TOPOLOGICAL ALGEBRAS WITH BASES

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By

SALEEM H. WATSON, B.Sc., M.Sc.

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AUTHOR: Saleem H. Watson, B. Sc. (Andrews University)

M. Sc. (McMaster University)

SUPERVISOR: Professor T. Husain

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ABSTRACT

Let A be a topological algebra. A (Schauder) basis $\{x_n\}$ in A is called an *orthogonal basis* if $x_n x_m = \delta_{nm} x_n$, $n,m \in \mathbb{N}$ (δ denotes Kronecker's delta). A basis in A of the form $\{z^n : n=0,1,\ldots\}$, $z \in A$, is called a *cyclic basis*. This thesis is concerned with the structure of topological algebras possesing bases of these types. It is shown how the existence of such bases determines algebraic and topological properties of A. An interesting connection between the dense maximal ideals and the topological dual of certain types of topological algebras having unconditional orthogonal bases is explored. These results are used to obtain characterizations of some important F-algebras in terms of the type of bases they possess.

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TABLE OF CONTENTS

INTRODUCTION

It has been known for a long time that every separable Hilbert space has a basis (a complete orthonormal set). Schauder [36] first introduced the concept of bases in complete normed spaces, which generalize Hilbert spaces, and constructed bases for many of the complete normed spaces encountered in analysis. In his treatise [4] on the theory of linear operators, Banach proved some of the fundamental properties of bases and posed the problem of whether every separable Banach space has a basis. This long standing problem (called the "basis problem", and finally settled in the negative in 1972 by Enflo [13]) generated considerable interest in the subject. Consequently, today there is a large number of results on the theory of bases. They reveal a very close connection between the existence of bases and the structure of topological vector spaces.

Many of the topological vector spaces studied in analysis are in fact topological algebras under some natural multiplication. An investigation of the behavior of the basis with respect to the algebra product in several of these examples leads to the observation that the bases behave in a particularly simple way. For example, the algebra H(D) of analytic functions on the open unit disc D (with pointwise operations and compact-open topology) has the sequence of functions $\{z^n : n=0,1,\ldots\}$ as a basis. Thus in this case the basis is multiplicatively generated by a single element. As another example, consider the Banach algebras $\ell^p(\mathbb{N})$, $1 \leq p < \infty$ (with pointwise operations). The unit vectors $e_n = \{\delta_{mn}\}$, $n=1,2,\ldots$ (where δ_{mn} denotes Kronecker's delta) form a basis in $\ell^p(\mathbb{N})$ in which each e_n is idempotent (i.e., $e_n^2 = e_n$) and $e_n e_m = 0$ for $n \neq m$. There are numerous other examples of topological algebras having bases.

In another direction the theory of topological algebras itself has undergone considerable development since the appearance of Gelfand's paper [14] on normed algebras. We are particularly interested in the theory of locally m-convex algebras (first introduced by Arens [2]) and in LC-algebras. These algebras afford a more natural setting than Banach algebras for the study of bases. This is so because, as we will show, in some cases a topological algebra with a basis cannot be normed. We shall be especially interested in complete metrizeable locally m-convex algebras (hereafter called F-algebras).

Bases in the context of topological algebras were first studied by Husain and Liang in [22] and [23]. They were interested in the question of the continuity of multiplicative linear functionals on F-algebras with bases (where the bases satisfy different multiplicative conditions). In this thesis we initiate a study of the structure of topological algebras with bases. We will consider bases satisfying the properties of the bases given in the above two examples. Bases which are generated by a single element will be called *cyclic* bases. A basis with the property that each element is idempotent and the product of two different elements is zero will be called an *orthogonal* basis. Naturally, some of our results about algebras with bases will generalize results known for specific algebras. Our methods will allow us to extend these results to any algebra with the same type of basis. For example we will see (Chapter II) that certain properties known about the closed ideals of the algebra H(D) are in fact shared by all algebras with orthogonal bases (such as ℓ^p or H^p, $1 < p < \infty$). Thus our definitions give us a convenient framework in which to study certain aspects of all these algebras at once.

In analogy with the theory of bases in topological vector spaces it is to be expected that the existence of a basis (cyclic or orthogonal) in a topological algebra would give valuable indications on the structure of the algebra or of its closed ideals. In fact it will be seen that properties of the bases determine such aspects of the topological algebra (F-algebra) as semisimplicity, the form of the closed ideals, the uniqueness of F-topologies, or the topological character of the maximal ideal space. In some cases the existence of a basis completely describes the algebra. In this way we get characterizations of certain important F-algebras.

Chapter I is mainly devoted to known results which will be needed later. We also introduce the notions of cyclic and orthogonal bases and prove some general results concerning topological algebras possessing such bases.

In Chapter II we study topological algebras with orthogonal bases. In particular, we describe the closed ideals of such algebras and use these results to give a characterization of complete locally m-convex algebras with orthogonal bases.

Chapter III deals with maximal ideals in LC-algebras A with unconditional orthogonal bases. We define a form of local invertibility for elements of A we call E-regularity. This enables us to describe the collection $\mathcal{M}(A)$ of all maximal ideals of A. We prove that $\mathcal{M}(A)$ with the hull-kernel topology is homeomorphic to $\beta M(A)$, the Stone-Čech compactification of M(A) (the maximal ideal space of A). We also use E-regularity to determine the ideal which is the intersection of the dense maximal ideals of A and we show the relationship of this ideal to the dual space A' of A.

Finally, Chapter IV deals with topological algebras having cyclic bases. We describe the spectrum of the element generating the basis and use our results to give a characterization of the F-algebra $H(\Omega)$ (of holomorphic functions on the simply connected domain Ω). We also prove that an F-algebra with a cyclic basis has unique F-algebra topology.

Chapter I

PRELIMINARIES

In this chapter we give the definitions and some of the fundamental properties of the structures to be studied in the following chapters. The chapter is divided into five sections. The first three deal respectively with topological vector spaces, bases in toplogical vector spaces, and topological algebras. We state the major theorems of these subjects in the form we will need them later. In general proofs are not given since there are many excellent books (e.g. [27], [37], and [30]) where they can be found. We prove, however, some results concerning bases which are simple generalizations to the setting of complete metric spaces of results known for Banach spaces. In Section 4 we define cyclic bases and briefly discuss the representation of a complete locally m-convex algebra with a cyclic basis as the inverse limit of Banach algebras. In the last section we define the concept of orthogonal bases and show that in a certain sense (see Theorem 1.18 for the exact statement) a topological algebra can have at most one such basis.

1. Topological Vector Spaces

A *topological vector space* (TVS) is a vector space E over $\mathbb{K}$ (the field of real or complex numbers) together with a Hausdorff topology τ such that the operation + is continuous from $E \times E$ into E and the scalar product is continuous from $\mathbb{K} \times E$ into E. Every TVS has a neighbourhood base for the topology at 0 consisting of circled and absorbing sets. A TVS is called *locally convex* if it possesses a neighbourhood base for the topology at 0 consisting of convex sets. The topology of a locally convex TVS can always be generated by a family of *seminorms* (subadditive, positive homogeneous, symmetric functions from E into $\mathbb{R}^+$) where the seminorms are taken to be the gauge functions of a family of convex 0-neighbourhoods whose positive multiples form a subbase at 0. The *gauge* p_V of a set V is defined by $p_V = inf\{\lambda>0 : x \in \lambda V\}$. A family of seminorms is *directed* if it is a directed set under the order $p(x) \leq q(x)$ for all $x \in$ E. A locally convex topology for E can always be generated by a directed family of seminorms.

Every TVS is a uniform space and its topology can be derived from a unique translation invariant uniformity. A TVS is *complete* if it is complete in this uniformity. Every TVS has a unique (up to isomorphism) completion.

A TVS is *metrizeable* if its topology is metrizeable. A locally convex TVS is metrizeable if, and only if, its topology can be defined by an increasing sequence of seminorms $\{p_n : n=1,2,\ldots\}$. In this case a translation invariant metric is given by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)} .$$

A complete metrizeable locally convex TVS is called an *F-space*. A TVS E is *normable* if its topology can be generated by a single *norm* (i.e. a seminorm which has the property that $p(x)=0$ iff $x=0$). A complete normed TVS is called a *Banach space*.

A *linear functional* f is a linear map f:E → $\mathbb{K}$. Let E be a locally convex TVS whose topology is generated by a directed family of seminorms $\{p_\alpha : \alpha \in \Lambda\}$. A linear functional f on E is continuous if, and only if, there exists an $\alpha \in \Lambda$ and $c \geq 0$ such that $|f(x)| \leq cp_\alpha(x)$ for all $x \in E$. If f is a linear functional defined and continuous on a subspace of E, then f has a continuous linear extension to all of E (the Hahn-Banach theorem). A collection Φ of linear maps from a TVS E into a TVS F is called *equicontinuous* if for each neighbourhood V of 0 in F, $\cap\{f^{-1}(V) : f \in \Phi\}$ is a neighbourhood of 0 in E. A collection of linear functionals Φ on a locally convex space E (with a directed family of seminorms $\{p_\alpha : \alpha \in \Lambda\}$ generating its topology) is equicontinuous iff there exists $\alpha \in \Lambda$ and $c \geq 0$ such that $|f(x)| \leq cp_\alpha(x)$, for all $x \in E$ and $f \in \Phi$. Now, let $\{T_n\}$ be a sequence of continuous linear maps from a complete metrizeable TVS E into a TVS F. If for each $x \in E$ the sequence $\{T_n(x)\}$ is bounded in F and is Cauchy for each x in a dense subset of E, then $\{T_n(x)\}$ is Cauchy for each $x \in E$ (the Banach-Steinhaus theorem). Also, if $\{T_n(x)\}$ converges to $T(x)$ for each $x \in E$, then $T(x)$ is linear and continuous.

If E, F are complete metrizeable spaces then a continuous linear map from E onto F is open (The open mapping theorem). A corollary to this result is the closed graph theorem: with E and F as above a linear map from E into F with closed graph is continuous.

We now prove a result which follows from the closed graph theorem and which we will use later. First, recall that a family of maps Φ on E is called *separating* if for every x and y in E with $x \neq y$ there is an $f \in \Phi$ with the property that $f(x) \neq f(y)$.

THEOREM 1.1 *Let E, F be complete metrizable TVS's, and let Φ be a separating family of continuous linear functionals on F. Then, a linear map $T:E \to F$ is continuous if, and only if, $f \circ T$ is continuous for every $f \in \Phi$*

Proof: ($\Longrightarrow$) clear.

($\Longleftarrow$) Let $\{a_\lambda : \lambda \in \Lambda\}$ be a net in E, $a_\lambda \to a$, and suppose that $T(a_\lambda) \to b$. Since each $f \in \Phi$ is continuous, it follows that $f(T(a_\lambda)) \to f(b)$. Also, since $f \circ T$ is continuous, we have that $(f \circ T)(a_\lambda) \to (f \circ T)(a)$ for all $f \in \Phi$; i.e., $f(T(a_\lambda)) \to f(T(a))$ for all $f \in \Phi$. Since the limit of a convergent net in $\mathbb{K}$ is unique, it follows that $f(b) = f(T(a))$ for all $f \in \Phi$. Now, since Φ is separating, we have that $b = T(a)$. Hence, T has closed graph and is thus continuous by the closed graph theorem. ////

We note that the above theorem (and hence other results in this thesis) is true if E and F are in any class of topological vector spaces for which the closed graph theorem holds (see [20] for a discussion of such spaces). However, we will restrict ourselves here to considering complete metrizeable TVS's only.

If E and F are topological vector spaces then their product $E \times F$ (their algebraic product endowed with the product topology) is a topological vector space. If E and F are locally convex, then so is their product $E \times F$. Also, $E \times F$ is complete if, and only if, both E and F are complete. We note here also that if E and F are locally convex then the injections i_E and i_F of E and F, respectively, into the product $E \times F$ are isomorphisms (topologically and algebraically) onto their respective images.

If E is a topological vector space and if H is a closed subspace of E then the quotient E/H (the algebraic quotient endowed with the quotient topology) is a topological vector space. If E is also locally convex, then so is E/H. Moreover, if E is locally convex and if $\{p_\alpha\}$ is a directed famil[t]y of seminorms generating the topology of E, then the family of seminorms $\{\tilde{p}_\alpha\}$ on E/H defined by

$$\tilde{p}_\alpha(\tilde{x}) = inf\ \{p_\alpha(y) : y \in \tilde{x}\}$$

(where $\tilde{x}$ is the image of x under the canonical map $\eta: E \to E/H$) is a directed family of seminorms generating the topology of E/H. It is not true in general that E/H is complete if E is complete. However, if E is an *F*-space, then so is E/H.

Finally, we consider the linear space E' of all continuous linear functionals on E. E' is called the *topological dual* (or simply *dual*) of E. If E is a locally convex space, then E' separates points of E. Several topologies can be introduced on the dual. We will be interested in the *weak topology* $\sigma(E',E)$ which is the topology of uniform convergence on finite subsets of E.

The definitions and results mentioned in this section can be found in any one of [27], [35], or [40].

2. Bases in Topological Vector Spaces

Let E be a TVS. A *basis* for E is a sequence $\{x_n\}$ in E with the property that for every $x \in E$, there is a unique sequence $\{\alpha_n\}$ in $\mathbb{K}$ such that $x = \Sigma_{n=1}^{\infty} \alpha_n x_n$, where the series converges in the topology of E. The *coefficient functionals* associated to the basis $\{x_n\}$ are the functionals x_n^* defined by $x_n^*(x) = \alpha_n$. A basis for which all the coefficient functionals are continuous is called a *Schauder basis*. Two fundamental results in this direction are: a basis in a complete metrizeable TVS (with a translation invariant metric) is a Schauder basis, and a *weak Schauder basis* (i.e. a Schauder basis in the weak topolgy) in a complete metrizeable TVS is a Schauder basis for that space.

A series $\Sigma_{n=1}^{\infty} x_n$ in a TVS E is called *unconditionally convergent* if for every permutation π of the natural numbers the series $\Sigma_{n=1}^{\infty} x_{\pi(n)}$ also converges in E. In a complete locally convex TVS the series $\Sigma_{n=1}^{\infty} x_n$ is unconditionally convergent iff for every sequence $\{\beta_n\}$ of scalars with $|\beta_n| \leq 1$, $n=1,2,\ldots$ the series $\Sigma_{n=1}^{\infty} \beta_n x_n$ also converges. This is also the case if, and only if, every subseries of $\Sigma_{n=1}^{\infty} x_n$ converges. A series $\Sigma_{n=1}^{\infty} x_n$ in a locally convex space is called *absolutely convergent* if for each continuous seminorm p on E, $\Sigma_{n=1}^{\infty} p(x_n)$ converges in $\mathbb{R}$. A basis $\{x_n\}$ in a TVS is called an *unconditional basis* if every convergent series of the form $\Sigma_{n=1}^{\infty} \alpha_n x_n$ converges unconditionally. A basis $\{x_n\}$ in a locally convex TVS is called an *absolute basis* if every convergent series of the form $\Sigma_{n=1}^{\infty} \alpha_n x_n$ converges absolutely.

Let $\{x_n\}$ be a basis in a TVS E. The sequence $\{x_n, x_n^*\}$ of basis elements and their associated coefficient functionals has the property that $x_n^*(x_m) = \delta_{nm}$, $n,m=1,2,\ldots$ A sequence $\{x_n; f_n\}$, $\{x_n\} \subseteq$ E, $\{f_n\} \subseteq$ E', with this latter property is called *biorthogonal*. We will need the following result on when the elements $\{x_n\}$ of a biorthogonal sequence form a basis in E. It is a simple extension of a result for Banach spaces [37].

First, for a given biorthogonal system $\{x_n, f_n\}$ we define the *partial sum operators* S_n, $n=1,2,\ldots$ by $S_n(x) = \Sigma_{k=1}^n f_k(x)\, x_k$. Also, for $F \subseteq E$ we will write $[F]$ for the linear span of F and $\overline{F}$ for the closure of F.

THEOREM 1.2 *Let E be a TVS and let $\{x_n, f_n\}$ be a biorthogonal sequence in E. The following are equivalent:*

(a) $\{x_n\}$ is a basis in E

(b) For every $x \in E$, $\lim_{n\to\infty} S_n(x) = x$.

If E is also complete and metrizeable then (a) and (b) are equivalent to:

(c) $\overline{[\{x_n\}]} = E$ and $\{S_n(x)\}$ is bounded for each $x \in E$.

Proof: If $\{x_n\}$ is a basis for E, then clearly $\lim_{n\to\infty} S_n(x) = x$ for each $x \in E$. If (b) holds then each $x \in E$ has the representation $x = \Sigma_{n=1}^\infty f_n(x)\, x_n$. This representation is unique because if $\Sigma_{n=1}^\infty f_n(x) x_n = 0$ then for each f_k we have $f_k\left(\Sigma_{n=1}^\infty f_n(x) x_n\right) = f_k(x) = 0$; i.e., $f_k(x) = 0$, $k=1,2,\ldots$ This shows (a) $\Longleftrightarrow$ (b).

Now, for every finite linear combination of the form

$$q = \sum_{i=1}^{n} \alpha_i x_i \qquad (*)$$

and for $m \geq n$ we have

$$S_m(q) = \sum_{j=1}^{m} f_j\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) x_j = \sum_{j=1}^{m}\ \sum_{i=1}^{n} \alpha_i \delta_{ij} x_i = \sum_{i=1}^{n} \alpha_i x_i = q.$$

Hence for each x of the form (*), $\{S_n(x)\}$ is Cauchy. Since the elements of this form are dense in E by (c), $\{S_n(x)\}$ is Cauchy by the Banach-Steinhaus theorem, and $S(x) = \lim_{n\to\infty} S_n(x)$ is continuous. But $S(x) = x$ on a dense set. Consequently, $S(x) = x$ for each $x \in E$; i.e. $\lim_{n\to\infty} S_n(x) = x$ for each $x \in E$. Thus each $x \in E$ has the representation

$$x = \sum_{n=1}^{\infty} f_n(x)x_n$$

and it is clear that this representation is unique. This completes the proof (since (a) $\Longrightarrow$ (c) is trivial). ////

In the next theorem we consider the existence of bases in the product of topological vector spaces with bases.

THEOREM 1.3 *Let E, F be TVS's with bases $\{x_n\}$, $\{y_n\}$, resp. Then the sequence $\{z_n\}$ in $E \times F$ defined by*

$$z_{2n-1} = (x_n, 0), \quad z_{2n} = (0, y_n), \qquad n=1,2,\ldots$$

is a basis for $E \times F$.

Moreover, if E, F are complete locally convex (resp., locally convex), and if the bases $\{x_n\}$, $\{y_n\}$ are unconditional (resp., absolute), then the basis $\{z_n\}$ is an unconditional (resp., absolute) basis for $E \times F$.

Proof: Suppose $(x,y) \in E \times F$ and $x = \Sigma_{i=1}^{\infty}\alpha_i x_i$, $y = \Sigma_{i=1}^{\infty}\beta_i y_i$. Now, in $E \times F$,

$$\sum_{i=1}^{n} \alpha_i(x_i, 0) = \left(\sum_{i=1}^{n} \alpha_i x_i, 0\right) \to (x, 0)$$

as $n \to \infty$. Similarly, $\Sigma_{i=1}^{n}\beta_i(0, y_i) \to (0, y)$, as $n \to \infty$. So,

$$(x, y) = (x, 0) + (0, y) = \sum_{i=1}^{\infty} \alpha_i(x_i, 0) + \sum_{i=1}^{\infty} \beta_i(0, y_i).$$

Therefore,

$$(x,y) = \sum_{j=1}^{\infty} \alpha_j z_{2j-1} + \sum_{j=1}^{\infty} \beta_j z_{2j}.$$

To show uniqueness of these expansions, suppose that

$$\sum_{j=1}^{n} \gamma_j z_j \to (x,y),$$

say. Then

$$\sum_{j=1}^{n} \gamma_{2j-1} x_j \to x \text{ and } \sum_{j=1}^{n} \gamma_{2j} y_j \to y.$$

Hence, if $\Sigma_{j=1}^{\infty}\gamma_j z_j = (0,0)$ then $\Sigma_{j=1}^{\infty}\gamma_{2j-1}x_j = 0$ and $\Sigma_{j=1}^{\infty}\gamma_{2j}y_j = 0$. But since $\{x_n\}$ and $\{y_n\}$ are bases in E and F respectively, it follows from the uniqueness of the representation there that $\gamma_j = 0,\ j=1,2,\ldots$ This completes the proof of the first assertion.

Now, suppose that the bases $\{x_n\}$ and $\{y_n\}$ are unconditional and suppose that $\Sigma_{j=1}^{\infty}\gamma_j z_j$ converges in E × F. Then the same as above we have that in this case $\Sigma_{j=1}^{\infty}\gamma_{2j-1}x_j$ and $\Sigma_{j=1}^{\infty}\gamma_{2j}y_j$ both converge in E and F respectively. Now, let $\{\beta_i\}$ be a sequence of scalars with the property that $|\beta_i| \leq 1,\ i=1,2,\ldots$ Since the bases in E and F are unconditional, it follows that $\Sigma_{j=1}^{\infty}\beta_{2j-1}\gamma_{2j-1}x_j$ and $\Sigma_{j=1}^{\infty}\beta_{2j}\gamma_{2j}y_j$ both converge in E and F respectively. Therefore the series

$$\sum_{j=1}^{\infty} \beta_j \gamma_j z_j$$

converges in E × F. Since this is true for every sequence $\{\beta_i\}$ bounded by 1, and since E × F is a complete locally convex TVS, it follows that the basis $\{z_n\}$ is unconditional.

Suppose now that the bases $\{x_n\}$ and $\{y_n\}$ are absolute and let p be a continuous seminorm on $E \times F$. Since the injection $i_E: E \to E \times F$ is an isomorphism onto $i_E(E)$, it follows that $p \circ i_E$ is a continuous seminorm on E. We also have that $p \circ i_E(x_n) = p(z_{2n-1})$, $n=1,2,\ldots$ Now suppose that the series $\Sigma_{j=1}^{\infty} \gamma_j z_j$ converges in $E \times F$. Then, the same as before, we have

$$\sum_{j=1}^{\infty} \gamma_j z_j = \sum_{j=1}^{\infty} \gamma_{2j-1} z_{2j-1} + \sum_{j=1}^{\infty} \gamma_{2j} z_{2j}$$

where the two series on the right converge. Applying p to this equation we have

$$\begin{aligned} p\Big(\sum_{j=1}^{\infty} \gamma_j z_j\Big) &\leq p\Big(\sum_{j=1}^{\infty} \gamma_{2j-1} z_{2j-1}\Big) + p\Big(\sum_{j=1}^{\infty} \gamma_{2j} z_{2j}\Big) \\ &\leq \sum_{j=1}^{\infty} |\gamma_{2j-1}| p(z_{2j-1}) + \sum_{j=1}^{\infty} |\gamma_{2j}| p(z_{2j}) \\ &= \sum_{j=1}^{\infty} |\gamma_{2j-1}| p \circ i_E(x_j) + \sum_{j=1}^{\infty} |\gamma_{2j}| p \circ i_F(y_j) \end{aligned}$$

where i_F is the injection of F into $E \times F$. Now since the series $\Sigma_{j=1}^{\infty} \gamma_{2j-1} x_j$ and $\Sigma_{j=1}^{\infty} \gamma_{2j} y_j$ converge (see above), and since $p \circ i_E$ and $p \circ i_F$ are continuous seminorms on E and F respectively, and since the bases $\{x_n\}$ and $\{y_n\}$ are absolute, it follows that the last two series above converge. Thus $\Sigma_{j=1}^{\infty} |\gamma_j| p(z_j)$ converges and this shows that the basis $\{z_n\}$ is absolute. ////

Next we prove a theorem on bases in the quotient of a topological vector space by the closed linear span of some of its basis elements. It is a generalization of a similar theorem known for Banach spaces[37].

THEOREM 1.4 *Let E be a TVS with a Schauder basis (resp., unconditional Schauder basis) $\{x_n\}$. Let $\{i_n\}$ be an increasing sequence of natural numbers and ~~let~~ $\{j_n\}$ be its complementary sequence. Let $F = \overline{[\{x_{i_n}\}]}$. If $\eta : E \to E/F$ is the canonical map and if we set $\tilde{x}_n = \eta(x_{j_n})$, $n=1,2,\ldots,$ then $\{\tilde{x}_n\}$ is a Schauder basis (resp., unconditional Schauder basis) for E/F.*

Moreover, if E is locally convex and if the basis $\{x_n\}$ is an absolute Schauder basis for E, then the basis $\{\tilde{x}_n\}$ is an absolute Schauder basis for E/F.

Proof: First note that if $x \in F$, then $x^*_{j_n}(x) = 0$ for all $n \in \mathbb{N}$. This is so because for each $x \in F$ we have $x = \lim_\lambda y_\lambda$ where $\{y_\lambda\} \subseteq [\{x_{i_n}\}]$. Hence, since the basis $\{x_n\}$ is a Schauder basis (i.e., each x^*_n is continuous) we have $x^*_{j_n}(x) = \lim_\lambda x^*_{j_n}(y_\lambda) = 0$ for all $n \in \mathbb{N}$.

