# THEORY AND APPLICATION OF SELF-TUNING REGULATORS

by

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# ABSTRACT

The theory of self-tuning regulators and related topics in linear stochastic control theory have been examined. A unifying treatment of the theory of self-tuning regulators has been presented, and an attempt made to clarify the confusion surrounding certain aspects of these regulators. The notation is that of Box and Jenkins (1970).

Self-tuning control of a steam jacketed stirred tank was successfully implemented. The experimental program was designed to illustrate points from the theory that have caused confusion in previous industrial and pilot plant implementations.

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# NOMENCLATURE

| | |
|---|---|
| $A(z^{-1})$ | numerator of process dynamic transfer function (2.6) |
| $a_t$ | random shock |
| $\bar{B}(z^{-1})$ | denominator of process dynamic transfer function (2.6) |
| b | number of whole periods of delay |
| $C(z^{-1})$ | polynomial in $z^{-1}$ |
| c | fractional period of delay |
| d | order of the pole lying on unit circle of disturbance model (2.4) |
| $e_{t+b}$ | b step ahead prediction error (2.15) |
| $E\{\}$ | mathematical expectation operator |
| $e_{t+b}$ | b step prediction error (2.10) |
| f | number of pure periods of delay |
| $\underline{f}(\underline{\theta})$ | vector of auto and cross covariances (3.64) |
| $G(\underline{\theta})$ | matrix of auto and cross covariances (3.65) |
| $\underline{K}(t)$ | weighting vector for recursive least squares (3.57) |
| $L_2(z^{-1})$, $L_3(z^{-1})$, $L_4(z^{-1})$ | polynomials in factorization of disturbance model (2.13, 2.14) |
| $\ell$ | order of $B(z^{-1})$ |
| $\ell o$ | order of demoninator of minimum variance controller (3.9) |
| m | order of $\alpha(z^{-1})$ |
| mo | order of numerator of minimum variance controller (3.8) |
| $N_t$ | disturbance affecting process output (2.4) |

| | |
|---|---|
| $\hat{N}_{t+b/t}$ | b step ahead forecast of disturbance (2.14) |
| (p,d,q) | orders of polynomials in disturbance model (2.4) |
| $\underline{P}(t)$ | weighting matrix for recursive least squares (3.58) |
| (r,s,b) | orders of the process dynamic transfer function (2.2) |
| $r_{yy}(k)$ | sample auto correlation at lag k |
| $r_{\nabla^d uy}(k)$ | sample cross correlation at lag k |
| $\underline{R}$ | covariance matrix of vi(t) |
| T | sampling interval, time |
| t | time |
| U | manipulated variable or process inputs (mean corrected) |
| $\hat{U}$ | manipulated variable (deviation variable) |
| $V(z^{-1})$ | process transfer function (2.2) |
| vi(t) | sequence of independent white noise variates |
| $\underline{X}$ | matrix of process inputs and outputs |
| $\underline{x}(t)$ | vector of process inputs and outputs (3.49) |
| Y | controlled variable or process output (setpoint corrected) |
| Y* | output from a dynamic system (2.1) |
| $\underline{Y}$ | vector of process outputs (B10) |
| z | feedforward variable |
| $z^{-1}$ | backward shift operator |
| $\alpha(z^{-1})$ | numerator of controller transfer function |
| $B(z^{-1})$ | denominator of controller transfer function |
| $\nu$ | constant term to be identified (3.20) |
| $\nu(z^{-1})$ | polynomial in $z^{-1}$ |
| $\varepsilon^0_{t+b}$ | equal to $L_4(z^{-1})a_{t+b}$, b step ahead forecast error (2.15) |

| | |
|---|---|
| $\underline{\theta}(t)$ | vector of controller parameters to be estimated (3.50) |
| $\theta(z^{-1})$ | moving average term in disturbance model (2.4) |
| $\phi(z^{-1})$ | autoregressive term in disturbance model (2.4) |
| $\Phi(t)$ | objective function for constrained control (3.39) |
| $\delta(z^{-1})$ | denominator of process dynamic transfer function (2.1) |
| $\lambda$ | forgetting factor for discounted least squares (3.59) |
| $\xi,\xi',\xi''$ | constraining weights |
| $\rho_{yy}(k)$ | theoretical autocorrelation at lag k |
| $\rho_{uy}(k)$ | theoretical cross correlation at lag k |
| $\sigma^2$ | variance |
| $\tau_d$ | continuous process deadtime |
| $\tau$ | continuous process time constant, lag indicator |
| $\psi(z^{-1})$ | disturbance transfer function (2.12) |
| $\omega(z^{-1})$ | numerator in process transfer function (2.1) |
| $\nabla$ | difference operator $(1-z^{-1})$ |

Superscripts

| | |
|---|---|
| ' | alternate polynomials |
| 0 | true value of the parameter |
| ^ | estimate of the particular valve |

# CHAPTER 1

## INTRODUCTION

An increasing number of industrial processes are being controlled by digital computers. In many instances, the control objective is to keep the process output as close to the desired setpoint as possible. Classical design techniques used to tune feedforward or feedback control loops, require the process output to have a satisfactory response when subject to a deterministic forcing function or satisfy certain specified stability margins. Such design criteria (as percentage overshoot, decay ratio, phase and gain margins), may ignore the nature of inherent process disturbances, resulting sometimes in stable but poor regulatory control. Proportional-Integral-Derivative type controllers may not adequately control processes characterized by long dead times and certain drifting stochastic disturbances.

The approach of Box and Jenkins (1962, 1963, 1970) and Aström (1967, 1970) was to design controllers to compensate for disturbances inherent to a particular process. Data was collected under open or closed loop, and models of the process dynamics and disturbances identified off-line. Feedforward or feedback controllers were then designed to minimize fluctuations of the process output from its target value.

The data collection and subsequent off-line analysis can be time consuming, and require considerable expertise which few industrial people have. Another possible drawback to this off-line design of the controller, is that the process dynamics and disturbances may change appreciably with time. This requires reidentification of the dynamic and disturbance models from new data.

This led Aström and Wittenmark (1973) to develop a self-tuning regulator, in which only those parameters that appear in the optimal regulator are identified on-line. The parameters of a model are estimated at each sampling interval by a recursive estimation technique, and used in a control law as if they were exactly known. If the parameter estimates converge, and several weak conditions are satisfied, the resulting controller is the same one that could have been designed off-line, had the process dynamic and stochastic models been known.

The self-tuning regulator overcomes the need for data collection experiments, and extensive off-line analysis and controller design. It is easily implemented with a minimum of specialized training. Minor modifications to the estimation algorithm allows the self-tuning regulator to track changing process and disturbance characteristics. It is not necessary to collect new data, reidentify the process dynamic and disturbance models and redesign the controller off-line.

There have been powerful theoretical techniques developed to study the convergence properties and stability of the self-tuning regulator. Extensive simulations of the self-tuning regulator have been reported. Industrial applications to the control of paper machines, an ore crusher, and a batch digestor have demonstrated that self-tuning regulators are quite robust to assumptions in their derivation.

The purpose of this thesis is to

(1) present a unifying treatment of the theory of self-tuning regulators and clarify the confusion surrounding certain

aspects of these controllers;

(2) to gain a familiarity with the self-tuning regulator by writing the necessary mini-computer software and implementing self-tuning control of a pilot plant process;

(3) to examine the limitations of the self-tuning regulator and to suggest areas that require further investigation.

The remainder of the thesis is outlined as follows:

Chapter Two: The representation, fitting and diagnostic checking of the process dynamic and disturbance models of Aström (1970) and Box and Jenkins (1970) is examined. The design of minimum variance controllers, constrained controllers, the sensitivity of the resulting closed loop systems, the choice of sampling interval and problems of closed loop identification are reviewed. This extensive background will aid in the understanding of self-tuning regulators.

Chapter Three: The theory of the self-tuning regulator is presented here using the notation of Box and Jenkins (1970). The theoretical developments of Ljung and Wittenmark (1974), the recursive estimation scheme, and problems of parameter identification are discussed. A proposed self-tuning constrained controller of Clarke and Gawthrop (1975) is shown to be in error. A simulation is presented to bring some of the concepts in this chapter together.

Chapter Four: The applications of self-tuning regulators to the control of industrial and pilot plant processes are reviewed. Insights, extensions and problems that have arisen in implementation are discussed.

Chapter Five: The self-tuning regulator is used to control a steam jacketed stirred tank heater. Different methods for eliminating offset, and the use of the sample auto and cross correlation functions as diagnostic tools are discussed. It is demonstrated that all the controller parameters may be estimated. The robustness of the self-tuning controller to different input disturbances is also considered.

Chapter Six: This chapter summarizes the most important aspects of this work, and gives suggestions for further work in this area.

CHAPTER 2

# A REVIEW OF LINEAR STOCHASTIC CONTROL THEORY

## 2.1 Representation of the Dynamic and Disturbance Models

Consider a process, Figure 2.1, where the opportunity exists to measure the process output and take control action at equispaced intervals of time, t=0, T, 2T,..., where T is the sampling interval.

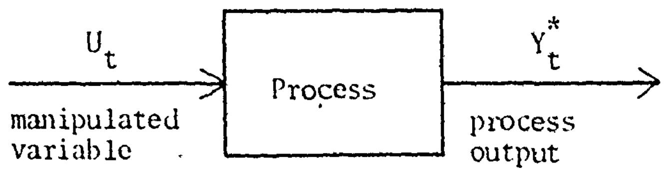


Figure 2.1: Representation of a Dynamic System

Aström (1967, 1970) and Box and Jenkins (1962, 1963, 1970) represent the discrete linear transfer function relating the process output $Y_t^*$, and the manipulated variable, $U_t$ as

$$Y_t^* - \delta_1 Y_{t-1}^* - \dots - \delta_r Y_{t-r}^* = \omega_0 U_{t-b} - \omega_1 U_{t-b-1} - \dots - \omega_s U_{t-b-s} \quad (2.1)$$

There are b whole periods of delay before the effect of a change in the manipulated variable is observed at the output (b includes the transport delay plus an additional period of delay for the sample and hold. This is discussed in section 2.6). The manipulated variable is held constant in the interval $nT < t \leq (n+1)T$. $Y_t^*$ and $U_t$ are deviation variables from their steady state values.

Defining an operator $z^{-k}$ such that $z^{-k} U_t = U_{t-k}$, (2.1) may be written more compactly as.

$$Y_t^* = V(z^{-1})\, U_t = \frac{\omega(z^{-1}) z^{-b}}{\delta(z^{-1})} U_t = \frac{\omega_0 - \omega_1 z^{-1} - \ldots - \omega_s z^{-s}}{1 - \delta_1 z^{-1} - \ldots - \delta_r z^{-r}} U_{t-b} \tag{2.2}$$

$V(z^{-1})$ is called a transfer function of order (r, s, b). The process is open loop stable if $\delta(z^{-1})$ has all its zeros (in $z^{-1}$) outside the unit circle, and is referred to as minimum phase if $\omega(z^{-1})$ has all its zeros outside the unit circle.

If the transfer function is represented by ratios of polynomials in $z^{-1}$, then the output from this dynamical system is an aggregate of past inputs and outputs. This representation provides a sensible class of transfer functions and is justified in its own right, without consideration of the underlying continuous process. In some instances it is possible to relate the discrete model to the continuous process, if the latter may be described by a linear differential equation of the form

$$\begin{aligned}&(1 - T_1 D - T_2 D^2 - \ldots - T_n D^n)\, Y^*(t)\\ &= (\xi_0 - \xi_1 D - \ldots - \xi_m D^m)\, U(t-\tau_d)\end{aligned} \tag{2.3}$$

This is discussed in Box and Jenkins (1970).

In essentially all processes there are disturbances or noise corrupting the process output, Figure 2.2.

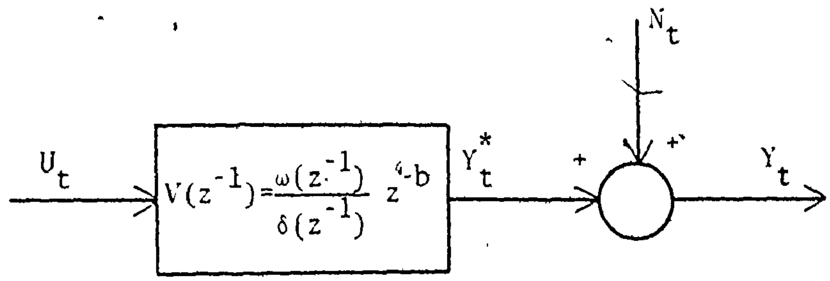


Figure 2.2: Representation of a dynamic system affected by disturbances

Here $N_t$ represents the total effect on the process output of all unobserved disturbances acting within the system, which in the absence of some compensating action would cause the process output to drift away from its target value. These load disturbances may not be of a deterministic nature such as step, ramp or acceleration functions. Åström (1970) and Box and Jenkins (1970) characterize these stochastic disturbances by auto-regressive integrated moving average (ARIMA (p,d,q)) time series models of the form

$$N_t = \frac{\theta(z^{-1})}{\nabla^d\,\phi(z^{-1})}\,a_t = \frac{1-\theta_1 z^{-1}-\ldots-\theta_q z^{-q}}{(1-z^{-1})^d(1-\phi_1 z^{-1}-\ldots-\phi_p z^{-p})}\,a_t \qquad (2.4)$$

The $\{a_t\text{'s}\}$ are a white noise sequence, or a series of random shocks, roughly normally distributed with mean zero and variance $\sigma^2_a$. The moving average term $\theta(z^{-1})$, and the autoregressive term $\phi(z^{-1})$ have all their roots outside the unit circle (in $z^{-1}$). The presence of d roots, usually 1 or 2, on the unit circle allows the disturbance to be of a drifting or nonstationary nature. The stochastic disturbance may be interpreted as the output from a filter $\psi(z^{-1})$ driven by white noise.

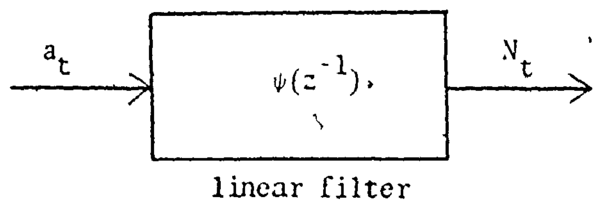


Figure 2.3: Interpretation of the disturbances as an output of a linear filter driven by white noise

The future behavior of a deterministic function can be exactly predicted from a record of its past history. This information is not sufficient to uniquely determine the future behavior of a stochastic function. However, the past history of a stochastic function is sufficient to predict its future probability distribution. Although one may predict the future value of a stochastic function there will be some uncertainty associated with this estimate.

Box and Jenkins (1970) representation of Figure 2.2 is

$$Y_t = \frac{\omega(z^{-1})}{\delta(z^{-1})} U_{t-b} + \frac{\theta(z^{-1})}{\nabla^d \phi(z^{-1})} a_t \qquad (2.5)$$

whereas Aström (1970) uses the notation

$$Y_t = \frac{\overline{B}(z^{-1})}{A(z^{-1})} U_{t-b} + \frac{C(z^{-1})}{A(z^{-1})} a_t \qquad (2.6)$$

Both forms are capable of providing an adequate representation of a dynamic-stochastic system. The Box and Jenkins notation provides more insight into the nature of the process dynamics and stochastic disturbances, as one may distinguish these separately. The common denominator mixes up the dynamic and stochastic models in Aström's representation, and it is more simply a mathematical representation of the output. It will be seen in the next section that the Box-Jenkins notation permits greater flexibility when identifying, fitting and checking dynamic and stochastic models from input/output data.

## 2.2 Identification, Fitting and Testing of Dynamic/Stochastic Models

Box and Jenkins (1970) propose an iterative procedure of

IDENTIFICATION → ESTIMATION → DIAGNOSTIC CHECKING
↑- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -|

to build dynamic and stochastic models. Cross correlation techniques are employed to provide a preliminary identification of the orders (r, s, b) of the transfer function, and initial parameter estimates. Tentative estimates of the orders (p, d, q) of the stochastic disturbances, and initial parameter estimates are obtained using statistical properties of the ARIMA time series models. Both models are fitted simultaneously to ensure that one obtains the best estimates of the parameters, using the maximum likelihood criterion. Diagnostic checks detect model inadequacies and may suggest model improvements.

A multiple regression approach is used by Aström (1970) to build dynamic and stochastic models. Models of increasing order are fitted, and discrimination between them is based on whether the reduction in sums of squares for the model of increased order is statistically significant. Diagnostic analysis of the model residuals is used to evaluate model adequacy.

The Box and Jenkins approach appears to be more flexible and appealing from an engineering point of view. The preliminary identification of the stochastic model requires that the effect of the process dynamics be removed from the data. Dynamic models developed from a theoretical analysis of the process may be used if available. One is not required to treat the process as a "black box". It was first thought that the techniques employed by Box and Jenkins (1970) and Aström (1970) required that the data be collected while the system is operated under open loop conditions. Safety and production constraints may prohibit this. It may be necessary to implement a feedback controller to keep the process variables in the operating region where one wants to identify their dynamic and stochastic behavior. If a time invariant, linear feedback controller is used during the period of data collection, Box and MacGregor (1974) have shown that the use of a noncasual method (cross correlation) to identify the process dynamics, will in fact identify the inverse of the implemented controller. If the data must be collected in closed loop, they suggest that a 'dither signal', uncorrelated with the process output be injected into the process input. Identification techniques and residual checks are outlined. Soderström et al.(1974, 1975, 1976) suggest that switches between several feedback laws will also insure identifiability.

Even if one has an apriori knowledge of the orders of the dynamic and stochastic models, and a time invariant, linear feedback controller with no external dither signal is used during the collection of data, the estimated parameters may not be unique. For different controllers, Soderström et al. (1974) and Box and MacGregor (1976) give necessary and sufficient conditions that must be satisfied if the estimation space is to be nonsingular.

The open loop estimation methods of Box and Jenkins (1970) and Aström (1970) may be used on closed loop data in a straightforward manner, ignoring the presence of a feedback controller (Soderström et al. (1974), Box and MacGregor (1976)) if: these necessary and sufficient conditions are satisfied, or the controller is time-varying, a nonlinear function of the process inputs and outputs, or if an external dither signal is employed in the feedback loop.

The methods of Box and Jenkins (1970), Box and MacGregor (1974) and Astrom (1970) provide a systematic approach for the identification, fitting and diagnostic checking of process dynamic and stochastic models. It is not necessary that the data be collected while the process is operated in an open loop manner if certain precautions are taken. Once the dynamic and stochastic models have been identified, a controller may be designed to compensate for the effect of the disturbances on the process output.

## 2.3 Design of a Minimum Variance Feedback Controller

The output of the system depicted in Figure 2.2 is

$$Y_{t+b} = \frac{\omega(z^{-1})}{\delta(z^{-1})} U_t + N_{t+b} \tag{2.7}$$

The total effect of the disturbance on the output would be cancelled if the control action

$$U_t = \frac{-\delta(z^{-1})}{\omega(z^{-1})} \cdot N_{t+b} \tag{2.8}$$

were taken. This is physically unrealizable, as the value of the disturbance $b$ steps in the future is unknown. It seems reasonable then to take control action based on some predicted value of the disturbance

$$U_t = \frac{-\delta(z^{-1})}{\omega(z^{-1})} \cdot \hat{N}_{t+b/t} \tag{2.9}$$

where $\hat{N}_{t+b/t}$ is the $b$ step ahead prediction of the disturbance based solely on information available at time t. Thus at time t+b the error at the output will be equal to the forecast error. The disturbance $N_{t+b}$ may be written as

$$N_{t+b} = e_{t+b} + \hat{N}_{t+b/t} \tag{2.10}$$

where $e_{t+b}$ is the prediction or forecast error. There are many ways one might predict the effect of the disturbance b steps in the future, but the most sensible predictor would be that which minimizes the variance of the b step ahead prediction error. This is equivalent then, to minimizing the variance of the process output. The criterion for designing the controller is

$$\min_{U_t} E\{Y^2_{t+b}\} \tag{2.11}$$

where E{} denotes the mathematical expectation operator. If $N_{t+b}$ is written as

$$N_{t+b} = \psi(z^{-1})a_{t+b} = (1 + \psi_1 z^{-1} + \ldots + \psi_{b-1} z^{-(b-1)} + \psi_b z^{-b} + \ldots)a_{t+b} \tag{2.12}$$

$$= L_4(z^{-1})a_{t+b} + L_3(z^{-1})a_t \tag{2.13}$$

where $L_4(z^{-1})$ is of order b-1 (the first b terms of 2.12).

It is shown in Box and Jenkins (1970) that

$$\hat{N}_{t+b/t} = L_3(z^{-1})a_t = \frac{L_2(z^{-1})a_t}{\nabla^d \phi(z^{-1})} \tag{2.14}$$

is the predictor which minimizes the variance of the b step ahead prediction error. This forecast error is given by

$$e_{t+h} = L_4(z^{-1})a_{t+h} \tag{2.15}$$

The minimum variance control strategy is therefore

$$U_t = \frac{-\delta(z^{-1})}{\omega(z^{-1})} \cdot L_3(z^{-1})a_t \tag{2.16}$$

It is more convenient to compute the control action based on the error $e_t$ the difference between the process output $Y_t$ and the setpoint $Y_{sp}$. Substitution of (2.15) into (2.16) gives the minimum variance strategy

$$U_t = \frac{-\delta(z^{-1})}{\omega(z^{-1})} \cdot \frac{L_3(z^{-1})}{L_4(z^{-1})} e_t = \frac{-\delta(z^{-1})}{\omega(z^{-1})} \cdot \frac{L_2(z^{-1})}{L_4(z^{-1})\phi(z^{-1})\nabla^d} e_t \tag{2.17}$$

From (2.12) it is seen that the $\{a_t$'s$\}$ may be interpreted as the one step ahead forecast error.

The output of the closed loop system, Figure 2.4 is

$$e_{t+b} = \frac{\psi(z^{-1})}{1 - D(z^{-1})V(z^{-1})} a_{t+b} \tag{2.18}$$

(In Figure 2.4 the sign of the error term on $e_t$ is opposite to conventional notation. The negative sign is included in the controller block).

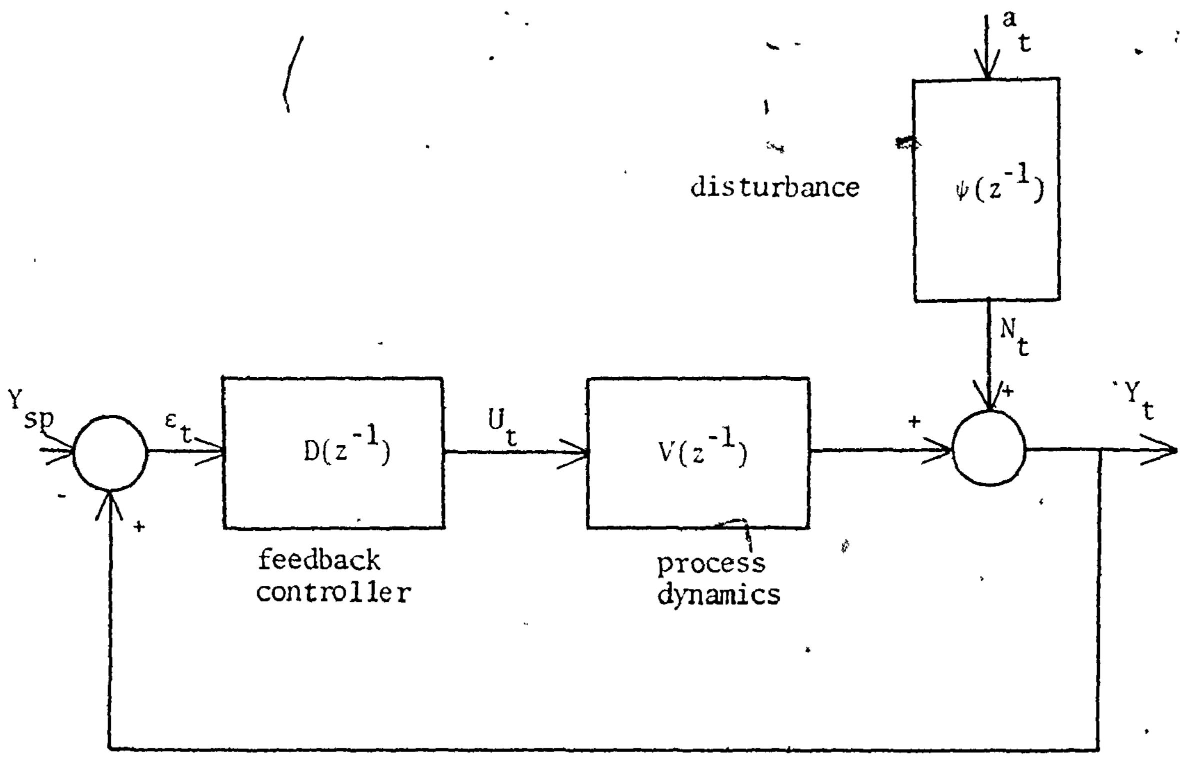


Figure 2.4: Feedback control scheme

If $D(z^{-1})$ is the minimum variance controller, (2.17) then the output of the closed loop system is a moving average process of order b-1, (2.15). The variance of the output is

$$\mathrm{var}(e_{t+b}) = \mathrm{var}(L_1(z^{-1})a_{t+b}) = (1+\psi_1^2 + \ldots + \psi_{b-1}^2)\sigma_a^2 \quad (2.19)$$

If the output is a moving average process order b-1, the autocorrelation function

$$\rho_{yy}(\tau) = \frac{E\{Y(t)Y(t+\tau)\}}{\sigma_y^2} \quad (2.20)$$

and the cross correlation function

$$\rho_{uy}(\tau) = \frac{E\{U(t)Y(t+\tau)\}}{\sigma_u \sigma_y} \tag{2.21}$$

vanish for $\tau \geq b$, where $\sigma^2_y$ is the variances of the process output and $\sigma^2_u$ is the variance of the manipulated variable. The sample auto and cross correlation function may be computed by (Box and Jenkins (1970))

$$r_{yy}(\tau) = \frac{\sum_{s=1}^{N-\tau} Y(s)Y(s+\tau)}{\sum_{s=1}^{N} Y^2(s)} \tag{2.22}$$

and

$$r_{uy}(\tau) = \frac{\sum_{s=1}^{N-\tau} U(s)Y(s+\tau)}{\sqrt{\sum_{s=1}^{N} U^2(s) \sum_{s=1}^{N} Y^2(s)}} \tag{2.23}$$

The variance of the estimated cross and auto correlations are given by Bartlett (1946, 1955). These functions can be computed to verify if the implemented controller is the minimum variance controller.

Suppose the process dynamics are represented by a first order transfer function with one whole period of delay

$$Y^*_t = \frac{\omega_o}{1-\delta z^{-1}} U_{t-1} \tag{2.24}$$

and the stochastic disturbance, by an ARIMA (1,1,0) model

$$N_t = \frac{1}{\nabla(1-\phi z^{-1})} a_t \qquad (2.25)$$

Then

$$Y_{t+1} = \frac{\omega_o}{1-\delta z^{-1}} U_t + \frac{1}{\nabla(1-\phi z^{-1})} a_{t+1} \qquad (2.26)$$

If no control action were taken, the presence of $N_t$ would cause the process output to drift away from its target valve. $N_{t+1}$ may be written as

$$N_{t+1} = a_{t+1} + \frac{\phi+1-\phi z^{-1}}{\nabla\ (1-\phi z^{-1})} a_t \qquad (2.27)$$

The minimum variance controller (2.17) is

$$U_t = \frac{-(1-\delta z^{-1})}{\omega_o} \cdot \frac{(\phi+1\ -\ \phi z^{-1})}{\nabla\ (1-\phi z^{-1})} e_t \qquad (2.28)$$

Taking $Y_{sp}=0$ with no loss of generality then

$$\nabla U_t = -\frac{(1+\phi)}{\omega_o} Y_t + \frac{\phi+\delta+\delta\phi}{\omega_o} Y_{t-1}$$

$$- \frac{\delta\phi}{\omega_o} Y_{t-2} + \phi\nabla U_{t-1} \qquad (2.29)$$

with this controller the variance of $Y_t$ is $\sigma^2_a$.

The presence of the $\nabla U_t$ term indicates that the controller has integral action. The minimum variance controller always has integral action if the disturbance model is nonstationary. The effect of the integral action is to eliminate offset. Thus the minimum variance controller will be able to satisfactorily handle the occasional set point change or deterministic load change.

## 2.4 Design of Constrained Controllers

It may happen that the variance of the manipulated variable is too large to implement if the minimum variance controller is used. This is true particularily if the sampling interval is too short compared with the process dynamics. In that case one wants to minimize the variance of the output, subject to a constraint on the variance of the manipulated variable. The design criterion is now to minimize

$$I_1 = E\{Y_{t+b}^2 + \xi U_t^2\} \tag{2.30}$$

where $\xi$ might be interpreted as cost per unit of control action taken.

The solution to (2.30) involves the solution of the discrete Wiener-Hopf equation. Wilson (1970) details a solution for this. Alternatively the dynamic, stochastic model may be transformed to state space form and (2.28) minimized by solving a Riccati equation (MacGregor (1973)). If the disturbance model is nonstationary then one must minimize

$$I_2 = E\{Y_{t+b}^2 + \xi(\nabla U_t)^2\} \tag{2.31}$$

as the variance of $U_t$ in (2.17) will be theoretically infinite due to the pole on the unit circle of the controller.

Recently Clarke et al. (1971, 1975) have proposed what appears to be a very simple solution to minimizing (2.30) or (2.31). $Y_{t+b}$ may be written as

$$Y_{t+b} = \hat{Y}_{t+b/t} + e_t \qquad (2.32)$$

Where $\hat{Y}_{t+b/t}$ is the b step ahead prediction and $e_t$ is the prediction error.

Since $\hat{Y}_{t+b/t}$ is determined from past data, $Y_{t+b/t}$ and $e_t$ are uncorrelated. Thus (2.30) may be written as

$$I_1 = E\{Y_{t+b}^2 + \xi U_t^2\} = \text{var } e_t + E\{\hat{Y}_{t+b/t}^2 + \xi U_t^2\} \qquad (2.33)$$

This equation is minimized by setting its derivative with respect to $U_t$ to zero and solving for $U_t$. Since $U_t$ is a combination of past inputs and outputs, all of which are known. Clarke et al. (1971, 1975) conclude that the expectation operator in (2.33) may be dropped. However, MacGregor and Tidwell (1976) show that this conclusion is incorrect. If the unconditional expectation operator, which is the integral over the probability density function of the random variable $U_t$, i.e.

$$E\{\hat{Y}_{t+b/t}^2 + \xi U_t^2\} = \int_{-\infty}^{\infty} (\hat{Y}_{t+b/t}^2 + \xi U_t^2)\, p(U_t)dU_t \qquad (2.34)$$

is dropped the function

$$I_3 = \hat{Y}_{t+b/t} + \xi U_t^2 \qquad (2.35)$$

is minimized. They call this a shortsighted optimal controller as it sets the instantaneous b step ahead prediction error to zero, subject to a constraint on the magnitude of the input signal. The optimal controller that minimizes (2.30) is averaged over the distribution function of possible control actions. The effect of $U_t$ on the variance of $Y_t$ at lead times greater than b is taken into account, whereas Clarke's algorithm chooses the control action that considers the effect on the output only to time t+b. As a result the optimal strategy (2.30) result in a smaller variance of the process output for the same reduction in the variance of the manipulated variable. When $\xi=0$ Clarke's solution reduces to the optimal minimum variance controller 2.16.

Although Clarke's solution does not minimize the stated objective function it does provide a sensible class of controllers. The design of a constrained controller by this optimal criterion is easier than the design based on the alternative optimal criterion (2.30, 2.31).

Minimizing (2.35) results in the control action

$$U_t = \frac{-L_3(z^{-1})}{L_4(z^{-1})} \cdot \frac{1}{\dfrac{\omega(z^{-1})}{\delta(z^{-1})} + \dfrac{\xi\,\psi(z^{-1})}{\omega_0 L_4(z^{-1})}}\, Y_t \qquad (2.36)$$

If this controller is implemented, the process output is a high order autoregressive moving average process

$$Y_t = L_4(z^{-1}) \cdot \frac{\dfrac{\xi}{\omega_0}\dfrac{\psi(z^{-1})}{L_4(z^{-1})} + \dfrac{\omega(z^{-1})}{\delta(z^{-1})}}{\dfrac{\xi}{\omega_0} + \dfrac{\omega(z^{-1})}{\delta(z^{-1})}}\, a_t \qquad (2.37)$$

For nonstationary disturbances, Clarke's algorithm can be modified to minimize

$$I_4 = \hat{Y}^2_{t+b/t} + \xi(\nabla U_t)^2 \quad (2.38)$$

The resulting controller is

$$\nabla U_t = \frac{-L_3(z^{-1})}{L_4(z^{-1})} \cdot \frac{1}{\frac{\omega(z^{-1})}{\nabla\delta(z^{-1})} + \frac{\xi}{\omega_o} \cdot \frac{\psi(z^{-1})}{L_4(z^{-1})}} Y_t \quad (2.39)$$

The process output is then

$$Y_t = L_4(z^{-1}) \frac{\frac{\xi}{\omega_o} \cdot \frac{\psi(z^{-1})}{L_4(z^{-1})} + \frac{\omega(z^{-1})}{\nabla\delta(z^{-1})}}{\frac{\xi}{\omega_o} + \frac{\omega(z^{-1})}{\nabla\delta(z^{-1})}} a_t \quad (2.40)$$

The constrained controllers can be expressed in the form

$$U_t = H(z^{-1})a_t \quad (2.41)$$

where $H(z^{-1})$ is a ratio of polynomials in $z^{-1}$. The variance of $U_t$ is then (Åström (1970))

$$\text{var}\ \{U_t\} = \frac{\sigma_a^2}{2\pi j} \oint_{-\pi}^{\pi} H(z^{-1})\, H(z)\, \frac{dz}{z} \quad (2.42)$$

where the path of integration is in the positive direction around the unit circle. Aström (1970) has detailed a solution to (2.42) that is suitable for machine or hand calculation. The variance of $Y_t$ is evaluated in a similar fashion.

The effect of increasing $\xi$ for the optimal constrained controller and Clarke's solution is to shift the poles of the closed loop system towards the zeros of $\delta(z^{-1})$.

To give an indication of the difference between the two methods consider the example from the previous section. The process is

$$Y_t = \frac{.2}{1-.9z^{-1}} U_{t-1} + \frac{a_t}{\nabla(1-.4z^{-1})} \tag{2.43}$$

The variance of $\nabla U_t$ for the unconstrained minimum variance controller is 78.3, and that of $Y_t$ is 1.0. Figure 2.5 shows the percentage decrease in the variance of $\nabla U_t$ that is possible for a given percentage increase in the variance of $Y_t$ for the controller designed to minimize criterion (2.31) by solution of the discrete Wiener-Hopf equation, and that designed to minimize (2.38) by Clarke's procedure. For this example, the difference between the two methods is quite small for increases in the output variance of less than five percent.

The curves in Figure 2.5 are produced by choosing values of $\xi$, ($\xi \geq 0$), and evaluating the integral (2.42) to obtain the variance of $\nabla^d U_t$ and the variance of $Y_t$. Clarke's procedure is straightforward as $\xi$ appears explicitly in the controller. It is more difficult to choose $\xi$ for criterion (2.31) as it does not appear explicitly in the final controller form obtained by solution of the discrete Wiener-Hopf equation. For both criteria, $\xi$ is adjusted until the variance of $Y_t$ and the variance

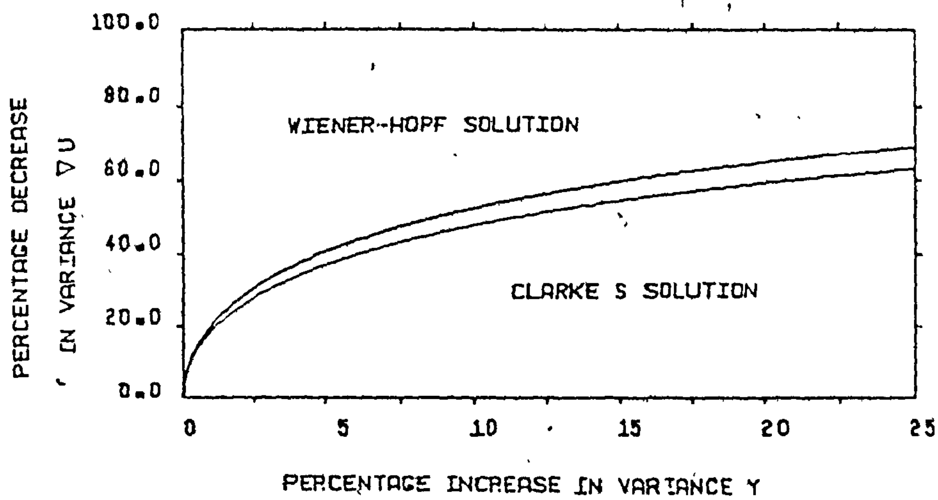


Figure 2.5: Variance $Y_t$ and the variance $\nabla U_t$ for controllers designed to minimize criterion (2.31) and criterion (2.38)

of $\nabla U_t$ are jointly acceptable.

