MATRIX AND B*-ALGEBRA

EIGENFUNCTION EXPANSIONS

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By

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ABSTRACT

Let M_n be the set of all n by n complex matrices and [a,b] be a finite real interval. Let $L^2(a,b;M_n)$ be the set of all functions f: $[a,b] \to M_n$ such that all entry functions belong to $L^2(a,b)$. Under the usual addition, scalar multiplication, and inner-product defined by $(F,G) = \int_a^b \pi(G^*(t)F(t))dt$, $L^2(a,b;M_n)$ is a Hilbert space. Consider the differential operator L defined by $LY = \sum_{s=o}^{m} (-1)^s [P_{m-s}(t)Y^{(s)}(t)]^{(s)}$, where the superscript (s) denotes s-times differentiation with respect to t, P_i and Y are n by n matrix functions and the domain of L is a linear subspace S of $L^2(a;b;M_n)$. Under very general hypotheses on the coefficients P_i, and an appropriate domain S given by end-point conditions, the operator L is symmetric, with finite and equal deficiency indices. The self-adjoint extensions L' of L are explicitly obtained and the following result holds: The self-adjoint operator L' has only a point spectrum, all its eigenvalues are of finite multiplicity, and every finite interval contains only a finite number of them.

The classical Inversion Theorem is also generalized to obtain a unitary mapping from $L^2(a,b;M_n)$ onto L^2_σ, where σ is an appropiate matrix distribution function.

Let now A be an $\mathfrak{M}$-algebra, that is a B^*-algebra for which there exists a positive linear functional π satisfying $\pi(x^*x) \geq 0$ and $\pi(x^*x) = 0$ if and only if $x = 0$ for all $x \in A$. Then the following result holds: Let A be an L^2-separable $\mathfrak{M}$-algebra, $[a,b]$ a finite interval and $q(x)$ an A-valued, continuous function on $[a,b]$ with spectrum $\sigma = \sigma(q(x))$ such that $\inf\{\sigma(q(x)) : x \in [a,b]\}$ is finite. Then the problem $y'' + (\lambda - q(x))y = 0$, $y(a,\lambda)\cos\mu - y'(a,\lambda)\sin\mu = 0$, $0 \leq \mu \leq \pi/2$, $y(b,\lambda) = 0$, has a non-trivial solution for only a countable number of real values λ_i of λ, and if z_i is the solution corresponding to λ_i, then for any A-valued, twice continuously differentiable function $g(x)$ on $[a,b]$ satisfying the above boundary conditions, we have $g(x) = \sum_i (g, z_i) z_i(x)$. An approximation theorem in the singular case for functions on $(-\infty,\infty)$ is also given.

To my wife

Emilia

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INTRODUCTION

The foundations for the study of "boundary value problems" for differential equations were laid by C. Sturm and J. Liouville c.1830. They studied the problem of finding solutions $y(x,\lambda)$ of the second order differential equations of the type

$$(0.1) \qquad Ly \equiv \frac{d}{dx}\left(K(x,\lambda)\frac{dy}{dx}\right) - G(x,\lambda)y = 0,$$

defined in a finite interval $[a,b]$, with K and G continuous with respect to x, and $K > 0$ for all x in $[a,b]$, which satisfy conditions of the form

$$(0.2) \qquad \begin{aligned} M\,y(a) - N\,y'(a) &= 0 \\ P\,y(b) - Q\,y'(b) &= 0. \end{aligned}$$

These conditions, because they depend on the end-points of the interval $[a,b]$, are called boundary conditions, and the problem (0.1) and (0.2) is called a boundary value problem.

In general, a solution may exist only for some values of the parameter λ, which are called the eigenvalues of the problem.

A modern formulation and study of the regular boundary value problem for differential equations of nth-order can be found in many textbooks, such as { 3 }, {10} etc. An important result that follows from this study, the "Expansion and Completeness Theorem", asserts that, under appropriate hypotheses the solutions of a regular boundary value problem, corresponding to all possible eigenvalues, constitute a basis for the Hilbert space $L^2(a,b)$.

The general theory of singular boundary value problems, i.e. when the interval is not finite or the coefficients are not smooth, was developed for the second order differential equation by H. Weyl in 1909 {20}. It can now be found in many treatises, in particular in { 3 } and {19}.

The next step was taken by I. M. Glazman and H. Kodaira who, independently, in 1949-50 developed the theory of singular differential operators on $L^2(0,\infty)$ from which they were able to give a new more general version of Weyl's Expansion and Completeness Theorem. These results can be found in { 4 }, { 5 }, { 6 } and {11}.

Regular boundary value problems for first order matrix differential equations were considered by G. D. Birkhoff and R. E. Langer (1923). The singular problem for this type of equations was solved by F. V. Atkinson in 1964, see { 2 }. In his treatment he makes use primarily

of classical analysis.

Results concerning boundary value problems on Hilbert spaces have been given by several authors, e.g. {7}, {15} and {16}. F. S. Rofe-Beketov has also made important contributions to the study of systems of differential equations by using operator theory methods. In his paper {17} several other references can be found.

The first aim of this thesis is to develop the theory of matrix differential operators of the type

$$(0.3) \qquad LY \equiv (-1)^m \frac{d^m}{dt^m}\left(P_0(t)\frac{d^m Y}{dt^m}\right) + \ldots + P_m(t)Y,$$

where the coefficients $P_0, \ldots, P_m$ and the function Y are n by n matrices belonging to a Hilbert space $L^2(a,b;M_n)$, which will be defined in Chapter 1, by using the theory of linear operators in Hilbert spaces.

The topology defined on $L^2(a,b;M_n)$ is such that for $n = 1$ the space $L^2(a,b;M_1)$ coincides with $L^2(a,b)$.

Here we obtain generalizations to $L^2(a,b;M_n)$ of some classical results in $L^2(a,b)$ and under similar hypotheses derive statements analogous to the scalar case, with complex-valued functions and constants replaced by n by n matrix-valued functions and constants.

The lack of the commutative property of multiplication in our case implies, among other things, that a generalization of the Wronskian is needed in order

to be able to obtain an explicit expression for the solution of the inhomogeneous differential equation

$$LY - \lambda Y = F,$$

where λ is complex, L is given by (0.3) and Y and F belong to $L^2(a,b;H_\mu)$. This generalization is discussed in Chapter 2.

An important advantage of confining ourselves to a symmetric operator L, defined by the expression (0.3), is that very simple explicit characterizations of all self-adjoint extensions L' of such operators can be obtained. Also in Chapter 2, by means of the generalized Wronskian, we prove that the resolvent operator R_λ of any self-adjoint extension L' is compact. Hence a straightforward application of operator theory gives the structure of the spectrum of L'.

This is used in Chapter 3 to obtain an Inversion Theorem which is not only an Expansion Theorem, as in {2}, but also a Completeness Theorem, thus generalizing the one given in {1}. The proof we give here is an extension of the one used by Levitan {12} in the classical case.

A second aim of this thesis is to obtain some expansion results in more general algebras than matrix algebras. This problem was suggested by Hille {8}. Thus in Chapter 4 we develop a theory of regular boundary

value problems for a particular class of B^*-algebras, including an Expansion Theorem for functions in these algebras. An approximation theorem for functions on $(-\infty,\infty)$ is also given.

CHAPTER 1

MATRIX DIFFERENTIAL OPERATORS

§ 1.1. Introduction. In this chapter we define the space of matrices $L^2(a,b;M_n)$ and we study some properties of differential operators with domain in this space. We also prove an existence and uniqueness theorem for solutions of a system of linear matrix differential equations which extends to matrices the theorem given for differential equations for example in {13}. It should be noted that a major fact in our case, since we are dealing with matrices, is the non-commutativity of multiplication.

§ 1.2. The space $L^2(a,b;M_n)$. Let M_n be the set of all n by n matrices with complex entries.

Definition 1.1. To each matrix $A = (a_{ij})$ in M_n we assign a matrix $A^* = (a^*_{ij})$, termed the conjugate of A, as follows:

$$a^*_{ij} = \bar{a}_{ji} \quad , \qquad \text{for } 1 \leq i,j \leq n,$$

where we denote by $\bar{a}_{ij}$ the complex conjugate of a_{ij}.

<u>Proposition 1.1</u>. The functional $\pi : M_n \to C$, defined by

$$\pi(A) = \sum_{i=1}^{n} a_{ii} \tag{1.1}$$

for each matrix $A = (a_{ij})$, $1 \leq i,j \leq n$, has the following properties:

(i) π is linear, i.e. for all complex λ, μ and A, B in M_n we have

$$\pi(\lambda A + \mu B) = \lambda\, \pi(A) + \mu\, \pi(B);$$

(ii) for all A, B in M_n

$$\pi(A^*B) = \overline{\pi(B^*A)};$$

(iii) $\pi(A^*A) \geq 0$, $\pi(A^*A) = 0$ only if $A = 0$.

<u>Proof</u>.

(i) Let $A = (a_{ij})$, $B = (b_{ij})$. Then

$$\lambda A + \mu B = (\lambda a_{ij} + \mu b_{ij}).$$

and

$$\pi(\lambda A + \mu B) = \sum_{i=1}^{n} (\lambda a_{ii} + \mu b_{ii}) =$$

$$= \lambda \sum_{i=1}^{n} a_{ii} + \mu \sum_{i=1}^{n} b_{ii} = \lambda\, \pi(a) + \mu\, \pi(B).$$

(ii) $\pi(A*B) = \pi(C)$ where $C = (c_{ij})$, $1 \leq i,j \leq n$,

and

$$c_{ij} = \sum_{k=1}^{n} a^*_{ij}\, b_{kj} \qquad \text{for all} \quad 1 \leq i,j \leq n.$$

Hence

$$\pi(A*B) = \sum_{i=1}^{n} \sum_{k=1}^{n} \bar{a}_{ki}\, b_{ki}.$$

Also

$$\pi(B*A) = \sum_{i=1}^{n} \sum_{k=1}^{n} \bar{b}_{ki}\, a_{ki}.$$

Clearly

$$\pi(B*A) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ki}\, a_{ki} = (A*B).$$

(iii) By taking $B = A^*$ in (ii), we have

$$\pi(A^*A) = \sum_{i=1}^{n} \sum_{k=1}^{n} \bar{a}_{ki}\, a_{ki} = \sum_{i,k=1}^{n} |a_{ki}|^2 > 0$$

if and only if $A \neq 0$.

<u>Definition 1.2</u>. Let (a,b) be an open or closed real interval. A function $F(t)$ from (a,b) to M_n is said to be continuous, differentiable, Lebesgue measurable, square integrable, etc., if and only if each of the functions $(F(t))_{ij}$, $a < t < b$, $1 \leq i,j \leq n$, is continuous, differentiable, Lebesgue measurable, square integrable, etc., respectively, on (a,b).

We shall denote by $L^2(a,b;M_n)$ the set of all square integrable functions $F:(a,b) \to M_n$.

<u>Theorem 1.1</u>. The set $L^2(a,b;M_n)$ becomes a linear space over the complex field if for each pair of elements F, G of the set and each complex number λ we define

$$\{(F + G)(t)\}_{ij} = \{F(t)\}_{ij} + \{G(t)\}_{ij},$$

$$\{(\lambda F)(t)\}_{ij} = \lambda \{F(t)\}_{ij},$$

for all $1 \leq i,j \leq n$.

The function $(\ ,\ ): L^2(a,b;M_n) \times L^2(a,b;M_n) \to C$ defined by

$$(1.2) \qquad (F,G) = \int_a^b \pi\{G^*(t)F(t)\}dt$$

is an inner-product and the space $L^2(a,b;M_n)$ is complete under the associated norm.

<u>Proof</u>. The linearity is obvious from the definition. To show that (1.2) is an inner-product we make use of Proposition 1.1.

$$\text{(i)} \qquad (\lambda F_1 + \mu F_2, G) = \int_a^b \pi\{G^*(t)(\lambda F_1(t) + \mu F_2(t)\}dt =$$

$$= \int_a^b \pi\{\lambda G^*(t)F_1(t) + \mu G^*(t)F_2(t)\}dt =$$

$$= \lambda\int_a^b \pi\{G^*(t)F_1(t)\}dt + \mu\int_a^b \pi\{G^*(t)F_2(t)\}dt =$$

$$\lambda(F_1,G) + \mu(F_2,G)$$

for all F_1, F_2, G in $L^2(a,b;M_n)$ and all λ,μ in C.

(ii) $$(F,G) = \int_a^b \pi\{G^*(t)F(t)\}dt = \int_a^b \overline{\pi\{F^*(t)G(t)\}}dt =$$

$$= \overline{\int_a^b \pi\{F^*(t)G(t)\}dt} = \overline{(G,F)},$$

for all F,G in $L^2(a,b;M_n)$;

(iii)

$$(F,F) = \int_a^b \pi\{F^*(t)F(t)\}dt > 0$$

if and only if $F(t) \neq 0$ a.e. on (a,b).

To show the completeness, let $\{F_p\}_1^\infty$ be a Cauchy sequence. Since

$$||F_p - F_q||^2 = \int_a^b \pi\{(F_p-F_q)^*(t)(F_p-F_q)(t)\}dt =$$

$$= \sum_{i,j=1}^n ||(F_p-F_q)_{ij}||^2_{L^2(a,b)},$$

it follows that $(F_p)_{ij}$ is Cauchy for all $1 \leq i,j \leq n$.

Therefore, there exists

$$F_{ij} = \lim_{p \to \infty} (F_p)_{ij}$$

in the $L^2(a,b)$-norm and, given $\varepsilon > 0$ there exists δ_{ij} such that

$$||F_{ij} - (F_p)_{ij}||^2 < \frac{\varepsilon}{n^2} \quad \text{for all } p > \delta.$$

Let us define $F = (F_{ij})$. Then, given $\varepsilon > 0$, there exists $\delta = \max \delta_{ij}$ such that for all $p > \delta$,

$$||F - F_p||^2 = \sum_{i,j=1}^{n} ||F_{ij} - (F_p)_{ij}||^2 < \sum_{i,j=1}^{n} \frac{\varepsilon}{n^2} = \varepsilon.$$

Thus $F = \lim_{p \to \infty} F_p$ in the $L^2(a,b;M_n)$-norm and this space is complete.

<u>Remark</u>. $L^2(a,b;M_n)$ is thus a Hilbert space and we shall denote it by H in chapters 1, 2 and 3.