Now, define $\phi_m : E/F \to \mathbb{K}$, $m=1,2,\ldots$ by $\phi_m(\eta(x)) = x^*_{j_m}(x)$. Then ϕ_m is well defined: For if $\eta(x) = \eta(y)$ then $x-y \in F$ and it follows (from the above) that $x^*_{j_m}(x-y) = 0$, whence $\phi_m(\eta(x)) = \phi_m(\eta(y))$. Now, clearly each ϕ_m is a linear functional of E/F. We show that each ϕ_m is continuous. To this end fix $m \in \mathbb{N}$, and consider the following commutative diagram:

$$\begin{tikzcd} & \mathbb{K} \\ E \arrow[ur, "x^*_{j_m}"] \arrow[r, "\eta"'] & E/F \arrow[u, "\phi_m"'] \end{tikzcd}$$

i.e., $x^*_{j_m} = \phi_m \circ \eta$. Let U be an open subset of $\mathbb{K}$ and set $W = x^{*-1}_{j_m}(U)$

$= \eta^{-1}(\phi_m^{-1}(U))$. Since $x^*_{j_m}$ is continuous, it follows that W is open in E. i.e., $\eta^{-1}(\phi_m^{-1}(U))$ is open. But η being an open map, we have $\phi_m^{-1}(U) = \eta(\eta^{-1}(\phi_m^{-1}(U))$ is open. Hence, ϕ_m is continuous. Note also that we have $\phi_m(\tilde{x}_n) = \phi_m(\eta(x_{j_n})) = x^*_{j_m}(x_{j_n}) = \delta_{mn}$, $m,n=1,2,\ldots$. Therefore $\{\tilde{x}_n, \phi_n\}$ is a biorthogonal sequence in E/F.

Now, let $\eta(x) \in E/F$. Then, since

$$x = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{k=1}^{n} x^*_k(x)x_k$$

we have (from the continuity of η)

$$\eta(x) = \lim_{n\to\infty} \eta(S_n(x))$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} x^*_k(x)\eta(x_k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \phi_k(\eta(x))\eta(x_{j_k})$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \phi_k(\eta(x))\tilde{x}_n$$

Hence, for each $\eta(x) \in E/F$ we have that $\eta(x) = \lim_{n\to\infty} S_n(\eta(x))$ where S_n is the partial sum operator associated with the biorthogonal system $\{\tilde{x}_n, \phi_n\}$. It follows from this by Theorem 1.2 that $\{\tilde{x}_n\}$ is a basis for E/F.

If the basis $\{x_n\}$ is unconditional, then for every series of the form $\Sigma_{n=1}^{\infty} x^*_n(x)x_n$ $(x \in E)$ and for every permutation π of the natural numbers, we have that the series $\Sigma_{n=1}^{\infty} x^*_{\pi(n)}(x)x_{\pi(n)}$ converges.

It is clear from this and from the above reasoning that in this case every series of the form $\Sigma_{n=1}^{\infty}\phi_{\pi(n)}\tilde{x}_{\pi(n)}$ is convergent in E/F. This shows that the basis $\{\tilde{x}_n\}$ is unconditional.

Suppose E is locally convex and the basis $\{x_n\}$ is absolute. Let $\{p_\alpha\}$ be a directed family of seminorms generating the topology of E. The family $\{\tilde{p}_\alpha\}$ defined by $\tilde{p}_\alpha(\tilde{x}) = inf\ \{p_\alpha(x) : x \in \tilde{x}\}$ $(\tilde{x} \in E/F)$ is a directed family of seminorms giving the topology of E/F. For $x \in E$ we have

$$\sum_{n=1}^{\infty} |\phi_n(\eta(x))|\,\tilde{p}_\alpha(\tilde{x}_n) \leq \sum_{n=1}^{\infty} |x_{j_n}^*(x)|\,p_\alpha(x_{j_n})$$

where the series on the right converges because the basis $\{x_n\}$ is an absolute basis. It follows that the series on the left converges and this shows that the basis $\{\tilde{x}_n\}$ is an absolute basis for E/F. ////

The standard reference for the subject of bases in Banach spaces is the book by that title of Singer [37]. The results in this section are generalizations of theorems known for Banach spaces. We include them here because we will need them later in this more general setting.

3. Topological Algebras

A *topological algebra* is an algebra A (always assumed to be commutative here) over $\mathbb{K}$ together with a Hausdorff topology τ such that (A,τ) is a topological vector space and such that the algebra product is a continuous function from $A \times A$ into A. A topological algebra which is also a locally convex space is called an *LC-algebra*. A convex subset U of A is called *m*-convex if $U^2 \subseteq U$. A topological algebra with a neighbourhood base for the topology at 0 consisting of m-convex sets is called a *locally m-convex algebra.*

If A and B are topological algebras (resp., LC-algebras, locally m-convex algebras) then $A \times B$ (the algebraic product with the product topology) is a topological algebra (resp., LC-algebra, locally m-convex algebra). If A is a topological algebra and if M is a closed ideal of A, then A/M (the algebraic quotient with the quotient topology) is also a topological algebra. Moreover, if A is an LC-algebra (locally m-convex algebra) then so is A/M.

Let A be a locally m-convex algebra. The topology of A is given by a directed family $\{p_\alpha : \alpha \in \Lambda\}$ of *submultiplicative* (i.e., $p(xy) \leq p(x)p(y)$, $x, y \in A$) seminorms. For $\alpha \in \Lambda$ let $K_\alpha = \{x \in A : p_\alpha(x) = 0\}$ be the kernel of p_α. K_α is an ideal of A, and so $A_\alpha = A/K_\alpha$ is a normed algebra with the norm $\| \pi_\alpha(x) \| = p_\alpha(x)$ where $\pi_\alpha : A \to A_\alpha$ is the natural homomorphism. We let $\tilde{A}_\alpha$ be the completion of A_α relative to this norm. If $p_\alpha \leq p_\beta$ then the map $\pi'_{\alpha\beta} : A_\beta \to A_\alpha$ defined by $\pi'_{\alpha\beta}(\pi_\beta(x)) = \pi_\alpha(x)$ induces a norm decreasing homomorphism $\pi_{\alpha\beta}$ from $\tilde{A}_\beta$ to $\tilde{A}_\alpha$. The family $(\tilde{A}_\alpha, \pi_{\alpha\beta})_{\alpha,\beta \in \Lambda}$ of algebras and homomorphisms is an inverse limit system and $\varprojlim (A_\alpha, \pi_{\alpha\beta})$ is algebraically and topologically isomorphic to the completion of A. This representation of a complete locally m-convex algebra as an inverse limit of complete normed algebras is discussed in [30].

The *maximal ideal space* of A is the space M(A) of all non-zero continuous complex valued homomorphisms of A with the relative $\sigma(A', A)$-topology (the *Gelfand topology*). There is a bijective correspondence between M(A) and the closed maximal ideals of codimension one of A given by $f \to M_f = ker(f)$. For each $x \in A$, the *Gelfand transform* of x is the continuous function $\hat{x}:M(A) \to \mathbb{C}$ defined by $\hat{x}(f) = f(x)$. The *Gelfand map* is the map $G:A \to C(M(A))$ which takes x to $\hat{x}$. The image of this map in $C(M(A))$ is denoted by $\hat{A}$. If A is locally m-convex, then the Gelfand map is one-one if, and only if A is *semisimple* in the sense that $\bigcap M(A) = \{0\}$. We also define $M^{\#}(A)$ to be the set of all complex-valued homomorphisms of A (continuous or not) and $\mathcal{M}(A)$ to be the set of all maximal ideals of A. We will call A *strongly semisimple* if for every x in A, $x \neq 0$, there exists an $f \in M(A)$ with $f(x) \neq 0$. If A is locally m-convex then the notions of semisimplicity and strong semisimplicity coincide[31].

If $x \in A$, the *spectrum* of x is $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is singular}\}$†. If A is a complete locally m-convex algebra then $\sigma(x) = \hat{x}(M(A)) = \hat{x}(M^{\#}(A))$. The *spectral radius* of x is $\rho(x) = sup\ \{|\lambda| : \lambda \in \sigma(x)\}$.

The same as for topological vector spaces, we call a complete metrizeable locally m-convex algebra an *F-algebra* and a complete normed algebra (with a submultiplicative norm) is called a *Banach algebra*. If A is an F-algebra then the Gelfand map is a continuous function from A into $C(M(A))$ [31]. Also for an F-algebra A, M(A) is *hemicompact* (i.e., there exists a sequence of compact subsets of M(A) whose union is M(A) and such that every compact subset of M(A) is contained in some member of this sequence) and each compact subset of M(A) is equicontinuous. If A is a Banach algebra with identity then M(A) is compact and $\sigma(x)$ is compact for each $x \in A$. If A is a commutative Banach algebra with identity then $\rho(x)$ is a norm on A

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† e is the identity of the algebra.

called the *spectral norm*. If the spectral norm of a commutative Banach algebra is equivalent to the original norm then the algebra is called a *function algebra*.

A topological algebra A (with identity e) is said to be *generated* by $F \subseteq A$ if the smallest closed subalgebra containing F (and e) is all of A. If A is a complete locally m-convex algebra generated by the single element z, then the map $f \to f(z)$ of M(A) onto $\sigma(z)$ is continuous. If A is a Banach algebra then this map is a homeomorphism. However, this map need not be a homeomorphism for F-algebras (see [9]).

A topological algebra A is called *functionally continuous* if $M(A) = M^{\#}(A)$; i.e., if every multiplicative linear functional on A is continuous. It is known that every Banach algebra is functionally continuous. However, it is a long standing problem (see [30]) whether the same is true for F-algebras. Arens [3] has proved that a finitely generated F-algebra is functionally continuous. Husain and Ng ([24] and [25]) have shown that certain classes of F-algebras are functionally continuous. Husain and Liang ([22] and [23]) have shown that $M(A) = M^{\#}(A)$ for certain classes of F-algebras with bases. We mention in this connection Michael's result [30] that a functionally continuous semisimple F-algebra has unique F-algebra topology.

Finally, we state some results about the functional calculus in locally m-convex algebras: let A be a complete locally m-convex algebra with identity and let $x \in A$. If U is an open set in $\mathbb{C}$ containing $\sigma(x)$, and if w is a complex-valued function defined and analytic on U, then there exists an element $y \in A$ with the property that $f(y) = w(f(x))$

for all $f \in M(A)$. Furthermore, if A is commutative and semisimple then y is unique. We note that in particular if $\sigma(x)$ is open then any function analytic on $\sigma(x)$ has the above property.

The results in this section that are not specifically referenced are standard results of topological algebras and can be found in any of [6], [17], [30], or [44].

4. Cyclic Bases

Let A be a topological algebra with a basis $\{x_n\}$. We connect the algebra structure of A and the basis by making the following definition:

DEFINITION 1.5 A basis $\{x_n\}$ in A is called a *cyclic basis* if there exists $z \in A$ such that $z^n = x_n$, $n=0, 1, \ldots$ where $z^0 = e$, the identity of A.

As might be expected, algebras with cyclic bases have a close connection to algebras of analytic functions. In fact, we will show later (Theorem 4.12) that under certain conditions an F-algebra with a cyclic basis is isomorphic to the algebra of analytic functions on an open subset of the complex plane.

Gelfand called a topological algebra an *analytic algebra* if for each $x \in A$, if $\hat{x}$ vanishes on an open subset of M(A) then $\hat{x}$ vanishes identically on M(A); see [44,p.164]. Now, suppose that A is an F-algebra with a cyclic basis $\{z^n\}$. Suppose further that M(A) is naturally homeomorphic to $\sigma(z)$. Then we will show that A is an analytic algebra in the above sense. First, we have noted in Section 3 that the Gelfand map $x \to \hat{x}$ is continuous. Hence

$$\begin{aligned}\hat{x}(f) &= \sum_{n=0}^{\infty} x_n^*(x)\hat{z}(f)^n \\ &= \sum_{n=0}^{\infty} x_n^*(x)f(z)^n\end{aligned}$$

Now, if $\hat{x}$ vanishes on an open subset of M(A) it follows from the fact that $f \to f(z)$ is a homeomorphism that the series $\Sigma_{n=0}^{\infty} x_n^*(x)t^n$ vanishes for all t in an open subset of the plane. It follows by the identity theorem for power series that all the coefficients are zero; i.e. $x_n^*(x) = 0$, $n=1,2,\ldots$ Whence $\hat{x}(f) = 0$ for all $f \in$ M(A). Thus A is an analytic algebra. The condition that M(A) be homeomorphic to $\sigma(z)$ occurs for a large class of algebras with cyclic bases (e.g. if the basis is unconditional as in theorems 3.7 and 3.15).

Another property of algebras of analytic functions shared by algebras with cyclic bases is the fact that the seminorms defining the topology are in fact all norms. To show this we need a lemma.

LEMMA 1.6 *If A is an F-algebra with a cyclic basis $\{z^n\}$ and with $\rho(z) > 0$, then $M(A)$ contains an infinite compact set.*

Proof: Suppose to the contrary that each compact subset of M(A) is finite. Then, because M(A) is hemicompact (see Section 3), it must be at most countable. This, however, is impossible since there is a one-one map between M(A) and $\sigma(z)$ and the latter is uncountable by Lemma 4.8. (Lemma 4.8 is proved independently of the material in this section). ////

THEOREM 1.7 *Let A be an F-algebra with a cyclic basis $\{z^n\}$ and let $\{p_n\}$ be an increasing family of seminorms defining the topology of A. If $\rho(z) > 0$, then there exists $k > 0$ such that p_n is a norm for all $n > k$.*

Proof: Let H be an infinite compact subset of $M(A)$; such exists by the above lemma. Since A is an F-algebra and H is compact, it follows that H is equicontinuous. This means that there exists $k \in \mathbb{N}$ and $c > 0$, such that

$$|f(x)| \leq cp_k(x) \qquad x \in A, f \in H. \qquad (*)$$

We will show that p_k is a norm. So, suppose $p_k(x) = 0$ for some $x \in$ A. Then from (*) we get

$$|f(x)| = |f(\sum_{n=0}^{\infty} x_n^*(x)z^n)| = |\sum_{n=0}^{\infty} x_n^*(x)f(z)^n| \leq cp_k(x)$$

for all $f \in H$. Since $p_k(x) = 0$, we have

$$|\sum_{n=0}^{\infty} x_n^*(x)f(z)^n| = 0, \qquad f \in H. \qquad (**)$$

Since H is compact and infinite, and since $f \to f(z)$ is one-one and continuous, it follows that the set $\{f(z): f \in H\}$ is an infinite compact subset of $\mathbb{C}$. Therefore, from (**) and the identity theorem for power series we have that $x_n^*(x) = 0$, $n=0,1,2,\ldots$ Since $\{z^n\}$ is a basis, the coefficients $x_n^*(x)$ are uniquely determined, and hence $x = 0$. Therefore p_k is a norm.

Now, since the sequence $\{p_n\}$ of seminorms is increasing, it follows that p_n is a norm for all $n \geq k$. ////

COROLLARY 1.8 *Under the hypothesis of the theorem, the topology of A is given by an increasing sequence of norms.* ////

The condition that $\rho(z) > 0$ is equivalent to A being semisimple as we will show later (Lemma 4.6).

Now, consider the representation of a complete locally m-convex algebra as the inverse limit of Banach algebras (see Section 3). The above corollary says that for an F-algebra A with a cyclic basis

$$A = \lim_{\leftarrow} \tilde{A}_n$$

where each $\tilde{A}_n$ is the completion of A in the norm p_n. Since $\{z^n\}$ is a basis for A, the polynomials in z are dense in A, hence also in each $\tilde{A}_n$. It follows that each $\tilde{A}_n$ is the completion of the algebra P of polynomials in the norm p_n. Thus A is the inverse limit of such algebras. Algebras which are completions of normed algebras of polynomials are discussed in [11].

5. Orthogonal Bases

Let A be a topological algebra and let $\{x_n\}$ be a basis for A. Again we connect the algebra structure of A and the basis by the following definition:

DEFINITION 1.9 The basis $\{x_n\}$ in A is called an *orthogonal basis* if each element of the basis is idempotent (i.e., $x_n^2 = x_n$, $n=1,2,..$) and $x_n x_m = 0$ for $n \neq m$. In other words, $x_n x_m = \delta_{nm} x_n$, $n,m=1,2,\ldots$

A few remarks about our definition of orthogonal bases are in order. F-algebras with orthogonal bases were first studied by Husain and Liang [21] in connection with the continuity of multiplicative linear functionals. The definition there differs from ours but we will show that they are equivalent. To this end consider the following conditions on a topological algebra A:

(a) A has a basis $\{x_n\}$ with $x_n x_m = 0$ for $n \neq m$;

(b) A has a basis $\{x_n\}$ with $x_n x_m = 0$ for $n \neq m$ and $x_n^2 \neq 0$;

(c) A has a basis $\{x_n\}$ with $x_n x_m = 0$ for $n \neq m$ and $x_n^2 = c_n x_n$, $c_n \neq 0$;

(d) A has a basis $\{x_n\}$ with $x_n x_m = \delta_{nm} x_n$,

where in each case $n,m=1,2,\ldots$

If A is a topological algebra satisfying (a) and if x and y are elements of A, $x = \Sigma_{n=1}^{\infty}\alpha_i x_i$, $y = \Sigma_{n=1}^{\infty}\beta_i x_i$, then by the continuity of the multiplication in A we have

$$xy = x\Big(\sum_{i=1}^{\infty} \beta_i x_i\Big) = \sum_{i=1}^{\infty} \beta_i x x_i = \sum_{i=1}^{\infty} \beta_i \Big(\sum_{j=1}^{\infty} \alpha_j x_j\Big) x_i = \sum_{i=1}^{\infty} \beta_i \alpha_i x_i^2.$$

We will make use of this fact without further mention.

THEOREM 1.10 *If A is a topological algebra, then (b), (c), and (d) above are all equivalent for A.*

Proof: (b) $\Longrightarrow$ (c) Since $\{x_n\}$ is a basis and since $x_n^2 \neq 0$, it follows that $x_n^2 = \Sigma_{i=1}^{\infty}\alpha_i x_i$, for some sequence of scalars $\{\alpha_i\}$. Thus, for $m \neq n$, we have

$$0 = x_m x_n^2 = x_m\Big(\sum_{i=1}^{\infty} \alpha_i x_i\Big) = \alpha_m x_m^2.$$

But $x_m^2 \neq 0$. Therefore, $\alpha_m = 0$ for all $m \neq n$. Thus, $x_n^2 = \alpha_n x_n$, and since $x_n^2 \neq 0$, $\alpha_n \neq 0$. This is (c).

(c) $\Longrightarrow$ (d) Suppose $\{x_n\}$ is a basis satisfying (c). Let $y_n = x_n/c_n$, $n=1,2,\ldots$ Clearly $\{y_n\}$ is a basis for A. Furthermore, for $n \in \mathbb{N}$,

$$y_n^2 = \frac{x_n^2}{c_n^2} = \frac{c_n x_n}{c_n^2} = \frac{x_n}{c_n} = y_n.$$

Also, for $n \neq m$, $y_n y_m = (x_n/c_n)(x_m/c_m) = 0$. Therefore $y_n y_m = \delta_{nm} x_n$, $n,m=1,2,\ldots$ This shows that (d) holds.

(d) $\Longrightarrow$ (b) obvious. ////

THEOREM 1.11 *If A is a topological algebra with an identity e, then (a), (b), (c), and (d) are all equivalent for A.*

Proof: In view of the preceeding theorem it is sufficient to show that (a) is equivalent to (b). So, suppose that $e \in$ A. Then $e = \Sigma_{i=1}^{\infty} \alpha_i x_i$. Now, if $x_j \in \{x_n\}$

$$x_j = x_j e = x_j\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = \alpha_j x_j^2.$$

Since $x_j \neq 0$ it follows that $x_j^2 \neq 0$. Since x_j was any basis element, we have that $x_i^2 \neq 0$, $i=1,2,\ldots$ This shows that (b) holds. It is clear that (b) implies (a) always. So, this completes the proof. ////

Now, suppose that A has a basis satisfying (a). We will show that there is a quotient algebra of A (modulo a closed ideal) satisfying (b), and hence all of (a)-(d). We first show the following theorem:

THEOREM 1.12 *Let A be a topological algebra with a Schauder basis* $\{x_n\}$ *satisfying* $x_n x_m = 0$ *for* $n \neq m$*;* $n,m=1,2,\ldots$ *The following are equivalent:*

(i) $x_n^2 \neq 0$, $n=1,2,\ldots$

(ii) A is strongly semisimple.

Proof: If *(i)* holds then A has an orthogonal basis by Theorem 1.10. Now, it is easy to see that the coefficient functionals x_n^* associated with this basis are multiplicative (see p.25). Since by hypothesis the basis is a Schauder basis, it follows that each coefficient functional x_n^* belongs to M(A). Now, if $x \neq 0$ then by the uniqueness of the basis representation there is an $n \in \mathbb{N}$ with $x_n^*(x) \neq 0$. This shows that A is strongly semisimple.

Conversely, suppose that $x_n^2 = 0$ for some basis element x_n. If $f \in$ M(A) then $f(x_n)^2 = f(x_n^2) = 0$. Therefore $f(x_n) = 0$ for every $f \in$ M(A). It follows that A is not strongly semisimple. ////

For the next theorem let N = $\{ x_i \in \{x_n\} : x_i^2 = 0 \}$; i.e., N is the set of all basis elements whose square is zero. It is clear that [N] is an ideal in A. Thus I = $\overline{[N]}$ is a closed ideal of A.

THEOREM 1.13 *If A is a topological algebra with a Schauder basis* $\{x_n\}$ *satisfying* $x_n x_m = o$ *for* $n \neq m$*,* $n,m=1,2,\ldots$ *then A/I (where I is the ideal defined above) has an orthogonal basis.*

Proof: Let $\{x_{i_n}\}$ be the sequence of basis elements in N and let $\{x_{j_n}\}$ be its complementary sequence in the basis $\{x_n\}$. Also, let η:A → A/I be the canonical map. By Theorem 1.4 the sequence $\{\tilde{x}_n =$

$\eta(x_{j_n}) : n=1,2,\ldots\}$ is a basis for A/I. We have for $m \neq n$, $m,n=1,2,\ldots$

$$\eta(x_{j_m})\eta(x_{j_n}) = \eta(x_{j_m}x_{j_n}) = 0 \text{ and } \eta(x_{j_n})\eta(x_{j_n}) = \eta(x^2_{j_n}) \neq 0$$

where $\eta(x^2_{j_m}) \neq 0$ because $x^2_{j_n} = cx_{j_n}$, $c \neq 0$, and x_{j_n} does not belong to I (see proof of (b) $\Longrightarrow$ (c) of Theorem 1.10). Thus the basis $\{\tilde{x}_n\}$ of A/I satisfies (b). It follows by Theorem 1.10 that A/I has an orthogonal basis. ////

Recall that the *radical* of A, *Rad*(A), is the intersection of all modular maximal ideals of A (see [30]). We will show that the ideal $I = \overline{[N]}$ defined above is in fact the radical of A whenever this radical is closed. First, let A be a topological algebra satisfying (a). Then, just as in the proof of Theorem 1.10 we have that for each basis element x_n, either $x^2_n = 0$ or $x^2_n = c_n x_n$. Let $\{y_n\}$ be the sequence in A defined by

$$y_n = \begin{cases} x_n/c_n & \text{if } x^2_n = c_n x_n \\ x_n & \text{if } x^2_n = 0. \end{cases}$$

Clearly $\{y_n\}$ is a basis for A satisfying the condition in (a). Moreover, $y^2_n = y_n$ if $x^2_n \neq 0$ and $y^2_n = 0$ if $x^2_n = 0$. Thus we have that $N = \{y_i \in \{y_n\}: y^2_i = 0\}$. Now, it is clear that if $x \in A$, and if x is of the form $\Sigma^{\infty}_{i=1}\alpha_i y_i$ where $\alpha_i = 0$ whenever $y_i \notin N$, then $x \in I = \overline{[N]}$. This is so because x is the limit of the partial sums $S_n(x) = \Sigma^{n}_{i=1}\alpha_i y_i$ and $S_n(x) \in [N]$ for each n. Now, suppose that $x \notin I$. Then there exists $k \in \mathbb{N}$ for which $y_k \notin N$ and such that $x^*_k(x) \neq 0$. Now it is clear from the properties of multiplication in A (see remarks before Theorem 1.10) that x^*_k is a multiplicative linear functional on A. Since $x_k^*(x) \neq 0$, it follows that $x \notin$ Rad(A). So, we have shown:

If A is a topological algebra with a basis $\{x_n\}$ satisfying $x_n x_m = 0$ for $n \neq m$, $n,m=1,2,\ldots$ Then $Rad(A) \subseteq I$.

