# HOSSZÚ'S FUNCTIONAL EQUATION

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## ABSTRACT

In this thesis we make an extensive study of the algebraic solutions of the functional equation

$$f(x+y-xy) + f(xy) = f(x) + f(y)$$

where the unknown function $f: R \to G$ maps a ring $R$ to an abelian group $G$.

After proving some general results about the solutions of the equation, we study it over rings generated by their units, over number rings, and over polynomial rings. We find that over a large class of rings, the equation is equivalent to Cauchy's functional equation, and we give ideal-theoretic criteria to specify when it is not.

Our methods involve a wide variety of techniques and results from algebra and algebraic number theory.

We complete our study with an examination of a class of functional equations which generalizes the above equation.

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## SYMBOLS NOT EXPLAINED IN THE TEXT

$\mathbb{Z}$ - the ring of rational integers

$\mathbb{R}$ - the field of real numbers

$\mathbb{C}$ - the field of complex numbers

$\mathbb{Q}$ - the field of rational numbers

$\mathbb{N}$ - the set of natural numbers

$\mathbb{F}_q$ - the finite field of $q$ elements ($q$ a prime power)

$\mathbb{Z}_n$ - the ring of rational integers modulo $n$

$\mathcal{O}_K$ - the ring of algebraic integers in the number field $K$

$K[x]$ - the polynomial ring in one indeterminate $x$ over the field $K$

The word "ideal" will always mean "two-sided ideal".

# CHAPTER 0

# INTRODUCTION

Functional equations have been studied by mathematicians ever since the theory of functions began in the eighteenth century. They appear in virtually every branch of mathematics, and every student of mathematics is no doubt familiar with the concept of a functional equation—an equation involving functions which are considered to be the unknown elements, and which must satisfy the equation for all values of the variables. Despite their frequent occurrence, the systematic study of functional equations as an organized subject in its own right began only recently.

One of the first equations to be studied extensively is the equation of additivity—Cauchy's functional equation:

$$f(x) + f(y) = f(x+y) .$$

Cauchy discovered in 1821 that the only continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of this equation are the linear functions: $f(x) = cx$, where $c \in \mathbb{R}$. (For this and other general references in this introduction, see the book by Aczél [1], which contains an extensive pre-1962 bibliography of

papers on functional equations.) Five years later, Abel solved Cauchy's equation for continuous functions on the complex numbers $f:\mathbb{C} \to \mathbb{C}$.

In 1875 Darboux improved Cauchy's result by showing that if a solution $f:\mathbb{R} \to \mathbb{R}$ of Cauchy's equation is continuous at any point, it must be continuous everywhere, and in 1880 he showed that the same conclusion follows if $f$ is non-negative (or non-positive) on any interval of the form $(0,\alpha)$ or $(-\alpha,0)$, where $\alpha>0$.

The concept of a Hamel basis (a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space) was invented by G. Hamel in 1905 to describe the general solution of Cauchy's equation for functions $f:\mathbb{R} \to \mathbb{R}$. He showed that any solution can be constructed by choosing arbitrary values for $f$ on a Hamel basis for $\mathbb{R}$, and then extending the function to all of $\mathbb{R}$ by linearity. The topology of the reals does not enter into this result—it is strictly algebraic in nature.

Pexider in 1903 investigated the following generalization of Cauchy's equation:

$$f(x+y) = g(x) + h(y) ,$$

where all three functions $f$, $g$, and $h$ are unknown. He showed that the solutions of this must satisfy

$$f(x) - f(0) = g(x) - g(0) = h(x) - h(0) ,$$

and that the function $\phi(x) = f(x) - f(0)$ must be a solution of Cauchy's equation.

If $A$ and $B$ are abelian groups, then the solutions $f:A \to B$ of Cauchy's equation are, by definition, the group homomorphisms from $A$ to $B$. So in this general setting, the study of Cauchy's equation is simply the study of the group $\mathrm{Hom}_{\mathbb{Z}}(A,B)$—an object of great importance in abelian group theory.

Cauchy's equation is an example of an *algebraic* functional equation. By this we mean an equation involving only known and unknown functions, variables, and algebraic operations (addition, subtraction, multiplication, division, and functional composition) applied to the functions and variables. A few other well-known examples of such functional equations are:

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \qquad \text{(Jensen's equation)}$$

$$f(x+y) + f(x-y) = 2f(x)f(y) \qquad \text{(D'Alembert's equation)}$$

$$f(x-y) = f(x)f(y) + g(x)g(y) \qquad \text{(cosine equation)}$$

$$F(x,F(y,z)) = F(F(x,y),F(x,z)) \qquad \text{(autodistributivity)}.$$

The (abbreviated) history of Cauchy's equation that we have traced illustrates the general development of the study of many algebraic functional equations. The equation first arises from another branch of mathematics or from some physical application, or perhaps to illustrate or clarify a general principle in the theory of functional equations itself. (The origins of Cauchy's equation lie

in the problems of the parallelogram of forces, the measurement of areas, the introduction of the normal probability distribution, and others.) The equation is then solved under some regularity assumptions on the unknown functions—perhaps differentiability, continuity, integrability, or measurability—whatever is needed for the applications at hand. As the range of applications increases, and as the equation develops its own mathematical interest, an effort is made to weaken the regularity assumptions under which solutions are obtained. The eventual goal is usually to obtain a complete, general algebraic solution of the equation (without regularity conditions), as such a solution depends only on the form of the equation itself and on the algebraic structure of the domain and range of the unknown functions. By generalizing the equation itself or by varying the domain and range of the unknown functions, we can discover which of these elements is most essential in determining the structure of the set of solutions.

In this thesis, we will study the following functional equation (known as *Hosszú's functional equation*) from the algebraic point of view that we have described above.

$$f(x) + f(y) = f(x+y-xy) + f(xy) \qquad (H)$$

This equation arises from the theory of measures,

and more generally, from the study of valuations in lattice theory. Let C be a collection of subsets of a set X which contains the empty set and is closed under the operations of finite union and complementation. Recall that a function $\mu$ defined on C and with values in $\mathbb{R}$ is called a *finitely additive measure* on X if it satisfies the equation

$$\mu(E) + \mu(F) = \mu(E \cup F)$$

for all $E, F \in C$ satisfying $E \cap F = \emptyset$. (See Bachman and Narici [3], pages 188-195 for an application of such measures to the so-called "measure problem".) This condition is equivalent to $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$, for all $E, F \in C$. The collection C has the structure of a complemented lattice, with the meet and join being intersection and union, respectively. In this context our equation becomes

$$\mu(E) + \mu(F) = \mu(E \vee F) + \mu(E \wedge F) .$$

For general lattices, functions $\mu$ satisfying this equation are called *valuations* (see Birkhoff [4], page 230). From any complemented lattice we can construct a ring with the same elements by setting $x+y = (x \vee y) \wedge (x \wedge y)^c$ $x \cdot y = x \wedge y$ for all elements $x, y$ of the lattice (where the superscript $^c$ denotes complementation). Then

$$E \vee F = E + F - EF ,$$

so that the equation defining a valuation on a lattice

becomes $\mu(E) + \mu(F) = \mu(E+F-EF) + \mu(EF)$. This is Hosszú's equation.

M. Hosszú first studied equation (H) for real-valued differentiable functions on $\mathbb{R}$. He showed that such solutions of (H) must be of the form $f(x) = ax+b$ (where $a,b \in \mathbb{R}$). We will call such functions *affine linear*. If we differentiate equation (H) with respect to $x$, and then set $x=1$, we get $f'(1) = f'(y)$—that is, $f$ has constant derivative, and hence is affine linear. At a conference on functional equations in Zakopane, Poland in 1967, Hosszú posed the problem of solving (H) for measurable real functions.

Fenyö [13] solved Hosszú's problem in 1969 by showing that the measurable real solutions of (H) are again affine linear. He later extended his investigations to the equation

$$f(a_0+a_1x+a_2y+a_3xy) + g(b_0+b_1x+b_2y+b_3xy) = h(x) + k(y)$$

with $f,g,h,k:\mathbb{R} \to \mathbb{R}$ unknown functions, and $a_i,b_i \in \mathbb{R}$, $(i=1,2,3,4)$ arbitary constants (Fenyö [14]). Solving this equation for measurable real functions turns out to be basically equivalent to solving equation (H).

In the meantime, Światak [20] had shown that the affine linear functions are the only solutions on $\mathbb{R}$ which are continuous at the pair of points $a$ and $-a+a^{-1}$ $(0 \neq a \in \mathbb{R})$. Daróczy [8] and Światak [21] later showed

that if a solution $f:\mathbb{R} \to \mathbb{R}$ is Lebesgue integrable on certain finite intervals, it must again be affine linear.

The first algebraic result on Hosszú's equation was the general solution of the equation (with no regularity restrictions) for functions $f:\mathbb{R} \to \mathbb{R}$. Blanuša [5] and Daróczy [9] showed independently that any such solution is of the form $f(x) = \phi(x) + f(0)$, where $\phi:\mathbb{R} \to \mathbb{R}$ is an arbitrary solution of Cauchy's functional equation. Such functions we will call *affine*.

Świątak [22] extended this result by showing that if $f:F \to G$ is a solution of Hosszú's equation, where $F$ is a field not of characteristic 2 or 3 and $G$ is an abelian group with certain restrictive properties, then $f$ must be affine. She posed the natural problem of dispensing with the conditions on $G$ required in her result. This problem was solved by Davison [11], who also dropped the characteristic condition on $F$ by proving the following theorem.

> *THEOREM. Let $F$ be a field with more than four elements, and let $G$ be an abelian group. A function $f:F \to G$ satisfies Hosszú's equation if and only if it is affine.*

It is clear that, assuming the known theory of Cauchy's functional equation, all the previously mentioned results concerning Hosszú's equation follow

as corollaries of this theorem.

Davison had solved Hosszú's equation for the two prime fields excluded in the above theorem in a previous paper [10], in which in fact he solved the equation for functions $f:\mathbb{Z} \to G$ and $f:\mathbb{Z}/n\mathbb{Z} \to G$. The results of this paper showed that on suitable rings there are solutions of Hosszú's equation which are not affine.

In this thesis we will continue the study of Hosszú's equation for functions defined on classes of rings more general than those on which it was previously studied. In doing so we will be able to see more clearly what aspects of the algebraic structure of the ring over which we are studying the equation affect the structure of the solutions of the equation. We have seen that over certain rings, the solutions are simply the affine functions, while over others, there are further solutions. We will try to illustrate the reasons for this in our study. We will also briefly consider generalizations of Hosszú's equation. Many of our results, apart from any interest they may have in themselves, also involve techniques that may be applicable to the algebraic study of other equations.

In Chapter 1, we study some general properties of the solutions $f:R \to G$ of Hosszú's equation, where $R$ is a ring (always with identity) and $G$ an abelian

group. Among other things, we prove three reduction theorems (Propositions 1.4.1, 1.5.1, and 1.5.2) which allow us to reduce our study of (H) over $R$ to a study over direct factors and, in some cases, homomorphic images of $R$.

In the second chapter, we study the equation over rings generated by their units, and we apply our results to local rings.

Chapter 3 deals with the study of Hosszú's equation over number rings. We give a complete solution of the equation over cyclotomic rings of integers, and over another special class of number rings described in Corollary 3.2.2. For all number rings, we describe those solutions $f$ which have the property that the function $\phi(x) = f(x) - f(0)$ is even or odd.

In Chapter 4, we consider the equation for functions defined on polynomial rings, and we use our results to determine the continuous solutions of the equation over singly generated topological algebras.

Finally, in Chapter 5, we examine various generalizations of Hosszú's equation.

# CHAPTER 1

## THE EQUATION

This thesis is devoted to the study of the functional equation

$$f(x) + f(y) = f(x+y-xy) + f(xy) \qquad \text{(H)}$$

which we call *Hosszú's functional equation* (or simply, equation (H)). In this chapter, we will assemble definitions, describe certain constructions, and prove some general results which we will require in the sequel.

We will study the equation for unknown functions $f:R \to G$ from a ring $R$ to an abelian group $G$. One can see from the form of the equation that this is a very natural setting in which to study it.

If $f:R \to G$ and $g:R \to G$ are two functions which satisfy (H), then $f+g$ (the pointwise sum) will clearly also satisfy it. Moreover, so will the function $\overline{f}:R \to G$ defined by $\overline{f}(r) = f(r) - f(0)$. Since this function has the property that $\overline{f}(0) = 0$, we see that every solution of (H) differs by a constant function from one whose value at $0$ is the zero element of the image group. We will therefore restrict our attention to solutions of

(H) with this property, and we will call such solutions *Hosszú functions* on $R$. Functions which satisfy the equation $f(x+y) = f(x) + f(y)$ (Cauchy's equation) will be called *additive* or *Cauchy* functions. Cauchy functions are clearly always Hosszú.

## 1.1 - ODD AND EVEN HOSSZÚ FUNCTIONS

In this section we will introduce two functional equations which characterize the odd and even Hosszú functions. By an odd function, of course, we mean one that satisfies $f(-x) = -f(x)$ for all $x$ in its domain, while an even function satisfies $f(-x) = f(x)$. Every additive function is clearly odd—if it is also even, then its image must be a group of exponent two.

Świątak [20] showed that any odd Hosszú function on $\mathbb{R}$ must be additive, and Blanuša [5] was able to use this fact to show that every Hosszú function on $\mathbb{R}$ is additive. We will find both the odd and even Hosszú functions useful in our study, particularly in Chapter 3.

*PROPOSITION 1.1.1. The function $f:R \to G$ is an odd Hosszú function if and only if $f(0) = 0$ and $f$ satisfies the functional equation*

$$f(x) + f(y) + f(xy) = f(x+y+xy). \qquad (H^O)$$

*PROOF.* If $f$ is an odd Hosszú function, then

$$f(-x) + f(-y) = f(xy) + f(-x-y-xy).$$

Using the oddness and rearranging terms, we obtain equation $(H^o)$.

Conversely, if $f$ satisfies $(H^o)$, setting $y = -1$ in the equation shows immediately that $f$ is odd. Now

$$f(-x) + f(-y) + f(xy) = f(-x-y+xy),$$

so using the oddness again, we obtain equation (H). |||

For the even Hosszú functions, we have a similar characterization, given in the following result.

*PROPOSITION 1.1.2. The function $g:R \to G$ is an even Hosszú function if and only if $g(0) = 0$ and $g$ satisfies the functional equation*

$$g(x) + g(y) = g(xy) + g(x+y+xy), \qquad (H^e)$$

*PROOF.* If we replace $f$ by $g$, the word "odd" by "even", and $(H^o)$ by $(H^e)$ in the proof of Proposition 1.1.1, we will obtain the desired conclusion. |||

If $f:R \to G$ is an arbitrary solution of $(H^o)$, then setting $x=y=0$ in the equation shows that $2f(0) = 0$. Thus the function $\psi(x) = f(x) - f(0)$ also satisfies $(H^o)$. Similarly, if $g:R \to G$ satisfies $(H^e)$, then so does the function $\chi(x) = g(x) - g(0)$. So as in the case of

equation (H), solutions of $(H^o)$ and $(H^e)$ will differ by constant solutions from solutions with the property that their value at zero is zero.

Because the techniques used in studying (H), $(H^o)$, and $(H^e)$ are often so similar, we will frequently make use of the following conventions. If the statement of a result is of the form *"for (odd, even) Hosszú functions, A (respectively B, C) holds"*, then we are to interpret it as three results; that is, the sentence or phrase A holds for Hosszú functions, while B and C hold for odd and even Hosszú functions respectively. Similarly, the statement *"for odd (even) Hosszú functions, A (respectively B) holds"* is to be interpreted as two results in one, in the obvious fashion. The proofs of such combined results will be joined together in the same way as their statements.

## 1.2 - THE HOSSZÚ GROUPS

We will now construct three Hosszú functions on any ring $R$ which will, in a sense, be the general solutions of equations (H), $(H^o)$, and $(H^e)$ respectively—that is, they will have the universal property for Hosszú functions described in Proposition 1.2.1.

Let $F(R)$ be the free abelian group on the set of elements of $R$, with the canonical embedding $c: R \to F(R)$.

Let $J(R)$ be the subgroup of $F(R)$ generated by the set $\{c(x+y-xy) + c(xy) - c(x) - c(y) : x,y \in R\} \cup \{c(0)\}$, and let $J^o(R)$ and $J^e(R)$ be the subgroups of $F(R)$ generated by the sets $\{c(x+y+xy) - c(xy) - c(x) - c(y) : x,y \in R\} \cup \{c(0)\}$ and $\{c(x+y+xy) + c(xy) - c(x) - c(y) : x,y \in R\} \cup \{c(0)\}$ respectively. (Note the relationships between the equations (H), ($H^o$), and ($H^e$), and the subgroups of $F(R)$ we have defined above.) Now let $\mathcal{H}(R) = F(R)/J(R)$, $\mathcal{H}^o(R) = F(R)/J^o(R)$, and $\mathcal{H}^e(R) = F(R)/J^e(R)$, with canonical maps $p:F(R) \to \mathcal{H}(R)$, $p_o:F(R) \to \mathcal{H}^o(R)$, and $p_e:F(R) \to \mathcal{H}^e(R)$.

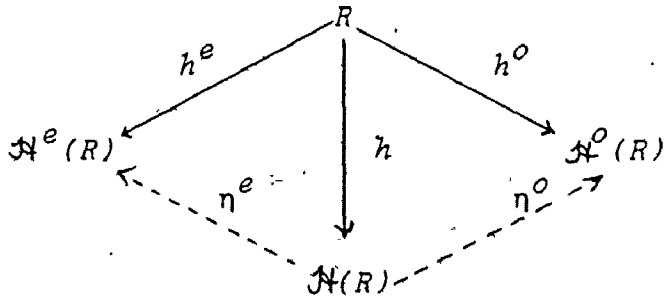
Define $h$, $h^o$, and $h^e$ by the compositions $h = p \circ c$ $h^o = p_o \circ c$, and $h^e = p_e \circ c$ respectively. The abelian group $\mathcal{H}(R)$ is called the *Hosszú group* of $R$, while $h:R \to \mathcal{H}(R)$ is called the *universal Hosszú function* on $R$. (The corresponding objects with the superscripts $^o$ and $^e$ will have the same names qualified by the adjectives *odd* and *even* respectively.)

