk_{l^n} H^2_{ét}(O^S_E, \mathbb{Z}/l^n\mathbb{Z}(k)) = 0$ for $i \equiv k \bmod d_n$, $k \geq 2$, and hence $rk_{l^n} K^{ét}_{2k-2}(O^S_E) = 0$. Similarly for the converse. The equivalence of (ii) and (iii) is obvious, since the order of $W^{(k-1)}_l(E_v)$ is bounded independently of $k-1 \bmod d_n$. □

Let us consider the Leopoldt conjecture more closely, i.e., the case $i = 0$ in the Schneider conjecture. Since for $i \neq 0$, $H^1(G_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i) = 0$, we have for

$n \geq 1$ and $\acute{d} := max(d_n, 2)$

$$H^0(G_\infty, \mathbb{Q}_l/\mathbb{Z}_l(\acute{d}_n)) \otimes_{\mathbb{Z}_l} \tfrac{1}{l^n}\mathbb{Z}_l/\mathbb{Z}_l \simeq H^1(G_\infty, \mathbb{Z}/l^n\mathbb{Z}(\acute{d}_n)) .$$

But $H^0(G_\infty, \mathbb{Q}_l/\mathbb{Z}_l(\acute{d}_n)) \simeq H^0_{\acute{e}t}(O_E^S, \mathbb{Q}_l/\mathbb{Z}_l(\acute{d}_n)) \simeq l\text{-}tor\, H^1_{\acute{e}t}(O_E^S, \mathbb{Z}_l(\acute{d}_n))$, and we get the commutative diagram with exact rows

$$\begin{array}{ccccccc} & H^0(G_\infty, \mathbb{Q}_l/\mathbb{Z}_l(\acute{d}_n)) \otimes_{\mathbb{Z}_l} \tfrac{1}{l^n}\mathbb{Z}_l/\mathbb{Z}_l & \tilde{\rightarrow} & H^1(G_\infty, \mathbb{Z}/l^n\mathbb{Z}(\acute{d}_n)) & & \\ & \downarrow & & \downarrow & & \\ 0 \rightarrow & K^{\acute{e}t}_{2\acute{d}_n-1}(O_E^S) \otimes_{\mathbb{Z}_l} \tfrac{1}{l^n}\mathbb{Z}_l/\mathbb{Z}_l & \rightarrow & H^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l^n\mathbb{Z}(\acute{d}_n)) & \rightarrow {}_{l^n}K^{\acute{e}t}_{2\acute{d}_n-2}(O_E^S) \rightarrow 0 . \end{array}$$

Since $l\text{-}tor\, K^{\acute{e}t}_{2\acute{d}_n-1}(O_E^S)$ contains an element of order l^n, we deduce the following

Proposition 7.12 *The number of independent cyclic extensions of E of degree l^n, which are unramified outside l, is given by*

$$rk_{l^n} H^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l^n\mathbb{Z}) = 1 + r_2(E) + rk_{l^n} K^{\acute{e}t}_{2\acute{d}_n-2}(O_E^S) ,$$

where $\acute{d}_n = max(d_n, 2)$, $d_n := [E(\zeta_{l^n}) : E]$.

For the l^n-rank of the higher wild kernel of E, we have the following

Proposition 7.13 *The number of independent cyclic extensions of E of degree l^n, which are locally embeddable into a $\mathbb{Z}_l$-extension, is given by*

$$rk_{l^n} \tilde{H}^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l^n\mathbb{Z}) = 1 + r_2(E) + rk_{l^n} WK^{\acute{e}t}_{2\acute{d}_n-2}(E) ,$$

where $\acute{d}_n = max(d_n, 2)$, $d_n := [E(\zeta_{l^n}) : E]$.

Proof: This is clear, since

$$0 \to K^{\acute{e}t}_{2d_n-1}(O_E^S) \otimes_{\mathbb{Z}_l} \tfrac{1}{l^n}\mathbb{Z}_l/\mathbb{Z}_l \to \tilde{H}^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l^n\mathbb{Z}(\acute{d}_n)) \to {}_{l^n}WK^{\acute{e}t}_{2d_n-2}(E) \to 0$$

is split exact and $\tilde{H}^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l^n\mathbb{Z})$ classifies such extensions, cf. [74]. □

The results in 7.11–7.13 are enough motivation for us to compute higher rank formulas for $K^{\acute{e}t}_{2i-2}(O_E^S)$ and $WK^{\acute{e}t}_{2i-2}(E)$. The following approach is due to M. Kolster in the case $K_2(O_E^S)$, cf. [52]. Since it goes through for all $i \geq 2$ as well as for the higher wild kernel, we consider it now briefly.

For $i \neq 0$ and $\nu \geq 1$, we have

$$H^0(G_\infty, \mathbb{Q}_l/\mathbb{Z}_l(i)) \otimes_{\mathbb{Z}_l} \tfrac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l \simeq H^1(G_\infty, \mathbb{Z}/l^n\mathbb{Z}(i)) ,$$

cf. above. Thus for $i \geq 2$ and $\nu \geq 1$, we get the split exact sequences

$$0 \to f_{\mathbb{Z}_l}\text{-}K^{\acute{e}t}_{2i-1}(O_E^S) \otimes_{\mathbb{Z}_l} \tfrac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l \to f\text{-}H^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to {}_{l^\nu}K^{\acute{e}t}_{2i-2}(O_E^S) \to 0$$

and

$$0 \to f_{\mathbb{Z}_l}\text{-}K^{\acute{e}t}_{2i-1}(O_E^S) \otimes_{\mathbb{Z}_l} \tfrac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l \to f\text{-}\tilde{H}^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to {}_{l^\nu}WK^{\acute{e}t}_{2i-2}(O_E^S) \to 0 ,$$

where $f_{\mathbb{Z}_l}\text{-}K^{\acute{e}t}_{2i-1}(O_E^S) := K^{\acute{e}t}_{2i-1}(O_E^S)/l\text{-}tor\, K^{\acute{e}t}_{2i-1}(O_E^S)$. Since we know the $\mathbb{Z}_l$-rank of $K^{\acute{e}t}_{2i-1}(O_E^S)$, we obtain the following

Lemma 7.14 *Let $i \geq 2$ and $\nu \geq 1$, then*

$$rk_{l^\nu} K^{\acute{e}t}_{2i-2}(O_E^S) = rk_{l^\nu} f\text{-}H^1_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}(i)) - k_i(E) ,$$

and

$$rk_{l^\nu} WK^{ét}_{2i-2}(O^S_E) = rk_{l^\nu} f\text{-}\tilde{H}^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) - k_i(E)\ ,$$

where

$$k_i(E) := \begin{cases} r_1(E) + r_2(E) & \textit{if } i \equiv 1 \textit{ mod } 2 \\ r_2(E) & \textit{if } i \equiv 0 \textit{ mod } 2 \end{cases} .$$