Now, it is clear that every element in [N] has the property that its square is zero. This shows $[N] \subseteq$ Rad(A). Hence, if Rad(A) is closed (e.g., if A is locally m-convex and complete [30]) then it follows that $\overline{[N]} = I \subseteq$ Rad(A). Alternatively, suppose that the basis $\{x_n\}$ is a Schauder basis. Then each element $x \in \overline{[N]}$ is of the form $x = \Sigma_{i=1}^{\infty}\alpha_i x_i$ where $\alpha_i = 0$ whenever $x_i \notin$ N, $i=1,2,\ldots$ [[For, suppose that $x_i \notin$ N and $\alpha_i \neq 0$. Since $x \in \overline{[N]}$, there is a net $\{x_\lambda\} \subseteq [N]$ such that $x_\lambda \to x$. Now, since x_i^* is continuous, we have that $x_i^*(x_\lambda) \to x_i^*(x)$. But $x_i^*(x_\lambda) = 0$ for all λ. This contradicts $x_i^*(x) = \alpha_i \neq 0$.]] It follows that each $x \in \overline{[N]}$ has the property that $x^2 = 0$ (see remarks before Theorem 1.10). Therefore, $\overline{[N]} = I \subseteq$ Rad(A) in this case also.

We have proved the following theorem:

THEOREM 1.14 *Let A be a topological algebra with a basis $\{x_n\}$ having the property that $x_n x_m = 0$ for $n \neq m$, $n,m=1,2,\ldots$. Then, if Rad(A) is closed or if the basis $\{x_n\}$ is a Schauder basis, then $Rad(A) = I$ (the ideal defined above).* ////

We have already noted that if A is a complete locally m-convex algebra Rad(A) is closed. Also we recall that a basis in a complete metrizeable topological vector space with a translation invariant metric is automatically a Schauder basis. From this we have the following corollary:

COROLLARY 1.15 *If A is a complete locally m-convex algera satisfying (a), or if A is a complete metrizeable algebra (with a translation invariant metric) satisfying (a), then $Rad(A) = I$ (the ideal defined above).* ////

The concepts of semisimplicity and strong semisimplicity are equivalent for complete locally m-convex algebras as we have mentioned before. The same is true for topological algebras with bases satisfying the condition in (a) as the following corollary (to Theorem 1.12) shows:

COROLLARY 1.16 *If A is a topological algebra with a Schauder basis $\{x_n\}$ satisfying $x_n x_m = 0$ for $n \neq m$, $n,m=1,2,\ldots$, then A is semisimple if, and only if, A is strongly semisimple.* ////

In particular, Rad(A) is closed for algebras with Schauder bases satisfying the condition in (a). We have the following restatement of Theorem 1.13:

THEOREM 1.17 *If A is a topological algebra with a Schauder basis $\{x_n\}$ satisfying $x_n x_m = 0$ for $n \neq m$, $n,m=1,2,\ldots$ then $A/Rad(A)$ has an orthogonal basis.* ////

In view of the above theorem and corollaries we will consider only topological algebras with orthogonal bases from now on. Moreover, all the natural examples (see Section 1 of Chapter II) of topological algebras satisfying (a) in fact have orthogonal bases.

An orthogonal basis in a topological algebra is unique in the sense that any two orthogonal bases for A are the same set. More precisely we have the following theorem:

THEOREM 1:18 *If* $\{x_i\}$ *and* $\{y_i\}$ *are orthogonal bases in the topological algebra A, then* $\{x_i\} = \{y_i\}$.

Proof: Let $x_n \in \{x_i\}$. Then, since $\{y_i\}$ is a basis for A, there is a sequence $\{\alpha_i\}$ of scalars such that

$$x_n = \sum_{i=1}^{\infty} \alpha_i y_i.$$

Now, there exists a $y_m \in \{y_i\}$ such that $x_n y_m \neq 0$ (for otherwise $x_n = 0$ which is impossible since no basis can contain 0). It follows that

$$x_n y_m = \Big(\sum_{i=0}^{\infty} \alpha_i y_i\Big) y_m = \alpha_m y_m. \qquad (*)$$

Multiplying both sides of this equation by x_n, we get $x_n y_m = \alpha_m x_n y_m$. This implies that $\alpha_m = 1$. Therefore from (*) we get

$$x_n y_m = y_m. \qquad (**)$$

By a similar argument, if we have $y_m = \Sigma_{i=1}^{\infty} \beta_i x_i$, then

$$x_n y_m = x_n\Big(\sum_{i=1}^{\infty} \beta_i x_i\Big) = \beta_n x_n \qquad (***)$$

Multiplying both sides of this equation by y_m we get $x_n y_m = \beta_n x_n y_m$. This implies that $\beta_n = 1$. Therefore from (***) we get

$$x_n y_m = x_n. \qquad (****)$$

From (**) and (****) we have that $x_n = y_m$. This shows that

$\{x_i\} \subseteq \{y_i\}$. Because of the symmetry of the situation we have the other inclusion also. Thus $\{x_i\} = \{y_i\}$. ////

Recall that the bases $\{x_i\}$ and $\{y_i\}$ of the TVS E are said to be *equivalent* if for a sequence $\{\alpha_i\}$ in $\mathbb{K}$ the convergence of $\Sigma_{i=1}^{\infty}\alpha_i x_i$ implies the convergence of $\Sigma_{i=1}^{\infty}\alpha_i y_i$ and conversely. The bases $\{x_i\}$ and $\{y_i\}$ are called *permutatively equivalent* if there is a permutation π of $\mathbb{N}$ such that $\{x_{\pi(i)}\}$ is a basis for E and such that the bases $\{x_{\pi(i)}\}$ and $\{y_i\}$ are equivalent. With these definitions we have the following corollary to Theorem 1.18:

COROLLARY 1.19 *If A is a topological algebra then any two orthogonal bases of A are permutatively equivalent.* ////

Chapter II

ORTHOGONAL BASES

Let A be a topological algebra and suppose that A has an orthogonal basis. We have shown in Chapter I (Theorem 1.18) that an orthogonal basis for a topological algebra is unique (up to a permutation), so, it is to be expected that the existence of such a basis to a large extent determines the structure of the algebra. In this chapter we study how the orthogonal basis describes the closed ideals of A. In fact, we show that a closed ideal in A is completely determined by the basis elements it contains. In particular, closed maximal ideals are determined in this way and we prove that every closed ideal of A is the intersection of closed maximal ideals. We also show that the maximal ideal space of A is homeomorphic to a countable discrete space.

If, in addition, A is locally *m*-convex and has an identity, then using the above characterization of the closed ideals of A we show that A is necessarily metrizeable. Moreover, if A is complete, then A is algebraically and topologically isomorphic to the *F*-algebra *S* of sequences.

Finally, we mention that some theorems in Section 2 generalize results in [33] about ideals in certain algebras of analytic functions. Section 3 extends results in [22] concerning the relationship between the existence of an identity and the seminorms defining the topology in *F*-algebras with orthogonal bases.

1. Examples and Permanence Properties

In this section we give several examples of topological algebras having orthogonal bases. Also, we give a general method of constructing a Banach algebra with an orthogonal basis from a given Banach space with an unconditional basis. If A and B are topological algebras (resp., Banach algebras) with orthogonal bases, then we prove that $A \times B$ (resp., $A \otimes B$ with the projective tensor norm) is a topological algebra (resp., Banach algebra) with an orthogonal basis.

EXAMPLE 2.1 The Banach space $\ell^1(\mathbb{N})$ is a Banach algebra under pointwise multiplication [40]. The sequence $\{e_n = (\delta_{nm})_{m=1}^{\infty} : n=1,2,\ldots\}$ is a basis for $\ell^1(\mathbb{N})$. Clearly $e_n e_m = \delta_{nm} e_n$ and thus this sequence is an orthogonal basis for the Banach algebra $\ell^1(\mathbb{N})$. This basis is an absolute basis.

EXAMPLE 2.2 For $1 \leq p < \infty$, the spaces $\ell^p(\mathbb{N})$ are Banach algebras under pointwise multiplication [40]. The sequence $\{e_n\}$ defined above is an unconditional orthogonal basis for $\ell^p(\mathbb{N})$.

EXAMPLE 2.3 The Banach algebra c_0 of sequences converging to 0 has $\{e_n\}$ as an unconditional basis [37]. This ~~is~~ clearly an orthogonal basis in our sense.

All the bases in the above examples are unconditional. In general we show below that any Banach space with an unconditional basis is a Banach algebra under "pointwise multiplication" according to the following definition:

DEFINITION 2.4 Let E be a Banach space with an unconditional basis $\{x_i\}$. If x and y are elements of E, $x = \Sigma_{i=1}^{\infty}\alpha_i x_i$, $y = \Sigma_{i=1}^{\infty}\beta_i x_i$, then define a product $*$ in E by

$$x*y = \sum_{i=1}^{\infty} \alpha_i \beta_i x_i \ .$$

This definition makes sense, because, without loss of generality, assume that the basis $\{x_i\}$ is *normalized* i.e., $\|x_i\|=1,\ i=1,2,\ldots$ Then, the convergence of $\Sigma_{i=1}^{\infty}\alpha_i x_i$ implies $\lim_{i\to\infty} \alpha_i = 0$ [37]. Thus the sequence $\{\alpha_i\}$ is bounded. Therefore, since $\Sigma_{i=1}^{\infty}\beta_i x_i$ converges unconditionally, it follows that $\Sigma_{i=1}^{\infty}\alpha_i\beta_i x_i$ converges in E. This shows that $x*y$ is a well-defined element of E. More is true:

THEOREM 2.5 *If E is a Banach space with an unconditional basis, then E is a Banach algebra with $*$ multiplication. Moreover, the basis in E is an orthogonal basis in the Banach algebra E.*

Proof: Without loss of generality assume that the basis is normalized and consider the following norm on E:

$$\|x\|_0 = \sup_{n\in\mathbb{N}} |x_n^*(x)|.$$

We will show that this norm is weaker than the original norm on E. To this end, consider the map

$$\sigma : E \to m,$$

where m is the Banach space of bounded sequences (with the supremum norm $\|..\|_\infty$) and where σ is defined by $\sigma(x) = (x_n^*(x))_{n=1}^{\infty}$. Let f_k be the k^{th}

coordinate functional on m. Then we have

$$(f_k \circ \sigma)(x) = f_k(\sigma(x)) = f_k[(x_n^*(x))_{n=1}^{\infty}] = x_k^*(x). \tag{*}$$

Since E is a Banach space, the functionals $x_k^*(x)$, $k=1,2,\ldots$ are continuous. Hence $f_k \circ \sigma$ is continuous for each $k \in \mathbb{N}$. Now, since the family $\{f_k\}$ is a separating family of functionals on m, it follows by Theorem 1.1 that σ is continuous; i.e., there exists $C_1 > 0$ such that

$$\|\sigma(x)\|_\infty \leq C_1 \|x\|, \qquad x \in \text{E}.$$

Since $\|x\|_0 = \|\sigma(x)\|_\infty$, we have that

$$\|x\|_0 \leq C_1 \|x\|, \qquad x \in \text{E}.$$

Now, consider the following norm on E,

$$|||x||| = sup \{ \sum_{n=0}^{\infty} |x_n^*(x)| |f(x_n)| : f \in \text{E}', \|f\| \leq 1 \}.$$

It can be shown [37, p.463] that $|||\,..\,|||$ is a norm on E equivalent to the original norm of E. Hence, there is a constant $C_2 > 0$ such that

$$\|x\| \leq C_2 |||x|||, \qquad x \in \text{E}.$$

We have

$$|||x*y||| = \sup_{\|f\| \leq 1, f \in \text{E}'} \sum_{n=0}^{\infty} |x_n^*(x)\ x_n^*(y)| |f(x_n)|$$

$$\leq \|x\|_0 \sup_{\|f\| \leq 1, f \in \text{E}'} \sum_{n=0}^{\infty} |x_n^*(y)| |f(x_n)|,$$

$$\leq C_1 \|x\|\ |||y||| \leq C_1 C_2 |||x|||\ |||y|||.$$

This shows that E is a Banach algebra with $\star$ multiplication in the norm $|||\cdot|||$ (see [41]) and hence is a Banach algebra in any equivalent norm.

It is clear that the basis $\{x_n\}$ has the property $x_n \star x_m = \delta_{nm} x_n$, for $n,m=1,2,\ldots$. ////

Note here that if, in addition, E' is separable, then $\{x_n^*\}$ is an unconditional basis for E' [28]. Applying the above theorem to E' we get that in this case E' is also a Banach algebra with an orthogonal basis in the multiplication $\star$ (where $\star$ is suitably defined).

To illustrate this theorem, consider the Banach spaces $L^p[0,2\pi]$ for $1 \leq p < \infty$. These spaces have the Haar functions as unconditional bases [37]. It follows that with the above product $\star$, these spaces are Banach algebras having orthogonal bases.

The product $\star$ also translates into a meaningful product in many examples of Banach spaces with unconditional bases:

EXAMPLE 2.6 For $1 < p < \infty$, let $L^p(\mathbb{T})$ be the usual L^p space on the circle group $\mathbb{T}$. It is known that the sequence of trigonometric polynomials $\{e_n : n=0,\pm1,\pm2,\ldots\}$, where $e_n(t) = t^n$ for $t \in \mathbb{T}$, form an unconditional basis for $L^p(\mathbb{T})$ [28]. Also, by a theorem of Zelazko [41], $L^p(\mathbb{T})$ is a Banach algebra with convolution multiplication. It is easy to see that with this multiplication the basis $\{e_n\}$ has the property $e_n \sigma e_m = \delta_{nm} e_n$. Thus $L^p(\mathbb{T})$ is a Banach algebra with an orthogonal basis. Moreover, note that convolution multiplication in $L^p(\mathbb{T})$ corresponds to the $\star$ product of Theorem 2.5.

The following closed subalgebras of $L^p(\mathbb{T})$ also have unconditional orthogonal bases:

Example 2.7 For $1 < p < \infty$, let $H^p(D)$ be the Hardy space on the open unit disc D. The sequence of functions $\{e_n : n=0,1,\dots\}$, where $e_n(x) = x^n$ for $x \in D$, is an unconditional basis for H^p. Also, it is easy to show [33, p.88] that H^p is a Banach algebra with the convolution product

$$f{*}g(x) = (2\pi i)^{-1} \int_{|z|=r} f(z)g(xz^{-1})z^{-1}dz \qquad (1)$$

where f and g are in H^p and $|x| < r < 1$. Now, since f and g are analytic on D, we have $f(x) = \Sigma_{n=0}^{\infty} a_n x^n$ and $g(x) = \Sigma_{n=0}^{\infty} b_n x^n$. A simple computation shows (see [33]) that with the convolution product (1),

$$f{*}g(x) = \sum_{n=0}^{\infty} a_n b_n x^n.$$

This is in agreement with the product $\star$ of Theorem 2.5.

Finally, we give two examples which play a special role in what follows.

Example 2.8 Let H(D) be the F-space of all functions holomorphic on the open unit disc D with the compact-open topology. H(D) is a topological algebra with the convolution product

$$f{*}g(\lambda) = (2\pi i)^{-1} \int_{\gamma} f(\zeta)g(\zeta^{-1}\lambda)\zeta^{-1}d\zeta$$

where $\lambda \in D$ and γ is a suitable closed curve in D enclosing 0 and λ. The sequence of functions $\{z_n : n=0,1,\dots\}$, where $z_n(\zeta) = \zeta^n$ for $\zeta \in D$, is a basis for H(D). If $f, g \in H(D)$, $f(\zeta) = \Sigma_{n=0}^{\infty} a_n \zeta^n$, $g(\zeta) = \Sigma_{n=0}^{\infty} b_n \zeta^n$, (i.e., $f = \Sigma_{n=0}^{\infty} a_n z_n$ and $g = \Sigma_{n=0}^{\infty} b_n z_n$ where these series converge in topology of H(D)), then

$$f\star g(\lambda) = (2\pi i)^{-1} \int_\gamma \Big(\sum_{n=0}^{\infty} a_n\zeta^n\Big)\Big(\sum_{n=0}^{\infty} b_n\zeta^{-n}\lambda^n\Big)\zeta^{-1}d\zeta$$

$$= (2\pi i)^{-1} \sum_{n=0}^{\infty} \int_\gamma \Big(\sum_{k=0}^{n} a_{n-k}b_k\zeta^{n-2k-1}\lambda^k\Big)d\zeta$$

$$= \sum_{n=0}^{\infty} a_n b_n \lambda^n$$

This shows that $\{z_n\}$ is an orthogonal basis for H(D).

We note that H(D) is not locally m-convex with this product. Also note that the function $e(\zeta) = (1-\zeta)^{-1}$ ($\zeta \in$ D) is an identity for H(D).

EXAMPLE 2.9 The algebra S of all sequences of complex numbers is an F-algebra in the topology of simple convergence in the coefficients. The sequence $\{e_n\}$ of Example 2.1 is an absolute orthogonal basis for S.

Other algebras with orthogonal bases can be constructed ~~from the~~ following two theorems.

THEOREM 2.10 *If A and B are topological algebras with orthogonal bases $\{x_n\}$ and $\{y_n\}$, respectively, then the basis $\{z_n\}$ (defined in Theorem 1.3) is an orthogonal basis for $A \times B$.*

Moreover, if A and B are LC-algebras (resp., complete LC-algebras) and if the bases $\{x_n\}$ and $\{y_n\}$ are absolute (resp., unconditional), then the basis $\{z_n\}$ is an absolute (resp., unconditional) orthogonal basis for $A \times B$.

Proof: It is clear that the basis $\{z_n\}$ defined in Theorem 1.3 has the property that $z_n^2 = z_n$, $n=1,2,\ldots$ Also, one can easily check (by considering the four cases depending of whether m and n are even or odd) that $z_m z_n = 0$ for $m \neq n$. Thus the basis $\{z_n\}$ is an orthogonal basis in $A \times B$. The remainder of the theorem follows directly from Theorem 1.3. ////

For the case of Banach algebras we have the following result.

THEOREM 2.11 *Let A and B be Banach algebras with orthogonal bases. Then,* $A \otimes_p B$, *the projective tensor product of A and B, is a Banach algebra with an orthogonal basis.*

Proof: Let $\{x_n\}$ and $\{y_n\}$ be the bases in A and B, respectively. It is known (see [15]) that the system of all products

$$\{x_n \otimes y_m : n,m \in \mathbb{N}\}$$

arranged in a certain way (which is not important for our present purposes) is a basis for the Banach space $A \otimes_p B$. Now, $A \otimes_p B$ is also a Banach algebra [6]. The product in $A \otimes B$ has the property that $(a \otimes b)(c \otimes d) = ac \otimes bd$, hence, $(x_n \otimes y_m)(x_n \otimes y_m) = x_n^2 \otimes y_m^2 = x_n \otimes y_m$. Also, we have $(x_n \otimes y_m)(x_j \otimes y_i) = x_n x_j \otimes y_m y_i$. Thus, if $n \neq j$ or if $m \neq i$, then this last product is 0. This shows that this basis is an orthogonal basis for the Banach algebra $A \otimes_p B$. ////

Theorem 2.10 shows that the class of topological algebras with orthogonal bases is closed under the formation of finite products. The above theorem shows that the class of Banach algebras having orthogonal bases is also closed under the formation of projective tensor products. We will show in the next section that the class of topological algebras with orthogonal Schauder bases is closed with respect to the formation of quotients by closed ideals. We will first need to study the structure of these ideals.

2. Closed Ideals

In this section we study the closed ideals of a topological algebra A with an orthogonal Schauder basis. It turns out that every closed ideal in A is the intersection of closed maximal ideals. Also, we will show that a closed ideal in A is precisely the closure of the linear span of the basis elements it contains. We use this fact to show that the quotient algebra of A by any of its closed ideals is a topological algebra with an orthogonal basis. These results will be used in the next section to prove a metrizeability result for locally m-convex algebras with orthogonal bases. We will also show in this section that the maximal ideal space M(A) of A is homeomorphic to a countable discrete space.

Unless otherwise stated, we assume A to be a topological algebra with an orthogonal Schauder basis $\{x_n\}$. We begin by proving two fundamental lemmas.

LEMMA 2.12 *Let A be a topological algebra with an orthogonal basis* $\{x_n\}$. *If* $x, y \in A$, $x = \Sigma_{n=1}^{\infty}\alpha_n x_n$, $y = \Sigma_{n=1}^{\infty}\beta_n x_n$, *then* $xy = \Sigma_{n=1}^{\infty}\alpha_n\beta_n x_n$. *In particular, A is commutative.*

Proof: Using the continuity of the multiplication in A we have

$$xy = x\Big(\sum_{i=1}^{\infty} \beta_i x_i\Big) = \sum_{i=1}^{\infty} \beta_i x x_i$$

$$= \sum_{i=1}^{\infty} \beta_i \Big(\sum_{j=1}^{\infty} \alpha_j x_j\Big) x_i = \sum_{i=1}^{\infty} \beta_i \alpha_i x_i .$$

////

LEMMA 2.13 *Let A be a topological algebra with an orthogonal basis* $\{x_n\}$. *If A has an identity e, then*

(a) $e = \Sigma_{n=1}^{\infty} x_i$

(b) an element $x = \Sigma_{n=1}^{\infty}\alpha_n x_n$ *of A is invertible if, and only if,* $\alpha_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $\Sigma_{n=1}^{\infty}\alpha_n^{-1}x_n$ *converges in A. In this case* $x^{-1} = \Sigma_{n=1}^{\infty}\alpha_n^{-1}x_n$.

Proof: (a) Since $\{x_n\}$ is a basis for A, then $e = \Sigma_{n=1}^{\infty}\alpha_n x_n$. Let x_n be any basis element, then

$$x_n = x_n e = x_n\Big(\sum_{m=1}^{\infty} \alpha_m x_m\Big) = \alpha_n x_n^2 = \alpha_n x_n .$$

This implies that $\alpha_n = 1$. Since x_n was any basis element, it follows that $\alpha_n = 1$, $n=1,2,\ldots$.

(b) If $x^{-1} = \Sigma_{n=1}^{\infty}\beta_n x_n$, then from ~~Lemma~~ 2,12 and part (a)

$$e = xx^{-1} = \sum_{n=1} \alpha_n \beta_n x_n = \sum_{n=1}^{\infty} x_n .$$

Since the coefficients are uniquely determined, it follows that $\alpha_n \neq 0$ and $\alpha_n\beta_n = 1$ (i.e., $\beta_n = \alpha_n^{-1}$)

Conversely, if $\Sigma_{n=1}^{\infty}\alpha_n^{-1}x_n$ converges in A, then clearly (from

Lemma 2.12) we have

$$\Big(\sum_{n=1}^{\infty}\alpha_n x_n\Big)\Big(\sum_{n=1}^{\infty}\alpha_n^{-1}x_n\Big) = \sum_{n=1}^{\infty}x_n = e.$$

////

We now describe the maximal ideal space of the algebra A.

THEOREM 2.14 *Let A be a topological algebra with an orthogonal basis* $\{x_n\}$. *If* $f \in M(A)$, *then* $f = x_n^*$ *for some* $n \in \mathbb{N}$. *If* $\{x_n\}$ *is a Schauder basis, then* $M(A) = \{x_n^* : n \in \mathbb{N}\}$ *and* $M(A)$ *is homeomorphic to a countable discrete space.*

Proof: Let $f \in M(A)$. Since $f \neq 0$, there exists an $x_n \in \{x_i\}$ such that $f(x_n) \neq 0$. Now, if $x_m \in \{x_i\}$ and $x_m \neq x_n$, then $0 = f(x_m x_n) = f(x_m)f(x_n)$. But $f(x_n) \neq 0$. Therefore $f(x_m) = 0$. Thus, $f(x_m) = 0$ for all $m \neq n$. So, let $x \in A$, then

$$f(x) = f\Big(\sum_{n=1}^{\infty}x_n^*(x)x_n\Big) = \sum_{n=1}^{\infty}x_n^*(x)f(x_n)$$

$$= x_n^*(x)f(x_n) = x_n^*(x).$$

The last equality being true by virtue of the fact that $f(x_n) = 1$ [[since $f(x_n) = f(x_n^2) = f(x_n)^2$ and $f(x_n) \neq 0$]].

From Lemma 2.12, it is clear that for each $n \in \mathbb{N}$, $x_n^*(xy) = x_n^*(x)x_n^*(y)$ for all $x,y \in A$. If the basis is a Schauder basis, then each x_n^* is continuous. Therefore, in this case $x_n^* \in M(A)$. This, combined with the first statement of the theorm gives that $M(A) = \{x_n^* : n \in \mathbb{N}\}$.

Finally, to show that M(A) is countable and discrete, consider the subbasic neighbourhood V of x_n^*

$$V = V(1/2, x_n, x_n^*) = \{x_k^* \in M(A) : |x_k^*(x_n) - x_n^*(x_n)| < 1/2\} = \{x_n^*\}.$$

This completes the proof. ////

We note that this theorem implies that an (infinite dimensional) Banach algebra (or, more generally a Q-algebra) with identity cannot have an orthogonal Schauder basis because the maximal ideal space of such algebras is compact.

COROLLARY 2.15 *A topological algebra with an orthogonal Schauder basis is semisimple.* ////

Husain and Liang [22] have shown that an F-algebra with an unconditional orthogonal basis is functionally continuous. According to a theorem of Michael [30] a commutative semisimple functionally continuous F-algebra has unique F-algebra topology. Combining these results with Corollary 2.15 and Lemma 2.12 (i.e., A is semisimple and commutative) we have: *an F-algebra with an unconditional orthogonal basis has unique F-algebra topology.*

DEFINITION 2.16 If A is a topological algebra with an orthogonal Schauder basis, then we set $M_k = \{x \in A : x_k^*(x) = 0\}$.