*PROPOSITION 1.2.1. (i) $h:R \to \mathcal{H}(R)$ is a Hosszú function, while $h^o$ and $h^e$ are odd and even Hosszú functions, respectively. (ii) If $f:R \to G$ is any (odd, even) Hosszú function, where $G$ is an abelian group, then there is a unique group homomorphism $f^*:\mathcal{H}(R) \to G$ (respectively $f^*:\mathcal{H}^o(R) \to G$, $f^*:\mathcal{H}^e(R) \to G$) such that $f = f^* \circ h$ (respectively $f = f^* \circ h^o$, $f = f^* \circ h^e$).*

*PROOF.* Part *(i)* is evident from the above constructions.

Suppose now that $f:R \to G$ is a Hosszú function. Since $F(R)$ is a free abelian group on the set of elements of $R$, there is a unique group homomorphism $\psi:F(R) \to G$ such that $f = \psi \circ c$. Since $f$ satisfies (H) and $f(0) = 0$, we know that $\psi(J(R)) = 0$. Thus $J(R)$ (the kernel of $p$) is contained in the kernel of $\psi$, so that $\psi$ factors uniquely through $p$—say $\psi = f^* \circ p$. Then $f = \psi \circ c = f^* \circ p \circ c = f^* \circ h$. This proves part *(ii)* for Hosszú functions, and the proofs in the odd and even cases are similar. |||

Since $h^o$ and $h^e$ are themselves Hosszú functions, there exist group homomorphisms $\eta^o:\mathcal{H}(R) \to \mathcal{H}^o(R)$ and $\eta^e:\mathcal{H}(R) \to \mathcal{H}^e(R)$ such that $h^o = \eta^o \circ h$ and $h^e = \eta^e \circ h$ (by part *(ii)* of the above proposition).




Since $h^e$ and $h^o$ are *dense* functions (that is, their images generate their ranges), it follows that $\eta^e$ and $\eta^o$ are epimorphisms, so that $\mathcal{H}^e(R)$ and $\mathcal{H}^o(R)$ are homomorphic images of $\mathcal{H}(R)$.

The construction used in the definition of the Hosszú group of $R$ and its universal Hosszú function

gives us a combinatorial method for finding directly the general solution of (H) over $R$. In general, however, it is impractical to use this direct method, which involves actually computing the subgroup $J(R)$ of $F(R)$. One case where it can be used is over the ring of rational integers.

*EXAMPLE*: (Davison [10])

$$\mathcal{H}(\mathbb{Z}) \simeq \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus C_2$$

$$h:\mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus C_2$$

$$:n \longmapsto (n,\ [-\tfrac{n}{2}],\ [-\tfrac{n}{3}],\ \lambda(n))$$

Here $[x]$ denotes the greatest integer less than or equal to $x$, while $\lambda:\mathbb{Z}\to C_2$ is the function which takes $n$ to $1 \in C_2$ if $n \equiv 2 \pmod 4$, and to $0$ otherwise. (For the proof of this result, see [10].) From the above description of $h:\mathbb{Z}\to\mathcal{H}(\mathbb{Z})$, we can readily show that $\mathcal{H}^o(\mathbb{Z}) \simeq \mathbb{Z}\oplus C_2$ and that $\mathcal{H}^e(\mathbb{Z}) \simeq \mathbb{Z}\oplus C_2$, with odd and even universal Hosszú functions given by

$$h^o:\mathbb{Z} \longrightarrow \mathbb{Z}\oplus C_2$$

$$:n \longmapsto (n,\ \lambda(n))$$

$$h^e:\mathbb{Z} \longrightarrow \mathbb{Z}\oplus C_2$$

$$:n \longmapsto (n+2[-\tfrac{n}{2}],\ \lambda(n)) .$$

## 1.3 - DARÓCZY ELEMENTS AND THE ADDITIVE NUCLEUS

We will use the universal Hosszú functions to define the following subsets of a ring $R$.

$$\mathfrak{D}(R) = \{d \in R : h(d-y) = h(d) - h(y) \text{ for all } y \in R\}$$

$$\mathfrak{A}(R) = \{a \in R : h(a+x) = h(a) + h(x) \text{ for all } x \in R\}$$

$$\mathfrak{A}^o(R) = \{b \in R : h^o(b+x) = h^o(b) + h^o(x) \text{ for all } x \in R\}$$

$$\mathfrak{A}^e(R) = \{b \in R : h^e(b+x) = h^e(b) + h^e(x) \text{ for all } x \in R\}$$

The set $\mathfrak{D}(R)$ is called the *set of Daróczy elements* of $R$, after Z. Daróczy [9], who first pointed out the significance of this set in the solution of (H). We call $\mathfrak{A}(R)$ the *additive nucleus* of $R$, while $\mathfrak{A}^o(R)$ and $\mathfrak{A}^e(R)$ are the *odd* and *even additive nuclei* of $R$, respectively.

It is clear that $\mathfrak{A}(R)$, $\mathfrak{A}^o(R)$, and $\mathfrak{A}^e(R)$ are subgroups of the additive structure of $R$. Moreover, if $d \in \mathfrak{D}(R)$ and $a \in \mathfrak{A}(R)$, then for all $y \in R$,

$$h(a+d-y) = h(a) + h(d-y)$$
$$= h(a) + h(d) - h(y) = h(a+d) - h(y),$$

so that $a+d \in \mathfrak{A}(R)$.

Also, if $d$ and $d'$ are both Daróczy elements of $R$, then for all $x \in R$,

$$h(d-d'+x) = h(d) - h(d'-x)$$
$$= h(d) - h(d') + h(x) = h(d-d') + h(x),$$

and so $d-d' \in \mathfrak{A}(R)$. These two facts show that if $\mathfrak{D}(R)$ is nonempty, it is an additive coset of $\mathfrak{A}(R)$ in $R$.

From Proposition 1.2.1, it follows that if $f:R \to G$ is an (odd, even) Hosszú function, and if $a \in \mathcal{A}(R)$ (respectively $a \in \mathcal{A}^o(R)$, $a \in \mathcal{A}^e(R)$), then $f(a+x) = f(a) + f(x)$ for all $x \in R$. Since $h^o$ and $h^e$ are Hosszú functions, this means that $\mathcal{A}(R) \subseteq \mathcal{A}^o(R)$ and $\mathcal{A}(R) \subseteq \mathcal{A}^e(R)$. Also, if $d \in \mathcal{D}(R)$, then $h^o(d+x) = h^o(d) - h^o(-x) = h^o(d) + h^o(x)$ for all $x \in R$ (since $h^o$ is odd), so that $d \in \mathcal{A}^o(R)$. Hence $\mathcal{D}(R) \subseteq \mathcal{A}^o(R)$.

Combining these facts, we see that

$$\mathcal{A}(R) \cup (\mathcal{A}(R) + \mathcal{D}(R)) \subseteq \mathcal{A}^o(R)$$

and $\mathcal{A}(R) \subseteq \mathcal{A}^o(R) \cap \mathcal{A}^e(R)$.

Of course, every (odd, even) Hosszú function on $R$ will be additive if and only if $\mathcal{A}(R) = R$ (respectively $\mathcal{A}^o(R) = R$, $\mathcal{A}^e(R) = R$), by definition of the additive nuclei. As we will see later, determining the structure of the additive nucleus of a ring will be our main tool in solving Hosszú's equation over that ring.

If $u$ is a unit (invertible element) of the ring $R$ with the property that $1-u$ is also a unit, we say that $u$ is an *exceptional unit*. In this case, $1-u^{-1} = (u-1)u^{-1} = -(1-u)u^{-1}$, which is invertible, so that $1-u^{-1}$ is also an exceptional unit. These units are important in the study of Hosszú's equation because their existence can tell us something about the contents of the set $\mathcal{D}(R)$

(and hence of $\mathcal{A}(R)$) as the following results show. The first proposition below is due to Davison [12] in the form given, although it was partially latent in the work of Daróczy [9].

*PROPOSITION 1.3.1. If $u$ is an exceptional unit in $R$, then $u+u^{-1} \in \mathcal{D}(R)$, and $u^{-1}+(1-u)^{-1} \in \mathcal{D}(R)$.*

*PROOF.* By using the equation (H), we see that

$$\begin{aligned} & h(x) + h(y) + h(z) \\ &= h(xy) + h(x+y-xy) + h(z) \\ &= h(xy) + h(xz+yz-xyz) + h(x+y+z-xy-xz-yz+xyz) \\ &= h(x+y-xy) + h(xy+z-xyz) + h(xyz) , \end{aligned}$$

where the last two of the above four expressions are obtained from the second by expanding it using (H) in two different ways.

Since the expression $h(x) + h(y) + h(z)$ is invariant under the permutation of variables which sends $x$ to $y$, $y$ to $z$, and $z$ to $x$, all four of the above expressions must be invariant under this permutation. This tells us that

$$h(xy) + h(xz+yz-xyz) = h(yz) + h(xy+xz-xyz), \text{ and}$$
$$h(x+y-xy) + h(xy+z-xyz) = h(y+z-yz) + h(yz+x-xyz).$$

Since $u$ is an exceptional unit, we can substitute $x = wu$, $y = u^{-1}$, and $z = (u^{-1}-1)^{-1}u^{-1}$ into the first of these two equations (where $w \in R$ is arbitrary) to obtain

$h(w) + h(u^{-1}(u^{-1}-1)^{-1}u^{-1} - w) = h(u^{-1}(u^{-1}-1)^{-1}u^{-1}) + h(0)$. Thus $u^{-1}(u^{-1}-1)^{-1}u^{-1} \in \mathcal{D}(R)$. But $u^{-1}(u^{-1}-1)^{-1}u^{-1} = (1-u)^{-1}u^{-1} = u^{-1} + (1-u)^{-1}$.

Now substitute $x = (u^{-1}-w)(1-u)^{-1}$, $y = u$, and $z = u^{-1}$ into the second equation of the above pair to get

$$h(u+u^{-1}-w) + h(w) = h(u+u^{-1}-1) + h(1) .$$

Setting $w = 0$, we see that $h(u+u^{-1}-1) + h(1) = h(u+u^{-1})$, and hence $h(u+u^{-1}-w) = h(u+u^{-1}) - h(w)$. Since $w$ is arbitrary, we have that $u+u^{-1} \in \mathcal{D}(R)$. ||

*COROLLARY 1.3.2. If R has at least one exceptional unit, then* $1 \in \mathcal{D}(R)$, *so that* $\mathcal{D}(R) = 1 + \mathcal{A}(R)$.

*PROOF.* If $u$ is an exceptional unit, then

$$1 = [(1-u)+(1-u)^{-1}] - [u^{-1}+(1-u)^{-1}] + [u+u^{-1}] \in \mathcal{D}(R),$$

using the above proposition and the fact that $\mathcal{D}(R)$ is a coset of $\mathcal{A}(R)$ in $R$. ||

Under the hypothesis of the above corollary, we can see that if we could prove $1 \in \mathcal{A}(R)$, then $\mathcal{D}(R)$ would have to be equal to $\mathcal{A}(R)$ (since $\mathcal{A}(R)$ is closed under addition). The following proposition gives a situation in which this is the case.

*PROPOSITION 1.3.3. If R has at least one exceptional unit, and if* $\mathcal{A}(R)$ *contains a unit of R, then* $\mathcal{A}(R) = \mathcal{D}(R)$.

*PROOF.* Let $a \in \mathcal{A}(R)$ be a unit, and let $x \in R$ be arbitrary.

Let $y = a^{-1}x$. Then $h(1-a+ay) = h((1-a) + y - (1-a)y)$

$= h(1-a) + h(y) - h((1-a)y)$ (using equation (H))

$= h(1-a) + h(y) + h(a) - [h(a) + h((1-a)y)]$

$= h(1-a) + h(y) + h(a) - h(a+y-ay)$ (since $a \in \mathcal{A}(R)$)

$= h(1-a) + h(y) + h(a) - [h(a) + h(y) - h(ay)]$

$= h(1-a) + h(ay)$.

Since $y = a^{-1}x$, this means that $h(1-a+x) = h(1-a)+h(x)$ for all $x \in R$, so that $1-a \in \mathcal{A}(R)$, and hence $(1-a)+a = 1 \in \mathcal{A}(R)$. So by Corollary 1.3.2, we see that $\mathcal{A}(R) = \mathcal{D}(R)$. |||

The preceding results enable us to find elements of $\mathcal{A}(R)$ if we know that $R$ has an exceptional unit $u$. For example, $1-u-u^{-1}$ will belong to $\mathcal{A}(R)$, and if this is again a unit, then $\mathcal{A}(R)$ will be equal to $\mathcal{D}(R)$.

*PROPOSITION 1.3.4. Every Hosszú function on $R$ is odd if and only if $\mathcal{A}(R) = \mathcal{D}(R)$.*

*PROOF.* By Proposition 1.2.1, every Hosszú function on $R$ is odd if and only if $h: R \to \mathcal{H}(R)$ is odd; that is, $h(-x) + h(x) = 0$ for all $x \in R$, or, rewriting this, $h(0-x) + h(x) = h(0)$. This means precisely that $0 \in \mathcal{D}(R)$. Since $\mathcal{D}(R)$ is a coset of $\mathcal{A}(R)$, which always contains $0$, it follows that $0 \in \mathcal{D}(R)$ if and only if $\mathcal{A}(R) = \mathcal{D}(R)$. |||

If every Hosszú function on $R$ is odd,

a function $f$ defined on $R$ (and satisfying $f(0)=0$) will satisfy (H) if and only if it satisfies ($H^o$) (by Proposition 1.1.1). This will turn out to be a useful fact, since the equation ($H^o$) is often easier to work with than (H).

From the preceding proposition we can also conclude that if $\mathcal{A}(R) = \mathcal{D}(R)$, then $\mathcal{A}(R) = \mathcal{A}^o(R)$.

*PROPOSITION 1.3.5. If $u$ is a unit in $R$, then, $2(u-1) \in \mathcal{A}^e(R) \cap \mathcal{A}^o(R)$.*

*PROOF.* For all $x, y \in R$,

$$h^o(x) + h^o(y) + h^o(xy) = h^o(x+y+xy), \text{ and}$$
$$h^o(x) + h^o(-y) + h^o(-xy) = h^o(x-y-xy).$$

Adding these two equations and using the oddness of $h^o$, we get the equation

$$2h^o(x) = h^o(x + (1+x)y) + h^o(x - (1+x)y).$$

If $u \in R$ is a unit and $z \in R$ is arbitrary, then setting $x = u-1$ and $y = u^{-1}(u-1+z)$ we see that

$$2h^o(u-1) = h^o(2(u-1) + z) + h^o(-z).$$

By letting $z=0$, we can show that $2h^o(u-1) = h^o(2(u-1))$, and so the preceding equation becomes

$$h^o(2(u-1)) + h^o(z) = h^o(2(u-1) + z).$$

Thus $2(u-1) \in \alpha^o(R)$.

We also know that

$$h^e(x) + h^e(y) = h^e(xy) + h^e(x+y+xy), \text{ and}$$
$$h^e(x) + h^e(-y) = h^e(-xy) + h^e(x-y-xy).$$

Subtracting these equations and using the even-ness of $h^e$, we see that

$$h^e(x + (1+x)y) = h^e(x - (1+x)y)$$

for all $x,y \in R$. Performing the same substitution as before, we get $h^e(2(u-1) + z) = h^e(-z) = h^e(z)$, and setting $z=0$, it follows that $h^e(2(u-1)) = 0$. Thus

$$h^e(2(u-1) + z) = h^e(2(u-1)) + h^e(z),$$

so that $2(u-1) \in \alpha^e(R)$. |||

Note that if $a \in \alpha^e(R)$, then $h^e(2a) = h^e(a)+h^e(a)$ $= h^e(a)+h^e(-a) = h^e(a-a) = h^e(0) = 0$.

Now, for any ring $R$, let $U(R)$ denote the set of those elements of $R$ which are sums of units. Clearly $U(R)$ is a subring of $R$. We will call it the *unit ring* of $R$.

*COROLLARY 1.3.6.* $4U(R) \subseteq \alpha^o(R) \cap \alpha^e(R)$.

*PROOF.* By the preceding proposition, for any unit $u \in R$, $2(u-1) \in \alpha^o(R) \cap \alpha^e(R)$. But $-u$ is also a unit, so that $2(-u-1)$ also belongs to this intersection. Hence

$-2(u-1)-2(-u-1) = 4 \in \mathcal{A}^o(R) \cap \mathcal{A}^e(R)$, and so $2(u-1) + 2(u-1) + 4 = 4u \in \mathcal{A}^o(R) \cap \mathcal{A}^e(R)$. Thus $4U(R) \subseteq \mathcal{A}^o(R) \cap \mathcal{A}^e(R)$. |||

## 1.4 - DIRECT PRODUCTS OF RINGS

If $R$ and $S$ are rings, then by their *direct product* $R \times S$, we mean the cartesian product of $R$ and $S$ with coördinate-wise multiplication and addition. If we have two functions $f:R \to X$ and $g:S \to X$ (where $X$ is an abelian group), then the *direct product* of $f$ and $g$ is the function $f \times g:R \times S \to X$ defined by $f \times g((r,s)) = f(r) + g(s)$.

*PROPOSITION 1.4.1. If $R$ and $S$ are rings, then any (odd, even) Hosszú function $f:R \times S \to G$ is the direct product of two (odd, even) Hosszú functions $f_1:R \to G$ and $f_2:S \to G$. Conversely, any such direct product of (odd, even) Hosszú functions will be an (odd, even) Hosszú function on $R \times S$.*

*PROOF.* The second statement is obviously true, so we will prove the first for Hosszú functions. (The proofs for the odd and even cases will be virtually identical.)

Let $(r,s) \in R \times S$, and let $f:R \times S \to G$ be Hosszú. Then

$$\begin{aligned} f((r,s)) &= f((r,0) + (0,s) - (r,0)\cdot(0,s)) \\ &= f((r,0)) + f((0,s)) - f((r,0)\cdot(0,s)) \\ &= f((r,0)) + f((0,s)). \end{aligned}$$

(where we have used equation (H)).

Now define $f_1: R \to G$ by $f_1(r) = f((r,0))$, and $f_2: S \to G$ by $f_2(s) = f((0,s))$. Clearly $f_1$ and $f_2$ are Hosszú functions (being restrictions of $f$), and so $f((r,s)) = f_1(r) + f_2(s)$ is the decomposition of $f$ required by the statement of the proposition. |||

This proposition will clearly generalize to any finite direct product of rings. This reduces the problem of solving Hosszú's equation over a ring to solving it over its direct factors.