It is also possible to 'tune' a controller on-line by adjusting $\xi$ and deciding whether the response of the input and output are satisfactory. This is done more easily with Clarke's algorithm as $\xi$ enters the controller explicitly.

## 2.5 Sensitivity of the Optimal Solution

The variance of $Y_t$ is very sensitive to variations in the parameters if $\omega(z^{-1})$ contains roots lying inside the unit circle (in $z^{-1}$ space), i.e., the dynamics are nonminimum phase (Aström (1970)). The effect of the controller $D(z^{-1})$ is to cancel the poles and zeros of the process dynamics and replace them with its own. If those zeros of $\omega(z^{-1})$, in the true process, lying inside the unit circle differ slightly from those identified in the model then imperfect cancellation will result. The closed loop transfer function (2.18) will then contain poles lying inside the unit circle and the variance of the controlled variable will increase rapidly. The controller (2.17) is still the minimum variance controller (if we have perfect cancellation), but the variance of the process output is very sensitive to changes in the system parameters.

The solution to this problem is to require that all the poles of the controller lie outside the unit circle. Aström's solution (1970) is to move the undesired poles to infinity. $\omega(z^{-1})$ may be factored as

$$\omega(z^{-1}) = \omega^{-}(z^{-1})\ \omega^{+}(z^{-1}) \qquad (2.44)$$

where $\omega^-(z^{-1})$ contains the $S^-$ zeros lying inside the unit circle and $\omega^+(z^{-1})$ contains the $S^+$ zeros of $\omega(z^{-1})$ lying outside the unit circle. $D(z^{-1})$ must not contain the zeros of $\omega^-(z^{-1})$. Therefore (2.13) is written as

$$\psi(z^{-1}) = L_4'(z^{-1}) + z^{-b} L_3'(z^{-1})\omega^-(z^{-1}) \qquad (2.45)$$

The order of $L_4'(z^{-1})$ is $b-1+S^-$. The resulting controller is

$$U_t = \frac{-\delta(z^{-1})}{\omega^+(z^{-1})\omega^-(z^{-1})} \cdot \frac{L_3'(z^{-1})\omega^-(z^{-1})}{L_4'(z^{-1})} Y_t$$

$$= \frac{\delta(z^{-1})}{\omega^+(z^{-1})} \cdot \frac{L_3'(z^{-1})}{L_4'(z^{-1})} Y_t \qquad (2.46)$$

The output of the closed loop system is then a moving average process of order $b-1+S^-$.

A move flexible approach however, is to use a constrained controller (minimize (2.30) or (2.35)). Increasing $\xi$ progressively shifts the roots of the controller transfer function from outside the unit circle towards infinity. If the control is constrained by solution of the Wiener-Hopf equation (or Riçatti solution) any nonzero value of $\xi$ will stabilize the variance of $U_t$, no matter how small. For the method of Clarke et al. (1971, 1975), there will be some 'threshold' value of $\xi$ greater than zero beyond which it will stabilize the variance of $U_t$.

If controller transfer function is stabilized by constraining the control, instead of by Aström's method, it is possible to design a controller that has a lower variance of the manipulated variable for the same output variance. However, the latter solution is much easier to implement.

## 2.6 Optimal Choice of the Sampling Interval

The prime consideration when choosing a sampling interval is that not too much should happen to the process between sampling intervals. Early work in digital control led to the following guidelines (Shinsky (1967))

| | Sampling Time (seconds) |
|---|---|
| Flow | 1 |
| Level and Pressure | 5 |
| Temperature | 20 |

These guidelines reflect the fact that some loops are faster than others. In Figure 2.4 $N_t$ represents the total effect on the output of all unobserved disturbances acting within the system. The presence of the disturbances are the reason for having the controller, and the sampling time should be chosen so that good control is maintained in the presence of these disturbances.

If a dynamic stochastic model is known at a sampling interval T, MacGregor (1976) has shown how the parameters of the process dynamic and stochastic model change if the sampling interval is changed to an integer multiple of T. New data is not required to investigate the effect of the sampling interval on the variance of the process input and output. It is shown that the variance of the process output increases very little as the sampling rate is decreased, until the sampling rate approaches the process dead time. If the sampling rate is decreased further the variance of the output increases rapidly. There will be a significant reduction in the variance of the manipulated variable as the sampling rate is decreased. If the dynamics are minimum phase at b=1, it is concluded that there is little to be gained by sampling at a faster rate (at least in so far as the stochastic disturbances are concerned).

The zeros of $\omega(z^{-1})$ that lie inside the unit circle can be introduced by the sampling of a continuous process at discrete time intervals. Suppose that the underlying continuous process may be described by a first order differential equation

$$\tau \frac{dY(t)}{dt} + Y(t) = g\, U(t-\tau_d) \tag{2.47}$$

where $\tau_d$ is the transport delay. If the process is sampled every T seconds then (2.47) may be written as

$$\tau \frac{dY(t)}{dt} + Y(t) = g\, U(t-(f+c)T) \tag{2.48}$$

Here f is the number of pure periods of delay and c is the fractional period of delay $(0 \le c < 1)$. If the inputs to the process are changed only at sampling times then it is easily shown that the discretized version of this process is given by

$$Y_t = \frac{g(1-\delta)\ (1-\nu+\nu z^{-1}) z^{-(f+1)}}{1-\delta z^{-1}} U_t$$

$$Y_t = \frac{g(1-\delta)\ (1-\nu+\nu z^{-1}) z^{-b}}{1-\delta z^{-1}}\ U_t \tag{2.49}$$

where

$$b = f + 1 \tag{2.50}$$

$$\delta = \exp(-T/\tau) \tag{2.51}$$

$$\nu = \frac{\delta - \delta^{1-c}}{1-\delta} \tag{2.52}$$

If $\nu$ is greater than 0.5 then the discrete transfer function may be nonminimum phase even though the underlying continuous process may be minimum phase.

If the sampling interval is short relative to the process dynamics then $\delta \to 1.0$ By L'Hopitals rule

$$\lim_{\delta \to 1} \nu = c \tag{2.53}$$

Equation (2.47) may be written as

$$Y_t = \frac{g(1-\delta)\ (1-c+cz^{-1})}{1-\delta z^{-1}}\ U_{t-b} \tag{2.54}$$

Thus only when the fractional period of delay is less than 0.5 will the discrete process be minimum phase. For higher order processes a similar analysis can be done, however, the effect of the fractional period of delay is not as clear. Thus the sampling interval must be selected with some care.

## 2.7 Discussion of Minumum Variance Control

Stochastic control appears to be well suited for control of industrial processes characterized by long dead times and drifting disturbances. The weakness of many modern control strategies is that they require an exhaustive knowledge of the process dynamics and disturbance statistics, yet provide no means of determining these. Box and Jenkins (1970) and Aström (1970) have provided techniques so that one can build dynamic and stochastic models from input/output data. These models are usually of low order. The orders of the noise model rarely exceed two. Dynamic models of a complex process are usually simple (Mosler et al. (1966)). The transfer function characterizes the major time constants and the delay handles the effect of the distributed parameters and transport lag. Where applicable, minimum variance feedforward control can be implemented.

There are very few reported industrial applications of minimum variance control. It has been used in the pulp and paper industry for control of basis weight and moisture content on a paper machine, Aström (1970), and for the viscosity control of a polymerization process, MacGregor and Tidwell (1977). On processes for which stochastic control is well suited, the process characteristics may be changing, Cegrell and Hedqvist (1974). If the parameters of the dynamic and stochastic models change very slowly, one might periodically re-estimate these. This is a laborious task and necessitates a plant experiment. One might try to estimate the parameters from closed loop operating data. The feedback controller must be nonlinear, time varying or an external "dither signal" supplied to insure that unique parameter estimates are obtained, unless certain necessary and sufficient conditions are satisfied.

To ensure that the controller is always optimal, one might try to update the parameters of the dynamic and stochastic models at every sampling interval, compute the minimum variance control signal and implement it. This concept of an adaptive algorithm is shown in Figure 2.6 (Wittenmark (1975)).

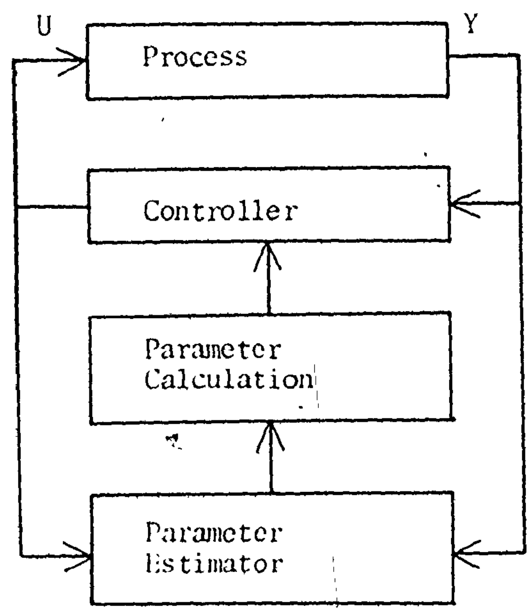


Figure 2.6: Adaptive control scheme

This strategy is fraught with the problems of parameter estimation in closed loop. The requirements necessary to insure consistent parameter estimates conflict with the primary goal-good control.

If one were willing to sacrifice quality of control, how would one estimate the parameters? The maximum likelihood method would be computationally too cumbersome to implement at every sampling invertal. If the purpose of building dynamic and stochastic models is to control the process, then one is not interested in current estimates of these parameters. Only those combinations of parameters which appear in the minimum variance controller need be well tuned. Aström and Wittenmark (1973) have developed an algorithm where only those parameters which appear in the minimum variance controller are estimated at each sampling interval. The model form is chosen so that these parameters may be estimated recursively. It is assumed that the parameters are constant, but a slight modification to the algorithm will allow the parameter estimator to track slowly drifting parameters.

The algorithm has good transient behavior and requires very little knowledge of the process dynamic and stochastic models. The theory of self-tuning regulators is discussed in the next chapter. The applications of self-tuning regulators to the control of industrial processes is examined in chapter four.

CHAPTER 3

THEORY OF SELF-TUNING REGULATORS

## 3.1 Introduction

The principle theoretical developments of self-tuning regulators are discussed in Aström and Wittenmark (1973), Wittenmark (1973), Ljung and Wittenmark (1974), and most recently by Clarke et al. (1975). It is presumed that the process may be described by a model of the form

$$Y_{t+b} = \frac{\overline{B}(z^{-1})}{A(z^{-1})} U_t + \frac{C(z^{-1})}{A(z^{-1})} a_{t+b} \qquad (3.1)$$

This notation is mathematically convenient to deal with, and from this convenience stems the motivation for the self-tuning regulator algorithm. However, it is felt that the Box and Jenkins (1970) description of the process dynamics and disturbances,

$$Y_{t+b} = \frac{\omega(z^{-1})}{\delta(z^{-1})} U_t + \frac{\theta(z^{-1})}{\nabla^d \phi(z^{-1})} a_{t+b} \qquad (3.2)$$

is more appealing from an engineering point of view. The separation of the process dynamics and stochastics, clearly indicates the form of the estimation model in an adaptive environment, and how a knowledge of the process dynamics may be used if they are known. It is readily seen how to modify the estimation scheme so that integral action is included in the controller. If the stochastic disturbances are nonstationary or the process is subject to deterministic load changes, direct application of the self-tuning regulator results in a controller that is sensitive to parameter uncertainities, and will not eliminate the resulting offset. Unless the structure of the self-tuning regulator correctly accounts for nonstationary disturbances, the self-tuning controller of Clarke et al. (1975) will not produce the stated results.

The theory of self-tuning regulators is discussed in this chapter using the notation of Box and Jenkins (1970). It is hoped to clarify aspects of the self-tuning regulators that have caused confusion, and it is felt these uncertainities are best resolved using this notation.

## 3.2 Theory of the Self-Tuning Regulator

Consider the Box and Jenkins representation of Figure 2.2,

$$Y_{t+b} = \frac{\omega(z^{-1})}{\delta(z^{-1})} U_t + \frac{\theta(z^{-1})}{\nabla^d \phi(z^{-1})} a_{t+b} \tag{3.2}$$

Using (2.14) the disturbance model can be expressed as

$$\frac{\theta(z^{-1})}{\nabla^d \phi(z^{-1})} = L_4(z^{-1}) + z^{-b} \frac{L_2(z^{-1})}{\nabla^d \phi(z^{-1})} \tag{3.3}$$

multiplying (3.2) by $L_4(z^{-1})$ and substituting (3.3) into the result yields

$$\delta(z^{-1})\theta(z^{-1})(Y_{t+b}-L_4(z^{-1})a_{t+b}) = \delta(z^{-1})L_2(z^{-1})Y_t$$

$$+ \omega(z^{-1})L_4(z^{-1})\phi(z^{-1})\nabla^d U_t \tag{3.4}$$

The prediction error $L_4(z^{-1})a_{t+b}$ is uncorrelated with $\{Y_t, Y_{t-1} \ldots\}$ and with $\{\nabla^d U_t, \ \nabla^d U_{t-1}, \ldots\}$ and may for convenience be replaced by $\varepsilon^0_{t+b}$. Equation (3.4) is of the form

$$\delta(z^{-1})\theta(z^{-1})(Y_{t+b}-\varepsilon^0_{t+b}) = \alpha^0(z^{-1})Y_t+B^0(z^{-1})\nabla^d U_t \quad (3.5)$$

where

$$\alpha^0(z^{-1}) = \alpha^0_0 + \alpha^0_1 z^{-1} + \ldots + \alpha^0_{mo} z^{-mo} \quad (3.6)$$

$$B^0(z^{-1}) = B^0_0 + B^0_1 z^{-1} + \ldots + B^0_{\ell o} z^{-\ell o} \quad (3.7)$$

It is easily verified that the orders of mo and $\ell o$ are

$$mo = r + \max(q-b, p+d-1) \quad (3.8)$$

$$\ell o = s + p + b - 1 \quad (3.9)$$

If the parameters of (3.5) were known, the control action

$$\nabla^d U_t = \frac{-\alpha^0(z^{-1})}{B^0(z^{-1})} Y_t = \frac{-\delta(z^{-1})}{\omega(z^{-1})} \cdot \frac{L_2(z^{-1})}{L_4(z^{-1})\ \phi(z^{-1})} Y_t \quad (3.10)$$

would minimize $E\{Y^2_{t+b}\}$.

Suppose that $\delta(z^{-1})\theta(z^{-1}) = 1.0$. It is seen from (3.5) that the process output is now expressed as a direct function of the minimum variance controller parameters. If these were unknown one might try to identify them from a model of the form

$$Y_{t+b} = \alpha(z^{-1})Y_t + B(z^{-1})\nabla^d U_t + \varepsilon_{t+b} \qquad (3.11)$$

where

$$\alpha(z^{-1}) = \alpha_0 + \alpha_1 z^{-1} + \ldots + \alpha_m z^{-m} \qquad (3.12)$$

and

$$B(z^{-1}) = B_0 + B_1 z^{-1} + \ldots + B_\ell z^{-\ell} \qquad (3.13)$$

The number of whole periods of delay, b and the order of the pole, d lying on the unit circle of the disturbance are presumed known. Equation (3.11) admits least squares estimation which may be expressed recursively. The parameters could be updated at every sampling interval and used in the control law

$$\nabla^d U_t = \frac{-\hat{\alpha}(z^{-1})}{\hat{B}(z^{-1})} Y_t \qquad (3.14)$$

as if they were exactly known. One is not trying to identify the process dynamic and stochastic parameters. Only those combinations which appear in the minimum variance controller are estimated.

This algorithm does not minimize $E\{Y^2_{t+b}\}$ as it fails to account for parameter uncertainities, (Aström and Wittenmark (1973)). Algorithms using parameter uncertainities in the computation of the control signal are known as 'cautious controllers', (Wittenmark (1975)). Large control actions are not permitted unless the controller parameters are well estimated.

The restriction that $\delta(z^{-1})\theta(z^{-1}) = 1.0$ appears to limit the usefulness of this scheme, known as a self-tuning regulator. However, Aström and Wittenmark (1973) have proved two theorems which show that this algorithm may have some desired asymptotic properties, irrespective of the product $\delta(z^{-1})\theta(z^{-1})$.

Theorem 1: If the parameter estimates $\hat{\alpha}_i$, i=0,1,...m and $\hat{B}_i$, i=0,1,...$\ell$ of (3.11) converge as $t\to\infty$, where m and $\ell$ are of arbitrary non zero order, and the closed loop system is such that the output is ergodic (in the second moments) then

$$\rho_{yy}(\tau) = \frac{E\{Y(t)Y(t+\tau)\}}{\sigma_y^2} = 0 \ , \ \tau = b,b+1,\ldots,b+m \qquad (3.15)$$

and

$$\rho_{\nabla^d uy}(\tau) = \frac{E\{\nabla^d U(t)Y(t+\tau)\}}{\sigma_{\nabla^d u}\sigma_y} = 0 \ , \ \tau = b,b+1,\ldots b+\ell \qquad (3.16)$$

Theorem 2: Assume that the system may be described by equation (3.2). If the self-tuning regulator is used with m=mo and $\ell=\ell o$, and the parameters estimates converge so that $\hat{\alpha}(z^{-1})$ and $\hat{B}(z^{-1})$ have no common factors, then the regulator will converge to the minimum variance controller.

Theorem 1 states that the use of least squares estimation and the control law (3.14) will reduce certain of the auto and cross correlations at the output to zero according to (3.15) and (3.16), if the parameter estimates of (3.11) converge. There is no requirement that $\delta(z^{-1})\theta(z^{-1}) = 1.0$. If $\rho_{yy}(\tau)$ and $\rho_{\nabla^d uy}(\tau)$ vanish for all $\tau \geq b$, Theorem 2 states that the orders of $\alpha(z^{-1})$ and $B(z^{-1})$ have not been under estimated and the controller (3.14) is the same one that could have been designed had the process dynamic and stochastic models (3.2) been known. There is no guarantee that the parameter estimates will converge. Ljung and Wittenmark (1974) have constructed an example where convergence of the controller parameters (3.11) to the optimal parameters is theoretically impossible.

The motivation for the self-tuning regulator stems from the fact that if $\delta(z^{-1})\theta(z^{-1}) = 1.0$, the process output may be expressed as an explicit function of the minimum variance controller parameters. A computationally efficient algorithm may be then used to estimate these parameters. Convergence of the controller parameters to the optimal values for an arbitrary $\delta(z^{-1})\theta(z^{-1})$ was a surprising result (Aström and Wittenmark (1973)) as the output of the process is no longer an explicit function of only the minimum variance controller parameters.

The following analysis is an attempt to show that the process may behave as though it were being generated by a model of the form (3.11) even though $\delta(z^{-1})\theta(z^{-1}) \neq 1.0$.

The system (3.5) may be written as

$$Y_{t+b} = (\alpha^0(z^{-1})Y_t + B^0(z^{-1})\nabla^d U_t)(1+\xi_1^0 z^{-1} + \ldots) + \varepsilon^0_{t+b} \qquad (3.17)$$

where

$$\delta^{-1}(z^{-1})\theta^{-1}(z^{-1}) = 1 + \xi_1^0 z^{-1} + \xi_2^0 z^{-2} + \ldots \qquad (3.18)$$

The controller parameters are estimated from the model (3.11) and the control action (3.14) taken at every sampling interval. If the control action (3.14) is substituted into (3.17) then

$$Y_{t+b} = \alpha^0(z^{-1})Y_t + B^0(z^{-1})\nabla^d U_t + \xi_1^0[\alpha^0(z^{-1})Y_{t-1}$$

$$- \frac{\hat{\alpha}(t-1,z^{-1})}{\hat{B}(t-1,z^{-1})} \cdot B^0(z^{-1})Y_{t-1}] + \xi_2^0[\alpha^0(z^{-1})Y_{t-2}$$

$$- \frac{\hat{\alpha}(t-2,z^{-1})}{\hat{B}(t-2,z^{-1})} \cdot B^0(z^{-1})Y_{t-2}] + \ldots + \varepsilon^0_{t+b} \qquad (3.19)$$

If the process is open loop stable, i.e. $\delta(z^{-1})$ has all its roots lying outside the unit circle, $\xi^0(z^{-1})$ is a convergent polynomial in $z^{-i}$, and the $\xi_i^0$ weights decrease with increasing i. The process output then depends to a decreasing extent on the successive terms in (3.19). Furthermore the terms in the square brackets in (3.19) tend to be small due to the control action (3.14). The output therefore appears to be nearly generated by a model of the form (3.11). Estimation of the minimum variance controller parameters with this model form might then be justified.

Aström and Wittenmark (1973) define a self-tuning or self-adjusting strategy as an algorithm using parameter estimates that are constant but unknown, and that converges to the optimal solution that could have been obtained had the parameters been known. They show that the strategy described above will not asymptotically minimize $E\{Y^2_{t+b}\}$ for an arbitrary controller form. (It will still reduce certain auto and cross correlations to zero as in (3.15) and (3.16)). If the order of the controller (3.14) is underestimated it is possible to design a regulator that results in a smaller output variance. Thus the controller (3.14) will only be self-tuning if its order has not been underestimated. The terminology self-tuning regulator has been used quite loosely in the literature to refer to an algorithm where the parameters of an arbitrary controller form, i.e. PID are tuned recursively.

Some corollories from Theorem 2 are (Aström and Wittenmark (1973)):

1) if $m > mo$ or $\ell > \ell o$ Theorem 2 still holds

2) if $m > mo$ and $\ell > \ell o$ then $\hat{\alpha}(z^{-1})$ and $\hat{B}(z^{-1})$ contain common factors if the estimates converge, and Theorem 2 does not hold.

There are two approaches to implementing a self-tuning regulator. If the process dynamic and stochastic models are known, $\ell$ and $m$ are known. Alternatively one can select values for $m$ and $\ell$, and implement the self-tuning regulator. If the parameter estimates converge certain auto and cross correlations will be zero. If these are not zero for all lags greater than the number of whole periods of delay, then the order of the controller should be increased. The use of diagnostic tools suggested by Theorem's 1 and 2 allows one by successive modification to select the correct order for the optimal controller if it is unknown.

## 3.3 Structure of the Estimation Model

The controller parameters are estimated from a model of the form

$$Y_{t+b} = \alpha(z^{-1})Y_t + B(z^{-1})\nabla^d U_t + \varepsilon_{t+b} \tag{3.11}$$

where the input and output sequence $(Y_t, Y_{t-1}, \ldots, U_t, U_{t-1} \ldots)$ are deviation variables from their mean values. It is unlikely that a real process is linear over the entire range where the input and output may vary.

One is trying to control the process about its steady state value and the parameters of (3.11) must reflect the behavior of the process and disturbances in this region. Consequently the input and output sequence are expressed as deviation variables.

The mean value of the controlled variable will be its set point. The mean value of the manipulated variable necessary to maintain the output at its set point, may not be known, and will change if the set point of the controlled variable is altered. If d=0, then (3.11) may be written as

$$Y_{t+b} = \alpha(z^{-1})Y_t + B(z^{-1})\tilde{U}_t + \nu + \varepsilon_{t+b} \qquad (3.20)$$

The sequence $\{\tilde{U}_t$'s$\}$ are deviations of the manipulated variable from some reference value, which is an apriori estimate of the steady state value of the manipulated variable. The difference between the chosen reference value and the true steady state value is reflected in $\nu$, an additional parameter to be estimated. If the steady state value is not exactly known and $\nu$ not estimated, the controlled variable will have offset.

If the disturbance is nonstationary, i.e. $d > 0$, the steady state value of the manipulated variable need not be known, since the control action (3.14) will be expressed only in terms of $\nabla^d U_t$. If the parameter estimates of (3.11) converge, the integral action in the controller (3.14) insures that there is no offset in the controlled variable. It is not necessary that the orders, m and $\ell$ of $\alpha(z^{-1})$ and $B(z^{-1})$ be equal to mo and $\ell$o to eliminate offset in this manner.

One proposal for overcoming offset in the face of nonstationary disturbances (Seborg et al. (1976)) is to use a non-integrating form of the controller (d=0 in (3.14)), but increase the order of $B(z^{-1})$. This allows the self-tuning algorithm to force an additional root of $B(z^{-1})$ towards the unit circle, thereby creating integral action in an indirect way. However, this seems pointless because it is known apriori that the optimal controller must be of the form (3.14) with $d > 0$.

The additional complication introduced by trying to estimate the extra pole of $B(z^{-1})$ near unity would usually not be trivial since it is known that ill-conditioning of the estimation space results when parameters lie near stability boundaries (Box and Jenkins (1970)). Furthermore, the variance of the controlled variable may be sensitive to parameter variations. The roots of the controller may bounce inside the unit circle and the controller transfer function will become unstable. The analysis of the sensitivity of the closed loop system to parameter uncertainities is analogous to the case where the process dynamics are non-minimum phase.

The number of whole periods of delay b, and the order of the pole d, (usually 0 or 1 depending on whether or not the disturbance is stationary) lying on the unit circle in the disturbance model are presumed known. It is reasonable to expect that b will be known, noting that it equals one, plus the integer portion of the transport delay divided by the sampling interval. The number of whole periods of delay is a fundamental characteristic of the process and should be known. However, there will be a rapid degredation in the ability to control the process if b is less than the true number of whole periods of delay. In situations where the number of whole periods of delay is unknown it is better to overestimate b rather than underestimate it (Wittenmark (1973)).

Box and Jenkins (1970) characterize process disturbances by low order ARIMA time series models of the form

$$N_t = \frac{\theta(z^{-1})}{\nabla^d \phi(z^{-1})} a_t \qquad (3.20a)$$

In most physical processes, d will usually be 0 or 1, (Box and Jenkins (1970)). If the process output drifts away from its target value, then the disturbance can be adequately characterized with d = 1. It is not unreasonable then to expect that d will be known.

## 3.4 Deterministic Disturbances

The disturbances (load or setpoint) affecting a process may be deterministic rather than stochastic. They may still be modelled by the methods of Box and Jenkins (1970) and Aström (1970) although the interpretation of $a_t$ is different. Instead of the disturbance $N_t$ being considered as the output of a linear filter driven continuously by white noise, (Figure 2.3) a deterministic disturbance is the output of a linear filter driven by one shock or impulse. For example step disturbances may be modelled as:

$$N_t = \frac{1}{1 - z^{-1}} a_t \qquad (3.21)$$

where $a_t$ is an impulse at time t, of magnitude equal to the size of the step change. If the disturbance is deterministic a controller based on the minimum variance strategy will remove the total effect of the disturbance within b sampling intervals, as the future behavior of the disturbance is exactly known from a record of its past history. These controllers are referred to as 'dead beat' controllers (Ragazzini and Franklin (1958)) rather than unconstrained minimum variance controllers.

The self-tuning regulator may converge to a dead beat controller if the process is subject to deterministic disturbances (Wittenmark (1973)). Analysis of the design of these controllers shows that the model form for estimation of the regulator parameters should be

$$Y_{t+b} = \alpha(z^{-1})Y_t + B(z^{-1})\nabla^d U_t + \varepsilon_{t+b} \qquad (3.22)$$

where d=1 for step disturbances and d=2 for ramp disturbances. For processes affected by deterministic and stochastic disturbances the controller (3.14) may converge to a regulator which is neither dead beat nor minimum variance, but whose values depended on the relative size of each disturbance.

The most useful model for estimation of the controller parameters is (3.11) with d=1. This is the correct structure for process subject to drifting types of stochastic disturbances, or to step disturbances. It is unnecessary to know the steady state value of the manipulated variable, and the integral action in the controller insures that the controlled variable is not biased from its target value.

### 3.5 Self-Tuning Feedforward-Feedback Control

Consider the situation in Figure 3.1. In addition

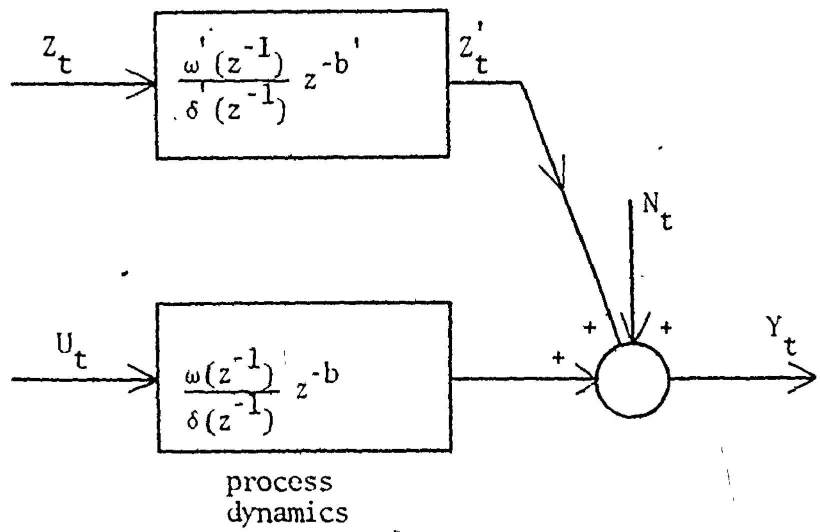


Figure 3.1: Combined feedforward and feedback control scheme

to the unobserved disturbances $N_t$, fluctuations in the process output may be attributed to a variable $Z_t$, which can be measured but not manipulated. If there were no unobserved disturbances $N_t$, and no control action taken, the process output would equal $Z'_t$, where

$$Z'_t = \frac{\omega'(z^{-1})}{\delta'(z^{-1})} Z_{t-b'} = \frac{\omega'_0 - \omega'_1 z^{-1} \ldots - \omega'_s z^{-s}}{1 - \delta'_1 z^{-1} - \ldots - \delta'_r z^{-r}} Z_{t-b'} \quad (3.23)$$

$Z_t$ is referred to as a feedforward variable. In the case where $b' \geq b$, (i.e. where the manipulated variable $U_t$, can compensate for the measured disturbance before it reaches the process output $Y_t$) the controller minimizing

$E\{Y^2_{t+b}\}$ is given by (Box and Jenkins (1970))

$$\nabla^d U_t = \frac{-\delta(z^{-1})}{\omega(z^{-1})} \cdot \left[\frac{\omega'(z^{-1})}{\delta'(z^{-1})}\hat{\nabla} dZ_{t-j} + \frac{L_2(z^{-1})}{\phi(z^{-1})L_4(z^{-1})} Y_t\right] \quad (3.24)$$

where

$$j = b' - b \quad (3.25)$$

$L_2(z^{-1})$ and $L_4(z^{-1})$ are defined as before. (If $b' < b$ then $\nabla dZ_{t-j}$ has not yet occurred at time t and the controller (3.24) is not physically realizable. A "minimum variance forecast" of $\nabla\hat{Z}'_{t-j/t}$ is made and substituted in (3.33) in place of $\frac{\omega'(z^{-1})}{\delta'(z^{-1})} \nabla Z_{t-j}$. This procedure is outlined by Box et al. (1974).)

If the parameters of (3.24) were unknown one might estimate them from a model of the form

$$Y_{t+b} = \alpha(z^{-1})Y_t + B(z^{-1})\nabla^d U_t + \nu(z^{-1})\nabla^d Z_{t-j} + \varepsilon_{t+b} \quad (3.26)$$

where

$$\alpha(z^{-1}) = \alpha_0 + \alpha_1 z^{-1} + \ldots + \alpha_m z^{-m} \quad (3.27)$$

$$B(z^{-1}) = B_0 + B_1 z^{-1} + \ldots + B_\ell z^{-\ell} \quad (3.28)$$

$$\nu(z^{-1}) = \nu_0 + \nu_1 z^{-1} + \ldots + \nu_n z^{-n} \quad (3.29)$$

and use them in the control law

$$\nabla^d U_t = -\left(\frac{\hat{\nu}(z^{-1})}{\hat{B}(z^{-1})} \nabla^d Z_{t-j} + \frac{\hat{\alpha}(z^{-1})}{\hat{B}(z^{-1})} Y_t\right) \quad (3.30)$$

as if they were exactly known. The development parallels the case outlined in (3.2), and only the result will be stated. If the parameter estimates of (3.26) converge then (Wittenmark (1973))

$$\rho_{yy}(\tau) = 0, \qquad \tau = b,\ b+1, \ldots b+m \tag{3.31}$$

$$\rho_{\nabla^d u y}(\tau) = 0, \qquad \tau = b,\ b+1, \ldots b+\ell \tag{3.32}$$

$$\rho_{\nabla^d z y}(\tau) = 0, \qquad \tau = b-j,\ b+1-j, \ldots b+n-j \tag{3.33}$$

The optimal values for $\ell$, m, and n are $\ell o$, mo and no, and are given by

$$\ell o = r' + s + p + b - 1 \tag{3.34}$$

$$mo = r' + r + \max(q-b,\ p+d-1) \tag{3.35}$$

$$no = s' + r + p + b - 1 \tag{3.36}$$

In addition if $\ell = \ell o$, m = mo, and n = no, and certain weak conditions are satisfied, then the controller (3.30) is the same one that could have been designed offline, had the disturbance and both dynamic models been known (Wittenmark (1973)).

Inclusion of the feedforward variable in the estimation model (3.26) may possibly increase the order of the terms $\alpha(z^{-1})$ and $B(z^{-1})$. More than one feedforward variable may be included in the estimation model (3.26) and controller (3.30) if they are available.

## 3.6 Decoupling in Multivariable Situations

Self-tuning regulators may be used to decouple a multivariable control problem. Consider the situation shown in Figure 3.2, where for simplicity a two input two output system is depicted.

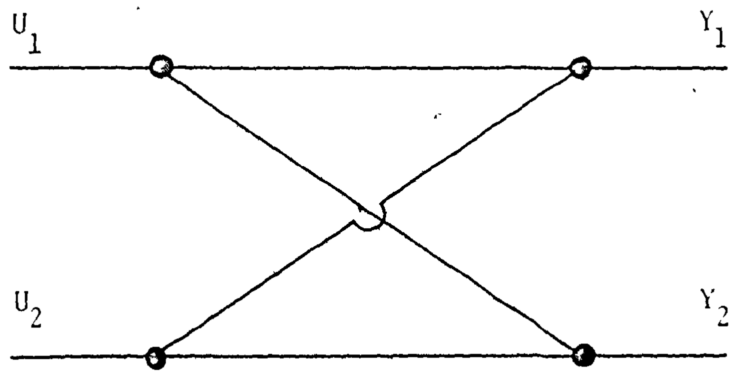


Figure 3.2: Representation of a multivariable process

There are four transfer functions, $U_1$ to $Y_1$, $U_1$ to $Y_2$, $U_2$ to $Y_1$ and $U_2$ to $Y_2$, each having their own dead times. If $U_1$ did not affect $Y_2$ then $Y_1$ is a two input one output process. (This is a common situation in control of paper machines, where the moisture content can be controlled by manipulating the input to the dry basis weight control loop, as well as the thick stock flow which does not influence the dry basis weight.) The output $Y_2$ is controlled by manipulating $U_2$. $U_2$ enters the other control loop as a feedforward variable and is handled by methods discussed in the previous section. Thus the self-tuning regulator may be used to decouple a multivariable process.