Thus in order to calculate higher rank formulas for $K^{ét}_{2i-2}(O^S_E)$ and $WK^{ét}_{2i-2}(E)$ we are left to determine the higher rank of $f\text{-}H^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$ and $f\text{-}\tilde{H}^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$. By Galois descent we have for $n \geq 1$ and $\nu \geq 1$,

$$f\text{-}H^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq f\text{-}H^1_{ét}(O^S_n, \mathbb{Z}/l^\nu\mathbb{Z}(i))^{G_n}$$

and

$$f\text{-}\tilde{H}^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq f\text{-}\tilde{H}^1_{ét}(O^S_n, \mathbb{Z}/l^\nu\mathbb{Z}(i))^{G_n}\ ,$$

cf. above, and if $\nu \leq e + n$, the cup-product induces isomorphisms

$$f\text{-}H^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq f\text{-}H^1_{ét}(O^S_n, \mathbb{Z}/l^\nu\mathbb{Z}(1))(i-1)^{G_n}$$

and

$$f\text{-}\tilde{H}^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq f\text{-}\tilde{H}^1_{ét}(O^S_n, \mathbb{Z}/l^\nu\mathbb{Z}(1))(i-1)^{G_n}\ .$$

If $i - 1 \not\equiv 0 \textit{ mod } l$ and $\nu = e + n$, then we can compute the l^ν-rank of $f\text{-}H^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$ and $f\text{-}\tilde{H}^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i))$ via the exact sequences, cf. 7.8,

$$0 \to \left(U^S_n/\mu_{l^\nu} \otimes \tfrac{1}{l^\nu}\mathbb{Z}/\mathbb{Z}(i-1)\right)^{G_n} \to f\text{-}H^1_{ét}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to {}_{l^\nu}A^S_n(i-1)^{G_n} \to 0$$

and

$$0 \to \left(ker\, g_n/\mu_{l^\nu} \otimes_{\mathbb{Z}_l} \tfrac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l(i-1)\right)^{G_n} \to f\text{-}\tilde{H}^1_{\acute{e}t}(O^S_E, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to {}_{l^\nu}C_n(i-1)^{G_n} \to 0\,.$$

Namely from 7.14 we get the following

Theorem 7.15 *Let E be a number field, l an odd prime, $F = E(\zeta_l)$ and F_∞ the cyclotomic $\mathbb{Z}_l$-extension of F with intermediate fields F_n, $n \geq 0$. We set $G_n := Gal(F_n/E)$, $G_0 = \Delta$, and let $l^e = \#W_l(F)$. Furthermore, let $i \geq 2$, and suppose that $i-1 \not\equiv 0 \mod l$. Then for $1 \leq \nu \leq e$,*

$$rk_{l^\nu} K^{\acute{e}t}_{2i-2}(O^S_E) = rk_{l^\nu}\left(U^S_0/\mu_{l^e} \otimes \tfrac{1}{l^\nu}\mathbb{Z}/\mathbb{Z}(i-1)\right)^\Delta - k_i(E) + rk_{l^\nu} A^S_0(i-1)^\Delta$$

and

$$rk_{l^\nu} WK^{\acute{e}t}_{2i-2}(E) = rk_{l^\nu}\left(ker\, g_0/\mu_{l^e} \otimes_{\mathbb{Z}_l} \tfrac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l(i-1)\right)^\Delta - k_i(E) + rk_{l^\nu} C_0(i-1)^\Delta\,.$$

For $\nu = e+n$, $n \geq 1$, we have

$$\begin{aligned} rk_{l^\nu} K^{\acute{e}t}_{2i-2}(O^S_E) = \;& rk_{l^\nu}\left(U^S_n/\mu_{l^\nu} \otimes \tfrac{1}{l^\nu}\mathbb{Z}/\mathbb{Z}(i-1)\right)^{G_n} - k_i(E) \\ & + rk_l\left({}_{l^\nu}A^S_n(i-1)^{G_n}/im\,\alpha^n_{n-1}(i-1)\right) \end{aligned}$$

and

$$\begin{aligned} rk_{l^\nu} WK^{\acute{e}t}_{2i-2}(E) = \;& rk_{l^\nu}\left(ker\, g_n/\mu_{l^\nu} \otimes_{\mathbb{Z}_l} \tfrac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l(i-1)\right)^{G_n} - k_i(E) \\ & + rk_l\left({}_{l^\nu}C_n(i-1)^{G_n}/im\,\gamma^n_{n-1}(i-1)\right)\,, \end{aligned}$$

where $k_i(E)$ is as in the above lemma and $\alpha^n_{n-1}(i-1) : {}_{l^{\nu-1}}A^S_{n-1}(i-1)^{G_{n-1}} \to {}_{l^\nu}A^S_n(i-1)^{G_n}$ resp. $\gamma^n_{n-1}(i-1) : {}_{l^{\nu-1}}C_{n-1}(i-1)^{G_{n-1}} \to {}_{l^\nu}C_n(i-1)^{G_n}$.

As mentioned earlier, these formulas were already proven for $K_2(O^S_E)$ in [52] as well as similar results for the prime $l = 2$. It is also remarked there, that these rank formulas are not the sum of two positive integers and an illustrating example due to J. Hurrelbrink for the case $l = 2$ is given. However for an odd prime l, we have the following

Corollary 7.16 *Let the notations be as in the previous theorem, and suppose that $\varepsilon_{1-i}H = 0$. There exists $n \geq 0$ independent $i \geq 2$ such that for $\nu = e+n$,*

$$rk_{l^\nu} K^{\acute{e}t}_{2i-2}(O^S_E) = rk_l \left({}_{l^\nu}A^S_n(i-1)^{G_n} / im\, \alpha^n_{n-1}(i-1) \right)$$

and

$$rk_{l^\nu} WK^{\acute{e}t}_{2i-2}(E) = rk_l \left({}_{l^\nu}C_n(i-1)^{G_n} / im\, \gamma^n_{n-1}(i-1) \right) \ .$$

Proof: Since for $n \geq 0$ large enough, $H(i-1)^{G_n} \simeq ker({}_{l^\nu}A^S_n(i-1)^{G_n} \to {}_{l^\nu}A^S_\infty(i-1)^{G_\infty}$, we get by assumption

$$\left(U^S_n / \mu_{l^\nu} \otimes \tfrac{1}{l^\nu}\mathbb{Z}/\mathbb{Z}(i-1) \right)^{G_n} \simeq {}_{l^\nu}\left(U^S_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1) \right)^{G_\infty} .$$