§ 1.3. <u>An existence and uniqueness theorem</u>. Before defining differential operators in H, we first have to study the existence and uniqueness of solutions of differential equations of the type

$$\ell(Y) = F,$$

where the differential expression $\ell(Y)$ is given by

$$(1.3) \qquad \ell(Y) = \sum_{s=o}^{m} (-1)^s \{P_{m-s}(t)Y(t)^{(s)}\}^{(s)},$$

where the superscript (s) denotes s-times differentiation with respect to t, and Y, P_i for $0 \leq i \leq m$ and F are n by n matrix functions of t over an interval (a,b).

In all that follows we assume that P_o^{-1} exists on (a,b) and that $P_o^{-1}, P_1, \ldots, P_m$ are all locally Lebesgue integrable on (a,b).

Following {1} we formally introduce the so-called "quasi-derivatives" as follows:

$$(1.4) \qquad \begin{aligned} Y^{\{o\}} &= Y \\ Y^{\{k\}} &= \frac{d^k Y}{dt^k}, \quad k = 1, \ldots, m-1, \\ Y^{\{m\}} &= P_o \frac{d^m Y}{dt^m}, \\ Y^{\{m+k\}} &= P_k \frac{d^{m-k} Y}{dt^{m-k}} - \frac{dY^{\{m+k-1\}}}{dt}, \quad k=1,\ldots,m. \end{aligned}$$

With this notation, (1.3) can be written as

$$\ell(Y) = Y^{\{2m\}}.$$

From the next theorem it follows that the set of all

functions Y in H, such that $Y^{(k)}$, $0 \leq k \leq 2m-1$, is absolutely continuous and $Y^{(2m)}$ is in H, is not empty.

<u>Theorem 1.2</u>. Let $A_{ij}(t)$, $1 \leq i,j \leq p$, $F_k(t)$, $1 \leq k \leq p$, be n by n Lebesgue measurable matrix functions, locally integrable on (a,b), and let C_s, $1 \leq s \leq p$, be arbitrary n by n constant matrices. Then the linear system of matrix differential equations

$$\frac{dY_1}{dt} = A_{11} Y_1 + \dots + A_{1p} Y_p + F_1$$

$$(1.5) \qquad \dots\dots\dots\dots$$

$$\frac{dY_p}{dt} = A_{p1} Y_1 + \dots + A_{pp} Y_p + F_p$$

has a unique solution of n by n matrix functions $Y_1, \dots, Y_p$ in (a,b) which satisfies at a given point $t_o \in (a,b)$ the initial conditions

$$(1.6) \qquad Y_s(t_o) = C_s \qquad \text{for } 1 \leq s \leq p.$$

<u>Proof</u>. Passing to vector notation, equation (1.5) can be written as

$$(1.7) \qquad \frac{d\bar{Y}}{dt} = A\bar{Y} + \bar{F},$$

where

$$\vec{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_p \end{pmatrix}, \qquad \vec{F} = \begin{pmatrix} F_1 \\ \vdots \\ F_p \end{pmatrix}$$

and

$$A = \begin{pmatrix} A_{11} & \cdots\cdots & A_{1p} \\ \cdots & \cdots\cdots & \cdots \\ A_{p1} & \cdots\cdots & A_{pp} \end{pmatrix}.$$

Then equation (1.7) has a unique solution if and only if the integral equation

$$\vec{Y}(t) = \vec{C} + \int_{t_o}^{t} \{A(\xi)\vec{Y}(\xi) + \vec{F}(\xi)\}\, d\xi, \tag{1.8}$$

where

$$\vec{C} = \begin{pmatrix} C_1 \\ \vdots \\ C_p \end{pmatrix},$$

has a unique solution.

The method of successive approximations can now be applied to (1.8). We define

$$\begin{aligned} \vec{Y}_o(t) &= \vec{C} \\ \vec{Y}_{s+1}(t) &= \vec{C} + \int_{t_o}^{t} \{A(\xi)\vec{Y}_s(\xi) + \vec{F}(\xi)\}d\xi, \end{aligned} \tag{1.9}$$

for all $t \in (a,b)$ and $s = 0, 1, \ldots$.

Obviously all the elements in the sequence $\{Y_s\}_1^\infty$ are absolutely continuous functions on (a,b).

For every p-dimensional vector $\vec{X}$ with components $X_k = (x_{ij}^k)$ in M_n we define the norm

$$|\vec{X}| = \max_{1\le k\le p} ||X_k||, \qquad \text{where} \quad ||X_k|| = \max_{1\le i,j\le n} |x_{ij}^k|,$$

and for every matrix A whose entries A_{rs} are matrices in M_n we define

$$|A| = \max_{1\le r,s\le p} ||A_{rs}|| , \qquad \text{where} \quad ||A_{rs}|| = \max_{1\le i,j\le n} |a_{ij}^{rs}| .$$

Consider $t \ge t_o$. Let

$$Q_s(t) = \sup_{t_o<\xi<t} |\vec{Y}_{s+1}(\xi) - \vec{Y}_s(\xi)|, \quad s = 0, 1, \ldots,$$

and

$$q(t) = \int_{t_o}^{t} |A(\xi)| d\xi.$$

From (1.9) we have that

$$\vec{Y}_{s+1}(t) - \vec{Y}_s(t) = \int_{t_o}^{t} A(\xi)\{\vec{Y}_s(\xi) - \vec{Y}_{s-1}(\xi)\}d\xi$$

for all $s = 1, 2, \ldots$.

We shall prove by induction that

$$Q_s(t) \leq \frac{q^s(t)}{s!} Q_o(t). \tag{1.10}$$

Indeed for $s = 1$ we have

$$|\vec{Y}_2(u) - \vec{Y}_1(u)| \leq \int_{t_o}^{t} |A(\xi)| |\vec{Y}_1(\xi) - \vec{Y}_o(\xi)| d\xi \leq$$

$$\leq Q_o(u) \int_{t_o}^{t} |A(\xi)| d\xi = Q_o(u)\, q(u) \leq Q_o(t)\, q(t),$$

for all $t_o \leq u \leq t$.

Thus

$$Q_1(t) = \sup_{t_o \leq u \leq t} |\vec{Y}_2(u) - \vec{Y}_1(u)| \leq Q_o(t)\, q(t).$$

Suppose that (1.10) is true for $s = k$. Then

$$|\vec{Y}_{k+2}(u) - \vec{Y}_{k+1}(u)| \leq \int_{t_o}^{u} |A(\xi)| Q_k(\xi) d\xi \leq$$

$$\leq \int_{t_o}^{t} |A(\xi)| \frac{q^k(\xi)}{k!} Q_o(\xi) d\xi \leq \frac{Q_o(u)}{k!} \int_{t_o}^{u} |A(\xi)| q^k(\xi) d\xi \leq$$

$$\leq \frac{Q_o(u)}{k!} \int_{t_o}^{u} q^k(\xi)\, d\int_{t_o}^{\xi} |A(\nu)|\, d\nu = \frac{Q_o(u)}{k!} \int_{t_o}^{u} q^k(\xi)\, dq(\xi) =$$

$$= \frac{Q_o(u)}{k!} \frac{q^{k+1}(u)}{k+1} \leq \frac{Q_o(t)}{(k+1)!} q^{k+1}(t).$$

Hence

$$Q_{k+1}(t) = \sup_{t_o \leq u \leq t} |\vec{Y}_{k+2}(u) - \vec{Y}_{k+1}(u)| \leq \frac{Q_o(t)}{(k+1)!} q^{k+1}(t),$$

and (1.10) holds for all $s = 1, 2, \ldots$.

Since

$$|\vec{Y}_{s+1}(t) - \vec{Y}_s(t)| \leq Q_s(t) \leq \frac{q^s(t)}{s!} Q_o(t)$$

and the series

$$Q_o(t) + \frac{q^1(t)}{1!} Q_o(t) + \ldots + \frac{q^s(t)}{s!} Q_o(t) + \ldots$$

is absolutely and uniformly convergent on $t_o \leq u \leq t$, it follows that

$$\vec{Y}_o(t) + \{\vec{Y}_1(t) - \vec{Y}_o(t)\} + \{\vec{Y}_2(t) - \vec{Y}_1(t)\} + \ldots$$

is also absolutely and uniformly convergent on the same interval. If we denote its sum by $\vec{Y}(t)$ then

$$\vec{Y}(t) = \lim_{s \to \infty} \vec{Y}_s(t),$$

and hence by the uniform convergence on $t_o \leq u \leq t$ we can take limits in (1.9) to obtain

$$\vec{Y}(u) = \vec{C} + \int_{t_o}^{u} \{A(\xi)\vec{Y}(\xi) + \vec{F}(\xi)\}d\xi \ , \ t_o \leq u \leq t < \infty \quad ,$$

so the existence of a solution is proved.

Suppose that we have two different solutions $\vec{Y}(t)$ and $\vec{Z}(t)$ of (1.8). Then, by substraction,

$$\vec{Y}(t) - \vec{Z}(t) = \int_{t_o}^{t} A(\xi)\{\vec{Y}(\xi) - \vec{Z}(\xi)\}d\xi.$$

Let

$$t_1 = \inf \{t : |\vec{Y}(t) - \vec{Z}(t)| \neq 0\}.$$

Then $\vec{Y}(t) = \vec{Z}(t)$ on $t_o \leq t \leq t_1$. Put

$$p(t) = \max_{t_1 \leq \xi \leq t} |\vec{Y}(\xi) - \vec{Z}(\xi)|.$$

Clearly, for $t > t_1$

$$0 < p(t) \leq \int_{t_1}^{t} |A(\xi)| p(t) d\xi$$

and therefore

$$1 \leq \int_{t_1}^{t} |A(\xi)|d\xi.$$

We have a contradiction since we can take t sufficiently near t_1 to obtain

$$\int_{t_1}^{t} |A(\xi)|d\xi < 1.$$

Thus the solution of (1.8), and hence of (1.5), is unique.

<u>Note</u>: By considering the components in the matrix system (1.5), Theorem 1.2 can be viewed as an existence and uniqueness theorem for systems of ordinary linear first order differential equations, and could thus be proved as in Theorem 1, {13,p51}.

<u>Theorem 1.3</u>. Let $P_o, \ldots, P_m, F$ be n by n matrix functions on (a,b) such that $P_o^{-1}, P_1, \ldots, P_m, F$ are Lebesgue measurable, locally integrable on (a,b), and let C_k, $0 \leq k \leq 2m-1$ be arbitrary constant matrices.

Then the matrix differential equation

$$\ell(Y) - \lambda Y = F \quad , \tag{1.11}$$

where λ is a complex number and $\ell(Y)$ is defined by (1.3), has a unique solution $Y(t)$ on (a,b) that

satisfies the initial conditions

$$Y^{\{k\}}(t_o) = C_k$$

for all $0 \leq k \leq 2m-1$ and given $t_o \in (a,b)$.

<u>Proof</u>. Equation (1.11) can be written as

$$Y^{\{2m\}} - \lambda Y = F$$

and therefore by (1.4) is equivalent to the system:

$$\frac{dY^{\{0\}}}{dt} = Y^{\{1\}}$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{dY^{\{m-1\}}}{dt} = P_o^{-1} Y^{\{m\}}$$

(1.12)
$$\frac{dY^{\{m\}}}{dt} = P_1 Y^{\{m-1\}} - Y^{\{m+1\}}$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{dY^{\{2m-1\}}}{dt} = P_m Y^{\{0\}} - \lambda Y^{\{0\}} - F.$$

If we denote by $\vec{Y}$ the vector solution

$$\vec{Y} = \begin{pmatrix} Y^{\{0\}} \\ \\ Y^{\{2m-1\}} \end{pmatrix}$$

then the system (1.12) can be written as

$$\vec{Y}' = A\vec{Y} + \vec{B},$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & & & & 0 & 0 \\ & \ddots & \ddots & \ddots & & \ddots & \ddots & \\ & & \ddots & 1 & 0 & 0 & & \\ & & & 0 & P_o^{-1} & & & \\ & & & P_1 & 0 & -1 & & \\ & \ddots & \ddots & & & \ddots & \ddots & \\ 0 & \ddots & & & & & \ddots & -1 \\ P_m-\lambda & & & & & & & 0 \end{pmatrix}$$

where the entry P_o^{-1} occurs in the mth-row and (m+1)th-column, and $\vec{B}$ is the vector

$$\vec{B} = \begin{pmatrix} 0 \\ \vdots \\ -F \end{pmatrix}$$

The hypotheses of Theorem 1.2 are fulfilled. Hence the system (1.12) and, as a consequence the matrix differential equation (1.11), have a unique solution $Y(t)$ satisfying the initial conditions

$$Y^{\{k\}}(t_o) = C_k, \qquad 0 \leq k \leq 2m-1.$$

<u>Definition 1.3</u>. Consider the solutions Y_k, $0 \leq k \leq 2m-1$, of the matrix differential equation

$$\ell(Y) - \lambda Y = 0, \quad \lambda \text{ complex}, \tag{1.13}$$

that satisfy the initial conditions at $t = t_o$, $a \leq t_o \leq b$,

$$Y_k^{\{j\}}(t_o) = I \cdot \delta_{jk}, \quad 0 \leq j \leq 2m-1, \tag{1.14}$$

where I is the n by n identity matrix and δ_{ij} the Kronecker delta, i.e. $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$. The set $Y_o, \ldots, Y_{2m-1}$ of these solutions of (1.13) is called the fundamental system at t_o of the differential equation (1.13).

The following theorem follows immediately from the linearity of (1.13) and the above definition.

<u>Theorem 1.4</u>. The general solution $Y(t)$ of (1.13) can

be written as

$$Y(t) = Y_o(t)\, C_o + \dots + Y_{2m-1}(t)\, C_{2m-1},$$

where $Y_o, \dots, Y_{2m-1}$ is the fundamental system at t_o of (1.13) and $C_o, \dots, C_{2m-1}$ are arbitrary constant n by n matrices.

<u>Definition 1.4</u>. Let E_{ij}, $1 \le i,j \le n$ be n by n matrices with entries

$$(1.15) \qquad (E_{ij})_{pq} = \delta_{ip}\, \delta_{jq}, \quad 1 \le p,q \le n.$$

If $Y_k(t)$, $0 \le k \le 2m-1$, is a fundamental system at t_o of (1.13), then we set

$$(1.16) \qquad Z_{kij}(t) = Y_k(t)\, E_{ij}.$$

<u>Remark</u>. From (1.14) we have that

$$Z_{kij}^{\{s\}}(t_o) = Y_k^{\{s\}}(t_o)\, E_{ij} = I\delta_{sk}E_{ij} = \delta_{sk}E_{ij}.$$

<u>Theorem 1.5</u>. The set of all solutions of (1.13) forms a linear manifold of dimension $2mn^2$ over the complex field, and the set Z_{kij}, $1 \le i,j \le n$, $0 \le k \le 2m-1$, is

one of its bases.