Clearly M_k is a maximal ideal since $M_k = ker(x_k^*)$. Also M_k is closed since x_k^* is continuous. The converse is also true (recall here that in this section A denotes a topological algebra with an orthogonal Schauder basis $\{x_n\}$):

THEOREM 2.17 *M is a closed maximal ideal of A if, and only if, $M = M_k$ for some $k \in \mathbb{N}$.*

Proof: Suppose that M is a closed maximal ideal of A and $M \neq M_k$ $k=1,2,\ldots$. Then, for each $k \in \mathbb{N}$, there is an $x \in M$ with $x_k^*(x) \neq 0$. Now, $x_k^*(x)^{-1} x_k x = x_k$, and so $x_k \in M$. Since this is true for each $k \in \mathbb{N}$, it follows that $\{x_n\} \subseteq M$. Since the polynomials in $\{x_n\}$ are dense in A and since M is closed, it follows that $M = A$. This contradicts the assumption that M is a maximal ideal.

Conversely, each M_k is a closed maximal ideal since it is the kernel of the continuous multiplicative linear functional x_k^*. ////

THEOREM 2.18 *If I is a closed ideal of A, then $I \subseteq M_k$ for some $k \in M$.*

Proof: Suppose that $I \not\subset M_k$ for $k=1,2,\ldots$ Then, just as in the above proof, I contains each basis element x_k and hence, being an ideal, contains the linear span of $\{x_n\}$. Thus, I is dense in A, contradicting the assumption that I is a closed ideal. ////

COROLLARY 2.19 *The closure of an ideal of A is an ideal in A if, and only if, it is contained in some M_k $(k \in \mathbb{N})$.* ////

We recall that an ideal I is called *prime* if either x or y belong to I whenever xy belongs to I.

THEOREM 2.20 *A closed prime ideal of A is a maximal ideal.*

Proof: Since I is closed, it follows by Theorem 2.18 that $I \subseteq M_k$

for some $k \in \mathbb{N}$. Let y be any element of M_k. Then $yx_k = 0$ and thus $yx_k \in M_k$. Now, since $I \subseteq M_k$, $x_k \notin I$. Since I is prime, it follows that $y \in I$. This shows that $M_k \subseteq I$. Thus, $I = M_k$. The result now follows from Theorem 2.17. ////

A closed ideal in A is completely determined by the basis elements it contains as the following theorem shows:

THEOREM 2.29 *Let I be a closed ideal in A. If we set*

$$D = \{x_k \in \{x_n\} : x_k \in I\}$$

then $I = \overline{[D]}$ *(i.e., I is the closure of the linear span of the basis elements it contains).*

Proof: Since I is an ideal, any finite linear combination of elements of D belongs to I. Hence $[D] \subseteq I$. Since I is closed, it follows from this that $\overline{[D]} \subseteq I$.

Now, suppose that $x \in I$. Then, $x = \Sigma_{n=0}^{\infty}\alpha_n x_n$ where $\alpha_n = 0$ whenever $x_n \notin D$ ⟦For, if $\alpha_n \neq 0$ and $x_n \notin D$, then, since clearly $x_n = (\alpha_n^{-1}x_n)x$ and since I is an ideal, it follows that $x_n \in I$. But this is a contradiction since $x_n \notin D$⟧. Also, for each $k \in \mathbb{N}$, $S_k(x) = \Sigma_{n=0}^{k}\alpha_n x_n \in I$ because $S_k(x) = x(\Sigma_{n=0}^{k}x_n)$. It follows that for every $x \in I$, $S_k(x) \in [D]$ (for $k \in \mathbb{N}$). Now, since $x = \lim_{k\to\infty} S_k(x)$, we have that $x \in \overline{[D]}$, and this shows that $I \subseteq \overline{[D]}$. ////

Note that if D is any subset of $\{x_n\}$ then $x = \Sigma_{n=0}^{\infty}\alpha_n x_n$ belongs to $\overline{[D]}$ if, and only if, $\alpha_n = 0$ whenever $x_n \notin D$ ⟦For, suppose that

for some $n \in \mathbb{N}$, $\alpha_n \neq 0$ and $x_n \notin D$. Then, since $x \in \overline{[D]}$, it follows that there is a net $\{x_\lambda : \lambda \in \Lambda\} \subseteq [D]$ such that $x_\lambda \to x$. This, however, is impossible since $x_n^*(x_\lambda) = 0$ for all $\lambda \in \Lambda$ and $x_n^*(x) = \alpha_n \neq 0$. Conversely, if x has this property, then clearly $S_k(x) \in [D]$ for all $k \in \mathbb{N}$. It follows from the fact that $x = \lim_{k\to\infty} S_k(x)$ that $x \in \overline{[D]}$]].
It is clear from this and from the definition of M_k that in this case

$$\overline{[D]} = \bigcap\{M_k : x_k \in D\}$$

So, from theorem 2.21 we have that for a closed ideal I of A, $I = \bigcap_{x_k \notin I} M_k$.
Thus we have the following corollary:

COROLLARY 2.22 *Every closed ideal of A is the intersection of the closed maximal ideals containing it.* ////

In Section 1 we considered when products and tensor products of topological algebras with orthogonal bases have orthogonal bases. We now use the above characterization (Theorem 2.21) of the closed ideals of A to show that the quotient algebra of A by any of its closed ideals is a topological algebra with an orthogonal basis.

THEOREM 2.23 *If A is a topological algebra with an (unconditional) orthogonal Schauder basis $\{x_n\}$ and if I is a closed ideal in A, then A/I has an (unconditional) orthogonal Schauder basis.*

Moreover, if A is an LC-algebra and if the basis $\{x_n\}$ is absolute, then A/I has an absolute orthogonal Schauder basis.

Proof: Let $\eta : A \to A/I$ be the canonical map. Since I is a closed ideal we have by Theorem 2.21 that $I = \overline{[\{x_{i_n}\}]}$ where $\{x_{i_n}\}$ is the sequence of basis elements contained in I. Let $\{x_{j_n}\}$ be the sequence of basis elements complementary to $\{x_{i_n}\}$. Then, by Theorem 1.4, the sequence $\{\tilde{x}_n = \eta(x_{j_n}) : n \in \mathbb{N}\}$ is a basis for A/I.

Now, for each $n \in \mathbb{N}$, we have (with the notation of Theorem 1.4) $\tilde{x}_n\tilde{x}_n = \eta(x_{j_n})\eta(x_{j_n}) = \eta(x_{j_n}^2) = \eta(x_{j_n}) = \tilde{x}_n$. Also, for $n \neq m$, we have $\tilde{x}_n\tilde{x}_m = \eta(x_{j_n})\eta(x_{j_m}) = \eta(x_{j_n}x_{j_m}) = 0$. This shows that the basis $\{\tilde{x}_n\}$ is an orthogonal basis.

The rest of the theorem now follows directly from Theorem 1.4.

////

To complete this discussion of ideals in A, we consider the principal ideals. Recall that a *principal ideal* is an ideal which is generated by a single element. We will denote the principal ideal generated by the element x by $<x>$.

LEMMA 2.24 *Let* $x \in A$ *and let* $D = \{x_k \in \{x_n\} : x_k^*(x) \neq 0\}$. *Then,* $\overline{<x>} = \overline{[D]}$.

Proof: Clearly the ideal $<x>$ contains only those basis elements which are in D. It follows that $\overline{<x>}$ also contains those same basis elements only. For, suppose that $x_{k_0} \notin D$ and $x_{k_0} \in \overline{<x>}$, then there is a net $\{y_\lambda\} \subseteq <x>$ with $y_\lambda \to x_{k_0}$. But this is impossible since then $x_{k_0}^*(y_\lambda) \to x_{k_0}^*(x_{k_0}) = 1$ and $x_{k_0}^*(y_\lambda) = 0$ for all λ.

Thus, $\overline{<x>}$ is a closed ideal containing only the basis elements in D. Therefore, the result follows by theorem 2.21. ////

THEOREM 2.25 *Let $x \in A$. Then, $y \in \overline{\langle x\rangle}$ if, and only if, $x_n^*(y) = 0$ whenever $x_n^*(x) = 0$ $(n \in \mathbb{N})$.*

Proof: ($\Longrightarrow$) Suppose that $y \in \overline{\langle x\rangle}$, then by the above lemma, $y \in \overline{[D]}$ where $D = \{x_k \in \{x_n\} : x_k^*(x) \neq 0\}$. Thus, $x_k^*(y) = 0$ whenever $x_k^*(x) = 0$ (see remarks following Theorem 2.21).

($\Longleftarrow$) Conversely, suppose that $x_k^*(y) = 0$ whenever $x_k^*(x) = 0$. Let $C = \{x_k \in \{x_n\} : x_k^*(x) \neq 0\}$ and let $D = \{x_k \in \{x_n\} : x_k^*(y) \neq 0\}$. It follows that $D \subseteq C$, hence $\overline{[D]} \subseteq \overline{[C]}$. But $y \in \overline{[D]}$ and $\overline{[C]} = \overline{\langle x\rangle}$ (Theorem 2.21). Therefore, $y \in \overline{\langle x\rangle}$. ////

COROLLARY 2.26 *If $x \in A$, then $\overline{\langle x\rangle} = \bigcap\{M_k : x_k^*(x) = 0\}$.*

Proof: This follows immediately from the above theorem since it is clear from the definition of M_k that $y \in \cap\{M_k : x_k^*(x) = 0\}$ if, and only if, $x_k^*(y) = 0$ whenever $x_k^*(x) = 0$ $(k \in \mathbb{N})$. One can also see this from the fact that $\overline{\langle x\rangle} = \overline{[D]}$ where $D = \{x_k \in \{x_n\} : x_k^*(x) \neq 0\}$ (Lemma 2.24) and from the fact that $\overline{[D]} = \cap\{M_k : x_k^*(x) = 0\}$ (see remarks following Theorem 2.21) ////

Finally; we consider principal ideals in *LC*-algebras with identity having unconditional orthogonal bases. Algebras with these properties will be the subject matter of Chapter III. Now we will show the following:

THEOREM 2.27 *Let A be a complete LC-algebra with identity e, and suppose that A has an unconditional orthogonal Schauder basis. Then, every closed ideal of A is a principal ideal.*

Proof: By Corollary 2.26, every closed ideal of A is the intersection of closed maximal ideals. Since each closed maximal ideal of A is M_k for some $k \in \mathbb{N}$ (Theorem 2.17), it is sufficient to show that every ideal of A which is the intersection of ideals of the form M_k ($k \in \mathbb{N}$) is a principal ideal. To this end let K be any subset of $\mathbb{N}$ and let $I = \cap \{M_k : k \in K\}$. Let $K^c = \mathbb{N} - K$ be the complement of K in $\mathbb{N}$. Now, define y by

$$y = \sum_{n=1}^{\infty} \chi_{K^c}(n) x_n$$

where χ_{K^c} is the characteristic function on K^c. Since $e = \Sigma_{n=1}^{\infty} x_n$ belongs to A (Lemma 2.15) and since the basis is unconditional (hence every series of the form $\Sigma_{n=1}^{\infty} \alpha_n x_n$ which is convergent is subsereis convergent because A is a complete locally convex space) it follows that $y \in A$. Now, clearly $y \in M_k$ for all $k \in K$, and so $y \in I$. Also, suppose $w \in I$, then $x_k^*(w) = 0$ for all $k \in K$. Hence $w = yw$ and therefore y generates the ideal I; i.e., $I = \langle y \rangle$. ////

In Chapter III we will give a necessary and sufficient condition for a principal ideal (in a topological algebra A satisfying the conditions in the hypothesis of Theorem 2.27) to be closed in terms of the concept of E-regularity to be defined there.

3. Locally M-Convex Algebras

In this section we consider complete locally m-convex algebras with orthogonal bases. It turns out that any two such algebras with identity are isomorphic. We also show that the topology of a locally

m-convex algebra with identity having an orthogonal basis must necessarily be metrizeable. Hence, if such an algebra is complete, it is an F-algebra, and, in fact, is isomorphic to the F-algebra S of sequences (see Theorem 2.34). In proving this theorem we will consider some relationships (for a locally m-convex topological algebra A having an orthogonal basis) between the existence of an identity for A, the space of coefficients associated to the orthogonal basis of A, and the seminorms defining the topology of A. Finally, we note that the results of this section extend certain results of Husain and Liang in [22].

We begin by proving a theorem on the continuity of the coefficient functional associated to the orthogonal basis of A. We recall here that the coefficient functionals associated with a basis in a complete metrizeable TVS are all continuous (see Chapter I and see Husain [26] for other classes of TVS's having this property). We will now show that orthogonal bases in locally m-convex algebras have this property:

THEOREM 2.28 *If A is a locally m-convex algebra, then any orthogonal basis for A is a Schauder basis.*

Proof: Let $\{x_n\}$ be an orthogonal basis in A and let $P = \{p_\alpha\}$ be a family of submultiplicative seminorms generating the topology of A. We need to show that the coefficient functionals associated with the basis $\{x_n\}$ are continuous. To this end let $x \in$ A. Then, for each $n \in \mathbb{N}$, $xx_n = x_n^*(x)x_n^2 = x_n^*(x)x_n$. It follows from this that for any submultiplicative seminorm $p_\alpha \in P$, and for each $n \in \mathbb{N}$,

$$|x_n^*(x)|p_\alpha(x_n) \leq p_\alpha(x)p_\alpha(x_n) \qquad (*)$$

Now, since A is Hausdorff, there exists $p_\beta \in P$ such that $p_\beta(x_n) \neq 0$. With this p_β, we get from (*),

$$|x_n^*(x)| \leq p_\beta(x)$$

for all $x \in A$. This shows that $x_n^*(x)$ is continuous for each $n \in \mathbb{N}$.

////

Note that if f is any multiplicative linear functional on A with $f(x_n) \neq 0$ for some $n \in \mathbb{N}$, then by the argument in the proof of Theorem 2.14 we get that $f = x_n^*$. Hence, by the above theorem f is continuous. This shows: *If* $f \in M^{\#}(A)$ *and if* $f(x_n) \neq 0$ *for some* $n \in \mathbb{N}$, *Then* $f \in M(A)$.

We now consider some relationships between the existence of an identity in A, the space of coefficients associated with the orthogonal basis of A, and the seminorms defining the topology of A. For this, we make the following definition:

DEFINITION 2.29 Let E be a topological vector space with a basis $\{x_n\}$, and let S be the algebra of all sequences of complex numbers. Define,

$$\sigma : E \longrightarrow S$$
$$x \rightsquigarrow (x_n^*(x))_{n=1}^{\infty}$$

i.e., $\sigma(x) = (x_n^*(x))_{n=1}^{\infty}$. σ is the *coefficient map associated with* $\{x_n\}$.

LEMMA 2.30 *Let E be a locally convex space with a basis $\{x_n\}$, and let P be a family of seminorms generating the topology of E.*

(a) If σ is onto S, then for every seminorm $p \in P$ there exists an $N > 0$ such that $p(x_n) = 0$ for $n > N$.

(b) If E is also complete, then the converse of (a) is also true.

Proof: (a) Suppose to the contrary that there exists $p \in P$ such that there exists arbitrarily large $n \in \mathbb{N}$ for which $p(x_n) \neq 0$. Define a sequence $\{\beta_n\}$ by

$$\beta_n = \begin{cases} p(x_n)^{-1} & \text{if } p(x_n) \neq 0 \\ 0 & \text{if } p(x_n) = 0, \end{cases}$$

$n=1,2,\ldots$. Since σ is onto S by assumption, it follows that $\Sigma_{n=1}^{\infty}\beta_n x_n$ converges in E to x, say; i.e., $S_n(x) \to x$ in E. This means that for every $q \in P$, $q(S_n(x)-x) \to 0$ as $n \to 0$. In particular, the sequence $\{S_n(x)\}$ is Cauchy in the topology generated on E by p. So, for every $\varepsilon > 0$, there exists $K > 0$ such that $n,m > K$ implies

$$p(S_n(x)-S_m(x)) < \varepsilon$$

In particular, for $n > K$, we have

$$p(S_n(x)-S_{n+1}(x)) < \varepsilon$$

Equivalently, $p(\beta_n x_n) < \varepsilon$ for all $n > K$. But there exists arbitrarily large $m \in \mathbb{N}$ for which $p(x_m) \neq 0$. So, choose such an $m > K$. Now, from the definition of β_m we have that $p(\beta_m x_m) = p(x_m)^{-1}p(x_m) = 1$. This, however, contradicts $p(\beta_m x_m) < \varepsilon$ (since ε was arbitrary).

(b) Suppose that E is complete and that each $p \in P$ has the property that $p(x_n) = 0$ for all sufficiently large natural numbers n. Let $\{\alpha_n\}$ be any sequence of complex numbers and set

$$y_m = \sum_{n=0}^{m} \alpha_n x_n.$$

Let $p \in P$. Then, for a given $\varepsilon > 0$, and for m,n sufficiently large, $(n > m)$, we have

$$p(y_m - y_n) = p\Big(\sum_{k=n}^{m} \alpha_k x_k\Big)$$

$$\leq \sum_{k=n}^{m} |\alpha_k| p(x_k) = 0 < \varepsilon.$$

This shows that the sequence $\{y_m\}$ is Cauchy in E. Since E is complete, it follows that $\{y_m\}$ converges in E to an element y, $y = \Sigma_{n=1}^{\infty} \alpha_n x_n$. Clearly $\sigma(y) = (\alpha_n)_{n=1}^{\infty}$. This shows that σ is onto S. ////

THEOREM 2.31 *Let A be a complete locally m-convex algebra with an orthogonal basis $\{x_n\}$. Then, A has an identity if, and only if, σ is onto S.*

Proof: ($\Longrightarrow$) If A has an identity e then by Lemma 2.13 we have that $e = \Sigma_{n=1}^{\infty} x_n$. Hence, for every $p \in P$, $p(S_n(e)-e) \to 0$ as $n \to \infty$ and consequently the sequence $\{S_n(e)\}$ is Cauchy with respect to the semi-norm p; i.e., for every $\varepsilon > 0$ there is a $K > 0$ such that for $m,n > K$,

$$p(S_n(e) - S_m(e)) = p\Big(\sum_{k=n}^{m} x_k\Big) < \varepsilon$$

In particular, for n sufficiently large, we have

$$p(x_n) < 1 \quad \text{or} \quad p(x_n)-1 < 0. \tag{*}$$

On the other hand we have $p(x_n) = p(x_n^2) \leq p(x_n)^2$. Hence, $p(x_n)^2-p(x_n) \geq 0$. It follows that

$$p(x_n)(p(x_n)-1) \geq 0. \tag{**}$$

We conclude from (*) and (**) that $p(x_n) \leq 0$ for all sufficiently large natural numbers n. Of course, we always have $p(x_n) \geq 0$. Therefore, we conclude that $p(x_n) = 0$ for n sufficiently large. Since p was arbitrary, it follows that for every $p \in P$, there exists $K_p > 0$ such that $p(x_n)=0$ for all $n > K_p$. Now, by Lemma 2.30(b) we conclude that σ is onto S.

($\Longleftarrow$) Conversely, suppose that σ is onto S. Then, for every sequence $\{\alpha_n\}$ of complex numbers, $\Sigma_{n=1}^{\infty}\alpha_n x_n$ converges in A. Hence, in particular, $\Sigma_{n=1}^{\infty}x_n$ belongs to A, and this is an identity for A by Lemma 2.13(a). ////

Note that the proof of this theorem implies that a normed algebra with identity cannot have an orthogonal basis. For, suppose that A is a normed algebra with identity e and suppose that $\{x_n\}$ is an orthogonal basis for A. Then, just as in the above proof, we have that $\|x_n\| < 1$ for n sufficiently large. Now, $\|x_n\| = \|x_n^k\| \leq \|x_n\|^k (k \in \mathbb{N})$. Thus, for n sufficiently large we have $\|x_n\| \leq \lim_{k\to\infty}\|x_n\|^k = 0$. This, however, is impossible since $x_n \neq 0$.

Now, let A be a topological algebra with an orthogonal basis. It is clear that σ is an algebraic (algebra) homomorphism of A into the algebra S (with pointwise operations). Also, because of the uniqueness of the coefficients in the basis expansion, σ is one-to-one. Hence, if A is also complete locally m-convex and has an identity, then it follows from Theorem 2.31 that σ is an algebraic isomorphism of A onto S. Recall that the algebra S is an F-algebra in its usual topology (the topology of simple convergence in the coefficients). It is natural to ask if σ is a continuous function from A into S. Now, if A is complete metrizeable (and hence the coefficient functionals associated to the basis are continuous), then a similar application of the closed graph theorem (via Theorem 1.1) as that used in the proof of Theorem 2.5 shows that σ is continuous. This same argument is valid for showing that σ is continuous in the case that A is a complete locally m-convex algebra with identity since such an algebra is necessarily metrizeable as the following theorem shows:

THEOREM 2.32 *Let A be a locally m-convex algebra with an orthogonal basis. If A has an identity, then A is metrizeable.*

Proof: Let $\{x_n\}$ be an orthogonal basis for A and let P be a family of submultiplicative seminorms generating the topology of A. We will show that a countable subfamily of P generates the topology of A. To this end let $p \in P$ and consider

$$p^{-1}(0) = \{x \in A : p(x) = 0\}.$$

This is a closed ideal in A. Now, for $p \in P$ let

$$N(p) = \{x_k \in \{x_n\} : p(x_k) = 0\}.$$

It follows by Theorem 2.21 that $p^{-1}(0) = \overline{[N(p)]}$. Now, since A has an identiy, it follows by the same proof as that of Theorem 2.31 ($\Longrightarrow$) that for $p \in P$ we have that $p(x_n) = 0$ for all but finitely many natural numbers n. It follows that each $N(p)$ contains all but finitely many of the basis elements $\{x_n\}$. Clearly then there can be only countably many different sets of the form $N(p)$ (i.e., $\{N(p) : p \in P\}$ is countable), call them

$$N_1, N_2, N_3, \ldots$$

where $N_i \neq N_j$ for $i \neq j$ $(i,j \in \mathbb{N})$. Now, set $K_n = \overline{[N_n]}$ $(n=1,2,\ldots)$ By the above discussion each K_n is the kernel of some seminorm $p \in P$ and the kernel of each seminorm $p \in P$ is some K_n. So, let

$$Q_n = \{p \in P : p^{-1}(0) = K_n\} \qquad (n=1,2,\ldots).$$

Now, suppose that $p,q \in Q_n$, then p and q are equivalent seminorms. For, let $\pi_n : A \to A/K_n$ be the canonical map. Then, A/K_n is a normed algebra with respect to the norm $\|\pi_n(x)\|_p = p(x)$. Similarly, A/K_n is a normed algebra with respect to the norm $\|\pi_n(x)\|_q = q(x)$. Now, clearly A/k_n is finite dimensional, and so, all norms on it are equivalent. It follows that $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent norms on A/K_n and thus p and q are equivalent seminorms on A (see [40]); i.e., p and q define the same topology on A.

Now, choose a seminorm q_n from each of the sets Q_n, so that

$$q_n \in Q_n \qquad (n=1,2,\ldots).$$

Since, by the above, all the seminorms in each Q_n are equivalent, it is clear that the collection $\{q_n\}$ of seminorms generates the topology of A. Since $\{q_n\}$ is countable, it follows that A is metrizeable. ////

COROLLARY 2.33 *If A is a complete locally m-convex algebra with identity, and if A has an orthogonal basis, then A is an F-algebra.* ////

This leads to the following characterization of complete locally *m*-convex algebras with identity posessing orthogonal bases:

THEOREM 2.34 *Let A be a complete locally m-convex algebra with identity. If A has an orthogonal basis, then A is algebraically and topologically isomorphic to the F-algebra S.*

Proof: We will show that the map σ (see Definition 2.29) is the desired isomorphism. We have already noted in the remarks following Theorem 2.31 that in this case σ is an algebraic isomorphism of A onto S. So, it remains to show that σ is a topological isomorphism.

Consider the coefficient functionals $\{f_n\}$ on S. These form a separating family of continuous linear functionals on S. Clearly the functionals $f_n \circ \sigma$ are continuous on A because they correspond to the coefficient functionals associated to the orthogonal basis in A and the latter are continuous by virtue of Theorem 2.28. Hence, by Theorem 1.1, σ is continuous. Now, by Corollary 2.33, A is an F-algebra. Thus, σ is a continuous linear map from the F-algebra A onto the F-algebra S. The open mapping theorem now shows that σ is open. Thus σ is bicontinuous and this completes the proof. ////

We summarize the results of this section in the following theorem which generalizes a theorem in [22]:

THEOREM 2.35 *Let A be a complete locally m-convex algebra with an orthogonal basis* $\{x_n\}$ *and let P be a family of submultiplicative seminorms defining the topology of A. The following are equivalent:*

(a) A has an identiy.

(b) σ *is onto S.*

(c) For every $p \in P$, $p(x_n) = 0$ *for all sufficiently large n.*

(d) A is algebraically and topologically isomorphic to S.

Proof: *(a)* and *(b)* are equivalent by Theorem 2.31. *(b)* and *(c)* are equivalent by Lemma 2.30. *(c)* implies *(d)* by Theorem 2.34 in view of the equivalence of *(a)* and *(c)* above. *(d)* implies *(a)* is obvious.