But $(U^S_\infty \otimes \mathbb{Q}_l/\mathbb{Z}_l(i-1))^{G_\infty} \simeq (\mathbb{Q}_l/\mathbb{Z}_l)^{k_i(E)} \oplus (tor_{\mathbb{Z}_l}\mathcal{Y}^S(-i)_{G_\infty})^*$, and the second summand is finite, independent of $i - 1 \not\equiv 0 \ mod\, l$. The proof for $WK^{\acute{e}t}_{2i-2}(E)$ carries over word by word. □

If we impose a further condition on i and H, we get an even better result for $WK^{\acute{e}t}_{2i-2}(E)$, but by a different method. Namely

Proposition 7.17 *Let the notations be as in 7.15. Furthermore assume that $\varepsilon_{i-1}H = 0$ and $\varepsilon_{1-i}H = 0$. Then there exists $n \geq 0$ independent of $i \geq 2$ such that for $\nu = e + n$,*

$$rk_{l^\nu} WK_{2i-2}^{\acute{e}t}(E) = rk_{l^\nu}\, l\text{-}tor\, C_n(i-1)^{G_n} .$$

Proof: Since $\varepsilon_{i-1}H = 0$, we get an isomorphism

$$WK_{2i-2}^{\acute{e}t}(E) \simeq l\text{-}tor\, C_\infty(i-1)^{G_\infty} ,$$

cf. 6.16. Moreover, the other assumption implies, that there exists $n \geq 0$ independent of $i \geq 2$, such that for $\nu = e + n$,

$${}_{l^\nu}C_\infty(i-1)^{G_\infty} \simeq {}_{l^\nu}C_n(i-1)^{G_n} ,$$

cf. the proof of 7.9. □

Remark 7.18 *If E is a totally real number field and $j \equiv 1 \bmod 2$, then $\varepsilon_j H = 0$, cf. chapter 5. Thus the extra conditions in 7.16 and 7.17 are satisfied for $i - 1 \equiv 1 \bmod 2$.*

The above rank formulas are obviously of a rather theoretical type. Let us add at least one corollary, from which we can deduce divisibility criteria for $K_{2i-2}^{\acute{e}t}(O_E^S)$ and $WK_{2i-2}^{\acute{e}t}(E)$.

Corollary 7.19 *Suppose that $i-1 \not\equiv 0 \bmod l$ and $[F:E]=2$, and let $1 \le \nu \le e$. Then for $i-1 \equiv 1 \bmod 2$*

$$rk_{l^\nu} K_{2i-2}^{\acute{e}t}(O_E^S) = m + rk_{l^\nu} ker(A_F^S \to A_E^S)$$

and

$$rk_{l^\nu} WK_{2i-2}^{\acute{e}t}(E) = m + rk_{l^\nu} ker(l\text{-}tor\, C_F \to l\text{-}tor\, C_E)\ ,$$

where m is the number of l-adic primes in E, which decompose in F, and for $i-1 \equiv 0 \bmod 2$,

$$rk_{l^\nu} K_{2i-2}^{\acute{e}t}(O_E^S) = g_l(E) - 1 + rk_{l^\nu} A_E^S$$

and

$$rk_{l^\nu} WK_{2i-2}^{\acute{e}t}(E) = g_l(E) - 1 + rk_{l^\nu} l\text{-}tor\, C_E\ ,$$

where $g_l(E)$ denotes the number of l-adic primes in E.

Proof: We just show the formulas for $K_{2i-2}^{\acute{e}t}(O_E^S)$. The proof for $WK_{2i-2}^{\acute{e}t}(E)$ again carries over word by word. Consider the surjective morphism

$$N_{F/E} \otimes id : U_F^S/\mu_{l^e} \otimes \tfrac{1}{l^\nu}\mathbf{Z}/\mathbf{Z} \to U_E^S \otimes \tfrac{1}{l^\nu}\mathbf{Z}/\mathbf{Z}\ .$$

Then for $i-1 \equiv 1 \bmod 2$, $(U_F^S/\mu_{l^e} \otimes \frac{1}{l^\nu}\mathbf{Z}/\mathbf{Z}(i-1))^\Delta \simeq \varepsilon_1(U_F^S/\mu_{l^e} \otimes \frac{1}{l^\nu}\mathbf{Z}/\mathbf{Z}) =$ $ker\, N_{F/E} \otimes id$, and thus

$$rk_{l^\nu}\left(U_F^S/\mu_{l^e} \otimes \tfrac{1}{l^\nu}\mathbf{Z}/\mathbf{Z}(i-1)\right)^\Delta = m + k_i(E)\ .$$

Similarly, $A^S_F(i-1)^\Delta \simeq ker(A^S_F \to A^S_E)$. For $i-1 \equiv 0 \; mod\, 2$, we have

$$_{l^\nu}A^S_F(i-1)^\Delta \simeq {_{l^\nu}A^S_F}^{\Delta} \simeq {_{l^\nu}A^S_E}$$

and

$$\left(U^S_F/\mu_{l^e} \otimes \tfrac{1}{l^\nu}\mathbb{Z}/\mathbb{Z}(i-1)\right)^\Delta \simeq (N_{F/E} \otimes id)\left(U^S_F/\mu_{l^e} \otimes \tfrac{1}{l^\nu}\mathbb{Z}/\mathbb{Z}\right) = U^S_E \otimes \tfrac{1}{l^\nu}\mathbb{Z}/\mathbb{Z}\,.$$

Hence in both cases, the corollary is an immediate consequence of 7.15. □

Example 7.20 *Let $d \in \mathbb{Z}$, $d \neq 1$ and square-free. Set $E = \mathbb{Q}(\sqrt{d})$, $K = \mathbb{Q}(\sqrt{-3d})$ and $F = E(\zeta_3) = \mathbb{Q}(\sqrt{d}, \sqrt{-3})$. Let $h(\cdot)$ resp. $h^S(\cdot)$ denote the ideal resp. S-ideal class number, and assume that $i-1 \not\equiv 0 \; mod\, l$. Then for $i-1 \equiv 1 \; mod\, 2$,*

$$3 \not| K^{et}_{2i-2}(O^S_E) \quad \text{if and only if} \quad d \not\equiv 6 \; mod\, 9 \text{ and } 3 \not| h^S(K)$$
$$\text{if and only if} \quad d \not\equiv 6 \; mod\, 9 \text{ and } 3 \not| h(K)\,,$$

and for $i-1 \equiv 0 \; mod\, 2$,

$$3 \not| K^{et}_{2i-2}(O^S_E) \quad \text{if and only if} \quad d \not\equiv 1 \; mod\, 3 \text{ and } 3 \not| h^S(E)$$
$$\text{if and only if} \quad d \not\equiv 1 \; mod\, 3 \text{ and } 3 \not| h(E)\,.$$

Of course there are analogous results for $WK^{et}_{2i-2}(E)$. But since the order of the idelé class group is rather unknown, we would not get any explicit statements.