## 3.7 Constrained Control

Recently Clarke et al. (1971, 1975) proposed a strategy claimed to minimize

$$I_1 = E\{Y^2_{t+b} + \xi''(\nabla^d U_t)^2\} \tag{3.37}$$

As pointed out by MacGregor and Tidwell (1976) they incorrectly account for an expectation operation and in fact minimize

$$I_2 = \hat{Y}^2_{t+b/t} + \xi'(\nabla^d U_t)^2 \tag{3.38}$$

which was termed short sighted optimal control (refer to section 2.4). Clarke et al. prove that minimizing (3.38) is equivalent to minimizing

$$I_3 = E\{(Y_{t+b} + \xi\nabla^d U_t)^2\} = E\{\phi^2_{t+b}\} \tag{3.39}$$

where

$$\xi = \frac{\xi'}{\omega_0} \tag{3.40}$$

This identity forms the basis for their derivation of a self-tuning controller to minimize (3.38). (If the disturbance is nonstationary it is necessary to minimize $E\{(Y_{t+b} + \xi\ \nabla^d U_t)^2\}$ because the variance of $U_t$ is theoretically infinite. The results of this section are modified from Clarke's algorithm to account for this.) By defining this new function $\phi_{t+b}$, it is seen that the system (2.2) may be written as

$$\delta(z^{-1})\theta(z^{-1})(\phi_{t+b} - \epsilon^0_{t+b}) = \alpha^0(z^{-1})Y_t +$$

$$[B^0(z^{-1}) + \xi\delta(z^{-1})\theta(z^{-1})]\nabla^d U_t \tag{3.41}$$

It the parameters of (3.41) were known the control action

$$\nabla^d U_t = \frac{-\alpha^0(z^{-1})}{B^0(z^{-1}) + \xi\delta(z^{-1})\theta(z^{-1})} Y_t$$
$$= \frac{-\delta(z^{-1})L_2(z^{-1})}{\omega(z^{-1})L_4(z^{-1})\phi(z^{-1}) + \xi\delta(z^{-1})\theta(z^{-1})} Y_t \tag{3.42}$$

would minimize (3.39). The effect of $\xi$ is to possibly increase the order of denominator of the controller. If the parameters of (3.41) are unknowns Clarke et al. (1975) suggest that they be estimated from a model of the form

$$\phi_{t+b} = \alpha(z^{-1})Y_t + B'(z^{-1})\ \nabla^d U_t + \epsilon_{t+b} \tag{3.43}$$

where the order of $B'(z^{-1})$ may be increased over the unconstrained minimum variance controller. If the process dynamic and stochastic models are unknown, the value of $\xi$ that will reduce the variance of $U_t$ by a given amount will be unknown and it will be necessary to search for this value by trial and error. There have been no reported applications of Clark's algorithm. Clarke's paper is fraught with many errors and a corrected version is given in Appendix C.

## 3.8 Nonminimum Phase Systems

Minimum variance and dead beat controllers are very sensitive to variations in the parameters of nonminimum phase dynamics. Strategies for 'detuning' these controllers are readily implemented in a nonadaptive environment (refer to Chapter 2.5). In an adaptive environment these systems are difficult to handle.

Sampling of a continuous minimum phase process may result in a discrete model which is nonminimum phase, i.e. some roots of $\omega(z^{-1})$ lie inside the unit circle. A discrete process may be minimum phase at one operating level and nonminimum phase at another, depending whether the fractional period of delay passes a certain critical value. Flow processes would have this problem. (A paper machine operating at different speeds is a common industrial example.) A solution is to increase the sampling interval so that this shift in and out of nonminimum phase does not occur. However, the sampling rate may be fixed due to hardware restrictions.

Aström and Wittenmark (1974) have proposed a strategy to asymptoticly minimize the variance of $Y_t$ subject to the constraint that the closed loop have all its zeros outside the unit circle. It is not a practical strategy as it involves a real time factorization of a polynomial in $z^{-1}$, and solution of a set of linear equations. The simplicity of the basic self-tuning algorithm is lost and problems may occur with estimation of the parameters. It is impossible to directly estimate the controller parameters since the process output cannot be expressed as an explicit function of the parameters we wish to estimate.

Alternatively, Aström (1974) suggests that one try to identify the parameters of the process dynamic and stochastic models,

$$Y_{t+b} = \frac{\omega(z^{-1})}{\delta(z^{-1})} U_t + \frac{\theta(z^{-1})}{\nabla^d \phi(z^{-1})} a_{t+b} \quad (3.2)$$

The parameters of (3.2) are estimated at every sampling interval using recursive maximum likelihood estimation or an equivalent method. These methods essentially require repeated application of least squares to identify the dynamic and stochastic parameters. The state variable representation of (3.2) is reconstructed at every sampling interval and a Ricatti equation solved to minimize $\text{var}\{Y_t\} + \xi\ \text{var}\{\nabla^d U_t\}$.

Estimation of the process dynamic and stochastic parameters requires that one have an apriori knowledge of the structure of both the process dynamics and disturbances. There is no function that can be computed to indicate optimality of the controller, as the process output will be a complex ARIMA time series. Identifiability problems should not arise as the controller will be nonlinear and time varying. This algorithm is computationally more time consuming than the ordinary self-tuning regulator.

Clarke's algorithm (1975) may be used although the value of $\xi$ that moves all the poles of the closed loop outside the unit circle will probably not be known. This is the most appealing method of dealing with nonminimum phase systems other than increasing the sampling interval.

## 3.9 Stability of the Closed Loop System

Ljung and Wittenmark (1975) have proven that the self-tuning regulator has a stabilizing property when $\delta(z^{-1})\theta(z^{-1}) = 1.0$. Suppose the output of the process may be expressed as

$$Y_{t+b} = \alpha^0(z^{-1})Y_t + B^0(z^{-1})U_t + v^0_{t+b} \quad (3.44)$$

The requirement that $v^0_{t+b}$ be a moving average process of order b-1, is relaxed. $v^0_{t+b}$ is any disturbance, stochastic or deterministic with the restriction that

$$\frac{1}{t} \sum_{s=1}^{t} v^2(s) < C_1 \quad (3.45)$$

The parameters of the minimum variance controller are estimated from the model

$$Y_{t+b} = \alpha(z^{-1})Y_t + B(z^{-1})U_t + \varepsilon_{t+b} \quad (3.46)$$

If the number of whole periods of delay is known, the orders of $\alpha(z^{-1})$ and $B(z^{-1})$ not having been underestimated, then the regulator

$$U_t = \frac{-\hat{\alpha}(z^{-1})}{\hat{B}(z^{-1})} Y_t \quad (3.11)$$

will stabilize the system (3.44) in the sense that

$$\frac{1}{t} \sum_{s=1}^{t} Y^2(s) < C_2 \text{ with probability one} \quad (3.47)$$

It is not necessary that the parameter estimates converge. If the system is minimum phase then as well

$$\frac{1}{t} \sum_{s=1}^{t} U^2(s) < C_3 \text{ with probability one} \tag{3.48}$$

The constants $C_i$ are independent of t, but dependent on the sequence $\{v_i\}$, $i = 1,2,\ldots t$. If $B_0$ is fixed then it must be chosen so that $.5\omega o < B_0 < \infty$, to insure stability of the closed loop.

This stabilizing property is important as it guarantees that the output of the closed loop will remain bounded (although this limit may be unsatisfactory from an operating standpoint) irrespective of the characteristics of the disturbances. The self-tuning regulator cannot be shown to have this stabilizing property for an arbitrary $\delta(z^{-1})\theta(z^{-1})$ polynomial, as the system cannot be written in the form (3.44).

This stabilizing property can be briefly described as follows. If the output of a process approaches instability, Ljung and Wittenmark (1975) show that the controller parameters of (3.11) quickly approach those of the minimum variance controller. The process output is then forced into the stability region by the control action

$$U_t = \frac{-\hat{\alpha}(z^{-1})}{\hat{B}(z^{-1})} Y_t \approx \frac{-\alpha^0(z^{-1})}{B^0(z^{-1})} Y_t \tag{3.14}$$

It is not necessary that the controller estimates (3.11) converge.

On a real process the self-tuning regulator may not have this stabilizing property, even if $\delta(z^{-1})\theta(z^{-1}) = 1.0$ due to nonlinearitites. The model of the process dynamics and disturbances can be severely strained if the output moves far from its steady state value and this stabilizing property may not be realized.

### 3.10 Least Squares Estimation

Introduce the vectors

$$\underline{x}(t) = (Y_t, Y_{t-1}, \ldots Y_{t-m}, \nabla^d U_t, \nabla^d U_{t-1}, \ldots \nabla^d U_{t-\ell})^T \quad (3.49)$$

$$\underline{\theta} = (\alpha_0, \alpha_1, \ldots \alpha_m, B_0, B_1, \ldots B_\ell)^T \quad (3.50)$$

The controller parameters are estimated from the model (3.11) which may be written as

$$Y_t = \underline{x}^T(t-b)\underline{\theta} + \varepsilon_t \quad (3.51)$$

The parameters of 3.51 are determined so that the least squares criterion

$$V_1(\underline{\theta}) = \sum_{s=1}^{t} \varepsilon^2(s) \quad (3.52)$$

is minimized. The solution to this is (Kendall and Stuart (1966))

$$\underline{\theta}(t) = (X^T X)^{-1} X^T \underline{Y} \quad (3.53)$$

where

$$(X^T X) = \sum_{s=b+1}^{t} \underline{x}(s-b)x(s-b)^T \quad (3.54)$$

$$X^T \underline{Y} = \sum_{s=b+1}^{t} \underline{x}(s-b)Y(s) \quad (3.55)$$

This may be expressed recursively, (Söderström et al. (1974a)) as

$$\hat{\underline{\theta}}(t) = \hat{\underline{\theta}}(t-1) + \underline{K}(t)[Y_t - \underline{x}^T(t-b)\hat{\underline{\theta}}(t-1)] \qquad (3.56)$$

where

$$\underline{K}(t) = \frac{\underline{\underline{P}}(t-1)\underline{x}(t-b)}{1 + \underline{x}^T(t-b)\underline{\underline{P}}(t-1)\underline{x}(t-b)} \qquad (3.57)$$

$$\underline{\underline{P}}(t) = \underline{\underline{P}}(t-1) - \frac{\underline{\underline{P}}(t-1)x(t-b)x^T(t-b)\underline{\underline{P}}(t-1)}{1 + \underline{x}^T(t-b)\underline{\underline{P}}(-1)\underline{x}(t-b)} \qquad (3.58)$$

The notation is that a double bar represents a matrix and a single bar a vector. $\underline{\underline{P}}(t)$ is the symmetric matrix $(\underline{\underline{X}}^T \underline{\underline{X}})^{-1}$ at time t. If (3.51) converges to the optimal solution, and b=1 (i.e. $\varepsilon_t$ is white noise) $\underline{\underline{P}}(t)$ is proportional to the variance-covariance matrix of the parameters. (See Appendix B).

Initial estimates $\underline{\theta}(0)$ and $\underline{\underline{P}}(0)$ are needed to start the recursion. From a Bayesian viewpoint $\underline{\theta}(0)$ represents the prior expectation of $\underline{\theta}$, and $\underline{\underline{P}}(0)$ a matrix proportional to the covariance matrix of the prior distribution of $\underline{\theta}$. The prior mean will strongly influence the estimates of the parameters at all future times if the elements of $\underline{\underline{P}}(0)$ are chosen small. In order that $\underline{\theta}(0)$ not unduly influence the recursive estimates, Wittenmark (1973) suggests that $\underline{\underline{P}}(0)$ be chosen as $10\alpha\underline{\underline{I}}$ to $100\alpha\underline{\underline{I}}$, where $\alpha$ is the variance of the output variance, Y, and $\underline{\underline{I}}$ is the unit matrix. However, the magnitude of $\underline{\theta}(0)$ and $\underline{\underline{P}}(0)$ depend upon the scaling (units) of both the $\{Y_t\text{'s}\}$ and the $\{\nabla^d U_t\text{'s}\}$ series. Choosing equal diagonal elements (variances) would imply having more prior information on the B parameters than the $\alpha$ parameters or vice-versa. Therefore, choice of a noninformative $\underline{\underline{P}}(0)$ should depend on both the variances of $Y_t$ and $\nabla^d U_t$

(It may therefore be advantageous to scale the process inputs and outputs such that their variances are roughly equal.)

Since the diagonal elements of $\underline{P}(t)$ represents $var(\theta_i(t)/var(\varepsilon_t))$ the choice of a noninformative $\underline{P}(0)$ should just involve insuring that these variances are sufficiently large that the prior confidence region on each $\theta_i(0)$ will include any even remotely possible value of $\theta_i$. The elements of $\underline{P}(0)$ may be chosen smaller if one feels that good prior information is available (e.g. from existing controller parameters) or one simply wants to restrict the movement of the parameters from the initial $\underline{\theta}(0)$.

It is important to note that equations (3.56) to (3.58) are derived for the estimation of parameters that are constant, but unknown. They are not capable of tracking changing parameter values as evident from the fact that $\underline{P}(t)$ decreases to the null matrix in a positive definite sense.

## 3.11 Time Varying Parameters

$\underline{P}(t)$ may be prevented from approaching the null matrix by using an exponentially discounted least squares approach. Instead of minimizing (3.52) the criterion is to minimize

$$v_2(\underline{\theta}) = \sum_{s=1}^{t} \lambda^{t-s} \varepsilon^2(s) \qquad (3.59)$$

where $0 << \lambda \leq 1$.

This leads again to equation 3.52, (Söderström et al. (1974a)), but with $\underline{K}(t)$ and $\underline{P}(t)$ defined as

$$\underline{K}(t) = \frac{\underline{P}(t-1)\underline{x}(t-b)}{\lambda + \underline{x}^T(t-b)\underline{P}(t-1)\underline{x}(t-b)} \tag{3.60}$$

and

$$\underline{P}(t) = \frac{\underline{P}(t-1)}{\lambda} - \frac{1}{\lambda} \cdot \frac{\underline{P}(t-1)\underline{x}(t-b)\underline{x}^T(t-b)\underline{P}(t-1)}{\lambda + \underline{x}^T(t-b)\underline{P}(t-1)\underline{x}(t-b)} \tag{3.61}$$

The use of $\lambda$, the discounting factor reduces the influence of past data on the current estimates, and they will then reflect the most recent characteristics of the data. Effectively $(\frac{1}{1-\lambda})$ data points are included in the estimation of the parameters. This quantity is referred to as the asymptotic sample length (Clarke et al. (1975)). $\lambda$ is usually in the range $.95 \leq \lambda \leq 1.0$. The smaller the value of $\lambda$ the faster the algorithm will track changing parameters, but the greater will be the variance of the parameter estimates. Hence the value of $\lambda$ is usually chosen to provide a compromise between the speed of tracking and the smoothness of the estimation sequence.

The choice of the noninformative prior $\underline{P}(0)$ will strongly influence the parameter estimates $\hat{\theta}(t)$ at all future times only if $\lambda = 1.0$. If $\lambda < 1.0$ then the effect of $\underline{P}(0)$ dies out and the consequences of underestimating the magnitude of element of $\underline{P}(0)$ may not be long lasting.

Alternatively, the controller parameters may be interpreted as the time varying states of a Kalman filter (Wieslander (1969)). In its simplest version, the estimated parameters are assumed to follow a random walk

$$\hat{\theta}_i(t) = \hat{\theta}_i(t-1) + v_i(t) \qquad i = 1, 2, \ldots, m+\ell+2 \tag{3.62}$$

$\{v_i(t)\}$ is a sequence of independent white noise variates with covariance matrix $\underline{R}$. This again leads to the recursive estimation scheme given in (3.56) with $\underline{K}(t)$ remaining as in (3.57), but $\underline{P}(t)$ now expressed recursively as

$$\underline{P}(t) = \underline{P}(t-1) - \frac{\underline{P}(t-1)\underline{x}(t-b)\underline{x}^T(t-b)\underline{P}(-1)}{1 + \underline{x}^T(t-b)\underline{P}(t-1)\underline{x}(t-1)} + \underline{R} \qquad (3.63)$$

$\underline{R}$ is positive semi-definite and prevents $\underline{P}(t)$ from approaching the null matrix. In this formulation the parameters may all be time varying at different rates. $\underline{R}$ would probably be made diagonal and the larger the elements the faster is the adaption for that parameter, but the noiser is the estimate. It is not obvious how to select the elements of $\underline{R}$ to reflect an asymptotic sample length. Interprelation of the controller parameters as the time varying states of a Kalman filter requires a more sophisticated understanding of the process dynamics and stochastics. In many instances this will be unknown. The estimation of time varying parameters seems to be handled most easily by the use of the discounting factor since its effect on the estimation scheme is more readily appreciated.

Self-tuning regulators were designed to control processes whose parameters were constant but unknown. It would seem reasonable to expect, with one of these above estimation schemes, that the self-tuning strategy would control processes whose parameters changed slowly relative to the process dynamics.

## 3.12 Parameter Estimation under Closed Loop Conditions

If a process is operating in closed loop, with a linear, time invariant feedback controller as discussed in Chapter 2, it is known that it may not be possible to uniquely identify the parameters of the process dynamic and stochastic models. If the parameter estimates of the model

$$Y_{t+b} = \alpha(z^{-1})Y_t + B(z^{-1})\nabla^d U_t + \varepsilon_{t+b} \qquad (3.11)$$

converge, the controller

$$\nabla^d U_t = \frac{-\hat{\alpha}(z^{-1})}{\hat{B}(z^{-1})} Y_t \qquad (3.14)$$

will reduce to a constant regulator. Under this condition it is shown in Appendix A that one parameter may always be expressed as a linear combination of the remaining ones if (3.14) is the minimum variance controller. Aström and Wittenmark (1973) suggest that one parameter, $\hat{B}_0$, be fixed. The parameters in the controller $\frac{\hat{B}_i}{\hat{B}_0}$, $i = 0,1,\ldots m$ and $\frac{\hat{B}_i}{\hat{B}_0}$, $i = 1,2,\ldots \ell$ are all ratios with respect to $\hat{B}_0$. By fixing $\hat{B}_0$ (an estimate of $\omega_0$) one simply will scale up or down the values of the parameters $(\hat{\alpha}_i, \hat{B}_i)$ estimated by the on-line estimation algorithm. Even though the choice of $\hat{B}_0$ is arbitrary in that the same ratio of controller parameters can theoretically be obtained, it has been shown that for stability of the closed loop system it is necessary that $.5\omega_0 < \hat{B}_0 < \infty$ (Ljung and Wittenmark (1974)). Although this result was derived for the case where $\delta(z^{-1})\theta(z^{-1}) = 1.0$, if the controller (3.14) is minimum variance the process output may behave as though $\delta(z^{-1})\theta(z^{-1}) = 1.0$, when in fact it is an arbitrary polynomial in $z^{-1}$. One might expect this restriction on $\hat{B}_0$ to hold under less restrictive conditions if the controller (3.14) is minimum variance. The rate of convergence of the estimated parameters is strongly influenced by the choice of $B_0$ and is most rapid when $\hat{B}_0 = \omega_0$ (Cegrell and Hedgvist (1975), Wittenmark (1973)).

$\hat{B}_0$ has been fixed in the reported applications of the self-tuning regulator. In most instances a reasonably adequate model of the process was available, and so an estimate of $\omega_0$ could be made. When this information has been unavailable difficulties in selecting a satisfactory value of $\hat{B}_0$ have been reported (Cegrell and Hedgvist (1976)).

However, it is not actually necessary to fix one parameter when employing the self-tuning regulator since the parameter estimates are non-linear, time varying functions of the input and output. If the estimates converge they become less and less time varying. However, the least squares criterion

$$\min V_1(\underline{\theta}) = \sum_{s=1}^{t} \varepsilon^2(s+b) \qquad (3.52)$$

includes the information from the process when the controller was nonlinear and time varying. There will not be a singularity in the estimation space although in the limit ($t\to\infty$) one will approach singularity (possibly giving problems on finite word minicomputers).

If a discounting factor is used then the parameter estimates will never really converge since they will be based on a finite sample length. The shorter the asymptotic sample length ($\frac{1}{1-\lambda}$) the more they will vary, and the singularity in the estimation space should never arise. In practice, if the true parameters are not time varying and the asymptotic sample length is long enough that the parameter estimates become essentially constant over a long period of time (after the initial transient has been discounted, then $\underset{\sim}{P}(t)$ could become very nearly singular. However, as $\underset{\sim}{P}(t)$ has been prevented from approaching the null matrix one would expect the parameter estimates to start varying due to the high correlations among them. It would then appear as though the process dynamic or disturbances were changing. In practice it would appear unnecessary to continue estimating $\hat{B}_0$ once a reasonable estimate had been obtained, and one might fix $\hat{B}_0$ after a short period of time (by zeroing the row and column of $\underset{\sim}{P}(t)$ corresponding to $\hat{B}_0$).

### 3.13 Start-Up Situations and Biased Estimation

If the self-tuning algorithm is started off with $\underline{\theta}(0) = \underline{0}$, the process output may drift away from its target value until better parameter estimates are available for use in the controller (3.14). If this initial transient in the process output is intolerable, then one might estimate the controller parameters by the recursive estimation scheme, but use an existing PID controller to compute the control signal instead of (3.14). When one has reasonable estimates of the parameters, the feedback controller (3.14), based on the estimated parameter would be used.

However, consistent (asymptotically unbiased) estimates of the minimum variance controller parameters can only be obtained from the estimation model (3.11) if the implemented controller is of the correct form, and the estimates converge to those of the minimum variance controller (Appendix B). It is impossible to obtain consistent estimates of the minimum variance controller parameters if an arbitrary feedback controller is used. If the parameter estimates are tracked, but not used initially in the computation of the control signal, it is important that $\lambda < 1$, otherwise one will always have biased estimates.

## 3.14 Convergence of the Parameter Estimates

The difference equations describing the parameter estimates are stochastic, nonlinear and time varying functions of the input and output, making analysis of the estimation situation extremely difficult. Simulations of the self-tuning regulator have been the primary tool for such analysis. Aström and Wittenmark (1973), Wittenmark (1973), Chang (1975), Sastry et al. (1976) and Clarke et al. (1975) have numerous examples of simulations examining the effect of $\underline{P}(0)$, $\underline{\theta}(0)$, $\lambda$ and $\hat{B}_0$ on the transient and asymptotic behavior of self-tuning regulators.

Ljung and Wittenmark (1974) have shown that a set of deterministic, ordinary differential equations may describe the expected trajectories of the parameters. Let the system be described by (3.1), and a model of the form (3.51)

$$Y_{t+1} = \underline{x}^T(t-b+1)\underline{\theta} + \varepsilon_{t+1} \tag{3.51}$$

is used to estimate the minimum variance controller parameters.

Introduce

$$\underline{f}(\underline{\theta}) = E\ \{\underline{x}(t-b+1)\cdot\ (Y_{t+1}-\underline{x}^T(t-b+1)\underline{\theta})\} \tag{3.64}$$

$$\underline{G}(\underline{\theta}) = E\ \{\underline{x}(t-b+1)\underline{x}^T(t-b+1)\} \tag{3.65}$$

Then for sufficiently large t and weak conditions (Ljung and Wittenmark (1974)) the ordinary differential equations

$$\frac{d\underline{\theta}(\tau)}{d\tau} = \underline{R}(\tau)\ \underline{f}(\underline{\theta}(\tau)) \tag{3.66}$$

$$\frac{d\underline{R}(\tau)}{d\tau} = \underline{R}(\tau) - \underline{R}(\tau)\ \underline{G}(\underline{\theta})\ \underline{R}(\tau) \tag{3.67}$$

will describe the expected trajectories of the estimated parameters. The ficticious time $\tau$ is related to t by

$$\tau \approx \ln(t) \tag{3.68}$$

if the discounting factor is one. Denote the vector of minimum variance controller parameters by $\underline{\theta}^0$, i.e.

$$\underline{\theta}^0 = (\alpha_0^0, \alpha_1^0, \ldots \alpha_{mo}^0, B_0^0, B_1^0, \ldots B_{\ell Q}^0)^T \tag{3.69}$$

If $\underline{f}(\underline{\theta}^0)$ is a globally asymptotic stationary point of (3.66), where $\underline{R}(\tau)$ is positive definite, then

$$\lim_{t\to\infty} \hat{\underline{\theta}}(t) \to \underline{\theta}^0 \text{ with probability one} \tag{3.70}$$

When one parameter is estimated it is possible to solve (3.66) and (3.67) analytically. Stability of the closed loop equations may be investigated as well as possible convergence points, which are the solutions to

$$\frac{d\theta}{d\tau} = \underset{\sim}{R}(\tau)\underline{f}(\underline{\theta}(\tau)) = \underline{0} \qquad (3.71)$$

For more complicated systems the differential equations may be linearized or numerically integrated to investigate the convergence properties of the algorithm. Ljung and Wittenmark (1974) were able to construct a system where the parameters of the minimum variance controller were not a globally asymptotically stable solution to (3.66). Thus the parameters of the self-tuning regulator could not converge to those of the minimum variance controller.

Analysis of the differential equations is difficult if more than one parameter is to be estimated. If n parameters are to be estimated at least $\frac{1}{2}n(n+3)$ simultaneous differential equations must be solved. The elements of $\underline{f}(\theta)$ and $\underset{\sim}{G}(\theta)$, the theoretical auto and cross correlations between the input and output are difficult to evaluate. Several examples are shown in Appendix D.

The parameter estimates $\hat{\theta}(t)$ may converge to a stationary solution of (3.66). Due to the presence of the disturbances there is a non zero probability that the solution will depart from this region unless it is globally asymptotically stable (Ljung and Wittenmark (1974)). As a result the controller parameters may not directly converge to those of the minimum variance controller but may jump between various convergence points. If the estimated parameters tend towards the minimum variance controller, they will not converge to another solution of (3.66) (Ljung and Wittenmark (1974)).

Theorems 1 and 2 of Section 3.1 stated that if the parameter estimates of the model (3.11) converged, and the order of $\alpha(z^{-1})$ and $B(z^{1-})$

were not underestimated then the resulting regulator would be the minimum variance controller. However, there is no guarantee of parameter convergence.

The rate of convergence is influenced by many factors. The relative rate of convergence between the $\hat{\alpha}$ and $\hat{B}$ parameters is influenced by the signal to noise ratio, $\mathrm{var}\{v(z^{-1})\}U_t/\mathrm{var}\{\psi(z^{-1})a_{t+b}\}$. If the parameter estimates are highly correlated the rate of convergence will be slow. The rate of convergence may be improved by the use of a discounting factor. This is readily seen by examining (3.17). If the estimation equation discounts data where $\varepsilon_{t+b}$ was far removed from $\varepsilon^0_{t+b}$, the rate of convergence of the controller parameters is increased. A typical strategy is to set $\lambda = .95$ at the start of the estimation and increase it after thirty to fifty sampling intervals.

The transient behavior of the estimation scheme is also influenced by $\underline{P}(0)$ and $\underline{\theta}(0)$, however, the long term influence of these values dies out if a discounting factor is employed. If $\hat{B}_0$ is fixed, convergence is most rapid when $B_0 = \omega_0$, (Cegrell and Hedqvist (1975)).

Although convergence of the regulator parameters to the minimum variance parameters is not assured, industrial applications and simulation examples indicate that control is very good within twenty sampling intervals, even if the parameter estimates have not reached their final values.

## 3.15 Simulation of a Self-Tuning Regulator

Let the process dynamics and disturbances be described as

$$Y_{t+1} = \frac{.2}{1 - .9z^{-1}} U_t + \frac{1}{(1 - .4z^{-1})\nabla} a_{t+1} \qquad (3.72)$$

This is the same example used in Chapter 2. This system was simulated using a sequence of N(0,1) $\{a_t\text{'s}\}$. The presence of the disturbance causes the process output $Y_t$ to drift away from its target value of 10.0, Figure 3.3.

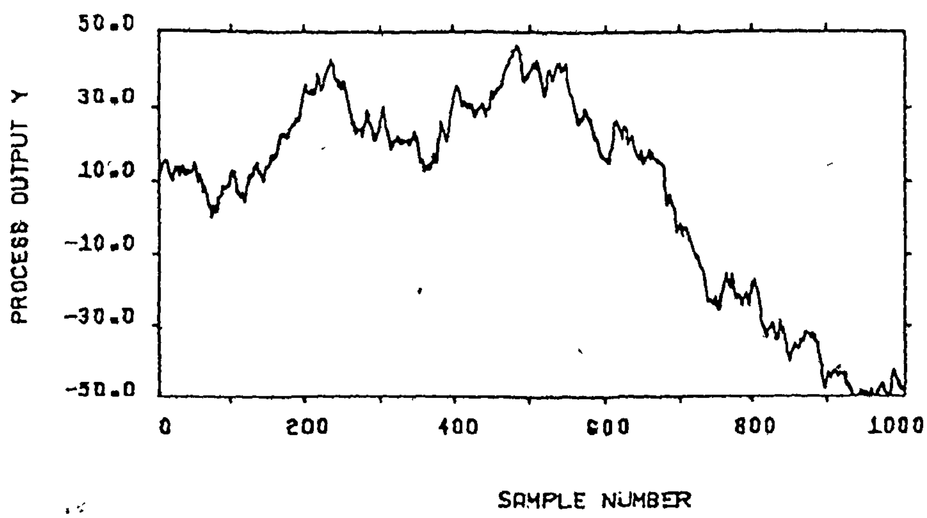


Figure 3.3: Open loop behavior of the process (3.72)

Since the disturbance is nonstationary a model of the form (3.11) with d=1

$$Y_{t+1} = \alpha(z^{-1})Y_t + B(z^{-1})\nabla U_t + \varepsilon_{t+1} \tag{3.11}$$

is used to estimate the controller parameters. We are not estimating the parameters of the process dynamics and stochastic disturbances.

The variance of $\nabla U_t$ is almost two orders of magnitude larger than the variance of $Y_t$ (Section 2.4) for the unconstrained minimum variance controller. Better control was obtained by making the transformation $\nabla U_t' \rightarrow \nabla U_t/10$. The minimum variance controller for (3.72) is now

$$\nabla U_t' = -\frac{(1.4 - 1.66z^{-1} + .36z^{-1})}{2 - .8z^{-1}} Y_t \tag{3.73}$$

or

$$\nabla U_t' = -.7Y_t + .83Y_{t-1} - .18Y_{t-2} + .4\nabla U_{t-1}' \tag{3.74}$$

A model of the form

$$Y_{t+1} = \alpha_0 Y_t + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + B_0 \nabla U_t' + B_1 \nabla U_{t-1}' + \varepsilon_{t+1} \tag{3.75}$$

was used to estimate the controller parameters, in the first simulation S1. $\underline{P}(0)$ was $10\underline{I}$ and $\underline{\theta}(0) = \underline{0}$. A discounting factor $\lambda = .95$ was used for the first 50 sampling intervals, after which $\lambda = .998$ was used. All the parameters were estimated and the ratios $\frac{\alpha_i}{B_0}$, $i = 0,1,2$, $B_0$ and $\frac{B_i}{B_0}$ are plotted in Figure 3.4. The controller ratios are not close to their correct values after 1000 sampling intervals, yet the control is very good. The auto and cross correlation function computed over the last 950 observations is plotted in Figure 3.5.

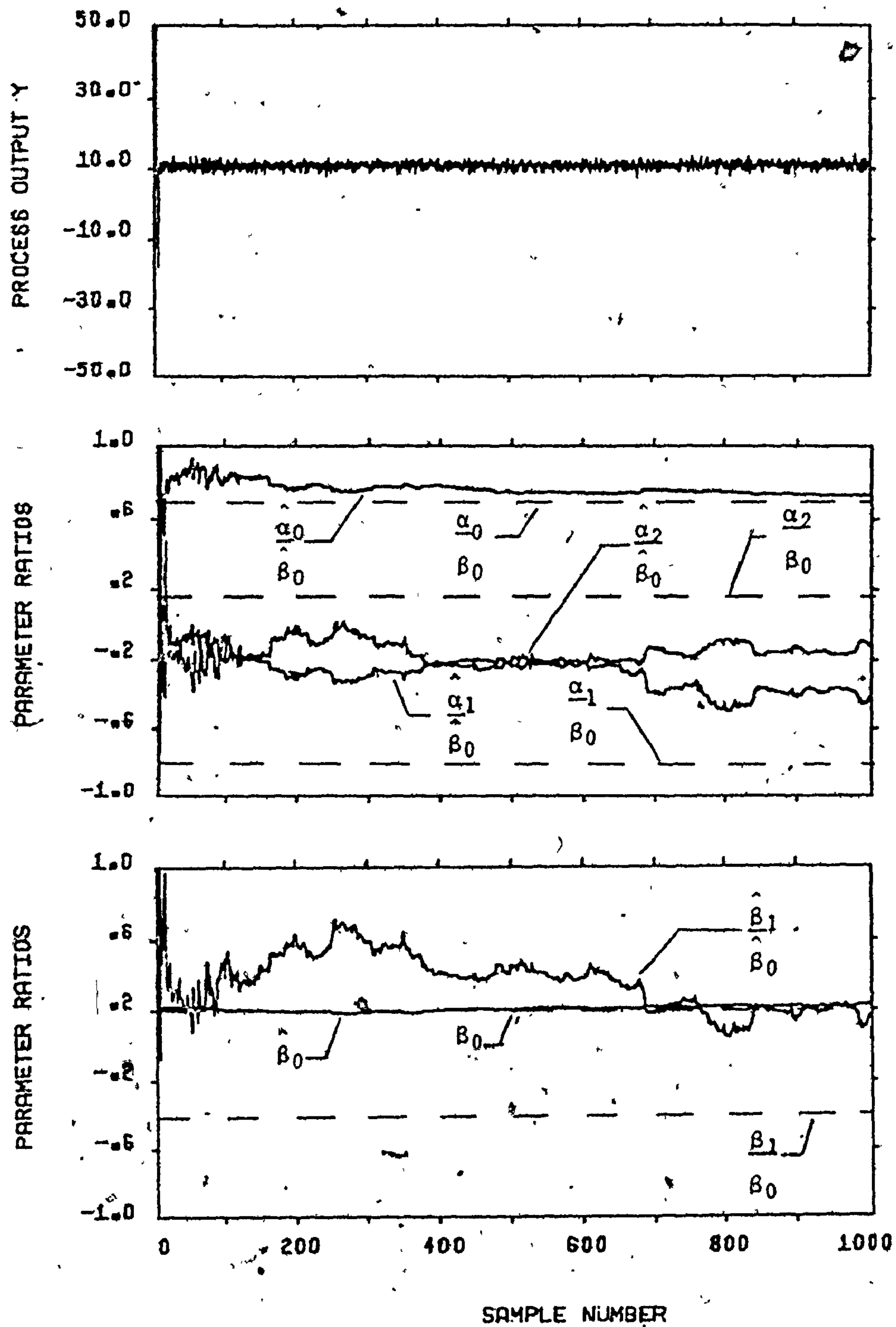


Figure 3.4: Process output and controller parameter ratios for the unconstrained self-tuning regulator simulation, S1

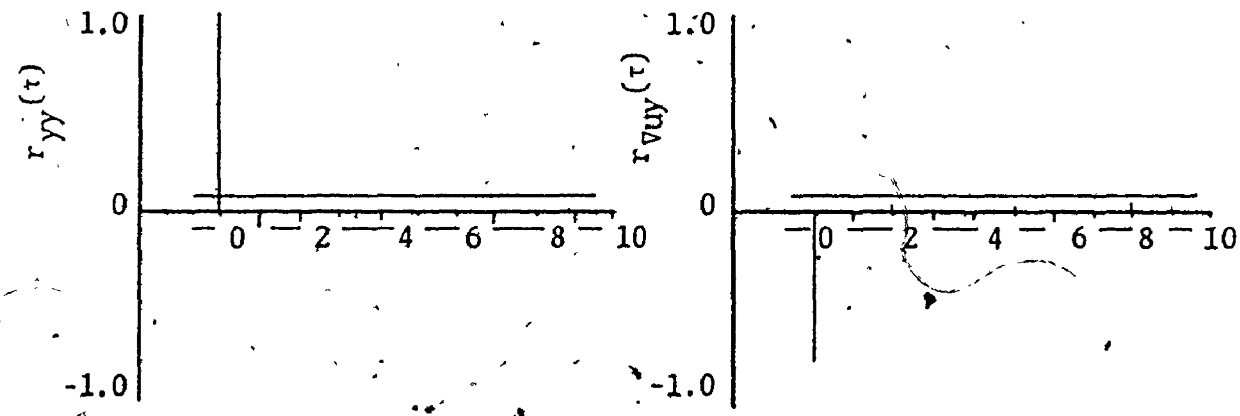


Figure 3.5: Sample auto and cross correlation function for simulation S1

All values beyond lag zero should be zero if the control is minimum variance. Since these functions are computed from a finite number of observations they will not be exactly zero. Bartlett (1946) has shown that for a moving average process of order b-1 the variance of the computed autocorrelations beyond lag b are given by

$$\mathrm{var}\ \{r_{yy}(\tau)\} \approx \frac{1}{N-\tau}\left(1 + 2\sum_{k=1}^{b-1} \rho^2_{yy}(K)\right),\ \tau \geq b \qquad (3.76)$$

where $\rho_{yy}(K)$ is the theoretical auto correlation at lag K. The variance of the cross correlations beyond lag b are approximately given by (Bartlett (1955)), as

$$\mathrm{var}\ \{r_{\nabla d_{uy}}(\tau)\} \approx \frac{1}{N-\tau}\left(1 + 2\sum_{k=1}^{b-1} \rho^2_{\nabla d_{uy}}(K)\right),\ \tau \geq b \qquad (3.77)$$

The approximate two standard deviation limits on the auto and cross correlations ($2/\sqrt{N}$ in this case as b=1) are shown in Figure 3.5. It is seen that the control appears to be minimum variance. The variance of $Y_t$ and the variance of $\nabla U_t$ are compared to their theoretical values in Table 3.1.