## 1.5 - IDEALS IN THE ADDITIVE NUCLEUS

If every Hosszú function on a ring $R$ is additive, then so is every Hosszú function on a homomorphic image $R/I$ of $R$ (where $I$ is an ideal in $R$). For if $f: R/I \to G$ is Hosszú, and if $\nu: R \to R/I$ is the canonical map, then $f \circ \nu$ will clearly be Hosszú on $R$, and hence additive. Now if $\bar{r}, \bar{s} \in R/I$, then

$$\begin{aligned} f(\bar{r} + \bar{s}) &= f \circ \nu(r+s) \\ &= f \circ \nu(r) + f \circ \nu(s) \\ &= f(\bar{r}) + f(\bar{s}), \end{aligned}$$

so that $f$ is also additive.

The statements of the above paragraph will also hold for odd Hosszú functions, since $\nu$ is an odd function. We will call $\tilde{f} = f \circ \nu$ the *lift* of $f$ to $R$.

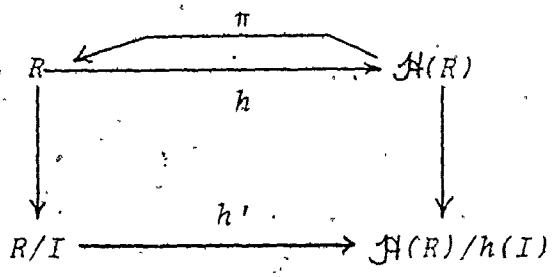
A partial converse to the above is provided by the following proposition.

*PROPOSITION 1.5.1. Suppose that $\mathcal{A}(R)$ (respectively $\mathcal{A}^o(R)$, $\mathcal{A}^e(R)$) contains the ideal $I$ of $R$. If every (odd, even) Hosszú function on $R/I$ (respectively $R/I$, $R/2I$) is additive, then every (odd, even) Hosszú function on $R$ is additive.*

*PROOF.* Since the identity function on $R$ is a Hosszú function, there exists (by Proposition 1.2.1) a group homomorphism $\pi:\mathcal{H}(R) \to R$ such that $\pi \circ h = id_R$ (the identity function on $h$). Define $h':R/I \to \mathcal{H}(R)/h(I)$ by

$$h'(r+I) = h(r) + h(I)$$

for all $\bar{r} = r+I \in R/I$. (Note that $h(I)$ is a subgroup of $\mathcal{H}(R)$ since $I \subseteq \mathcal{A}(R)$.) The map $h'$ is well-defined since $I \subseteq \mathcal{A}(R)$, and is Hosszú on $R/I$ since $h$ is Hosszú on $R$.




By hypothesis, $h'$ (being Hosszú on $R/I$) will be additive. Thus $h'(r+s+I) = h'(r+I) + h'(s+I)$ for all $r,s \in R$, and so by definition of $h'$, $h(r+s) + h(I) = h(r) + h(s) + h(I)$. The fact that these two cosets in $\mathcal{H}(R)$ are equal means that there is an $i \in I$ such that

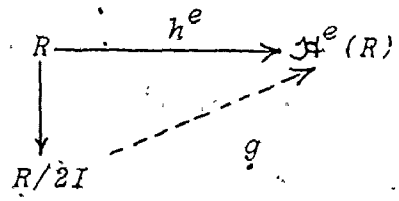
$h(r+s) + h(i) = h(r) + h(s) + h(0)$. Applying the function $\pi$ to this equation, we see that $r+s+i = r+s$ (using $\pi \circ h = id_R$), and so $i=0$. Thus $h(r+s) = h(r) + h(s)$—that is, $h$ is additive. Hence any Hosszú function on $R$ is additive.

The same proof can clearly be used for the case of odd Hosszú functions, but since the identity function is not (in general) even, we have to use a different argument in the even case.

As we have seen in Section 1.3, $h^e(2a)=0$ for all $a \in \mathcal{A}^e(R)$. Thus $h^e(2I)=0$, and so

$$\begin{aligned} h^e(2i+x) &= h^e(2i) + h^e(x) \\ &= h^e(x) \end{aligned}$$

for all $i \in I$ and for all $x \in R$. This shows that $h^e$ is constant on the cosets of $R$ modulo $2I$, and hence there is a unique function $g$ making the following diagram commute.




Obviously $g$ is an even Hosszú function, and so by hypothesis must be additive. Hence $h^e$ is additive, and the proposition is proved. |||

If the ideal $I$ of the above proposition is in fact

a direct summand of $R$ (summand as an abelian group), then we can describe all Hosszú functions on $R$ in terms of the Hosszú functions on $R/I$.

*PROPOSITION 1.5.2. Suppose that $\alpha(R)$ (respectively $\alpha^o(R)$, $\alpha^e(R)$) contains the ideal $I$ of $R$, and that $R = B \oplus I$ for some subgroup $B$ of $R$. Then every (odd, even) Hosszú function on $R$ is the sum of an additive function and the lift of an (odd, even) Hosszú function on $R/I$.*

*PROOF.* Let $f:R \to G$ be a Hosszú function, and let $\rho:R \to B$ be the projection of $R$ onto the subgroup $B$. Define $f_1:R \to G$ by $f_1(r) = f(\rho(r))$. We will show that $f_1$ is a Hosszú function.

Let $r,s$ be arbitrary elements of $R$, and suppose that $r = b+i$, $s = c+j$, where $b,c \in B$ and $i,j \in I$. Then

$$\begin{aligned} f_1(r) + f_1(s) &= f(\rho(r)) + f(\rho(s)) \\ &= f(b) + f(c) \\ &= f(b+c-bc) + f(bc). \end{aligned}$$

Let $k = bc - \rho(bc)$. It is clear that $k \in I$. Then

$$\begin{aligned} f(b+c-bc) + f(bc) &= f(b+c-bc+k) + f(bc-k) \\ &= f(b+c-\rho(bc)) + f(\rho(bc)) \\ &= f(\rho(b)+\rho(c)-\rho(bc)) + f(\rho(bc)) \end{aligned}$$

$$= f(\rho(r)+\rho(s)-\rho(rs)) + f(\rho(rs)$$

$$= f(\rho(r+s-rs)) + f(\rho(rs))$$

$$= f_1(r+s-rs) + f_1(rs)$$

(where we have used the fact that $k \in \mathcal{A}(R)$, and that $\rho(rs) = \rho((b+i)(c+j)) = \rho(bc+ic+bj+ij) = \rho(bc)$). This shows that $f_1$ is Hosszú.

It is clear from its definition that $f_1$ is constant on the cosets of $R$ modulo $I$, and hence is the lift of a Hosszú function on $R/I$.

Now let $f_2 = f - f_1$. Then

$$f_2(r+s) = f(r+s) - f_1(r+s)$$

$$= f(b+i+c+j) - f(b+c)$$

$$= f(b+c) + f(i+j) - f(b+c)$$

$$= f(i) + f(j)$$

$$= f(b+i) - f(b) + f(c+j) - f(c)$$

$$= f(r) - f_1(r) + f(s) - f_1(s)$$

$$= f_2(r) + f_2(s).$$

Hence $f_2$ is additive, and $f = f_1 + f_2$ is the required decomposition of $f$. |||

# CHAPTER 2

# RINGS GENERATED BY THEIR UNITS

The class of rings which are additively generated by their units includes fields, local rings in general, division rings, rings of matrices over division rings not of characteristic two (Wolfson [25] and Zelinsky [26]), and others (see, for example, Fisher and Snider [15]). In this chapter we will develop some general techniques for dealing with the study of Hosszú's equation over this class of rings, and then we will examine Hosszú functions on local rings in more detail.

## 2.1 - GENERAL RESULTS

*THEOREM 2.1.1. If $R$ has at least one exceptional unit and is additively generated by its units, and if $\mathcal{Q}(R)$ contains a unit, then every Hosszú function on $R$ is additive if and only if every Hosszú function on $R/2R$ is additive.*

*PROOF.* As we have seen in Section 1.5, if every Hosszú function on $R$ is additive, then so is every Hosszú function on $R/2R$. Suppose therefore that every Hosszú function on $R/2R$ is additive. Since $\mathcal{Q}(R)$ contains a unit and $R$ has an exceptional unit, $\mathcal{Q}(R) = \mathcal{D}(R)$ by Proposition 1.3.3,

and so $\mathcal{A}^{\circ}(R) = \mathcal{A}(R)$. This means that $1 \in \mathcal{A}(R)$ (Corollary 1.3.2); and $2(u-1) \in \mathcal{A}(R)$ for all units $u \in R$ (Proposition 1.3.5). Thus $2u = 2(u-1) + 1 + 1 \in \mathcal{A}(R)$, and since every element of $R$ is a sum of units, $2x \in \mathcal{A}(R)$ for all $x \in R$. Thus $2R \subseteq \mathcal{A}(R)$, and the theorem follows by Proposition 1.5.1. |||

For the odd and even Hosszú functions, we have a similar but simpler theorem, which is an immediate consequence of Corollary 1.3.6 and Proposition 1.5.1.

*THEOREM 2.1.2. If $R$ is additively generated by its units, then every odd (even) Hosszú function on $R$ is additive if and only if every odd (even) Hosszú function on $R/4R$ (respectively $R/8R$) is additive.* |||

If $2$ is a unit of $R$, then $2R = 4R = 8R = R$; so we have the following corollary of the above two theorems.

*COROLLARY 2.1.3. If $R$ is additively generated by its units, and if $2$ is a unit of $R$, then every odd (even) Hosszú function on $R$ is additive. If moreover $R$ has at least one exceptional unit and $\mathcal{A}(R)$ contains a unit, then every Hosszú function on $R$ is additive.* |||

*EXAMPLE.* If $F$ is a field not of characteristic 2, then by the above corollary, every odd (even) Hosszú function on $F$ is additive. If $V$ is a finite-dimensional vector

space over $F$, then by Corollary 7 of Fisher and Snider [15], every element of $R = End_F(V)$ (the ring of $F$-vector space endomorphisms of $V$) is a sum of two units. Hence every odd and even Hosszú function on $R$ as well will be additive.

If we assume moreover that the characteristic of $F$ is not equal to 3, either, then both $2$ and $3$ are units in $F$ (as well as in $R$), so that $-3 \cdot 2^{-1} = 1 - (2+2^{-1})$ belongs to the additive nucleus of both $F$ and $R$, by Proposition 1.3.1 and Corollary 1.3.2. So by Corollary 2.1.3, every Hosszú function on both $F$ and $R$ is additive.

## 2.2 - LOCAL RINGS

All our results until now have been valid for non-commutative as well as commutative rings. From now on, however, we will restrict our attention to commutative rings.

Recall that a local ring is a commutative ring with exactly one maximal ideal. We will normally write such rings as a pair $(R,m)$, where $R$ denotes the ring and $m$ is its maximal ideal. (Note that $R \neq m$, by definition of a maximal ideal, unless $R = 0$.) The quotient ring $R/m$ will be a field, and we denote the cardinality of this field by $N(m)$, and we will write $N(m) = \infty$ if $R/m$ is infinite (with the convention that $\infty > n$ for all natural numbers $n$). We call $N(m)$ the *norm* of $m$.

Let $(R,m)$ be a local ring. Obviously, all the elements of $R$ not in $m$ are units, since otherwise $m$ would not be maximal. If $x \in m$, then $1+x \notin m$, so $x = (1+x)+(-1)$ exhibits $x$ as a sum of two units. This shows that $R$ is additively generated by its units, and so we can apply the theorems of the preceding section to local rings.

*PROPOSITION 2.2.1. If $(R,m)$ is a local ring with $N(m) > 2$, then $R$ has at least one exceptional unit. If $N(m) > 4$, then $Q(R)$ contains a unit.*

*PROOF.* Clearly if $r \in R$ does not belong to $m$ or $1+m$, then $r$ is an exceptional unit, so if $N(m) > 2$, $R$ will have exceptional units. Now suppose that $N(m) > 4$, and consider the equation $x^3-2x^2+2x-1 = 0$. This equation will have at most three roots in the field $R/m$. Since $R/m$ has more than four elements, there will be a nonzero element (say $w$) of $R/m$ which is not a root of the equation. Suppose that $w = u+m$, where $u \in R$. Then $w \neq 0$ means that $u \notin m$, so that $u$ is a unit of $R$.

Since $w$ was chosen not to be a root of the above equation, we know that $u^3-2u^2+2u-1 \notin m$, so that this latter expression is also a unit of $R$. Now

$$u^3-2u^2+2u-1 = (1-u)(-u^2+u-1)$$
$$= u^{-1}(1-u)(1-u-u^{-1}) .$$

Since this is a unit, all its factors must be units; in particular, $1-u$ is a unit, so that $u$ is an exceptional unit and hence $1-u-u^{-1} \in Q(R)$ (by Proposition 1.3.1 and Corollary 1.3.2). But this is also a unit, so that $Q(R)$ contains a unit. |||

*THEOREM 2.2.2. If $(R,m)$ is a local ring, and if $N(m) > 4$ and $2 \notin m$, then every Hosszú function on $R$ is additive.*

*PROOF.* This theorem now follows immediately from Corollary 2.1.3 and Proposition 2.2.1. |||

We now turn to the more complicated case when $2 \in m$ (so that $2$ is not a unit in $R$). In this case we will only consider finite local rings—the solution of Hosszú's equation over infinite local rings in which $2$ is not a unit remains an open problem.

For any ring $R$, let $A(R)$ be the additive subgroup of $R$ generated by the set $\{u+u^{-1}: u \text{ is an exceptional unit in } R\}$. If $R$ is a local ring with maximal ideal of norm greater than 4, then Proposition 2.2.1 together with Propositions 1.3.1 and 1.3.3 shows that $A(R) \subseteq Q(R)$.

*THEOREM 2.2.3. If $(R,m)$ is a finite local ring with $N(m) > 4$, then every Hosszú function on $R$ is additive.*

*PROOF.* By Theorem 2.2.2, we can assume that $2 \in m$. Thus $R/m$ must be a field of characteristic 2, and hence, since $N(m) > 4$, the field must have at least eight elements (i.e. $N(m) \geq 8$). Suppose that $N(m) = n_1$ and that $m$ has $n_2$ elements. Clearly every element of $R$ not belonging to the cosets $m$ and $1+m$ will be an exceptional unit of $R$, so that $R$ has $(n_1-2)n_2$ exceptional units.

Suppose now that $x+x^{-1} = y+y^{-1}$ for two exceptional units $x, y \in R$. Then $x^2y+y = xy^2+x$, so that $(xy-1)(x-y) = 0$. Since all zero-divisors in $R$ must lie inside $m$, either $x-y \in m$ or $xy-1 \in m$ (which means that $x-y^{-1} \in m$). Suppose that $x-y \in m$. Then $y = x+\mu$, where $\mu \in m$, so that $(x^2-1+x\mu)(\mu) = 0$. Now if $x^2-1 \in m$, then $x$ would have to belong to $1+m$, which it does not since it is an exceptional unit. This means that $x^2-1$ is a unit, and hence so is $x^2-1+x\mu$. We can therefore conclude from the last equation above that $\mu = 0$ (by multiplying by $(x^2-1+x\mu)^{-1}$), and so $x=y$. The possibility that $x-y^{-1} \in m$ will lead in a similar fashion to the conclusion that $x=y^{-1}$. Thus $x+x^{-1} = y+y^{-1}$ if and only if $x=y$ or $x=y^{-1}$, for exceptional units $x, y \in R$.

This fact allows us to conclude that since $R$ has $(n_1-2)n_2$ exceptional units, there will thus be exactly $\frac{1}{2}(n_1-2)n_2$ elements in $R$ of the form $u+u^{-1}$ (for $u$ an exceptional unit) and so $A(R)$ will have at least $\frac{1}{2}(n_1-2)n_2$

distinct elemtents. Since $A(R) \subseteq Q(R)$, this means that $Q(R)$ will have at least this many as well. Thus the index of the subgroup $Q(R)$ in $R$ will satisfy

$$[R: Q(R)] \leq \frac{n_1 n_2}{\frac{1}{2}(n_1-2)n_2} = \frac{2n_1 n_2}{n_1 n_2 - 2n_2}$$

$$= \frac{2n_1}{n_1-2} = 2 + \frac{4}{n_1-2} .$$

Now, we know that $n_1 \geq 8$, so that

$$\frac{4}{n_1-2} \leq \frac{2}{3} < 1 .$$

Hence $[R: Q(R)] < 3$, and so $[R: Q(R)] = 1$ or $2$. If it is $1$, then $Q(R) = R$, and the proof is complete. If it is $2$, then there is an element $\theta$ of $R$ such that $R = Q(R) \cup (\theta + Q(R))$.

We know by Proposition 2.2.1 and the results of Section 1.3 that every Hosszú function on $R$ is odd, and that $1$ (and hence $-1$) belong to $Q(R)$. Thus

$$h(x) + h(-1) = h(-x) + h(2x-1) \quad \text{(using (H))}$$

$$= h(-x) + h(2x) + h(-1).$$

Rearranging terms we see that $h(2x) = h(x) - h(-x) = 2h(x)$, for all $x \in R$.

Now let $y$ be an arbitrary element of $R$. Since $R = Q(R) \cup (\theta + Q(R))$, either $y \in Q(R)$ or $y = \theta + \alpha$ for some $\alpha \in Q(R)$. Then

$$\begin{aligned} h(y) + h(\theta) &= h(\theta+a) + h(\theta) \\ &= h(a) + 2h(\theta) \\ &= h(a) + h(2\theta) \\ &= h(a + 2\theta) \\ &= h(y + \theta). \end{aligned}$$

This shows that $\theta \in \alpha(R)$, and hence $\alpha(R) = R$. |||

From the two preceding theorems, we can immediately conclude that all Hosszú functions are additive on all finite fields of order greater than four, and on all infinite fields not of characteristic two. Davison [11] showed that this is also true for infinite fields of characteristic two.

In the following examples we give the Hosszú group and the universal Hosszú function for each field with four or fewer elements, showing that the hypothesis $N(m) > 4$ in the preceding theorems is necessary.

*EXAMPLE 1.* $\mathcal{H}(\mathbb{F}_2) \simeq \mathbb{Z}$, $h : \mathbb{F}_2 \longrightarrow \mathbb{Z}$

$: 0 \longmapsto 0$

$: 1 \longmapsto 1$

*EXAMPLE 2.* $\mathcal{H}(\mathbb{F}_3) \simeq \mathbb{Z}$, $h : \mathbb{F}_3 \longrightarrow \mathbb{Z}$

$: 0 \longmapsto 0$

$: 1 \longmapsto 2$

$: 2 \longmapsto 1$

*EXAMPLE 3.* $\mathbb{F}_4 = \{0, 1, \omega, 1+\omega : \omega^2 = \omega+1\}$

$$\mathcal{H}(\mathbb{F}_4) \simeq \mathbb{Z} \times C_2 , \qquad h : \mathbb{F}_4 \longrightarrow \mathbb{Z} \times C_2$$

$$: 0 \longmapsto (0,0)$$

$$: 1 \longmapsto (0,1)$$

$$: \omega \longmapsto (1,0)$$

$$: 1+\omega \longmapsto (1,1)$$

The calculations involved in these examples are quite straightforward and are left to the reader.