8 (l,i)-regular fields and Galois descent

Several authors have studied the arithmetic of l-regular fields, cf. [26], [27], [45] and [71]. Following a suggestion by T. Nguyen Quang Do we introduce the notion of (l,i)-regular fields, and relate this property to étale K-theory, which then leads to the problem of Galois descent.

As usual let E be a number field, l an odd prime, $F = E(\zeta_l)$ and F_∞ the cyclotomic $\mathbb{Z}_l$-extension of F with intermediate fields F_n, $F_0 := F$. For $n \leq \infty$, we set $G_n := Gal(F_n/E)$, $G_0 := \Delta$.

For any number field K, let K^S be the maximal pro-l-extension of K, which is unramified outside S and $X_K^S := Gal(K^S/K)^{ab}$. Then

$$X_K^S \simeq H^1_{\acute{e}t}(O_K^S, \mathbb{Q}_l/\mathbb{Z}_l)^* ,$$

where $*$ stands for the Pontryagin-dual, and thus

$$X_K^S \simeq \mathbb{Z}_l^{1+r_2(K)+\delta_K^{Leop}} \oplus T_K^S ,$$

where T_K^S is the (finite) $\mathbb{Z}_l$-torsion submodule of X_K^S. Moreover for all $\nu \geq 1$,

$$_{l^\nu}T_K^S \simeq \left(coker : H^1_{\acute{e}t}(O_K^S, \mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\times l^\nu} H^1_{\acute{e}t}(O_K^S, \mathbb{Q}_l/\mathbb{Z}_l)\right)^*$$

and there is an exact sequence

$$0 \to (_{l^\nu}T_K^S)^* \to H^2_{\acute{e}t}(O_E^S, \mathbb{Z}/l^\nu\mathbb{Z}) \to \left(ker : H^2_{\acute{e}t}(O_K^S, \mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\times l^\nu} H^2_{\acute{e}t}(O_K^S, \mathbb{Q}_l/\mathbb{Z}_l)\right) \to 0 .$$

$$\left(ker : H^2_{\acute{e}t}(O_K^S, \mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\times l^\nu} H^2_{\acute{e}t}(O_K^S, \mathbb{Q}_l/\mathbb{Z}_l)\right) \simeq (\mathbb{Z}/l^\nu\mathbb{Z})^{\delta_K^{Leop}}$$

As a special case we get the following result, quoted as the wrong duality in [10] and [55].

Proposition 8.1 *Let $1 \le \nu \le e+n$, and suppose that the Leopoldt conjecture is valid for F_n. Then there is a pairing*

$$ {}_{l^\nu}T_n^S \times K_2^{\acute{e}t}(O_n^S) \otimes_{\mathbb{Z}_l} \tfrac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l(-1) \to \mu_{l^\nu} , $$

where $T_n^S := T_{F_n}^S$.

Proof: By assumption there is a pairing

$$ {}_{l^\nu}T_n^S \times H^2_{\acute{e}t}(O_n^S, \mathbb{Z}/l^\nu\mathbb{Z}) \to \mathbb{Z}/l^\nu\mathbb{Z} . $$

Since $\zeta_{l^\nu} \in F_n^*$, we get

$$ {}_{l^\nu}T_n^S \times H^2_{\acute{e}t}(O_n^S, \mathbb{Z}/l^\nu\mathbb{Z}(2))(-1) \to \mathbb{Z}/l^\nu\mathbb{Z}(1) = \mu_{l^\nu} , $$

and the assertion follows from the isomorphism

$$ H^2_{\acute{e}t}(O_n^S, \mathbb{Z}/l^\nu\mathbb{Z}(2)) \simeq K_2^{\acute{e}t}(O_n^S) \otimes_{\mathbb{Z}_l} \tfrac{1}{l^\nu}\mathbb{Z}_l/\mathbb{Z}_l , $$

cf. 2.20. □

Of course there is also the notion of the right duality. Namely, let K be a number field containing ζ_l, then we have exact sequences

$$ \begin{array}{ccccccccc} 0 \to & H^1_{cont}(O_K^S, \mathbb{Z}_l)/l \cdot H^1_{cont}(O_K^S, \mathbb{Z}_l) & \to & H^1_{\acute{e}t}(O_K^S, \mathbb{Z}/l\mathbb{Z}) & \to & {}_lH^2_{\acute{e}t}(O_K^S, \mathbb{Z}_l) & \to 0 \\ & & & \downarrow\wr & & & \\ 0 \to & H^1_{cont}(O_K^S, \mathbb{Z}_l(2))/l \cdot H^1_{cont}(O_K^S, \mathbb{Z}_l(2)) & \to & H^1_{\acute{e}t}(O_K^S, \mathbb{Z}/l\mathbb{Z}(2)) & \to & {}_lK_2^{\acute{e}t}(O_K^S) & \to 0 , \end{array} $$

where the vertical isomorphism is induced by the cup-product. Let $\tilde{K}$ be the compositum of all $\mathbf{Z}_l$-extension of K. Then

$$\begin{aligned} A_K &:= H^1_{cont}(O_K^S, \mathbf{Z}_l)/l \cdot H^1_{cont}(O_K^S, \mathbf{Z}_l) \\ &= \left\{\tfrac{1}{l}\, mod\, \mathbf{Z} \otimes a \in \Delta_K^{(1)}(-1) :\ K(\sqrt[l]{a}) \subseteq \tilde{K}\right\} \end{aligned}$$

and

$$\begin{aligned} C_K &:= H^1_{cont}(O_K^S, \mathbf{Z}_l(2))/l \cdot H^1_{cont}(O_K^S, \mathbf{Z}_l(2)) \\ &= \left\{\tfrac{1}{l}\, mod\, \mathbf{Z} \otimes c \in \Delta_K^{(1)}(1) :\ \{\zeta_l, c\} = 1 \ in\ K_2(O_K^S)\right\} . \end{aligned}$$

In [12] J.Coates posed the question, whether

$$A_K = C_K ,$$

in other words

$$\begin{aligned} {}_lK_2^{\acute{e}t}(O_K^S) &\simeq {}_lH^2_{\acute{e}t}(O_K^S, \mathbf{Z}_l) \\ &\simeq (T_K^S/l \cdot T_K^S)^* \end{aligned}$$

and thus a pairing (the right duality)

$$T_K^S/l \cdot T_K^S \times {}_lK_2^{\acute{e}t}(O_K^S)(-1) \to \mu_l .$$

But as R.Greenberg showed, this is wrong in general, cf. [29]. He also proved that Coates's question has an affirmative answer for the field F_n, provided $n \geq 0$ is large enough and the Leopoldt conjecture is valid for F_n.