<u>Proof</u>. If $C_k = (c_{k,ij})$, $1 \leq i,j \leq n$, for all $k = 0, \ldots, 2m-1$, then from Theorem 1.4 and Definition 1.4 it follows that if $Y(t)$ is a solution of (1.13),

$$Y(t) = \sum_{k=0}^{2m-1} Y_k(t)\, C_k = \sum_{k=0}^{2m-1} \sum_{i,j=1}^{n} Y_k(t)\, E_{ij}\, c_{k,ij} =$$

$$= \sum_{\substack{1 \leq i,j \leq n \\ 0 \leq k \leq 2m-1}} Z_{kij}\, c_{k,ij} ,$$

and the representation is clearly unique.

§ 1.4. <u>The Lagrange identity and the matrix differential operator L</u>. In the sequel $[a,b]$ will be a finite interval and the coefficients $P_0, \ldots, P_m$, in addition to satisfying the conditions given in Theorem 1.3, will be assumed to be hermitian, with P_0 positive on $[a,b]$.

<u>Definition 1.5</u>. We shall denote by T the linear operator in $H = L^2(a,b;M_n)$ defined by

$$(1.17) \qquad T\,Y = \ell(Y),$$

whose domain $\mathcal{D}(T)$ is the set of all functions Y in H

for which $\ell(Y)$ is defined and belongs to H, i.e. $\mathcal{D}(T)$ is the set of all Y in H such that $Y^{\{k\}}$, for $k = 0, \ldots, 2m-1$, is absolutely continuous and $Y^{\{2m\}}$ is in H. We shall denote by L the restriction of T to the set

$$(1.18) \qquad \{Y \in \mathcal{D}(T) : Y^{\{k\}}(a) = Y^{\{k\}}(b) = 0,\ 0 \leq k \leq 2m-1\}.$$

The next theorem is the basis for the development of the theory of linear matrix differential operators in H.

<u>Theorem 1.6. (Lagrange identity)</u>. Let Y and Z be two n by n matrix functions in $\mathcal{D}(T)$. Then we have

$$(1.19) \qquad Z^* \ell(Y) - \ell(Z)^* Y = \frac{d}{dt}[Y,Z],$$

where

$$(1.20) \qquad [Y,Z] = \sum_{k=1}^{m} \{Z^{\{2m-k\}*} Y^{\{k-1\}} - Z^{\{k-1\}*} Y^{\{2m-k\}}\}.$$

<u>Proof</u>. From (1.4) we have

$$\frac{dY^{\{2m-k\}}}{dt} = P_{m-k+1} Y^{\{k-1\}} - Y^{\{2m-k+1\}},\ k=1,\ldots, m,$$

$$\frac{dY^{\{k-1\}}}{dt} = Y^{\{k\}}, \qquad k = 1,\ldots, m-1,$$

$$\frac{dY^{\{m-1\}}}{dt} = P_o^{-1}\, Y^{\{m\}} .$$

Hence

$$\frac{d}{dt}\,[Y,Z] = \sum_{k=1}^{m} \{(Z^{\{k-1\}*}\, P_{m-k+1} - Z^{\{2m-k+1\}*})\, Y^{\{k-1\}} -$$

$$- Z^{\{k-1\}*}(P_{m-k+1}\, Y^{\{k-1\}} - Y^{\{2m-k+1\}})\} +$$

$$+ \sum_{k=1}^{m-1} \{Z^{\{2m-k\}*}\, Y^{\{k\}} - Z^{\{k\}*}\, Y^{\{2m-k\}}\} +$$

$$+ Z^{\{m\}*}\, P_o^{-1}\, Y^{\{m\}} - Z^{\{m\}*}\, P_o^{-1}\, Y^{\{m\}} =$$

$$= Z^{\{0\}*}\, Y^{\{2m\}} - Z^{\{2m\}*}\, Y^{\{0\}} = Z^{*}\, \ell(Y) - \ell(Z)^{*}\, Y.$$

The following corollary can be considered as the integral form of the Lagrange identity.

<u>Corollary</u>. Let Y and Z be in $\mathcal{D}(T)$. Then,

$$(1.21) \qquad (TY,Z) - (Y,TZ) = \pi[Y,Z]_a^b,$$

where we define

$$[Y,Z]_a^b = [Y,Z](b) - [Y,Z](a).$$

Proof. This follows immediately from the definition of π, (1.19) and (1.2).

The following two theorems are needed to establish an important relationship between the operators L and T.

Theorem 1.7. Let $F \in \mathcal{H}$. The equation

$$\ell(Y) = F \tag{1.22}$$

has a solution $Y(t)$ in $\mathcal{D}(L)$ if and only if F is orthogonal to the linear manifold $\mathcal{N}$ of solutions of the homogeneous equation $\ell(Y) = 0$.

Proof. Let $Y(t)$ be the solution of (1.22) defined by the initial conditions

$$Y^{\{k\}}(a) = 0, \quad k = 0,\ldots, 2m-1. \tag{1.23}$$

Let $Y_0,\ldots, Y_{2m-1}$ be a fundamental system at b of $\ell(Y) = 0$. From Theorem 1.5 we know that Z_{kij}, $1 \le i,j \le n$, $0 \le k \le 2m-1$, is a basis of $\mathcal{N}$. Thus $F \in \mathcal{N}^{\perp}$ is equivalent to $(F, Z_{kij}) = 0$ for all $1 \le i,j \le n$,

$0 \leq k \leq 2m-1$.

The Lagrange identity (1.21) applied to Y and Z_{kij} gives

$$(F, Z_{kij}) = (TY, Z_{kij}) = (Y, TZ_{kij}) + \pi[Y, Z_{kij}]_a^b =$$

$$= \pi[Y, Z_{kij}](b)$$

for all $1 \leq i,j \leq n$, $0 \leq k \leq 2m-1$, using $TZ_{kij} = 0$ and (1.23).

But by (1.20)

$$\pi[Y, Z_{kij}](b) = \pi \sum_{s=1}^{m} \{Z_{kij}^{\{2m-s\}*}(b)\, Y^{\{s-1\}}(b) -$$

$$- Z_{kij}^{\{s-1\}*}(b)\, Y^{\{2m-s\}}(b)\} =$$

$$\pi \sum_{s=1}^{m} \{\delta_{k,2m-s}\, E_{ji}\, Y^{\{s-1\}}(b) - \delta_{k,s-1}\, E_{ji}\, Y^{\{2m-s\}}(b)\} =$$

$$= \begin{cases} \pi \sum_{s=1}^{m} -\delta_{k,s-1} E_{ji} Y^{\{2m-s\}}(b) = -\pi(E_{ji} Y^{\{2m-k-1\}}(b)), & 0 \leq k \leq m-1, \\ \pi \sum_{s=1}^{m} \delta_{k,2m-s} E_{ji} Y^{\{s-1\}}(b) = \pi(E_{ji} Y^{\{2m-k-1\}}(b)), & m \leq k \leq 2m-1. \end{cases}$$

If we denote by $(Y^{\{2m-k-1\}}(b))_{pq}$ the entry in the pth-row and qth-column of the matrix $Y^{\{2m-k-1\}}(b)$, then we have

$$\pi[Y,Z_{kij}](b) = \begin{cases} -(Y^{\{2m-k-1\}}(b))_{ij} & \text{for } 0 \le k \le m-1 \\ (Y^{\{2m-k-1\}}(b))_{ij} & \text{for } m \le k \le 2m-1, \end{cases}$$

since $\pi(E_{ji}(a_{pq})) = a_{ij}$, $1 \le p,q \le n$.

Hence

$$(1.24) \qquad (F,Z_{kij}) = \begin{cases} -(Y^{\{2m-k-1\}}(b))_{ij} & \text{for } 0 \le k \le m-1 \\ (Y^{\{2m-k-1\}}(b))_{ij} & \text{for } m \le k \le 2m-1, \end{cases}$$

and the statement of the theorem follows.

<u>Remark</u>. Theorem 1.7 can also be formulated as follows

$$(1.25) \qquad H = R(L) \oplus N,$$

where $R(L)$ is the range of L and $\oplus$ the direct sum for subspaces. Here N is closed since it is of finite dimension.

<u>Theorem 1.8.</u> For arbitrary matrices $A_0, \ldots, A_{2m-1}$ and $B_0, \ldots, B_{2m-1}$, there is a function $Y \in \mathcal{D}(T)$ satisfying the boundary conditions

$$(1.26) \qquad Y^{\{k\}}(a) = A_k, \qquad Y^{\{k\}}(b) = B_k,$$

for $k = 0, \ldots, 2m-1$.

<u>Proof</u>. Let Z_{kij} be as in (1.16) and let F be a function in N satisfying

$$(1.27) \qquad (F, Z_{kij}) = \begin{cases} -(B_{2m-k-1})_{ij} & \text{for } 0 \leq k \leq m-1 \\ (B_{2m-k-1})_{ij} & \text{for } m \leq k \leq 2m-1. \end{cases}$$

This function can be found as follows: Let

$$F = \sum_{r,p,s} \alpha_{rps} Z_{rps}.$$

Then for each k, i, j,

$$(F, Z_{kij}) = \sum_{r,p,s} \alpha_{rps} (Z_{rps}, Z_{kij}),$$

and the linear system

$$\sum_{r,p,s} \alpha_{rps}(Z_{rps}, Z_{kij}) = \begin{cases} -(B_{2m-k-1})_{ij}, & 0 \le k \le m-1 \\ (B_{2m-k-1})_{ij}, & m \le k \le 2m-1, \end{cases}$$

for all $1 \le i,j \le n$, has a unique solution, since the determinant of the coefficients is the Gram determinant of the basis $\{Z_{kij}\}$.

Let V be the solution of (1.22) which satisfies the initial conditions

$$V^{\{k\}}(a) = 0, \qquad k = 0, \ldots, 2m-1. \tag{1.28}$$

Then, from (1.22), (1.24), (1.27) and (1.28) we have that

$$V^{\{k\}}(b) = B_k, \qquad k = 0, \ldots, 2m-1. \tag{1.29}$$

Similarly, there is a solution U of (1.22) such that

$$U^{\{k\}}(a) = A_k, \quad U^{\{k\}}(b) = 0, \quad k = 0, \ldots, 2m-1. \tag{1.30}$$

Then (1.28), (1.29) and (1.30) imply that the function $Y = U + V$ belongs to $\mathcal{D}(T)$ and satisfies (1.26).

The next theorem lists some properties of the operators L and T.

<u>Theorem 1.9</u>. (i) The domain $\mathcal{D}(L)$ is dense in H; (ii) $L^* = T$; (iii) $L = L^{**}$, i.e. L is closed.

<u>Proof</u>. (i) Let $G \in \mathcal{D}(L)^{\perp}$, i.e. $(G,Y) = 0$ for all $Y \in \mathcal{D}(L)$. Let Z be a solution of $\ell(Z) = G$. Then, by the Lagrange identity, and remembering that $L = T\big|_{\mathcal{D}(L)}$ and that $\pi[Y,Z]_a^b = 0$, since $Y^{\{k\}}(a) = Y^{\{k\}}(b) = 0$ for all $k = 0, \ldots, 2m-1$, we have that

$$(TY,Z) - (Z,LY) = \pi[Y,Z]_a^b = 0$$

and hence

$$(Z,LY) = (TZ,Y) = (G,Y) = 0.$$

Thus $Z \in R(L)^{\perp}$ and by (1.25) $Z \in M$. But then $LZ = 0$ and therefore $G = 0$.

(ii) Clearly $T \subset L^*$. Indeed, let $Y \in \mathcal{D}(T)$. Then, by the Lagrange identity,

$$(TY,Z) - (Y,LZ) = \pi[Y,Z]_a^b = 0$$

for all $Z \in \mathcal{D}(L)$, and so $T \subset L^*$.

To show that $L^* \subset T$, let $Y \in \mathcal{D}(L^*)$ and $G = L^*Y$. Then, by definition,

$$(1.31) \qquad (G,Z) = (L^*Y,Z) = (Y,LZ)$$

for all $Z \in \mathcal{D}(L)$. Let U be a particular solution of $\ell(Y) = G$, i.e. $TU = G$. Then, for all $Z \in \mathcal{D}(L)$,

$$(1.32) \qquad (G,Z) = (TU,Z) = (U,LZ) = 0,$$

since $\pi[U,Z]_a^b = 0$, and therefore (1.31) and (1.32) give $(Y - U,LZ) = 0$.

Hence $Y - U \in R(L)^{\perp} = M \subset \mathcal{D}(T)$. Since $U \in \mathcal{D}(T)$ it follows that $Y \in \mathcal{D}(T)$ and consequently that

$$L^*Y = G = TU = T(U + \{Y - U\}) = TY.$$

Thus $L^* \subset T$ and so $L^* = T$.

(iii) Since L is symmetric, we have in general

$$(1.33) \qquad L \subset L^{**} \subset L^*.$$

We only need to show that $L^{**} \subset L$ or, by (ii),

that $T^* \subset L$. Let $Y \in \mathcal{D}(T^*)$. Then, for all $Z \in \mathcal{D}(T)$,

$$(T^*Y,Z) = (Y,TZ).$$

By (ii) and (1.33) we have

$$T^* = L^{**} \subset L^* = T.$$

Hence $Y \in \mathcal{D}(L^*)$, $L^*Y = TY$ and

$$(L^*Y,Z) = (TY,Z) = (Y,TZ),$$

for all $Z \in \mathcal{D}(T)$, where the last equality follows from the hypothesis $Y \in \mathcal{D}(T^*)$.

Thus the Lagrange identity implies

$$\pi[Y,Z]_a^b = 0,$$

for all $Z \in \mathcal{D}(T)$.

By Theorem 1.8 it follows that $Y^{\{k\}}(a) = Y^{\{k\}}(b) = 0$, for all $k = 0, \ldots, 2m-1$, and so $Y \in \mathcal{D}(L)$ and

$$L^{**}Y = T^*Y = TY = LY.$$

<u>Theorem 1.10</u>. The deficiency indices of the operator L are $(2mn^2, 2mn^2)$.