To check the simulation the self-tuning regulator was also started off with $\underline{P}(0) = .001\ \underline{I}$ and the minimum variance controller parameters. The estimates remained very close to their optimal values.

The self-tuning algorithm of Clarke et al. (1975) was simulated in the second simulation S2. The model form for identification of the controller parameters was

$$(Y_{t+1} + .5\nabla U_t') = \alpha_0 Y_t + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + B_0 \nabla U_t'$$

$$+ B_1 \nabla U_{t-1}' + \varepsilon_{t+1} \qquad (3.78)$$

The theoretical constrained controller is (2.39)

$$\nabla U_t' = \frac{-(1.4 - 1.66z^{-1} - .36z^{-2})}{2.5 - 1.65z^{-1}} Y_t \qquad (3.79)$$

or

$$\nabla U_t' = .56Y_t + .664Y_{t-1} - .144Y_{t-2} + .66\nabla U_{t-1}' \qquad (3.80)$$

remembering that the transformation $\nabla U_t' \rightarrow \nabla U_{t/10}$ changes $\omega_0$ from .2 to 2.0. The controller ratios $\frac{\hat{\alpha}_i}{\hat{B}_0}$, i = 0,1,2, $\hat{B}_0$ and $\frac{\hat{B}_i}{\hat{B}_0}$ are shown in Figure 3.6, and the sample auto correlation function of $\underline{\phi}(t)$, $r_{\phi\phi}(\tau)$ is plotted in Figure 3.7.

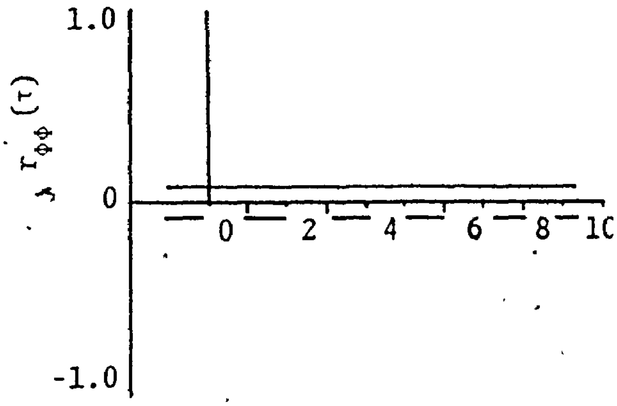


Figure 3.7: Sample auto correlation function
for simulation S2

If the 95% confidence interval is taken as $2/\sqrt{N}$ then the resulting control appears to be optimal. The variances of $Y_t$ and $\nabla U_t'$ are compared to their theoretical values in Table 3.1.

| Simulation | var $Y_t$ | var $\nabla U_t'$ | $\bar{Y}$ |
|---|---|---|---|
| unconstrained | 1.00 | 0.79 | 10.00 |
| S1 | 1.02 | 0.90 | 9.98 |
| constrained | 1.10 | 0.470 | 10.00 |
| S2 | 1.13 | 0.479 | 9.98 |

Table 3.1: Simulation results for the constrained and unconstrained self-tuning regulator

There is a 9.7% increase in the var of $Y_t$ compared to the first simulation with a corresponding decrease in the variance of $\nabla U_t'$ of 47%. Referring

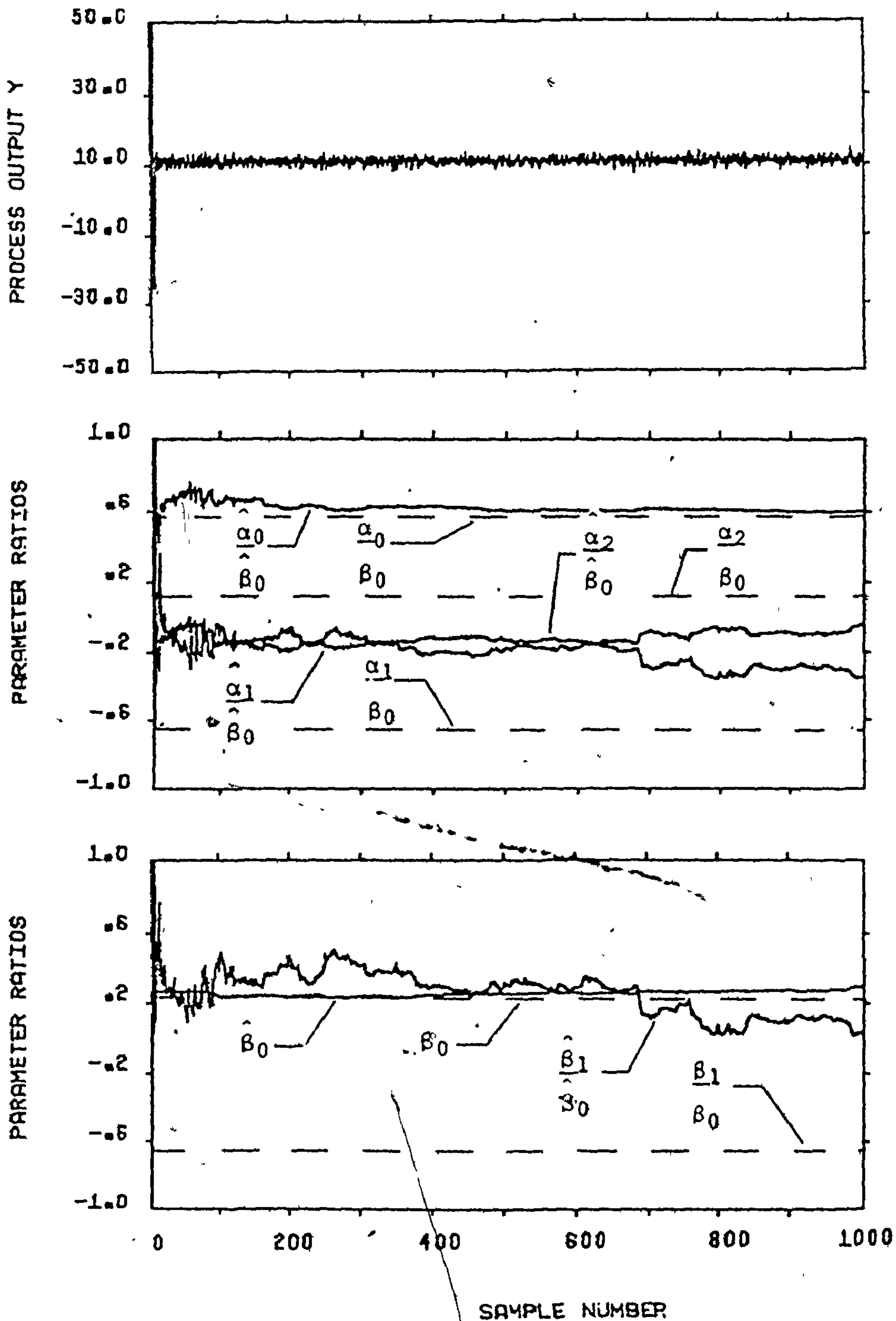


FIGURE 3.6: Process output and controller parameter ratios for the constrained self-tuning regulator simulation S2

to Figure 2.5 for this system, it is seen that for a 10.0% increase in the variance of $Y_t$ it is expected that there will be a 47% decrease in the variance of $\nabla U_t$. The simulation results agree well with these expected results.

## 3.16 Summary

The parameters of the model

$$Y_{t+b} = \alpha(z^{-1})Y_t + B(z^{-1})\nabla^d U_t + \varepsilon_{t+b} \tag{3.11}$$

are estimated at every sampling interval by recursive least squares, and used in the control law

$$\nabla^d U_t = \frac{-\hat{\alpha}(z^{-1})}{\hat{B}(z^{-1})} Y_t \tag{3.14}$$

as if they were exactly known. If the parameter estimates converge, the orders of $\alpha(z^{-1})$ and $B(z^{-1})$ have not been underestimated, and several weak conditions are met, the resulting regulator is the minimum variance controller. This is the same one that could have been designed had the process dynamic and stochastic models been known. However, there is no guarantee that the parameter estimates will converge.

The basic self-tuning algorithm of Aström and Wittenmark (1973) was discussed in this chapter. It was shown how deterministic disturbances, feedforward variables, multivariable decoupling and constrained control are incorporated in the framework of self-tuning regulators. There was considerable discussion concerning the structure of the estimation model, and the importance of including differencing if the disturbances are nonstationary. The choice of the sampling interval is important as difficulties in direct implementation of the self-tuning regulator may occur if the process dynamic model is nonminimum phase. Least squares estimation, methods of handling time varying parameters and convergence properties of the parameter estimates were also topics discussed. The next chapter will review applications of the self-tuning regulator to the control of industrial and pilot plant processes.

# CHAPTER 4

## LITERATURE REVIEW OF PROCESS APPLICATIONS

There have been several practical implementations of self-tuning regulators to control pilot plant and industrial processes. The purpose of this chapter is to review these applications, indicating insights, extensions of the basic theory, and to identify some of the problems that have occurred in its implementation.

In "Adaptive Control of a Paper Machine", Wittenmark (1974) examined the feasibility of implementing a self-tuning regulator for the moisture control loop on an industrial paper machine. Feedback control was combined with a feedforward signal from an upstream part of the process. The controller was of the form

$$\nabla U_t = \frac{\hat{\alpha}_1 + \hat{\alpha}_1 z^{-1} + \hat{\alpha}_3 z^{-2}}{B_0(1 + \hat{B}_1 z^{-1} + \hat{B}_2 z^{-2} \pm \hat{B}_3 z^{-3})} Y_t$$

$$+ \frac{1}{B_0(1 + \hat{B}_1 z^{-1} + \hat{B}_2 z^{-2} + \hat{B}_3 z^{-3})} \nabla z_t \qquad (4.1)$$

where $z_t$ was the feedforward signal. The structure of the controller and choice of $\hat{B}_0$ (which was fixed) were based on a reasonably good knowledge of the process dynamic and stochastic models.

The self-tuning regulator had good transient behavior and within fifteen sampling intervals the regulator was providing good control. The controller required a large number of parameters to maintain good control as the process had a long transport delay and large stochastic disturbances.

A similar application was reported by Cegrell and Hedqvist (1975) in "Successful Adaptive Control of a Paper Machines", where the control objective was to minimize fluctuations in the moisture content and basis weight. The process was a input-two output coupled system, the dynamics of which were well known. Information from the basis weight loop was used in the estimation of the controller parameters and computation of the moisture content control signal, essentially decoupling the system.

The transient behavior of the controller was good and when a forgetting factor was carefully chosen, the parameters of the regulator were close to their optimal values within twelve sampling intervals. The auto correlation of the process output indicated the optimality of the controller. The performance of the self-tuning regulator was compared to existing control algorithms-discrete proportional integral controllers. The difference between a well tuned PI controller and the self-tuning regulator was small during steady-state operation. PI controllers were seldom well tuned though, because the dynamcis of the paper machine changed whenever a different grade of paper was manufactured. If the process was noisy, and the self-tuning regulator was implemented, fewer paper breaks occurred, resulting in increased production. At the time the paper was written, the self-tuning regulator had been in continuous operation for several months.

Cegrell and Hedqvist (1974) discuss the application of a self-tuning regulator to control the Kappa-number (indication of wood pulp delignification) and a number of subprocesses on a continuous digestor. This paper primarily describes the development of a mechanistic model to describe the rate of delignification. The unknown dynamic parameters of this model are estimated and used to determine an "optimal" temperature set point, ignoring the effect of the process disturbances on this calculation. The results of the proposed Kappa-number control scheme are not presented. Only self-tuning control of the chip level in the digestor is examined, although the results are rather confusing because the controller is designed to minimize the change in chip level fluctuations.

Borisson and Syding (1976) have described "Self-Tuning Control of an Ore Crusher" where the objective of the study was to evaluate the economics of installing a digital computer for process control. (The process control computer was at the Lund Institute of Technology, Sweden, 1800 kilometers from the industrial process.) High production rates on the ore crusher were difficult to maintain due to long transport delays, changing ore characteristics and wear of the crusher jaws. A self-tuning regulator was used to control the crusher power. An asymptotic sample length of one hour was used in the estimation scheme, and $\hat{B}_0$ was fixed, although values between 1 and 100 all gave good results. This apparent insensitivity of $\hat{B}_0$ was due to the truly time varying nature of the process. So much 'new' information was being made available to the estimation scheme at every sampling interval that the absolute value of $\hat{B}_0$ was not crucial and it could easily have been estimated.

The reduction in the variance of the crusher power realized by implementing the self-tuning regulator, as compared to conventional analog PI controllers, meant that the set point of the crusher power could be moved closer to the valve at which thermal overload occurred. A ten percent increase in production could be realized and it was concluded that there was significant economic incentive to install a digital control system to implement a self-tuning regulator.

Industrial applications of the self-tuning regulator have been reported only in Europe, and many of the people reporting these applications have had a close affiliation with the Lund Institute of Technology. The processes on which the self-tuning regulator has been implemented were characterized by dynamics with long transport delays, stochastic disturbances and in some instances, time varying parameters.

Application of self-tuning regulators for the control of an inherently disturbance free processes subject to deterministic load changes have also been investigated at the University of Alberta. In "An Application of a Self-Tuning Regulator to a Binary Distillation Column", Sastry et al. (1976) controlled the top product composition by manipulating the reflux flow rate, for load and set point changes. Neither an integrator nor a constant term was included in the controller and therefore offset was observed in the controlled variable, which they were unable to explain. For feedrate and step disturbances it was found that the self-tuning regulator gave improved transient response over conventional PI controllers.

Chang (1975) investigated the application of a self-tuning regulator to control a pilot scale double effect evaporator. The concentration of product from the second effect was controlled by manipulating the stream flow. Interaction between the self-tuning loop and level controllers produced severe oscillations in a number of process variables, necessitating the reduction in level control gains, when the input stream was subject to load disturbances. Changes in setpoint produced offset which may be explained again by the fact that neither an integrator nor a constant term was included in the controller. The controller attempted to remove the effect of offset by moving the roots of $B(z^{-1})$ very close to the unit circle producing a high gain controller, which might explain the necessity of reducing the level controller gains. The auto correlation function of the output was never plotted to check for controller optimality.

The following critical comments can be made about some of the above applications. A clear understanding of the theory of nonadaptive minimum variance controllers would probably have removed much of the confusion in the application of these controllers such as to the ways of eliminating offset. In all applications $\hat{B}_0$ was fixed, although the reasons for doing so were not always understood. The controller parameters (3.11) were identified in every application although some authors implied that they were estimating the parameters of the process dynamic model. There was also no discussion concerning the selection of sampling interval.

The application of self-tuning regulators to control industrial processes have shown these regulators have good transient as well as asymptotic behavior. It has been possible to modify the structure of the controller to include feedforward terms, and to automatically decouple an inherently multivariable control system. Not only have most applications been technically successful, but there has been economic incentive to support their permanent installation.

# CHAPTER 5

## SELF-TUNING CONTROL OF A STEAM JACKETED STIRRED TANK HEATER

### 5.1 Introduction

This chapter will describe an application of the self-tuning regulator for temperature control of a jacketed steam heated continuous stirred tank. A schematic of the apparatus is shown in Figure 5.1 along with typical operating conditions, and is described in detail by Huynh (1974). The control objective was to maintain the water temperature in the second tank at a desired valve by adjusting the steam flow to this tank. Temperature disturbances were artificially introduced into the inlet water temperature by manipulating the steam flow to the first tank. The flow rate of water into the first tank was regulated with a digital PI controller. Control action on this flow loop was taken every five seconds.

Assuming total condensation of the steam and negligible heat losses, an energy balance on the second tank gives

$$\tau \frac{dY}{dt} + Y = K' U \qquad (5.1)$$

where

$$Y = T_{OUT} - \bar{T}_{OUT} \qquad (5.2)$$

$$U = V^2 - \bar{V}^2 \qquad (5.3)$$

$$K' = \frac{\lambda_s K}{F C_p} \qquad (5.4)$$

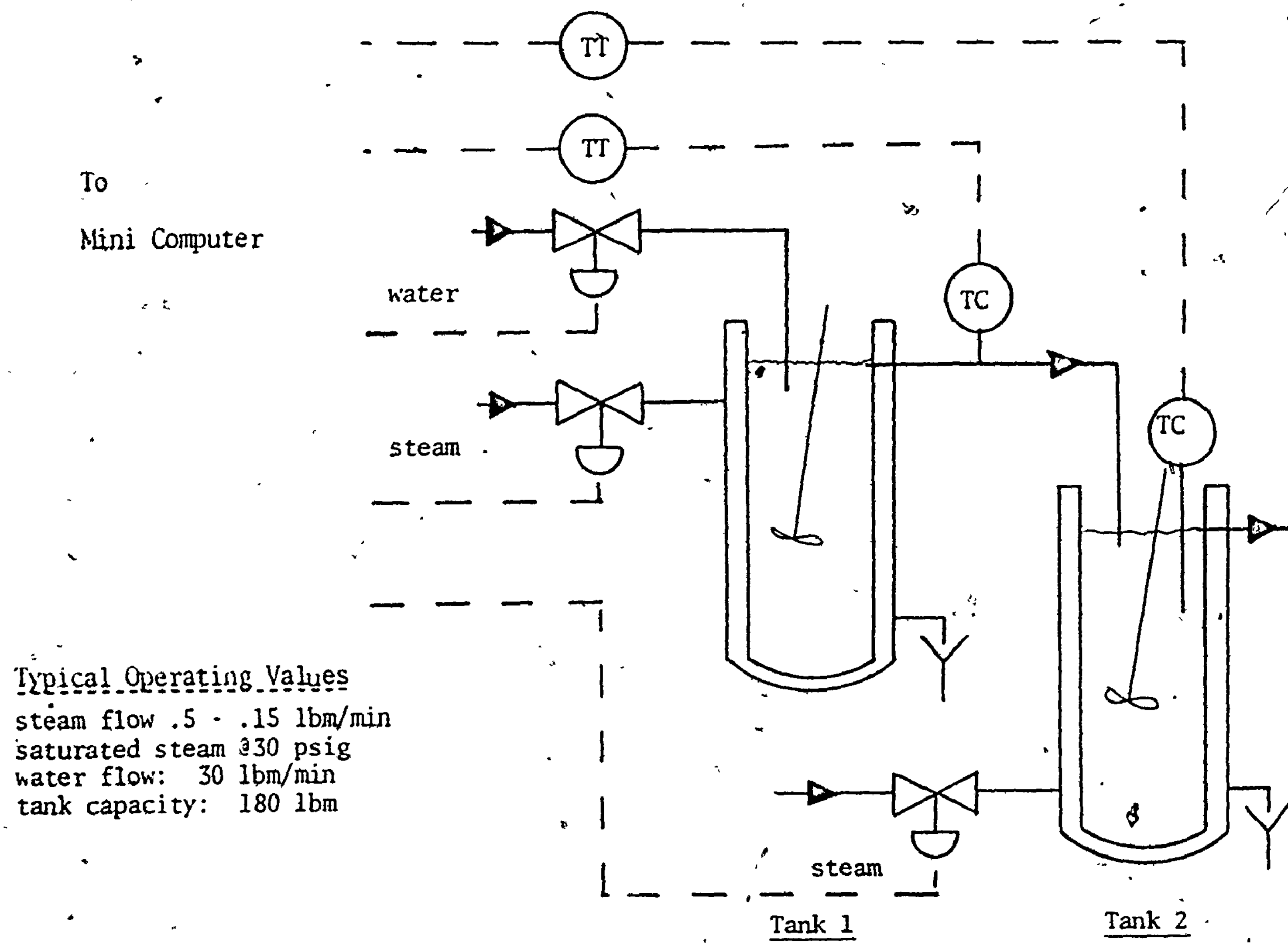


Figure 5.1: Schematic of the Steam Jacketed Stirred Tank Heater

and

$$\tau = \frac{M}{F} \tag{5.5}$$

Here T is the outlet temperature, degrees Celcius, V is the voltage to the valve transducer, $\lambda_s$ is the heat of vaporization of steam at supply conditions, M is the mass of water in the tank, F is the mass flow rate of water into the tank and $C_p$ is the heat capacity of water. It is known that the steam flow is roughly proportional to the square of the voltage applied to the valve transducer, for the second tank (Huynh (1974)). Therefore

$$F_s = KV^2 \tag{5.6}$$

The steady state values of T and V are denoted by $\bar{T}$ and $\bar{V}$. Substitution of numerical valves into (5.1) gives

$$6.0\,\frac{dY}{dt} + Y = .68\,\hat{U} \tag{5.7}$$

The tank may be modelled by a first order transfer function. The purpose of this chapter is <u>not</u> simply to evaluate the ability of the self-tuning regulator to control this process, as it is controlled adequately by a well tuned digital PI controller. This apparatus provides a safe process on which to gain a familiarity with the implementation of self-tuning regulator. The use of the diagnostic tools (Theorem's 1 and 2 of Section 3.2) for verification of the correct model order, the ability to estimate $B_0$, different methods for estimating offset and the ability of the estimation routine to track time varying parameters are aspects of the self-tuning regulator that will be investigated.

## 5.2 Computer Hardware and Software

The apparatus was interfaced to a dual processor-shared Disk System. A 256 K word fixed head disk was shared by Data General Corporation NOVA 2/10 and NOVA 1200 computers. This arrangement is described in more detail by Tremblay (1975). All analog to digital (A/D) and digital to analog (D/A) processing, output and logging routines and operator communications with the computer were handled by a Generalized Operating System Executive (GOSEX), written by Tremblay (1975). All application programs were written in Data General Corporation Fortran IV with the exception of one Assembly language program which linked the user's program to GOSEX.

The programs for the self-tuning regulator are very compact, consisting primarily of a series of subroutine calls. The recursive least squares subroutine is very short. Much of the matrix multiplication that would appear to be required (Equations 3.56-3.58) is eliminated by recognizing quadratic forms and the symmetry of some matrices. A flow diagram for the self-tuning package is outlined in Appendix E, with source listing of the important subroutines.

There was one modification to the apparatus over that used by Huynh (1974). An air to close control valve (Minim flow control valve, trim D) and an electropneumatic transducer (Fisher Type 546) were installed so that the water flow rate could be manipulated from the computer. The flow rate was measured with an orifice meter and a differential pressure transducer.

In order to introduce unknown disturbances into the temperature of the second tank stochastic variations were introduced into the valve of the first tank. The square of the voltage applied to the valve transducer on the first tank was determined every four minutes from the auto regressive model

$$V_{1,t}^2 = .75\, V_{1,t-1}^2 - .50\, V_{1,t-2}^2 + a_t \quad , \qquad (5.8)$$

where $V_1$ is the mean corrected voltage applied to the valve transducer.

The $\{a_t\}$ were a sequence of normally distributed random numbers with mean zero, and variance 144 volts$^4$. This variance was chosen so that 95% of the valve settings would be in the range 1-81 volts squared, the operational range of the valve transducer.

The process measurements were filtered every five seconds by a first order digital filter

$$W_t = \frac{.75}{1 - .25z^{-1}} z_t \quad , \qquad (5.9)$$

where $z_t$ is the raw data and $W_t$ is the filtered data. A filter constant of .25 implies that twenty-five percent of the current filtered value is obtained from past information.

It was stressed in Chapter 2 that $N_t$ (Equation 2.4) represents the total effect observed at the output if no control action were taken, including sensor noise, A/D noise as well as inherent disturbances within the process. The effect of a high frequency measurement noise superimposed on a low frequency process disturbance, is to mask out the latter. This reduces the forecastability of the process disturbance and our ability to control the process. Filtering removes some of the high frequency components from the measured variables and reduces the variance of the measured variable. This insures that the control action is based on a signal that is well known.

The parameters of the controller were updated and control action taken every two minutes. For the input disturbance (5.8), Huynh (1974) showed that the variance of the temperature for tank two increased rapidly if the sampling interval was larger than two minutes.

## 5.3 Experimental Program

The results from a series of self-tuning controller experiments on the steam jacketed stirred tank will be analyzed in this section. For convenient reference the important characteristics of each run are summarized in Table 5.1. These experiments were chosen to illustrate certain aspects of the theory from Chapter 2. The correct structure of the estimation model was one area that was discussed extensively, and the first few experiments will examine the effect of different model forms on our ability to control the process.

For run A001 the model structure was

$$Y_{t+1} = \alpha_0 Y_t + \alpha_1 Y_{t-1} + B_0 \tilde{U}_t + B_1 \tilde{U}_{t-1} + \varepsilon_{t+1} \qquad (5.10)$$

The $\{\tilde{U}'s\}$ were deviations of water temperature from the setpoint of 73.6°C. The steady-state value of the $\{\tilde{U}'s\}$ was estimated to be 25 volts squared.

| Run Number | Number of Parameters Identified $A(z^{-1})$ | $B(z^{-1})$ | | Differencing | Comments |
|---|---|---|---|---|---|
| A001 | 2 | 2 | no | no | $P(0) = 100\underline{I}$<br>no preliminary identification<br>$\lambda = 1.0$<br>step change at sample number 48 |
| A002 | 2 | 2 | yes | no | $\underline{P}(0) = 100\underline{I}$<br>no preliminary identification<br>$\lambda = 1.0$ |
| A003 | 2 | 2 | yes | no | $\underline{P}(0) = 100\underline{I}$<br>preliminary identification for 8 samples<br>$\lambda = .95$ to record 30<br>$\lambda = .98$ record 31 to end<br>step change to 65°C at sample number 71 |
| A004 | 2 | 2 | no | yes | $\underline{P}(0) = 100\underline{I}$<br>preliminary identification for 6 samples<br>$\lambda = .95$ to record 30<br>$\lambda = 1.0$ record 31 to end<br>step change to 65°C at sample number 71 |
| A005 | 3 | 3 | no | yes | $P(0) = 100\underline{I}$<br>preliminary identification for 6 samples<br>$\lambda = .95$ to record 30<br>$\lambda = 1.0$ record 31 to end<br>step change to 65°C at sample number 71 |
| A006 | - | - | - | - | open loop |

TABLE 5.1: Experimental Conditions for runs A001 - A006

$\underline{P}(0)$ was 100 $\underline{I}$ and $\underline{\theta}(0)$ was $\underline{0}$. The parameters of (5.10) were estimated, and the control signal

$$\tilde{U}_t = \frac{-\hat{\alpha}_0}{\hat{B}_0} Y_t - \frac{\hat{\alpha}_1}{\hat{B}_0} Y_{t-1} - \frac{\hat{B}_1}{\hat{B}_0} \tilde{U}_{t-1} \qquad (5.11)$$

implemented every two minutes. The setpoint of the controlled variable was changed to 65°C at sample number forty-eight.

The controlled temperature, the voltages squared to the valve transducer and the temperature of the inlet feed are plotted in Figure 5.2. The pointer (↑) on each of the plots indicates when the estimated parameters were used in the computation of the control signal. The mean value of the control signal did not turn out to be 25 volts squared, and since neither integral action or a constant term appear in the controller, the temperature deviates significantly from its setpoint. The plot of the controller parameter ratios, Figure 5.3, indicates that the controller was trying to eliminate offset by moving the root of $\hat{B}(z^{-1})$ close to the unit circle. This accounts for the bang-bang nature of the control signal, Figure 5.2.

Theorem 1 of Chapter 3.2 stated that if the parameter estimates $\{\hat{\alpha}_0, \hat{\alpha}_1, \hat{B}_0, \hat{B}_1\}$ converged then $\hat{\rho}_{yy}(\tau)$ and $\rho_{\nabla^d uy}(\tau)$, $\tau = 1,2$ would have been zero. Had these parameters converged, non zero values of $\rho_{yy}(\tau)$ and $\rho_{\nabla^d uy}(\tau)$ beyond $\tau = 2$ indicate that the order of the controller must be increased.

The sample auto correlation $r_{yy}(\tau)$ and cross correlation $r_{uy}(\tau)$, Figure 5.4 (based on input/output data to record number 44) do not lie within the approximate 95% confidence intervals for these estimates, and

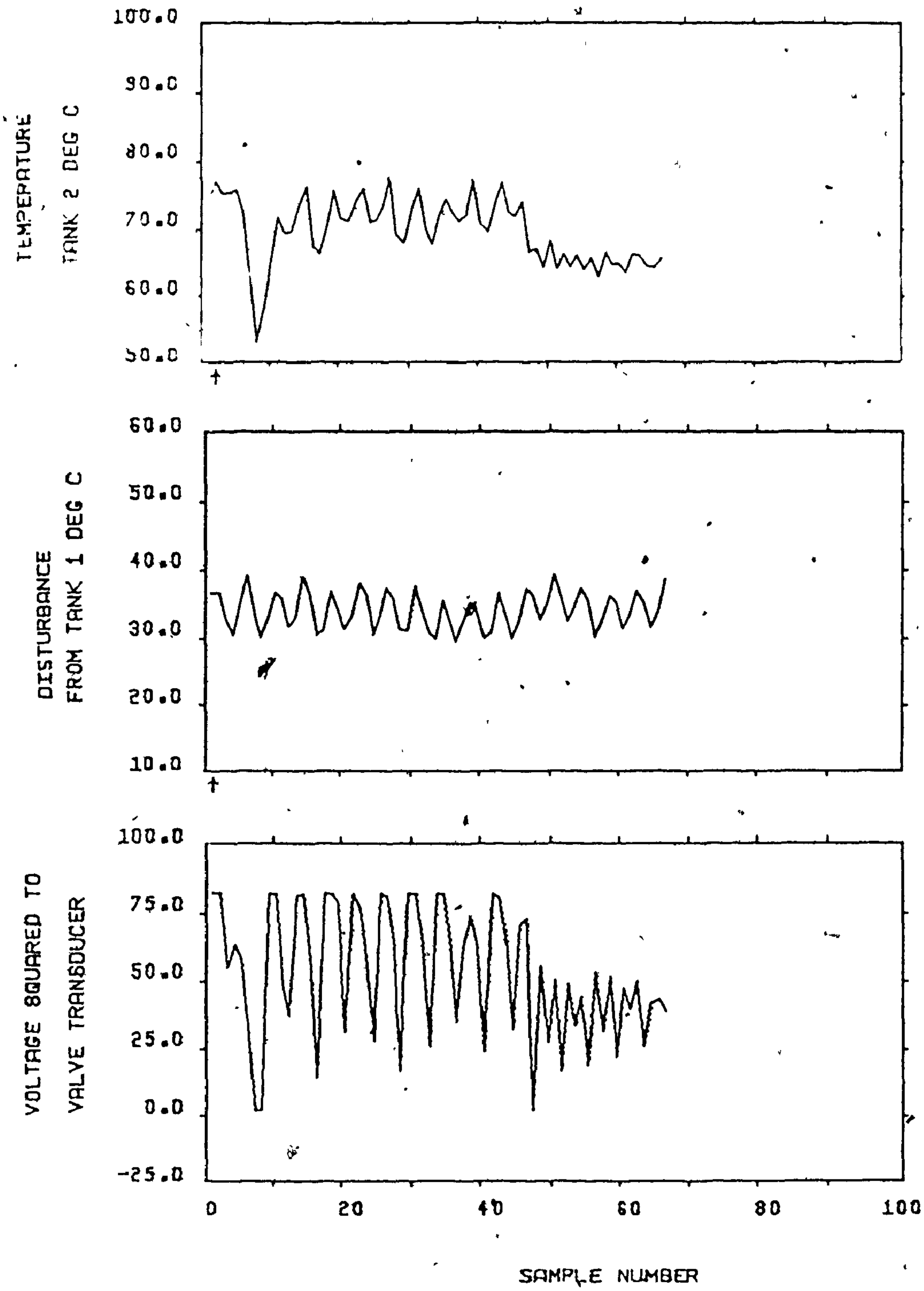


Figure 5.2: Process output, input disturbance and manipulated variable sequence for run A001

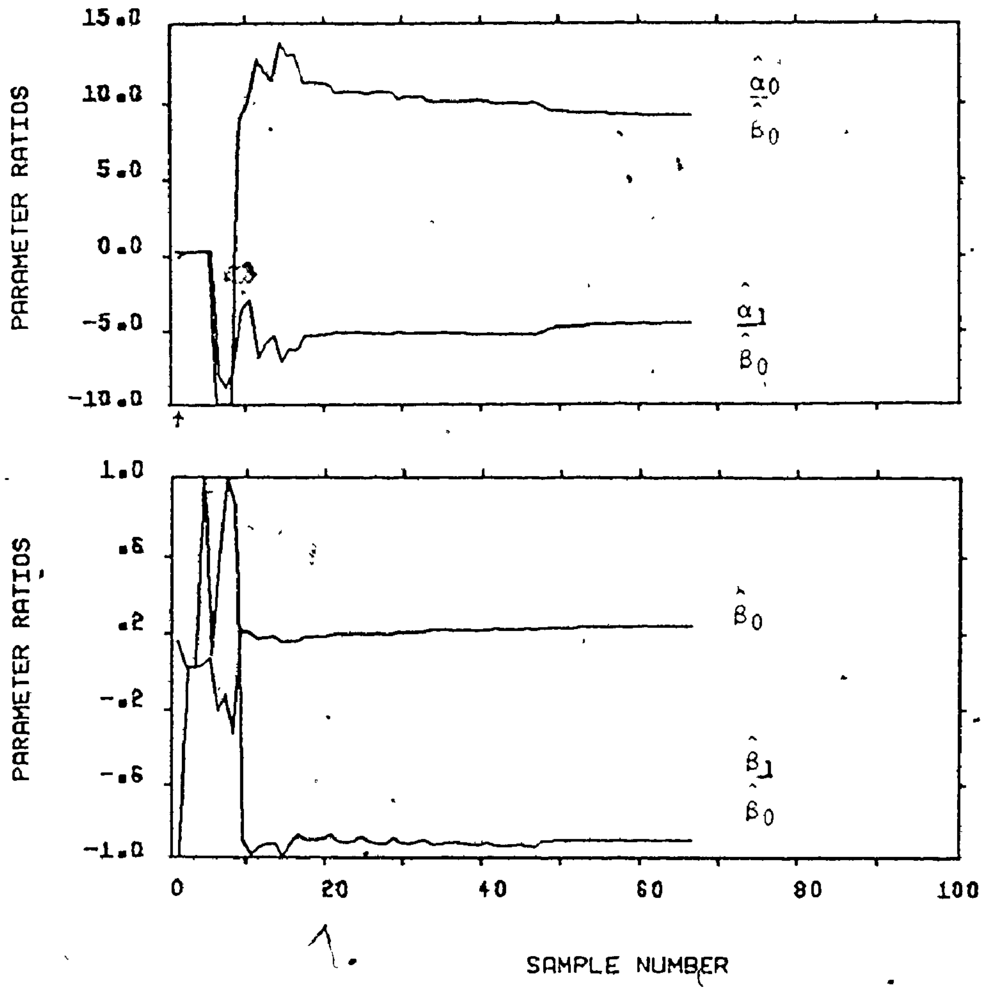


Figure 5.3: Controller parameter ratios, for run A001

are taken as non zero. This indicates

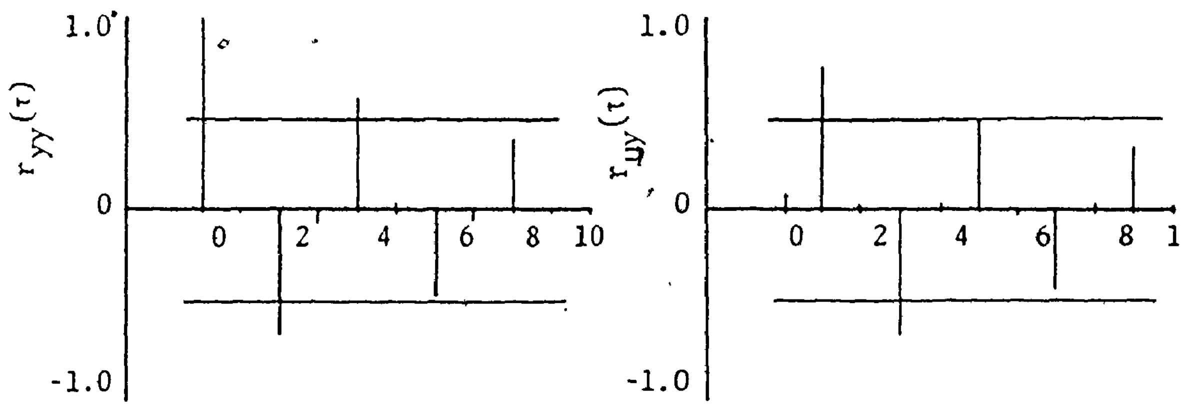


Figure 5.4: Sample auto and cross correlation function for run A001

that the parameters have yet to converge although they appear to be changing very slowly. (It is to be noted that the auto and cross correlations were computed about the mean value of the input and output sequence and not the reference values.)