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Chapter III

UNCONDITIONAL ORTHOGONAL BASES

We have already seen in the previous chapter how the existence of an orthogonal basis in a topological algebra determines certain structural properties of the topological algebra. In this chapter we will consider the situation where the orthogonal basis is unconditional. Specifically, we will study complete *LC*-algebras A with identity having unconditional orthogonal bases.

In Chapter II we proved that the maximal ideal space of A is homeomorphic to a countable discrete space. We will show here that in this case $\mathcal{M}(A)$ with the hull-kernel topology is homeomorphic to $\beta M(A)$, the Stone-Čech compactification of M(A). This is done by associating an ultrafilter on $\mathbb{N}$ to each maximal ideal of A using a concept of local invertibility for elements of A we call E-regularity. This also enables us to describe the ideal which is the intersection of the dense maximal ideals of A. This ideal is of special interest since we will show that under certain conditions it is (linear space) isomorphic to the dual space A' of A.

Throughout this chapter we make the blanket assumption that A is a complete LC-algebra with identity e and that $\{x_n\}$ is an unconditional orthogonal Schauder basis in A.

1. E-Regularity

In the next section we will prove that $\mathcal{M}(A)$ equipped with the hull-kernel topology is homeomorphic to $\beta M(A)$. Here, we will lay the ground-work for this result by defining a form of local invertability for elements of A we call E-regularity and developing some of its fundamental properties. We use this concept to associate to each ideal of A a filter on $\mathbb{N}$ ($\simeq M(A)$). This correspondence is useful for several purposes which will be dealt with in the following sections.

We begin by proving some fundamental facts about complete *LC*-algebras with identity having unconditional orthogonal bases.

LEMMA 3.1 *Let A be as above. Then,*

(a) If $\{\alpha_n\}$ is a bounded sequence of complex numbers then $\Sigma_{n=1}^{\infty}\alpha_n x_n$ converges in A.

(b) If $x = \Sigma_{n=1}^{\infty}\alpha_n x_n \in A$ and if $\{\alpha_n\}$ is a sequence of complex numbers which is bounded away from 0 (i.e., $\inf\{|\alpha_n| : n \in \mathbb{N}\} > 0$), then x is invertible in A.

Proof: (a) Since A has an identity e, it follows by Lemma 2.13(a) that $e = \Sigma_{n=1}^{\infty}x_n$. Since the basis $\{x_n\}$ is unconditional and since A is a locally convex space it follows that $\Sigma_{n=1}^{\infty}\alpha_n x_n$ converges in A for any bounded sequence $\{\alpha_n\}$ of complex numbers.

(b) If $\{\alpha_n\}$ is bounded away from 0, then $\{\alpha_n^{-1}\}$ is a bounded sequence. Hence, by part *(a)*, $\Sigma_{n=1}^{\infty}\alpha_n^{-1}x_n \in A$ and this is the inverse of x by Lemma 2.13(b). ////

LEMMA 3.2 *Let A be as above. If* $x = \Sigma_{n=1}^{\infty}\alpha_n x_n \in A$, *then* $\Sigma_{n=1}^{\infty}\overline{\alpha}_n x_n$ *and* $\Sigma_{n=1}^{\infty}|\alpha_n| x_n$ *both converge in A.*

Proof: Let $\beta_n = sgn\,\overline{\alpha}_n$ $(n=1,2,\ldots)$. Clearly $\{\beta_n\}$ is a bounded sequence. So, since the basis $\{x_n\}$ is unconditional (or by the previous lemma) we have

$$\sum_{n=1}^{\infty}\beta_n\alpha_n x_n = \sum_{n=1}^{\infty}|\alpha_n| x_n \in A.$$

To show the other part, note that for a complex number α, $\overline{\alpha} = |\alpha| sgn\,\overline{\alpha}$ Now, since $\{sgn\,\overline{\alpha}_n\}$ is a bounded sequence and since $\Sigma_{n=1}^{\infty}|\alpha_n| x_n \in A$ (by the above), if follows that

$$\sum_{n=1}^{\infty}|\alpha_n|(sgn\,\overline{\alpha}_n)x_n = \sum_{n=1}^{\infty}\overline{\alpha}_n x_n \in A.$$

////

This lemma allows us to make the following definition:

DEFINITION 3.3 Let A be as above. For $x \in A$, $x = \Sigma_{n=1}^{\infty}\alpha_n x_n$, we will write $|x| = \Sigma_{n=1}^{\infty}|\alpha_n| x_n$ and $\overline{x} = \Sigma_{n=1}^{\infty}\overline{\alpha}_n x_n$.

The following lemma shows that each ideal in A is "absolutely convex" and "self-adjoint"

LEMMA 3.4 *Let A be as above. If I is an ideal in A and if* $x \in I$, *then* $|x| \in I$ *and* $\overline{x} \in I$.

Proof: Since $\{x_n\}$ is a basis for A, we have $x = \Sigma_{n=1}^{\infty}\alpha_n x_n$ for some sequence of complex numbers $\{\alpha_n\}$. Just as in the proof of Lemma 3.2

let $\beta_n = sgn\, \overline{\alpha}_n$ $(n=1,2,\dots)$. Since $\{\beta_n\}$ is bounded, it follows by Lemma 3.1 that $y = \Sigma_{n=1}^{\infty} \beta_n x_n \in A$. Since I is an ideal, we have that $xy \in I$. But

$$xy = \left(\sum_{n=1}^{\infty} \alpha_n x_n\right)\left(\sum_{n=1}^{\infty} \beta_n x_n\right)$$

$$= \sum_{n=1}^{\infty} \alpha_n \beta_n x_n = \sum_{n=1}^{\infty} |\alpha_n| x_n = |x|.$$

The other part is proved similarly. ////

To study ideals more closely we make the following definitions:

DEFINITION 3.2 Let A be as above and let $x \in A$, $x = \Sigma_{n=1}^{\infty} \alpha_n x_n$. If E is any subset of $\mathbb{N}$, then we define x_E by

$$x_E = \sum_{n=1}^{\infty} \chi_E(n) \alpha_n x_n$$

where χ_E is the characteristic function on E.

Note that since the basis $\{x_n\}$ in A is unconditional and since A is a complete *LC*-algebra (i.e., A is a locally convex space), then every convergent series of the form $\Sigma_{n=1}^{\infty} \alpha_n x_n$ is subseries convergent. Thus x_E is an element of A for each $x \in A$ and for each subset E of $\mathbb{N}$.

DEFINITION 3.6 Let $x \in A$ and let E be a non-empty subset of $\mathbb{N}$. We will say that x is E-*regular* if there exists a $y \in A$ such that $yx_E = e_E$ (recall that e is the identity element in A). We will call x E-*singular* if it is not E-regular. By x ϕ-regular (where ϕ denotes the empty set) we will mean that x is non-invertible.

Note that if x is invertible in A, then x is E-regular for every nonempty subset E of $\mathbb{N}$. If x is not E-regular for any subset $E \neq \phi$ of $\mathbb{N}$, then $x = 0$, for if $x_n^*(x) \neq 0$ for some $n \in \mathbb{N}$ then clearly x is $\{n\}$-regular.

Also, we note that a similar proof as that of Lemma 3.1(b) ~~shows~~ that if $x = \Sigma_{n=1}^{\infty}\alpha_n x_n$ and if the sequence $\{\alpha_n : n\varepsilon E\}$ is bounded away from 0, then x is E-regular ($E \subseteq \mathbb{N}$).

We will use this definition to associate a filter on $\mathbb{N}$ to each ideal of A, but first we give a characterization of E-regularity in terms of principal ideals (see Chapter II, section 2). Recall that we have shown (Theorem 2.27) that every closed ideal in A is a principal ideal.

THEOREM 3.7 *Let A be as above, let $x \in A$ and let E be a nonempty subset of $\mathbb{N}$. Then, x is E-regular if, and only if, the ideal generated by x_E is closed in A.*

Proof: ($\Longrightarrow$) If x is E regular, then

$$x_E = \sum_{n \in E} \alpha_n x_n$$

where $\alpha_n \neq 0$ for every $n \in E$. Also, there exists $y \in A$ such that $yx_E = e_E$ and thus $e_E \in \langle x_E\rangle$, the principal ideal generated by x_E. Now, if $z \in A$, then $ze_E = z_E \in \langle x_E\rangle$. So, $\langle x_E\rangle$ contains all elements $z \in A$ for which $x_n^*(z) = 0$ for $n \in E^c$ (E^c denotes the complement of E in $\mathbb{N}$). It follows that

$$\langle x_E\rangle = \bigcap\{M_k : k \in E^c\}$$

(see Definition 2.16). Therefore $\langle x_E\rangle$ is closed being the intersection of the closed maximal ideals M_k (see Theorem 2.17).

($\Longleftarrow$) Conversely, suppose that the ideal $\langle x_E \rangle$ is closed. Then, by Corollary 2.26, we have

$$\langle x_E \rangle = \bigcap \{M_k : k \notin E\}$$

Since $e \in A$, and since the basis is unconditional, it follows that $e_E \in A$. Clearly $e_E \in M_k$ whenever $k \notin E$. Therefore $e_E \in \langle x_E \rangle$. Thus, there exists $y \in A$ such that $yx_E = e_E$. This shows that x is E-regular. ////

In the next two lemmas we prove some fundamental properties of E-regularity. We will write $|x| \leq |y|$ if $|x_n^*(x)| \leq |x_n^*(y)|$, $n=1,2,\ldots$

LEMMA 3.8 *Let A be as above, let $x, y \in A$, and let E and F be subsets of $\mathbb{N}$. Then*

(a) *If x is E-regular and if $F \subseteq E$, then x is also F-regular.*

(b) *If x is E-regular and F-regular, then x is $E \cup F$-regular.*

(c) *If x is E-regular, then $|x|$ is also E-regular.*

(d) *If $|x| \leq |y|$ and if x is E-regular, then y is E-regular.*

Proof: (a) Clear.

(b) There exists $y, z \in A$ such that $yx_E = e_E$ and $zx_F = e_F$. Consider the following sets:

$$G_1 = E \cap F, \quad G_2 = E - E \cap F, \quad G_3 = F - E \cap F$$

(where E-X denotes the complement of X in E). We have, $G_i \cap G_j = \phi$ for

$i \neq j$ $(i,j=1,2,3)$. Let $w = y_{G_1} + y_{G_2} + z_{G_3}$ where y_{G_1}, y_{G_2}, z_{G_3} all belong to A because the basis is unconditional (see the remark following Definition 3.5). Now, $wx = y_{G_1}x + y_{G_2}x + z_{G_3}x = e_{G_1} + e_{G_2} + e_{G_3} = e_{G_1 \cup G_2 \cup G_3} = e_{E \cup F}$. Therefore, x is $E \cup F$-regular.

Parts *(c)* and *(d)* are clear. ////

LEMMA 3.9 *Let A be as above. Let $x, y \in A$, and let E and F be subsets of $\mathbb{N}$. Then,*

(a) If x is E-regular and y is F-regular, then xy is $E \cap F$-regular.

(b) If x is E-regular and y is F-regular, then $|x|+|y|$ is $E \cup F$-regular.

Proof: (a) There exists $w, z \in A$ such that $wx_E = e_E$ and $zy_F = e_F$. So, clearly $wx_{E \cap F} = e_{E \cap F}$ and $zy_{E \cap F} = e_{E \cap F}$. Therefore we have $xy\,(wz)_{E \cap F} = (wx_{E \cap F})(zy_{E \cap F}) = (e_{E \cap F})^2 = e_{E \cap F}$. It follows that xy is $E \cap F$-regular.

(b) Clearly $|x|+|y| \geq |x|$. But $|x|$ is E-regular by Lemma 3.8(c). Therefore, by Lemma 3.8(d), $|x| + |y|$ is E-regular. Now, since $|x|+|y| \geq |y|$ then by a similar argument we have that $|x|+|y|$ is also F-regular. It follows by Lemma 3.8(b) that $|x|+|y|$ is $E \cup F$-regular. ////

DEFINITION 3.10 Let A be as above. For $x \in A$, set

$$Z[x] = \{E \subseteq \mathbb{N} : x \text{ is } E^c\text{-regular}\},$$

where E^c is the complement of E in $\mathbb{N}$.

LEMMA 3.11 *Let A be as above and let $x,y \in A$. Then,*

(a) $Z[xy] \subseteq Z[x] \cap Z[y]$

(b) $Z[|x|+|y|] \supseteq Z[x] \cup Z[y]$

(c) If $|x| \leq |y|$, *then* $Z[x] \subseteq Z[y]$.

Proof: (a) If xy is E^c-regular, then from the definition of E-regularity it is clear that both x and y are also E^c-regular

(b) If x is E^c-regular then by Lemma 3.8(a) and Lemma 3.9(b), $|x|+|y|$ is also E^c-regular. Since the situation is symmetric with respect to x and y, the result follows.

(c) This follows directly from Lemma 3.8(d). ////.

Recall that a *filter* on a set X is a collection $\mathscr{F}$ of subsets of X satisfying the following properties: (i) If F, G $\in \mathscr{F}$, then $F \cap G \in \mathscr{F}$. (ii) If $F \in \mathscr{F}$, and $G \supseteq F$, then $G \in \mathscr{F}$. (iii) $\phi \notin \mathscr{F}$.

THEOREM 3.12 *Let A be as above and let $x \in A$. If x is non-invertible then $Z[x]$ is a filter on $\mathbb{N}$, and conversely.*

Proof: If x is noninvertible then clearly $\phi \notin Z[x]$. If $E,F \in Z[x]$, then x is E^c-regular and F^c-regular, hence, by Lemma 3.8(b), x is $E^c \cup F^c$-regular. But $E^c \cup F^c = (E \cap F)^c$. Therefore x is $(E \cap F)^c$-regular and this shows that $E \cap F \in Z[x]$.

Suppose that $E \in Z[x]$, then x is E^c-regular. If $F \supseteq E$, then $F^c \subseteq E^c$, and so, by Lemma 3.8(a), x is F^c-regular; i.e., $F \in Z[x]$. This completes the proof that $Z[x]$ is a filter.

Conversely, suppose that x is invertible in A. Then clearly

$\phi \in Z[x]$ (i.e., x is $\mathbb{N}$-regular) and therefore $Z[x]$ is not a filter on $\mathbb{N}$. ////

DEFINITION 3.13 Let A be as above. If I is a subset of A, then we define Z[I] by

$$Z[I] = \bigcup \{Z[x] : x \in I\}$$

THEOREM 3.14 *Let A be as above. If I is an ideal in A, then Z[I] is a filter on* $\mathbb{N}$.

Proof: Since I is an ideal it consists of noninvertable elements only. It follows by Theorem 3.12 that $\phi \notin Z[x]$ for each $x \in I$. Therefore, $\phi \notin Z[I]$.

Suppose that $E, F \in Z[I]$. Then, there exists $x, y \in I$ such that x is E^c-regular and y is F^c-regular. Since I is an ideal, it follows by Lemma 3.4 that $w = |x|+|y| \in I$. By Lemma 3.9(b), w is $E^c \cup F^c$-regular: i.e., w is $(E \cap F)^c$-regular. This shows that $E \cap F \in Z[I]$.

Finally, if $E \in Z[I]$, then $E \in Z[x]$ for some $x \in I$. Thus, if $F \supseteq E$, then by Lemma 3.12, $F \in Z[x]$. Therefore $F \in Z[I]$. ////

Note that the function Z preserves inclusion; i.e., If I,J are ideals in A and if $I \subseteq J$, then $Z[I] \subseteq Z[J]$. We will use this fact in the next section.

2. The Space $\mathcal{M}(A)$

In this section we use the concepts developed in Section 1 to show that the map $M \to Z[M]$ gives a one-to-one correspondence between the set of maximal ideals of A ($\mathcal{M}(A)$) and the collection of ultrafilters on $\mathbb{N}$. Also, it will be shown that if $\mathcal{M}(A)$ is equipped with the hull-kernel topology, then it is a compact space in which M(A) ($\simeq \mathbb{N}$) is densely embedded. Moreover, in this case the above map is a homeomorphism between the space $\mathcal{M}(A)$ and $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$. (Note that the collection of ultrafilters on $\mathbb{N}$ can be topologized so as to be $\beta\mathbb{N}$, [16]).

Recall that an *ultrafilter* on a set X is a filter on X such that there is no filter on X which properly contains it. A filter $\mathcal{F}$ on X is an ultrafilter if, and only if, $G \in \mathcal{F}$ whenever $G \cap F \neq \phi$ for every $F \in \mathcal{F}$, [7]. Recall also that here A denotes a complete *LC*-algebra with identity e and $\{x_n\}$ is an unconditional orthogonal Schauder basis in A.

THEOREM 3.15 *Let A be as above. If M is a maximal ideal in A, then Z[M] is an ultrafilter on* $\mathbb{N}$.

Proof: Let G be any subset of $\mathbb{N}$ and suppose that $G \cap F \neq \phi$ for every $F \in Z[M]$. We will show that in this case $G \in Z[M]$ and this shows that Z[M] is an ultrafilter. To this end let $H = G^c$ and consider

$$e_H = \sum_{n=1}^{\infty} \chi_H(n) x_n .$$

Now, suppose that $e_H \notin M$. Then, M being a maximal ideal has the property that $e \in M + \langle e_H \rangle$; i.e., there exists $y \in M$ and $z \in A$ such that $e = y + ze_H$. Hence $y = e - ze_H$, so that $ye_G = ee_G - ze_H e_G$

$= e_G$. It follows from this that $e_G \in M$. Now, $G^c = H \in Z[e_G]$. Hence $H \in Z[M]$, contradicting the fact that $G \cap F \neq \Phi$ for every $F \in Z[M]$. This contradiction shows that $e_H \in M$, and thus $G \in Z[M]$. ////

We now consider the inverse function (considered as a function on sets) of the function Z.

DEFINITION 3.16. Let A be as above, and let $\mathscr{F}$ be a filter on $\mathbb{N}$. Define $Z^+[\mathscr{F}]$ by

$$Z^+[\mathscr{F}] = \{x \in A : Z[x] \subseteq \mathscr{F}\}$$

We aim to show that $Z^+[\mathscr{F}]$ is a maximal ideal of A whenever $\mathscr{F}$ is an ultrafilter on $\mathbb{N}$. First, we need two lemmas:

LEMMA 3.17 *Let A be as above, let $x \in A$, and let E be a subset of $\mathbb{N}$. If $E \cap F \neq \Phi$ for every $F \in Z[x]$, then x is E-singular.*

Proof: Suppose that x is E-regular. Then, x is $(E^c)^c$-regular and hence $E^c \in Z[x]$. Since $E \cap E^c = \Phi$, this contradicts the hypothesis of the lemma. ////

We recall here that we have shown in Section 2 of Chapter II that M(A), the maximal ideal space of A, is homeomorphic to a countable discrete space. In fact, $M(A) = \{x_n^* : n \in \mathbb{N}\}$. Thus, we can identify M(A) with $\mathbb{N}$ via the map $x_n^* \to n$. In this section we will consider the Gelfand transform $\hat{x}$ of an element $x \in A$ to be the function on $\mathbb{N}$ given by $\hat{x}(n) = x_n^*(x)$.

We also recall that if $\mathscr{F}$ is an ultrafilter on a set X and if f is a mapping of X into a set Y, then $f(\mathscr{F})$ is an ultrafilter base on the set Y [7].

LEMMA 3.18 *Let A be as above, $x, z \in A$, and let $\mathscr{F}$ be an ultrafilter on $\mathbb{N}$. If $\lim_{\mathscr{F}}(xz)\hat{} = 0$ for every $z \in A$, then x is E-singular for every $E \in \mathscr{F}$.*

Proof: Suppose that x is E-regular for some $E \in \mathscr{F}$. Then, there exists $z \in A$ such that $xz_E = e_E$ and hence 1 is a cluster point of $(xz)\hat{}(\mathscr{F})$ (because, since $E \cap F \neq \phi$ for each $F \in \mathscr{F}$, it follows that $1 \in (xz)\hat{}(F)$ for each $F \in \mathscr{F}$). But 0 is also a cluster point of $(xz)\hat{}(\mathscr{F})$ by hypothesis. This contradicts the fact that $(xz)\hat{}(\mathscr{F})$ is an ultrafilter base. ////

THEOREM 3.19 *Let A be as above. If $\mathscr{F}$ is an ultrafilter on $\mathbb{N}$, then $Z^+[\mathscr{F}]$ is a maximal ideal of A.*

Proof: Let $M = Z^+[\mathscr{F}]$. If $x \in M$, then $Z[x] \subseteq \mathscr{F}$ and so, by Theorem 3.12, x is noninvertible. Thus, M consists of noninvertible elements only.

Let $x \in M$ and $y \in A$. Then, by Lemma 3.11(a), $Z[xy] \subseteq Z[x]$. Therefore, $xy \in M$.

If $x, y \in M$, then $Z[x] \subseteq \mathscr{F}$ and $Z[y] \subseteq \mathscr{F}$. Hence, by Lemma 3.17 x and y are E-singular for every $E \in \mathscr{F}$. It follows from this that if z is any element of A, then xz and yz are also E-singular for every $E \in \mathscr{F}$. Therefore, by Lemma 3.1(b) (modified to apply to E-regularity), we have that $inf\{(xz)\hat{}(n) : n \in E\} = 0$ for every $E \in \mathscr{F}$, and thus 0 is a cluster point of the filter base $(xz)\hat{}(\mathscr{F})$. Similarly, 0 is a cluster point of the filter base $(yz)\hat{}(\mathscr{F})$. Now, since $\mathscr{F}$ is an ultrafilter, it follows that $(xz)\hat{}(\mathscr{F})$ and $(yz)\hat{}(\mathscr{F})$ are ultrafilter bases on $\mathbb{C}$. Therefore, $(xz)\hat{}(\mathscr{F})$ and $(yz)\hat{}(\mathscr{F})$ both converge to 0 for every $z \in A$; i.e., $lim_{\mathscr{F}}(xz)\hat{} = lim_{\mathscr{F}}(yz)\hat{} = 0$ for every $z \in A$. Now, because of the continuity of addition in $\mathbb{C}$, we have that $lim_{\mathscr{F}}((xz)\hat{}+(yz)\hat{}) = lim_{\mathscr{F}}(xz)\hat{} + lim_{\mathscr{F}}(yz)\hat{}$ (see [7,p.375]). It follows from this that

$$lim_{\mathscr{F}}((x+y)z)\hat{} = 0 \qquad (z \in A)$$

(since $((x+y)z)\hat{} = (xz+yz)\hat{} = (xz)\hat{}+(yz)\hat{}$). From this and Lemma 3.18 we conclude that $x+y$ is E-singular for every $E \in \mathscr{F}$. Therefore, $Z[x+y] \subseteq \mathscr{F}$, and consequently $x+y \in M$. This shows that M is an ideal in A.

Now, suppose that N is an ideal in A containing M. Then, $Z[N] \supseteq \mathscr{F}$, and since $\mathscr{F}$ is an ultrafilter, we have that $Z[N] = \mathscr{F}$. So, if $x \in N-M$, then $Z[x] \subseteq \mathscr{F}$. But then $x \in M$ by definition. Therefore $M = N$. This shows that M is a maximal ideal. ////

We have already noted that Z preserves inclusion of ideals and it is clear from the definition of Z^+ that it preserves inclusion of filters. From these facts we get the following theorem:

THEOREM 3.20 *For $\mathscr{F}$ an ultrafilter on $\mathbb{N}$, and for M a maximal ideal in A, the following relations hold:*

$$Z[Z^+[\mathscr{F}]] = \mathscr{F} \quad \text{and} \quad Z^+[Z[M]] = M$$

Proof: Let $N = Z^{+}[\mathscr{F}]$. N is a maximal ideal by Theorem 3.19. From the definitions of Z and Z^{+} we see that Z[N] is a filter containing $\mathscr{F}$. But since $\mathscr{F}$ is an ultrafilter, the result follows.

The second part is proved similarly. ////

This shows that the mapping $M \to Z[M]$ gives a one-to-one correspondence between the set of all maximal ideals of A and the set of ultrafilters on $\mathbb{N}$.

Now, let $\mathscr{F}(\mathbb{N})$ be the collection of all ultrafilters on $\mathbb{N}$, and consider the collection of subsets of $\mathscr{F}(\mathbb{N})$ consisting of sets of the form

$$E^{*} = \{\mathscr{F} \varepsilon \mathscr{F}(\mathbb{N}) : E \varepsilon \mathscr{F}\},$$

for $E \subseteq \mathbb{N}$. It is well known (see [16]) that the collection

$$\mathscr{B} = \{E^{*} : E \subseteq \mathbb{N}\}$$

is a *base for the closed sets* (i.e., the closed sets are the intersections of members of $\mathscr{B}$) for a topology on $\mathscr{F}(\mathbb{N})$ and in this topology $\mathscr{F}(\mathbb{N})$ is the *Stone-Čech compactification* of $\mathbb{N}$.

Also, there is a natural topology on the set $\mathcal{M}(R)$ of maximal ideals of any ring R with identity. This is the *hull-kernel topology*. A base for the closed sets in this topology is given by the collection of sets of the form

$$H(x) = \{M \varepsilon \mathcal{M}(R) : x \varepsilon M\}$$

for $x \varepsilon A$ (see [16]). However, for our algebras A a very much smaller collection forms a base for the closed sets in the hull-kernel topology on $\mathcal{M}(A)$.