<u>Proof</u>. Let $Im(\lambda) > 0$. The first deficiency index of L is the number of linearly independent elements of H such that

$$(L^* - \lambda I)Z = 0,$$

i.e. the number of all linearly independent solutions of

$$\ell(Z) - \lambda Z = 0.$$

From Theorem 1.5 it follows that this number is $2mn^2$. Similarly for $Im\ \lambda < 0$.

CHAPTER 2.

SELF-ADJOINT EXTENSIONS OF MATRIX DIFFERENTIAL OPERATORS. THE RESOLVENT OPERATOR.

§ 2.1. Introduction. The minimal operator L, of Definition 1.5 is clearly seen to be symmetric. In this chapter we give an explicit expression for all the self-adjoint extensions L' of L. By an appropiate generalization of the Wronskian we also show that the resolvent operator R'_λ of L' is an integral operator with a Hilbert-Schmidt kernel. From this we deduce the structure of the spectrum of all self-adjoint extensions L' of L.

§ 2.2. Self-adjoint extensions of L. In order to obtain self-adjoint extensions of the symmetric matrix differential operator L, we shall make use of the following general characterization.

Proposition 2.1. Let L be a symmetric linear operator on a Hilbert space H. Then an extension L' of L is self-adjoint if and only if it is symmetric and every function Z in $D(L^*)$ which satisfies

$$(2.1) \qquad (LY,Z) = (Y,L^*Z),$$

for all $Y \in \mathcal{D}(L')$, belongs to $\mathcal{D}(L')$.

Proof. If L' is self-adjoint, then it is symmetric and condition (2.1) is obviously satisfied.

Conversely, let L' be a symmetric extension of L that satisfies (2.1). Then

$$L \subset L' \subset L'^* \subset L^*.$$

It remains to show that $\mathcal{D}(L'^*) \subset \mathcal{D}(L')$. Let $Z \in \mathcal{D}(L'^*)$. Then $Z \in \mathcal{D}(L^*)$ and so, for all $Y \in \mathcal{D}(L')$,

$$(L'Y,Z) = (Y,L'^*Z) = (Y,L^*Z).$$

Thus, by (2.1), it follows that $Z \in \mathcal{D}(L')$, and therefore $\mathcal{D}(L'^*) \subset \mathcal{D}(L')$.

Remark. Proposition 2.1 is given in { 1 } without proof.

By means of the Lagrange identity, the above proposition can be written as follows:

Proposition 2.2. An extension L' of a symmetric operator L is self-adjoint if and only if the following two

conditions hold:

(i) For all Y, Z in $\mathcal{D}(L')$,

$$\pi[Y,Z]_a^b = 0; \tag{2.2}$$

(ii) Every function Z in $\mathcal{D}(L^*)$ which satisfies

$$\pi[Y,Z]_a^b = 0, \tag{2.3}$$

for all $Y \in \mathcal{D}(L')$, belongs to $\mathcal{D}(L')$.

<u>Proof</u>. L' is symmetric if and only if condition (i) holds. Also (2.1) holds if and only if $\pi[Y,Z]_a^b = 0$.

The following theorem characterizes the self-adjoint extensions of the differential symmetric operator L.

<u>Theorem 2.1</u>. An extension L' of a matrix differential symmetric operator L on H is self-adjoint if and only if the following conditions hold:

(i) There exist n by n matrices $A_{ik}, A_{i,2m-k-1}, B_{ik}, B_{i,2m-k-1}$, $1 \leq i \leq 2mn^2$, $0 \leq k \leq m-1$, satisfying

$$\pi \sum_{k=0}^{m-1} (A_{j,2m-k-1} A^*_{ik} - A_{jk} A^*_{i,2m-k-1}) =$$

(2.4)

$$= \pi \sum_{k=0}^{m-1} (B_{j,2m-k-1} B^*_{ik} - B_{jk} B^*_{i,2m-k-1}),$$

for all $1 \leq i,j \leq 2mn^2$.

(ii) The boundary conditions

$$\pi \sum_{k=0}^{m-1} (A_{ik} Y^{\{k\}}(a) + B_{ik} Y^{\{k\}}(b)) = 0 \tag{2.5}$$

for $1 \leq i \leq 2mn^2$, are linearly independent in $\mathcal{D}(L^*)$, i.e. the equation

$$\sum_{i=1}^{2mn^2} \mu_i \pi \sum_{k=0}^{2m-1} (A_{ik} Y^{\{k\}}(a) + B_{ik} Y^{\{k\}}(b)) = 0,$$

for some numbers $\mu_1, \ldots, \mu_{2mn^2}$ and all $Y \in \mathcal{D}(L^*)$, implies $\mu_1 = \ldots = \mu_{2mn^2} = 0$.

(iii)

$$\mathcal{D}(L') = \{Y \in \mathcal{D}(L^*) : \pi \sum_{k=0}^{2m-1} (A_{ik} Y^{\{k\}}(a) + B_{ik} Y^{\{k\}}(b)) = 0, 1 \leq i \leq 2mn^2\}. \tag{2.6}$$

<u>Proof</u>. Assume that L' is self-adjoint. Since $\dim \mathcal{D}(L') = 2mn^2 \bmod \mathcal{D}(L)$, there exist matrix functions $W_1, \ldots, W_{2mn^2}$.

in $\mathcal{D}(L')$, linearly independent modulo $\mathcal{D}(L)$, such that every element Y' in $\mathcal{D}(L')$ can be written as

$$Y' = Y + \sum_{j=1}^{2mn^2} \alpha_j W_j,$$

where $\alpha_1, \ldots, \alpha_{2mn^2}$ are complex numbers and $Y \in \mathcal{D}(L)$.

Condition (2.2) implies that

$$\pi[W_i, W_j]_a^b = 0, \tag{2.7}$$

for all $1 \leq i,j \leq 2mn^2$, and condition (2.3) that for all $Y' \in \mathcal{D}(L')$,

$$\pi[Y', W_j]_a^b = 0, \tag{2.8}$$

for all $1 \leq j \leq 2mn^2$.

Let

$$\begin{cases} A_{ik} = W_i^{\{2m-k-1\}*}(a) \\ A_{i,2m-k-1} = -W_i^{\{k\}*}(a) \\ B_{ik} = -W_i^{\{2m-k-1\}*}(b) \\ B_{i,2m-k-1} = W_i^{\{k\}*}(b) \end{cases} \tag{2.9}$$

for $1 \leq i \leq 2mn^2$ and $0 \leq k \leq m-1$.

If we make use of (1.20) and (2.9), then (2.7) can be written as

$$\pi \sum_{k=0}^{m-1} (W_j^{\{2m-k-1\}*}(b)\ W_i^{\{k\}}(b) - W_j^{\{k\}*}(b)\ W_i^{\{2m-k-1\}}(b)) =$$

$$\pi \sum_{k=0}^{m-1} (W_j^{\{2m-k-1\}*}(a)\ W_i^{\{k\}}(a) - W_j^{\{k\}*}(a)\ W_i^{\{2m-k-1\}}(a)),$$

and therefore as

$$\pi \sum_{k=0}^{m-1} (A_{j,2m-k-1}\ A^*_{ik} - A_{jk}\ A^*_{i,2m-k-1}) =$$

$$= \pi \sum_{k=0}^{m-1} (B_{j,2m-k-1}\ B^*_{ik} - B_{jk}\ B^*_{i,2m-k-1}),$$

for all $1 \leq i,j \leq 2mn^2$.

Suppose now that the boundary conditions (2.5) are not linearly independent with respect to $\mathcal{D}(L^*)$. Then there exist numbers $\mu_1, \ldots, \mu_{2mn^2}$, not all zero, such that

$$\sum_{i=1}^{2mn^2} \mu_i\ \pi \sum_{k=0}^{2m-1} (A_{ik}\ Y^{\{k\}}(a) + B_{ik}\ Y^{\{k\}}(b)) = 0,$$

for all $Y \in \mathcal{D}(L^*)$.

By taking into account (2.9) we obtain

$$\pi[Y, \sum_{i=1}^{2mn^2} \mu_i W_i]_a^b = 0,$$

for all $Y \in \mathcal{D}(L^*)$, and by the same argument as used in Theorem 1.9, part (iii), we conclude that

$$\sum_{i=1}^{2mn^2} \mu_i W_i \in \mathcal{D}(L).$$

Hence we have a contradiction since, by hypothesis, the W_i, $1 \leq i \leq 2mn^2$, are linearly independent modulo $\mathcal{D}(L)$. Thus the linear independence of the boundary conditions (2.5) in $\mathcal{D}(L^*)$ follows.

To prove (iii) we observe that

$$\mathcal{D}(L') \subset \left\{ Y \in \mathcal{D}(L^*) : \pi \sum_{k=0}^{2m-1} (A_{ik} Y^{\{k\}}(a) + B_{ik} Y^{\{k\}}(b)) = 0, 1 \leq i \leq 2mn^2 \right\} =$$

$$= \left\{ Y \in \mathcal{D}(L^*) : \pi[Y, W_i]_a^b = 0, \ 1 \leq i \leq 2mn^2 \right\}.$$

Indeed, since L' is symmetric,

$$0 = (L'Y, W_i) - (Y, L'W_i) = \pi[Y, W_i]_a^b,$$

for all $i = 1, \ldots, 2mn^2$.

To show the opposite conclusion, let $Y \in \mathcal{D}(L^*)$ such that $\pi[Y,W_i]_a^b = 0$ for $i = 1, \ldots, 2mn^2$. For all $Z \in \mathcal{D}(L)$ we have also $\pi[Y,Z]_a^b = 0$. Hence, since every element $Y' \in \mathcal{D}(L')$ can be written as

$$Y' = Z + \sum_{i=1}^{2mn^2} \alpha_i W_i,$$

it follows that

$$\pi[Y,Y']_a^b = \pi[Y,Z + \sum_{i=1}^{2mn^2} \alpha_i W_i]_a^b =$$

$$\pi[Y,Z]_a^b + \sum_{i=1}^{2mn^2} \pi[Y,\alpha_i W_i]_a^b = 0,$$

for all $Y' \in \mathcal{D}(L')$.

From Proposition 2.2, part (ii), it follows that $Y \in \mathcal{D}(L')$, and so (iii) holds.

Conversely, let L' be an extension of L that satisfies (i), (ii) and (iii). By Theorem 1.8 there exist functions $W_1, \ldots, W_{2mn^2}$ in $\mathcal{D}(L^*)$ defined by equation (2.9). From (ii) it follows that they are linearly independent modulo $\mathcal{D}(L)$. Indeed, if there were numbers $\mu_1, \ldots, \mu_{2mn^2}$, not all zero, such that

$$\sum_{i=1}^{2mn^2} \mu_i W_i \in \mathcal{D}(L),$$

then, for all $Y \in \mathcal{D}(L^*)$,

$$\pi[Y, \sum_{i=1}^{2mn^2} \mu_i W_i]_a^b = 0$$

and hence

$$\sum_{i=1}^{2mn^2} \bar{\mu}_i \pi[Y, W_i]_a^b = 0.$$

But by (2.9) this equation can be written as

$$\sum_{i=1}^{2mn^2} \bar{\mu}_i \pi \sum_{k=0}^{2m-1} (A_{ik} Y^{\{k\}}(a) + B_{ik} Y^{\{k\}}(b)) = 0.$$

Since by hypothesis they are linearly independent in $\mathcal{D}(L^*)$, this contradiction shows that $W_1, \ldots, W_{2mn^2}$ are linearly independent modulo $\mathcal{D}(L)$.

By (2.4) and (iii) it follows that W_i, $1 \leq i \leq 2mn^2$, belong to $\mathcal{D}(L')$ and obviously we also have $\mathcal{D}(L) \subset \mathcal{D}(L')$. Thus

$$\left\{Y \in \mathcal{D}(L^*) \; ; \; Y = Z + \sum_{i=1}^{2mn^2} \mu_i W_i, \; Z \in \mathcal{D}(L)\right\} \subset \mathcal{D}(L')$$

and since the first set is of dimension $2mn^2$ mod $\mathcal{D}(L)$, the two sets coincide, i.e.

$$\mathcal{D}(L') = \left\{ Y \in \mathcal{D}(L^*) \ : \ Y = Z + \sum_{i=1}^{2mn^2} \mu_i \, W_i, \ Z \in \mathcal{D}(L) \right\} .$$

But then conditions (i) and (ii) of Proposition 2.2 are fulfilled and hence L' is self-adjoint.

§ 2.3. <u>The resolvent of self-adjoint matrix differential operators</u>. In this section we show that the resolvent $R'_\lambda = (L' - \lambda I)^{-1}$ of the self-adjoint operator L' is an integral operator with a Hilbert-Schmidt kernel. This will follow from a suitable generalization of the Wronskian.

Indeed, let Y and Θ be solutions of $\ell(Y) = 0$ and $\ell^\dagger(\Theta) = 0$, respectively, where $\ell(Y)$ is given by (1.3) and

$$\ell^\dagger(\Theta) = \sum_{s=0}^{m} (-1)^s \{\Theta(t)^{(s)} P_{m-s}(t)\}^{(s)}, \tag{2.10}$$

with the same notation as in chapter 1. Clearly, an obvious modification of Theorem 1.3 gives an existence and uniqueness theorem for the differential equation

$$\ell^\dagger(\Theta) - \lambda\Theta = F. \tag{2.11}$$

The following theorem can now be established:

<u>Theorem 2.2</u>. Let Y and Θ be solutions of $\ell(Y) = 0$

and $\ell^{\dagger}(\Theta) = 0$, respectively. Then the function $W(\Theta,Y)$ defined by

$$(2.12) \qquad W(\Theta,Y) = -\Theta^{\{0\}}Y^{\{2m-1\}} - \dots - \Theta^{\{m-1\}}Y^{\{m\}} + \\ +\Theta^{\{m\}}Y^{\{m-1\}} + \dots + \Theta^{\{2m-1\}}Y^{\{0\}},$$

is constant for all $t \in (a,b)$.