## 2.3 - ODD AND EVEN HOSSZÚ FUNCTIONS ON LOCAL RINGS

If $(R,m)$ is a local ring with $2 \notin m$, then $2$ is a unit in $R$, and so by Corollary 2.1.3, every odd or even Hosszú function on $R$ will be additive. If $(R,m)$ is a finite local ring with $2 \in m$, then we have already shown that except when $R/m$ is isomorphic to $\mathbb{F}_2$ or $\mathbb{F}_4$, every Hosszú function on $R$ is additive. We therefore examine the odd and even Hosszú functions in these two exceptional cases.

Since the defining generators of $A(R)$ belong to $\mathcal{D}(R)$, and since $\alpha^o(R)$ contains $\mathcal{D}(R)$ (see section 1.3), we know that $A(R) \subseteq \alpha^o(R)$.

Suppose now that $(R,m)$ is a finite local ring with $N(m) = 4$. Then $R$ will consist of four cosets modulo $m$, which we may write $m$, $1+m$, $\omega+m$, and $1+\omega+m$, where $\omega^2+\omega+1 \in m$ and $2\omega$, $2 \in m$.

Every element of the cosets $\omega+m$ and $1+\omega+m$ will be an exceptional unit of $R$. Moreover, if $x \in \omega+m$, then $x^{-1} \in 1+\omega+m$, and conversely, so that if $x$ is an exceptional unit, $x+x^{-1} \in 1+m$.

As in the proof of Theorem 2.2.3, we can show that if $x+x^{-1} = y+y^{-1}$ for exceptional units $x$ and $y$, then $x=y$ or $x=y^{-1}$. Thus the cardinality of the set

$$V = \{u+u^{-1} : u \text{ is an exceptional unit in } R\}$$

will be exactly one-half of the cardinality of the set of exceptional units, which in our case is half the cardinality of the ring itself. Thus $V$ contains one quarter of the elements of $R$. But we have seen that $V \subseteq 1+m$, and the cardinality of $1+m$ also is a quarter of that of $R$. Thus $V = 1+m$. Since $V \subseteq A(R) \subseteq \alpha^{o}(R)$, we have $1+m \subseteq \alpha^{o}(R)$, and hence also $m = 1+(1+m) \subseteq \alpha^{o}(R)$. This means that the (additive) index of $\alpha^{o}(R)$ in $R$ is at most 2, and so we can show, as in the proof of Theorem 2.2.3, that $\alpha^{o}(R) = R$. We have proved the following theorem.

*THEOREM 2.3.1. If $(R,m)$ is a finite local ring with $N(m) > 2$, then every odd Hosszú function on $R$ is additive.* |||

If $N(m) = 2$, then $R$ may or may not have non-additive odd Hosszú functions, as the following examples show.

*EXAMPLE 1.* $\mathbb{F}_2$

Every odd function on $\mathbb{F}_2$ is clearly additive.

*EXAMPLE 2.* $C_4$.

Define $f:C_4 \to C_2$ by $f(2)=1$, $f(0)=f(1)=f(3)=0$. Then $f$ is an odd Hosszú function, but $f$ is not additive, since $f(1) + f(1) \neq f(2)$.

*EXAMPLE 3.* $\mathbb{F}_2[X]/(X^2)$

Any function from this ring into $C_2$ will be an odd Hosszú function, and clearly not all such functions are additive.

Any local ring $(R,m)$ with $N(m) = 2$ and with less than five elements will be isomorphic to one of these three examples.

Note from the examples of the preceding section that if $(R,m)$ is local with $N(m) = 2$ or $4$, then $R$ will have non-additive even Hosszú functions. This fact, coupled with the remarks at the beginning of this section, proves the following result.

*THEOREM 2.3.2. If $(R,m)$ is a finite local ring, then every even Hosszú function on $R$ is additive if and only if $N(m) \neq 2$ or $4$.* |||

We will need these two theorems in Chapter 3.

# CHAPTER 3

## NUMBER RINGS

By the term "number ring", we mean the ring of integers in a finite extension of the rational field $\mathbb{Q}$—that is, the ring of those elements in such an extension which satisfy a monic polynomial over the ring of rational integers $\mathbb{Z}$. Number rings form an important and extensively studied class of commutative rings, and we will see in this chapter that the study of Hosszú's equation (and the associated equations $(H^o)$ and $(H^e)$) over such rings gives us a deeper insight into why all Hosszú functions on some rings are additive, while over other rings there are non-additive ones.

Throughout this chapter we will make free use of well-known facts about number rings—facts which can be found in most algebraic number theory texts (see, for example, Weiss [24] or Narkiewicz [18]). We have already seen the general solution of (H) over $\mathbb{Z}$, the simplest number ring, in the example in Section 1.2.

As we have seen from the preceding chapters, the units of a ring, and especially the exceptional units, play an important role in solving Hosszú's equation

over the ring. In number rings it is often difficult to find exceptional units. However, we do know enough about the ordinary units of such rings to enable us to determine whether or not they have non-additive odd or even Hosszú functions, and this we do in the first section of this chapter. In the second section, we prove some results about all Hosszú functions on number rings and then apply these results to cyclotomic number rings, for which we do know a lot about the exceptional units. In the third section we introduce a technique for studying Hosszú's equation over rings which are finitely generated as abelian groups. Finally, in Section 4 we use these techniques to completely describe the non-additive odd and even Hosszú functions on all number rings.

## 3.1 - ODD AND EVEN HOSSZÚ FUNCTIONS ON NUMBER RINGS

In view of Corollary 1.3.6, we will first determine some properties of the unit ring of a number ring.

Let $K$ be an extension of $\mathbb{Q}$ of degree $n$, and let $\mathcal{O}_K$ be the ring of integers in $K$. Let $L$ be the smallest subfield of $K$ containing $U(\mathcal{O}_K)$.

If $x$ is any element of $\mathcal{O}_K$, then $x$ will satisfy a monic integral polynomial, say

$$x^k + a_1 x^{k-1} + \ldots + a_k = 0,$$

where $a_i \in \mathbb{Z}$, $i=1,2,\ldots,k$, and $a_k \neq 0$. Then

$$x(x^{k-1}+a_1x^{k-2}+\ldots+a_{k-1}) = -a_k ,$$

so that

$$x\cdot\left[\frac{(x^{k-1}+a_1x^{k-2}+\ldots+a_{k-1})}{(-a_k)}\right] = 1 .$$

This shows that the inverse (in $K$) of any element $x$ of $\mathcal{O}_K$ is a rational multiple of sums and differences of powers of $x$. This in turn means that if $R$ is a subring of $\mathcal{O}_K$, the smallest subfield of $K$ containing $R$ is simply $\mathbb{Q}\cdot R$ (this product being taken in $K$). Hence the rank of $R$ as an abelian group, and the degree over $\mathbb{Q}$ of the smallest subfield of $K$ containing $R$, will be identical. In particular, the rank of $U(\mathcal{O}_K)$ will be equal to the degree of $L$ over $\mathbb{Q}$.

*PROPOSITION 3.1.1. The rank of $U(\mathcal{O}_K)$ is equal to the rank of $\mathcal{O}_L$, and the degree of the extension $[K:L]$ is either 1 or 2. Moreover, $[K:L] = 2$ only if all the embeddings of $L$ into $\mathbb{C}$ are real, and $K$ is an imaginary quadratic extension of $L$.*

*PROOF:* The first statement is clear from the above remarks, since $U(\mathcal{O}_K)$ and $\mathcal{O}_L$ both have the same field of fractions in $K$ (namely $L$).

Now recall that $[K:\mathbb{Q}] = n$ and suppose that

$[L:\mathbb{Q}] = m$. Let $r_1$ and $2r_2$ be the number of real and imaginary embeddings, respectively, of $K$ into $\mathbb{C}$, and let $s_1$ and $2s_2$ be the analogous constants for $L$. Then it is well known that $n = r_1+2r_2$, and $m = s_1+2s_2$.

By the Dirichlet-Minkowski Unit Theorem (see Narkiewicz [18], page 100) the rank of the group of units of $\mathcal{O}_K$ is $r_1+r_2-1$, while that of $\mathcal{O}_L$ is $s_1+s_2-1$. But by definition of $L$, $\mathcal{O}_K$ and $\mathcal{O}_L$ have the same group of units. Thus $r_1+r_2 = s_1+s_2$.

Suppose now that $K \neq L$. Then $[K:L] \geq 2$, so that $n \geq 2m$. Thus $r_1+2r_2 \geq 2s_1+4s_2$. But $2r_1+2r_2 = 2s_1+2s_2$, and so subtracting, we get $-r_1 \geq 2s_2$. Since $r_1$ and $s_2$ are both non-negative, this means that $r_1=s_2=0$. Hence $r_2=s_1$, and so $n=2m$, and the proposition is proved. |||

*LEMMA 3.1.2. If $M$ is an extension of $\mathbb{Q}$ of degree $\ell$, and if $A$ is an (additive) subgroup of $\mathcal{O}_M$ of rank $\ell$, then $A$ contains a nonzero ideal of $\mathcal{O}_M$.*

*PROOF.* Since $[M:\mathbb{Q}] = \ell$, $\mathcal{O}_M$ is a free abelian group of rank $\ell$, so that $A$ is a subgroup of $\mathcal{O}_M$ of the same rank. Thus there will exist a basis $[a_1, \ldots, a_\ell]$ of $\mathcal{O}_M$ as a $\mathbb{Z}$-module, and positive rational integers $m_1, m_2, \ldots, m_\ell$ having the property that $m_i$ divides $m_{i+1}$ (for $i = 1, 2, \ldots, \ell-1$) and $[m_1a_1, \ldots, m_\ell a_\ell]$ is a $\mathbb{Z}$-module basis for $A$ (see Fuchs [16], page 78). Then $m_\ell \mathcal{O}_M$ will be an ideal

of $\mathcal{O}_M$ contained in $A$. |||

What we intend to do is to show that $a^o(\mathcal{O}_K)$ and $a^e(\mathcal{O}_K)$ are of rank $n$ (where $n=[K:\mathbb{Q}]$) and thus conclude by the above lemma that these additive subgroups of $\mathcal{O}_K$ contain a nonzero ideal. This will allow us to use Proposition 1.5.1. Note that if $I$ is a nonzero ideal of $\mathcal{O}_K$, then $\mathcal{O}_K/I$ is a finite ring.

By Corollary 1.3.6 we know that

$$4U(\mathcal{O}_K) \subseteq a^o(\mathcal{O}_K) \cap a^e(\mathcal{O}_K) .$$

From Proposition 3.1.1 we know that the rank of $U(\mathcal{O}_K)$, and hence the rank of $4U(\mathcal{O}_K)$, is equal to the rank of $\mathcal{O}_L$, so by the above lemma, $4U(\mathcal{O}_K)$ will contain a non-zero ideal of $\mathcal{O}_L$. This ideal in turn will contain a non-zero principal ideal of $\mathcal{O}_L$ generated by a rational integer $\alpha$. Thus

$$\alpha\mathcal{O}_L \subseteq 4U(\mathcal{O}_K) \subseteq a^o(\mathcal{O}_K) \cap a^e(\mathcal{O}_K).$$

If $K=L$, then this shows that $a^o(\mathcal{O}_K)$ and $a^e(\mathcal{O}_K)$ both contain a nonzero ideal of $\mathcal{O}_K$. We will show that this is also true when $K\neq L$, in which case by Proposition 3.1.1, $[K:L]=2$ and $K$ is an imaginary quadratic extension of the field $L \subseteq \mathbb{R}$. This means that $K = L(\xi)$, where $\xi^2$ is some negative element of $\mathcal{O}_L$—that is, $\xi^2 + b = 0$ for some $b \in \mathcal{O}_L$. Thus $R = \mathcal{O}_L + \mathcal{O}_L\xi$ is a subring of $\mathcal{O}_K$ of rank $n$, so that

there exists $\delta \in \mathbb{N}$ such that if $x+y\xi \in \mathcal{O}_K$ $(x,y \in L)$, then $\delta x \in \mathcal{O}_L$ and $\delta y \in \mathcal{O}_L$.

If $f$ is an odd Hosszú function on $\mathcal{O}_K$, then from the equation $(H^o)$ it follows that for any $x_1, x_2 \in \mathcal{O}_L$,

$$f(x_1\xi) + f(\alpha x_2\xi) + f(-\alpha x_1 x_2 b) = f(x_1\xi+\alpha x_2\xi-\alpha x_1 x_2 b) .$$

Using the fact that $\alpha\mathcal{O}_L \subseteq \mathcal{A}^o(\mathcal{O}_K)$, we can conclude that

$$f(x_1\xi) + f(\alpha x_2\xi) = f(x_1\xi+\alpha x_2\xi) \qquad (*)$$

for all $x_1, x_2 \in \mathcal{O}_L$.

Now let $x+y\xi$ be an arbitrary element of $\mathcal{O}_K$ $(x,y \in L)$. Then, using equation $(H^o)$,

$$f(\alpha\delta\tau\xi) + f(x+y\xi) + f(\alpha\delta\tau(x\xi-by)) = f(\alpha\delta\tau\xi+x+y\xi+\alpha\delta\tau(x\xi-by))$$

for all $\tau \in \mathcal{O}_L$. Now use the fact that $\alpha\delta\tau by \in \alpha\mathcal{O}_L \subseteq \mathcal{A}^o(\mathcal{O}_K)$ (since $\delta y \in \mathcal{O}_L$) and the equation $(*)$ to conclude that

$$f(x+y\xi) + f(\alpha\delta\tau(x+1)\xi) = f(x+y\xi+\alpha\delta\tau(x+1)\xi) \qquad (**)$$

for all $x+y\xi \in \mathcal{O}_K$ and for all $\tau \in \mathcal{O}_L$. (We can use $(*)$ since $\delta x \in \mathcal{O}_L$.)

Choose one nonzero element from each of the (finitely many) cosets in $\mathcal{O}_L$ of the ideal $\alpha\delta\mathcal{O}_L$. Call this set of coset representatives $S$, and let

$$\pi = \prod_{s \in S} s .$$

If $w_1+w_2\xi$ is an arbitrary element of $\mathcal{O}_K$, then $\delta(w_1+1)$ belongs to some coset in $\mathcal{O}_L$ of $\alpha\delta\mathcal{O}_L$ (since $\delta(w_1+1) \in \mathcal{O}_L$), so that there will exist $z \in \mathcal{O}_L$ and $s_1 \in S$ such that $\delta(w_1+1) = s_1 + \alpha\delta z$. Hence $\delta((w_1-\alpha z)+1) = s_1$, so if we set $x = w_1-\alpha z$, $y=w_2$, and $\tau=v\pi s_1^{-1}$ ($v \in \mathcal{O}_L$ arbitrary) in equation (**) (noting that $s_1$ divides $\pi$), we see that

$$f(w_1-\alpha z+w_2\xi) + f(\alpha\pi v\xi) = f(w_1+w_2\xi-\alpha z+\alpha\pi v\xi).$$

But $\alpha z \in \alpha\mathcal{O}_L \subseteq \mathcal{A}^o(\mathcal{O}_K)$, so that

$$f(w_1+w_2\xi) + f(\alpha\pi v\xi) = f(w_1+w_2\xi+\alpha\pi v\xi).$$

Since $w_1+w_2\xi$ is an arbitrary element of $\mathcal{O}_K$, this equation shows that $\alpha\pi\mathcal{O}_L\xi \subseteq \mathcal{A}^o(\mathcal{O}_K)$. Let $\beta$ be a rational integer in $\alpha\pi\mathcal{O}_L$. Then $\beta\mathcal{O}_L\xi \subseteq \alpha\pi\mathcal{O}_L\xi \subseteq \mathcal{A}^o(\mathcal{O}_K)$, and $\beta\mathcal{O}_L \subseteq \alpha\mathcal{O}_L \subseteq \mathcal{A}^o(\mathcal{O}_K)$, so that $\beta\mathcal{O}_L + \beta\mathcal{O}_L\xi$ is a subgroup of $\mathcal{A}^o(\mathcal{O}_K)$ of rank $n$, and hence by Lemma 3.1.2, $\mathcal{O}_K$ has a nonzero ideal $I$ contained in $\mathcal{A}^o(\mathcal{O}_K)$.

Now if we had chosen $f$ to be even Hosszú rather than odd, and then used equation ($H^e$) instead of ($H^o$) while following through the steps of the above argument, we would also have arrived at equation (*) for our even $f$. So if in equation (*) we replace $x_1$ by $-\alpha z_1$ and $x_2$ by $z_2+z_1$, and if we then rearrange terms and use the evenness of $f$, we see that

$$f(\alpha z_2\xi) - f(\alpha z_1\xi) = f(\alpha z_1\xi+\alpha z_2\xi),$$

for all $z_1, z_2 \in \mathcal{O}_L$.

If we now continue the argument as before, still using $(H^e)$ instead of $(H^o)$, but this time using the preceding equality for our new, even $f$, instead of the equality (*) as before, we will again show that $f$ satisfies (**). Thus we can show as above that $\beta\mathcal{O}_L + \beta\mathcal{O}_L\xi \subseteq a^e(\mathcal{O}_K)$, and so we have proved the following proposition.

*PROPOSITION 3.1.3. There is a nonzero ideal $I$ of $\mathcal{O}_K$ with the property that $I \subseteq a^o(\mathcal{O}_K) \cap a^e(\mathcal{O}_K)$.* |||

We are now in a position to prove the main theorems of this section.

*THEOREM 3.1.4. Every odd Hosszú function on the number ring $\mathcal{O}_K$ is additive if and only if every ideal of $\mathcal{O}_K$ is of norm greater than 2.*

*PROOF.* Recall that the norm of an ideal $J$ in the ring $R$ is the cardinality of the factor ring $R/J$.