LEMMA 3.21 *Let A be as above. The collection of subsets of* $\mathcal{M}(A)$,

$$\mathcal{C} = \{H(e_E) : E \subseteq \mathbb{N}\}$$

is a base for the closed sets in the hull-kernel topology on the set $\mathcal{M}(A)$ *of maximal ideals of A.*

Proof: It is sufficient to show that for every $x \in A$, there exists $E \subseteq \mathbb{N}$ such that for every maximal ideal M of A, $x \in M$ implies $e_E \in M$. To show this, let M be a maximal ideal of A, $x \in M$, and consider

$$E = \{n \in \mathbb{N} : |x_n^*(x)| > 1\}$$

Clearly x_E is E-regular by (a modification of) Lemma 3.1(b). So, there is an element $z \in A$ such that $xz_E = e_E$. Now, since $x \in M$, it follows that $e_E \in M$. This is what we wanted to show. ////

We now prove the result mentioned in the introduction to this section; i.e., that the map $M \to Z[M]$ is topological.

THEOREM 3.22 *Let A be as above. The map* $M \to Z[M]$ *is a homeomorphism from* $\mathcal{M}(A)$ *with the hull-kernel topology onto* $\beta\mathbb{N}$, *the Stone-Čech compactification of* $\mathbb{N}$.

Proof: We have already shown that this map, call it Ψ, is one-to-one and onto. We will now show that the map Ψ takes a base for the closed sets in $\mathcal{M}(A)$ onto a base for the closed sets in $\beta\mathbb{N}$. This will show that Ψ is a homeomorphism. To this end, let $E \subseteq \mathbb{N}$ and consider

$$H(e_E) = \{M \in \mathcal{M}(A) : e_E \in M\}$$

If $M \in H(e_E)$, then $E^c \in Z[M]$. Conversely, if $E^c \in Z[M]$, then there exists $x \in M$ such that x is E-regular and therefore (as in the proof of Lemma 3.31) $e_E \in M$; i.e., Ψ takes $H(e_E)$ to $(E^c)^*$. It follows that Ψ takes a member of the base for the closed sets (defined above) for the topology of $\mathcal{M}(A)$ to a member of the base for the closed set for the topology of $\beta\mathbb{N}$ (see above). Now, each member of the base for the closed sets in $\beta\mathbb{N}$ is of the form E^* ($E \subseteq \mathbb{N}$), and by the above we have that Ψ takes $H(e_{E^c})$ to E^*. It follows that Ψ maps the base $\mathcal{C} = \{H(e_E) : E \subseteq \mathbb{N}\}$ (see Lemma 3.21) onto the base $\mathcal{B} = \{E^* : E \subseteq \mathbb{N}\}$ of $\beta\mathbb{N}$. This completes the proof. ////

We have already shown (Chapter II, Theorem 2.14) that M(A) is homeomorphic to $\mathbb{N}$. From this and the above theorem we conclude that $\mathcal{M}(A)$ with the hull-kernel topology is homeomorphic to $\beta M(A)$, the Stone-Čech compactification of M(A). Also, it is clear that under the map Ψ (see the proof of Theorem 3.22), the closed maximal ideals of A (these are ideals of the form M_k by Theorem 2.17) correspond to the points of $\mathbb{N}$ (i.e., the *fixed ultrafilters* on $\mathbb{N}$ — that is, ultrafilters $\mathcal{F}$ satisfying $\bigcap \mathcal{F} \neq \phi$) and the dense maximal ideals of A correspond to the points of $\beta\mathbb{N} - \mathbb{N}$ (i.e., the *free ultrafilters* on $\mathbb{N}$. These are ultrafilters $\mathscr{F}$ satisfying $\bigcap \mathscr{F} = \phi$). ▫

In Section 2 of Chapter II we characterized the maximal ideals of A corresponding (under the map Ψ) to the points of $\hat{\mathbb{N}}$. In the next section we will consider the maximal ideal corresponding to the points of $\beta\mathbb{N} - \mathbb{N}$.

3. Maximal Ideals and the Dual Space

We continue the study of the maximal ideals of A by describing the intersection of the dense maximal ideals of A. It turns out that this intersection is a dense ideal of A we will call J(A). We also define an ideal we call K(A) which, if A is also metrizeable, is isomorphic (as a linear space) to the topological dual A' of A. It turns out that under certain conditions (see Theorem 3.31) the ideals J(A) and K(A) are equal. Therefore, in this case the dual space A' is isomorphic to the intersection of the dense maximal ideals of A. Moreover, the algebras H(D) and S of Examples 2.8 and 2.9, respectively, satisfy this condition.

We recall that A denotes a complete *LC*-algebra with identity e and with an unconditional orthogonal Schauder basis $\{x_n\}$. We begin by defining the ideal J(A).

DEFINITION 3.23 Let A be as above. Define the subset J(A) of the algebra A by

$$J(A) = \{x \in A : \lim_{n\to\infty} x_n^*(xy) = 0 \text{ for all } y \in A\}$$

Note that since $e \in A$, then each $x \in J(A)$ has the property that $\lim_{n\to\infty} x_n^*(x) = 0$. Note also that in the algebras of Examples 2.8 and 2.9, $J(A) \neq A$.

LEMMA 3.24 *If A is as above, then $J(A)$ is a dense ideal in A.*

Proof: Suppose that $x, y \in J(A)$, and let z be any element of A. Then, $\lim_{n\to\infty} x_n^*((x+y)z) = \lim_{n\to\infty} x_n^*(xz) + \lim_{n\to\infty} x_n^*(yz) = 0$. Thus, $x+y \in J(A)$. Clearly if $x \in J(A)$ then $\lambda x \in J(A)$ for $\lambda \in \mathbb{C}$. Now, if $x \in J(A)$ and $y \in A$, then $\lim_{n\to\infty} x_n^*(xyz) = \lim_{n\to\infty} x_n^*(x(yz)) = 0$, hence, $xy \in J(A)$. This shows that J(A) is an ideal in A.

Clearly any finite linear combination of elements from $\{x_n\}$ belongs to J(A). Since $\{x_n\}$ is a basis in A, it follows that J(A) is dense in A. ////

We aim to show that J(A) is precisely the intersection of the dense maximal ideals of A. We need the following characterization of the elements of J(A).

LEMMA 3.25 *Let A be as above. Then, $x \in J(A)$ if, and only if, for every infinite subset E of $\mathbb{N}$, x is not E-regular.*

Proof: ($\Longrightarrow$) Suppose that there exists an infinite subset E of $\mathbb{N}$ for which x is E-regular. Then, there exists $y \in A$ such that $xy_E = e_E$. But then $\lim_{n\to\infty} x_n^*(xy_E)$ is clearly not 0. Therefore $x \notin J(A)$.

($\Longleftarrow$) Conversely, suppose that $x \notin J(A)$. Then, there exists $y \in A$ such that $\lim_{n\to\infty} x_n^*(xy)$ is not 0 (possibly this limit does not exist). So, let $r = \overline{\lim}\, |x_n^*(xy)|$, $r \neq 0$. Thus, there is a subsequence $\{n_k\}$ of

$\mathbb{N}$ such that $\lim_{k\to\infty} x^*_{n_k}(xy) = r$. Without loss of generality, we can choose $\{n_k\}$ in such a way that $x^*_{n_k}(xy) \neq 0$ for all $k \in \mathbb{N}$ ⟦because this can happen only finitely often, so, drop these terms from the sequence $\{n_k\}$⟧. It follows by Lemma 3.1(b) (applied only on the sequence $\{n_k\}$) that xy is $\{n_k\}$-regular. Hence, x is also $\{n_k\}$-regular; i.e., x is E-regular for an infinite set E $(=\{n_k\})$. ////

To prove the next theorem we use the correspondence between the dense maximal ideals of A and the free ultrafilters on $\mathbb{N}$ established in Section 2.

THEOREM 3.26 *Let A be as above. Then, we have that*

$$J(A) = \bigcap\{M : M \in \mathcal{M}(A) - M(A)\}$$

i.e., the ideal J(A) is the intersection of the dense maximal ideals of A.

Proof: "$\subseteq$" It will suffice for this to show that for each dense maximal ideal M, the ideal J(A)+M is proper. To this end let $M \in \mathcal{M}(A)-M(A)$, and let W = J(A)+M. Suppose that W is not proper; i.e., suppose that $e \in W$. Then, there exists $x \in M$, $y \in J(A)$, such that $x + y = e$. Since $y \in J(A)$, we have that $\lim_n x^*_n(y) = 0$, and so, $x^*_n(y) = 1$ at most finitely often, say,

$$x^*_{m_1}(y) = x^*_{m_2}(y) = \ldots = x^*_{m_k}(y) = 1.$$

Now, let $w = \sum_{i=1}^{k} \frac{1}{2} x_{m_i}$. Since M is a dense ideal, $\{x_n\} \subseteq M$ by Theorem

2.17, and hence $\omega \in M$. For the same reason, $\omega \in J(A)$. It follows that if we set $y' = y-\omega$ and $x' = x+\omega$, then $x' \in M$ and $y' \in J(A)$. Also, we have that $x_n^*(y') \neq 1$ $(n=1,2,\ldots)$. Now, $x'+y' \in W$ and we have that

$$x'+y' = x+\omega+y-\omega$$

$$= x+y = e.$$

This implies that

$$x' = e-y' \qquad (*)$$

However, by the way y' was chosen, we have that $x_n^*(e-y') \neq 0$ $(n \in \mathbb{N})$ Also, we have

$$\lim_{n\to\infty} x_n^*(e-y') = \lim_{n\to\infty} x_n^*(e) - \lim_{n\to\infty} x_n^*(y') = 1$$

(because y', being an element of J(A) has the property that $\lim_{n\to\infty} x_n^*(y') = 0$). Therefore, by Lemma 3.1(b), $e-y'$ is invertible. But from (*) we have that $x' = e-y'$. It follows that x' is invertible. This, however, contradicts the fact that $x' \in M$. This contradiction stems from the assumption that the $e \in W$. Therefore, $e \notin W$, and hence $W = M+J(A)$ is a proper ideal of A. Since M is a maximal ideal, it follows that J(A) is contained in M.

" $\supseteq$ " Suppose that $x \in \bigcap\{M : M \in \mathcal{M}(A)-M(A)\}$. Let E be an infinite subset of $\mathbb{N}$ and let $\mathscr{F}$ be a free ultrafilter with $E \in \mathscr{F}$. [[This is always possible. For example, let $\mathscr{G} = \{F \subseteq E : F$ has finite complement in E$\}$. Then $\mathcal{G}$ is a filter base and is therefore contained in an ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Clearly $\bigcap \mathcal{G} = \phi$. Therefore $\bigcap \mathcal{F} = \phi$ also. This shows that $\mathcal{F}$ is a free ultrafilter on $\mathbb{N}$ and clearly $E \in \mathcal{F}$]. Now, suppose that x is E-regular. Then $E^c \in Z[x]$. Since $E \in \mathcal{F}$, it follows that $Z[x] \not\subset \mathcal{F}$. Hence $x \notin Z^+[\mathcal{F}]$. But since $\mathcal{F}$ is a free ultrafilter, $Z^+[\mathcal{F}]$ is a dense maximal ideal of A (see Theorem 3.19 and the remarks at the end of Section 2). It follows that $x \notin \bigcap\{M : M \in \mathcal{M}(A)-M(A)\}$, contradicting our hypothesis. Therefore the assumption that x is E-regular for some infinite set E leads to a contradiction. It follows that x is not E-regular for any infinite subset E of $\mathbb{N}$. So, from Lemma 3.25, we conclude that $x \in J(A)$.

////

To study A', the topological dual of A, we make the following definition:

DEFINITION 3.28 Let A be as above. We define a subset K(A) of the algebra A by

$$K(A) = \{x \in A : \sum_{n=1}^{\infty} x_n^*(xy) \text{ converges for all } y \in A\}$$

Note that if $x \in K(A)$, the $\Sigma_{n=1}^{\infty} x_n^*(x)$ converges since we have $\Sigma_{n=1}^{\infty} x_n^*(x) = \Sigma_{n=1}^{\infty} x_n^*(xe)$. Now, since the basis $\{x_n\}$ is unconditional, we have by Lemma 3.1(a) that $\Sigma_{n=1}^{\infty} \beta_n x_n = y \in A$ for any bounded sequence $\{\beta_n\}$ of complex numbers. Therefore, if $x \in K(A)$, we have that $\Sigma_{n=1}^{\infty} \beta_n x_n^*(x) = \Sigma_{n=1}^{\infty} x_n^*(xy)$ converges in A. Since this is, true for any bounded sequence $\{\beta_n\}$ of complex numbers, it follows that for each $x \in K(A)$, $\Sigma_{n=1}^{\infty} x_n^*(x)$ converges absolutely.

Lemma 3.29 *If A is as above, then K(A) is a dense ideal in A.*

Proof: If $x, y \in K(A)$, then clearly $x+y \in K(A)$. Suppose that $y \in A$ and $x \in K(A)$, and let z be any element of A. Then, $\Sigma_{n=1}^{\infty} x_n^*(xyz)$ converges (because $x \in K(A)$). Since this is true for all $z \in A$, it follows that $xy \in K(A)$.

Clearly all finite linear combinations of elements from $\{x_n\}$ belong to K(A). Since $\overline{[\{x_n\}]} = A$, it follows that $\overline{K(A)} = A$; i.e., K(A) is dense in A. ////

In view of this lemma and the remarks preceeding it, we have the following alternative description of K(A):

$$K(A) = \{x \in A : \sum_{n=1}^{\infty} |x_n^*(xy)| < \infty \text{ for all } y \in A\}.$$

Note also that $K(A) \subseteq J(A) \subseteq A$ so that $\overline{K(A)} = \overline{J(A)} = A$.

For the next theorem we need the additional assumption that A is metrizeable; i.e., A is a complete metrizeable *LC*-algebra. Algebras of this type are called B_0-*algebras* [43].

Theorem 3.30 *Let A be a B_0-algebra with identity and suppose that A has an unconditional orthogonal basis $\{x_n\}$. Then, A' is (linear space) isomorphic to the ideal K(A).*

Proof: Consider the map $T : K(A) \to A'$ defined by $y \to f_y$

where $f_y \in A'$ is defined by

$$f_y(x) = \sum_{n=1}^{\infty} x_n^*(xy) \qquad (x \in A).$$

Note that the series on the right above is convergent because $x \in K(A)$. It is clear that f_y is linear and homogeneous for each $y \in K(A)$. To show that f_y is continuous, consider

$$\begin{aligned} f_{S_n(y)}(x) &= \sum_{k=1}^{\infty} x_k^*(x) x_k^*(S_n(y)) \\ &= \sum_{k=1}^{n} x_k^*(x) x_k^*(y) \end{aligned} \qquad (*)$$

for $x \in A$ (recall that S_n is the partial sum operator $S_n(x) = \Sigma_{k=1}^n x_k^*(x) x_k$). Clearly $f_{S_n(y)}$ is a continuous linear functional for each $y \in K(A)$ since it is a finite linear combination of the continuous functionals x_k^* (these are continuous because A is an F-space). Now, let $y \in K(A)$, then from (*) we have that for each $x \in A$

$$f_{S_n(y)}(x) \to f_y(x) = \sum_{k=1}^{\infty} x_n^*(xy) \quad \text{as} \quad n \to \infty$$

Since A is a complete metrizeable space, it follows by the Banach-Steinhaus theorem that f_y is a continuous linear functional for each $y \in K(A)$, and thus defines an element of A'.

Now, suppose that $y_1, y_2 \in K(A)$, $y_1 \neq y_2$. Then, there exists $n \in \mathbb{N}$ such that $x_n^*(y_1) \neq x_n^*(y_2)$. But $f_{y_1}(x_n) = x_n^*(y_1)$ and $f_{y_2}(x_n) = x_n^*(y_2)$. Hence $f_{y_1} \neq f_{y_2}$ and this shows that T is one-to-one.

To show that T is onto A', suppose that $f \in A'$, and let $\alpha_n = f(x_n)$, $n=1,2,\ldots$ Since $e \in A$ ($e = \Sigma_{n=1}^{\infty} x_n$ by Lemma 2.13(a)) it follows that (since f is continuous)

$$f(e) = \sum_{n=1}^{\infty} f(x_n) = \sum_{n=1}^{\infty} \alpha_n$$

and this series converges. Therefore $\{\alpha_n\}$ is a bounded sequence and thus $\Sigma_{n=1}^{\infty} \alpha_n x_n$ converges in A (by Lemma 3.1(a)). So, let

$$w = \sum_{n=1}^{\infty} \alpha_n x_n .$$

Now, if $x \in$ A, then since f is continuous we have

$$f(x) = \sum_{n=1}^{\infty} x_n^*(x) f(x_n) = \sum_{n=1}^{\infty} x_n^*(x) \alpha_n$$

$$= \sum_{n=1}^{\infty} x_n^*(x) x_n^*(w) = \sum_{n=1}^{\infty} x_n^*(xw).$$

Since this last series converges for every $x \in$ A, it follows that w is an element of K(A). Clearly we have that $f = f_w$ and this shows that T is onto A'. This completes the proof. ////

In Examples 2.8 and 2.9 (Chapter II) it is easy to show that K(A) = J(A), so that in these cases A' is isomorphic (by the isomorphism T above) to the intersection of the dense maximal ideals of A (=J(A) by Theorem 3.26). In Example 2.8 J(A) is the algebra of analytic functions with radius of convergence greater than 1 (see [27]). In Example 2.9 J(A) is the algebra of finite sequences (see [39]). However, we do not know whether it is true in general (for the algebras we are considering here) that K(A) = J(A).

It was seen in Lemma 3.1(a) that if $\{\beta_n\}$ is a bounded sequence of complex numbers, then $\Sigma_{n=1}^{\infty}\beta_n x_n$ converges in A. The converse of this statement, however, is always false for the type of algebras we are considering here (i.e., complete metrizeable LC-algebras A with e having unconditional orthogonal bases). This is so because if every convergent series of the form $\Sigma_{n=1}^{\infty}\alpha_n x_n$ in A has the property that $\{\alpha_n\}$ is bounded, then a similar application of the closed graph theorem (via Theorem 1.1) as that used in the proof of theorem 2.34 shows that the map σ (see Definition 2.29) is an algebraic and topological isomorphism of A onto m, the Banach algebra of bounded sequences. This, however, is impossible since m is not separable but A is separable (since it has a basis). Therefore, for topological algebras of this type there always exists $x \in A$, $x = \Sigma_{n=1}^{\infty}\alpha_n x_n$, with $\{\alpha_n\}$ unbounded. We require a slightly stronger condition to show that the ideals K(A) and J(A) are equal.

THEOREM 3.31 *Let A be a topological algebra with an identity and let $\{x_n\}$ be an orthogonal basis in A. If there exists an invertible element $x \in A$ such that $\Sigma_{n=1}^{\infty}|x_n^*(x)| < \infty$, then $J(A) = K(A)$.*

Proof: We have already noted that the inclusion $K(A) \subseteq J(A)$ always holds. So, we show the other inclusion. Let $x \in A$ be invertible with $\Sigma_{n=1}^{\infty}|x_n^*(x)|$ convergent and let $y \in J(A)$. Then, for every $z \in A$, we have that $\lim_{n\to\infty} x_n^*(yzx^{-1}) = 0$. Since $x^{-1} = \Sigma_{n=1}^{\infty}x_n^*(x)^{-1}x_n$ (by Lemma 2.13(b)), it follows from this that $\lim_{n\to\infty} x_n^*(yz)x_n^*(x)^{-1} = 0$.

Therefore we also have

$$\lim_{n\to\infty} \left|\frac{x_n^*(yz)}{x_n^*(x)}\right| = 0 \qquad (z \in A).$$

Hence, for n sufficiently large

$$\left|\frac{x_n^*(yz)}{x_n^*(x)}\right| \leq 1;$$

i.e., $|x_n^*(yz)| \leq |x_n^*(x)|$ $(z \in A)$. Since $\Sigma_{n=1}^{\infty} |x_n^*(x)| < \infty$, it follows that the series

$$\sum_{n=1}^{\infty} x_n^*(yz)$$

converges for all $z \in A$. This shows that $y \in K(A)$. ////

Chapter IV

CYCLIC BASES

In this chapter we study F-algebras with cyclic bases. Special attention will be paid to the form of the spectrum of the element generating the basis. It turns out that the properties of this spectrum are intimately connected with certain properties of the algebra. For example we will show that the algebra is semisimple if, and only if, the spectrum of the element generating the basis has nonempty interior. In fact, if this spectrum is open (as a subset of $\mathbb{C}$) then the algebra is (algebraically and topologically) the F-algebra of holomorphic functions on the spectrum. This fact can be considered a characterization of the algebra of holomorphic functions on a simply connected domain in terms of the Taylor series expansion. We also show that an F-algebra with a cyclic basis has unique F-algebra topology. Finally, we will briefly consider topological algebras with unconditional cyclic bases.

1. The Spectrum of the Generator

We begin by giving some examples of topological algebras with cyclic bases, and developing some of the fundamental properties of these algebras. Also in this section we will describe the spectrum of the element generating the basis. The results of this section are essential to what follows.

EXAMPLE 4.1 The Banach algebra $\ell^1(\mathbb{N})$ (with convolution product) has a cyclic basis $\{z^n : n=0,1,\dots\}$ where $z = (0,1,0,0,\dots)$. This basis is an absolute basis.

EXAMPLE 4.2 The algebra H(D) of functions holomorphic on the open unit disc D with pointwise operations is an F-algebra in the compact-open topology [39]. It is an easy consequence of the Taylor series expansion that the function $z(t)=t$, $t \in$ D, generates a cyclic basis for H(D).

More generally, let Ω be any simply connected open (proper) subset of $\mathbb{C}$ and let $w=\psi(z)$ be a conformal map from Ω onto the unit disc D. The set $\{(\psi(z))^n : n=0,1,\dots\}$ forms a cyclic basis for $H(\Omega)$ because if f is holomorphic in Ω then

$$f(z) = f(\psi^{-1}(w)) = \sum_{n=0}^{\infty} \alpha_n w^n = \sum_{n=0}^{\infty} \alpha_n (\psi(z))^n$$

where the series converges in the topology of $H(\Omega)$. The uniqueness is a consequence of the Taylor series.

In this connection we mention that the algebra $H(\overline{D})$ of functions holomorphic in (a neighbourhood of) the closed unit disc $\overline{D}$ is a topological algebra with the usual inductive limit topology [17]. Moreover, it is easy to see [32] that the sequence $\{z^n : n=0,1,\dots\}$ is a basis in $H(\overline{D})$. This is clearly a cyclic basis in our sense.

EXAMPLE 4.3 The algebra $\mathbb{C}[[X]]$ of formal power series in the indeterminate X is an F-algebra in the topology of simple convergence in the coefficients [39]. The indeterminate X generates a cyclic basis for the algebra $\mathbb{C}[[X]]$.

EXAMPLE 4.4 The spaces $\ell^p(\mathbb{N})$, $0 < p < 1$, are complete metrizeable topological algebras with the convolution multiplication [42]. These algebras have cyclic bases generated by the element $z = (0,1,0,0,\ldots)$.

Other examples of topological algebras with cyclic bases can be constructed as we will see in the sequel. Now, however, we will prove some of the fundamental properties of such algebras. *Throughout this section A will denote a complete locally m-convex algebra with a cyclic Schauder basis* $\{z^n : n=0,1,\ldots\}$

LEMMA 4.5 *Let A be as above and let x,y be elements of A. If* $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$ *and* $y = \Sigma_{n=0}^{\infty}\beta_n z^n$, *then*

$$xy = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\alpha_{n-k}\beta_k\Big)z^n.$$

Proof: We will show that for every $k \in \mathbb{N}$, the coefficient functionals x_k^* have the property that

$$x_k^*(xy) = \sum_{l=0}^{k}\alpha_{k-l}\beta_l$$

and this will prove the result since we have that $xy = \Sigma_{n=0}^{\infty}x_n^*(xy)z^n$. To this end note that

$$xy = x\Big(\sum_{n=0}^{\infty}\beta_n z^n\Big) = \sum_{n=0}^{\infty}\beta_n xz^n = \sum_{n=0}^{\infty}\beta_n\Big(\sum_{i=0}^{\infty}\alpha_i z^{n+i}\Big)$$

(We have used the continuity of multiplication). Now, set $w_n = \beta_n(\Sigma_{i=0}^{\infty}\alpha_i z^{n+i})$ so that $xy = \Sigma_{n=0}^{\infty}w_n$. Since x_k^* is a continuous functional for each $k \in \mathbb{N}$, it follows that $x_k^*(xy) = \sum_{n=0}^{\infty} x_k^*(w_n)$. However, for $n > k$, w_n contains no z^k terms and so $x_k^*(w_n) = 0$ for $n > k$. It follows that

$$x_k^*(xy) = \sum_{n=0}^{\infty} x_k^*(w_n) = \sum_{n=0}^{k} x_k^*(w_n)$$

$$= \sum_{n=0}^{k} x_k^*\Big(\beta_n\Big(\sum_{i=0}^{\infty} \alpha_i z^{n+i}\Big)\Big)$$

$$= \sum_{n=0}^{k} \alpha_{k-n}\beta_n .$$

This completes the proof. ////

This representation of the product of two elements in A allows us to more easily decide when an element is invertible, and so more easily describe the spectrum of z. This we do in the next few lemmas.