<u>Note</u>: $\Theta^{\{k\}}$, $0 \le k \le 2m-1$, are defined as follows:

$$(2.13) \quad \begin{cases} \Theta^{\{0\}} = \Theta \\ \Theta^{\{k\}} = \dfrac{d^k\Theta}{dt^k}, & k = 1,\dots, m-1 \\ \Theta^{\{m\}} = \dfrac{d^m\Theta}{dt^m} P_0 \\ \Theta^{\{m+k\}} = \dfrac{d^{m-k}\Theta}{dt^{m-k}} P_k(t) - \dfrac{d\Theta^{\{m+k-1\}}}{dt}, & k = 1,\dots, m. \end{cases}$$

<u>Proof of the theorem</u>. From (2.13) and (1.4), we obtain

$$(2.14) \quad \begin{cases} \dfrac{dY^{\{k\}}}{dt} = Y^{\{k+1\}}, & k = 0,\dots, m-2, \\ \dfrac{dY^{\{m-1\}}}{dt} = P_0^{-1}(t)\, Y^{\{m\}} \\ \dfrac{dY^{\{m+k\}}}{dt} = P_{k+1}(t)\, Y^{\{m-k-1\}} - Y^{\{m+k+1\}}, & k=0,\dots m-1, \end{cases}$$

and

$$(2.15)\quad \begin{cases} \dfrac{d\Theta^{\{k\}}}{dt} = \Theta^{\{k+1\}}, & k = 0, \ldots, m-2 \\[2ex] \dfrac{d\Theta^{\{m-1\}}}{dt} = \Theta^{\{m\}}\, P_0^{-1}(t) & \\[2ex] \dfrac{d\Theta^{\{m+k\}}}{dt} = \Theta^{\{m-k-1\}}\, P_{k+1}(t) - \Theta^{\{m+k+1\}}, & k=0,\ldots,m-1. \end{cases}$$

By differentiation of (2.12) and taking (2.14) and (2.15) into account, we obtain

$$\frac{dW(\Theta,Y)}{dt} = \Theta\, \ell(Y) - \ell^{\dagger}(\Theta)\, Y = 0,$$

and hence $W(\Theta,Y) = \text{const.}$

We can now obtain a useful expression for the general solution of the inhomogeneous matrix differential equation $\ell(Y) = F$, by considering the fundamental systems Y_k and Θ_k, $0 \le k \le 2m-1$, at a of $\ell(Y) = 0$ and $\ell^{\dagger}(\Theta) = 0$, respectively. The "Wronskian" $W(\Theta,Y)$ applied to pairs of functions of these fundamental systems gives, for all $t \in (a,b)$,

$$(2.16)\quad W(\Theta_i, Y_j) = \begin{cases} -\delta_{i,2m-j-1} I & \text{for } 0 \le i \le m-1, \\[2ex] \delta_{i,2m-j-1} I & \text{for } m < i < 2m-1, \end{cases}$$

and all $j = 0, \ldots, 2m-1$.

<u>Theorem 2.3</u>. The general solution of the inhomogeneous matrix differential equation $\ell(Y) = F$, can be written as

$$(2.17) \qquad Y(t) = \sum_{j=0}^{2m-1} Y_j(t)\, C_j + \sum_{j=0}^{2m-1} \int_a^t Y_j(t) Z_j(s) F(s)\, ds,$$

where C_j, $0 \le j \le 2m-1$, are arbitrary n by n constant matrices and the functions $Z_j(s)$ are defined in the proof.

<u>Proof</u>. We use the method of variation of parameters to obtain a particular solution of $\ell(Y) = F$. Indeed, if the matrix functions $V_j(t)$, $0 \le j \le 2m-1$, satisfy the system

$$\begin{aligned}
Y_0 \quad V_0' + \ldots + Y_{2m-1} \quad V_{2m-1}' &= 0 \\
\cdots\cdots\cdots\cdots \\
Y_0^{\{2m-2\}} V_0' + \ldots + Y_{2m-1}^{\{2m-2\}} V_{2m-1}' &= 0 \\
Y_0^{\{2m-1\}} V_0' + \ldots + Y_{2m-1}^{\{2m-1\}} V_{2m-1}' &= F
\end{aligned}$$

then the function

$$Y(t) = Y_0(t)V_0(t) + \ldots + Y_{2m-1}(t)V_{2m-1}(t)$$

is a particular solution of $\ell(Y) = F$.

In order to solve the system we multiply on the left each of the first m equations by $\Theta_i^{\{2m-1\}}, \ldots, \Theta_i^{\{m\}}$, respectively, and each of the last m by $-\Theta_i^{\{m-1\}}, \ldots, -\Theta_i^{\{0\}}$, respectively. Summing we obtain

$$\sum_{j=0}^{2m-1} W(\Theta_i, Y_j)\ V_j'(t) = -\Theta_i^{\{0\}} F(t),$$

and so, by (2.16),

$$-\delta_{i,2m-j-1}\, V_j' = -\Theta_i F \qquad \text{if } 0 \le i \le m-1,$$

$$\delta_{i,2m-j-1}\, V_j' = -\Theta_i F \qquad \text{if } m \le i \le 2m-1.$$

Hence

$$V_j(t) = \begin{cases} -\int_a^t \Theta_{2m-j-1}(s)\ F(s)\ ds, & \text{if } 0 \le j \le m-1, \\ \int_a^t \Theta_{2m-j-1}(s)\ F(s)\ ds, & \text{if } m \le j \le 2m-1. \end{cases}$$

The general solution of $\ell(Y) = F$ can thus be written in the form (2.17), where

$$(2.18) \qquad Z_j = \begin{cases} -\Theta_{2m-j-1}, & \text{for} \quad 0 \le j \le m-1 \\ \\ \Theta_{2m-j-1}, & \text{for} \quad m \le j \le 2m-1. \end{cases}$$

Let L' be an arbitrary self-adjoint extension of the operator L. Then we have the following theorem concerning the resolvent:

<u>Theorem 2.4</u>. Let $\lambda \in \rho(L')$, the resolvent set of L'. Then the resolvent R'_λ of L' is an integral operator with a Hilbert-Schmidt kernel K, i.e. it satisfies

$$\int_a^b \pi(K^*(t,s,\lambda)\ K(t,s,\lambda))dt\ ds < \infty.$$

<u>Proof</u>. It is clear that for each $F \in H$, $Y = R'_\lambda F$ if and only if Y satisfies the inhomogeneous equation

$$(2.19) \qquad \ell(Y) - \lambda Y = F$$

and the boundary-conditions (2.5), i.e. Y is defined by (2.17) where $\{Y_j\}$, $\{\Theta_j\}$, $0 \le j \le 2m-1$ are the fundamental system at a of the homogeneous equations

$$\ell(Y) - \lambda Y = 0, \qquad \ell^\dagger(\Theta) - \lambda\Theta = 0,$$

respectively, and satisfies the boundary conditions

$$(2.20) \qquad \pi[Y,W_j]_a^b = 0, \qquad 1 \le j \le 2mn^2.$$

Thus the substitution of (2.17) into (2.20) will determine the unique values of the constant matrices C_j, $1 \le j \le 2mn^2$.

We then obtain

$$\pi\left\{ \sum_{i=0}^{2m-1} [Y_i,W_j]_a^b C_i + \sum_{i=0}^{2m-1} [Y_i,W_j](b) \int_a^b Z_i(s)F(s)\, ds \right\} = 0$$

for each $1 \le j \le 2mn^2$.

Hence we have the linear system

$$(2.21) \quad \pi \sum_{i=0}^{2m-1} [Y_i,W_j]_a^b C_i = -\pi \sum_{i=0}^{2m-1} [Y_i,W_j](b) \int_a^b Z_i(s)F(s)\, ds$$

for all $1 \le j \le 2mn^2$.

Set

$$C_i \doteq (c_{i,rs}), \quad 1 \le r,s \le n, \quad 0 \le i \le 2m-1,$$

and

$$[Y_i,W_j]_a^b = V_{ij} = (v_{ij,pq}), \; 1 \le p,q \le n, \; 0 \le i \le 2m-1, \; 1 \le j \le 2mn^2.$$

Then equation (2.21) becomes

$$(2.22)\quad \sum_{i=0}^{2m-1} \sum_{r,s=1}^{n} v_{ij,rs}\, c_{i,sr} = -\pi \sum_{i=0}^{2m-1} [Y_i, W_j](b) \int_a^b Z_i(s) F(s)\, ds,$$

$1 \leq j \leq 2mn^2$.

This is a linear system of $j = 1, \ldots, 2mn^2$ equations with $2mn^2$ unknowns $c_{i,rs}$, $0 \leq i \leq 2m-1$, $1 \leq r,s \leq n$, the determinant of whose coefficients is not zero. Otherwise, the system

$$\sum_{i=0}^{2m-1} \sum_{r,s=1}^{n} v_{ij,rs}\, c_{i,sr} = 0, \quad 1 \leq j \leq 2mn^2,$$

would have a non-trivial solution $\{c_{i,rs}\}$, and so we would have matrices C_i, $0 \leq i \leq 2m-1$, such that

$$\pi \sum_{i=0}^{2m-1} [Y_i, W_j]_a^b\, C_i = 0, \quad \text{for all } 1 \leq j \leq 2mn^2.$$

This would mean

$$\pi \left[\sum_{i=0}^{2m-1} Y_i C_i, W_j\right]_a^b = 0, \quad \text{for all } 1 \leq j \leq 2mn^2,$$

and therefore, by (iii) of Theorem 2.1

$$\sum_{i=0}^{2m-1} Y_i C_i \in \mathcal{D}(L').$$

Also, since

$$\ell(Y_i) = \lambda Y_i, \qquad \text{for all} \quad 0 \le i \le 2m-1,$$

we would have

$$L' \sum_{i=0}^{2m-1} Y_i C_i = \lambda \sum_{i=0}^{2m-1} Y_i C_i$$

and this is a contradiction because λ by hypothesis is not in the spectrum of L'.

Instead of (2.22) we may equivalently solve the system

$$\pi \sum_{i=0}^{2m-1} [Y_i, W_j]_a^b G_i(s) = -\pi \sum_{i=0}^{2m-1} [Y_i, W_j](b) Z_i(s), \quad 1 \le j \le 2mn^2.$$

Indeed by the above discussion the determinant is different from zero and the solutions $G_i(s)$, $0 \le i \le 2m-1$, are continuous on $[a,b]$, since the Z_i are continuous there. Hence

$$C_i = \int_a^b G_i(s) F(s)\, ds, \qquad 0 \le i \le 2m-1,$$

and equation (2.17) takes the form

$$Y(t) = \int_a^b K(t,s,\lambda) F(s)\, ds,$$

where the kernel K is defined as

$$K(t,s,\lambda) = \begin{cases} \sum_{i=0}^{2m-1} Y_i(t)\ G_i(s)\ , & a \le t < s \\ \sum_{i=0}^{2m-1} Y_i(t)\ [G_i(s) + Z_i(s)]\ , & s < t \le b. \end{cases}$$

Since all the functions Y_i, G_i, Z_i, $0 \le i \le 2m-1$, are continuous with respect to (t,s), it follows that

$$\int_a^b \pi(K^*(t,s,\lambda)\ K(t,s,\lambda))\ dtds < \infty$$

and so R'_λ is a Hilbert-Schmidt operator.

The next two theorems show that an integral operator on H, with a Hilbert-Schmidt kernel, is compact.

<u>Theorem 2.5</u>. Let (a,b) be a finite or infinite interval on the real line. Then there exists an isometric mapping from H onto ℓ^2.

<u>Proof</u>. H is separable because each entry in a matrix function in this space belongs to the space $L^2(a,b)$ which is separable. Let $\{U_k\}_1^\infty$ be a complete orthonormal set in H. To each element $x = \{x_j\}_1^\infty$ in ℓ^2 we

associate the element $F \in H$ defined by

$$F = \sum_{j=1}^{\infty} x_j U_j. \tag{2.23}$$

Clearly F belongs to H because

$$\int_a^b \pi(F^*(t)F(t))\, dt = \sum_{j=1}^{\infty} |x_j|^2 < \infty. \tag{2.24}$$

This same formula shows that (2.23) is an isometry. The inverse mapping is given by

$$x_j = \int_a^b \pi(U_j^*(t)F(t))dt. \tag{2.25}$$

Thus (2.23) defines an isometric mapping from ℓ^2 onto H.

We now give a condition for compactness of integral operators in H:

<u>Theorem 2.6</u>. Let the kernel of an integral operator on H be a Hilbert-Schmidt kernel. Then this operator is compact.

<u>Proof</u>. The isometry from ℓ^2 onto H defined by (2.23) makes the integral operator

$$G(t) = \int_a^b K(t,s)F(s)\, ds \tag{2.26}$$

on H correspond to the operator

$$y = A\,x, \qquad y_i = \sum_{j=1}^{\infty} a_{ij}\, x_j ,$$

where $A = (a_{ij})$ is a matrix defined by

$$a_{ij} = \int_a^b \int_a^b \pi(U_i^*(t) K(t,s) U_j(s))\, dt\, ds ,$$

in such a way that if

$$x_j = \int_a^b \pi(U_j^*(t) F(t))\, dt ,$$

then

$$y_i = \sum_{j=1}^{\infty} a_{ij}\, x_j = \int_a^b \pi(U_i^*(t) G(t))\, dt.$$

Clearly

$$\sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty ,$$

since

$$\int_a^b \int_a^b \pi(K^*(t,s) K(t,s))\, dt\, ds < \infty ,$$

and it follows that the operator A is compact. Consequently, the integral operator (2.26) is compact.

Theorem 2.7. Let $\lambda \in \rho(L')$ and real. Then the self-adjoint operator $L' - \lambda I$ has purely a point spectrum, all its eigenvalues are of finite multiplicity and every finite interval contains only a finite number of them.

Proof. It follows immediately from {18, Thm. 2, p.557} and Theorems 2.4 and 2.6.

Theorem 2.8. The self-adjoint operator L' has only a point spectrum, all its eigenvalues are of finite multiplicity and every finite interval contains only a finite number of them.

Proof. From the proof of Theorem 2.4 it follows that λ is in the resolvent set of L' if and only if λ is not an eigenvalue. Since H is separable, only a countable set of real numbers can be eigenvalues. Otherwise we would have a basis with an uncountable number of vectors, which would be in contradiction with the separability of H. Therefore there exists a real number in $\rho(L')$ and the conclusion follows from the previous theorem.

CHAPTER 3.

THE INVERSION THEOREM.

§ 3.1. Introduction. In this chapter we prove an extension of the classical Inversion Theorem, presented for instance in {13, p.111}, to matrix functions. The proof follows that of Levitan, see {12}.