By Proposition 3.1.3, $\mathcal{O}_K$ has a nonzero ideal $I$ contained in $a^o(\mathcal{O}_K)$. Suppose that $I$ has the prime decomposition $I = \mathfrak{p}_1^{n_1}\mathfrak{p}_2^{n_2}\cdots\mathfrak{p}_k^{n_k}$, where the $\mathfrak{p}_i$'s are distinct prime ideals of $\mathcal{O}_K$. Then by the Chinese Remainder Theorem,

$$\mathcal{O}_K/I \cong \mathcal{O}_K/\mathfrak{p}_1^{n_1} \times \mathcal{O}_K/\mathfrak{p}_2^{n_2} \times \cdots \times \mathcal{O}_K/\mathfrak{p}_k^{n_k}.$$

We know by Proposition 1.5.1 that every odd Hosszú function

on $\mathcal{O}_K$ is additive if and only if every odd Hosszú function on $\mathcal{O}_K/I$ is additive, which in turn is true (by Proposition 1.4.1) if and only if every odd Hosszú function on each of the factors $\mathcal{O}_K/P_i^{n_i}$ is additive. Note that these factors are finite local rings with maximal ideal $P_i/P_i^{n_i}$, and that the norm of this maximal ideal is equal to the norm of $P_i$ in $\mathcal{O}_K$. Thus by Theorem 2.3.1, if every ideal in $\mathcal{O}_K$ has norm greater than 2, then every odd Hosszú function on $\mathcal{O}_K/P_i^{n_i}$ will be additive, and hence every odd Hosszú function on $\mathcal{O}_K$ will be additive.

On the other hand, if $\mathcal{O}_K$ does have an ideal $P$ of norm 2 ($P$ is necessarily prime), then the norm of $P^2$ will be 4, so that $\mathcal{O}_K/P^2$ will be a local ring with four elements. This means that it is isomorphic either to Example 2 or Example 3 of Section 2.3, and thus will have non-additive odd Hosszú functions, so that $\mathcal{O}_K$ has non-additive odd Hosszú functions as well. |||

For the even Hosszú functions we have a similar theorem.

*THEOREM 3.1.5. Every even Hosszú function on $\mathcal{O}_K$ is additive if and only if $\mathcal{O}_K$ has no ideals of norm 2 or 4.*

*PROOF.* If $\mathcal{O}_K$ has an ideal of norm 2 or 4, then it will have $\mathbb{F}_2$ or $\mathbb{F}_4$ as a homomorphic image, so from the examples

of Section 2.2, we can see that $\mathcal{O}_K$ will have nonadditive even Hosszú functions.

By Proposition 3.1.3, $\mathcal{O}_K$ has a nonzero ideal $I$ contained in $\mathcal{A}^e(\mathcal{O}_K)$. Let $2I = Q_1^{m_1} Q_2^{m_2} \cdots Q_t^{m_t}$, where the $Q_i$'s are distinct prime ideals of $\mathcal{O}_K$. Then using Proposition 1.5.1 and the same argument as before in the odd case, we can show that every even Hosszú function on $\mathcal{O}_K$ is additive if every even Hosszú function on $\mathcal{O}_K/Q_i^{m_i}$ is additive for $i=1,2,\ldots,t$. This will be true if no ideal of $\mathcal{O}_K$ has norm 2 or 4, by Theorem 2.3.2. |||

If $K$ is a quadratic extension of $\mathbb{Q}$, then the ideal $2\mathcal{O}_K$ (which has norm 4) either will be prime, or else will factor into two prime ideals each of norm 2. The former case occurs if the discriminant of $K$ (written $d(K)$) is congruent to 5 modulo 8, while the latter case occurs for all other discriminants (see Narkiewicz [18], page 166). So the following result is an immediate consequence of the two preceding theorems.

*COROLLARY 3.1.6. If $\mathcal{O}_K$ is a quadratic number ring, then there will exist non-additive even Hosszú functions on $\mathcal{O}_K$, and every odd Hosszú function on $\mathcal{O}_K$ will be additive if and only if $d(K) \equiv 5 \pmod 8$.* |||

For general number rings $\mathcal{O}_K$ we will give in Section 3.4 a characterization of the odd and even Hosszú functions

that occur when $\mathcal{O}_K$ does have an ideal of norm 2 or 4.

## 3.2 - HOSSZÚ FUNCTIONS ON NUMBER RINGS

For certain number rings we can apply our knowledge of the odd and even Hosszú functions to come to some conclusions about all Hosszú functions on those rings. From Theorem 3.1.4, we see that if there exist non-additive odd Hosszú functions on $\mathcal{O}_K$, then there is an ideal of norm 2 in $\mathcal{O}_K$. But then by Theorem 3.1.5, $\mathcal{O}_K$ will also have non-additive even Hosszú functions, and from Example 1 of Section 2.2, it is clear that not all of these will be odd. So if we can show that every Hosszú function on $\mathcal{O}_K$ is odd, we will already have shown that every Hosszú function on $\mathcal{O}_K$ is in fact additive. This fact, coupled with Proposition 1.3.4, gives us the next result.

*THEOREM 3.2.1. If $a(\mathcal{O}_K) = \mathcal{D}(\mathcal{O}_K)$, then every Hosszú function on $\mathcal{O}_K$ is additive.* |||

The following corollary is a consequence of the above theorem and the remarks immediately preceding Proposition 1.3.4.

*COROLLARY 3.2.2. If $\mathcal{O}_K$ contains an exceptional unit $u$ with the property that $1-u-u^{-1}$ is also a unit, then every Hosszú function on $\mathcal{O}_K$ is additive.* |||

As an application of this corollary, we will study Hosszú functions on cyclotomic number rings.

Let $\zeta_m$ denote a primitive $m$-th root of unity, and let $\mathcal{O}_m$ be the ring of integers in the cyclotomic field $\mathbb{Q}(\zeta_m)$. It is well known (see Weiss [24], page 267) that if $m$ is a power of a prime number, then $(1-\zeta_m^a)/(1-\zeta_m)$ is a unit in $\mathcal{O}_m$, while if $m$ is composite, $1-\zeta_m^a$ is a unit in $\mathcal{O}_m$, where $a$ is any natural number relatively prime to $m$.

Now suppose that $m$ is a positive integer not of the form $2^r 3^s$, where $r, s \in \mathbb{N}$. Then $m$ will have a prime divisor $p$ not equal to 2 or 3, and $\mathcal{O}_p$ will thus be a subring of $\mathcal{O}_m$. Since $p$ is odd,

$$1 + \zeta_p = \frac{1 - \zeta_p^2}{1 - \zeta_p}$$

will be a unit in $\mathcal{O}_p$, and so $-\zeta_p$ will be an exceptional unit in $\mathcal{O}_p$. We will now show that $1-(-\zeta_p)-(-\zeta_p)^{-1}$ is also a unit.

$$\begin{aligned} 1+\zeta_p+\zeta_p^{-1} &= 1+\zeta_p+\zeta_p^{p-1} \\ &= 1+\zeta_p+(-1-\zeta_p-\zeta_p^2-\ldots-\zeta_p^{p-2}) \\ &= -\zeta_p^2(1+\zeta_p+\zeta_p^2+\ldots+\zeta_p^{p-4}) \\ &= -\zeta_p^2\left(\frac{1-\zeta_p^{p-3}}{1-\zeta_p}\right) \end{aligned}$$

Since $p$ is relatively prime to 3, $p$ will also be relatively

prime to $p-3$, so that the last expression above is a unit in $\mathcal{O}_p$, and hence also in $\mathcal{O}_m$. This shows that $\mathcal{O}_m$ satisfies the hypotheses of Corollary 3.2.2, and hence every Hosszú function on $\mathcal{O}_m$ is additive.

Let us now suppose that $m = 2^r 3^s$, where both $r$ and $s$ are natural numbers greater than or equal to 2. Then $\zeta_m^6$ will be a primitive $m/6$-th root of unity, so that $1-\zeta_m^6$ will be a unit in $\mathcal{O}_{m/6}$ (since $m/6$ is composite), and hence in $\mathcal{O}_m$. But $1-\zeta_m^6 = (1-\zeta_m^3)(1+\zeta_m^3)$, so that $1+\zeta_m^3$ is a unit in $\mathcal{O}_m$. Also, $\zeta_m^2$ will be a primitive $m/2$-th root of unity, so that $1-\zeta_m^2$ will be a unit in $\mathcal{O}_{m/2}$ (and hence in $\mathcal{O}_m$).

Now, $1-\zeta_m$ is a unit in $\mathcal{O}_m$, and hence an exceptional unit. We will now show that $1-(1-\zeta_m)-(1-\zeta_m)^{-1}$ is a unit.

$$\begin{aligned} 1-(1-\zeta_m)-(1-\zeta_m)^{-1} &= \zeta_m-(1-\zeta_m)^{-1} \\ &= -\left(\frac{\zeta_m^2-\zeta_m+1}{1-\zeta_m}\right) \\ &= -\left(\frac{1+\zeta_m^3}{1-\zeta_m^2}\right) \end{aligned}$$

This will be a unit, since it is the quotient of two units. So again, by Corollary 3.2.2, every Hosszú function on $\mathcal{O}_m$ is additive.

If $m$ is of the form $2^r$, $3^s$, $2\cdot 3^s$, or $2^r\cdot 3$, where $r, s \in \mathbb{N}$, then $\mathcal{O}_m$ will have a prime ideal of norm 2, 3, or

4 (see Weiss [24], pages 262-263), so that there will exist non-additive Hosszú functions on $\mathcal{O}_m$ (as we can see from the examples of Section 2.2). We summarize our results in the following theorem.

*THEOREM 3.2.3. Every Hosszú function on the cyclotomic number ring $\mathcal{O}_m$ is additive if and only if m is not of the form $2^r$, $3^s$, $2\cdot 3^s$, or $2^r\cdot 3$, where $r,s \in \mathbb{N}$.* |||

*COROLLARY 3.2.4. If the number ring $\mathcal{O}_K$ contains a primitive m-th root of unity, where m is not of the form $2^r$, $3^s$, $2\cdot 3^s$, or $2^r\cdot 3$ ($r,s \in \mathbb{N}$), then every Hosszú function on $\mathcal{O}_K$ is additive.*

*PROOF.* $\mathcal{O}_K$ will contain $\mathcal{O}_m$, which as we have seen in the proof of the above theorem contains a unit with the property required in Corollary 3.2.2. |||

## 3.3 - DIVISIBLE HOSSZÚ FUNCTIONS

Divisible groups are abelian groups with the property that given any element $a$ of the group, and any non-zero integer $n$, there exists an element $x$ in the group such that $nx=a$. The divisible groups are precisely the injective abelian groups (see Fuchs [16], page 107). If $G$ is any abelian group, then we can embed $G$ into a divisible group $DG$, called the *divisible hull* of $G$.

It is minimal in the sense that if $E$ is another divisible group containing $G$, then there is a monomorphism from $DG$ to $E$ leaving the elements of $G$ fixed. Moreover, $DG$ is unique up to isomorphism. (See Fuchs [16], pages 106-107.)

Let $d:G \to DG$ denote a fixed embedding of $G$ into $DG$. If $R$ is any ring, and $f:R \to G$ any Hosszú function, then clearly $d \circ f:R \to DG$ is also a Hosszú function. If $f$ is odd or even, then $d \circ f$ will also be odd or even respectively. Moreover, if $f$ and $g$ are two different Hosszú functions from $R$ to $G$, then clearly $d \circ f$ and $d \circ g$ will be different Hosszú functions from $R$ to $DG$ (since $d$ is a monomorphism). This means that the set of Hosszú functions from $R$ to $G$ is in one-to-one correspondence with the set of those Hosszú functions from $R$ to $DG$ whose images lie entirely in $d(G)$.

If $f:R \to G$ is a Hosszú function, then a *corestriction* of $f$ is a Hosszú function $g:R \to H$ with the property that there exists a group monomorphism $\phi:H \to G$ such that $\phi \circ g = f$. We will also say that $f$ is a *coextension* of $g$. The discussion of the preceding paragraphs tells us that every Hosszú function on $R$ is a corestriction of a Hosszú function from $R$ into a divisible group. Let us call a Hosszú function $f:R \to E$ whose range $E$ is a divisible group a *divisible Hosszú function*.

Let $Qdiv(R) = \{a \in R : f(a)+f(x)=f(a+x)$ *for all* $x \in R$, *and for all divisible Hosszú* $f$ *on* $R\}$,

and let $\mathcal{A}^o div(R)$ and $\mathcal{A}^e div(R)$ denote the analogous objects for the odd and even divisible Hosszú functions respectively.

*PROPOSITION 3.3.1.*

*(a) Every (odd, even) Hosszú function on the ring $R$ is the corestriction of an (odd, even) divisible Hosszú function on $R$.*

*(b) Let $d: \mathcal{H}(R) \to D\mathcal{H}(R)$ be a fixed embedding of $\mathcal{H}(R)$ into its divisible hull. If $f: R \to E$ is a divisible Hosszú function, then there exists a group homomorphism $\psi: D\mathcal{H}(R) \to E$ such that $\psi \circ d \circ h = f$ (where $h$ is the universal Hosszú function on $R$).*

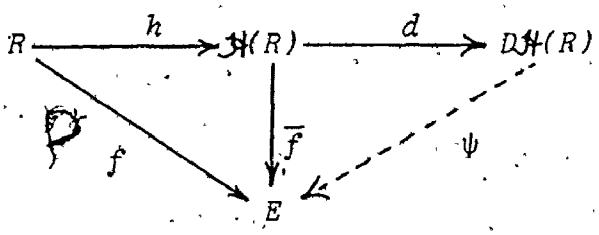
*(c)* $\mathcal{A}(R) = \mathcal{A} div(R)$

$\mathcal{A}^o(R) = \mathcal{A}^o div(R)$

$\mathcal{A}^e(R) = \mathcal{A}^e div(R)$ .

*PROOF.* Part (a) has just been shown above.

(b) Let $\overline{f}: \mathcal{H}(R) \to E$ be the unique group homomorphism satisfying $\overline{f} \circ h = f$ (Proposition 1.2.1).




Since divisible groups are injective, and since $d$ is a monomorphism, there exists a group homomorphism

$\psi: D\mathcal{H}(R) \to E$ such that $\psi \circ d = \overline{f}$. So $\psi \circ d \circ h = \overline{f} \circ h = f$.

(c) Clearly $\mathcal{Q}(R) \subseteq \mathcal{Q}div(R)$ (by part (b) and the definition of $\mathcal{Q}(R)$). On the other hand, if $x \in \mathcal{Q}div(R)$, then $d \circ k(x+y) = d \circ h(x) + d \circ h(y)$ for all $y \in R$, since $d \circ h$ is a divisible Hosszú function. Hence $d(h(x+y)) = d(h(x)+h(y))$, and so $h(x+y) = h(x) + h(y)$, since $d$ is one-to-one. Hence $x \in \mathcal{Q}(R)$, and so $\mathcal{Q}(R) = \mathcal{Q}div(R)$.

The proofs of the other two parts of (c) will be similar. |||

This proposition tells us that if we can find all the divisible Hosszú functions on $R$, we can characterize all Hosszú functions on $R$ as corestrictions of these.

Recall from Section 1.5 that if $J$ is an ideal of $R$, the lift $\tilde{f}: R \to G$ of a function $f: R/J \to G$ is defined by $\tilde{f}(r) = f(r+J)$, for all $r \in R$. The next theorem provides a strengthening of Propositions 1.5.1 and 1.5.2 for rings which are finitely generated free $\mathbb{Z}$-modules.

*THEOREM 3.3.2. If $R$ (in its additive structure) is a finitely generated free abelian group, and if $I$ is an ideal of $R$ contained in $\mathcal{Q}(R)$ (respectively $\mathcal{Q}^o(R)$), then every divisible Hosszú function (respectively odd Hosszú function) $f: R \to D$ is the sum of an additive function $f_1: R \to D$ and the lift of a Hosszú function*

*(respectively odd Hosszú function)* $f_2: R/I \to D$.

*PROOF.* Since $I$ is a subgroup of the finitely generated free abelian group $R$, we can find a basis $[a_1, a_2, \ldots, a_n]$ of $R$ as a $\mathbb{Z}$-module (where $n$ is the rank of $R$) such that $[m_1a_1, m_2a_2, \ldots, m_na_n]$ is a basis for $I$, for some non-negative integers $m_1, \ldots, m_n$ (see Fuchs [16] page 78).

Let $f: R \to D$ be a divisible Hosszú function. For $i=1,2,\ldots,n$, choose $x_i \in D$ such that $m_ix_i = f(m_ia_i)$ (using the fact that $D$ is divisible). Define $f_1: R \to D$ by

$$f_1\left(\sum_{i=1}^{n} k_ia_i\right) = \sum_{i=1}^{n} k_ix_i ,$$

for all $k_i \in \mathbb{Z}$. Let $f_2: R \to D$ be the function $f-f_1$. Obviously $f_1$ is additive, and hence Hosszú, so that $f_2$ will also be Hosszú.

Let $\alpha \in I$, so that $\alpha = l_1m_1a_1 + l_2m_2a_2 + \ldots + l_nm_na_n$ for some list of integers $[l_1, l_2, \ldots, l_n]$. Now, since $m_ia_i \in I \subseteq Q(R)$ for $i=1,2,\ldots,n$, we know that

$$\begin{aligned} f(\alpha) &= \sum_{i=1}^{n} l_i f(m_ia_i) \\ &= \sum_{i=1}^{n} l_im_ix_i \\ &= f_1\left(\sum_{i=1}^{n} l_im_ia_i\right) \\ &= f_1(\alpha) . \end{aligned}$$

Thus if $y \in R$, $\tilde{f}_2(y+\alpha) = f(y+\alpha) - f_1(y+\alpha) = f(y)+f(\alpha)-f_1(y)-f_1(\alpha)$

$= f(y) - f_1(y) = \tilde{f}_2(y)$. Hence the function $\tilde{f}_2$ is constant on the cosets of $I$ in $R$, and so is the lift of a Hosszú function $f_2: R/I \to D$. Thus $f = f_1 + \tilde{f}_2$ is the decomposition of $f$ required in the statement of the theorem.

The odd case is proved in the same way. |||

*COROLLARY 3.3.3. If $R$ and $I$ are as in the preceding theorem, then $x+I \in \mathcal{A}(R/I)$ (respectively $\mathcal{A}^o(R/I)$) implies $x \in \mathcal{A}(R)$ (respectively $\mathcal{A}^o(R)$).*

*PROOF.* Suppose that $x+I \in \mathcal{A}(R/I)$. Using the notation of the preceding proof, if $f: R \to D$ is any divisible Hosszú function, then for any $y \in R$,

$$\begin{aligned} f(x+y) &= f_1(x+y) + \tilde{f}_2(x+y) \\ &= f_1(x) + f_1(y) + \tilde{f}_2(x+y) \end{aligned}$$

Now, $\tilde{f}_2(x+y) = f_2(x+y+I) = f_2(x+I) + f_2(y+I)$ (since $x+I \in \mathcal{A}(R/I)$), so that $\tilde{f}_2(x+y) = \tilde{f}_2(x) + \tilde{f}_2(y)$. Thus

$$\begin{aligned} f(x+y) &= f_1(x) + f_1(y) + \tilde{f}_2(x) + \tilde{f}_2(y) \\ &= (f_1+\tilde{f}_2)(x) + (f_1+\tilde{f}_2)(y) \\ &= f(x) + f(y). \end{aligned}$$

This means that $x \in \mathcal{A}(R)$.