Next we make the following

Definition 8.2 *Let E be a number field and l an odd prime. For $i \in \mathbb{Z}$, E is called (l,i)-regular if $H^2_{ét}(O^S_E, \mathbb{Z}/l\mathbb{Z}(i)) = 0$.*

From the above sequence we immediately get

Corollary 8.3 *E is $(l,0)$-regular if and only if the Leopoldt conjecture is valid for E and $T_E = 0$.*

Corollary 8.4 *E is $(l,1)$-regular if and only if $g_l(E) = 1$ and $l \nmid h^S(E)$, where $g_l(E)$ is the number of l-adic primes in E and $h^S(E) := \# A^S_E$.*

Proof: Since $H^2_{ét}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(1)) \simeq (\mathbb{Q}_l/\mathbb{Z}_l)^{g_l(E)-1}$ and $U^S_E \otimes \mathbb{Q}_l/\mathbb{Z}_l$ is the maximal divisible subgroup of $H^1_{ét}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(1))$ the assertion is obvious from the exact sequence

$$0 \to U^S_E \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^1_{ét}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(1)) \to A^S_E \to 0 .$$

□

The property (l,i)-regular depends just on $i \bmod d$, $d := [F : E]$, and in the view of the previous corollaries we can assume without loss of generality that $i \geq 2$.

Corollary 8.5 *Let $i \geq 2$, then E is (l,i)-regular if and only if $K^{ét}_{2i-2}(O^S_E) = 0$.*

Proof: Since $K^{ét}_{2i-2}(O^S_E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$ is the maximal divisible subgroup of $H^1_{ét}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i))$, the assertion follows again by the exact sequence

$$0 \to K^{ét}_{2i-1}(O^S_E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \to H^1_{ét}(O^S_E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to K^{ét}_{2i-2}(O^S_E) \to 0 .$$

□

In [27] and [71] $(l,0)$-regular fields are called l-rational, and E is called l-regular if E is $(l,0)$-regular and $\mathbb{Q}(\zeta_l + \zeta_l^{-1}) \subseteq E$. From the above and the rank formulas in chapter 6, the equivalences for l-rational resp. l-regular fields stated in [27] are now all obvious.

Remark 8.6 *There is also an analogous result for the vanishing of* $WK_{2i-2}^{et}(E)$, $i \geq 2$. *Namely for* $i \in \mathbb{Z}$, E *is called* $(\tilde{l,i})$*-regular, if the kernel of* $\pi_l^{(1)}(i)$ *is minimal, i.e.,* $\ker \pi_l^{(1)}(i) = H^1_{cont}(O_E^S, \mathbb{Z}_l(i)) \otimes_{\mathbb{Z}_l} \frac{1}{l}\mathbb{Z}_l/\mathbb{Z}_l$. *Then* X_E^S *has to be replaced by the so-called Bertrandias-Payan module* $\tilde{X}_E^S$, *cf. [75], and the corollaries 8.3-8.5 as well as their proof carry over word by word. In particular since* $\pi_l^{(1)}(1)$ *is the trivial map, the properties* $(l,1)$*-regular and* $(\tilde{l,1})$*-regular are just the same. It would be interesting to know, whether one could modify the property* $(\tilde{l,i})$*-regular in such a way that it remains the same for* $i \neq 1$ *and* $(\tilde{l,1})$*-regular is equivalent to the vanishing of* l*-tor* C_E.

For a finite Galois l-extension L/E with Galois group G, G. Gras studied Galois descent for the torsion submodule T_L^S, which certainly requires a well-defined G-action on T_L^S. This can be done in the following way. Let S_{ram} be the union of S and all ramified primes in L/E. Then $T_L^{S_{ram}}$ is a G-module in a natural way, and by [74] there is an exact sequence

$$0 \to \bigoplus_{\mathfrak{p} \in S_{ram} - S} \mu(L_{\mathfrak{p}}) \to T_L^{S_{ram}} \to T_L^S \to 0 .$$

Thus T_L^S becomes a G-module. Under the assumption that the Leopoldt conjecture is valid for L (and hence for E) G. Gras then proved a genus formula for $(T_L^S)^G/T_E^S$ involving the so-called Gras logarithm, cf. [26]. He also introduced the notion of primitive ramification, cf. [26], and then together with J.-F. Jaulent proved the following, cf. [27]

Theorem 8.7 *Let E be a number field containing $\mathbb{Q}(\zeta_l + \zeta_l^{-1})$, where l is an odd prime, and L a finite Galois l-extension of E. Then L is $(l,0)$-regular if and only if E is $(l,0)$-regular and L/E is primitively ramified.*

If E contains $\mathbb{Q}(\zeta_l + \zeta_l^{-1})$ and so does L, the property $(l,0)$-regular is equivalent to the vanishing of $K_2^{\acute{e}t}(O_E^S)$ resp. $K_2^{\acute{e}t}(O_L^S)$, cf. 8.5. Hence the above theorem has to be seen as a going-up lemma for the triviality of $K_2^{\acute{e}t}(\cdot)$. We see that the generalization of 8.7 to (l,i)-regular fields depends on Galois descent for étale K-theory, which we consider now.

Let L/E be a finite Galois extension with Galois group G, S the set of infinite and l-adic primes, and $T = S_{ram}$ the union of S and all ramified primes in L/E. To begin with we consider Galois descent for the odd-dimensional étale K-theory.