3.2. The Inversion Theorem.

Definition 3.1. A matrix function

$$\sigma(\lambda) = (\sigma_{ij}(\lambda)),$$

$-\infty < \lambda < \infty$, $1 \leq i,j \leq m$, is called a matrix distribution function if it satisfies the following properties:

(i) For all λ, $\sigma(\lambda)$ is hermitian;

(ii) For $\lambda < \mu$, the difference $\sigma(\mu) - \sigma(\lambda)$ is a positive semidefinite matrix;

(iii) All the functions $\sigma_{ij}(\lambda)$ are right-continuous.

Let F denote the set of all bounded piecewise continuous vector functions $f(\lambda) = (f_1(\lambda), \ldots, f_m(\lambda))$,

which vanish outside a finite interval. The mapping $(\ ,\ )$ from $E \times E$ into C given by

$$(f,g) = \sum_{i,j=1}^{m} \int_{-\infty}^{\infty} f_i(\lambda)\, \overline{g_j(\lambda)}\, d\sigma_{ij}(\lambda),$$

for all f, g in E, defines an inner-product in E, if we adopt the convention $f \equiv g$ if $(f-g, f-g) = 0$.

<u>Definition 3.2</u>. We denote by L^2_σ the Hilbert space completion of $(E, (\ ,\ ))$.

<u>Theorem 3.1</u>. Let (a,b) be a finite or infinite interval and assume that the coefficients P_k, $0 \le k \le m$ of the differential expression $\ell(Y)$ given by (1.3) have continuous derivatives up to the $(m-k)$th-order inclusive on (a,b), and are also hermitian with P_0 positive on (a,b).

Let $t_0 \in (a,b)$ and let U_j, $0 \le j \le 2m-1$, be the solutions of the matrix differential equation

$$\ell(Y) = \lambda Y,$$

where $\ell(Y)$ is given by (1.3), satisfying the initial conditions

$$(3.1) \qquad U_j^{(k)}(t_0,\lambda) = \delta_{jk} I, \quad 0 \le j,k \le 2m-1,$$

where I is the identity matrix and δ_{jk} is the Kronecker delta. Then, there exists a matrix distribution function $\sigma(\lambda) = (\sigma_{jk}(\lambda))$, $0 \leq j,k \leq 2mn^2$, such that the formulae

$$(3.2) \qquad \phi_i(\lambda) = \int_a^b \pi(Z_i^*(t,\lambda)F(t)\,dt, \quad 1 \leq i \leq 2mn^2,$$

where

$$Z_{jrs} = U_j E_{rs}, \qquad 0 \leq j \leq 2m-1,\ 1 \leq r,s \leq n,$$

Z_i, $1 \leq i \leq 2mn^2$, an ordering of the Z_{jrs} in some fixed way, E_{rs} is defined by (1.15), and

$$(3.3) \qquad F(t) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2mn^2} \phi_j(\lambda) Z_k(t,\lambda)\, d\sigma_{jk}(\lambda),$$

define mutually inverse mappings of $L^2(a,b;M_n)$ onto L^2_σ and of L^2_σ onto $L^2(a,b;M_n)$, respectively, which are isometries, i.e.

$$\int_a^b \pi(F^*(t)F(t)\,dt = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2mn^2} \phi_j(\lambda)\overline{\phi_k(\lambda)}\, d\sigma_{jk}(\lambda)$$

for each $F \in L^2(a,b;M_n)$ and $\phi \in L^2_\sigma$ related by (3.2) and (3.3).

<u>Proof</u>. Let $F(t)$ be a 2m-times continuously differentiable matrix function which vanishes outside a finite interval

$[\alpha,\beta] \subset (a,b)$, and let L' be a self-adjoint operator defined by the differential expression ℓ and an appropiate set of self-adjoint boundary conditions at α and β.

Let $\{W_r\}$, $r = 1, \ldots$ be a complete orthonormal system of eigenfunctions, and $\{\lambda_r\}$ the corresponding eigenvalues of the operator L'. Then,

$$(3.4) \qquad W_r(t) = \sum_{\substack{0\le k\le 2m-1 \\ 1\le i,j\le n}} \alpha_{rkij}\, Z_{kij}(t,\lambda),$$

where the α_{rkij} are complex-valued functions of λ.

By the Parseval equality,

$$(3.5) \qquad \int_a^b \pi(F^*(t)F(t))\,dt = \sum_{r=1}^{\infty} \left| \int_\alpha^\beta \pi(W_r^*(t)F(t))\,dt \right|^2 =$$

$$= \sum_{|\lambda_r|\le\mu} + \sum_{|\lambda_r|>\mu}$$

If $\lambda_r \neq 0$, then $\lambda_r^{-1}L' W_r = W_r$. Hence, by the Lagrange identity,

$$\int_\alpha^\beta \pi(W_r^*(t)F(t))\,dt = \lambda_r^{-1}\int_\alpha^\beta \pi([L'W_r(t)]^*F(t))\,dt =$$

$$= \lambda_r^{-1}\int_\alpha^\beta \pi(W_r^*(t)\,(L'F)(t))\,dt.$$

So

$$\sum_{|\lambda_r|>\mu} \left| \int_\alpha^\beta \pi(W_r^*(t)F(t))\,dt \right|^2 \le \mu^{-2} \sum_{|\lambda_r|>\mu} \left| \int_\alpha^\beta \pi(W^*(t)L'F(t))\,dt \right|^2$$

$$\le \mu^{-2} \|L'F\|^2 ,$$

and therefore the second term in the right-hand side of (3.5) tends to 0 as μ tends to infinity.

For the first term in the right-hand side of (3.5) we make use of (3.4) as follows:

$$(3.6) \qquad \sum_{|\lambda_r|\le\mu} \left| \int_\alpha^\beta \pi(W_r^*(t)F(t))\,dt \right|^2 = \sum_{|\lambda_r|\le\mu} (F,W_r)\overline{(F,W_r)} =$$

$$= \sum_{|\lambda_r|\le\mu} \sum_{i,j=1}^{2mn^2} \bar{\alpha}_{ri}\, \alpha_{rj}\, (F,Z_i)\overline{(F,Z_j)} =$$

$$= \sum_{|\lambda_r|\le\mu} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\phi_j(\lambda)}\,\bar{\alpha}_{ri}\alpha_{rj} =$$

$$= \int_{-\mu}^{\mu} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\phi_j(\lambda)}\, d\sigma_{ij}(\lambda;\alpha\ \beta),$$

where we denote by α_{rs}, $1 \le s \le 2mn^2$, an element of the set $\{\alpha_{rkij},\ 1 \le i,j \le n,\ 0 \le k \le 2m-1\}$, in some ordering,

$$(3.7) \qquad \phi_i(\lambda) = \int_\alpha^\beta \pi(Z_i^*(t,\lambda)F(t))\,dt,$$

for $1 \le i \le 2mn^2$, $\sigma_{ij}(0;\alpha,\beta) = 0$ and

$$(3.8) \qquad \sigma_{ij}(\lambda;\alpha,\beta) = \begin{cases} -\sum\limits_{0\ge\lambda_r>\lambda} \bar{\alpha}_{ri}\,\alpha_{rj}, & \lambda < 0 \\ \sum\limits_{0<\lambda_r\le\lambda} \bar{\alpha}_{ri}\,\alpha_{rj}, & \lambda > 0. \end{cases}$$

The following lemma will permit us to pass to the limit as α tends to a and β tends to b:

<u>Lemma 3.1</u>. For fixed μ, the variations of the functions $\sigma_{ij}(\lambda;\alpha,\beta)$ in the interval $(-\mu,\mu)$ are uniformly bounded relative to α and β.

<u>Proof of the lemma</u>. Let V_k, $k = 1, \ldots, 2m$, be $2m$-times continuously differentiable n by n matrix functions, which vanish outside the open interval (t_0, t_0+h), where $t_0 \in (a,b)$, and such that every entry v_{kij}, $1 \le i,j \le n$, of the matrix V_k satisfies

$$v_{kij}(t,h) \ge 0; \quad \int_{t_0}^{t_0+h} v_{kij}(t,h)\,dt = 1, \quad \text{for all } k = 1,\ldots,2m.$$

Let k be a fixed number, $0 \le k \le 2m-1$. Then

$$\left| \int_{t_0}^{t_0+h} \pi(W_s^{*(k)}(t)V_{k+1}(t,h))\,dt \right|^2 =$$

$$= \left| \sum_{\substack{0\le r\le 2m-1\\ 1\le i,j\le n}} \bar{\alpha}_{srij} \int_{t_o}^{t_o+h} \pi(Z^{*(k)}_{rij}(t,\lambda_s)\ V_{k+1}(t,h))dt \right|^2 =$$

$$= \left| \sum_{p=1}^{2mn^2} \bar{\alpha}_{sp} \int_{t_o}^{t_o+h} \pi(Z^{*(k)}_{p}(t,\lambda_s)\ V_{k+1}(t,h))dt \right|^2 \ge$$

$$\ge \sum_{1\le i,j\le n} |\alpha_{skij}|^2 \left| \int_{t_o}^{t_o+h} \pi(Z^{*(k)}_{kij}(t,\lambda_s)\ V_{k+1}(t,h))dt \right|^2 +$$

$$+ 2Re\left\{ \sum_{p>q} \bar{\alpha}_{sp}\alpha_{sq}\ J_p\ \bar{J}_q \right\} = J, \text{ say,}$$

and where

$$J_p = \int_{t_o}^{t_o+h} \pi(Z^{*(k)}_{p}(t,\lambda_s)\ V_{k+1}(t,h))dt .$$

Since, by hypothesis (3.1), $U_j^{(k)}(t_o) = \delta_{jk}I$ we have

$$u^{(k)}_{kii}(t_o) = 1, \qquad u^{(k)}_{kij}(t_o) = 0 \quad \text{for } i \ne j,$$

and it is easy to check that

$$\pi(Z^{*(k)}_{rij}V_s) = \sum_{q=1}^{n} \bar{u}^{(k)}_{riq}\ v_{sqj} .$$

Consequently,

$$\lim_{h\to 0} \left| \int_{t_0}^{t_0+h} u_{k,ii}^{(k)}(t,\lambda_s)\ v_{k+1,qj}(t,h)dt \right| = 1,$$

and

$$\lim_{h\to 0} \left| \int_{t_0}^{t_0+h} u_{k,iq}^{(k)}(t,\lambda_s)\ v_{k+1,qj}(t,h)dt \right| = 0$$

if $q \neq i$. Indeed, for the first limit, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$| u_{kii}^{(k)}(t,\lambda_s) - 1 | < \varepsilon,$$

for all $t \in (t_0, t_0+\delta)$.

Let $h < \delta$. Then

$$\left| \left| \int_{t_0}^{t_0+h} \bar{u}_{k,ii}^{(k)}(t, \lambda_s)\ v_{k+1,ij}(t,h)dt \right| - \right.$$

$$\left. - \left| \int_{t_0}^{t_0+h} v_{k+1,ij}(t,h)dt \right| \right| \leq$$

$$\leq \int_{t_0}^{t_0+h} |\bar{u}_{k,ii}^{(k)}(t,\lambda_s) - 1|\ v_{k+1,ij}(t,h)dt <$$

$$< \varepsilon \int_{t_0}^{t_0+h} v_{k+1,ij}(t,h)dt = \varepsilon.$$

Similarly for the second limit. Hence

$$I \equiv \left| \int_{t_o}^{t_o+h} \sum_{q=1}^{n} \bar{u}_{k,ij}^{(k)}(t,\lambda_s)\ v_{k+1,qj}(t,h)\,dt \right| \geq$$

$$\geq \left| \int_{t_o}^{t_o+h} \bar{u}_{k,ii}^{(k)}(t,\lambda_s)\ v_{k+1,ij}(t,h)\,dt \right| -$$

$$- \sum_{\substack{q=1\\ q\neq i}}^{n} \left| \int_{t_o}^{t_o+h} \bar{u}_{k,iq}^{(k)}(t,\lambda_s)\ v_{k+1,qj}(t,h)\,dt \right| \geq \frac{1}{2},$$

for sufficiently small h, and also

$$I \leq \left| \int_{t_o}^{t_o+h} \bar{u}_{k,ii}^{(k)}(t,\lambda_s)\ v_{k+1,ij}(t,h)\,dt \right| +$$

$$+ \sum_{\substack{q=1\\ q\neq i}}^{n} \left| \int_{t_o}^{t_o+h} \bar{u}_{k,iq}^{(k)}(t,\lambda_s)\ v_{k+1,qj}(t,h)\,dt \right| < \frac{3}{2},$$

for some other sufficiently small h. These two inequalities can be put together, i.e., there exists an $h > 0$ such that

$$\frac{1}{2} < \left| \int_{t_o}^{t_o+h} \sum_{q=1}^{n} \bar{u}_{k,iq}^{(k)}(t,\lambda_s)\ v_{k+1,qj}(t,h)\,dt \right| < \frac{3}{2},$$

and also

$$\left| \int_{t_o}^{t_o+h} \sum_{q=1}^{n} \bar{u}_{r,iq}^{(k)}(t,\lambda_s)\ v_{k+1,qj}(t,h)dt \right| < \frac{1}{8m(2mn^2-1)}$$

for $r \neq k$.