LEMMA 4.6 *z is noninvertible.*

Proof: Suppose that z is invertible. It follows that there is a sequence $\{\alpha_n\}$ of scalars such that

$$z^{-1} = \sum_{n=0}^{\infty} \alpha_n z^n .$$

We have

$$e = zz^{-1} = z\Big(\sum_{n=0}^{\infty} \alpha_n z^n\Big)$$

$$= \sum_{n=0}^{\infty} \alpha_n z^{n+1} = \alpha_0 z + \alpha_1 z^2 + \ldots$$

This, however, is impossible since e (the identity of A) is itself a member of the basis and thus has the unique representation $e = 1\cdot e + 0\cdot z + 0\cdot z^2 + \ldots$. ////

This lemma shows that $0 \in \sigma(z)$.

LEMMA 4.7 *If* $\lambda \in \mathbb{C}$ *then* $z-\lambda e$ *is invertible if, and only if,* $\Sigma_{n=0}^{\infty}(z/\lambda)^n$ *converges in A. In this case*

$$(z-\lambda e)^{-1} = -\frac{1}{\lambda}\left(\sum_{n=0}^{\infty}\left(\frac{z}{\lambda}\right)^n\right).$$

Proof: ($\Longrightarrow$) Suppose that $z-\lambda e$ is invertible. If $\lambda=0$, then by Lemma 4.6 z is not invertible. Hence, assume that $\lambda\neq 0$. Now, since $\{z^n\}$ is a basis for A, there is a sequence $\{\alpha_n\}$ of scalars such that $(z-\lambda e)^{-1} = \Sigma_{n=0}^{\infty}\alpha_n z^n$. It follows that

$$\begin{aligned} e &= (z-\lambda e)\sum_{n=0}^{\infty}\alpha_n z^n \\ &= \sum_{n=0}^{\infty}\alpha_n z^{n+1} - \sum_{n=0}^{\infty}\lambda\alpha_n z^n \\ &= -\lambda\alpha_0 e + (\alpha_0-\lambda\alpha_1)z + (\alpha_1-\lambda\alpha_2)z^2 + \ldots \\ &\qquad \ldots + (\alpha_n-\lambda\alpha_{n+1})z^{n+1} + \ldots \end{aligned}$$

Because of the uniqueness of the basis representation we have that $-\lambda\alpha_0=1$ and $\alpha_0-\lambda\alpha_1 = \alpha_1-\lambda\alpha_2 = \ldots = \alpha_n-\lambda\alpha_{n+1} = \ldots = 0$. This implies that $\alpha_0 = -(1/\lambda)$ and $\alpha_{n+1} = \alpha_n/\lambda$, $n=0,1,\ldots$ By induction on n we can easily conclude that $\alpha_n = -(1/\lambda)^{n+1}$, $n=0,1,\ldots$ Therefore, $(z-\lambda e)^{-1} = \Sigma_{n=0}^{\infty}\left(\frac{z^n}{\lambda^{n+1}}\right)$.

($\Longleftarrow$) Conversely, suppose $\Sigma_{n=0}^{\infty}(z/\lambda)^n$ converges in A. Then clearly $-\Sigma_{n=0}^{\infty}(z^n/\lambda^{n+1})$ also converges in A. Now, this last series is the inverse of $z-\lambda e$ because

$$\left(-\sum_{n=0}^{\infty}\frac{z^n}{\lambda^{n+1}}\right)(z-\lambda e) = -\sum_{n=0}^{\infty}\frac{z^{n+1}}{\lambda^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{\lambda^n} = e.$$

////

LEMMA 4.8 *If $\lambda \notin \sigma(z)$ then $|\lambda| \geq \rho(z)$.*

Proof: Since $\lambda \notin \sigma(z)$, $z-\lambda e$ is invertible and therefore, by Lemma 4.7,

$$(z-\lambda e)^{-1} = -\frac{1}{\lambda}\left(\sum_{n=0}^{\infty}\left(\frac{z}{\lambda}\right)^n\right). \tag{*}$$

Now, let $f \in M(A)$ be a continuous multiplicative linear functional. Then from (*) we have

$$f((z-\lambda e)^{-1}) = -\frac{1}{\lambda} f\left(\sum_{n=0}^{\infty}\left(\frac{z}{\lambda}\right)^n\right) = -\frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{f(z)}{\lambda}\right)^n < \infty.$$

Since this last series converges we must have that $|f(z)/\lambda| < 1$. Hence $|f(z)| < |\lambda|$ for all $f \in M(A)$. It follows that $sup\ \{|f(z)|: f \in M(A)\} \leq |\lambda|$. Since $\rho(z) = sup\{|f(z)|: f \in M(A)\}$, the result follows. ////

Let $D_r = \{t \in \mathbb{C} : |t|<r\}$ and $\overline{D}_r = \{t \in \mathbb{C} : |t| \leq r\}$ be the open and closed discs of radius r about 0, respectively. Since it is clear that if $|\lambda| > \rho(z)$ then $\lambda \notin \sigma(z)$, it follows from the above lemma that the spectrum of z is contained in $\overline{D}_{\rho(z)}$ and contains $D_{\rho(z)}$; i.e., $D_{\rho(z)} \subseteq \sigma(z) \subseteq \overline{D}_{\rho(z)}$.

In the following lemma we give a different description of $\sigma(z)$ which we will need later. For this purpose, for each $x \in A$, $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$, set $S_x = \{\lambda \in \mathbb{C} : \Sigma_{n=0}^{\infty}\alpha_n\lambda^n \text{ converges}\}$.

LEMMA 4.9 *With the above notation,* $\sigma(z) = \bigcap\{S_x : x \in A\}$.

Proof: Suppose $\lambda \in \sigma(z)$. Then there exists $f \in M(A)$ such that $f(z) = \lambda$. Let $x \in A$ with $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$. Then

$$f(x) = \sum_{n=0}^{\infty} \alpha_n f(z)^n = \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

and this series converges. Thus $\lambda \in S_x$. But x was arbitrary. Therefore $\lambda \in \bigcap\{S_x : x \in A\}$.

Conversely, suppose that $\lambda \in \bigcap\{S_x : x \in A\}$. Then $f(z) = \lambda$ defines a multiplicative linear functional on A; i.e., for every $x \in A$, $x=\Sigma_{n=0}^{\infty}\alpha_n z^n$, we define $f(x) = \Sigma_{n=0}^{\infty}\alpha_n f(z)^n = \Sigma_{n=0}^{\infty}\alpha_n\lambda^n$ (this series converges by the definition of S_x). So, $f \in M^{\#}(A)$. But $\sigma(z) = \hat{z}(M^{\#}(A))$. Therefore $\lambda \in \sigma(z)$.

////

For $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$ an element of A, S_x is the domain of convergence of the power series $\Sigma_{n=0}^{\infty}\alpha_n t^n$. So, if we let r be the radius of convergence of this power series, then $D_r \subseteq S_x \subseteq \overline{D}_r$. Now, since $\sigma(z) = \bigcap_{x \in A} S_x$, it follows that there exists $s \in \mathbb{C}$ such that $D_s \subseteq \sigma(z) \subseteq \overline{D}_s$. Also, if $\rho(z) = \infty$ then each S_x must contain complex numbers of arbitrarily large moduli. But each S_x is the domain of convergence of a power series. Hence $S_x = \mathbb{C}$ for each $x \in A$. It follows that in this case $\sigma(z) = \mathbb{C}$ (by the above lemma).

In Lemma 4.6 we showed that 0 always belongs to $\sigma(z)$. We now show that if $\sigma(z)$ contains points other than 0 then A is necessarily semisimple.

THEOREM 4.10 *A is semisimple if, and only if,* $\rho(z) > 0$.

Proof: ($\Longleftarrow$) First, recall that $\mathrm{Rad}(A) = \bigcap\{ker(f) : f \in M(A)\}$ since

A is a complete locally m-convex algebra. Now, suppose that $x \in \mathrm{Rad}(A)$. We will show that $x = 0$. To this end suppose that $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$. As before, for $f \in M(A)$, we have

$$f(x) = f\left(\sum_{n=0}^{\infty}\alpha_n z^n\right) = \sum_{n=0}^{\infty}\alpha_n f(z)^n.$$

This last series is a complex power series in the complex numbers $f(z)$. Now, since $x \in \mathrm{Rad}(A)$, $f(x) = 0$ for all $f \in M(A)$; i.e., the power series (*) converges to 0 for each $f \in M(A)$. But since $\sigma(z) = \{f(z) : f \in M(A)\}$, it follows that the series $\Sigma_{n=0}^{\infty}\alpha_n t^n$ converges to zero for every $t \in \sigma(z)$. By Lemma 4.8 (and the remarks following it) the open disc $D_{\rho(z)}$ is contained in $\sigma(z)$ (because $\rho(z) > 0$). Hence the power series $\Sigma_{n=0}^{\infty}\alpha_n t^n$ is zero on $D_{\rho(z)}$, $\rho(z) > 0$. So, by the identity theorem for power series, the coefficients are all zero; i.e., $\alpha_n = 0$, $n=0,1,\ldots$ Therefore, by the uniqueness of the basis representation, $x = 0$.

($\Longrightarrow$) Conversely, suppose that $\rho(z) = 0$. Then $\sigma(z) = \{0\}$, and so, if $f \in M(A)$, $f(z) = 0$. Now, since z generates A, each continuous multiplicative linear functional is completely determined by its value at z. So, there is only one such functional, namely $f(x) = x_0^*(x)$. It follows that in this case $\mathrm{Rad}(A) \neq \{0\}$. Thus A is not semisimple. ////

2. The F-Algebra H(Ω)

Let Ω be an open subset of the complex plane $\mathbb{C}$, and let

$$H(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is analytic on } \Omega\};$$

i.e., H(Ω) is the algebra (with pointwise operations) of holomorphic functions on Ω. If this algebra is equipped with the topology of uniform convergence on compact subsets of Ω (the compact-open topology) it then becomes an F-algebra [39]. In particular, if Ω is an open disc with center at t_0, then the function $z(t)=t-t_0$, $t \in \Omega$, generates a cyclic basis for $H(\Omega)$ (see Example 4.2). We aim to show that these two properties characterize $H(\Omega)$ among F-algebras (the exact statement is Theorem 4.12). This will lead us to consider the possibility of A posessing different cyclic bases, and we will show how any two such bases are related.

For the next theorem we need to show that $\sigma(z)$ is homeomorphic to M(A). This is true in general for Banach algebras (i.e., if the algebra is generated by a single element)[44]. However, this is not true in general for F-algebras [8]. For algebras with cyclic bases we have the following result:

LEMMA 4.11 *Let A be a complete locally m-convex algebra with a cyclic basis* $\{z^n\}$. *If* $\sigma(z)$ *is open, then M(A) is homeomorphic to* $\sigma(z)$.

Proof: Consider the map $\phi: M(A) \to \sigma(z)$ defined by $\phi(f) = f(z)$. Since z generates A, it is well known and easy to show [8], that ϕ is continuous and onto. Also ϕ is one-to-one since clearly $f(z) = g(z)$ implies $f = g$. It remains to show that ϕ is open.

Let V be a subbasic neighbourhood of f_0 in the topology of M(A). Then,

$$V = V(\varepsilon, x, f_0) = \{f \in M(A) : |f(x)-f_0(x)| < \varepsilon\} \qquad (\varepsilon > 0).$$

Now, $x = \sum_{n=0}^{\infty} \alpha_n z^n$ for some sequence of scalars $\{\alpha_n\}$. Therefore, we can rewrite V as follows

$$V = \{f \varepsilon M(A) : |\sum_{n=0}^{\infty} \alpha_n f(z)^n - \sum_{n=0}^{\infty} \alpha_n f_0(z)^n| < \varepsilon\}.$$

Let $a(t)$ be the function defined on $\sigma(z)$ by $a(t) = \Sigma_{n=0}^{\infty}\alpha_n t^n$ and set $t_0 = f_0(z)$. It is clear that $a(t)$ converges for every $t \varepsilon \sigma(z)$ and since $\sigma(z)$ is open, $a(t)$ is analytic on $\sigma(z)$. It follows that $a(t)$ defines a continuous function on $\sigma(z)$. Now,

$$\phi(V) = \{f(z) : f \varepsilon V\}$$

$$= \{f(z) : |\sum_{n=0}^{\infty} \alpha_n f(z)^n - \sum_{n=0}^{\infty} \alpha_n f_0(z)^n| < \varepsilon\}$$

$$= \{t \varepsilon \sigma(z) : |a(t) - a(t_0)| < \varepsilon\}$$

$$= \{t : |a(t) - a(t_0)| < \varepsilon\} \cap \sigma(z)$$

$$= a^{-1}(D_\varepsilon(a(t_0))) \cap \sigma(z)$$

This last set is open being the intersection of two open sets. It follows that ϕ is open. ////

We now prove the theorem mentioned in the introduction to this section.

THEOREM 4.12 *Let A be an F-algebra with a cyclic basis $\{z^n\}$. If $\sigma(z)$ is open, then A is algebraically and topologically isomorphic to $H(\sigma(z))$, the F-algebra of holomorphic functions on $\sigma(z)$.*

Proof: Let $\phi:A \to \hat{A}$ be the Gelfand map. Also, for each $x \in A$, define the function $\tilde{x}$ on $\sigma(z)$ by $\tilde{x}(t) = \hat{x}(\phi^{-1}(t))$ for $t \in \sigma(z)$, where ϕ is the homeomorphism between M(A) and $\sigma(z)$ defined in Lemma 4.11. For the sake of simplicity, we set $f_t = \phi^{-1}(t)$ so that $f_t(z) = t$. Now, if $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$ then for $t \in \sigma(z)$ we have

$$\tilde{x}(t) = \hat{x}(\phi^{-1}(t)) = \hat{x}(f_t) = f_t(x)$$

$$= f_t\left(\sum_{n=0}^{\infty} \alpha_n z^n\right) = \sum_{n=0}^{\infty} \alpha_n f_t(z)^n$$

$$= \sum_{n=0}^{\infty} \alpha_n t^n.$$

Since this last series converges for every $t \in \sigma(z)$, it follows that each $\tilde{x}$ is a holomorphic function on $\sigma(z)$. Now, let $\tilde{A}$ be the algebra consisting of all the functions $\tilde{x}$ equipped with the topology of uniform convergence on compact subsets of $\sigma(z)$. It follows that $\tilde{A}$ is a subalgebra of $H(\sigma(z))$. Now, since M(A) is homeomorphic to $\sigma(z)$, it is clear from the definition of $\tilde{A}$ that the map $\Lambda:\hat{A} \to \tilde{A}$ by $\hat{x} \to \tilde{x}$ is an algebraic and topological isomorphism onto $\tilde{A}$.

Let $\Delta = \Lambda \circ \Gamma$, so Δ maps A onto $\tilde{A} \subseteq H(\sigma(z))$. We will show that Δ is one-to-one and onto $H(\sigma(z))$:

First, since $\sigma(z)$ is open, we must have that $\rho(z) > 0$. So, by Lemma 4.10, A is semisimple. It follows that the Gelfand map Γ is one-to-one. But Λ is one-to-one. Therefore Δ is one-to-one.

Now, suppose that g is a function which is holomorphic on $\sigma(z)$; i.e., $g \in H(\sigma(z))$. By the functional calculus for locally m-convex algebras there exists $y \in A$ with the property that

$$\hat{y}(f) = g(\hat{z}(f)) \qquad f \in M(A).$$

This is equivalent to

$$g(t) = \hat{y}(\phi^{-1}(t)) \qquad t \in \sigma(z).$$

It follows from the definition of $\tilde{y}$ that $g = \tilde{y}$. Therefore Δ is onto $H(\sigma(z))$.

It remains to show that Δ is bicontinuous. To show this we observe that since A is a locally convex space its topology is the topology of uniform convergence on equicontinuous subsets of its dual A' [35]. But, since A is an F-space, each compact subset of M(A) is equicontinuous. It follows that the topology of A is finer than the topology of $\hat{A}$ if we identify A and $\hat{A}$ via the algebraic isomorphism Γ. This is the same as saying that Γ is continuous. It follows that Δ is continuous being a composition of continuous maps. Thus Δ is a continuous linear map from the F-space A onto the F-space $\tilde{A}$. From the open mapping theorem we conclude that Δ is open. Therefore Δ is a topological isomorphism. This completes the proof. ////

The algebra of holomorphic functions on an open subset of $\mathbb{C}$ has been characterized in several ways. For example, Arens [3] characterized $H(\Omega)$ in terms of the existence of derivations satisfying a condition similar to the Cauchy estimate; Rudin [34] characterized $H(\Omega)$ in terms of the "maximum modulus principle"; Meyers [29] used the Montel theorem to describe the algebra $H(\Omega)$; and Birtel [5] characterizes the algebra of entire functions in terms of Liouville's theorem. In this spirit the above theorem can be considered a characterization of $H(\Omega)$ (for simply connected Ω) in terms of the Taylor series expansion for analytic functions.

Recall (see the remarks after Lemma 4.9) that if $\rho(z) = \infty$, then $\sigma(z) = \mathbb{C}$. We have the following special case of Theorem 4.12:

COROLLARY 4.13 *Let A be an F-algebra with a cyclic basis* $\{z^n\}$. *If* $\rho(z) = \infty$ *then A is algebraically and topologically isomorphic to the F-algebra* $\mathcal{E}$ *of entire functions.* ////

In connection with this we have the following:

COROLLARY 4.14 *Let A be an F-algebra with a basis* $\{z^n\}$. *If there is an* $\alpha \neq 0$ *such that* $z-\alpha e$ *also generates a cyclic basis for A, then A is algebraically and topologically isomorphic to* $\mathcal{E}$.

Proof: By the definition of $\sigma(z)$, we have

$$\begin{aligned}\sigma(z-\alpha e) &= \{f(z-\alpha e) : f \in M(A)\}\\ &= \{f(z)-\alpha : f \in M(A)\}\\ &= \sigma(z)-\alpha\end{aligned}$$

Now, by the remarks following Lemma 4.8, both $\sigma(z)$ and $\sigma(z-\alpha e)$ are discs in the plane centered about 0. Since $\alpha \neq 0$, it is clear that the only way $\sigma(z-\alpha e)$ can equal $\sigma(z)-\alpha$ is for $\sigma(z) = \mathbb{C}$. The result now follows from Corollary 4.13. ////

In the algebra $\mathcal{E}$ of entire functions the function $z(t) = t$, $t \in \mathbb{C}$, generates a cyclic basis. Since every entire function has a Taylor series expansion about any point in the plane, it follows that the function $z(t) = t - t_0$ also generates a cyclic basis for $\mathcal{E}$. Thus, a topological algebra A can have many different cyclic bases. We will now consider all the possible different cyclic bases of A and their relationship to a given cyclic basis $\{z^n\}$.

THEOREM 4.15 *Let A be a complete locally m-convex algebra with a cyclic basis $\{z^n\}$ with $0 < \rho(z) < \infty$. If $\{w^n\}$ is another cyclic basis for A, then*

$$w = \lambda(z-ae)(r^2e-\bar{a}z)^{-1}$$

where $a \in \mathbb{C}$, $|a| < r = \rho(z)$, and $|\lambda| = \rho(w)\rho(z)$.

Proof: First note that $r^2e-az = \bar{a}(\frac{r^2}{\bar{a}}e - z) = -\bar{a}(z - \frac{r^2}{\bar{a}}e)$ and $\frac{r^2}{|\bar{a}|} > \rho(z)$. Hence $r^2e-\bar{a}z$ is invertible and thus w is well defined.

Now, since $\{z^n\}$ is a basis, we must have that $w = \Sigma_{n=0}^{\infty}\beta_n z^n$ for some sequence of scalars $\{\beta_n\}$. Consider the map $\psi:\sigma(z) \to \sigma(w)$ defined by $\psi(t_f) = f(w)$, where t_f is the point of $\sigma(z)$ corresponding to the continuous multiplicative linear functional f; i.e., $t_f = f(z)$. We note the following about ψ:

(i) ψ is one-to-one: For, suppose $\psi(t_f) = \psi(t_g)$, then $f(w) = g(w)$ and this implies that $f = g$ because a continuous multiplicative linear functional is completely determined by its value at w (because w generates A).

(ii) ψ is onto: This is clear from the fact that $\sigma(w)=\{f(w):f \in M(A)\}$

(iii) ψ is analytic on $\sigma(z)°$ (° denotes interior): We have

$$\psi(t_f) = f(w) = f\left(\sum_{n=0}^{\infty} \beta_n z^n\right)$$

$$= \sum_{n=0}^{\infty} \beta_n f(z)^n = \sum_{n=0}^{\infty} \beta_n t_f^n .$$

But $\sigma(z)$ is a disc and the series is convergent on $\sigma(z)°$ and thus defines an analytic function there.

(iv) ψ^{-1} is also analytic on $\sigma(w)°$: This is shown exactly as in (iii) because of the symmetry of the situation in z and w; i.e., since $\{w^n\}$ is a basis we have $z = \Sigma_{n=0}^{\infty} \gamma_n w^n$, etc....

Now, from the Riemann open mapping theorem of complex analysis we have that $\psi(\sigma(z)°)$ is open. By this same theorem $\psi^{-1}(\sigma(w)°)$ is open. It follows that $\psi(\sigma(z)°) = \sigma(w)°$ [[because, if $\psi(\sigma(z)°)$ is properly contained in $\sigma(w)°$, then, since ψ is one-to-one, $\psi^{-1}(\sigma(w)°)$ properly contains $\sigma(w)°$ i.e., we would have $\sigma(z)° \subsetneq \psi^{-1}(\sigma(w)°) \subseteq \sigma(z)$. Thus $\psi^{-1}(\sigma(w)°)$ cannot be open contradicting the fact that ψ^{-1} is analytic]].

From Lemma 4.8 $\sigma(z)°$ and $\sigma(w)°$ are open discs centered about 0 of radii $\rho(z)$ and $\rho(w)$, respectively. By a well known corollary to Schwarz's Lemma [10,p.128] a one-to-one analytic map of the unit disc onto itself is of the form $\xi(t) = c\,\frac{t-a}{1-\bar{a}t}$, where $|c| = 1$ and $|a| < 1$. It is not hard to deduce from this that any one-to-one analytic map of a disc of radius r onto a disc of radius s (both centered at 0) is of the form $\xi(t)=\lambda\frac{t-a}{r^2-\bar{a}t}$ where $|a| < r$ and $|\lambda| = rs$. In our case $|a| < r = \rho(z)$ and $|\lambda| = \rho(w)\,\rho(w)$, and

$$\psi(t_f) = \lambda\,\frac{t_f - a}{r^2 - \bar{a}t_f}\,, \qquad t_f \in \sigma(z)$$

This implies that

$$\psi(t_f) = \lambda \frac{f(z) - a}{r^2 - \bar{a}f(z)}, \qquad f \in M(A)$$

$$= \lambda\, f(z-ae)\; f((r^2e-\bar{a}z)^{-1}) \qquad f \in M(A)$$

$$= f(\lambda\,(z-ae)\,(r^2e-\bar{a}z)^{-1}) \qquad f \in M(A).$$

Now, by definition, $\psi(t_f) = f(w)$. Hence for every $f \in M(A)$ we have from the above equation that $f(w) = f(\lambda\,(z-ae)(r^2e-az)^{-1})$. Now, since $\rho(z) > 0$, A is semisimple by Lemma 4.10. It follows that $w = \lambda\,(z-ae)(e-\bar{a}z)^{-1}$. This completes the proof. ////

Returning to the case where $\rho(z) = \infty$, we can prove a stronger result.

THEOREM 4.16 *Let A be an F-algebra with a cyclic basis $\{z^n\}$ and with $\rho(z) = \infty$. If $\{w^n\}$ is any other cyclic basis for A, then there is $\lambda \neq 0$, $a \in \mathbb{C}$ with $w = \lambda z - ae$. Conversely, if w is of this form then $\{w^n\}$ is a cyclic basis for A.*

Proof: ($\Longrightarrow$) This is proved exactly as in Theorem 4.15 after we note that any one-to-one entire function of $\mathbb{C}$ onto $\mathbb{C}$ is of the form $\xi(t) = \lambda t - a$, $\lambda, a \in \mathbb{C}, \lambda \neq 0$. This is so because by Picard's theorem any entire function which is not a polynomial assumes each complex number as value (with the possible exception of one) infinitely many times hence cannot be one-to-one. A polynomial of degree n assumes each complex number (with the possible exception of n numbers) as value n times.

Hence a one-to-one entire function must be a linear polynomial; i.e., $\xi(t) = \lambda t - a$.

($\Longleftarrow$) The fact that any such ω generates a cyclic basis for A follows from Corollary 4.13. ////

3. Unconditional Cyclic Bases

In this section we briefly consider the situation where the basis is unconditional. It will be seen that in this case $\sigma(z)$ is always either open or compact and that M(A) is always homeomorphic with $\sigma(z)$. We will also show that a function algebra with an unconditional cyclic basis is isomorphic to the Banach algebra $\ell^1(\mathbb{N})$.