The controller parameters would appear to be well estimated. The effect of the control action is to induce temperature oscillations, providing good information for the estimation of the controller parameters. The structure of the controller though appears to be incorrect due to the presence of high auto and cross correlations at high lags (T · 2).

The temperature into the second tank during this run was not the desired AR(2) process due to a mistaken implementation of (5.8). This was uncorrected for the first series of experiments. Nevertheless this represents a disturbance acting on the system. A sampling interval greater than two minutes might be more appropriate for this disturbance, as the process would damp out the high frequency fluctuations.

For Run A002 the model structure was

$$Y_{t+1} = \alpha_0 Y_t + \alpha_1 Y_{t-1} + B_0 \tilde{U}_t + B_1 \hat{U}_1 + \nu + \varepsilon_{t+1} \qquad (5.12)$$

The reference value of the manipulated variable was again 25 volts squared. The offset in $Y_t$ should be eliminated by the inclusion of the $\nu$ term as it will compensate for the difference between the estimated steady value of the manipulated variable and its true value. The input and output are plotted in Figure 5.5. The input does not have the bang-bang behavior seen in run A001, and the roots of $\hat{B}(z^{-1})$ are not close to the unit circle (Figure 5.6). The controller parameters estimates have not converged as indicated by the non zero auto and cross correlations at lag 1, 2, Figure 5.7.

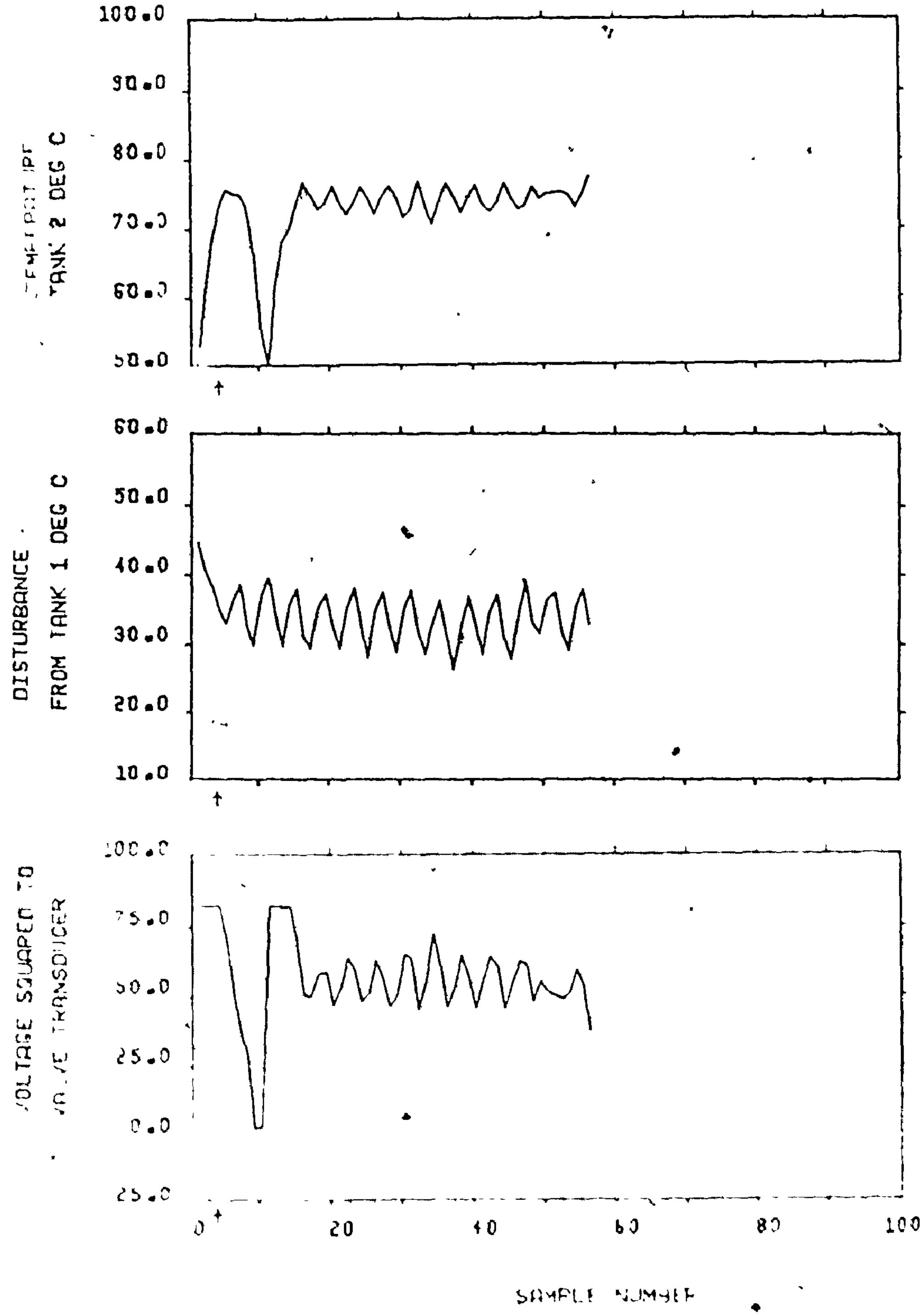


Figure 5.5: Process output, input disturbance and manipulated variable sequence for run A002

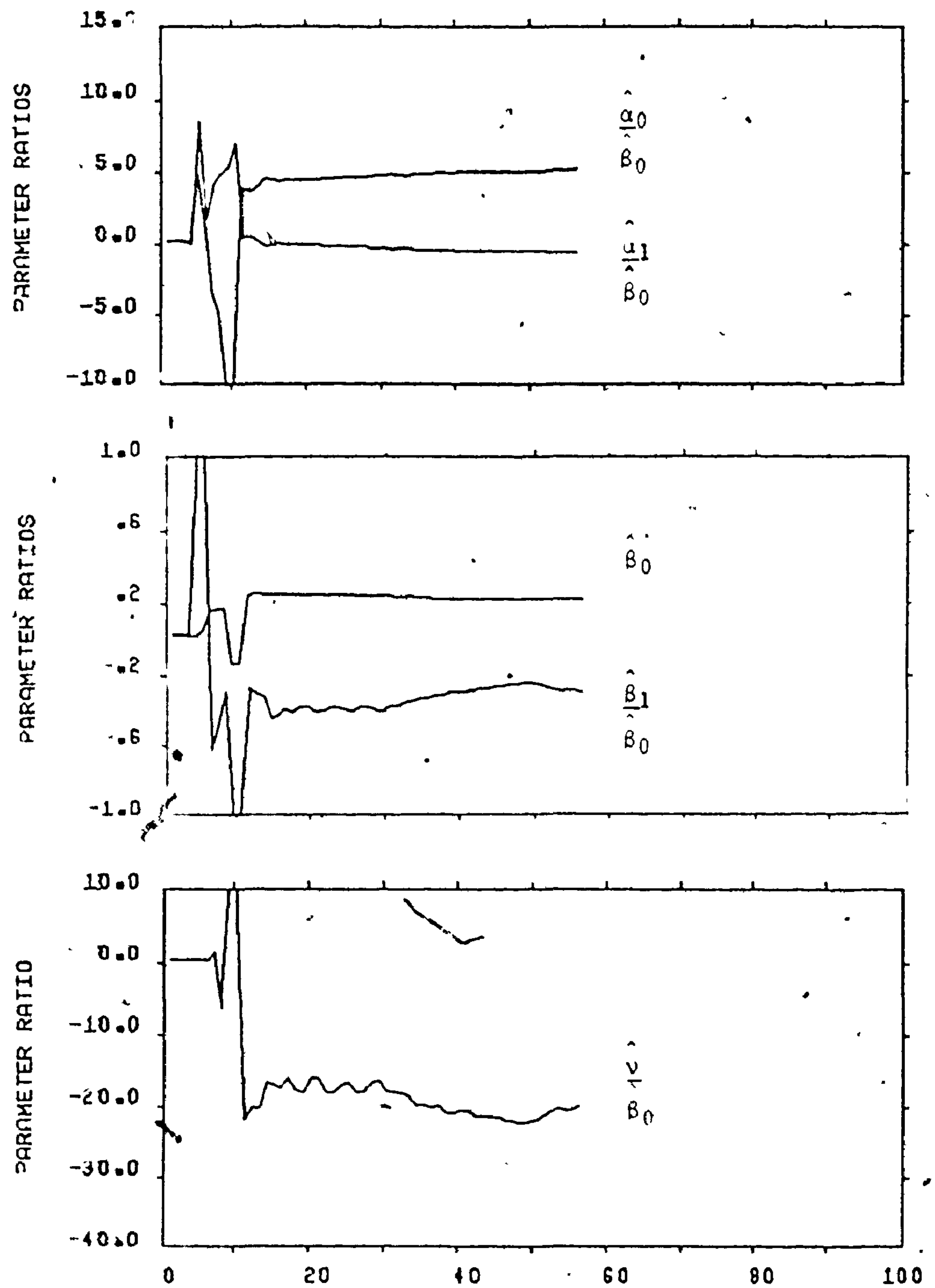


Figure 5.6: Controller parameter ratios for run A002

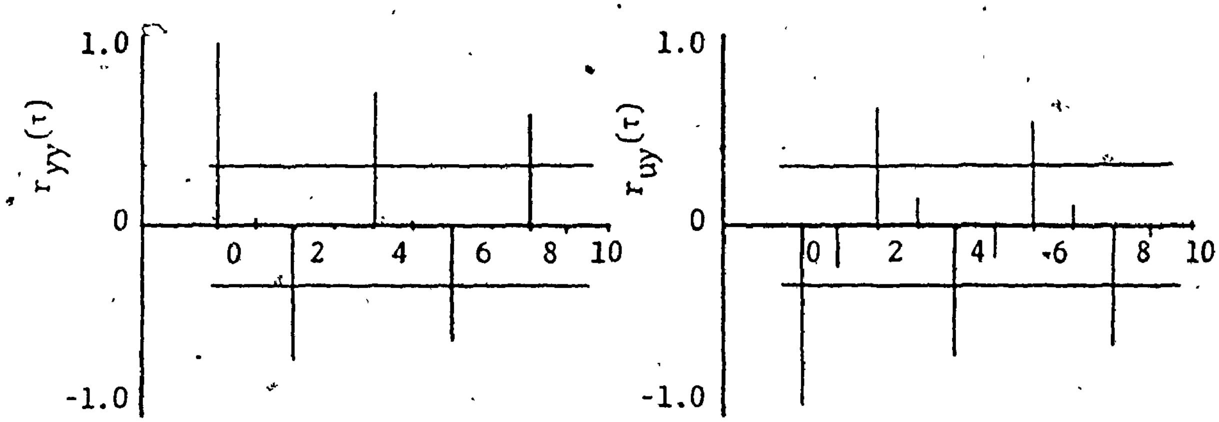


Figure 5.7: Sample auto and cross correlation function for run A002.

A summary of the results for the first two runs are shown in Table 5.2. The initial transient in the output was not included in the calculation of these quantities.

| Run | $\sigma_y^2$ | $\sigma_u^2$ | $\bar{Y}$ |
|---|---|---|---|
| A-001 | 8.4 | 471. | 72.1 |
| A-002 | 2.6 | 67.0 | 73.9 |

Table 5.2: Comparison of results for runs A001, A002

The inclusion of $\nu$ has almost eliminated the offset ($Y_{sp}$ = 73.7).

We would not expect that the offset would be totally removed unless the controller parameters have converged. Since b=1, the minimum variance controller should produce an output which is white noise.

Poor parameter estimates initially, resulted in a control signal that moved the process output to 50°C (Figures 5.2 and 5.5). The stochastic disturbances are masked by what appears to be a major load disturbance, and the parameter estimates may be those of a deadbeat controller. After this initial excursion the process output reflects the stochastic disturbances, and the controller parameters should move away from the deadbeat controller towards the minimum variance controller. The convergence will be slow because of the decrease in magnitude of the elements of $\underline{P}(t)$ due to large excursion initially.

In runs A001 and A002 the estimation was started off with $\underline{\theta}(0) = \underline{0}$. In run A003 we again used $\underline{\theta}(0) = \underline{0}$, however, the parameters were identified for eight sampling intervals, prior to being used to calculate the control signal. A discounting factor of $\lambda = .95$ was used until sampling interval 30, after which $\lambda = .98$ was used. A change in setpoint was made after 70 samples. The parameters were estimated from the same model structure as in A002.

$$Y_{t+1} = \alpha_0 Y_t + \alpha_1 Y_{t-1} + B_0 \tilde{U}_t + B_1 \tilde{U}_{t-1} + \nu + \epsilon_{t+1} \quad (5.13)$$

The input and output sequence, Figure 5.8 indicate that the transient response was much improved as compared to the first two runs. The initial parameter estimates from the preliminary identification stage were not close to their final controller values, Figure 5.9. There is no advantage

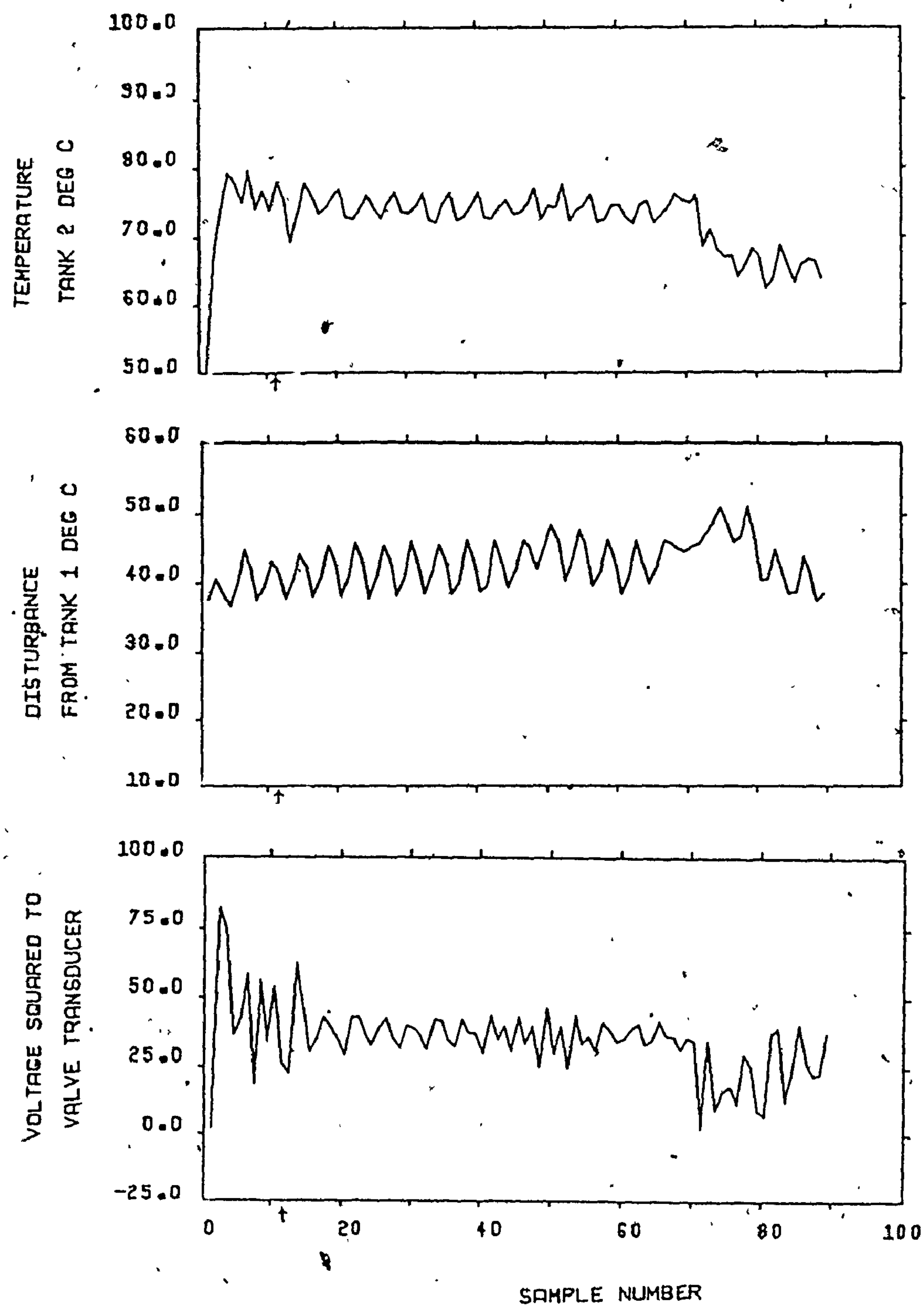


Figure 5.8: Process output, input disturbance and manipulated variable sequences for run A003

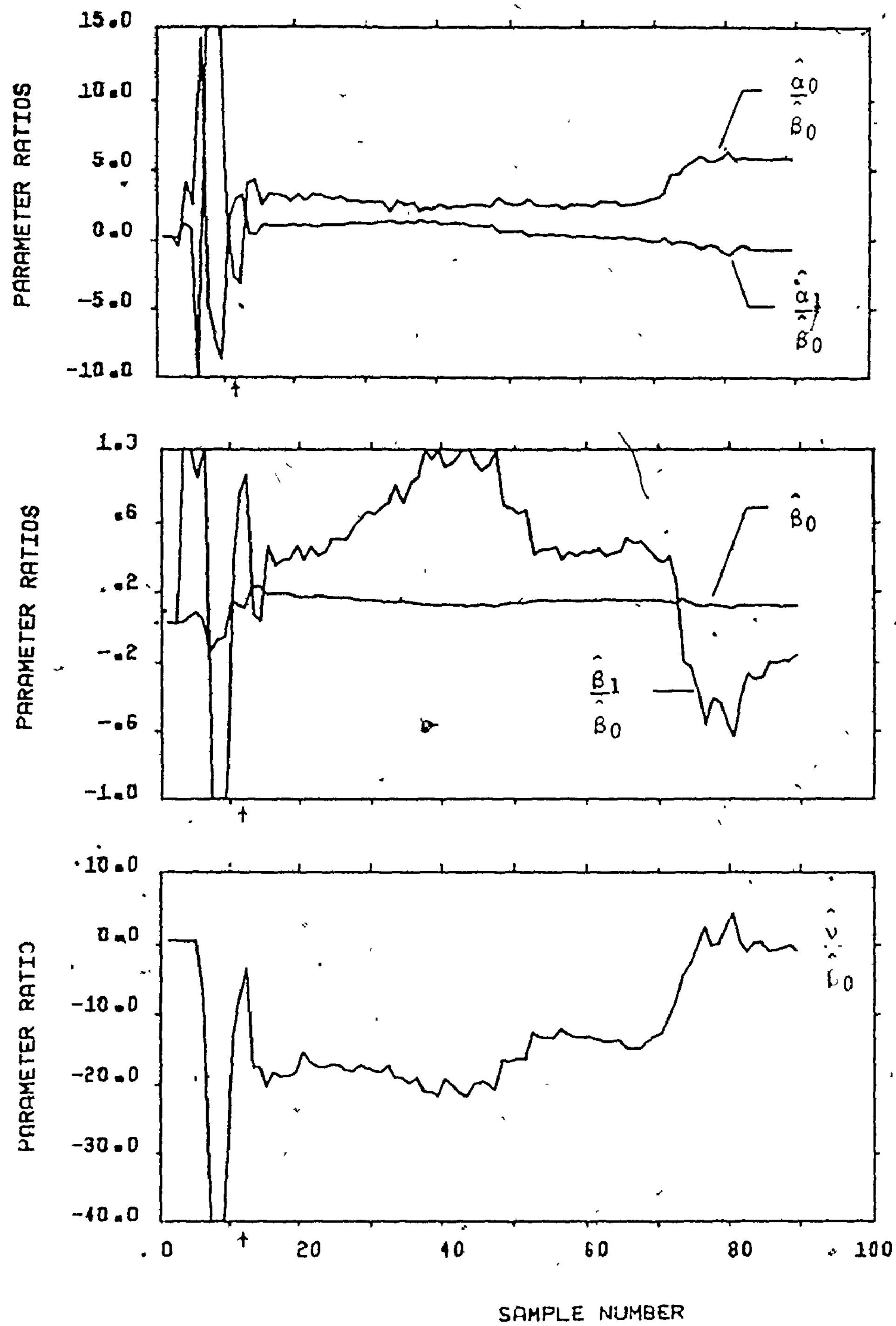


Figure 5.9: Controller parameter ratios for run A003

in continuing the length of the preliminary identification stage, prior to implementing the self-tuning controller, to twenty or thirty sampling intervals as one is not identifying the desired parameters.

$\hat{B}_1$ is very poorly estimated, and it is not until sample 45 that it appears to settle down. In spite of the poor estimation of $\hat{B}_1$ the control signal and process output are not adversely affected. The controller parameters have not converged as indicated by the fact that $r_{yy}(\tau)$, and $r_{uy}(\tau)$ are non zero for $\tau = 1,2$.

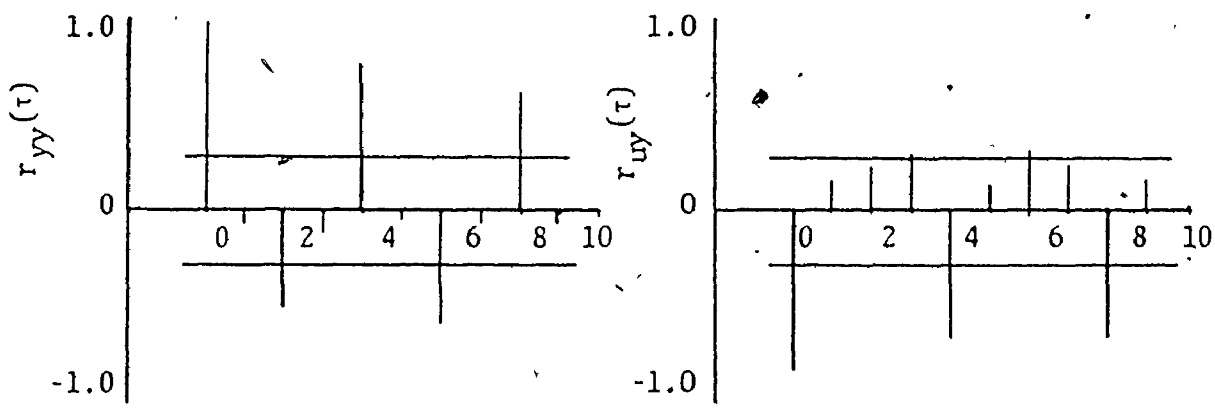


Figure 5.10: Sample auto and cross correlation function for run A003

The parameter estimates fluctuate more than was observed in runs A001 and A002, which is the result of including a forgetting factor in the estimation. The long term influence of the forgetting factor is not seen as the asymptotic sample length of 50, is not far removed from the length of the experiment. Over heating of the stirrer motors prevented the experiments from continuing past about three hours.

Instead of including a $\nu$ term to eliminate offset in the controlled variable, integral action may be included in the controller. The minimum variance controller will only contain integral action if the disturbance is nonstationary. In run A003 the input temperature was drifting upwards and integral action may be justified.

A model of the form

$$Y_{t+1} = \alpha_0 Y_t + \alpha_1 Y_{t-1} + B_0 \nabla U_t + B_1 \nabla U_{t-1} + \varepsilon_{t+1} \qquad (5.14)$$

was used for identification of the controller parameters. The parameters were estimated, but not used in the computation of the control signal for the first six sampling intervals. A forgetting factor of $\lambda$ = .95 was used for the first 30 sampling intervals, after which $\lambda$ = 1.0 was used. Again a setpoint change is made at t = 70. The temperature and voltage squared to the valve transducer are shown in Figure 5.11. The controller ratios are plotted in Figure 5.12. Although the estimates seem to be changing very little, the auto and cross correlations indicate that they have yet to converge, Figure 5.13.

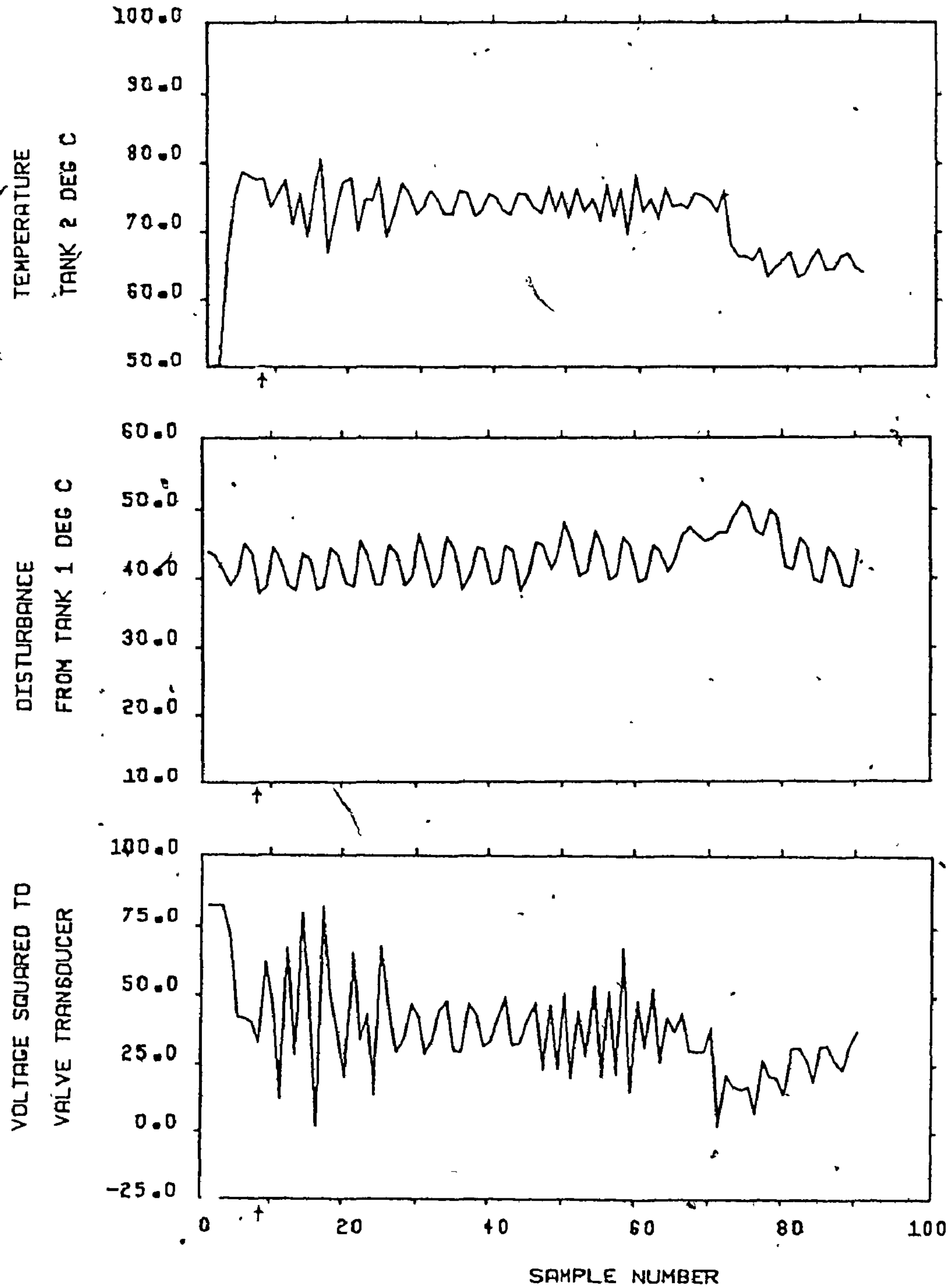


Figure 5.11: Process output, input disturbance and manipulated variable sequences for run A004

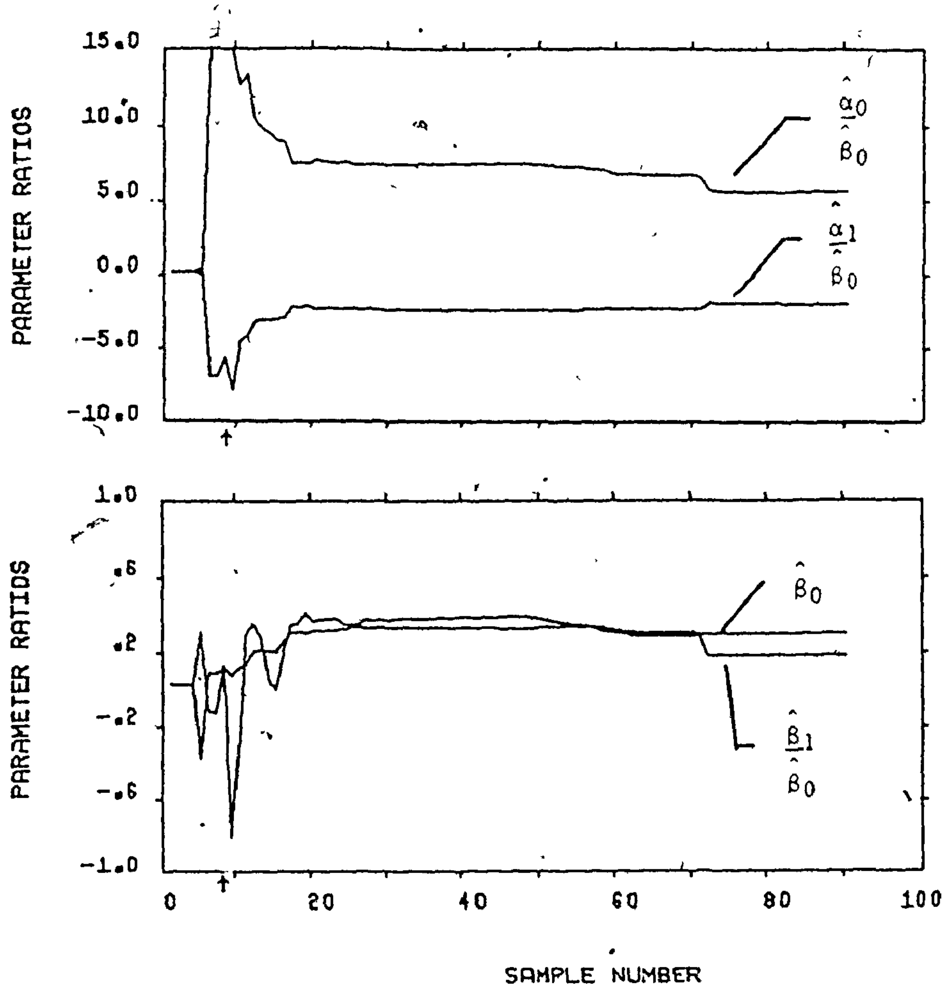


Figure 5.12: Controller parameter ratios for run A004

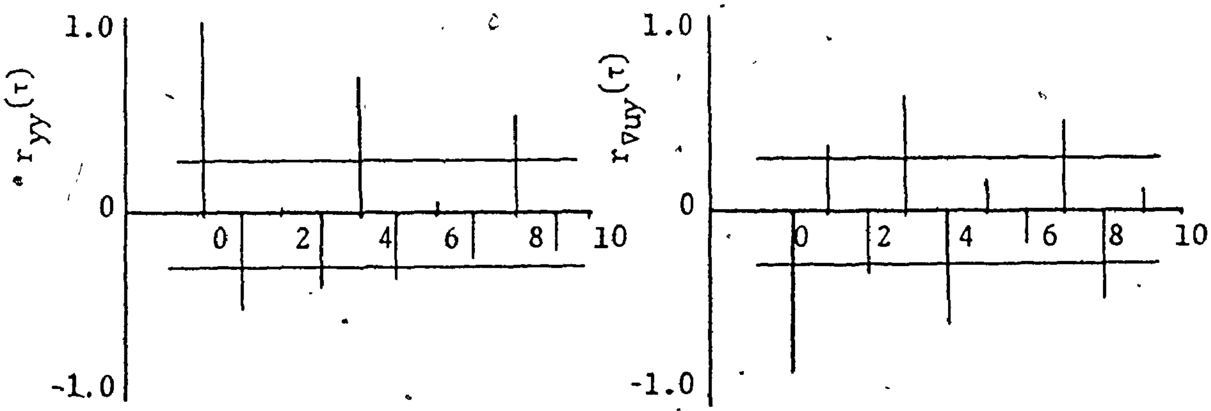


Figure 5.13: Sample auto and cross correlation function for run A004

It appears that the estimates have converged after record 63. It would have been interesting to have continued the experiment for another thirty sampling intervals at the same operating conditions to evaluate the optimality of the controller at what appears to be converged parameter values.

The inclusion of integral action has completely eliminated the offset in the controlled variable. The results from run A003 and A004 are shown in Table 5.3 (up to the setpoint changes).

It appears as though the elements of $\underline{P}(0) = 100\ \underline{I}$ were chosen too small for the B parameters, as the variance of $\nabla^d U_t$ is of same order of magnitude as the initial diagonal element. The inclusion of a forgetting factor of $\lambda$ = .95 for approximately 25 sampling intervals has removed the effect of choosing $\underline{P}(0)$ = 100 $\underline{I}$ by sample number 30.

| Run | $\sigma_y^2$ | $\sigma_u^2$ | $\bar{Y}$ |
|---|---|---|---|
| A003 | 2.4 | 25.7 | 73.8 |
| A004 | 2.8 | 162.3 | 73.7 |

Table 5.3: Comparison of results for runs A003, A004

The parameter estimates have not converged, and the auto and cross correlations provide no guidelines as to whether the controller structure is correct. It was indicated in Chapter 2 that there might be high correlation among the parameters if $\hat{B}_0$ was estimated. This high correlation could result in slow convergence. P(t) is proportional to the variance-covariance matrix of the parameters $\underline{\theta}(t)$, and the correlation matrix of the parameters $\underline{\Gamma}(t)$ is approximated by normalizing $\underline{P}(t)$. Refer to Appendix B for more details. Table 5.4 shows $\underline{\Gamma}$ at sample number 70 for run A004.

$$\Gamma = \begin{array}{c} \\ \end{array} \begin{array}{cccc} \hat{\alpha}_1 & \alpha_2 & \hat{B}_0 & \hat{B}_1 \end{array}$$

$$\Gamma = \begin{bmatrix} 1 & & & \\ -.94 & 1 & & \\ .98 & -.91 & 1 & \\ .76 & -.62 & .84 & 1 \end{bmatrix}$$

Table 5.4: Correlation among parameters for run A004 at sample number 70

The correlation between the parameters is fairly high, and this may contribute to the slow convergence of the parameters.

Slow convergence may also result if the model form is incorrect, as discussed in Section 3.2. It was decided to increase the order of $\alpha(z^{-1})$ and $B(z^{-1})$ by one. The controller parameters were estimated from a model of the form

$$Y_{t+1} = \alpha_0 Y_t + \alpha_1 Y_{t-1} + \alpha_3 Y_{t-3} + B_0 \nabla U_t + B_1 \nabla U_{t-1}$$

$$+ B_2 \nabla U_{t-2} + \varepsilon_{t+1} \qquad (5.15)$$

Although the model structure of run A003 resulted in better control than run A004, with much less control action required, the parameters fluctuated much more. In addition, it was felt that integral action in the controller was a more appealing way of eliminating offset. A forgetting factor $\lambda$ = .95 was used until sampling interval 30, afterwards $\lambda$ = 1.0. The parameters were tracked for six sampling intervals prior to being used in the computation of the control signal. The tank temperature and voltage squared applied to the valve transducer are plotted in Figure 5.14. The output looks like it may be white noise. The parameter ratios, Figure 5.15 have converged and appear to be optimal as indicated by the auto and cross correlation functions, Figure 5.16.