The odd case again has identical proof. |||

For the even Hosszú functions, we have two slightly different counterparts of the above theorem and corollary.

*PROPOSITION 3.3.4. If the ideal $I$ of the ring $R$ is contained in $\mathcal{Q}^e(R)$, then every even Hosszú function $g:R \to G$ is the lift of an even Hosszú function $\bar{g}:R/2I \to G$.*

*PROOF.* As we have seen in the proof of Proposition 1.5.1, $g$ will be constant on the cosets modulo $2I$ in $R$. From this the result follows immediately. |||

*THEOREM 3.3.5. If $R$ (in its additive structure) is a finitely generated free abelian group, and if the ideal $I$ of $R$ is contained in $\mathcal{Q}^e(R)$, then every even divisible Hosszú function $g:R \to D$ is the sum of an additive function $g_1:R \to D$ and the lift of a divisible Hosszú function $g_2:R/I \to D$. Moreover, if $x+I \in \mathcal{Q}^e(R/I)$, then $x \in \mathcal{Q}^e(R)$.*

*PROOF.* This is proved in the same way that we proved Theorem 3.3.2 and Corollary 3.3.3. |||

If the ring $R$ is isomorphic to a quotient $S/J$ of the ring $S$, which is a finitely generated free abelian group, then every Hosszú function on $R$ lifts to a Hosszú function on $S$. Hence we can apply the preceding methods and results to prove the following corollary.

*COROLLARY 3.3.6. Theorems 3.3.2. and 3.3.5., and Corollary 3.3.3 remain true if $R$ is a quotient ring*

*of a ring which is a finitely generated free abelian group in its additive structure.* |||

## 3.4 - DIVISIBLE ODD AND EVEN HOSSZÚ FUNCTIONS ON NUMBER RINGS

Since number rings are finitely generated free abelian groups, we will be able to use the results of the preceding section to determine all odd and even divisible Hosszú functions on such rings. By doing so, we will (in view of Proposition 3.3.1) characterize all odd and even Hosszú functions on number rings.

We will first study the odd divisible Hosszú functions. From Theorem 3.1.4 we know that the number ring $\mathcal{O}_K$ will have non-additive odd Hosszú functions if and only if $\mathcal{O}_K$ has an ideal of norm 2. We also know from Proposition 3.1.3 that $\mathcal{O}_K$ has a nonzero ideal $I$ contained in $a^o(\mathcal{O}_K)$. Suppose that

$$I = P_1^{n_1} P_2^{n_2} \cdots P_k^{n_k} Q_1^{m_1} Q_2^{m_2} \cdots Q_l^{m_l}$$

where the exponents are natural numbers, and $P_i, Q_j$ are prime ideals of $\mathcal{O}_K$ with $N(P_i) = 2$, $i=1,2,\ldots,k$, and $N(Q_j) > 2$, $j=1,2,\ldots,l$. Then

$$\mathcal{O}_K/I \simeq \mathcal{O}_K/P_1^{n_1} \times \ldots \times \mathcal{O}_K/P_k^{n_k} \times \mathcal{O}_K/Q_1^{m_1} \times \ldots \times \mathcal{O}_K/Q_l^{m_l}$$

by the Chinese Remainder Theorem. From the results of

Section 1.4, we see that any odd Hosszú function on $\mathcal{O}_K/I$ is the direct product of odd Hosszú functions on each of the direct factors in the above decomposition of $\mathcal{O}_K/I$. But each of these factors is a finite local ring, and the norm of the maximal ideal $Q_i/Q_i^m$ in $\mathcal{O}_K/Q_i^m$ will be greater that 2, for $i=1,2,\ldots,l$. Thus by Theorem 2.3.1, every odd Hosszú function on $\mathcal{O}_K/Q_i^m$ is additive, so that the ideal $J = \{0\}\times\{0\}\times\ldots\times\{0\}\times \mathcal{O}_K/Q_1^{m_1}\times\ldots\times\mathcal{O}_K/Q_l^m$ of $\mathcal{O}_K/I$ will be contained in $\mathcal{A}^o(\mathcal{O}_K/I)$. This means (by Corollary 3.3.3) that the inverse image of $J$ under the canonical map $\nu: \mathcal{O}_K \to \mathcal{O}_K/I$ will be in $\mathcal{A}^o(\mathcal{O}_K)$. Clearly $\nu^{-1}(J) = P_1^{n_1}P_2^{n_2}\ldots P_k^{n}$. Theorem 3.3.2 now tells us that every odd divisible Hosszú function on $\mathcal{O}_K$ is the sum of an additive function and the lift of an odd divisible Hosszú function on $\mathcal{O}_K/P_1^{n_1}\ldots P_k^{n} \simeq \mathcal{O}_K/P_1^{n_1} \times \mathcal{O}_K/P_2^{n_2} \times \ldots \times \mathcal{O}_K/P_k^{n}$. This means that to solve the problem of finding all odd divisible Hosszú functions on $\mathcal{O}_K$, we must determine all the odd divisible Hosszú functions on the finite local rings $\mathcal{O}_K/P^s$, where P is a prime ideal of norm 2 in $\mathcal{O}_K$. To this end, we first prove the following lemma about the structure of such rings. (It is a special case of a theorem of Clark and Drake [7].)

In the proof of the lemma, we make use of the well-known *Nakayama Lemma*, which says that if $R$ is a commutative ring, $M$ a finitely generated $R$-module, $N$ a

submodule of $M$, and $I$ an ideal contained in the Jacobson radical of $R$, then $M = IM+N$ implies $M=N$ (see Atiyah and Macdonald [2] pages 21-22).

*LEMMA 3.4.1. Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_K$ of norm 2. Let $S$ be the subring of $R = \mathcal{O}_K/\mathfrak{P}^s$ generated by 1, the identity element of $R$. Then there exists $\theta \in R$ such that $R = S + S\theta + S\theta^2 + \ldots + S\theta^{s-1}$, and $\theta^s = 0$.*

*PROOF.* Let $m = \mathfrak{P}/\mathfrak{P}^s$ be the maximal ideal of $R$. We first show that $m$ is principal. Since $R/m \simeq \mathcal{O}_K/\mathfrak{P}$, which has two elements, it follows that $m/m^2$ will contain two elements. Let $\theta \in m\backslash m^2$, and consider the composite map $R\theta \to m \to m/m^2$, where the first map is the embedding and the second is the natural map. Since $\theta \notin m^2$, the image of $\theta$ under this composite map will be nonzero in the group $m/m^2$, so that the map is onto (since $m/m^2$ contains only two elements). Thus $m = m^2+R\theta$. By the Nakayama Lemma (taking $N=R$, $M=I=m$) we see that $m=R\theta$. Now, every element of $R$ is either in $m$ or $1+m$, so that $R = S+m = S+R\theta$. Thus

$$\begin{aligned} R &= S + R\theta \\ &= S + (S+R\theta)\theta \\ &= S + S\theta + R\theta^2 \\ &= S + S\theta + (S+R\theta)\theta^2 \\ &= \ldots = S + S\theta + S\theta^2 + \ldots + S\theta^{s-1} + R\theta^s. \end{aligned}$$

But $\theta^s \in m^s = 0$, and so the lemma is proved. |||

We will now use this structure lemma to determine $\mathcal{A}^o(R)$, where $R = \mathcal{O}_K/\mathfrak{P}^s$ and $N(\mathfrak{P}) = 2$. Since $R$ is generated by its units (being local), it follows from Corollary 1.3.6 that $4R \subseteq \mathcal{A}^o(R)$. Moreover, since all elements of $1+m$ are units, we can use Proposition 1.3.5 to conclude that $2((1+m)-1) = 2m \subseteq \mathcal{A}^o(R)$.

Now let us suppose that $s \geq 3$. We will show that $\theta^{s-1}$ (where we use the notation of the above lemma) belongs to $\mathcal{A}^o(R)$.

Every element of $R$ is either in $m$ ( $= Rm = S\theta+S\theta^2+\ldots+S\theta^{s-1}$) or in $1+m$. This means that every element in $R$ can be written in the form $P(\theta)$ or $1+P(\theta)$, where $P(X)$ is a polynomial with coefficients in $S$ and with constant term $0$.

Let $f$ be any odd Hosszú function on $R$. Then, using equation $(H^o)$, we get

$$f(\theta^{s-2}) + f(\theta^{s-1}) + f(0) = f(\theta^{s-2}+\theta^{s-1}) . \qquad (a)$$

(Note that $s \geq 3$ is crucial at this step.) Using $(H^o)$ again, we also have

$$f(\theta^{s-2}) + f(1+\theta) + f(\theta^{s-2}+\theta^{s-1}) = f(2\theta^{s-2}+\theta^{s-1}+\theta+1).$$

Using (a) and the fact that $2\theta^{s-2} \in 2m \subseteq \mathcal{A}^o(R)$, we obtain

$$f(\theta^{s-1}) + f(1+\theta) = f(1+\theta+\theta^{s-1}) . \qquad \text{(b)}$$

Now $f(\theta^{s-1}+1) + f(\theta^{s-1}+\theta+1) + f(2\theta^{s-1}+\theta+1) = f(4\theta^{s-1}+2\theta+3)$, so $f(\theta^{s-1}+1)+f(\theta^{s-1})+f(\theta+1)+f(2\theta^{s-1})+f(\theta+1)=f(4\theta^{s-1})+f(2\theta+3)$, (from (b) and the fact that $2m \subseteq \mathcal{A}^{o}(R)$). Collecting terms and adding $f(1)$ to both sides, we see that

$$f(\theta^{s-1}+1) + f(1) + f(\theta+1) + f(\theta+1) = f(\theta^{s-1}) + f(2\theta+3) + f(1).$$

But note that $f(1) + f(\theta+1) + f(\theta+1) = f(2\theta+3)$ (by $(H^{o})$, so that

$$f(\theta^{s-1}+1) = f(\theta^{s-1}) + f(1) . \qquad \text{(c)}$$

Now let $P(X)$ be a polynomial with coefficients in $S$ and with constant term $0$, as described above. Then $f(\theta^{s-1}+1) + f(\theta^{s-1}+P(\theta)+1) + f(2\theta^{s-1}+P(\theta)+1) = f(4\theta^{s-1}+2P(\theta)+3)$, so using (c), $f(\theta^{s-1}) + f(1) + f(\theta^{s-1}+P(\theta)+1) + f(2\theta^{s-1}) + f(P(\theta)+1) = f(4\theta^{s-1}) + f(2P(\theta)+3)$. If we collect terms, add $f(P(\theta)+1)$ to both sides of this equation, and note that $f(1) + f(P(\theta)+1) + f(P(\theta)+1) = f(2P(\theta)+3)$, we see immediately that

$$f(\theta^{s-1}+P(\theta)+1) = f(\theta^{s-1}) + f(P(\theta)+1).$$

Furthermore, it is obvious that

$$f(\theta^{s-1}) + f(P(\theta)) = f(\theta^{s-1}+P(\theta)),$$

since $\theta^{s-1}P(\theta) = 0$. This shows that $\theta^{s-1} \in \mathcal{A}^{o}(R)$, and

hence $m^{s-1} = S\theta^{s-1} \leq \mathcal{A}^0(R)$.

Theorem 3.3.2 and Corollary 3.3.6 now tell us that every odd divisible Hosszú function on $R$ is the sum of an additive function and the lift of an odd divisible Hosszú function on $R/m^{s-1} \simeq \mathcal{O}_K/P^{s-1}$. So we have reduced the problem of finding all the odd divisible Hosszú functions on $\mathcal{O}_K/P^s$ to the same problem on $\mathcal{O}_K/P^{s-1}$. Now we can use the same procedure to reduce the problem to $\mathcal{O}_K/P^{s-2}$, and so on, showing in fact that every odd divisible Hosszú function on $\mathcal{O}_K/P^s$ is the sum of an additive function and the lift of an odd divisible Hosszú function on $\mathcal{O}_K/P^2$ (recall the restriction $s \geq 3$ in the above reduction procedure).

We summarize our results in the following theorem.

*THEOREM 3.4.2. If the number ring $\mathcal{O}_K$ has the prime ideals $P_1, P_2, \ldots, P_k$ of norm 2, then every odd divisible Hosszú function on $\mathcal{O}_K$ is the sum of an additive function and lifts of odd divisible Hosszú functions on the factor rings $\mathcal{O}_K/P_i^2$.* |||

As we have seen in Section 2.3, the only local rings of cardinality 4 and with maximal ideal of norm 2 are (up to isomorphism) $C_4$ and $\mathbb{F}_2[X]/(X^2)$. Thus each of the rings $\mathcal{O}_K/P_i^2$ appearing in the above theorem must be isomorphic to one of these two rings. We give their

Hosszú groups and universal Hosszú functions in the following proposition.

*PROPOSITION 3.4.3.*

(a) $\mathcal{N}(C_4) \simeq \mathbb{Z} \times C_2 \times C_2 \times C_2$

$h: C_4 \longrightarrow \mathbb{Z} \times C_2 \times C_2 \times C_2$

$: 1 \longmapsto (1,0,1,0)$

$: 2 \longmapsto (0,1,0,0)$

$: 3 \longmapsto (1,0,0,1)$ , *and* $h(0) = 0$.

(b) $\mathcal{N}(\mathbb{F}_2[X]/(X^2)) \simeq \mathbb{Z} \times C_2 \times C_2 \times C_2$

$h: \mathbb{F}_2[X]/(X^2) \longrightarrow \mathbb{Z} \times C_2 \times C_2 \times C_2$

$: I \longmapsto (1,0,1,0)$

$: X \longmapsto (0,1,0,0)$

$: X+1 \longmapsto (1,0,0,1)$ , *and* $h(0) = 0$.

*PROOF.* Both parts of the proposition can be proved easily directly from the definition of Hosszú group and universal Hosszú function given in Section 1.2. |||

The above proposition combined with the preceding theorem completely solves the problem of finding all odd divisible Hosszú functions on number rings.

Let us now consider the case of even divisible Hosszú functions. Again by Proposition 3.1.3, $\mathcal{O}_K$ will have a nonzero ideal $I$ contained in $\mathcal{Q}^e(\mathcal{O}_K)$. If $2I$ has the prime decomposition $2I = P_1^{s_1} P_2^{s_2} \ldots P_r^{s_r}$, then by

Proposition 3.3.4, every even Hosszú function on $\mathcal{O}_K$ is the lift of an even Hosszú function on

$$\mathcal{O}_K/2I \simeq \mathcal{O}_K/P_1^{s_1} \times \ldots \times \mathcal{O}_K/P_r^{s_r},$$

and by Proposition 1.4.1, every even Hosszú function on this ring is the sum of even Hosszú functions on the direct factors $\mathcal{O}_K/P_i^{s_i}$. By Theorem 2.3.2, every even Hosszú function on $\mathcal{O}_K/P_i^{s_i}$ is additive if and only if $N(P_i) \neq 2$ or $4$, and so we are reduced to finding all even divisible Hosszú functions on the rings $\mathcal{O}_K/P^s$, where $N(P) = 2$ or $4$.

If $N(P) = 2$, then we can refine and suitably modify the argument used in the odd case to show that every even divisible Hosszú function on $\mathcal{O}_K/P^s$ is the sum of an additive function and the lift of a Hosszú function on $\mathcal{O}_K/P^2$. Let us therefore consider those rings $\mathcal{O}_K/P^s$ where $N(P) = 4$. We will write $R = \mathcal{O}_K/P^s$ and $m = P/P^s$, the maximal ideal of $R$. Since $N(P) = 4$, $R/m \simeq \mathbb{F}_4$.

Let $f$ be an even divisible Hosszú function on $R$, and let $d \in \mathcal{A}(R)$. Then $f(d+x) = f(d)-f(-x) = f(d)-f(x)$, for $x \in R$. So if $d_1, d_2 \in \mathcal{A}(R)$, then

$$\begin{aligned} f(d_1+d_2+x) &= f(d_1) - f(d_2+x) \\ &= f(d_1) - f(d_2) + f(x) \\ &= f(d_1+d_2) + f(x), \end{aligned}$$

for all $x \in R$. In particular, since $R$ has exceptional

units, $1 \in \mathcal{D}(R)$ (by Corollary 1.3.2), so that $f(1+d+x) = f(1+d)+f(x)$ for all $d \in \mathcal{D}(R)$ and all $x \in R$; that is, $1 + \mathcal{D}(R) \subseteq \mathcal{A}^e(R)$. If we let

$$V = \{u+u^{-1} : u \text{ is an exceptional unit in } R\},$$

then $V \subseteq \mathcal{D}(R)$ (by Proposition 1.3.1). Thus $1+V \subseteq \mathcal{A}^e(R)$. But we have seen in Section 2.3 (page 38) that $V = 1+m$. Thus $m = 1+V \subseteq \mathcal{A}^e(R)$, and so, by Theorem 3.3.5 and Corollary 3.3.6, every even divisible Hosszú function on $R = \mathcal{O}_K/P^s$ is the sum of an additive function and the lift of a Hosszú function on $R/m = \mathcal{O}_K/P$. We therefore have the following theorem.

*THEOREM 3.4.4. If the number ring $\mathcal{O}_K$ has the prime ideals $P_1,\ldots,P_k$ of norm 2 and the prime ideals $Q_1,\ldots,Q_r$ of norm 4, then every even divisible Hosszú function on $\mathcal{O}_K$ is the sum of an additive function and lifts of Hosszú functions on the factor rings $\mathcal{O}_K/P_i^2$ and $\mathcal{O}_K/Q_j$.* |||

Proposition 3.4.3 gives us all the Hosszú functions on $\mathcal{O}_K/P_i^2$ and Example 3 of Section 2.2 gives us all the Hosszú functions on $\mathcal{O}_K/Q_j$.

So we have described all the odd and even divisible Hosszú functions on number rings, and the corestrictions of these will give us all odd and even Hosszú functions on such rings, by Proposition 3.3.1.

To apply these theorems to any given number ring $\mathcal{O}_K$, we simply have to find all the prime ideals of norm 2 or 4 in the ring, and then, for each prime ideal P of norm 2, determine whether $\mathcal{O}_K/P^2$ is isomorphic to $C_4$ or $F_2[X]/(X^2)$. In other words, we have to see whether 4 or 2 (respectively) is the smallest rational integer in $P^2$. We can do this by the following method.