Therefore, since

$$2 \sum_{p>q} |\alpha_{sp}|\ |\alpha_{sq}| \leq (2mn^2-1) \sum_{p=1}^{2mn^2} |\alpha_{sp}|^2 ,$$

we have

$$J \geq \frac{1}{2} \sum_{1\leq i,j\leq n} |\alpha_{skij}|^2 - \frac{3}{2} \frac{1}{8m(2mn^2-1)}\ 2 \sum_{p>q} |\alpha_{sp}||\alpha_{sq}| \geq$$

$$\geq \frac{1}{2} \sum_{1\leq i,j\leq n} |\alpha_{skij}|^2 - \frac{3}{16m} \sum_{p=1}^{2mn^2} |\alpha_{sp}|^2 .$$

By the Parseval equality and integration by parts we obtain

$$\int_{t_o}^{t_o+h} \pi(V_{k+1}^{*(k)}(t,h)V_{k+1}^{(k)}(t,h))dt = \sum_{s=1}^{\infty} \left| \int_{t_o}^{t_o+h} \pi(W_s^*(t)V_{k+1}^{(k)}(t,h)dt \right|^2 =$$

$$= \sum_{s=1}^{\infty} \left| \int_{t_o}^{t_o+h} \pi(W_s^{*(k)}(t)V_{k+1}(t,h)dt \right|^2$$

and by the last inequality,

$$\int_{t_o}^{t_o+h} \pi(V_{k+1}^{*(k)}(t,h)\ V_{k+1}^{(k)}(t,h))dt \geq \frac{1}{2} \sum_{|\lambda_s|\leq\mu} \sum_{1\leq i,j\leq n} |\alpha_{skij}|^2 -$$

$$- \frac{3}{16} \sum_{|\lambda_s| \le \mu} \sum_{p=1}^{2mn^2} |\alpha_{sp}|^2 .$$

By adding these inequalities from $k = 0$ up to $k = 2m-1$, we have

$$\sum_{k=0}^{2m-1} \int_{t_o}^{t_o+h} \pi(V_{k+1}^{*(k)}(t,h) V_{k+1}^{(k)}(t,h))\,dt \ge$$

$$\ge \frac{1}{2} \sum_{|\lambda_s| \le \mu} \sum_{1 \le i,j \le n} \left(\sum_{k=0}^{2m-1} |\alpha_{skij}|^2 \right. -$$

$$- \frac{3}{16m} \sum_{k=0}^{2m-1} \left(\sum_{|\lambda_s| \le \mu} \sum_{p=1}^{2mn^2} |\alpha_{sp}|^2 \right) = \left(\frac{1}{2} - \frac{3}{8}\right) \sum_{|\lambda_s| \le \mu} \sum_{p=1}^{2mn^2} |\alpha_{sp}|^2,$$

and so

$$\sum_{p=1}^{2mn^2} \sum_{|\lambda_s| \le \mu} |\alpha_{sp}|^2 \le 8 \sum_{k=0}^{2m-1} \int_{t_o}^{t_o+h} \pi(V_{k+1}^{*(k)}(t,h) V_{k+1}^{(k)}(t,h))\,dt.$$

Therefore, since the right-hand side is independent of α and β, the total variation of $\sigma_{ii}(\lambda;\alpha,\beta)$ is uniformly bounded in $(-\mu,\mu)$ with respect to α and β.

The Cauchy-Schwarz inequality implies now the uniform boundedness of $\sigma_{ij}(\lambda;\alpha,\beta)$ for $i \ne j$, and the proof of the lemma is complete.

We now need Helly's first and second theorems (see {18, vol. V, Sect. 12 and 13}). From the first we deduce the

existence of sequences $\{\alpha_p\}$, $\{\beta_p\}$ converging to a and b, respectively, such that the function

$$\sigma_{ii}(\lambda;a,b) = \lim_{p\to\infty} \sigma_{ii}(\lambda;\alpha_p,\beta_p)$$

exists.

Taking limits through this sequence in (3.6), we obtain

$$\sum_{|\lambda_r|\le\mu} \left| \int_a^b \pi(W_r^*(t)F(t))dt \right|^2 = \int_{-\mu}^{\mu} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\phi_j(\lambda)}d\sigma_{ij}(\lambda).$$

Therefore, by (3.5), it follows that

$$(3.9) \qquad \int_a^b \pi(F^*(t)F(t))dt = \sum_{|\lambda_r|\le\mu} + \sum_{|\lambda_r|>\mu} =$$

$$\int_{-\mu}^{\mu} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\phi_j(\lambda)}d\sigma_{ij}(\lambda) + \sum_{|\lambda_r|>\mu}.$$

Since the left-hand side is independent of μ and the second term in the right-hand side tends to zero as μ tends to infinity, it follows that we can take limits in (3.9) and obtain

$$(3.10) \qquad \int_a^b \pi(F^*(t)F(t)dt = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\phi_j(\lambda)}d\sigma_{ij}(\lambda).$$

Since the left-hand side of (3.10) is $\|F\|^2$ in $L^2(a,b;M_n)$ and the right-hand side is $\|\Phi\|^2$ in L^2_σ, $\Phi(\lambda) = (\phi_1(\lambda), \ldots, \phi_{2mn^2}(\lambda))$, we have that the mapping defined by (3.7) is an isometry.

Let F be an arbitrary matrix function in $L^2(a,b;M_n)$. Since the 2m-times continuously differentiable functions with compact support in (a,b) are dense in $L^2(a,b;M_n)$, there exists a sequence $\{F_p\}$ of such functions converging to F in the $L^2(a,b;M_n)$-norm.

Then, because of (3.10), the corresponding sequence of transforms Φ_p, where $\Phi_p(\lambda) = (\phi_{p,1}(\lambda), \ldots, \phi_{p,2mn^2}(\lambda))$ and

$$\phi_{pi}(\lambda) = \int_a^b \pi(Z_i^*(t,\lambda)F_p(t))\,dt, \tag{3.11}$$

$1 \le i \le 2mn^2$, is a Cauchy sequence in L^2_σ.

If we put

$$\phi_i(\lambda) = \lim_{p\to\infty} \phi_{pi}(\lambda)$$

for $1 \le i \le 2mn^2$, from (3.11) it is clear that

$$\phi_i(\lambda) = \int_a^b \pi(Z_i^*(t,\lambda)F(t))\,dt, \tag{3.12}$$

for $1 \le i \le 2mn^2$, and

$$(3.13)\qquad \int_a^b \pi(F^*(t)F(t))\,dt = \int_a^b \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\phi_j(\lambda)}\,d\sigma_{ij}(\lambda).$$

Thus the mapping from $L^2(a,b;M_n)$ to L^2_σ, defined by (3.12), is an isometry.

It remains to show that the inverse mapping is given by (3.3) and that the mapping (3.12) is onto.

By the polarization identity,

$$(F,G) = \frac{1}{4}(\|F+G\|^2 - \|F-G\|^2 + i\|F+iG\|^2 - i\|F-iG\|^2),$$

it follows that for all F and G in $L^2(a,b;M_n)$ and its corresponding transforms Φ and Ψ, given by (3.12), we have

$$(3.14)\qquad \int_a^b \pi(G^*(t)F(t))\,dt = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\psi_j(\lambda)}\,d\sigma_{ij}(\lambda),$$

where $\Phi(\lambda) = (\phi_1(\lambda),\ldots,\phi_{2mn^2}(\lambda))$ and $\Psi(\lambda) = (\psi_1(\lambda),\ldots,\psi_{2mn^2}(\lambda))$.

Let F be in $L^2(a,b;M_n)$ and let Φ be its transform by (3.12). If $\Delta = (\mu,\nu]$, we define

$$(3.15)\qquad F_\Delta(t) = \int_\Delta \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)z_j(t,\lambda)\,d\sigma_{ij}(\lambda).$$

Let G be in $L^2(a,b;M_n)$ of compact support in (a,b), and Ψ its corresponding transform by (3.12). From (3.15) and Fubini's theorem, since the limits of integration are finite and the integrand continuous, we

obtain

$$\int_a^b \pi(G^*(t)F_\Delta(t))\,dt = \int_a^b \pi\Big(G^*(t)\int_\Delta \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)Z_j(t,\lambda)\,d\sigma_{ij}(\lambda)\Big)\,dt =$$

$$= \int_\Delta \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\ \Big\{\int_a^b \pi(G^*(t)Z_j(t,\lambda)\,dt\ \Big\}d\sigma_{ij}(\lambda) =$$

$$= \int_\Delta \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\psi_j(\lambda)}\,d\sigma_{ij}(\lambda).$$

Also, from (3.14),

$$\int_a^b \pi(G^*(t)F(t))\,dt = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\psi_j(\lambda)}\,d\sigma_{ij}(\lambda).$$

Hence by the Cauchy-Schwarz inequality, and (3.12) applied to G,

$$(3.16)\quad \int_a^b \pi(G^*(t)[F(t)-F_\Delta(t)]\,dt = \int_\nabla \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\psi_j(\lambda)}\,d\sigma_{ij}(\lambda) \le$$

$$\le \Big\{\int_\nabla \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\phi_j(\lambda)}\,d\sigma_{ij}(\lambda)\Big\}^{\frac{1}{2}} \Big\{\int_\nabla \sum_{i,j=1}^{2mn^2} \psi_i(\lambda)\overline{\psi_j(\lambda)}\,d\sigma_{ij}(\lambda)\Big\}^{\frac{1}{2}} \le$$

$$\le \Big\{\int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\phi_j(\lambda)}\,d\sigma_{ij}(\lambda)\Big\}^{\frac{1}{2}} \int_a^b \pi(G^*(t)G(t))\,dt,\quad \nabla=(-\infty,\infty)-\Delta.$$

Let $[\alpha,\beta] \subset (a,b)$ be a finite interval and

define

$$G(t) = \begin{cases} 0 & \text{if} \quad a < t < \alpha \\ F(t) - F_\Delta(t) & \text{if} \quad \alpha \le t \le \beta \\ 0 & \text{if} \quad \beta < t < b . \end{cases}$$

Then, by (3.16),

$$\int_\alpha^\beta \pi([F(t)-F_\Delta(t)]^*[F(t)-F_\Delta(t)])dt \le \left\{\int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\phi_j(\lambda)}d\sigma_{ij}(\lambda)\right\}.$$

Since the right-hand side does not depend on α or on β, we can take limits for $\alpha \to a$ and $\beta \to b$ to obtain

$$(3.17) \qquad F(t) = \lim_{\Delta \to (a,b)} F_\Delta(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda) Z_j(t,\lambda) d\sigma_{ij}(\lambda) .$$

It remains now to show that the isometry (3.12) is onto. This will be done by way of a classical argument, see { 3}.

We need the following lemma.

<u>Lemma</u>. Let $\Phi = \{\phi_i(\lambda)\}$, $1 \le i \le 2mn^2$, and

$$F_\Delta(t) = \int_\Delta \sum_{i,j=1}^{2mn^2} \phi_i(\lambda) Z_j(t,\lambda) d\sigma_{ij}(\lambda) ,$$

$$F(t) = \lim_{\Delta \to (-\infty,\infty)} F_\Delta(t) ,$$

in the sense of the $L^2(a,b;M_n)$-norm, exists.

<u>Proof</u>. Let $\Delta_1 \subset \Delta_2 \subset \ldots \to (-\infty,\infty)$, and let G belong to $L^2(a,b;M_n)$ of finite support, i.e. there exists a finite interval $[\alpha,\beta]$ such that $G(t) = 0$ for all $t \notin [\alpha\ \beta]$. If Ψ is the transform of G and p and q, $p > q$, are positive integers, then

$$\int_a^b \pi(G^*(t)[F_{\Delta_p}(t) - F_{\Delta_q}(t)]dt =$$

$$= \int_a^b \pi(G^*(t)\int_{\Delta_p-\Delta_q} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda) Z_j(t,\lambda) d\sigma_{ij}(\lambda))dt =$$

$$\int_{\Delta_p-\Delta_q} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\left\{\int_\alpha^\beta \pi(G^*(t) Z_j(t,\lambda)dt\right\}d\sigma_{ij}(\lambda) =$$

$$\int_{\Delta_p-\Delta_q} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\psi_j(\lambda)}d\sigma_{ij}(\lambda).$$

By means of the Cauchy-Schwarz inequality and letting

$$G(t) = \begin{cases} 0 & \text{if } a < t < \alpha \\ F_{\Delta_p}(t) - F_{\Delta_q}(t) & \text{if } \alpha \le t \le \beta \\ 0 & \text{if } \beta < t < b, \end{cases}$$

we have, in the same way as in (3.16),

$$\int_\alpha^\beta \pi([F_{\Delta_p}(t)-F_{\Delta_q}(t)]^*[F_{\Delta_p}(t)-F_{\Delta_q}(t)])\,dt \le \int_{\Delta_p-\Delta_q} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\phi_j(\lambda)}\,d\sigma_{ij}(\lambda).$$

Since the right-hand side does not depend on α or β we may let $\alpha \to a$ and $\beta \to b$ to obtain

$$\int_a^b \pi([F_{\Delta_p}(t)-F_{\Delta_q}(t)]^*[F_{\Delta_p}(t)-F_{\Delta_q}(t)]\,dt \le \int_{\Delta_p-\Delta_q} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda)\overline{\phi_j(\lambda)}\,d\sigma_{ij}(\lambda).$$

Letting Δ_p and Δ_q tend to $(-\infty,\infty)$ it follows that $\{F_{\Delta_p}\}$ is a Cauchy sequence in $L^2(a,b;M_n)$. Thus, if $\Phi(\lambda) = (\phi_1(\lambda), \ldots, \phi_{2mn^2}(\lambda))$ is in L^2_σ, then

$$\int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda) z_j(t,\lambda)\,d\sigma_{ij}(\lambda)$$

belongs to $L^2(a,b;M_n)$, and the proof of the lemma is complete.

Let $\Psi(\lambda) = (\psi_i(\lambda))$, $1 \le i \le 2mn^2$, be an arbitrary function in L^2_σ. To show now that the isometric mapping (3.12) is onto, we have to prove that the function

$$F(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} \psi_i(\lambda) z_j(t,\lambda)\,d\sigma_{ij}(\lambda), \tag{3.18}$$

which belongs to $L^2(a,b;M_n)$ by the previous lemma, is

such that

$$\phi_i(\lambda) = \int_a^b \pi(Z_i^*(t,\lambda)F(t))dt$$

coincides with $\psi_i(\lambda)$ for all $1 \leq i \leq 2mn^2$.

From (3.17) and the fact that L^2_σ functions with compact support are dense in L^2_σ it follows that, without loss of generality, we may assume that F, and ϕ_i, ψ_i, for $1 \leq i \leq 2mn^2$, are of compact support.

Hence the function $\chi(\lambda) = \psi(\lambda) - \phi(\lambda)$ is in L^2_σ, of compact support, and we have to show that $\|\chi\|$ in L^2_σ is zero.

Since the operator L is symmetric, the imaginary unit i is in the resolvent set of all self-adjoint extensions L' of L. Hence it follows that the function

$$F_1(t) = (L' - i)^{-1}F(t)$$

is in $L^2(a,b;M_n)$ and has the same support as F.