The level of the disturbance was changing in this experiment, Figure 5.14 indicating that integral action would appear in the optimal controller.

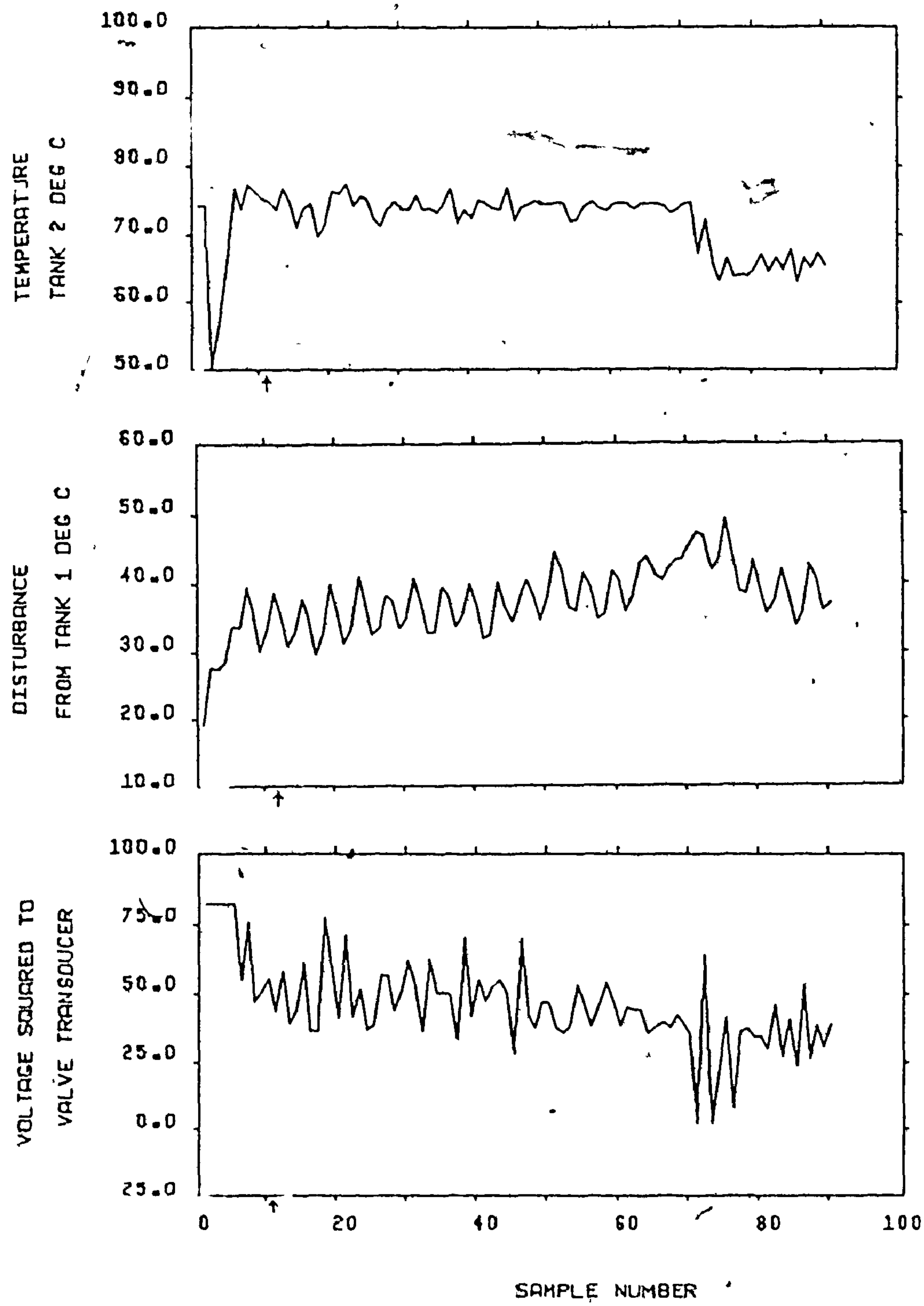


Figure 5.14: Process output, input disturbance and manipulated variable sequences for run A005

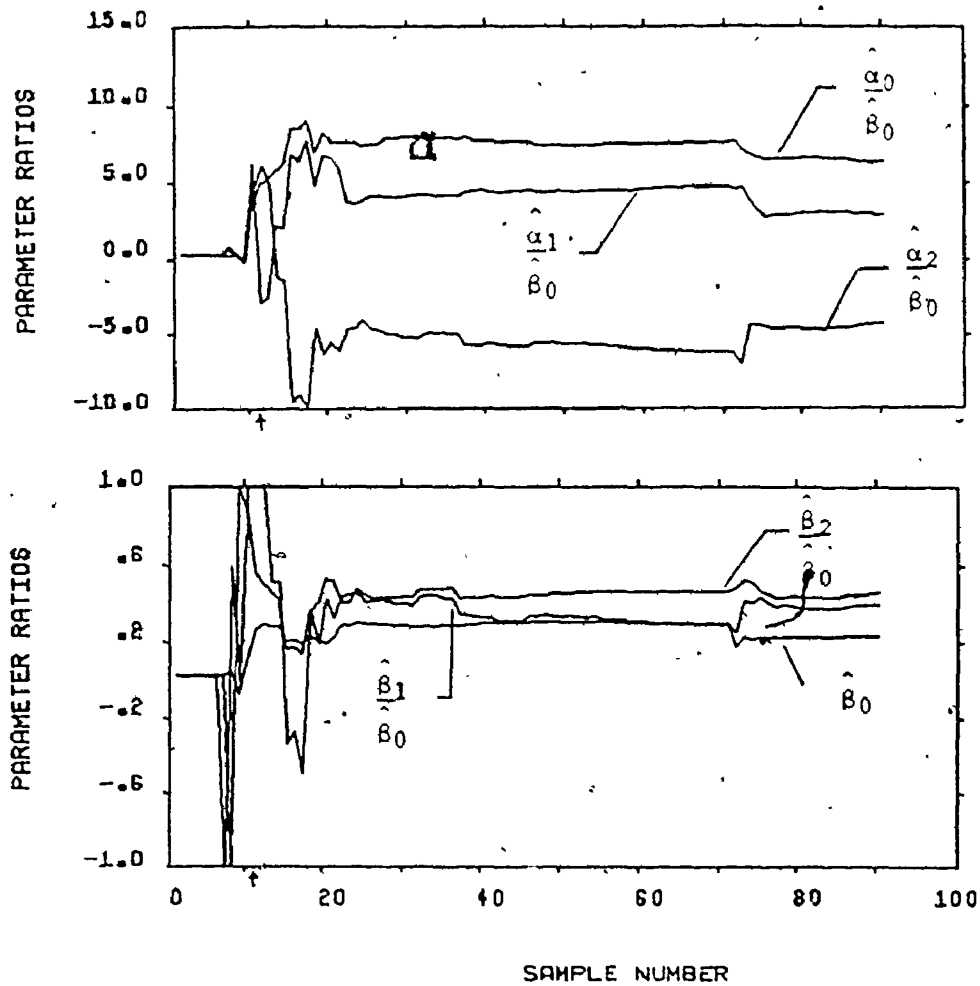


Figure 5.15: Controller parameter ratios for run A005

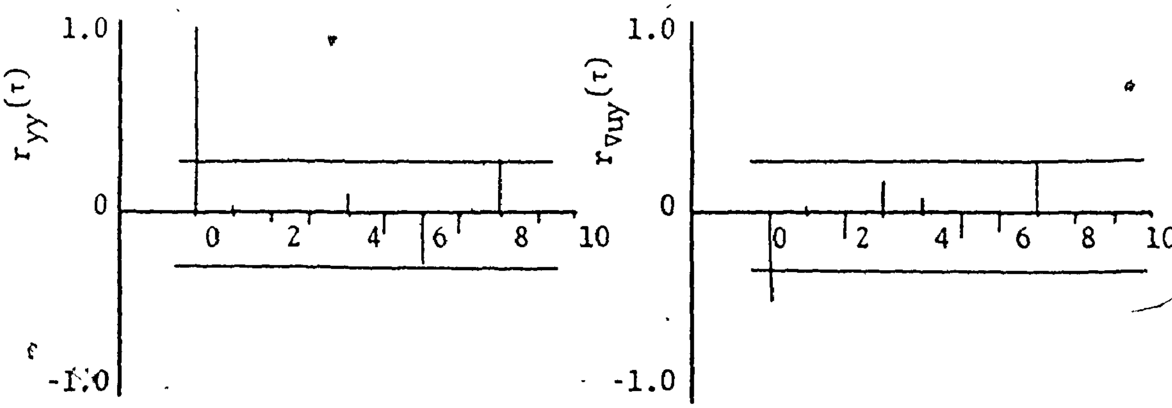


Figure 5.16: Sample auto and cross correlation function for run A005

For run A005 the variance of the output was 1.76, the mean value of the output 73.8 and the variance of the manipulated variable 93.3. The voltage squared to the valve transducer was well inside the operation limits (1-81 volts squared) indicating that it would not be necessary to constrain the variance of $\nabla U_t$. The controller parameter ratios, variances of the manipulated variable and process output and mean value of the process output are summarized in Table 5.6, for runs A001-A005.

The controller parameter ratios from run to run are different as would be expected, since the final controller valves are dependent on the structure of the estimation model and nature of the incoming disturbance. The temperature disturbance entering the second tank was not

| Run | $\hat{B}_0$ | $\frac{\hat{\alpha}_1}{\hat{B}_0}$ | $\frac{\hat{\alpha}_2}{\hat{B}_0}$ | $\frac{\hat{\alpha}_3}{\hat{B}_0}$ | $\frac{\hat{B}_1}{\hat{B}_0}$ | $\frac{\hat{B}_2}{\hat{B}_0}$ | $\frac{\hat{v}_0}{\hat{B}_0}$ | $\sigma^2_y$ $(°C)^2$ | $\sigma^2_u$ $V^4$ | $\bar{y}$ °C ($y_{sp}$= 73.7°C) |
|---|---|---|---|---|---|---|---|---|---|---|
| A001 | .20 | 9.8 | -5.5 | 0. | -.941 | 0. | 0. | 8.4 | 471 | 72.1 |
| A002 | .21 | 4.8 | -.66 | 0. | -.32 | 0. | -21.0 | 2.6 | 57.0 | 73.9 |
| A003 | .13 | 2.7 | .04 | 0. | .39 | 0. | -13.1 | 2.4 | 25.7 | 73.8 |
| A004* | .28 | 6.5 | -2.5 | 0. | .27 | 0. | 0. | 2.8 | 162.3 | 73.7 |
| A005* | .28 | 7.3 | -6.1 | 4.4 | .30 | .45 | 0. | 1.8 | 93.3 | 73.8 |
| A006† | - | - | - | - | - | - | - | 4.5 | - | - |

* integral action in controller

† open loop response

Table 5.6 & 5.7: Summary of results for Runs A001-A006 the same in each run although the same voltage sequence was sent to the valve transducer of tank one. The open loop response of the process is shown in Figure 5.17. In this experiment the disturbance is stationary over most of the experiment.

The disturbance entering tank two in run A005, Figure 5.14 appears to be the sum of a slowly drifting disturbance, upon which is superimposed the generated higher frequency AR(2) disturbance. The disturbance entering the second tank may be modelled by

$$\tilde{T}_{in,t} = \frac{1}{\nabla} a_t + \frac{1}{(1-\phi_1 z^{-1} - \phi_2 z^{-2})} a_t' \qquad (5.16)$$

where $\{a_t\}$ and $\{a_t'\}$ are two mutually independent white noise sequences, mean zero, variances $\sigma_a^2$ and $\sigma_a^{2'}$. The sum of the two stochastic processes, (5.16) may be written as, Box and Jenkins (1970).

$$\tilde{T}_{in,t} = \frac{\xi(z^{-1})}{\nabla\mu(z^{-1})} a_t'' = \frac{1 - \xi_1 z^{-1} - \xi_2 z^{-1}}{\nabla(1 - \mu_1 z^{-1} - \mu_2 z^{-2})} a_t'' \qquad (5.17)$$

The disturbance at the output, $N_t$ (2.4) will reflect the characteristics of this input disturbance, plus any internal disturbances. Thus the minimum variance controller

$$\nabla U_t = \frac{-\delta(z^{-1})}{\omega(z^{-1})} \cdot \frac{L_2(z^{-1})}{L_4(z^{-1})\phi(z^{-1})} Y_t \qquad (2.14)$$

may be very well include three past inputs and outputs, as did the controller for run A005.

The mistake mentioned previously in implementation of the temperature

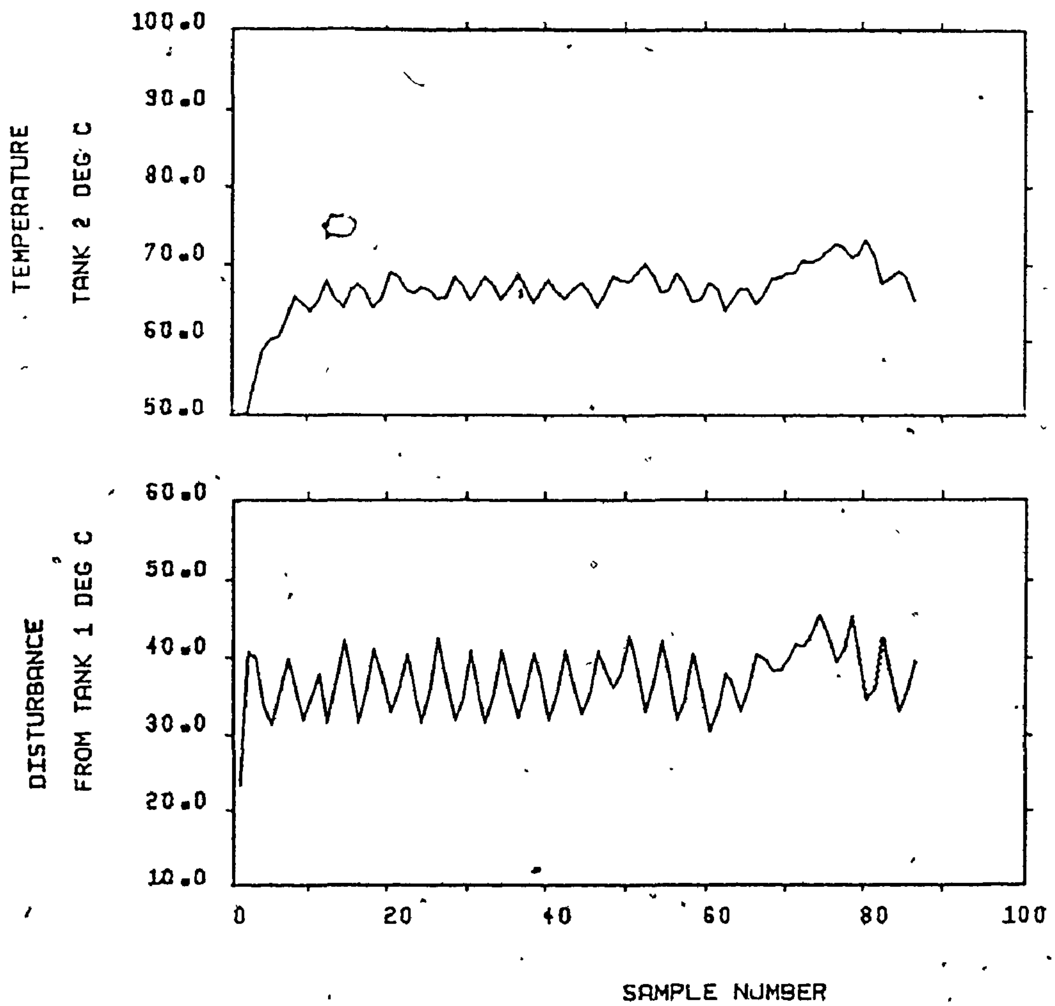


Figure 5.17: Open loop response of the process, run A006

disturbance in the first tank was corrected. The open loop response of the process to this new disturbance is shown in Figure 5.18. Over 20-30 sampling intervals the disturbance appears nonstationary. The controller parameters were estimated from a model of the form

$$Y_{t+1} = \alpha_0 Y_t + \alpha_1 Y_{t-1} + B_0 \nabla U_t + B_1 \nabla U_{t-1} + \;_{t+1} \qquad (5.18)$$

The voltage squared to the valve transducer, and the output temperature are shown in Figure 5.19. The controller ratios, Figure 5.20, are those of the minimum variance controller as indicated by the auto and cross-correlation function, Figure 5.21. The parameter ratio $\hat{B}_1/\hat{B}_0$ has converged near zero, indicating that the minimum variance controller may well be the PI controller

$$\nabla U_t = -8.0\ Y_t + 4.0\ Y_{t-1} \qquad (5.19)$$

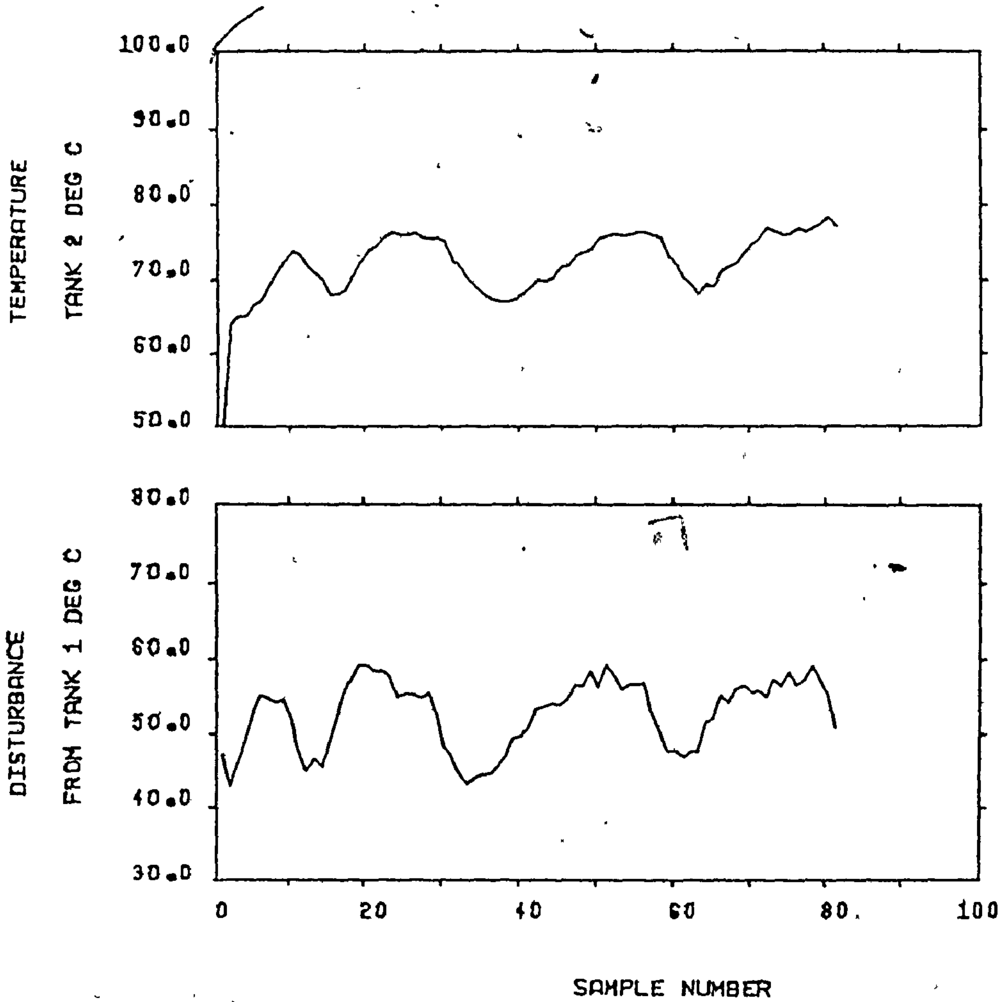


Figure 5.18: Open loop response of the process, run B001

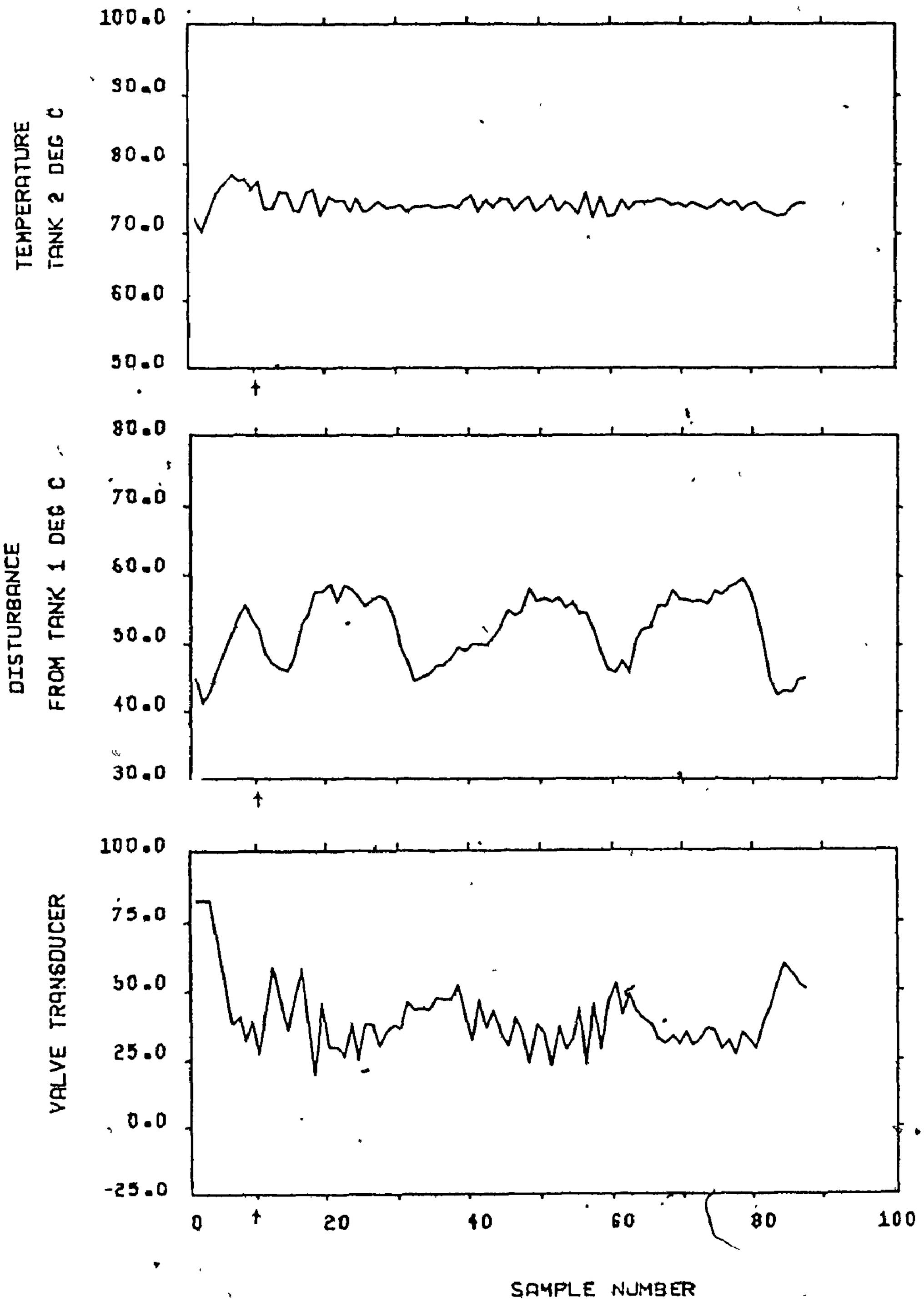


Figure 5.19: Process output, input disturbance and manipulated variable sequences for run B002

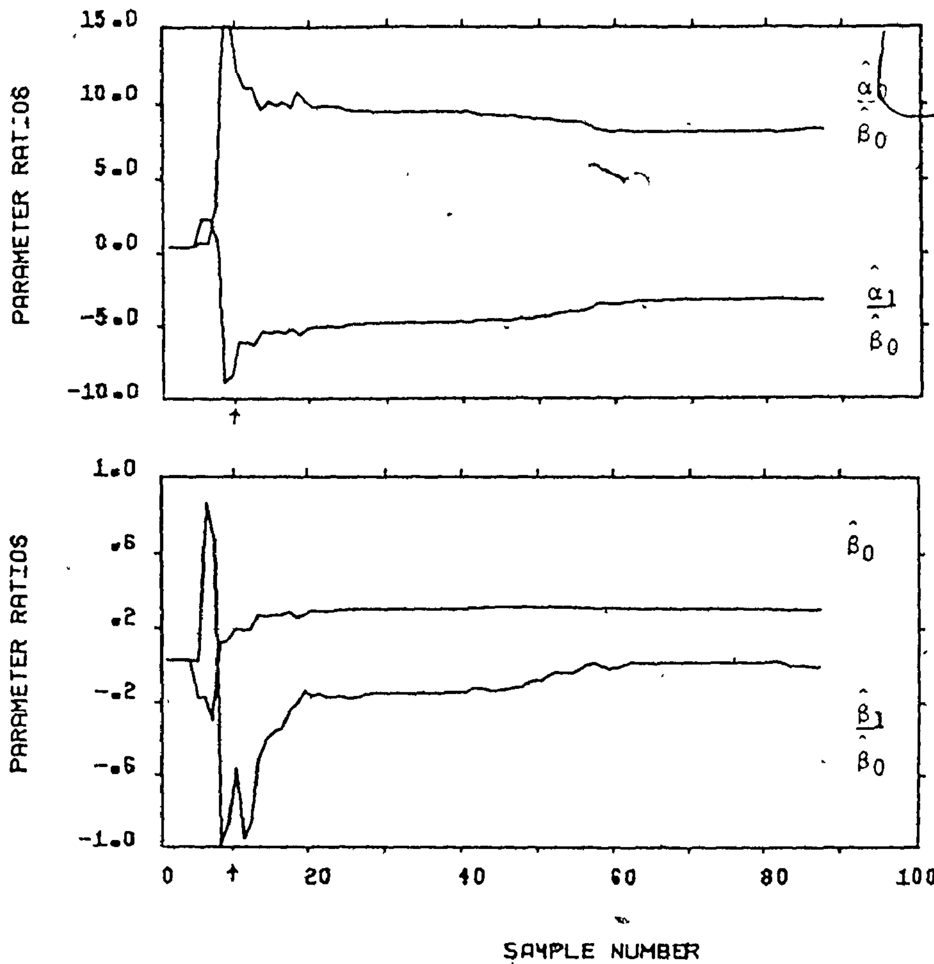


Figure 5.20: Controller parameter ratios for run B002

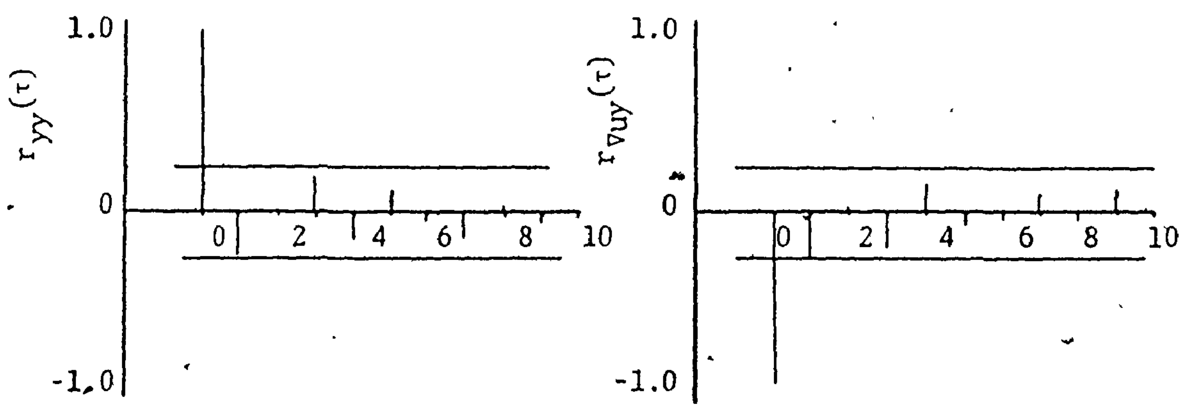


Figure 5.21: Sample auto and cross correlation function for run B002

The last experiment, B003, was to investigate whether the minimum variance controller would remain optimal if the process dynamics changed. The set point of the water flow rate remained unchanged at 30 lb/min for 30 sampling intervals, after which it was ramped down twenty percent over forty sampling intervals and stepped back to its original valve. The controller form was identical to that in Run B002. The initial parameter estimates $\underline{\theta}(0)$, and variance-covariance matrix $\underline{P}(0)$, were those at the conclusion of run B002. The forgetting factor was changed to .98 to allow the estimation routine to track time varying parameters. The input $\nabla U_t$, and output $Y_t$ are shown in Figure 5.22, and the controller ratios in Figure 5.23. The measured flow rate is shown in Figure 5.24. The controller ratios change negligibly over the first sixty sampling intervals. At sample sixty, a large disturbance affected the flowrate, which the digital PI controller was unable to handle. This rapid change in flow rate excited the parameters and they started to drift away from their previously unchanged valves. At sample number seventy, the flowrate was stepped back to 30 lbm/min. at which point the controller parameters changed rapidly, reflecting an increase in information.

The rate of change in the enthalpy input to the second tank was actually very small during the ramping of the flow rate. The decrease in flow rate was offset by an increase in temperature of the water from the first tank. In hindsight, one would not expect the controller parameters to change significantly. At sample number seventy, there was a significant increase in the rate of enthalpy input to the second tank and the controller parameters changed, reflecting this new information. Experimentally it was impossible to implement a twenty percent increase in water flow rate, as the signal required to realize this change saturated the valve transducer. The auto and cross correlations, Figure 5.25, show that the controller was optimal for the duration of the experiment.

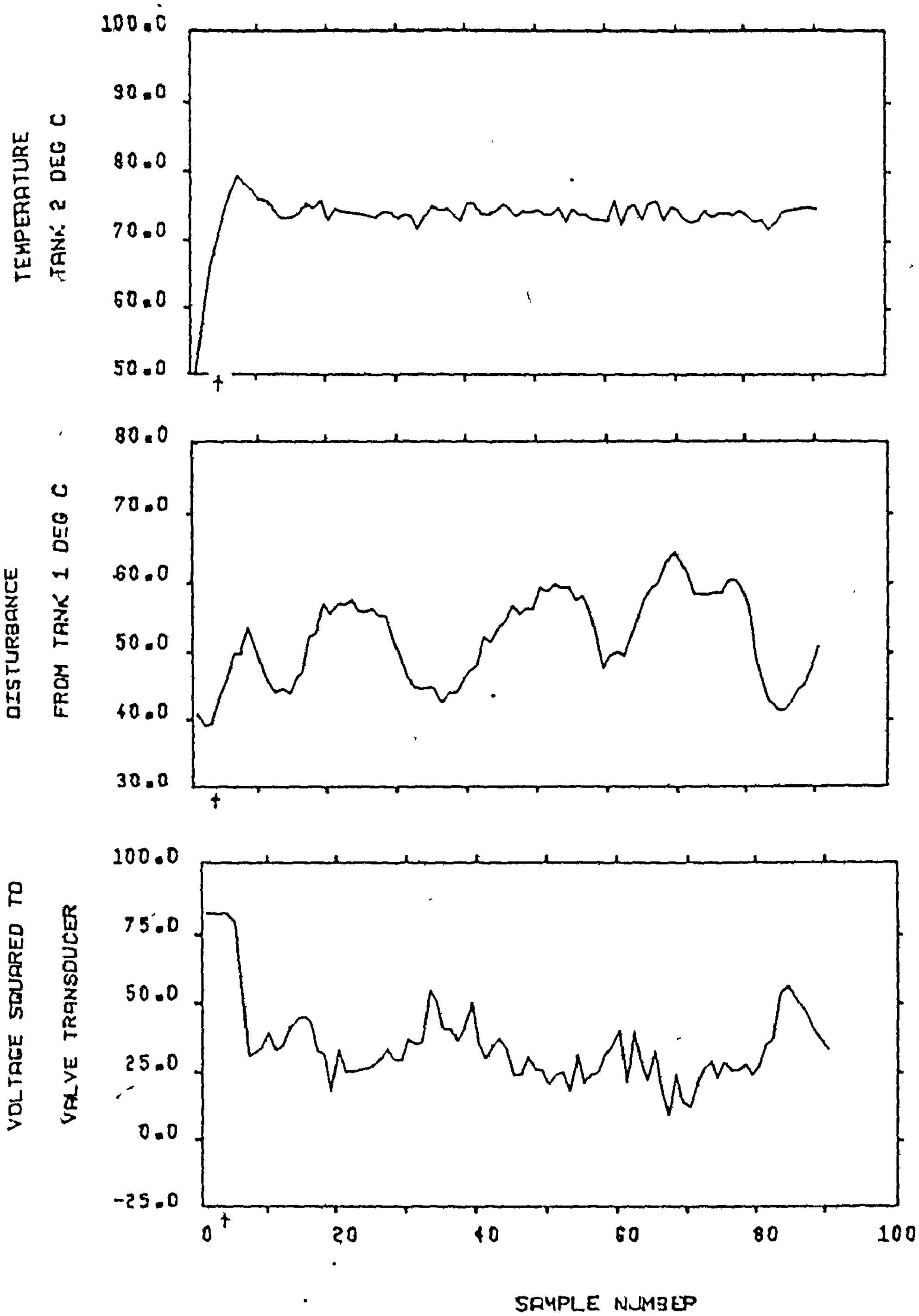


Figure 5.22: Process output, input disturbance and manipulated variable sequences for run B003