Lemma 8.8 *For $i \geq 2$, there is an isomorphism*

$$K_{2i-1}^{\acute{e}t}(E) \simeq K_{2i-1}^{\acute{e}t}(L)^G .$$

Proof: Since $Spec(L) \to Spec(E)$ is clearly an étale Galois covering, we have the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q_{cont}(L, \mathbb{Z}_l(i))) \Rightarrow H^{p+q}_{cont}(E, \mathbb{Z}_l(i))\ .$$

Since $H^0_{cont}(\cdot, \mathbb{Z}_l(i)) = 0$ for $i \neq 0$ by 1.17 and $cd_l L \leq 2$ resp. $cd_l E \leq 2$,

$$E_2^{p,q} = 0 \text{ for } q \neq 1, 2\ .$$

Hence we obtain an isomorphism

$$H^1_{cont}(E, \mathbb{Z}_l(i)) \simeq H^1_{cont}(L, \mathbb{Z}_l(i))^G\ .$$

The assertion follows now directly from 2.20, cf. also 2.23. □

For the odd-dimensional étale K-theory of rings of S-integers, we proceed as follows. By 1.20 we have a commutative diagram

$$\begin{array}{ccc} H^1_{cont}(O_L^S, \mathbb{Z}_l(i)) & \xrightarrow{\sim} & H^1_{cont}(L, \mathbb{Z}_l(i)) \\ \downarrow \wr & & \downarrow \wr \\ H^1_{et}(O_L^S, \mathbb{Z}_l(i)) & \xrightarrow{\sim} & H^1(L, \mathbb{Z}_l(i)) \end{array}$$

and we define the G-action on $H^1_{cont}(O_L^S, \mathbb{Z}_l(i))$ (and hence on $K^{et}_{2i-1}(O_L^S)$) via the isomorphism

$$H^1_{cont}(O_L^S, \mathbb{Z}_l(i)) \xrightarrow{\sim} H^1_{cont}(L, \mathbb{Z}_l(i))\ .$$

This is justified by the following. By 1.19 we have the exact sequence

$$0 \to H^1_{et}(O_L^S, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to H^1(L, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to \bigoplus_{\mathfrak{P} \notin S} H^0(k_{\mathfrak{P}}, \mathbb{Z}/l^\nu\mathbb{Z}(i-1))\ ,$$

where $k_{\mathfrak{P}}$ is the residue field of $L_{\mathfrak{P}}$ corresponding to $\mathfrak{P}$. There is a natural G-module structure on $\bigoplus_{\mathfrak{P}\notin S} H^0(k_{\mathfrak{P}}, \mathbb{Z}/l^\nu\mathbb{Z}(i-1))$, which is functorial in the coefficients, cf. chapter 6. Thus passing to the projective limit gives a natural G-module structure on $H^1_{\acute{e}t}(O_L^S, \mathbb{Z}_l(i))$, which by the above commutative diagram is compatible with the G-action on $H^1_{cont}(O_L^S, \mathbb{Z}_l(i))$. Summing it up we get

Lemma 8.9 *For $i \geq 2$, there is a natural G-module structure on $K^{\acute{e}t}_{2i-1}(O_L^S)$ and an isomorphism*

$$K^{\acute{e}t}_{2i-1}(O_E^S) \xrightarrow{\sim} K^{\acute{e}t}_{2i-1}(O_L^S)^G .$$

The situation for the even-dimensional étale K-theory is more complex. Since we are mainly interested in a going-up lemma for the triviality of $K^{\acute{e}t}_{2i-2}(O_E^S)$ resp. $K^{\acute{e}t}_{2i-2}(O_L^S)$, we assume from now on that L/E is a cyclic l-extension. Since

$$Spec(O_L^T) \to Spec(O_E^T)$$

is an étale Galois covering by construction, we can apply a spectral sequence argument and get the following

Lemma 8.10 *Let L/E be a cyclic l-extension with Galois group G and T the union of S and all ramified primes in L/E. Then for $i \geq 2$, there is an exact sequence*

$$0 \to H^1(G, K^{\acute{e}t}_{2i-1}(O_L^T)) \to K^{\acute{e}t}_{2i-2}(O_E^T) \to K^{\acute{e}t}_{2i-2}(O_L^T)^G \to H^2(G, K^{\acute{e}t}_{2i-1}(O_L^T)) \to 0$$

and

$$\#K^{\acute{e}t}_{2i-2}(O^T_E) = \#K^{\acute{e}t}_{2i-2}(O^T_L)^G .$$

Proof: Consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q_{cont}(O^T_L, \mathbb{Z}_l(i))) \Rightarrow H^{p+q}_{cont}(O^T_E, \mathbb{Z}_l(i)) .$$

Then $H^q_{cont}(\cdot, \mathbb{Z}_l(i)) = 0$ for $q \neq 1, 2$, cf. proof of 8.8, and hence we get the exact sequence

$$\begin{aligned} 0 \quad \to \quad & H^1(G, H^1_{cont}(O^T_L, \mathbb{Z}_l(i))) \quad \to H^2_{cont}(O^T_E, \mathbb{Z}_l(i)) \to H^2_{cont}(O^T_L, \mathbb{Z}_l(i))^G \\ \to \quad & H^2(G, H^1_{cont}(O^T_L, \mathbb{Z}_l(i))) \quad \to 0 , \end{aligned}$$

and for $p \geq 3$, isomorphisms

$$H^{p-2}(G, H^2_{cont}(O^T_L, \mathbb{Z}_l(i))) \xrightarrow{\sim} H^p(G, H^1_{cont}(O^T_L, \mathbb{Z}_l(i))) .$$

Since G is cyclic, we obtain for $p \geq 3$

$$H^{p-2}(G, H^2_{cont}(O^T_L, \mathbb{Z}_l(i))) \xrightarrow{\sim} H^{p-2}(G, H^1_{cont}(O^T_L, \mathbb{Z}_l(i))) ,$$

and the second assertion follows from the first and the fact that the Herbrand quotient of a finite G-module is trivial. □

Corollary 8.11 *Let the notations be as in the previous lemma. Then for* $i \geq 2$, $K^{\acute{e}t}_{2i-2}(O^T_L)$ *satisfies Galois co-descent, i.e., the transfer* Tr *induces an isomorphism*

$$Tr : K^{\acute{e}t}_{2i-2}(O^T_L)_G \xrightarrow{\sim} K^{\acute{e}t}_{2i-2}(O^T_E) .$$

Proof: Since $cd_l O_E^T \leq 2$, the transfer $Tr : H^2_{\acute{e}t}(O_L^T, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \to H^2_{\acute{e}t}(O_E^T, \mathbb{Z}/l^\nu\mathbb{Z}(i))$ is surjective, cf. [85]. But $H^2_{cont}(O_L^T, \mathbb{Z}_l(i))$ is finite, and thus $H^2_{\acute{e}t}(O_L^T, \mathbb{Z}/l^\nu\mathbb{Z}(i)) \simeq H^2_{cont}(O_L^T, \mathbb{Z}_l(i))$ for $\nu \geq 1$ large enough. The corollary follows now from 8.10. □