Let $P_1, P_2, \ldots, P_k$ be all the ideals of norm 2 in $\mathcal{O}_K$, so that $2\mathcal{O}_K = P_1^{n_1} P_2^{n_2} \cdots P_k^{n_k} \cdot J$, where $J$ is relatively prime to each $P_i$, and the $n_i$ are positive integers. If $n_i \geq 2$, then $P_i^2$ divides $2\mathcal{O}_K$, so that $P_i^2$ contains 2. If $n_i = 1$, then $P_i^2$ divides $(2\mathcal{O}_K)^2 = 4\mathcal{O}_K$ but $P_i^2$ does not divide $2\mathcal{O}_K$, so that $P_i^2$ contains 4 but not 2.

(For the following examples, consult Narkiewicz [18] page 166 for the relevant facts about the factorization of $2\mathcal{O}_K$.)

*EXAMPLE 1 - QUADRATIC NUMBER RINGS.*

Let $K$ be a quadratic extension of $\mathbb{Q}$ with discriminant $d$. If $d \equiv 5 \pmod 8$, then $2\mathcal{O}_K$ is a prime ideal (of norm 4) so there are no prime ideals of norm 2. Hence every odd divisible Hosszú function on $\mathcal{O}_K$ is additive, and every even divisible Hosszú function on $\mathcal{O}_K$ is the sum of an additive function and the lift of a Hosszú function on $\mathcal{O}_K/2\mathcal{O}_K$.

If $d \equiv 1 \pmod 8$, then $2\mathcal{O}_K = P_1 P_2$ for two different prime ideals $P_1$ and $P_2$ of norm 2. $\mathcal{O}_K$ will have no prime ideals of norm 4, and $\mathcal{O}_K/P_i^2 \simeq C_4$, $i=1,2$. Every odd and even divisible Hosszú function on $\mathcal{O}_K$ will be the sum of an additive function and lifts of Hosszú functions on $\mathcal{O}_K/P_1^2$ and $\mathcal{O}_K/P_2^2$.

If $d \equiv 0 \pmod 4$, then $2\mathcal{O}_K = P^2$, where P is the only prime ideal of $\mathcal{O}_K$ of norm 2. $\mathcal{O}_K$ will have no prime ideals of norm 4, and $\mathcal{O}_K/P^2 \simeq \mathbb{F}_2[X]/(X^2)$. Every odd and even divisible Hosszú function will be the sum of an additive function and the lift of a Hosszú function on $\mathcal{O}_K/P^2$.

*EXAMPLE 2 - PURE n-TH DEGREE NUMBER RINGS.*

Let $K = \mathbb{Q}(\sqrt[n]{a})$, where $n \geq 3$ and $a \in \mathbb{Z}$, $a \equiv 2 \pmod 4$. Then $2\mathcal{O}_K = P^n$, where P is the only prime ideal of $\mathcal{O}_K$ of norm 2, and there is no prime ideal in $\mathcal{O}_K$ of norm 4. Then $\mathcal{O}_K/P^2 \simeq \mathbb{F}_2[X]/(X^2)$, and every odd and even Hosszú function on $\mathcal{O}_K$ is the sum of an additive function and the lift of a Hosszú function on $\mathcal{O}_K/P^2$.

# CHAPTER 4

## POLYNOMIAL RINGS

We now turn our attention to a somewhat different sort of ring—namely, polynomial rings over a field $F$. The rings of integers which we considered in the last chapter have the property that their units generate an additive subgroup of rank at least half the rank (as a $\mathbb{Z}$-module) of the ring itself. We were able to use this fact to advantage in proving that $\mathcal{A}^o(\mathcal{O}_K)$ and $\mathcal{A}^e(\mathcal{O}_K)$ contain proper ideals of $\mathcal{O}_K$. We will not be able to use the units of polynomial rings in the same way, since in polynomial rings, the units form too 'sparse' a set. We will therefore have to use substantially different techniques in solving Hosszú's equation over these rings.

Polynomial rings are of interest in the study of Hosszú's equation not only for the new techniques that will be introduced, but also because every $F$-algebra is the homomorphic image of a polynomial ring over $F$ (in a sufficiently large number of indeterminates). As we have seen in Section 1.5, if we have solved Hosszú's equation over a ring $R$, then we can easily obtain the solutions over a homomorphic image of $R$. So the solutions over all the polynomial rings over $F$ would give us the solution

for all $F$-algebras. However, we will consider only the polynomial rings in one indeterminate, $F[X]$, which will limit our conclusions to singly-generated $F$-algebras.

## 4.1 - GENERAL RESULTS

In this section we prove some general results about $\mathcal{A}(F[X])$.

Let $F$ be a field containing more than four elements. There is a unique $F$-algebra embedding of $F$ into its polynomial ring $F[X]$, so we will identify $F$ with its image in $F[X]$ under this embedding. The units (nonzero elements) of $F$ will be precisely the units of $F[X]$.

Examining the proof of Proposition 2.2.1, we see that $F$ (and hence $F[X]$) has at least one exceptional unit $u$ such that $1-u-u^{-1}$ is also a unit. But by Proposition 1.3.1 and Corollary 1.3.2, this means that $\mathcal{A}(F[X])$ contains a unit, so by Proposition 1.3.3, $\mathcal{A}(F[X]) = \mathcal{H}(F[X])$. Hence every Hosszú function on $F[X]$ is odd (Proposition 1.3.4) and so will satisfy equation ($H^o$). Moreover, $A(F[X])$ (defined on page 34) will be contained in $\mathcal{A}(F[X])$, by Proposition 1.3.1.

Our first step will be to show that $F \subseteq \mathcal{A}(F[X])$. Since $A(F[X]) = A(F)$, we need only show that $A(F) = F$.

*PROPOSITION 4.1.1. (Davison [11]) If $F$ has more than four elements, then $A(F) = F$.*

*PROOF.* Note that since every element of $F$ except $0$ and $1$ is an exceptional unit, we can conclude from the equation in the proof of Corollary 1.3.2 that $1 \in A(F)$. Also, $A(F)$ is closed under multiplication, for if $x$ and $y$ are exceptional units ($x \neq y$, $x \neq y^{-1}$), then

$$\begin{aligned}(x+x^{-1})(y+y^{-1}) &= xy + x^{-1}y + xy^{-1} + x^{-1}y^{-1} \\ &= [(xy) + (xy)^{-1}] + [(xy^{-1}) + (xy^{-1})^{-1}] \\ &\in A(F).\end{aligned}$$

If $x \neq -1$, then $(x+x^{-1})(x+x^{-1}) = x^2+(x^2)^{-1}+1+1 \in A(F)$, and finally, if $-1$ is an exceptional unit, then we have $[(-1)+(-1)]\cdot[(-1)+(-1)] = 4 = 1+1+1+1 \in A(F)$.

If $x \in A(F)$, $x \neq 0, 1$, then $x$ is an exceptional unit, so $x+x^{-1} \in A(F)$. Hence $(x+x^{-1})-x = x^{-1} \in A(F)$. This means $A(F)$ is also closed under inversion, and so is a subfield of $F$.

We now show that for all $x \in F$, $x-x^2 \in A(F)$. This is clearly true for $x=0$ or $1$. So suppose that $x$ is an exceptional unit. Then $1+(x+x^{-1})+[(1-x)+(1-x)^{-1}] \in A(F)$. But this is equal to $(x-x^2)^{-1}$, so that $x-x^2 \in A(F)$ since $A(F)$ is a field.

Now suppose that $A(F)$ contains only the two elements $0$ and $1$. Then for all $x \in F$, $x-x^2=0$ or $1$. But these two quadratic equations can have a total of at most four solutions in $F$, which contradicts the fact that

$$= f(aX) + f(cX) + f(b+d)$$
$$= f(ca^{-1}) + f(aX) + f(cX) + f(b+d-ca^{-1})$$
$$= f(ca^{-1} + aX + cX) + f(b+d-ca^{-1})$$
$$= f(aX+cX+b+d) = f(P(X) + Q(X))$$

(where we have used Corollary 4.1.2 and the fact that $f$ satisfies $(H^{\circ})$). |||

## 4.2 - POLYNOMIAL RINGS OVER ALGEBRAICALLY CLOSED FIELDS

In this section we will show that if $F$ is algebraically closed, then every Hosszú function on $F[X]$ is additive. This will allow us to conclude the same result for singly generated algebras over such fields.

*THEOREM 4.2.1. If $F$ is an algebraically closed field, then every Hosszú function $f: F[X] \to G$ is additive.*

*PROOF.* Since $F$ is algebraically closed, it has infinitely many elements, so we may use the results of the preceding section. Recall that $h: F[X] \to \mathcal{H}(F[X])$ is the universal Hosszú function on $F[X]$. We will show that if $P(X)$, $Q(X) \in F[X]$, then $h(P(X)) + h(Q(X)) = h(P(X)+Q(X))$, by using induction on the maximum of the degrees of $P$ and $Q$.

If $\max\{\deg P, \deg Q\} \leq 1$, then by Proposition 4.1.3, $h(P(X)) + h(Q(X)) = h(P(X)+Q(X))$.

So let us assume (induction hypothesis) that if $\max\{\deg P, \deg Q\} \leq n$, then $h(P(X)) + h(Q(X)) = h(P(X)+Q(X))$.

If $a,b \in F$, and $a \neq 0$, then $h(aX^{n+1}) + h(bX^{n+1}) = h(ba^{-1}) + h(aX^{n+1}) + h(bX^{n+1}) - h(ba^{-1}) = h(ba^{-1}+aX^{n+1}+bX^{n+1}) - h(ba^{-1}) = h(aX^{n+1}+bX^{n+1})$, where we have used Corollary 4.1.2 and equation $(H^{\circ})$. Thus

$$h(aX^{n+1}) + h(bX^{n+1}) = h(aX^{n+1}+bX^{n+1}) \qquad (1)$$

for all $a,b \in F$, since the equation is trivially true when $a = 0$.

Now, if $a \in F$ $(a \neq 0)$, then for all nonzero $x_1 \in F$, $h(\frac{X}{x_1}) + h(ax_1X^n) + h(aX^{n+1}) = h(aX^{n+1}+ax_1X^n+\frac{X}{x_1})$, using equation $(H^{\circ})$. Thus

$$h(ax_1X^n+\frac{X}{x_1}) + h(aX^{n+1}) = h(aX^{n+1} + ax_1X^n + \frac{X}{x_1}), \qquad (2)$$

by the induction hypothesis.

Furthermore, for all nonzero $x_2 \in F$,

$$h(\frac{X}{x_2}) + h(x_2[aX^n+ax_1X^{n-1}+\frac{1}{x_1}]) + h(aX^{n+1}+ax_1X^n+\frac{X}{x_1})$$

$$= h(aX^{n+1}+a(x_1+x_2)X^n+ax_1x_2X^{n-1}+(\frac{1}{x_1}+\frac{1}{x_2})X+\frac{x_2}{x_1}).$$

So by the induction hypothesis and the fact that $\frac{x_2}{x_1} \in F \subseteq A(F[X])$, we get

$$h(ax_2X^n+ax_1x_2X^{n-1}+\frac{X}{x_2}) + h(aX^{n+1}+ax_1X^n+\frac{X}{x_1})$$

$$= h(aX^{n+1}+a(x_1+x_2)X^n+ax_1x_2X^{n-1}+(\frac{1}{x_1}+\frac{1}{x_2})X). \qquad (3)$$

Adding equations (2) and (3), and again using the induction hypothesis, we see that

$$h(a(x_1+x_2)X^n+ax_1x_2X^{n-1}+(\frac{1}{x_1}+\frac{1}{x_2})X)+h(aX^{n+1})$$
$$= h(aX^{n+1}+a(x_1+x_2)X^n+ax_1x_2X^{n-1}+(\frac{1}{x_1}+\frac{1}{x_2})X). \quad (4)$$

Thus we have shown that the following formula holds for $i=1$ and 2 (where $x_1, x_2, \ldots, x_i$ are arbitrary nonzero elements of $F$, $i \leq n$).

$$h\Big\{a[(x_1+x_2+\ldots+x_i)X^n+(x_1x_2+x_1x_3+\ldots+x_{i-1}x_i)X^{n-1}+$$
$$(x_1x_2x_3+\ldots+x_{i-2}x_{i-1}x_i)X^{n-2}+\ldots+(x_1x_2x_3\ldots x_i)X^{n-i+1}]$$
$$+(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_i})X\Big\}+h\Big\{aX^{n+1}\Big\}$$
$$= h\Big\{\text{the sum of these two arguments}\Big\} \quad (5)$$

(The coefficients of the polynomial in $X$ which appears inside the square brackets in (5) are the elementary symmetric functions in the $i$ variables $x_1, x_2, \ldots, x_i$.)

It is clear that if we set $i=1$ and 2 in (5), we will get equations (2) and (4), respectively, which we know to be true. Let us assume that (5) holds for $i \leq n$, and from this derive the same formula with $i$ replaced by $i+1$, by introducing a new variable $x_{i+1}$, in the same way that we derived (4) from (2) by introducing the variable $x_2$. By the principle of mathematical induction,

we will then have proved that (5) is true for all $i \leq n+1$ and for all nonzero $x_1, x_2, \ldots, x_i \in F$.

For all nonzero $x_{i+1}$ in $F$, we have (using (H$_2$)),

$$h\left(\frac{\cdot X}{x_{i+1}}\right) + h\Big(ax_{i+1}[X^n + (x_1+x_2+\ldots+x_i)X^{n-1} + (x_1x_2+\ldots+x_{i-1}x_i)X^{n-2} + \ldots + (x_1x_2x_3\ldots x_i)X^{n-i} + \ldots + (x_1x_2\ldots x_i)X^{n-i+1}] + \frac{1}{x_1}X\Big) + h\Big(a[X^{n+1} + (x_1+x_2+\ldots+x_i)X^n + (x_1x_2+\ldots+x_{i-1}x_i)X^{n-1} + \ldots + (x_1x_2x_3\ldots x_i)X^{n-i+1}] + \frac{1}{x_1}X\Big) = h\Big(\text{the sum of these three arguments}\Big).$$

If we add this equation to equation (5) and use the induction hypothesis and Corollary 4.1.2 as before, we see that

$$h\Big(a[(x_1+x_2+\ldots+x_{i+1})X^n + (x_1x_2+\ldots+x_ix_{i+1})X^{n-1} + \ldots + (x_1x_2x_3\ldots x_{i+1})X^{n-i}] + \left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_{i+1}}\right)X\Big) + h\Big(aX^{n+1}\Big) = h\Big(\text{the sum of these two arguments}\Big). \quad (6)$$

This is equation (5) with $i$ replaced by $i+1$. As we have remarked before, this means that (5) holds for all $i \leq n+1$. So let us set $i=n+1$ in (5), and let us define

$$S_1 = x_1 + x_2 + \ldots + x_{n+1}$$

$$S_2 = x_1x_2 + x_1x_3 + \ldots + x_nx_{n+1}$$

$$S_3 = x_1x_2x_3 + x_1x_2x_4 + \ldots + x_{n-1}x_nx_{n+1}$$

$$S_n = x_1 x_2 \ldots x_n + \ldots + x_2 x_3 \ldots x_{n+1}$$

$$S_{n+1} = x_1 x_2 x_3 \ldots x_n x_{n+1}$$

(so that $S_j$ is the $j$-th elementary symmetric function in the variables $x_1, x_2, \ldots, x_{n+1}$).

Then, after using the fact that $aS_{n+1} \in F \subseteq \alpha(F[X])$, equation (5) with $t=n+1$ becomes

$$h\left(a[S_1 X^n + S_2 X^{n-1} + S_3 X^{n-2} + \ldots + S_n X] + \frac{S_n}{S_{n+1}} X\right) + h\left(aX^{n+1}\right)$$

$$= h\left(a[X^{n+1} + S_1 X^n + S_2 X^{n-1} + \ldots + S_n X] + \frac{S_n}{S_{n+1}} X\right). \quad (7)$$

Now let $B(X) = b_1 X^n + b_2 X^{n-1} + \ldots + b_n X$ be an arbitrary polynomial of degree $n$ with zero constant term. Let us try to write $B(X)$ in the form

$$B(X) = a[S_1 X^n + S_2 X^{n-1} + \ldots + S_n X] + \frac{S_n}{S_{n+1}} X$$

(for suitable values of the $S_i$'s). This requires solving the following system of equations for $S_1, S_2, \ldots, S_{n+1}$ (and hence for $x_1, x_2, \ldots, x_{n+1}$).

$$\begin{aligned} aS_1 &= b_1 \\ aS_2 &= b_2 \\ &\ldots\ldots \\ aS_{n-1} &= b_{n-1} \\ aS_n + S_n/S_{n+1} &= b_n \end{aligned}$$

If we could set $S_{n+1} = 1/a$, $S_n = b_n/2a$, and $S_i = b_i/a$ for $i \leq n$, then this system would have a solution. But clearly if we let the $x_i$'s be the negatives of the roots of the polynomial

$$X^{n+1} + (b_1/a)X^n + (b_2/a)X^{n-1} + \ldots + (b_{n-1}/a)X^2 + (b_n/2a)X + (1/a),$$

then the $S_i$'s will have the required values (recall that $F$ is algebraically closed). Clearly none of the $x_i$'s will be zero, since their product is $1/a$, which is not zero. So substituting these values into (7), we get

$$h(B(X)) + h(aX^{n+1}) = h(aX^{n+1} + B(X)). \tag{8}$$

Note that this is also (trivially) true if $a=0$.

Now suppose that $P(X) = p_{n+1}X^{n+1}+p_nX^n+\ldots+p_1X+p_0$ and $Q(X) = q_{n+1}X^{n+1}+q_nX^n+\ldots+q_1X+q_0$ are two arbitrary polynomials in $F[X]$ of degree $\leq n+1$. Then

$$h(P(X)) + h(Q(X)) = h(P(X)-p_0) + h(Q(X)-q_0) + h(p_0+q_0)$$

(by Corollary 4.1.2)

$$= h(p_{n+1}X^{n+1}) + h(q_{n+1}X^{n+1}) + h(p_nX^n+p_{n-1}X^{n-1}+\ldots+p_1X) + h(q_nX^n+q_{n-1}X^{n-1}+\ldots+q_1X) + h(p_0+q_0)$$

(by (8))

$$= h((p_{n+1}+q_{n+1})X^{n+1}) + h((p_n+q_n)X^n+\ldots+(p_1+q_1)X) + h(p_0+q_0)$$

(by (1) and the induction hypothesis)

$= h(P(X)+Q(X))$ (by (8) and Corollary 4.1.2).