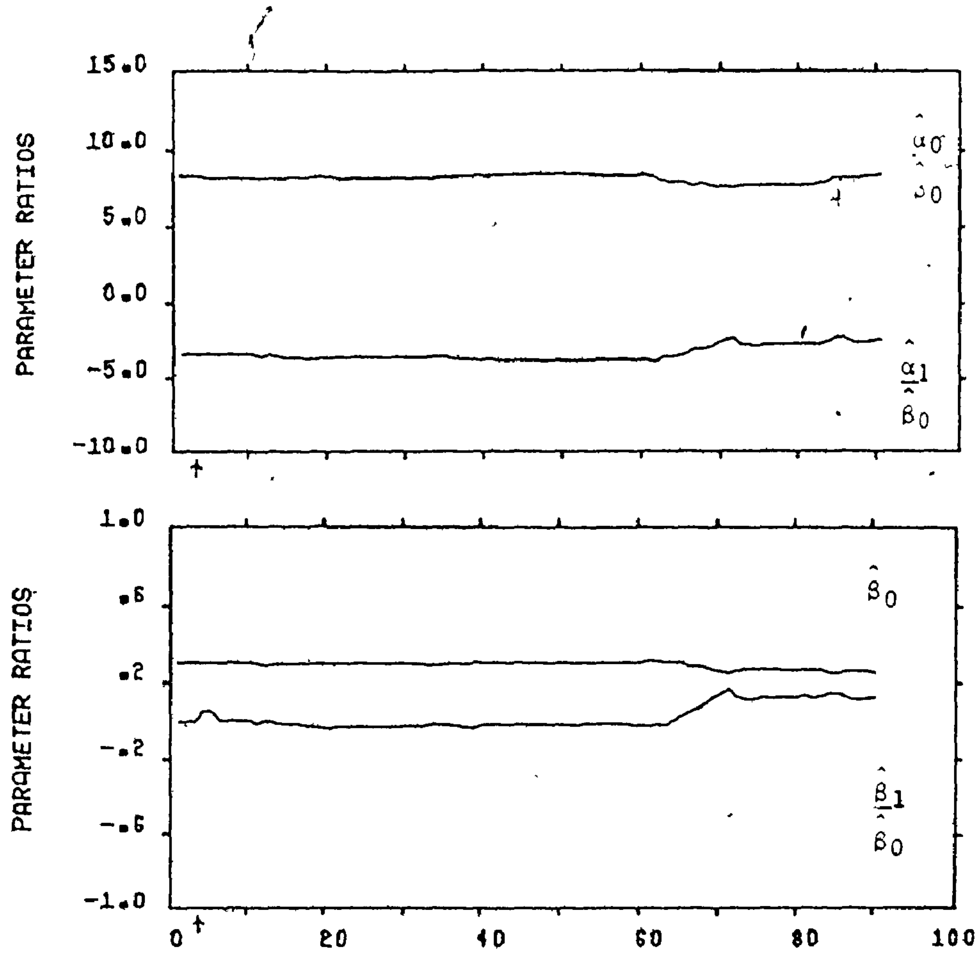


Figure 5.23: Controller parameter ratios for run B003

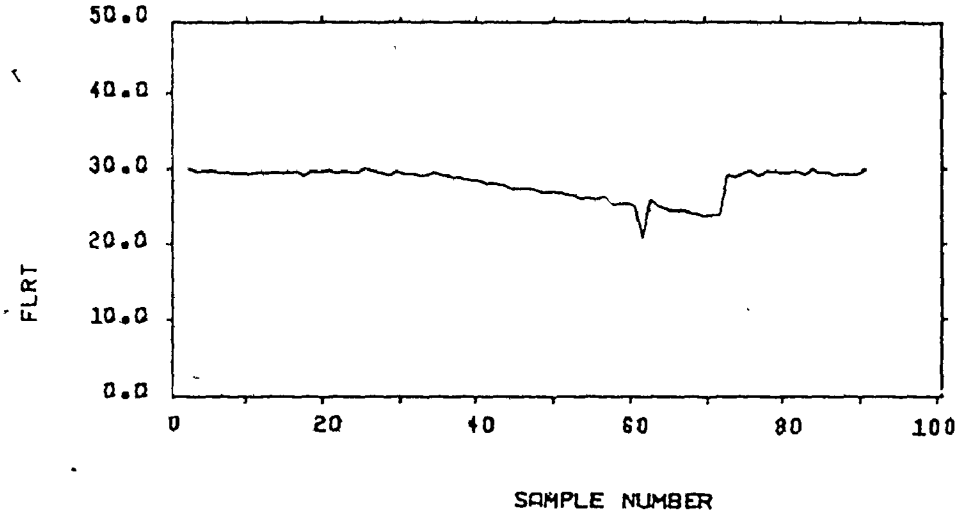


Figure 5.24: Input flowrate for run B003

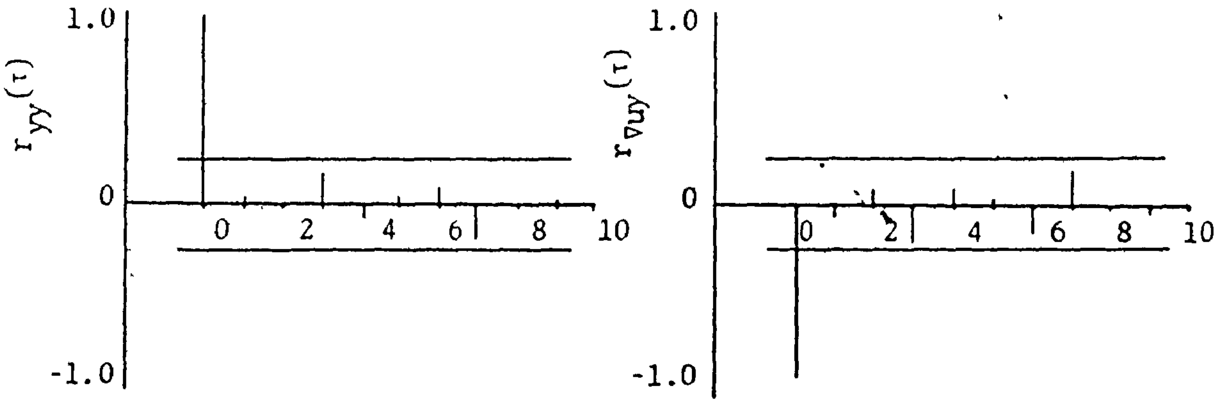


Figure 5.24: Sample auto and cross correlation function for run B003

Results for experimental runs B001, B002 and B003 are summarized in Table 5.7

| Run | var $Y_t$ | var $\nabla U_t$ | $\bar{Y}$ |
|---|---|---|---|
| B001 | 10.32 | 0.0 | 72.4 |
| B002 | 0.97 | 97.1 | 73.6 |
| B003 | 0.88 | 63.0 | 73.7 |

Table 5.7: Summary of results for runs B001, B002 and B003

## 5.4 Use of the Auto and Cross Correlation Function as Diagnostic Tools

Theorem 1 of Chapter 3.2 stated that if the parameter estimates of the model

$$Y_{t+b} = \alpha_0 Y_t + \alpha_1 Y_{t-1} + \ldots + \alpha_m Y_{t-m} + B_0 \nabla^d U_t + \ldots +$$

$$B_\ell \nabla^d U_{t-\ell} + \varepsilon_{t+b} \qquad (3.11)$$

converged and the implemented feedback controller was

$$\nabla^d U_t = -\frac{1}{\hat{B}_0}(\hat{\alpha}_0 Y_t + \ldots + \hat{\alpha}_m Y_{t-m} + B_1 \nabla^d U_{t-1} + \ldots +$$

$$B_\ell \nabla^d U_{t-\ell}) \qquad (3.8)$$

then

$$\rho_{yy}(\tau) = 0, \quad \tau = b+1, \ldots b+m$$

and

$$\rho_{\nabla^d y}(\tau) = 0 \quad \tau = b+1, \ldots b+\ell$$

There were no assumptions concerning the nature of the underlying process dynamic or stochastic models. One of the purposes of this experimental program was to examine the utility of this Theorem in aiding in the selection of the correct controller structure.

The sample auto and cross correlation function damped out quickly when the minimum variance controller form was correctly chosen. (See for example experiments A005 and B002.) For experiments A001 to A004 the controller ratios changed at a rate which did not appear to be significantly slower than runs A005 and B002. Yet, $r_{yy}(\tau)$, $\tau = 1,2$ and $r_{\nabla uy}(\tau)$ $\tau = 1,2$ were not zero, indicating that the controller parameters had not converged.

Consequently at the conclusion of these experiments one could not decide whether the optimal controller form had been correctly chosen.

Convergence of the controller parameters may be slow if its structure is not optimal, as indicated in Chapter 3.6. The output of the process will be a couple ARIMA time series, and $\varepsilon_{t+b}$ in (3.7) will be correlated with $\{Y_t, Y_{t-1},...\}$ and $\{U_t, U_{t-1},...\}$. Since the controller structure is not optimal this 'residual correlation' will persist and this may result in slow convergence of the controller parameters. The experimental results are in good agreement with this explanation. Simulations of self-tuning regulators have also shown that the auto and cross correlations damp out quickly if the minimum variance structure has been chosen. The experimental auto and cross correlations were computed with about forty pairs of input, output data. Use of the auto and cross correlation functions are very useful in testing for controller optimality, requiring relatively few samples to indicate whether the proper structure has been chosen. If the optimal structure has not been chosen, convergence of the controller parameters may be slow. If the parameter estimates have not converged, although they may appear to be changing very slowly, the auto and cross correlation functions provide no information as to how to modify the controller structure to make it optimal.

## 5.5 Set Point Changes

The response of the tank temperature to a change in set point was examined in runs A001, A003, A004 and A005. Near the conclusion of these runs, the set point of the controlled variable was changed from 73.7 °C to 65°C, and held constant for twenty sampling intervals.

In run A001, the set point was changed at sample number forty-eight. Neither an integrator or constant term were included in the controller. The performance of the controller at this new set point is much better, Figure 5.2, as the mean level of the manipulated variable is closer to the reference valve that was used in the estimation scheme.

If a constant term is identified, as was done in run A003, and the process is linear over the range of operation one would only expect that a change in set point would change this constant term. Possibly due to correlation among the parameters and nonlinearities in the process, all of them actually change, Figure 5.9.

If the controller has integral action one would expect that a change in set point would cause the controller parameters to move away from their previously identified valves, returning soon afterwards if the process were linear over the entire range. Unfortunately in runs A003, A004 and A005, the change in set point coincided with a change in the characteristics of the disturbance, Figures 5.8, 5.11 and 5.14. This makes it impossible to comment on the effect of set point changes on the parameter estimates.

The mean values of the controlled variable after the change in set point are shown in Table 5.8. After an initial very short transient period the controllers rapidly established good control over the temperature at the new set point (65°C) with no apparent offset.

| Run | $\underline{\bar{Y}}$ °C |
|---|---|
| A001 | 64.9 |
| A002 | 66.3 |
| A003 | 65.6 |
| A005 | 65.5 |

Table 5.8: Mean value of the tank temperature after set point change to 65°C

## 5.6 Precision of the Parameter Estimates

The controller parameters are estimated from a model of the form

$$Y_{t+1} = \alpha(z^{-1})Y_t + B(z^{-1})\nabla^d U_t + \varepsilon_{t+1} \qquad (3.11)$$

and used in the computation of the control signal

$$\nabla^d U_t = \frac{-\hat{\alpha}(z^{-1})}{\hat{B}(z^{-1})} Y_t \qquad (3.14)$$

as if they were exactly known. This section will examine the precision of the parameter estimates, how this influences the estimation scheme and the effect of estimating $\hat{B}_0$.

In run B002 the controller parameters were estimated from the model

$$Y_{t+1} = \alpha_0 Y_t + \alpha_1 Y_{t-1} + B_0 \nabla U_t + B_1 \nabla U_{t-1} + \varepsilon_{t+1} \qquad (5.15)$$

The two standard deviation confidence limits of the parameter estimates are shown in Table 5.9 at several points in time. The controller parameters are poorly estimated in spite of their being the optimal controller valves. If it were possible to reduce the variance of the estimated parameters, the rate of convergence might improve.

| Parameter | Sample Number | |
|---|---|---|
| | 40 | 70 (end) |
| $\hat{\alpha}_0$ | 2.58±1.09 | 2.20±1.03 |
| $\hat{\alpha}_1$ | -1.43±0.89 | -1.00±0.80 |
| $\hat{B}_0$ | 0.28±0.13 | 0.28±0.12 |
| $\hat{B}_1$ | -0.05±0.07 | -0.01±0.06 |

Table 5.9: Approximate two standard deviation confidence intervals for parameters of run B002

The large variances in the estimated parameters is due largely to the presence of high correlations among them, Table 5.10 as the joint confidence region of the parameters is large.

$$
\begin{array}{c} \hat{\alpha}_0 \\ \hat{\alpha}_1 \\ \hat{B}_0 \\ \hat{B}_1 \end{array}
\begin{bmatrix} 1 & & & \\ -.88 & 1 & & \\ .96 & -.88 & 1 & \\ .54 & -.78 & -.52 & .1 \end{bmatrix}
$$

Table 5.10: Correlation matrix of the estimated parameters at sample 70 for run B002

Even though there exists correlation among some of the parameters, it has been feasible to estimate all the controller parameters.

If $b > 1$, the residual $\epsilon_t^0$ will contain information about the controller parameters (see Equation (3.4)). In least squares estimation this information is not used. Convergence of the controller parameters might be improved by estimating them from a model of the form

$$Y_{t+b} = \alpha(z^{-1})Y_t + B(z^{-1})\nabla^d U_t + \pi(z^{-1})a_{t+b} \qquad (5.20)$$

where

$$\pi(z^{-1}) = 1 + \pi_1 z^{-1} + \ldots + \pi_{b-1} z^{-b+1} \qquad (5.21)$$

This requires that recursive maximum likelihood estimation or an equivalent method be used.

Huynh (1974) identified the process dynamic and stochastic models for the steam jacketed stirred tank as

$$Y_{t+1} = \frac{.168}{1-.91z^{-1}} U_t + \frac{1}{\nabla(1-.30z^{-1}-.17z^{-1})} a_{t+1} \qquad (5.22)$$

The disturbance was generated by (5.8). The minimum variance controller for (5.22) can be written as

$$\nabla U_t = -7.7Y_t + 5.47Y_{t-1} + .64Y_{t-2} \qquad (5.23)$$

where terms involving $Y_{t-3}$, $Y_{t-4}$, etc. are small. The self-tuning controller parameters in run B002 (5.19) have converged to values close to those in (5.23).

$\hat{B}_0$ is an estimate of $\omega_0$ in the process dynamics. Identification of the complete set of controller parameters has shown that $\omega_0$ may have shifted (the confidence limit on $\hat{B}_0$ = .28±.13). In this implementation had $\omega_0$ shifted to 0.10 and we fixed $\hat{B}_0$ at 0.17 convergence would have been exceedingly slow. The parameters of the controller may be well tuned even though the characteristics of the process dynamics or disturbances change. The performance of the self-tuning regulator should not be jeoparidized by a poor apriori estimate of $\hat{B}_0$, when it can be readily estimated.

## 5.7 Summary

The self-tuning regulator was successfully implemented to control the temperature of a jacketed-steam heated stirred tank. The necessity of estimating a constant term, or including differencing in the estimation model to eliminate offset was examined in runs A001-A004. It was found that the sample auto and cross correlation function were very useful in testing for controller optimality and parameter convergence. The rate of convergence of the controller parameters was most rapid when the minimum variance structure had been chosen which agrees well with the hypothesis in Chapter 3. The self-tuning regulator gave a smooth response to set point changes with no resulting offset. It was demonstrated that it was possible to estimate all the controller parameters, and it was not necessary to fix one.

Future work on the stirred tanks might include application of the self-tuning controller of Clarke and Gathrop (1975). The next chapter reviews the most important aspects of this work, examines some limitations of the self-tuning regulator and suggests areas that require further investigation.

# CHAPTER 6

## SUMMARY AND CONCLUSIONS

An attempt has been made in this thesis to provide a unifying treatment or overview, of the theory of self-tuning regulators. Topics in linear stochastic control theory were reviewed, followed by an extensive discussion of self-tuning regulators. A critical review of process applications indicated some problems that have occurred in several implementations of these regulators. It is felt that this overview is important, as it has brought together most of the relevant theory and related topics. Some of the confusion surrounding aspects of self-tuning regulators stem from the fact that similarities and relationships between different topics are not fully understood.

The Box and Jenkins (1970) representation of dynamic and stochastic processes was compared and contrasted to that proposed by Aström (1970). In the latter's representation there is no provision for modelling nonstationary disturbances. Consequently, minimum variance controllers have no integral action and the controlled variable may have offset. Elimination of offset was one of the problems encountered in application of self-tuning regulators. When the theory is presented using the notation of Box and Jenkins integral action enters the controller in an obvious manner when the disturbances (stochastic or deterministic) are nonstationary. By correctly accounting for nonstationary disturbances in the estimation model, one avoids the nontrivial problems of estimating an additional parameter lying near stability boundaries of the process. As a result the self-tuning regulator should be less sensitive to process parameter variations.

The discussion on the selection of the sampling interval has previously been neglected, but it is important that it be selected with some care. The number of parameters to be estimated increases, if the process is sampled at a fast rate compared to the process deadtime. However, if the disturbances are stochastic little improvement in control is achieved by choosing the sampling interval to be much shorter than the process deadtime. Sampling of a continuous process may result in a discrete dynamic model that is non minimum phase, and the self-tuning regulator algorithm must be modified to account for this. The complexities of the modified algorithms require a more sophisticated knowledge of the underlying process dynamics and stochastics. The most appealing method of handling non-minimum phase systems is to change the sampling interval if this is possible.

There is considerably confusion in the literature concerning the estimability of all the controller parameters. In fact, it is possible to estimate all the controller parameters since the self-tuning controller is a time varying function of the process input and output. This is an important result, since if one parameter is fixed, the stability of the process, and rate of convergence of the remaining parameter is dependent on how close the value of the fixed parameter is to its true (but probably unknown) value.

There was a long discussion on self-tuning constrained control, least squares estimation, convergence of the parameter estimates, stability of the closed loop system, and the incorporation of feedforward variables and multivariable decoupling. Self-tuning constraining control will only produce the desired results if the estimation model correctly accounts for nonstationary disturbances, if they are present. Selection of the sampling interval, and structure of the estimation model affect the stability of the closed loop. By considering all these topics in this thesis it is possible to see their interdependence.

Another objective of this thesis was to gain a familiarity with implementing a self-tuning regulator to control a pilot plant process. The self-tuning regulator successfully controlled the temperature of a steam jacketed stirred tank. Although this is an easy process to control, the application to a real process allows one to investigate topics in the theory that have led to problems, or caused confusion in previous applications. Diagnostic tools used to check for controller optimality and parameter convergence were found to be very useful. Estimation of a constant term and the inclusion of integral action in the estimation model were methods examined for eliminating offset. It was also found possible to estimate all the controller parameters. The rate of convergence of the controller parameters was most rapid when the minimum variance structure was chosen. This result agrees well with an intuitive explanation presented in Chapter 3. There have been several reported applications of self-tuning regulators to control industrial and pilot plant systems processes, so this portion of the thesis is not unique. Nor was the process difficult to control. However, the necessary software was written and with the familiarity gained from this project, the application of a self-tuning regulator to control more difficult and challenging process should proceed smoothly.

The self-tuning regulator would appear to be a very powerful means of controlling processes with nonstationary (deterministic or stochastic) disturbances and large deadtimes, as the controller includes deadtime compensation. Where there is strong economic incentive to maintain product quality within a specified range, minimum variance controllers are a sensible class of controllers. However, design of minimum variance and constrained minimum variance controllers requires a plant experiment, and extensive and sophisticated off-line analysis of data. If these controllers are to be effective new process data from design experiments must be collected if there are shifts in operating level (i.e. grade or selectivity changes) or the process characteristics change with time (i.e. decaying catalyst). The self-tuning regulator by contrast, is simple to use, requiring a minimum of experimental effort and can continuously tune the controller parameters, thus tracking slowly changing process characteristics.

There may be difficulty in applying self-tuning regulators to control processes that are nonminimum phase or that have changing process deadtimes. The latter is the more serious problem. If, for example, the number of whole periods of delay shifts from three to four, the delay used in the estimation model and the dimension of the controller will be underestimated. As well the process may move in and out of nonminimum phase with shifting deadtimes. These phenomena could lead to process instabilities. Many of these problems can be solved by a judicious choice of the sampling interval. It may not be necessary to sample the process so rapidly that there are three or four whole periods of delay to realize good control. By increasing the sampling interval the effect of changing deadtimes can be reduced.

The self-tuning regulator has not been used to control extremely non-linear processes or those having problems with changing deadtime.

further investigation into these areas is required. One of the unrealized objectives of this thesis was to use the self-tuning regulator to control a catalytic packed bed reactor carrying out the extremely temperature sensitive hydrogenolysis of butane reactions. This process is difficult to control, having high heats of reaction, radial and axial temperature gradients and extremely non-linear behavior. It was hoped to try different self-tuning regulator configurations and compare the results with PID algorithms and multivariate linear quadratic control studies that have been completed (Jutan (1976)). A lengthy mini-computer breakdown prevented these studies from being completed for this thesis. However, this project is being pursued.

An obvious extension of the univariate self-tuning regulator is to multivariate self-tuning regulators to account for process interreactions. Preliminary work (Borisson (1975)) suggests that the multivariate dynamic and stochastic models be represented as

$$A(z^{-1})\underline{Y}_t = B(z^{-1})\underline{U}_{t-b} + C(z^{-1})\underline{a}_t \qquad (6.1)$$

where $A(z^{-1})$, $B(z^{-1})$ and $C(z^{-1})$ are matrix polynomials and $\underline{Y}_t$ and $\underline{U}_t$ are vectors of process inputs and outputs. A rather straightforward extension of the self-tuning regulator concept was proposed there in which the parameters of the multivariate minimum variance controller were to be estimated from a model of the form

$$\underline{Y}_{t+b} = \alpha(z^{-1})\underline{Y}_t + B(z^{-1})\underline{U}_t + \underline{\varepsilon}_{t+b} \qquad (6.2)$$

and used in the control law

$$\underline{U}_t = B_0^{-1}(-\alpha(z^{-1})\underline{Y}_t + B_1\underline{U}_{t-1} + \ldots + B_m\underline{U}_{m-t}) \qquad (6.3)$$

at every sampling interval as if they were exactly known. It is doubtful that with this approach plant or industrial applications will be realized. Even for two inputs and two outputs a large number of parameters must be estimated. Convergence can be excruiatingly slow due to poor conditioning among the parameters. For the multivariate self-tuning regulators the fewest number of parameters must be identified that account for most of the variation in the process outputs. This requires that statistical techniques such as model reduction be used. Obviously this requires a sophisticated understanding of the process dynamics and stochastics. However, without application of some model reduction techniques the multivariate self-tuning regulators appear too cumbersome to use.

This thesis has tried to present a unifying approach to the theory of self-tuning regulators. With the understanding gained from this work it is felt that more challenging applications and extensions of the basic theory can be realized.

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## APPENDIX A

## PARAMETER ESTIMATION IN CLOSED LOOP

Most techniques for estimation of the parameters and identification of the structure of process dynamic and stochastic models require perturbation of the process input while the process is run under open-loop conditions. For a variety of reasons this mode of operation may be unsatisfactory. Only recently have the consequences of identification of process dynamic and stochastic models been thoroughly examined, Box and MacGregor (1974, 1976) and Soderström et al. (1974, 1975, 1976). Necessary and sufficient conditions are given so that one may obtain unique estimates of the process dynamic and stochastic parameters when the feedback controller is linear and time invariant.

In the literature on self-tuning regulators when trying to estimate the parameters of the controller $(\frac{\hat{\alpha}_0}{\hat{\beta}_0}, \frac{\hat{\alpha}_1}{\hat{\beta}_0} \ldots, \frac{\hat{\beta}_1}{\hat{\beta}_0} \ldots)$, there is a great deal of confusion as to whether these can be obtained by estimating all the parameters $(\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\beta}_0, \ldots)$ from equation (3.11) since this contains one redundant parameter. If one parameter is fixed (i.e. $\hat{\beta}_0$) the stability of the closed loop and rate of convergence of the remaining parameters is dependent on how close this fixed value is to the true valve. Aström and Wittenmark (1973) gave the following example to show that one may not be able to uniquely estimate all the parameters $(\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\beta}_0, \hat{\beta}_1, \ldots)$.

Let the process dynamic and stochastic model be

$$Y_t = aY_{t-1} + bU_t + a_t \tag{A1}$$

In this particular example the parameters in the estimation equation are also the process parameters. The minimum variance controller parameter is (a/b). Suppose the parameters of (A1) are obtained by minimizing the least squares criterion

$$V_1(a,b) = \sum_{s=1}^{N} (Y_t - aY_{t-1} - bU_t)^2 \tag{A2}$$

Suppose the feedback controller

$$U_t = kY_t \tag{A3}$$

is implemented during the collection of the data. Now (A3) may be written as

$$-c(U_t - kY_t) = 0 \tag{A4}$$

where c is any scalar. Adding this expression to the quantity inside the brackets of (A2) then

$$V_1(a,b) = \sum_{s=1}^{N} (Y_s + (ck-a)Y_{s-1} - (b+c)\,U_{s-1})^2 \tag{A5}$$

There is no unique solution that minimizes (A2) as it is seen that

$$V_1(a,b) = V_1(ck-a,\ b+c) \tag{A6}$$

also minimizes (A2). From this example it was concluded that all the parameters of the model (A1) could not be uniquely estimated when the control was generated by a linear feedback law.

This example does not really describe the estimation situation that one encounters with the self-tuning regulator. Unless $\delta(z^{-1})\theta(z^{-1}) = 1.0$ and b=1, the parameters of the estimation equation (3.11) are not those of the process dynamic and stochastic models. As well, if the controller parameter estimates are being used in the computation of the control signal, the control law is a nonlinear, time varying function of the input and output sequence. Thus <u>all</u> the parameters of the model (3.11) may be estimated.

By way of illustration, suppose that one is trying to estimate the parameters of the minimum variance controller from a model of the form

$$Y_{t+1} = \alpha_0 Y_t + B_0 \nabla U_t + \varepsilon_{t+1} \tag{A7}$$

At time t, the least squares estimates of $\hat{\alpha}_0$ and $\hat{B}_0$ are given by (conditional upon initial effects)

$$\begin{pmatrix} \hat{\alpha}_0 \\ \hat{B}_0 \end{pmatrix} = \begin{bmatrix} \sum_{s=1}^{N} Y_{s-1}^2 & \sum_{s=1}^{N} U_{s-1} Y_{s-1} \\ \sum_{s=1}^{N} Y_{s-1} U_{s-1} & \sum_{s=1}^{N} \nabla U_{s=1}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{s=1}^{N} Y_s Y_{s-1} \\ \sum_{s=1}^{N} Y_s Y_{s-1} \end{bmatrix} \tag{A8}$$

If the control law

$$\nabla U_t = kY_t \tag{A9}$$

is used then (A8) may be written as

$$\begin{bmatrix} \hat{\alpha}_0 \\ \\ \hat{B}_0 \end{bmatrix} = \frac{\sum_{s=1}^{N} Y_s Y_{s-1}}{\sum_{s=1}^{N} Y_{s=1}^2} \begin{bmatrix} 1 & k \\ \\ k & k^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \\ k \end{bmatrix} \tag{A10}$$

$\hat{\alpha}_0$ and $\hat{B}_0$ are perfectly correlated and there is a singularity in the estimation space. If the control law

$$\nabla U_t = k(Y_t, \nabla U_t)Y_t \tag{A11}$$

is used at every sampling interval then it is obvious that $\hat{\alpha}_1$ and $\hat{B}_0$ may be uniquely estimated, which is the case when

$$k(Y_t, \hat{\nabla} U_t) = \frac{\hat{\alpha}(z^{-1},t)}{\hat{B}(z^{-1},t)} \tag{A12}$$

If the parameter estimates converge then

$$k(Y_t, \nabla U_t) \rightarrow k \tag{A13}$$

However; the least squares criterion

$$\min V(\underline{\theta}) = \sum_{s=1}^{N} \varepsilon^2(s) \qquad \text{(A14)}$$

still contains information from earlier on when the controller was time varying and nonlinear.

Let us now examine whether or not one parameter in the estimation equation (3.11) will always be a linear combination of the others or to what extent this previous analysis depends or to what extent this previous analysis depends on the structure of the process dynamic and stochastic models. This is an asumptotic analysis and presumes that the controller parameters $\underline{\theta}$ estimated from the model

$$Y_{t+b} = \underline{x}^T(t)\underline{\theta} + \varepsilon_{t+b} \qquad \text{(A15)}$$

have converged. The least squares solution for $\underline{\theta}$ gives (Appendix B)

$$(\underline{X}_t^T \underline{X}_t)\hat{\underline{\theta}} = \underline{X}_t^T \underline{Y}_{t+b} \qquad \text{(A16)}$$

If $\hat{\underline{\theta}}$ converges so that $\underline{Y}_{t+b}$ is a moving average process of order b-1 then

$$(\underline{X}_t^T \underline{X}_t)E\{\underline{\theta}\} = 0 \qquad \text{(A17)}$$

where E{} denotes mathematical expectation. Since $\hat{\underline{\theta}} \neq \underline{0}$, $(\underline{X}_t^T \underline{X}_t)$ is not of full rank (Noble (1969)). Bohlin (1971) and Box and MacGregor (1976) have shown though, that the parameters of the minimum variance controller $(\frac{\alpha_0^0}{B_0^0}, \frac{\alpha_1^0}{B_0^0}, \ldots \frac{B_1^0}{B_0^0}, \ldots)$ are identifiable if the control is minimum variance optimal as soon as the transport delay is known. This implies that the rank of $(\underline{X}_t \; \underline{X}_t)$ is just one less than the number of parameters being estimated. (Implicit here is the assumption that the orders of $\hat{\alpha}(z^{-1})$ and $\hat{B}(z^{-1})$ have not both been overestimated, in which case $\hat{\alpha}(z^{-1})$ and $\hat{B}(z^{-1})$ would contain common factors (Section 3.2)). Thus, asymptotically only one parameter can be expressed as a linear combination of the remaining ones.

If $\underline{Y}_{t+b}$ is not a moving average process of order b-1, and the parameters $\hat{\underline{\theta}}$ converge to some values, then it is easily seen that at least one parameter can be expressed as a linear combination of the remaining ones and this is all that can be said.

It is seen that asymptotically, one parameter may always be expressed as a linear combination of the remaining ones. In practice though, all the parameters of the model (3.11) can be uniquely identified the feedback controller based on the parameter estimates is a nonlinear, time varying function of the input and output sequence.

# APPENDIX B

# CONSISTENCY AND EFFICIENCY OF THE LEAST SQUARES ESTIMATES

## B1 Introduction

Statistical aspects of the least squares estimation scheme will be examined in this Appendix. Consistency of the controller parameters will be examined, and there will be a brief discussion concerning the efficiency of the least squares estimates.

## B2 Parameter Consistency

The process dynamics and disturbances are described by a model of the form

$$Y_{t+b} = \frac{\omega(z^{-1})}{\delta(z^{-1})} U_t + \frac{\theta(z^{-1})}{\nabla^d \phi(z^{-1})} a_{t+b} \tag{B1}$$

where the $\{a_t\text{'s}\}$ are a sequence of normally distributed mean zero, variance $\sigma_a^2$ random variables. They have covariance structure

$$E\{a_t\, a_{t+j}\} = 0,\ j \neq 0 \tag{B2}$$

$$= \sigma_a^2,\ j = 0 \tag{B3}$$

The output of the process may be written as

$$Y_{t+b} = (\alpha^0(z^{-1})Y_t + B^0(z^{-1})\nabla^d U_t)(1 + \xi_1^0 z^{-1} + \ldots) + L_4(z^{-1})a_{t+b} \qquad \text{(B4)}$$

where

$$\alpha^0(z^{-1}) = \delta(z^{-1})L_2(z^{-1}) \qquad \text{(B5)}$$

$$B^0(z^{-1}) = \omega(z^{-1})L_4(z^{-1})\phi(z^{-1}) \qquad \text{(B6)}$$

$$\xi^0(z^{-1}) = \delta^{-1}(z^{-1})\theta^{-1}(z^{-1}) \qquad \text{(B7)}$$

$\alpha^0(z^{-1})$ is the order mo and $B(z^{-1})$ is of order $\ell o$. $L_2(z^{-1})$ and $L_4(z^{-1})$ are defined from

$$\frac{\theta(z^{-1})}{\nabla^d\phi(z^{-1})} = L_4(z^{-1}) + \frac{L_2(z^{-1})}{\nabla^d\phi(z^{-1})} \qquad \text{(B8)}$$

The total history of the process to time t, may be expressed compactly as

$$\underline{Y}_{t+b} = (\underline{X}_t\ \underline{\theta}^0)(1 + \xi_1^0 z^{-1} + \ldots) + \underline{\varepsilon}^0_{t+b} \qquad \text{(B9)}$$

where

$$\underline{Y}_{t+b} = (Y_{t+b}\ Y_{t+b-1} \ldots)^T \qquad \text{(B10)}$$

and $\underline{\theta}^0$ is the vector of minimum variance controller parameters, i.e.

$$\underline{\theta}^0 = (\alpha_0^0, \alpha_1^0 \ldots \alpha_{mo}^0, B_0^0, B_1^0 \ldots B_{\ell o}^0)^T \qquad \text{(B11)}$$

Åström and Wittenmark in the self-tuning regulator problem suggest that one try to estimate the minimum variance controller parameters $\underline{\theta}^0$ from a model of the form

$$Y_{t+b} = \alpha(z^{-1})Y_t + B(z^{-1})\nabla^d U_t + \epsilon_{t+b} \qquad \text{(B12)}$$

If the orders of $\alpha(z^{-1})$ and $B(z^{-1})$ are correct, then the total history of the model output to time t may be written as

$$\underline{Y}_{t+b} = \underline{X}_t\ \underline{\theta}(t) + \underline{\varepsilon}_{t+b} \qquad \text{(B13)}$$

where

$$\underline{\theta}(t) = (\alpha_0,\ \alpha_1 \ldots.\ \alpha_{mo},\ B_0,\ B_1 \ldots\ B_{\ell o})^T \qquad \text{(B14)}$$

It is assumed in this analysis that the correct orders of $\alpha(z^{-1})$ and $B(z^{-1})$ are known. The parameters of (B13) are estimated by least squares. The estimates $\hat{\underline{\theta}}(t)$ are given by (Kendall and Stuart (1966))

$$\hat{\underline{\theta}}(t) = (X_t^T\ X_t)^{-1}\ X_t^T\ \underline{Y}_{t+b} \qquad \text{(B15)}$$

$(X_t^T\ X_t)$ must be nonsingular and this requires that the implemented controller be time varying or nonlinear. The estimates $\hat{\underline{\theta}}(t)$ are unbiased if

$$E\{\hat{\underline{\theta}}(t)\} = \underline{\theta}^0 \qquad \text{(B16)}$$

where E{ } denotes mathematical expectation. Substitution of (B15) into (B16) gives

$$E\{\underline{\theta}(t)\} = E\{(X_t^T X_t)^{-1} X_t^T \underline{Y}_{t+b}\} \tag{B17}$$

In most regression analysis, the elements of $X$ are related to the settings of independent variables, i.e. pressure or temperature. These are assumed to be fixed quantities. The expectation operator in (B17) would then pass through these valves as $E\{kY\} = kE\{Y\}$ where k is a constant. Now, the elements of $X$ are functions of the random variables $\{Y_t, Y_{t-1}, \ldots \nabla^d U_t, \nabla^d U_{t-1} \ldots\}$. One must take a conditional expectation of (B17). It is assumed that at any time t, that $Y_{t+b}$ is a random variable and that $\{Y_t, Y_{t-1}, \ldots \nabla^d U_t, \nabla^d U_{t-1} \ldots\}$ are fixed since they have occurred. Thus (B19) is written as

$$E\{\hat{\underline{\theta}}(t)\} = (X_t^T X_t)^{-1} X_t^T E\{\underline{Y}_{t+b}\} \tag{B18}$$

When the output of the process (B13) is substituted for $\underline{Y}_{t+b}$, then

$$E\{\hat{\underline{\theta}}(t)\} = (X_t^T X_t)^{-1} X_t^T E\{X_t \underline{\theta}_0 (1 + \xi_1^0 z^{-1} + \ldots)$$

$$+ \underline{\varepsilon}^0_{t+b} \tag{B19}$$

$\varepsilon^0_{t+b}$ is a moving average process of order b-1

$$\varepsilon^0_{t+b} = (1 + \psi_1 z^{-1} + \ldots + \psi_{b-1} z^{-b+1}) a_{t+b} \tag{B20}$$

$$= L_4(z^{-1}) a_{t+b} \tag{B21}$$

$\varepsilon^0_{t+b}$ is normally distributed with mean zero, variance $(1 + \psi_1^2 + \ldots + \psi_{b-1}^2)\sigma_a^2$ and covariance structure

$$\mathrm{cov}\{\varepsilon^0_t \varepsilon^0_{t+k}\} = \sigma_a^2 \sum_{j=k}^{b-1} \psi_j \psi_{b+1-j}, \quad 0 \leq k \leq b-1$$

$$= 0 \quad , \quad k \geq b \tag{B22}$$

where $\psi_0 = 1.0$. A time t+b, $\varepsilon^0_{t+b}$ is a random variable with expectation equal to zero, and (B19) may be written as

$$E\{\hat{\underline{\theta}}(t)\} = (X_t^T X_t)^{-1} X_t^T \{X_t \underline{\theta}^0 (1 + \xi_1^0 z^{-1} + \ldots)\}$$

$$= \underline{\theta}^0 + (X_t^T X_t)^{-1} X_t [\xi_1^0 X_{t-1} \underline{\theta}^0$$

$$+ \xi_2^0 X_{t-2} \underline{\theta}^0 + \ldots] \tag{B24}$$

If $\delta(z^{-1})\theta(z^{-1}) = 1.0$ then $E\{\hat{\underline{\theta}}(t)\} = \underline{\theta}^0$, and the estimates of $\hat{\underline{\theta}}(t)$ are <u>always</u> unbiased if the correct model structure is chosen. If $\delta(z^{-1})\theta(z^{-1}) \neq 1.0$, as would be expected in most cases, then if the implemented controller is of the minimum variance form, Åström and Wittenmark (1973) showed that the estimates of $\underline{\theta}(t)$ will be consistent (asymptotically unbiased) if the estimates converge. The term in the square brackets of (B24) is equal to zero by virtue of the fact that for all time (t-j), j ≥ 1, the self-tuning regulator has set $X_{t-j} \hat{\underline{\theta}}(t) = X_{t-j} \underline{\theta}^0 = \underline{0}$.