Since $Spec(O_L^S) \to Spec(O_E^S)$ is not an étale Galois covering, unless L/E is unramified outside S, we have to argue differently for the Galois descent of $K^{\acute{e}t}_{2i-2}(O_L^S)$. Recall the localization sequence in étale K-theory, cf. 2.30,

$$0 \to K^{\acute{e}t}_{2i-2}(O_L^S) \to l\text{-}tor\, K^{\acute{e}t}_{2i-2}(L) \to \bigoplus_{\mathfrak{P} \notin S} K^{\acute{e}t}_{2i-3}(k_{\mathfrak{P}}) \to 0\,.$$

Hence for $S \subseteq T$, we get the exact sequence of G-modules

$$0 \to K^{\acute{e}t}_{2i-2}(O_L^S) \to K^{\acute{e}t}_{2i-2}(O_L^T) \to \bigoplus_{\mathfrak{P} \in T-S} K^{\acute{e}t}_{2i-3}(k_{\mathfrak{P}}) \to 0$$

as well as a corresponding sequence

$$0 \to H^2_{cont}(O_L^S, \mathbb{Z}_l(i)) \to H^2_{cont}(O_L^T, \mathbb{Z}_l(i)) \to \bigoplus_{\mathfrak{P} \in T-S} H^2_{cont}(L_{\mathfrak{P}}, \mathbb{Z}_l(i)) \to 0\,,$$

cf. chapter 6 for a discussion on the G-module structure. Note, that for $\mathfrak{P} \not| l$ and $i \geq 2$,

$$\begin{aligned} H^0(k_{\mathfrak{P}}, \mathbb{Q}_l/\mathbb{Z}_l(i-1)) &\simeq H^1(L_{\mathfrak{P}}, \mathbb{Q}_l/\mathbb{Z}_l(i)) \\ &\simeq H^2_{cont}(L_{\mathfrak{P}}, \mathbb{Z}_l(i))\,. \end{aligned}$$

Identifying $H^0(E_{\mathfrak{p}}, \mathbb{Q}_l/\mathbb{Z}_l(i))$ with its image under the diagonal

$$H^0(E_{\mathfrak{p}}, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to \bigoplus_{\mathfrak{P}|\mathfrak{p}} H^0(L_{\mathfrak{P}}, \mathbb{Q}_l/\mathbb{Z}_l(i))$$

gives a spectral sequence

$$E_2^{p,q} = H^p(G, \bigoplus_{\mathfrak{P}|\mathfrak{p}} H^q(L_{\mathfrak{P}}, \mathbf{Q}_l/\mathbf{Z}_l(i))) \Rightarrow H^{p+q}(E_{\mathfrak{p}}, \mathbf{Q}_l/\mathbf{Z}_l(i)) .$$

In the same manner as above, cf. 8.10, we get the exact sequences

$$\begin{aligned} 0 \;\to\; & H^1(G, \oplus_{\mathfrak{P}|\mathfrak{p}} H^0(L_{\mathfrak{P}}, \mathbf{Q}_l/\mathbf{Z}_l(i))) \;\to H^1(E_{\mathfrak{p}}, \mathbf{Q}_l/\mathbf{Z}_l(i)) \to \left(\oplus_{\mathfrak{P}|\mathfrak{p}} H^1(L_{\mathfrak{P}}, \mathbf{Q}_l/\mathbf{Z}_l(i))\right)^G \\ \to\; & H^2(G, \oplus_{\mathfrak{P}|\mathfrak{p}} H^0(L_{\mathfrak{P}}, \mathbf{Q}_l/\mathbf{Z}_l(i))) \;\to 0 \end{aligned}$$

and

$$\begin{aligned} 0 \;\to\; & H^1(G, \oplus_{\mathfrak{P}|\mathfrak{p}} H^1_{cont}(L_{\mathfrak{P}}, \mathbf{Z}_l(i))) \;\to H^2_{cont}(E_{\mathfrak{p}}, \mathbf{Z}_l(i)) \to \left(\oplus_{\mathfrak{P}|\mathfrak{p}} H^2_{cont}(L_{\mathfrak{P}}, \mathbf{Z}_l(i))\right)^G \\ \to\; & H^2(G, \oplus_{\mathfrak{P}|\mathfrak{p}} H^1_{cont}(L_{\mathfrak{P}}, \mathbf{Z}_l(i))) \;\to 0 . \end{aligned}$$

Since G is cyclic, kernel and cokernel of $H^2_{cont}(E_{\mathfrak{p}}, \mathbf{Z}_l(i)) \to (\oplus_{\mathfrak{P}|\mathfrak{p}} H^2_{cont}(L_{\mathfrak{P}}, \mathbf{Z}_l(i)))^G$ are of the same order, and we obtain from 8.10

Theorem 8.12 *Let L/E be a cyclic l-extension with Galois group G, S the set of infinite and l-adic primes and T the union of S and all ramified primes in L/E. Then for $i \geq 2$,*

$$\frac{\# K^{et}_{2i-2}(O^S_L)^G}{\# K^{et}_{2i-2}(O^S_E)} = \# coker \left(H^1_{cont}(O^T_E, \mathbf{Z}_l(i))/im\, Tr \to \bigoplus_{\mathfrak{p}\in T-S} H^1_{cont}(E_{\mathfrak{p}}, \mathbf{Z}_l(i))/im\; N_G \right) ,$$

where $Tr : H^1_{cont}(O^T_L, \mathbf{Z}_l(i)) \to H^1_{cont}(O^T_E, \mathbf{Z}_l(i))$ is the transfer and N_G : $\oplus_{\mathfrak{P}|\mathfrak{p}} H^1_{cont}(L_{\mathfrak{P}}, \mathbf{Z}_l(i)) \to \oplus_{\mathfrak{P}|\mathfrak{p}} H^1_{cont}(L_{\mathfrak{P}}, \mathbf{Z}_l(i))$ is the cohomological norm map.

Even if the order of $\oplus_{\mathfrak{p}\in T-S} H^1_{cont}(E_{\mathfrak{p}}, \mathbf{Z}_l(i))/im\; N_G$ can be easily computed, cf. 8.14, the above formula is practically inaccessible, which is not surprising since primitive ramification is of the same type. But the theorem gives us at least some information, which generalizes a remark in [26]

Corollary 8.13 *Let the notations be as in the previous theorem. Suppose that* $\#K^{et}_{2i-2}(O^S_L)^G = \#K^{et}_{2i-2}(O^S_E)$. *Then the number of primes, which are ramified outside* S, *is less or equal to* $1 + r_1(E) + r_2(E)$ *if* $i - 1 \equiv 0 \mod 2$, *and to* $1 + r_2(E)$ *if* $i - 1 \equiv 1 \mod 2$.