THEOREM 4.17 *Let A be a complete locally m-convex algebra with an unconditional cyclic basis $\{z^n\}$. If $\sigma(z)$ has a boundary point then it is compact.*

Proof: Suppose $x \in A$, $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$. Since this series converges unconditionally, it follows that for every $f \in M(A)$, $f(x) = \Sigma_{n=0}^{\infty}\alpha_n f(z)^n$ converges absolutely (because unconditional and absolute convergence are equivalent for series of complex numbers). Now, it is clear that if a power series converges absolutely at a point b, then it converges absolutely on the circle with radius $|b|$. Suppose then that $b \in \sigma(z)$ and that b is a boundary point of $\sigma(z)$. From Lemma 4.9 we have that $\sigma(z) = \bigcap\{S_x : x \in A\}$, and so $b \in S_x$ for all $x \in A$. Hence for each $x \in A$, $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$, $\Sigma_{n=0}^{\infty}\alpha_n b^n$ converges absolutely. So, for every d with $|d| = |b|$, $\Sigma_{n=0}^{\infty}\alpha_n d^n$ converges absolutely; i.e., $\{t : |t| = |b|\} \subseteq S_x$ for every $x \in A$. Since b is a boundary point of $\sigma(z)$ it follows by

Lemma 4.8 that $|b| = \rho(z)$. So, by the remarks following Lemma 4.8 we have that $\sigma(z) = \overline{D}_{\rho(z)}$. Therefore $\sigma(z)$ is compact. ////

COROLLARY 4.18 *With A as in the theorem, $\sigma(z)$ is either open or compact.* ////

As was stated earlier, it is not always the case that the spectrum of the generator of a complete locally m-convex algebra is homeomorphic to its maximal ideal space. We show, however, that this is always true for such algebras with unconditional cyclic bases.

THEOREM 4.19 *If A is a complete locally m-convex algebra with an unconditional cyclic basis $\{z^n\}$, then M(A) is homeomorphic to $\sigma(z)$.*

Proof: By the above Corollary, $\sigma(z)$ is either open or compact. If $\sigma(z)$ is open then this is exactly Lemma 4.11.

Suppose that $\sigma(z)$ is compact and consider the map $\phi : M(A) \to \sigma(z)$ given by $f \to f(z)$. This map is one-to-one onto and continuous and thus identifies these two spaces as the same set. So, we can consider each $\hat{x}$ defined also on $\sigma(z)$ via ϕ. Now, note that since each $\hat{x}(f) = \Sigma_{n=0}^{\infty} \alpha_n f(z)^n$ is an absolutely convergent power series on $\sigma(z)$, it defines a continuous function on $\sigma(z)$ [19]. Now, the topology of M(A) is the weak topology generated by the Gelfand transforms $\hat{x}$. So, the topology of $\sigma(z)$ is stronger than the topology of M(A) (i.e., since all the $\hat{x}$'s are also continuous on $\sigma(z)$). But since ϕ is continuous, the topology of $\sigma(z)$ is weaker than the topology of M(A). Hence M(A) is homeomorphic to $\sigma(z)$. ////

We now consider absolute cyclic bases. It is well known that a Banach space with an absolute basis is isomorphic to the Banach space $\ell^1(\mathbb{N})$ [12]. We prove a similar theorem for Banach algebras with absolute cyclic bases. First, we recall that a *function algebra* is a Banach algebra in which the spectral norm is equivalent to the original norm.

THEOREM 4.20 *If A is a function algebra with an absolute cyclic basis, then A is isomorphic to the Banach algebra* $\ell^1(\mathbb{N})$.

Proof: Let $\|..\|$ and $\|..\|_s$ denote the original norm of A and the spectral norm of A, respectively. We will show (equivalently) that $(A, \|..\|_s)$ is isomorphic to $\ell^1(\mathbb{N})$. First we note that since the basis is absolute and $\|..\|_s \leq \|..\|$, it follows that for $x = \Sigma_{n=0}^{\infty}\alpha_n z^n \in A$,

$$\|\sum_{n=0}^{\infty}\alpha_n z^n\|_s \leq \sum_{n=0}^{\infty}|\alpha_n| \; \|z^n\|_s$$

$$\leq \sum_{n=0}^{\infty}|\alpha_n| \; \|z^n\| \; < \infty$$

and thus the sequence $(\alpha_n\|z^n\|_s)_{n=0}^{\infty}$ belongs to $\ell^1(\mathbb{N})$. Let $T:A \to \ell^1(\mathbb{N})$ be the map defined by $T(x) = (\alpha_n\|z^n\|_s)_{n=0}^{\infty}$ for $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$. T is clearly one-to-one. Also, T is onto because if $(\alpha_i)_{i=0}^{\infty} \in \ell^1(\mathbb{N})$ then $x = \Sigma_{n=0}^{\infty}(\alpha_i/\|z^n\|_s)\, z^n$ converges in A and $T(x) = (\alpha_i)_{i=0}^{\infty}$.

T is linear and homogeneous. To show that T is multiplicative, let $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$ and $y = \Sigma_{n=0}^{\infty}\beta_n z^n$ be elements of A. Then

$$T(xy) = T\Big(\big(\sum_{n=0}^{\infty}\alpha_n z^n\big)\big(\sum_{n=0}^{\infty}\beta_n z^n\big)\Big).$$

Thus, by Lemma 4.1 we have

$$\begin{aligned}
T(xy) &= T\Big(\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\alpha_{n-k}\beta_k\Big)\ z^n\Big)\\
&= \Big(\Big(\sum_{k=0}^{n}\alpha_{n-k}\beta_k\Big)\|z^n\|_s\Big)_{n=0}^{\infty}\\
&= \Big(\Big(\sum_{k=0}^{n}\alpha_{n-k}\beta_k\Big)\|z\|_s^n\Big)_{n=0}^{\infty}\\
&= \Big(\sum_{k=0}^{\infty}\alpha_{n-k}\|z\|_s^{n-k}\ \beta_k\|z\|_s^k\Big)_{n=0}^{\infty}\\
&= (\alpha_n\|z^n\|_s)_{n=0}^{\infty} \star (\beta_n\|z^n\|_s)_{n=0}^{\infty}\\
&= T(x)\ T(y)
\end{aligned}$$

where we have used the fact that $\|z^n\|_s = \|z\|_s^n$ and $\star$ is the multiplication in $\ell^1(\mathbb{N})$. Thus, T is an algebraic isomorphism from the Banach algebra A onto the semisimple Banach algebra $\ell^1(\mathbb{N})$ and is therefore also a topological isomorphism [44]. ////

Finally, we show that an unconditional cyclic basis in a function algebra is always an absolute basis.

THEOREM 4.21 *If A is a function algebra with an unconditional cyclic basis $\{z^n\}$, then $\{z^n\}$ is an absolute basis.*

Proof: Since $\sigma(z)$ is compact, it follows by Lemma 4.8 (and the remarks following it) that $\sigma(z) = \{t \in \mathbb{C} : |t| \leq \rho(z)\}$. Thus $\rho(z) \in \sigma(z)$. So, there is an $f \in M(A)$ with $f(z) = \rho(z)$. Let $x \in A$, $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$. Then $f(x) = \Sigma_{n=0}^{\infty}\alpha_n f(z)^n$. Since the basis is unconditional, every permutation of this series converges and therefore the series converges absolutely; i.e., $\Sigma_{n=0}^{\infty}|\alpha_n|f(z)^n < \infty$. But

$$\sum_{n=0}^{\infty}|\alpha_n|f(z)^n = \sum_{n=0}^{\infty}|\alpha_n|\rho(z)^n = \sum_{n=0}^{\infty}|\alpha_n|\rho(z^n).$$

Thus every series of the form $\Sigma_{n=0}^{\infty}\alpha_n z^n$ converges absolutely with respect to the spectral norm. Since this norm is equivalent to the original norm, it follows that the series converges absolutely. This shows that the basis $\{z^n\}$ is an absolute basis. ////

This result allows us to improve on Theorem 4.20.

COROLLARY 4.22 *If A is a function algebra with an unconditional cyclic basis, then A is isomorphic to the Banach algebra $\ell^1(\mathbb{N})$.*

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4. Uniqueness of F-Algebra Topology

Let A be a commutative semisimple F-algebra. Michael [30] has shown that if A is functionally continuous then it has unique F-algebra topology. By a theorem of Arens [3] a finitely generated F-algebra is functionally continuous. It follows that a finitely generated semisimple F-algebra has unique F-algebra topology. Now, if A is an F-algebra with a cyclic basis $\{z^n\}$ then A is finitely generated. If, moreover, $\rho(z) > 0$ then by Theorem 4.10 A is semisimple. So, it follows that in this case A has unique F-algebra topology. If $\rho(z) = 0$, then A is not semisimple (see Theorem 4.10). However, we will show that under certain conditions A has unique F-algebra topology in this case also. Our result generalizes the fact that the F-algebra $\mathbb{C}[[X]]$ of formal power series has unique

F-algebra topology.

Now, let A be a complete locally m-convex algebra having a cyclic basis $\{z^n\}$ with $\rho(z) = 0$. In this case $\sigma(z) = \{0\}$ and since $\sigma(z) = \hat{z}(M(A))$ it follows that $f(z) = 0$ for every $f \in M(A)$. (recall that $M(A) \neq \phi$, [43]). Now, since A is generated by z, every continuous multiplicative linear functional on A is completely determined by its value at z. Hence, A has a unique continuous multiplicative linear functional, namely, $f(z) = 0$. In fact, this functional is x_0^*, for if $x \in A$, then we have

$$f(x) = f\Big(\sum_{n=0}^{\infty} x_n^*(x)z^n\Big) = \sum_{n=0}^{\infty} x_n^*(x)f(z)^n = x_0^*(x).$$

It follows from this also that $ker(x_0^*) = \{x \in A : x_0^*(x) = 0\}$ is a closed maximal ideal of A.

Consider now the ideal $Az = \{xz : x \in A\}$. Clearly $Az \subseteq ker(x_0^*)$. Moreover, Az is dense in $ker(x_0^*)$ since all the polynomials in z are in Az. In the sequel we will make the assumption that Az is closed in A. This will allow us to factor z from series of the form $\Sigma_{n=1}^{\infty}\alpha_n z^n$. To show this, suppose that Az is closed, then by the above it follows that $Az = ker(x_0^*)$. Hence, if $x \in A$, $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$, and $\alpha_0 = 0$ (i.e., $x \in ker(x_0^*)$), then $x = zy$ for some $y \in A$. From the uniqueness of the basis representation it is clear that $y = \Sigma_{n=1}^{\infty}\alpha_n z^{n-1}$ and this is what we were to show. We note also here that the assumption that Az is closed in A implies that

$$Az^n = \{x \in A : x_k^*(x) = 0,\ k=0,1,\dots,n-1\}$$

[[It is clear that Az is contained in this set. Conversely, if x has the property that $x_k^*(x)=0$, $k=0,1,\dots,n-1$, then by the above we have $x=z^n y$, $y \in A$]]

We will use these facts without further mention.

Throughout this section, unless otherwise stated, A *will denote a complete locally m-convex algebra with a cyclic basis* $\{z^n\}$ *with* $\rho(z)=0$ *and* Az *closed.* We begin by considering the closed ideals of A.

LEMMA 4.23 *Let A be as above and let $x \in A$, $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$. Then x is invertible if, and only if, $\alpha_0 \neq 0$.*

Proof: If x is invertible then $0 \notin \sigma(x)$. Now, $\rho(z) = 0$ means that there is only one $f \in M(A)$; i.e., $f = x_0^*$. So, $\sigma(x) = \{x_0^*(x)\} = \{\alpha_0\}$. Therefore $\alpha_0 \neq 0$.

Conversely, if $\alpha_0 \neq 0$ then $\sigma(x) = \{\alpha_0\}$; i.e., $0 \notin \sigma(x)$, and so x is invertible. ////

For the next lemma we define the *order* of an element $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$ of A to be the smallest n for which $\alpha_n \neq 0$.

LEMMA 4.24 *Every ideal in A is of the form Az^n for some $n \in \mathbb{N}$.*

Proof: Let I be an ideal in A. Then there is an $x \in$ I, $x = \Sigma_{n=0}^{\infty}\alpha_n z^n$, which is of minimal order. Let n be the order of x. Thus we can write (see remarks at the begining of this section)

$$x = z^n\left(\sum_{i=n}^{\infty}\alpha_i z^{i-n}\right), \qquad \alpha_i \neq 0.$$

But $\Sigma_{i=n}^{\infty}\alpha_i z^{i-n}$ is invertible by Lemma 4.23. Therefore $z^n \in$ I. It follows that I = Az^n. ////

We now give a necessary and sufficient condition for every ideal of A to be closed.

THEOREM 4.25 *Every ideal of A is closed if, and only if, the basis $\{z^n\}$ is a Schauder basis.*

Proof: ($\Longrightarrow$) Suppose that every ideal of A is closed. It follows by Lemma 4.23 that Az^n is closed for each $n \in \mathbb{N}$. It is clear that $Az^{n+1} \subsetneq ker(x_n^*)$ (recall that x_n^* is the coefficient functional associated with the basis $\{z^n\}$). Therefore we can factor x_n^* as follows:

$$A \xrightarrow{\pi_n} A/Az^{n+1} \xrightarrow{\psi_n} \mathbb{C}$$

where π_n is the quotient map and ψ_n is a linear functional on the finite-dimensional space A/Az^{n+1}. It follows that ψ_n is continuous. Now $x_n^* = \psi_n \circ \pi_n$. Thus x_n^* is continuous being a composition of continuous maps. This shows that the basis $\{z^n\}$ is a Schauder basis.

($\Longleftarrow$) It is clear that $Az^n = \bigcap_{i=0}^{n-1} ker(x_i^*)$, $n \geq 1$. Thus Az^n is closed being the intersection of closed sets. ////

To show uniqueness of F-topology we consider how the ideals of A behave under any F-topology.

LEMMA 4.26 *Let τ be any F-algebra topology on A. Then, either*

(a) all the ideals of A are τ-closed, or

(b) all the ideals of A are τ-dense in A.

Proof: By Lemma 4.24 all the ideals of A are of the form Az^n for some $n \in \mathbb{N}$. Suppose that for some $k \in \mathbb{N}$, Az^k is closed. Then clearly Az^m is closed for $1 < m \leq k$. In particular, Az is a closed maximal ideal. Now $Az = ker(x_0^*)$. It follows that x_0^* is a continuous linear functional. Consider now the map $T:A \to Az$ defined by $T(x) = zx$. T is continuous, linear, one-to-one, and onto Az. Since Az is closed in A, it is an F-space. Hence by the open mapping theorem T^{-1} is continuous. Now,

$$x_k^*(x) = x_{k-1}^*(T^{-1}(x-x_0^*(x)e))$$

for $k \geq 1$. It follows by induction on k, that x_k^* is continuous for all $k \in \mathbb{N}$. By the same argument as that in the proof of Theorem 4.25 we have that each ideal Az^n is closed. So, this shows that if any one ideal of A is closed then they all are.

Now, suppose that no ideal of A is closed and let I be an ideal of A. Since either $\overline{I} = A$ or $\overline{I}$ is a closed ideal of A, it follows that in this case each ideal of A is dense in A. ////

We now improve on this lemma by ruling out the second alternative; i.e., it is impossible for all ideals of A to be dense in A. We will use methods of Allan [1]. The proof is added here for the sake of completeness.

THEOREM 4.27 *Let* τ *be any F-algebra topology on A. Then each ideal of A is* τ*-closed.*

Proof: By Lemma 4.26, either all the ideal of A are closed or they they are all dense in A. By Lemma 4.24, each ideal of A is of the form Az^n for some $n \in \mathbb{N}$. So, we will show that $\overline{Az^n} = A$, $n=1,2,\ldots$ is impossible. To this end suppose that $\overline{Az^n} = A$, $n=1,2,\ldots$ and consider the maps $f_{n,m}:A \to A$ defined for $n \leq m$, $n,m \in \mathbb{N}$, by $f_{n,m}(x) = xz^{m-n}$; i.e.,

$$f_{n,m}: A \longrightarrow A$$
$$x \rightsquigarrow xz^{m-n}$$

Clearly each $f_{n,m}$ is continuous and $f_{n,n}$ is the identity function on A.

It is also easy to check that for $n \leq m \leq k$, $f_{n,m} \circ f_{m,k} = f_{n,k}$. Thus $(A, f_{n,m})$ is an inverse limit system.

Now, for $n < m$, $f_{n,m}(A) = Az^{m-n}$, and for all $k \geq m$, $f_{n,k}(A) = Az^{k-n}$. We have that $Az^{k-n} \subseteq Az^{m-n}$, and $\overline{Az^{k-n}} = \overline{Az^{m-n}} = A$. So, for all $k \geq m$, $f_{n,k}(A)$ is dense in $f_{n,m}(A)$. By this and the fact that A is metrizeable, it follows from a theorem of Arens [3, Theorem 2.4] on inverse limits of complete metric spaces that if $X = \underline{lim}(A, f_{n,m})$ and if π_n is the $n^{\underline{th}}$ projection from X into A, then $\pi_n(X)$ is dense in $f_{n,m}(A)$ for all $m \geq n$; i.e., $\overline{\pi_n(X)} \supseteq f_{n,m}(A)$. But $\overline{f_{n,m}(A)} = A$ and so $\overline{\pi_n(X)} = A$. Hence for each $n \in \mathbb{N}$, $\pi_n(X)$ is dense in A.

Now, suppose that $\tilde{x} = (x_0, x_1, x_2, \ldots) \in X$. Then from the definition of inverse limit $x_0 = f_{0,n}(x_n)$, $n=0,1,\ldots$; i.e., $x_0 = x_n z^n$ for all $n \in \mathbb{N}$. It follows that $x_0 \in \bigcap_{n=0}^{\infty} Az^n$. Therefore, $\pi_0(X) \subseteq \bigcap_{n=0}^{\infty} Az^n$. Since we have already shown that $\pi_0(X)$ is dense in A, it follows that $\bigcap_{n=0}^{\infty} Az^n$ is also dense in A. But this is impossible since $\bigcap_{n=0}^{\infty} Az^n = \{0\}$ (and A is Hausdorff). ////

We now combine the above theorems with the closed graph theorem to prove that A has unique F-algebra topology.

THEOREM 4.28 *Let A be an F-algebra with a cyclic basis* $\{z^n\}$ *satisfying* $\hat{\rho}(z) = 0$. *If Az is closed in A, then A has unique F-algebra topology.*

Proof: Let τ be any F-algebra topology on A and let $i:(A,\tau) \to A$ be the identity map. Now, the family $\{x_n^* : n=0,1,\ldots\}$ forms a separating family of continuous linear functionals on A (they are continuous because every basis in an F-space is a Schauder basis). By Theorem 4.27, each ideal of (A,τ) is τ-closed. Therefore, just as in the proof of Theorem 4.25, the functionals $x_n^* \circ i$ are continuous for $n=0,1,\ldots$ It follows by Theorem 1.1 that i is continuous. The open mapping theorem now yields that i is a homeomorphism. ////

We have already remarked at the begining of this section that if $\rho(z) > 0$, then A has unique F-algebra topology. Combining this with the above theorem we see that this theorem is true with no restrictions on $\rho(z)$.

In Example 4.3 it was shown that the algebra $\mathfrak{F} = \mathbb{C}[[X]]$ of formal power series in the indeterminate X has a cyclic basis. The ideal $\mathfrak{F}X$ is a (in fact the) closed maximal ideal of A. It follows by the above theorem that $\mathfrak{F}$ has unique F-algebra topology. This is a well known fact.

REFERENCES

[1] Allan, G. R. *Embedding the Algebra of Formal Power Series in a Banach Algebra.* Proc. London Math. Soc. (3) 25 (1972), 329-340.

[2] Arens, R. F. *The Space L^ω and Convex Topological Rings.* Bull. Amer. Math. Soc. 52(1946), 931-935.

[3] Arens, R. F. *Dense Inverse Limit Rings.* Mich. Math. J. 5(1958), 169-182.

[4] Banach, S. *Théorie des Opérations Linéaires.* Warszawa 1932.

[5] Birtel, F. T. *Singly-generated Liouville F-algebras.* Mich. Math. J. 11(1964), 89-94.

[6] Bonsall, F. F. and J. Duncan. *Complete Normed Algebras.* Springer-Verlag, New York, 1973.

[7] Bourbaki, N. *General Topology I.* Addison-Wesley, Reading, Mass., 1966.

[8] Brooks, R. M. *On the Spectrum of Finitely-generated Locally M-Convex Algebras.* Studia Math. 29(1968), 143-150.

[9] Brooks, R. M. *A Ring of Analytic Functions.* Studia Math. 24(1964), 191-209.

[10] Conway, J. B. *Functions of One Complex Variable.* Springer-Verlag, New York, 1973.

[11] Dales, H. G. and J. P. McClure. *Completion of Normed Algebras of Polynomials.* J. Austral. Math. Soc. 20 series (A)(1975), 504-510.

[12] Day, M. M. *Normed Linear Spaces.* Springer-Verlag, New York, 1973.

[13] Enflo, P. *A Counterexample to the Approximation Problem in Banach Spaces.* Acta Math. 130(1973), 309-317.

[14] Gelfand, I. *Normierte Ringe.* Rec. Math. (Math. Sbornik) N. S. 9 (1941), 3-24.

[15] Gil de Lamadrid, J. *Bases of Tensor Products of Banach Spaces.* Pacific J. Math. 11(1961), 1281-1286.

[16] Gillman, L. and M. Jerison. *Rings of Continuous Functions.* Van Nostrand Reinhold, New York, 1960.

[17] Guichardet, A. *Leçons sur Certaines Algèbres Topologiques.* Gordon and Breach, Paris, 1967.

[18] Herstein, I. N. *Topics in Algebra.* Blaisdell Publishing Company, Waltham, Massachusetts, 1964.

[19] Hille. E. *Analytic Function Theory I.* Chelsea, New York, 1976.

[20] Husain, T. *The Open Mapping and Closed Graph Theorems in Topological Vector Spaces.* Oxford Math. Monographs, Clarendon Press, Oxford, 1965.

[21] Husain, T. *Introduction to Topological Groups.* W. B. Saunders, Philadelphia, 1966.

[22] Husain, T. and J. Liang. *Multiplicative Functionals on Frèchet Algebras with Bases.* Can. J. Math. 29 (1977), 270-276.

[23] Husain, T. and J. Liang. *Continuity of Multiplicative Linear Functionals on Frèchet Algebras with Bases.* Bull. Soc. Royale Sc. de Liege 46 (1977), 8-11.

[24] Husain, T. and S. B. Ng. *On the boundedness of Multiplicative and Positive Functionals.* J. Austral. Math. Soc. 21 (Series A) (1976), 498-503.

[25] Husain, T and S. B. Ng. *On Continuity of Algebra Homomorphisms and Uniqueness of Metric Topology.* Math. Z. 139 (1974), 1-4.

[26] Husain, T. *On Bases in Locally Convex Spaces.* Aligarh Bull. Math. 1(1971), 1-7.

[27] Köthe, G. *Topologische Lineare Raüme I.* Springer-Verlag, Berlin 1966.

[28] Marti, J. T. *Introduction to the Theory of Bases.* Springer-Verlag, New York, 1969.

[29] Meyers, W. E. *Montel Algebras on the Plane.* Canad. J. Math. 22 (1970), 116-122.

[30] Michael, E. A. *Locally Multiplicatively-Convex Topological Algebras.* Amer. Math. Soc., Memoirs 11(1952).

[31] Morris, P. D. and D. E. Wulbert. *Functional Representations of Topological Algebras.* Pacific J. Math. 22(1967), 327-337.

[32] Newns, W. F. *On the Representation of Analytic Functions by Infinite Series.* Phil. Trans. Royal Soc. London (A)245(1953), 429-468.

[33] Porcelli, P. *Linear Spaces of Analytic Functions.* Rand McNally, Chicago, 1966.

[34] Rudin, W. *Analyticity, and the Maximum Modulus Principle.* Duke Math. J. 20(1953), 449-457.

[35] Schaefer, H. H. *Topological Vector Spaces.* Springer-Verlag, New York, 1970.

[36] Schauder, M. J. *Zur Theorie Stetiger Abbildungen in Funktionalraümen.* Math. Zeitschr. 26(1927), 47-65.

[37] Singer, I. *Bases in Banach Spaces I.* Springer-Verlag, New York, 1970.

[38] Toeplitz, O. *Die Linearen Vollkommenen Raüme der Funktionentheorie.* Comment. Math. Helv. 23(1949), 222-242.

[39] Treves, F. *Topological Vector Spaces, Distributions and Kernels.* Academic Press, New York, 1967.

[40] Wilansky, A. *Functional Analysis.* Blaisdell Press, New York, 1964.

[41] Zelazko, W. *On the Algebras L_p of Locally Compact Groups.* Coll. Math. 8(1961), 115-120.

[42] Zelazko, W. *A Theorem on the Discrete Groups and Algebras L^p.* Coll. Math. 8(1961), 205-207.

[43] Zelazko, W. *Topological Algebras.* Lecture Notes, McMaster Univ., 1975.

[44] Zelazko, W. *Banach Algebras.* Elsevier, Amsterdam, 1973.