This means that $h$ is additive on polynomials of degree $\leq n+1$, so by induction, $h$ is additive on $F[X]$. |||

Suppose now that $A$ is a singly generated algebra over the algebraically closed field $F$; that is, there is an element $x \in A$ with the property that for every element $y \in A$ there exists at least one polynomial $P(X) \in F[X]$ such that $y=P(x)$. Then the map $\phi: F[X] \to A$ defined by $F$-linear extension of $X^i \mapsto x^i$ $(i=0,1,2,\ldots)$ is clearly an $F$-algebra epimorphism, so that $A$ is a homomorphic image of $F[X]$. By the above theorem and the discussion at the beginning of Section 1.5, we therefore have the following result.

*COROLLARY 4.2.2. If $A$ is a singly generated algebra over the algebraically closed field $F$, then every Hosszú function on $A$ is additive.* |||

## 4.3 - SINGLY GENERATED TOPOLOGICAL ALGEBRAS

Recall that a *topological algebra* $T$ over the topological field $F$ is an $F$-algebra endowed with a topology in which the addition, multiplication, and scalar multiplication are jointly continuous functions (see Michael [17]). A topological algebra $T$ is *singly generated*

if it contains an element $x$, called a *generator* of $x$, with the property that the smallest subalgebra of $T$ containing $x$ is dense in $T$. In this section we apply the results of the previous section to show that every continuous Hosszú function $f:T \to G$, where $T$ is a singly generated topological algebra over the algebraically closed field $F$, and $G$ is a Hausdorff topological abelian group, will satisfy Cauchy's equation.

If $x$ is a generator of $T$, then the subalgebra $T_x$ of $T$ defined by $T_x = \{y \in T: y=P(x) \text{ for some } P(X) \in F[X]\}$ is clearly the smallest subalgebra of $T$ containing $x$. It is singly-generated in the algebraic sense of the preceding section, and so every Hosszú function on $T$ is additive when restricted to $T_x$. Also, $T_x$ is dense in $T$, by definition of a singly generated topological algebra.

*THEOREM 4.3.1. If $f:T \to G$ is a continuous Hosszú function from the singly generated topological algebra $T$ (over the algebraically closed topological field $F$) to the Hausdorff topological abelian group $G$, then $f$ is additive.*

*PROOF.* Let $x$ be a generator of $T$. Since $T_x$ is dense in $T$, $T_x \times T_x$ will be dense in $T \times T$. Thus, if $y$ and $z$ are arbitrary elements of $T$, there will be a net $((y_\alpha, z_\alpha))_{\alpha \in D}$ of elements of $T_x \times T_x$ converging to the point $(y,z) \in T \times T$,

where $D$ is some directed set. (See Thron [23] for the relevant topological definitions and facts.)

This means that $\lim_D y_\alpha = y$ and $\lim_D z_\alpha = z$. If $f:T \to G$ is a continuous Hosszú function, then

$$\begin{aligned} f(y+z) &= f(\lim_D (y_\alpha + z_\alpha)) \\ &= \lim_D f(y_\alpha + z_\alpha) \\ &= \lim_D [f(y_\alpha) + f(z_\alpha)] \\ &= \lim_D f(y_\alpha) + \lim_D f(z_\alpha) \\ &= f(\lim_D y_\alpha) + f(\lim_D z_\alpha) \\ &= f(y) + f(z). \end{aligned}$$

(We have used here the continuity of $f$ and of the addition in both $T$ and $G$, as well as the additivity of $f$ on $T_x$ and the fact that $G$ is Hausdorff.) |||

The following examples contain some important singly generated topological $F$-algebras in the case $F=\mathbb{C}$ (with its usual topology).

*EXAMPLE 1.*

The algebra $\mathbb{C}[a,b]$ of continuous complex-valued functions on the real interval $[a,b]$, with the supremum norm and pointwise operations, is singly generated (with generator $f:[a,b] \to \mathbb{C}$ defined by $f(t)=t$), by the Weierstrass approximation theorem.

*EXAMPLE 2.*

on the open, simply connected domain $\Omega$ in $\mathbb{C}$, with pointwise operations and compact-open topology. By the Riemann mapping theorem, there exists a conformal map $\psi:\Omega \to U$ of $\Omega$ onto the open unit disc $U$. Thus if $f \in H(\Omega)$, the function $f\circ\psi^{-1}$ will be analytic on the unit disc, and so will equal its Maclaurin series expansion in $U$:

$$f(\psi^{-1}(\omega)) = \sum_{n=0}^{\infty} \alpha_n \omega^n ,$$

by Taylor's theorem. Setting $\omega = \psi(z)$, we see that

$$f(z) = \sum_{n=0}^{\infty} \alpha_n [\psi(z)]^n ,$$

for all $z \in \Omega$, and so $H(\Omega)$ is singly generated, with generator $\psi(z)$.

Any standard functional analysis text will contain these and other examples of singly generated topological algebras over $\mathbb{C}$.

# CHAPTER 5

## GENERALIZATIONS AND CONCLUSIONS

In her paper on the solution of Hosszú's equation over fields not of characteristic 2 or 3, Świątak [22] introduced the following generalization of Hosszú's equation:

$$f(x+y-F(x,y)) + f(F(x,y)) = f(x) + f(y) , \qquad (GH_F)$$

where $B$ and $G$ are abelian groups, $f:B \to G$ is the unknown function, and $F:B\times B \to B$ is some given function. If $F$ is a multiplication which turns $B$ into a ring, then we have the usual Hosszú equation, so that we are justified in calling this equation a generalization of (H). If $F$ is identically zero, then we have the equation for affine functions mentioned in the Introduction. If the group $B$ has the property that for all $b \in B$, the equation $2x=b$ has exactly one solution (which we call $\frac{1}{2}b$), then if we set $F(x,y) = \frac{1}{2}(x+y)$, the equation $(GH_F)$ becomes Jensen's functional equation:

$$2f(\tfrac{1}{2}(x+y)) = f(x) + f(y) .$$

(See Aczél [1], where this classical equation is discussed extensively.)

Świątak [22] posed the problem of finding sufficient conditions on the known function $F$ to guarantee that the solutions of $(GH_F)$ are exactly the affine functions.

The purpose of the first part of this chapter is to show that some of the results which we obtained for the equation (H) do not really require that $F(x,y)=xy$. We will briefly indicate what properties of $F$ will allow us to extend certain results for (H) to the equation $(GH_F)$, thereby giving a partial solution to the problem of Świątak.

## 5.1 - THE GENERALIZED HOSSZÚ EQUATION

As was the case for Hosszú's equation, every solution of $(GH_F)$ will differ by a constant function from a solution $f$ having the property that $f(0)=0$, so we again will only be considering solutions with this property. Such solutions we will call $F$-Hosszú functions.

It is clear that the constructions carried out in Chapter 1 in defining $\mathcal{A}(R)$, $\mathcal{D}(R)$, and $h:R\to\mathcal{H}(R)$ can also be performed on an abelian group $B$ using the equation $(GH_F)$ instead of (H). We will denote the corresponding objects in the generalized situation by $\mathcal{A}_F(B)$, $\mathcal{D}_F(B)$, and $h_F:B\to\mathcal{H}_F(B)$ (since the construction depends on the known function $F$).

If $F(-x,-y)=F(x,y)$ for all $x,y\in B$, then using the techniques of Section 1.1 we can show that the odd

and the even $F$-Hosszú functions are precisely the solutions (with $f(0)=0$) of the equations

$$f(x) + f(y) + f(F(x,y)) = f(x+y+F(x,y)) \qquad (GH_F^o)$$

and $$f(x) + f(y) = f(F(x,y)) + f(x+y+F(x,y)) \qquad (GH_F^e)$$

respectively. In this case, we can construct the odd and the even analogues of $\mathcal{A}_F(B)$ and $h_F: B \to \mathcal{H}_F(B)$ just as we did for equation (H) in Chapter 1, and we will designate these by the superscripts $^o$ and $^e$ as we did before.

As in Section 1.3, we can show that if $\mathcal{D}_F(B)$ is nonempty, it is a coset in $B$ of $\mathcal{A}_F(B)$. Moreover, if $F(x,y) = F(-x,-y)$ for all $x,y \in B$, then $\mathcal{A}_F(B) \cup \mathcal{D}_F(B) \subseteq \mathcal{A}_F^o(B)$, and if $\mathcal{A}_F(B) = \mathcal{D}_F(B)$ for such an $F$, then every $F$-Hosszú function on $B$ is odd.

Suppose now that $f: B \to G$ is an $F$-Hosszú function, and that $\phi: B \to B$ is an automorphism of the abelian group $B$. Then $f = f\circ\phi^{-1}\circ\phi$, so that

$$f\circ\phi^{-1}(\phi(x)) + f\circ\phi^{-1}(\phi(y)) = f\circ\phi^{-1}(\phi(x)+\phi(y)-\phi(F(x,y))) + f\circ\phi^{-1}(\phi(F(x,y))),$$

for all $x,y \in B$. Setting $u=\phi(x)$, $v=\phi(y)$, we see that

$$f\circ\phi^{-1}(u) + f\circ\phi^{-1}(v) = f\circ\phi^{-1}(u+v-\phi(F(\phi^{-1}(u),\phi^{-1}(v)))) + f\circ\phi^{-1}(\phi(F(\phi^{-1}(u),\phi^{-1}(v)))),$$

for all $u,v \in B$. Define $F': B\times B \to B$ by

$$F'(u,v) = \phi(F(\phi^{-1}(u),\phi^{-1}(v))) ,$$

and let us say that two functions from $B\times B$ to $B$ are *similar* if they are related by some automorphism $\phi$ of $B$ in the same way that $F'$ and $F$ are. (Similarity is clearly an equivalence relation.) We have shown that if $f$ is an $F$-Hosszú function, then $f\circ\phi^{-1}$ is an $F'$-Hosszú function. Conversely, if $g:B\to G$ is an $F'$-Hosszú function, then $g\circ\phi$ will be an $F$-Hosszú function. We have therefore shown the following.

*PROPOSITION 5.1.1. If $F:B\times B\to B$ and $F':B\times B\to B$ are similar, then every $F$-Hosszú function on $B$ is additive if and only if every $F'$-Hosszú function on $B$ is additive.* ||

As an example, let us consider the case when $B=R$, a ring, and $F$ is the multiplication on $R$. If $u$ is a unit of $R$, then the function $\phi(x) = u^{-1}x$ is an automorphism of the additive group structure of $R$. Thus $F'(x,y) = \phi(F(\phi^{-1}(x),\phi^{-1}(y))) = u^{-1}(uxuy) = xuy$, and so by the above proposition, if every Hosszú function on $R$ is additive, then so is every solution $f$ (with $f(0)=0$) of the equation

$$f(x+y-xuy) + f(xuy) = f(x) + f(y) .$$

Since group homomorphisms are odd, the above

proposition can be proved for odd $F$- and $F'$-Hosszú functions as well.

Although the proofs of Proposition 1.3.1 and Corollary 1.3.2 depend strongly on the fact that multiplication in a ring is associative and distributive over addition, we can prove an analogue of Proposition 1.3.5 for odd $F$-Hosszú functions without imposing any strong conditions on the function $F$. Let us suppose simply that $F(x,-y) = F(-x,y) = -F(x,y)$ for all $x,y \in B$, and let $f:B \to G$ be an odd $F$-Hosszú function. Then

$$f(x) + f(y) + f(F(x,y)) = f(x+y+F(x,y)) ,$$

and $$f(x) + f(-y) + f(-F(x,y)) = f(x-y-F(x,y)) .$$

Adding these equations, we see that

$$2f(x) = f(x+y+F(x,y)) + f(x-y-F(x,y)) \qquad (*)$$

for all $x,y \in B$.

Suppose that $B$ has an element $u$ with the property that the function $g(y) = F(u,y) + y$ maps $B$ onto itself. This means that for any $z \in B$, we can find an element $z_0 \in B$ such that $F(u,z_0) + z_0 = u + z$. Substituting $x=u$, $y=z_0$ into (*), we see that $2f(u) = f(2u+z) + f(-z)$ for all $z \in B$. Setting $z=0$, it follows that $2f(u) = f(2u)$, and so $f(2u) + f(z) = f(2u+z)$. We have therefore shown that the following generalization of Proposition 1.3.5 holds.

*PROPOSITION 5.1.2. If $F:B\times B \to B$ satisfies $F(x,-y) = F(-x,y) = -F(x,y)$ for all $x,y \in B$, and if $u$ is an element of $B$ with the property that $F(u,y)+y$ is onto $B$ (as a function of $y$), then $2u \in \mathcal{A}_F^o(B)$.* |||

(As in Section 1.3, we can also prove the even analogue of this result—that $2u \in \mathcal{A}_F^e(B)$.)

For example, if we let $F:\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be the function $F(x,y) = P(x)Q(y)$, where $P$ is an odd function on $\mathbb{R}$ and $Q$ is an odd polynomial function on $\mathbb{R}$, then for all $z \in \mathbb{R}$, either $u=z$ or $u=-z$ will have the property that $P(u)Q(y)+y$ maps $\mathbb{R}$ onto itself (as a function of $y$). This means (by the above proposition) that for all $z \in \mathbb{R}$, either $2z$ or $-2z \in \mathcal{A}_F^o(\mathbb{R}) \cap \mathcal{A}_F^e(\mathbb{R})$, so that, since 2 is a unit and since $\mathcal{A}_F^o(\mathbb{R}) \cap \mathcal{A}_F^e(\mathbb{R})$ is closed under negation (being a subgroup of $\mathbb{R}$), we have $\mathbb{R} \subseteq \mathcal{A}_F^o(\mathbb{R}) \cap \mathcal{A}_F^e(\mathbb{R})$. Hence every odd and every even $F$-Hosszú function on $\mathbb{R}$ is additive (for this particular $F$).

Although the next result reduces to the trivial conclusion that $0 \in \mathcal{A}(R)$ for the usual Hosszú functions, it can be useful in other situations.

*PROPOSITION 5.1.3. If $F:B\times B \to B$ has the property that for some $x_o \in B$, the function $g(y) = F(x_o,y)$ is identically equal to the constant $c$, then $x_o - c \in \mathcal{A}_F(B)$.*

*PROOF.* If $F(x_o,y) = c$ for all $y \in B$, then

*B/S is additive.*

*PROOF.* This result can be proved in the same way as Proposition 1.5.1. |||

As a final simple illustration of these results, let $F:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ be defined by $F(x,y) = nxy-n$, where $n$ is a fixed integer. Then $F(0,y)\equiv -n$, so that $n \in Q_p(\mathbb{Z})$ by Proposition 5.1.3, and hence $n\mathbb{Z}\subseteq Q_p(\mathbb{Z})$. Clearly $n\mathbb{Z}$ is $F$-stable, and since $\overline{F}:\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}$ is identically zero, it follows that every $F$-Hosszú function on $\mathbb{Z}/n\mathbb{Z}$ is additive. Hence by Proposition 5.1.4 every solution $f:\mathbb{Z}\to\mathbb{Z}$ (with $f(0)=0$) of the equation

$$f(x+y-nxy+n) + f(nxy-n) = f(x) + f(y)$$

is additive, so that $f(x) = xf(1)$.

Before concluding this section on generalizations, we will briefly examine the following generalization of Hosszú's original equation, which was considered by Fenyö [13], who solved it for distributions on $\mathbb{R}$.

$$f_1(x+y-xy) + f_2(xy) = f_3(x) + f_4(y) \qquad (F)$$

The unknown functions $f_i$ $(i=1,2,3,4)$ again have domain a ring $R$ and range an abelian group $G$. If we define $\overline{f}_i(x)=f_i(x)-f_i(0)$, then clearly $\overline{f}_i(0)=0$ and

$$\overline{f}_1(x+y-xy) + \overline{f}_2(xy) = \overline{f}_3(x) + \overline{f}_4(y) .$$

Setting $x=0$ and $y=0$ in turn, we see that $\overline{f}_1(y) = \overline{f}_4(y)$, and that $\overline{f}_1(x) = \overline{f}_3(x)$. Now, setting $y=1$, it follows that $\overline{f}_1(1) + \overline{f}_2(x) = \overline{f}_3(x) + \overline{f}_4(1)$, so that $\overline{f}_2(x) = \overline{f}_3(x)$. Hence $\overline{f}_1 = \overline{f}_2 = \overline{f}_3 = \overline{f}_4$, and so the general solution of (F) is given by: $f_i(x) = f(x) + c_i (i=1,2,3,4)$, where $f$ is an arbitrary solution of (H), and the $c_i$ are arbitrary elements of $G$ satisfying $c_1 + c_2 = c_3 + c_4$.

## 5.2 - CONCLUSIONS AND QUESTIONS

In this thesis we have shown how a great variety of algebraic techniques and results can be used to determine the solutions of Hosszú's functional equation in various settings. In this chapter, we have demonstrated that some of our techniques can be applied to a more general class of equations, although Hosszú's equation is nonetheless still of particular interest.

In Chapter 2, we solved Hosszú's equation over a wide class of rings generated by their units, including local rings of odd or zero characteristic, and over all finite local rings. For the later applications of this chapter, we only needed to know the solutions for finite local rings, but it would nevertheless be interesting to close the gap in our results—to find the solutions also for infinite local rings in which 2 is not a unit.

We describe all odd and even Hosszú functions on

rings of algebraic integers in Chapter 3. We also give a condition on the units of a number ring (Corollary 3.2.2) which, if satisfied, guarantees that the ring has only additive Hosszú functions. This result enables us to specify those cyclotomic number rings on which every Hosszú function is additive. An interesting but apparently difficult problem is the complete determination of all Hosszú functions on all number rings. In view of Corollary 3.2.2 and Proposition 1.3.1, it is clear that a knowledge of the exceptional units in number rings would be very helpful in the solution of this problem. At present the only results in this direction seem to be the theorem of Chowla [6] which says that every number ring has at most finitely many exceptional units, and a theorem of Newman [19] which, among other things, guarantees for every natural number $n$ the existence of a number ring of degree $n$ with at least $n$ exceptional units.

We have shown in Chapter 4 that on polynomial rings over algebraically closed fields, every Hosszú function is additive. The problem of extending this result to a wider class of fields is open.

Theorems 3.1.4 and 3.1.5 show that on a number ring every odd and every even Hosszú function is additive if and only if the ring has no ideals of norm 2 or 4.

Theorem 2.2.3 tells us that every Hosszú function on a finite local ring is additive if and only if the ring has no ideals of norm 2, 3, or 4. It seems likely, on the basis of this and other empirical evidence, that future results on Hosszú's equation will continue to reveal a close connection between the ideal structure of a ring (particularly the presence of ideals of norm 2, 3, or 4) and the structure of its Hosszú group.

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