This analysis has some implications for startup situations. If during startup of the self-tuning algorithm, the parameters of (B13) are identified, and an existing controller, i.e. PID, is used to compute the control signal, one will not in general be estimating the minimum variance controller parameters. To prevent this initial unbiasedness from influencing future estimates, the identification stage should be kept as short as possible. The use of a discounting factor ($\lambda$) less than one will also insure that future estimates are not unduly influenced from data which occurred when the parameter estimates were far removed from their optimal valves.

## B3 Efficiency of the Least Squares Estimates

The most efficient estimate of a parameter may be defined as that one which has the smallest variance, (Kendall and Stuart (1966)). The variance-covariance matrix of the controller parameter estimates is given by

$$\mathrm{cov}\{\hat{\theta}(t)\} = E\{(\underline{\hat{\theta}}(t) - E\{\underline{\hat{\theta}}(t)\})\ (\underline{\hat{\theta}}(t) - E\{\underline{\hat{\theta}}(t)\})^T \qquad \text{(B25)}$$

Substituting (B15) and B17) into (B25) then

$$\mathrm{cov}\{\theta(t)\} = (X_t^T X_t)^{-1} X_t^T E\{(\underline{Y}_{t+b} - E(\underline{Y}_{t+b}))$$

$$(\underline{Y}_{t+b} - E(\underline{Y}_{t+b}))^T \cdot X_t (X_t^T X_t)^{-1} \qquad \text{(B26)}$$

If the parameter estimates $\underline{\theta}(t)$ have converged to the minimum variance controller parameters, then asymptotically

$$E\{(\underline{Y}_{t+b} - E\{\underline{Y}_{t+b}\})(\underline{Y}_{t+b} - E\{\underline{Y}_{t+b}\})^T$$

$$= E\{\underline{\varepsilon}^0_{t+b}\,\underline{\varepsilon}^{0T}_{t+b}\} \tag{B27}$$

$\varepsilon_{t+b}$ is a moving average process of order b-1, with covariance structure (B22). Thus (B27) may be written as

$$E\,\underline{\varepsilon}^0_{t+b}\,\underline{\varepsilon}^{0T}_{t+b} = \underline{M} \tag{B28}$$

where the elements $M_{ij}$ are given by

$$M_{ij} = \alpha_a^2 \sum_{k=|i-j|}^{b-1} \psi_{|i-j|}\,\psi_{b+1-|i+j|} \qquad 0 \leq |i+j| \leq b-1 \tag{B29}$$

$$= 0 \qquad b \geq |i-j| \tag{B30}$$

The variance-covariance matrix of the parameter estimates is therefore given by

$$\mathrm{cov}\{\underline{\hat{\theta}}(t)\} = (\underline{X}_t^T\,\underline{X}_t)^{-1}\,\underline{X}_t^T\,\underline{M}\,\underline{X}_t(\underline{X}_t^T\,\underline{X}_t)^{-1} \tag{B31}$$

when b=1 (B31) reduces to

$$\mathrm{cov}\{\underline{\hat{\theta}}(t)\} = (\underline{X}_t^T\,\underline{X}_t)^{-1}\,\upsilon_a^2 \tag{B32}$$

the usual least squares result, and the correlation matrix of the parameters is given by $\underline{\Gamma}(t)$, the elements of which are

$$\Gamma_{ij} = \frac{c_{ij}}{\sqrt{c_{ii}\, c_{jj}}} \tag{B33}$$

The $c_{ij}$'s are the elements (ij) of $(\underline{X}_t^T \underline{X}_t)^{-1}$. Since one is not identifying the moving average parameters it is impossible to determine the true variance-covariance matrix of the parameters of $b > 1$.

The least squares estimates of the controller parameters may not be the most efficient estimates. If $b > 1$ information about the controller parameters is contained in $\epsilon^0_{t+b}$. It may be possible to reduce the variance of the estimates by simultaneous identification of the controller parameters and the moving average parameters $L_4(z^{-1})$ by say a recursive maximum likelihood procedure (Soderström et al. (1975)).

# APPENDIX C

## CONSTRAINED CONTROL PROCEDURE OF CLARKE AND GAWTHROP (1975)

The self-tuning controller of Clark and Gawthrop (1975) is rederived in this Appendix. It is shown that if the controllers parameters converge then the self-tuning controller will minimize

$$\hat{Y}^2_{t+b/t} + \xi'(\nabla^d U_t)^2 \qquad \text{(C1)}$$

where $\hat{Y}_{t+b/t}$ is the b-step ahead forecast of the output and $\xi' \geq 0$, and that the same strategy will minimize

$$E\{\phi^2_{t+b}\} = E\{(Y_{t+b} + \xi\nabla^d U_t)^2\} \qquad \text{(C2)}$$

It will not minimize the cost function claimed by Clarke and Gawthrop (1975), namely

$$\text{var } Y_t + \xi'' \text{ var } \nabla^d U_t \qquad \text{(C3)}$$

It is important to demonstrate that minimization of (C1) is equivalent to minimization of (C2) because this forms the basis of the development of the self-tuning controller to minimize (C2) and because (C2) by itself does not appear to be a sensible criterion.

Consider the representation of the closed-loop system in Figure 2.4

using the notation of Box and Jenkins (1970)

$$\delta(z^{-1})\theta(z^{-1})\{Y_{t+b} + L_4(z^{-1})a_{t+b}\} =$$

$$\delta(z^{-1})L_2(z^{-1})Y_t + \omega(z^{-1})L_4(z^{-1})\phi(z^{-1})\nabla^d U_t \quad \text{(C4)}$$

(C4) may be written as

$$\delta(z^{-1})\theta(z^{-1})\{Y_{t+b} + \xi\nabla^d U_t\} = \delta(z^{-1})L_2(z^{-1})Y_t$$

$$+ \{\omega(z^{-1})L_4(z^{-1})\phi(z^{-1}) + \xi\delta(z^{-1})\theta(z^{-1})\}\nabla^d U_t$$

$$+ \delta(z^{-1})\theta(z^{-1})L_4(z^{-1})a_{t+b} \quad \text{(C5)}$$

or

$$\phi_{t+b} = \frac{\alpha_0^{0'}(z^{-1})}{\delta(z^{-1})\theta(z^{-1})}Y_t + \frac{B^{0'}(z^{-1})}{\delta(z^{-1})\theta(z^{-1})}\nabla^d U_t$$

$$+ \varepsilon^0_{t+b} \quad \text{(C6)}$$

where

$$\varepsilon^0_{t+b} = L_4(z^{-1})a_{t+b} \quad \text{(C7)}$$

The control strategy minimizing $E\{\phi^2_{t+b}\}$ is obtained as follows

$$\phi^2_{t+b} = [\frac{B^{0'}(z^{-1})\nabla^d U_t + \alpha^{0'}(z^{-1})Y_t}{\delta(z^{-1})\theta(z^{-1})}]$$

$$+ [\frac{B^{0'}(z^{-1})\nabla^d U_t + \alpha^{0'}(z^{-1})Y_t}{\delta(z^{-1})\theta(z^{-1})}]$$

$$L_4(z^{-1})a_{t+b} + [L_4(z^{-1})a_{t+b}]^2 \quad \text{(C8)}$$

Taking mathematical expectations of (C8)

$$E\{\phi^2_{t+b}\} = E\{[\frac{B^{0'}(z^{-1})\nabla^d U_t + \alpha^{0'}(z^{-1})Y_t}{\delta(z^{-1})\phi(z^{-1})}]^2\}$$

$$+ (1 + L^2_{4,1} + L^2_{4,2} + \ldots + L^2_{4,b-1})\sigma^2_a \qquad \text{(C9)}$$

where $\sigma^2_a$ is the variance of $\{a_t\text{'s}\}$. They are distributed $N(0, {}^2_a)$ with covariance structure

$$E\{a_t\ a_{t+j}\} = 0, \quad j > 0$$
$$= \sigma^2_a, \quad j = 0 \qquad \text{(C10)}$$

The cross product between $[B^{0'}(z^{-1})\nabla^d U_t + \alpha^{0'}(z^{-1})Y_t]$ and $L_4(z^{-1})a_{t+b}$ vanishes as the latter is the b step ahead forecast error which is completely uncorrelated with all information at time t. The variance of the forecast error $(1 + L^2_{4,1} + L^2_{4,2} + \ldots + L^2_{4,b-1})\sigma^2_a$ is fixed, independent of any control action that may be taken. The expression

$$E\{[\frac{B^{0'}(z^{-1})\nabla^d U_t + \alpha^{0'}(z^{-1})Y_t}{\delta(z^{-1})\theta(z^{-1})}]^2\} \qquad \text{(C11)}$$

is greater than or equal to zero and the control action

$$\nabla^d U_t = \frac{-\alpha^{0'}(z^{-1})}{B^{0'}(z^{-1})} Y_t = \frac{-\delta(z^{-1})L_2(z^{-1})}{\omega(z^{-1})L_4(z^{-1})\phi(z^{-1})+\xi\delta(z^{-1})\theta(z^{-1})} Y_t$$

(C12)

will minimize (C8).

If the parameters of the controller (C12) are unknown, Clarke proposes that the parameters of the model

$$\phi_{t+b} = B'(z^{-1})\nabla^d U_t + \alpha'(z^{-1})Y_t + \varepsilon_{t+b} \qquad \text{(C13)}$$

be estimated at every sampling interval and used in the controller

$$\nabla^d U_t = \frac{\alpha'(z^{-1})}{B'(z^{-1})} Y_t \qquad \text{(C14)}$$

as if they were exactly known. If the controller (C14) is optimal in the sense of minimizing $E\{\phi^2_{t+b}\}$ then

$$E\{\phi(t)\phi(t+\tau)\} = 0, \ \tau \geq b \qquad \text{(C15)}$$

The resulting controller is the same one that could have been designed had the process dynamic and stochastic models been known. This is not shown formally by Clarke and Gawthrop (1975), but analogies are made to the unconstrained self-tuning regulator since equations (C6) and (C13) are duals of (3.5) and (3.11).

On first glance the cost function to be minimized, $E\{(Y_{t+b} + \xi\nabla^d U_t)^2\}$ does not appear reasonable. Expansion of (C2) gives

$$E\{\phi^2_{t+b}\} = \text{var } Y_{t+b} + \xi^2 \text{var } \nabla^d U_t + 2 \text{ cov } (Y_{t+b} \nabla^d U_t) \qquad \text{(C16)}$$

There is no reason in general why the design criterion for the controller should involve a covariance between the input and output.

Consider for the moment the development of a strategy for the minimization of an alternate cost function. The process dynamic and stochastic models may be written in the form

$$Y_{t+b} = \frac{\omega(z^{-1})}{\delta(z^{-1})\nabla^d} \nabla^d U_t + \hat{N}_{t+b/t} + \varepsilon^0_{t+b} \qquad \text{(C17)}$$

$$= \hat{Y}_{t+b/t} + \varepsilon^0_{t+b} \qquad \text{(C18)}$$

Here again $\varepsilon^0_{t+b}$ is the minimum variance forecast ~~error of~~ $Y_{t+b}$ and it's completely uncorrelated with all information up to time t. $\hat{Y}_{t+b/t}$ is the minimum variance forecast of $Y_{t+b}$ based solely on information up to time t. Squaring (C18) and adding $\xi' U^2_t$ to both sides, then

$$\hat{Y}^2_{t+b} + \xi'(\nabla^d U_t)^2 = \hat{Y}_{t+b/t} + \xi'(\nabla^d U_t)^2 + \hat{Y}_{t+b/t}$$

$$\cdot\, \varepsilon^0_{t+b} + (\varepsilon^0_{t+b})^2 \qquad \text{(C19)}$$

Taking mathematical expectations of (C19) remembering that $\varepsilon^0_{t+b}$ is uncorrelated with $(Y_t, Y_{t-1}, \ldots, \nabla^d U_t, \nabla^d U_{t-1}, \ldots)$ then one gets

$$\text{var } Y_{t+b} + \xi \text{var } \nabla^d U_t = E(\hat{Y}_{t+b} + \xi'(\nabla^d U_t)^2 + \sigma^2_{\varepsilon 0} \qquad \text{(C20)}$$

where $\sigma^2_{\varepsilon 0}$ is the variance of the forecast error. Clarke and Gawthrop (1975) differentiated (C20) with respect to $\nabla^d U_t$ and set the result to zero, to find the control strategy which they thought minimized var $Y_{t+b}$ + $\xi'$ var $\nabla^d U_t$. However, they ignored the expectation operator when they took this derivative, and as shown by MacGregor and Tidwell (1976), minimize the objective function

$$\hat{Y}^2_{t+b/t} + \xi'(\nabla^d U_t)^2 \quad \text{(C3)}$$

MacGregor and Tidwell refer to this as an 'instaneous' or 'shortsighted' optimal controller as it does not take into account the effect of the control action on the output at lead times greater than b, whereas the Weiner-Hopf solution, which minimizes var $Y_t$ + $\xi''$ var $\nabla^d U_t$ accounts for this. Consequently the increase in the variance of $Y_t$, for a given reduction in the variance of $\nabla^d U_t$, will be larger if the controller is designed by Clarke's algorithm (Refer to Section 2.5 for an example).

(C1) may be written as

$$\hat{Y}^2_{t+b} + \xi'(\nabla^d U_t)^2 = \left(\frac{\omega(z^{-1})}{\delta(z^{-1})\nabla^d}\nabla^d U_t + \frac{L_2(z^{-1})}{\phi(z^{-1})\nabla^d}a_t\right)^2 + \xi'(\nabla^d U_t)^2 \quad \text{(C21)}$$

Taking derivatives with respect to $\nabla^d U_t$ and setting the result to zero then

$$\left(\frac{\omega(z^{-1})}{\delta(z^{-1})\nabla^d}\nabla^d U_t + \frac{L_2(z^{-1})a_t}{\phi(z^{-1})\nabla^d}\right)\left(\omega_0 + \frac{d}{d\nabla^d U_t}\left[\frac{L_2(z^{-1})}{\phi(z^{-1})\nabla^d}a_t\right]\right) + \xi'\nabla^d U_t = 0 \quad \text{(C22)}$$

$\frac{L_2(z^{-1})}{\phi(z^{-1})\nabla^d}$ is the b step ahead forecast of $Y_{t+b}$ given only the information at time t, and it is not a function of $\nabla^d U_t$.

The control action minimizing (C1) is then

$$\nabla^d U_t = - \frac{\dfrac{L_2(z^{-1})}{\phi(z^{-1})\nabla^d}}{\dfrac{\omega(z^{-1})}{\delta(z^{-1})\phi^d} + \dfrac{\xi'}{\omega_0}} a_t \qquad \text{(C23)}$$

Expressing $a_t$ in terms of $Y_t$ by substituting the control action (C23) into

$$Y_t = \frac{\omega(z^{-1})}{\delta(z^{-1})\nabla^d} \nabla^d U_t + [L_4(z^{-1}) + z^{-b} L_3(z^{-1})] a_t \qquad \text{(C24)}$$

and then substituting that result back into (C23) gives

$$\nabla^d U_t = \frac{-L_3(z^{-1})}{L_4(z^{-1})} \frac{1}{\dfrac{\omega(z^{-1})}{\nabla^d \delta(z^{-1})} + \dfrac{\xi'}{\omega_0} (1+z^{-b} \dfrac{L_3(z-1)}{L_4(z^{-1})}} Y_t \qquad \text{(C25)}$$

Substituting

$$L_3(z^{-1}) = \frac{L_2(z^{-1})}{\phi(z^{-1})\nabla^d} \qquad \text{(C26)}$$

then (C25) may be written as

$$\nabla^d U_t = - \frac{\delta(z^{-1}) L_2(z^{-1})}{\omega(z^{-1}) L_4(z^{-1}) \phi(z^{-1}) + \dfrac{\xi'}{\omega_0} \delta(z^{-1}) \phi(z^{-1})} Y_t \qquad \text{(C27)}$$

Comparing (C12) and (C30) it is seen that minimization of $E\{(Y_{t+b} + \xi\nabla^d U_t)^2\}$ is equivalent to minimization of $\hat{Y}^2_{t+b} + \frac{\xi}{\omega_0}(\nabla^d U_t)^2$.

If the disturbance is nonstationary it is necessary to minimize $\hat{Y}^2_{t+b} + \xi(\nabla^d U_t)^2$ where $d > 0$. The variance of $U_t$ is theoretically infinite if $d > 0$ due to the pole of order $d$ on the unit circle in the controller. Controllers designed to minimize $\hat{Y}^2_{t+b/t} + \xi U^2_t$ or $E\{(Y_{t+b} + \frac{\xi}{\omega_0} U_t)^2\}$ will not in general stabilize the variance of $Y_t$. The results in this Appendix are corrected to account for this.

The cost function $\hat{Y}^2_{t+b/t} + \xi(\nabla^d U_t)^2$ that Clarke and Gathrop (1975) minimizes, leads to useful controller designs. Their self-tuning controller is a clever extension of the basic self-tuning algorithm and provides an easy means of constraining the magnitude of the control action.

## APPENDIX D

## METHODS OF CONVERGENCE ANALYSIS

In this Appendix the ordinary differential equations, which may describe the expected trajectories of the recursively estimated parameters, will be examined. A simple one parameter example is solved, and it is shown that the complexity of the differential equations increases rapidly if more parameters are to be estimated.

It is assumed that the process may be described by a model of the form

$$Y_{t+b} = \frac{\bar{B}(z^{-1})}{A(z^{-1})} U_t + \frac{C(z^{-1})}{A(z^{-1})} a_{t+b} \tag{D1}$$

The parameters of the minimum variance controller are estimated from a model of the form

$$Y_{t+b} = \underline{x}^T(t)\underline{\theta} + \varepsilon_{t+b} \tag{D2}$$

Under weak conditions (Ljung and Wittenmark (1974)), the ordinary differential equations

$$\frac{d\underline{\theta}}{d\tau} = \underline{R}(\tau)\underline{f}(\underline{\theta}) \tag{D3}$$

$$\frac{d\underline{R}(\tau)}{d\tau} = \underline{R}(\tau) - \underline{R}(\tau)\ \underline{G}(\theta)\ \underline{R}(\tau) \qquad \text{(D4)}$$

may describe the expected trajectories of $\hat{\theta}(t)$, where

$$\underline{f}(\underline{\theta}) = E\ \{\underline{x}(t)(Y_{t+b} - \underline{x}^T(t)\underline{\theta})\} \qquad \text{(D5)}$$

and

$$\underline{G}(\underline{\theta}) = E\{\underline{x}(t)\ \underline{x}^T(t)\} \qquad \text{(D6)}$$

$\tau$ is related to t by

$$\tau \approx \ell n(t) \qquad \text{(D7)}$$

if the discounting factor ($\lambda$) is one. If the vector of minimum variance controller parameters, $\underline{\theta}^0$, is a globally asymptotic stationary solution to (D3) then (Ljung and Wittenmark (1974))

$$\lim_{t\to\infty} \underline{\theta}(t) \to \hat{\underline{\theta}}^0 \quad \text{with probability one} \qquad \text{(D8)}$$

Ljung and Wittenmark examine quantitatively under what conditions $\underline{\theta}^0$ may or may not be a globally asumptotic solution to (D3). Their results may be described briefly as follows.

$\underline{R}(\tau)$ is taken as the unit matrix. The differential equation (D3) is linearized about $\underline{\theta}^0$ to give

$$\frac{d\underline{\theta}}{d\tau} = \underline{f}(\underline{\theta}^0) + (\underline{\theta} - \underline{\theta}^0) \left. \frac{d\underline{f}(\underline{\theta})}{d\underline{\theta}} \right|_{\underline{\theta}^0} \quad \text{(D9)}$$

From (D5) $\underline{f}(\underline{\theta}^0) = \underline{0}$. (D9) may be written as

$$\frac{d(\underline{\theta} - \underline{\theta}^0)}{d\tau} = (\underline{\theta} - \underline{\theta}^0)\underline{M} \quad \text{(D10)}$$

where

$$\underline{M} = \left. \frac{d\underline{f}(\underline{\theta})}{d\underline{\theta}} \right|_{\underline{\theta}^0} \quad \text{(D11)}$$

$\underline{M}$ is a function of auto and cross covariances. By a judicious choice of values for $A(z^{-1})$, $\bar{B}(z^{-1})$ and $C(z^{-1})$, Ljung and Wittenmark (1974) were able to make the trace of $\underline{M}$ positive, indicating that at least one eigenvalue had a positive real part. Thus $\underline{\theta}^0$ was not a globally asumptotic solution to (D3) and the self-tuning regulator should not converge. This was verified with a similation.

As an example of the solution of the simultaneous ordinary differential equations (D3) and (D4), consider the system

$$Y_{t+b} = \frac{b}{1+az^{-1}} U_t + \frac{1+cz^{-1}}{1+az^{-1}} a_{t+1} \quad \text{(D12)}$$

This example is discussed in Wittenmark (1973). The minimum variance controller is

$$U_t = \frac{a-c}{b} Y_t \quad \text{(D13)}$$

The parameters of the minimum variance controller are estimated from the model

$$Y_{t+1} = \alpha Y_t + B_0 U_t + \varepsilon_{t+1} \tag{D14}$$

To reduce the complexity of the differential equations consider $B_0$ fixed to 1.0. Then

$$\underline{f}(\theta) = E\{Y_t \, Y_{t+1}\} = \nu_{yy}(1) \tag{D15}$$

and

$$\underline{G}(\theta) = E\{Y_t \, Y_t\} = \nu_{yy}(0) \tag{D16}$$

where the auto covariances are computed for the closed loop system

$$Y_{t+1} = \frac{-b}{1+az^{-1}} \cdot \hat{\alpha} Y_t + \frac{1+cz^{-1}}{1+az^{-1}} a_{t+1} \tag{D17}$$

or

$$Y_t = \frac{1+cz^{-1}}{1+(a+b\alpha)z^{-1}} a_t \tag{D18}$$

If $Y_t = H(z^{-1})a_t$, then the auto correlation at lag k is given by (Aström (1970))

$$\nu_{yy}(k) = \frac{\sigma_a^2}{2\pi i} \oint z^k H(z) H(z^{-1}) \frac{dz}{z} \tag{D19}$$

The path of integration is in the positive direction. Letting d = a+b then

$$\nu_{yy}(1) = \frac{\sigma_a^2}{2\pi i} \oint \frac{(1+cz^{-1})(1+cz)}{(1+dz^{-1})(1+dz)} \, dz \tag{D20}$$

$$= \frac{\sigma_a^2}{2\pi i} \oint \frac{(z+c)(1+cz)}{(z+d)(1+dz)} dz \qquad \text{(D21)}$$

Letting $\sigma_a^2 = 1.0$, (D21) is evaluated, using residue calculus (Jenson and Jeffreys (1963)) as,

$$\nu_{yy}(1) = \frac{(z+c)(1+cz)}{1+dz}\bigg|_{z=-d} \qquad \text{(D22)}$$

$$\nu_{yy}(1) = \frac{[c-(a+b\hat{\alpha})]\ [1-c(a+b\hat{\alpha})]}{1+(a+b\hat{\alpha})^2} \qquad \text{(D23)}$$

In a similar fashion $\nu_{yy}(0)$ is evaluated as

$$\nu_{yy}(0) = 1 + \frac{[c-(a+b\hat{\alpha})]^2}{1-(a+b\hat{\alpha})^2} \qquad \text{(D24)}$$

The ordinary differential equations (D3) and (D4) become

$$\frac{d\hat{\alpha}}{d\theta} = R \cdot \frac{[c-(a+b\hat{\alpha})]\ [1-c(a+b\hat{\alpha})]}{1-(a+b\hat{\alpha})^2} \qquad \text{(D25)}$$

$$\frac{dR}{d\tau} = R - R^2 \cdot \frac{(1+[c-(a+b\hat{\alpha})]^2)}{1-(a+b\hat{\alpha})^2} \qquad \text{(D26)}$$

These simultaneous nonlinear differential equations were solved with a fourth order Runge-Kutta method and an initial step size of .001. a, b and c were taken as (-0.95, 1.0, -0.45). The minimum variance control is (D13).

$$U_t = -0.50\, Y_t \qquad \text{(D27)}$$

Figure D1 shows the trajectories of $\hat{\alpha}$ for R(0) = 5.0 and different starting values of $\hat{\alpha}(0)$.

FIGURE D1: Trajectories of $\hat{\alpha}$ for R(0)=5.0 and $\hat{\alpha}(0)$=0.5 and -1.0

The differential equations can be readily solved when only a few parameters are to be estimated. The next example shows the difficulties encountered when several parameters are to be estimated. Let the system be

$$Y_{t+1} = \frac{b_0+b_1z^{-1}}{1 + az^{-1}} U_t + \frac{1 + cz^{-1}}{1 + az^{-1}} a_{t+1} \tag{D28}$$

The minimum variance controller for (D28) is

$$U_t = \frac{(a-c)}{b_0+b_1z^{-1}} Y_t \tag{D29}$$

The controller parameters are estimated from the model

$$Y_{t+1} = \alpha Y_t + B_0U_t + B_1U_{t-1} + \varepsilon_{t+1} \tag{D30}$$

Considering $B_0$ as a parameter to be estimated $\underline{f}(\underline{\theta})$ is

$$\underline{f}(\underline{\theta}) = E\{\underline{x}(t)Y_{t+1}\} = (\nu_{yy}(1) \;\; \nu_{uy}(1) \;\; \nu_{uy}(2))^T \tag{D31}$$

and

$$\underline{G}(\underline{\theta}) = E\{\underline{x}(t)\underline{x}^{T}(t)\} = \begin{pmatrix} \nu_{yy}(0) & \nu_{uy}(0) & \nu_{uy}(-1) \\ \nu_{uy}(0) & \nu_{uu}(0) & \nu_{uu}(1) \\ \nu_{uy}(-1) & \nu_{uu}(1) & \nu_{uu}(0) \end{pmatrix} \tag{D32}$$

The difficulty in solving the differential equations lies in evaluating the auto and cross covariances. $Y_t$ and $U_t$ are high order ARIMA time series. It is not feasible to evaluate the auto covariances by residue calculus, but Aström (1970) details a method for finding auto covariances which are of the form

$$\nu_{yy}(0) = \frac{\sigma_a^2}{2\pi i} \oint \frac{P(z)P(z^{-1})}{Q(z)Q(z^{-1})} \cdot \frac{dz}{z} \qquad \text{(D33)}$$

Crowe (1976) has extended this method for auto covariances at lag k, showing that

$$\nu_{yy}(k) = \frac{\sigma_a^2}{4\pi i} \left( \oint \frac{P_1(z)P_1(z^{-1})}{Q(z)Q(z^{-1})} \cdot \frac{dz}{z} - \oint \frac{P_2(z)P_2(z^{-1})}{Q(z)Q(z^{-1})} \cdot \frac{dz}{z} \right) \qquad \text{(B34)}$$

where

$$P_1(z) = P(z)\ (\cos(\tfrac{5\pi}{4})\ z^k + \cos(\tfrac{3\pi}{4})) \qquad \text{(D35)}$$

and

$$P_2(z) = P(z)(\sin(\tfrac{5\pi}{4})z^k + \sin(\tfrac{3\pi}{4})) \qquad \text{(D36)}$$

if

$$Y_t = H_y(z^{-1})a_t \qquad \text{(D37)}$$

and

$$U_t = H_u(z^{-1})a_t \qquad \text{(D38)}$$

then it can be easily shown that the covariance between $Y_t$ and $U_t$ is given by

$$\nu_{uy}(k) = \frac{\sigma_a^2}{2\pi i} \oint z^k H_y(z^{-1})H_u(z^{-1}) \frac{dz}{z} \qquad \text{(D39)}$$

when k=o Aström's solution is easily modified so that these cross co-variances can be reasily calculated. The cross covariances at lag k,

$k \neq 0$ are not easily evaluated using these methods. Alternatively, for low order auto regressive-moving average processes, the auto and cross correlations $\nu_{uu}(k)$ and $\nu_{uy}(k)$ may be evaluated by solving a system of simultaneous linear equations. This is detailed in Watts and MacCormick (1970).

For processes involving estimation of more than one or two parameters, simulation will probably remain the tool for analysis of convergence. This is not to say that the differential equations are of no use. Ljung and Wittenmark (1974), Wittenmark (1973) and Aström and Wittenmark (1973) have used these differential equations in simple cases to examine, the effect of fixing $\hat{B}_0$ on stability of the closed loop, and convergence points to (D3) if the number of whole periods of delay for the model (D2) is different from the true process delay. Convergence of the estimated parameters to those of the minimum variance controller has been shown not to be assumed. Examination of the differential equations has provided insights that would not be apparent if simulation were the sole tool of analysis.

# APPENDIX E

## Calling Sequence for the Self-Tuning Regulator Algorithm

IN THIS APPENDIX THE CALLING ALGORITHM FOR THE SELF-TUNING REGULATOR IS OUTLINED A BRIEF DESCRIPTION OF THE SUBROUTINES IS FOLLOWED BY A SAMPLE CALLING ROUTINE WITH LISTINGS OF THE SUBROUTINES

SUBROUTINE UYINCR

PURPOSE TO KEEP TRACK OF THE PAST 10 VALUES OF THE MANIPULATED AND CONTROLLED DEVIATION VARIABLES, (U-USSP) IN THE VECTOR US10 AND (Y-YSP) IN THE VECTOR YS10

USAGE CALL UYINCR(Y, YSP, US10, YS10)

SUBROUTINE FORM

PURPOSE TO COMPOSE THE VECTOR X(T-K) OF NA PAST VALUES OF (Y-YSP) AND NB PAST VALUES OF (U-USSP) IF A CONSTANT TERM IS IDENTIFIED THEN THE TOTAL NUMBER OF PARAMETERS IN THE CONTROLLER IS NP=NA+NB+1, OTHERWISE NP=NA+NB THIS VECTOR IS USED FOR THE PARAMETER UPDATING ROUTINE, IN WHICH CASE K=B (THE NUMBER OF WHOLE PERIODS OF DELAY), OR IN THE COMPUTATION OF THE CONTROL SIGNAL, FOR WHICH K=0 THE INFORMATION PLACED IN THIS VECTOR COMES FROM US10 AND YS10

USAGE CALL FORM(X, YS10(1+K), US10(1+K), NA, NB, NP)

SUBROUTINE RLS

PURPOSE TO UPDATE RECURSIVELY THE PARAMETERS, THETA, OF THE MODEL

DEV = THETA*X(T-B) + E

WHERE DEV IS THE OBJECTIVE FUNCTION

DEV = YS10(1) + ZETA*US10(B)

THE INCLUSION OF ZETA, THE CONSTRAINING FACTOR ALLOWS FOR CONSTRAINED CONTROL

USAGE CALL RLS(DEV, X, THETA, LAMBDA, P, K, S, NP)

SUBROUTINE STRCL

PURPOSE TO COMPUTE THE ABSOLUTE CONTROL SIGNAL (I E NOT THE DEVIATION SIGNAL) BASED ON THE ESTIMATED PARAMETERS

USAGE CALL STRCL(U, USSP, THETA, X, NP, NA)

```
C        THIS IS A TYPICAL CALLING SEQUENCE FOR THE USE OF THE
C        SELF-TUNING REGULATOR PROGRAMS




C
C        GET NEW OBSERVATION Y
C        UPDATE VECTORS OF INPUTS AND OUTPUTS
         CALL UYINCR(Y,YSP,US10,YS10)
C
C
         FORM X(T-B) FOR RECURSIVE LEAST SQUARES
         CALL FORM(X,YS10(1+B),US10(1+B),NA,NB,NP)
C
C
C        FORM OBJECTIVE FUNCTION FOR RECURSIVE LEAST SQUARES
         DEV=YS10(1)+ZETA*US10(1+B)
C
C
C        CALL RECURSIVE LEAST SQUARES
         CALL RLS(DEV,X,THETA,LMBDA,P,K,S,NP)
C
C
C        FORM X(T) FOR COMPUTATION OF CONTROL SIGNAL
         CALL FORM(X,YS10(1),US10(1),NA,NB,NP)
C
C
C        COMPUTE CONTROL SIGNAL
         CALL STRCL(U,USSP,THETA,X,NP,NA)
C
C
C        UPDATE US10(1)
         US10(1)=U-USSP
C
C
C        IF CONTROLLER HAS INTEGRAL ACTION CHANGE USSP
         IF(INTEGRAL ACTION)USSP=U
C
C
C        FINISHED
```

```
      SUBROUTINE RLS(DEV,X,THETA,LAMBDA,P,K,S,NP)
C
C     RECURSIVE LEAST SQUARES IDENTIFICATION ALGORITHM FROM
C     A COMPARATIVE STUDY OF RECURSIVE IDENTIFICATION
C     METHODS,PAGE 20,BY
C     SODERSTRUM,T , LJUNG, L , GUSTAFSUN, I
C     REPORT 7427,LUND INSTITUE OF TECHNOLOGY, LUND
C     SWEDEN
C
C     DEV - DEVIATION FROM TARGET
C     X - VECTOR OF INPUTS AND OUTPUTS
C     THETA - CURRENT PARAMETER ESTIMATES
C     LAMBDA - FORGETTING FACTOR
C     P - COVARIANCE MATRIX OF PARAMETERS
C     K - VECTOR OF WEIGHTING FACTORS
C     S - WORKING AREA
C     NP - # PARAMETERS TO BE ESTIMATED
C
      DIMENSION P(NP,NP),X(NP),THETA(NP),S(NP),K(NP)
      REAL K,LAMBDA
C
C     GET DENOMINATOR FOR UPDATING P AND K MATRICES
      SUM=0
      DO 1 I=1,NP
      DO 1 J=1,NP
      SUM=SUM+X(I)*X(J)*P(I,J)
    1 CONTINUE
      DEN=SUM+LAMBDA
C
C     UPDATE K MATRIX, GET PREDICTION ERROR
      ERRS=0.
      DO 3 I=1,NP
      SUM=0
      DO 2 J=1,NP
      SUM=SUM+P(I,J)*X(J)
    2 CONTINUE
      S(I)=SUM
      K(I)=SUM/DEN
      ERRS=ERRS+X(I)*THETA(I)
    3 CONTINUE
      ERRS=DEV-ERRS
C
C     UPDATE P MATRIX AND GET PARAMETER ESTIMATES
      DO 5 I=1,NP
      DO 4 J=1,NP
      P(I,J)=(P(I,J)-S(I)*S(J)/DEN)/LAMBDA
    4 CONTINUE
      THETA(I)=THETA(I)+K(I)*ERRS
    5 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE UYINCR(Y,YSP,US10,YS10)
C     THIS SUBROUTINE KEEPS TRACK OF THE LAST 10 VALUES
C     OF U-USSP AND Y-YSP IE DEVIATION VARIABLES
C     U - CURRENT VALUE OF THE CONTROL SIGNAL
C     Y - CURRENT VALUE OF THE DEPENDENT VARIABLE
C     USSP - REFERENCE VALUE FOR THE CONTROL SIGNAL
C     YSP - REFERENCE VALUE FOR THE DEPENDENT VARIABLE
C     US10(10) - STORAGE VECTOR
C     YS10(10) - STORAGE VECTOR
C
C
      DIMENSION US10(10),YS10(10)
      DO 9 J=1,9
      I=11-J
      US10(I)=US10(I-1)
      YS10(I)=YS10(I-1)
 9    CONTINUE
      US10(1)=0 0
      YS10(1)=Y-YSP
      RETURN
      END
```

```
      SUBROUTINE FORM(X,YS10,US10,NA,NB,NP)
      DIMENSION YS10(NA),US10(NB),X(NP)
      X(NP)=1 0
      DO 9 I=1,NA
      X(I)=YS10(I)
 9    CONTINUE
      DO 19 I=1,NB
      X(NA+I)=US10(I)
 19   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE STRCL(U,USSP,THETA,X,NF,NA)
      DIMENSION X(NF),THETA(NF)
      SUM=0
C     EVALUATE CONTROL SIGNAL
      DO 10 I=1,NF
      SUM=SUM+X(I)*THETA(I)
   10 CONTINUE
      U=USSP-SUM/THETA(NA+1)
      RETURN
      END
```