Let us consider the case that E is a totally real number field and $i - 1 \equiv 1 \mod 2$. Thus $H^1_{cont}(O^T_E, \mathbb{Z}_l(i)) \simeq H^0_{et}(O^T_E, \mathbb{Q}_l/\mathbb{Z}_l(i))$ is finite and cyclic. Firstly, we need the following

Lemma 8.14 *Let* L/E *be a cyclic* l-*extension of local fields over* $\mathbb{Q}_p$, $p \neq l$, *and* G *the Galois group. The ramification index and residue degree are denoted by* e *and* f. *Furthermore, let* $q = \#\bar{E}$ *be the order of the residue field. Then for* $i \geq 2$,

$$\begin{aligned} \#H^2(G, H^0(L, \mathbb{Q}_l/\mathbb{Z}_l(i))) &= \#coker(H^1(E, \mathbb{Q}_l/\mathbb{Z}_l(i)) \to H^1(L, \mathbb{Q}_l/\mathbb{Z}_l(i))^G) \\ &= l^{min\{v_l(e), v_l(q^{i-1}-1)\}} . \end{aligned}$$

Proof: Let $H \subseteq G$ be the inertia group, and K its fixed field. Since $l \neq p = char(\bar{E})$, L/K is tamely ramified and cyclic, thus $e | q^f - 1$. Furthermore $H^j(L, \mathbb{Q}_l/\mathbb{Z}_l(i)) = H^j(K, \mathbb{Q}_l/\mathbb{Z}_l(i))$ for $j = 1, 2$, i.e., $H^j(L, \mathbb{Q}_l/\mathbb{Z}_l(i))$ is a trivial H-module. Hence

$$H^1(K, \mathbb{Q}_l/\mathbb{Z}_l(i)) \overset{\times e}{\twoheadrightarrow} H^1(L, \mathbb{Q}_l/\mathbb{Z}_l(i))^H = H^1(K, \mathbb{Q}_l/\mathbb{Z}_l(i)) ,$$

and taking G/H-invariants yields

$$H^1(E,\mathbb{Q}_l/\mathbb{Z}_l(i)) \stackrel{\times e}{\to} H^1(L,\mathbb{Q}_l/\mathbb{Z}_l(i))^G = H^1(E,\mathbb{Q}_l/\mathbb{Z}_l(i)) .$$

Since $H^1(E,\mathbb{Q}_l/\mathbb{Z}_l(i)) \simeq K^{et}_{2i-3}(\bar{E})$ is cyclic of order $l^{v_l(q^{i-1}-1)}$, the lemma follows. Note, if L/E is ramified, then

$$min\{v_l(e), v_l(q^{i-1}-1)\} \geq 1 .$$

□

For each finite prime $\mathfrak{p} \subseteq E$, let $e_\mathfrak{p}$ be the ramification index in L/E and $q_\mathfrak{p} = \#k_\mathfrak{p}$ the order of the residue field of $E_\mathfrak{p}$. Since for $j = 1,2$,

$$H^j(G, \bigoplus_{\mathfrak{P}|\mathfrak{p}} H^0(L_\mathfrak{P},\mathbb{Q}_l/\mathbb{Z}_l(i))) \simeq H^j(G_\mathfrak{p}, H^0(L_\mathfrak{P},\mathbb{Q}_l/\mathbb{Z}_l(i)))$$

non-canonically, where $G_\mathfrak{p} \subseteq G$ is the decomposition group with respect to $\mathfrak{p}$, the above lemma enables one to compute the order and structure of the cokernel of

$$\bigoplus_{\mathfrak{p}\in T-S} H^1(E_\mathfrak{p},\mathbb{Q}_l/\mathbb{Z}_l(i)) \to \bigoplus_{\mathfrak{p}\in T-S} (\bigoplus_{\mathfrak{P}|\mathfrak{p}} H^1(L_\mathfrak{P},\mathbb{Q}_l/\mathbb{Z}_l(i)))^G .$$

If L/E is a cyclic l-extension of totally real number fields and $i-1 \equiv 1 \; mod 2$, one can then calculate the order of

$$H^2(G, H^0(L,\mathbb{Q}_l/\mathbb{Z}_l(i))) \to \bigoplus_{\mathfrak{p}\in T-S} H^2(G, \bigoplus_{\mathfrak{P}|\mathfrak{p}} H^0(L_\mathfrak{P},\mathbb{Q}_l/\mathbb{Z}_l(i)))$$

and thus of $\#K^{et}_{2i-2}(O^S_L)^G/\#K^{et}_{2i-2}(O^S_E)$. We omit the details here and rather consider the case when $K^{et}_{2i-2}(O^S_L)$ satisfies Galois co-descent.

Proposition 8.15 *Let L/E be a cyclic l-extension of totally real number fields and $i-1 \equiv 1 \bmod 2$. Then*

$$\frac{\#K^{et}_{2i-2}(O^S_L)^G}{\#K^{et}_{2i-2}(O^S_E)} = 1$$

if and only if one of the following is satisfied

1. *L/E is unramified outside S.*
2. *There is exactly one prime $\mathfrak{p}$, which is ramified outside S, and $w^{(i)}_l(E) = w^{(i)}_l(E_\mathfrak{p})\ (\neq 0)$.*

In particular, L is (l,i)-regular if and only if E is (l,i)-regular and one of the above conditions is satisfied.

Proof: Assume that $\#K^{et}_{2i-2}(O^S_L)^G = \#K^{et}_{2i-2}(O^S_E)$. Then

$$H^0(E, \mathbb{Q}_l/\mathbb{Z}_l(i))/Tr(H^0(L, \mathbb{Q}_l/\mathbb{Z}_l(i))) \to \bigoplus_{\mathfrak{p}\in T-S} H^0(E_\mathfrak{p}, \mathbb{Q}_l/\mathbb{Z}_l(i))/Tr(H^0(L_\mathfrak{P}, \mathbb{Q}_l/\mathbb{Z}_l(i)))$$

is surjective. Thus $T = S$ or there is exactly one prime $\mathfrak{p} \in T - S$ and $w^{(i)}_l(E) = w^{(i)}_l(E_\mathfrak{p})$ by 8.14. The converse is also clear. Since $K^{et}_{2i-2}(O^S_L)^G = 0$ if and only if $K^{et}_{2i-2}(O^S_L) = 0$, the second part follows directly from the first and 8.5. □

Example 8.16 *Let L be the decomposition field of $f = X^3 - 12X + 14$ and $E = \mathbb{Q}(\sqrt{5})$. Then L and E are $(3,2) = (3,0)$-regular.*

Proof: E is $(3, 2)$-regular by 7.20, and the conditions of the above proposition are satisfied. □

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