Since

$$L'Z_i(t,\lambda) = \lambda Z_i(t,\lambda),$$

for all $1 \leq i \leq 2mn^2$, and $F(t)$ is given by (3.18), it is easy to check that

$$F_1(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} (\lambda - i)^{-1} \psi_i(\lambda) Z_j(t,\lambda) d\sigma_{ij}(\lambda).$$

The same argument with F_1 instead of F gives, for $F_2(t) = (L' - i)^{-1} F_1(t)$,

$$F_2(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} (\lambda - i)^{-2} \psi_i(\lambda) Z_j(t,\lambda) d\sigma_{ij}(\lambda).$$

Hence for any positive integer p we have

$$F_p(t) = (L' - i)^{-p} F_{p-1}(t)$$

and

$$F_p(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} (\lambda - i)^{-p} \psi_i(\lambda) Z_j(t,\lambda) d\sigma_{ij}(\lambda).$$

This process can also be performed with the function

$$\tilde{F}(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} \phi_i(\lambda) Z_j(t,\lambda) d\sigma_{ij}(\lambda)$$

to obtain

$$\tilde{F}_p(t) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} (\lambda - i)^{-p} \phi_i(\lambda) Z_j(t,\lambda) d\sigma_{ij}(\lambda).$$

Thus, for any positive integer p,

$$(3.19) \qquad \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} (\lambda - i)^{-p} \chi_i(\lambda) Z_j(t,\lambda)\, d\sigma_{ij}(\lambda) = 0.$$

Set

$$\Theta_j(\lambda,s) = \int_{t_o}^{t_o+s} \pi(Z_j^*(t,\lambda))\,dt,$$

for $1 \leq j \leq 2mn^2$. Integrating (3.19) from t_o to t_o+s and using Fubini's theorem we obtain

$$(3.20) \qquad \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} (\lambda - i)^{-p} \chi_i(\lambda) \overline{\Theta_j(\lambda,s)}\, d\sigma_{ij}(\lambda) = 0.$$

The two functions Θ and χ are in L^2_σ. Hence the complex-valued function H defined by

$$H(z) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} (\lambda - z)^{-1} \chi_i(\lambda) \overline{\Theta_j(\lambda,s)}\, d\sigma_{ij}(\lambda)$$

is analytic for $Im\ \lambda > 0$.

By (3.20) all its derivatives are 0 at $z = i$, so $H(z) = 0$ for all $Im\ z > 0$, and the same result holds for $Im\ z < 0$.

If we put $z = x + iy$, we have

$$\frac{1}{\lambda - z} - \frac{1}{\lambda - \bar{z}} = \frac{2iy}{(\lambda - x)^2 + y^2},$$

and since $H(z) - H(\bar{z}) = 0$,

$$(3.21) \qquad \int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} \frac{\chi_i(\lambda)\overline{\Theta_j(\lambda,s)}}{(\lambda - x)^2 + y^2}\, d\sigma_{ij}(\lambda) = 0 .$$

If λ_1 and λ_2 are points of continuity of σ and (3.21) is integrated with respect to x from λ_1 to λ_2, then, letting $y \to 0$, it follows that

$$\int_{\lambda_1}^{\lambda_2} \sum_{i,j=1}^{2mn^2} \chi_i(\lambda)\overline{\Theta_j(\lambda,s)}\, d\sigma_{ij}(\lambda) = 0.$$

By differentiation with respect to s,

$$\int_{\lambda_1}^{\lambda_2} \sum_{i,j=1}^{2mn^2} \chi_i(\lambda)\pi(Z_j^*(t,\lambda))\, d\sigma_{ij}(\lambda) = 0$$

and if derivatives are taken and evaluated at $t = t_o$, it follows from (3.1) and from the fact that

$$Z_{jrs}^{(k)}(t_o) = U_j^{(k)}(t_o)E_{rs} = \delta_{jk}E_{rs},$$

i.e.

$$\pi(Z_{jrs}^{(k)}(t_o)) = \delta_{jk}\delta_{rs},$$

that

$$\int_{\lambda_1}^{\lambda_2} \sum_{i,j=1}^{2mn^2} \chi_i(\lambda)\, d\sigma_{ij}(\lambda) = 0,$$

for $1 \le j \le 2mn^2$.

If γ_j, $1 \leq j \leq 2mn^2$, are arbitrary step functions, since λ_1 and λ_2 are also arbitrary, we have

$$\int_{-\infty}^{\infty} \sum_{i,j=1}^{2mn^2} \chi_i(\lambda)\gamma_j(\lambda)d\sigma_{ij}(\lambda) = 0$$

and consequently $\chi_i(\lambda) = 0$, for all $1 \leq i \leq 2mn^2$.

CHAPTER 4.

EIGENFUNCTION EXPANSIONS IN SOME B*-ALGEBRAS.

§ 4.1. Introduction. In this chapter we develop a theory of boundary value problems in some B*-algebras. This includes an Expansion Theorem and we also state and prove a generalization of a well-known theorem of Weyl, the so-called "Nested Circles Theorem". E. Hille initiated the study of this problem in his work {8, pp. 566-573}.

§ 4.2. $\mathcal{H}$-algebras. In this chapter we shall consider functions with values in algebras Λ satisfying certain special properties:

Definition 4.1. Λ is said to be a B*-algebra with identity e if Λ is a complex Banach algebra which is also a *-algebra, i.e. it satisfies conditions (i) - (iv), (v) and (vi):

(i) For each $x \in \Lambda$ there exists a unique $x^* \in \Lambda$ and $x^{**} = x$ holds for all $x \in \Lambda$,

(ii) $(x + y)^* = x^* + y^*$, for all x, y in Λ,

(iii) $(\alpha x)^* = \bar{\alpha} x^*$, for all $x \in \Lambda$, $\alpha \in \mathbb{C}$,

(iv) $(xy)^* = y^* x^*$, for all x, y in Λ,

(v) $\|x^*x\| = \|x\|^2$ for all $x \in A$.

(vi) There exists $e \in A$ such that $ex = xe = x$, for all $x \in A$.

In the sequel we shall always assume A to be a B^*-algebra with identity.

<u>Definition 4.2</u>. Let $x \in A$. Then the set of complex numbers

$$\rho(x) = \{\lambda : (\lambda e - x)^{-1} \text{ exists}\}$$

is called the resolvent set of x. Its complement $\sigma(x)$ is called the spectrum of x.

<u>Definition 4.3</u>. An element $x \in A$ is said to be hermitian if $x = x^*$. An hermitian element is said to be positive if $\sigma(x)$ is contained in the non-negative reals.

The following propositions belong to the general theory of Banach algebras, (see for instance {14} and {9}).

<u>Proposition 4.1</u>. Every element x of the algebra can be represented in a unique way as

$$x = h + ik,$$

where h and k are hermitian. h is called the real part of x and k the imaginary part.

Proposition 4.2. Let x be a positive element in A. Then there exists a unique positive element y in A, called the (principal) square root of x, such that $y^2 = x$.

Proposition 4.3. (i) x is positive if and only if $x = y^*y$ for some $y \in A$; (ii) The set P of all positive elements in A is a convex cone.

Proposition 4.4. If $x = h + ik$, $h = h^*$, $k = k^*$ and k is strictly positive, or negative, then x is non-singular, i.e. x^{-1} exists.

Definition 4.4. A linear functional π on a B^*-algebra is said to be positive if $\pi(x) \geq 0$ for all positive elements x in A.

Definition 4.5. If there exists a positive linear functional π such that

(4.1) $\pi(x^*x) \geq 0$ and $\pi(x^*x) = 0$ if and only if $x = 0$

for all $x \in A$, where A is a B^*-algebra with identity, then the algebra A is called an $\mathcal{H}$-algebra.

In the sequel A is assumed to be an $\mathcal{H}$-algebra.

<u>Theorem 4.1</u>. An $\mathcal{H}$-algebra A is a pre-Hilbert space if endowed with the inner-product $(x,y)_\pi$ defined by

$$(4.2) \qquad (x,y)_\pi = \pi(y^*x),$$

for all x,y in A, and corresponding norm defined by $\|x\|_\pi = \pi(x^*x)^{\frac{1}{2}}$.

<u>Proof</u>. Since every positive linear functional π satisfies

$$\pi(x^*) = \overline{\pi(x)},$$

for all $x \in A$, then

$$(x,y)_\pi = \pi(y^*x) = \pi((x^*y)^*) = \overline{\pi(x^*y)} = \overline{(y,x)_\pi}.$$

Also, from the linearity of π we have

$$(\alpha x+\beta y,z)_\pi = \pi(z^*(\alpha x+\beta y)) = \alpha\pi(z^*x) + \beta\pi(z^*y) =$$

$$= \alpha(x,z)_\pi + \beta(y,z)_\pi.$$

Finally,

$$(x,x)_\pi = \pi(x^*x) \geq 0$$

since π is positive, and

$$(x,x)_\pi = 0 \quad \text{implies} \quad x = 0$$

by (4.1).

We shall denote by $C(a,b;A)$ the set of all A-valued, continuous functions $f(t)$ over the interval $[a,b]$. Unless explicitly stated otherwise, we shall assume $[a,b]$ to be finite.

<u>Theorem 4.2</u>. The set $C(a,b;A)$ is a pre-Hilbert space under the following inner-product,

$$(4.3) \qquad ((f,g))_\pi = \int_a^b \pi(g^*(t)f(t))dt,$$

for all f and g in $C(a,b;A)$.

<u>Proof</u>. This follows from Theorem 4.1 since we have that

$$((f,g))_\pi = \int_a^b (f(t),g(t))_\pi \, dt.$$

Therefore, we can define a norm $|||f|||_\pi = ((f,f))_\pi^{\frac{1}{2}}$ on C, and

$$(4.4) \qquad |||f|||_\pi = \left\{ \int_a^b \pi(f^*(t)f(t))dt \right\}^{1/2}$$

Thus we can consider $C(a,b;A)$ as a dense subspace of some Hilbert space $H = L^2(a,b;A)$.

If $k(t,\xi)$ is an A-valued, continuous function on $[a,b] \times [a,b]$, we define an integral operator. $K: H \to H$ as follows:

$$(4.5) \qquad Kf(t) = \int_a^b k(t,\xi) f(\xi) d\xi,$$

for all functions $f \in C(a,b;A)$.

In analogy with the classical case, we have the following theorem concerning the symmetry of the operator K:

<u>Theorem 4.3</u>. If the continuous function in two variables $k(t,\xi)$ also satisfies $k^*(t,\xi) = k(\xi,t)$, i.e. if k is hermitian symmetric, then the operator K is symmetric.

<u>Proof</u>. By the continuity of $k(t,\xi)$ we can apply Fubini's Theorem to obtain

$$((Kf,g))_\pi = \int_a^b \pi(g^*(t)Kf(t))dt = \int_a^b \pi(g^*(t)\int_a^b k(t,\xi)f(\xi)d\xi)dt =$$

$$\int_a^b \pi\{\int_a^b g^*(t)k(t,\xi)dt\}d\xi = \int_a^b \pi(\{\int_a^b k(\xi,t)g(t)dt\}^* f(\xi))d\xi =$$

$$= ((f, Kg))_{\pi} .$$

<u>Definition 4.6</u>. An $\mathcal{A}$-algebra is called L^2-separable if $H = L^2(a,b;\mathcal{A})$ is a separable Hilbert space.

The next two theorems show the usefulness of the above definition. They can be proved as in Theorems 2.5 and 2.6 respectively.

<u>Theorem 4.4</u>. Let $[a,b]$ be a finite interval on the real line and let $\mathcal{A}$ be an L^2-separable $\mathcal{H}$-algebra. Then there exists a unitary mapping from H to ℓ^2.

<u>Theorem 4.5</u>. Let $\mathcal{A}$ be an L^2-separable $\mathcal{H}$-algebra. Then the integral operator K defined in (4.5), with an hermitian symmetric kernel, is compact.

<u>Example 4.1</u>. Let H be a separable Hilbert space and let $\{e_j\}$, $1 \leq j < \infty$, be an orthonormal basis. Then the algebra $B(H)$ of bounded linear operators on H is clearly an L^2-separable $\mathcal{H}$-algebra if π is chosen as the linear functional defined by

$$\pi(T) = \sum_{j=1}^{\infty} \frac{1}{j^2} (Te_j, e_j) ,$$

for every T in $B(H)$.

We have that π is positive, since

$$\pi(T^*T) = \sum_{j=1}^{\infty} \frac{1}{j^2}(T^*Te_j, e_j) = \sum_{j=1}^{\infty} \frac{1}{j^2}(Te_j, Te_j) =$$

$$= \sum_{j=1}^{\infty} \frac{1}{j^2} \|Te_j\|^2 \geq 0.$$

Also $\pi(T^*T) = 0$ if and only if $Te_j = 0$ for all $j = 1, \ldots,$ i.e. if and only if $T = 0$. By Definition 4.5 $B(H)$ is thus an $\mathcal{H}$-algebra.

Now $f \in L^2(a, b; B(H))$ if and only if for every pair of positive integers i, j the complex-valued function $f_{ij}(t) = (f(t)e_i, e_j)$ belongs to $L^2(a, b)$. Since this last space is separable and we must consider a countable number of them corresponding to $1 \leq i, j < \infty$, the L^2-separability of $B(H)$ follows.

<u>Example 4.2</u>. Let HS be the algebra of Hilbert-Schmidt operators on a Hilbert space H, i.e. T is in HS if and only if

$$\sum_{\alpha \in \Lambda} \|Te_j\|^2 < \infty,$$

where $\{e_\alpha\}$, $\alpha \in \Lambda$, is any orthonormal basis of H.

If we define

$$\pi(T) = \sum_{\alpha\varepsilon\Lambda} (Te_\alpha, e_\alpha)$$

then (HS,π) is clearly an $\mathcal{H}$-algebra since

$$\pi(T^*T) = \sum_{\alpha\varepsilon\Lambda} \| Te_\alpha \|^2 \geq 0$$

and $\pi(T^*T) = 0$ if and only if $Te_\alpha = 0$ for all α, i.e. $T = 0$. If we assume H to be separable, then HS becomes an L^2-separable $\mathcal{H}$-algebra.

<u>Remark</u>. HS is an example of an algebra without identity. This is not a restriction since any B*-algebra $\mathcal{A}$ can be isometrically embedded in another B*-algebra $\mathcal{A} \times \mathbb{C}$ with identity, see {14}. If $\mathcal{A}$ is an L^2-separable $\mathcal{H}$-algebra, then $\mathcal{A} \times \mathbb{C}$ is also an L^2-separable $\mathcal{H}$-algebra under the inner-product defined by $((x,\alpha),(y,\beta)) = (x,y)_\pi + \bar{\beta}\alpha$.

§ 4.3. <u>The boundary value problem and the expansion theorem.</u> We now proceed to study integral operators related to second order differential equations of the type

$$y'' + (\lambda - q(x))y = 0, \qquad , \lambda \text{ complex},$$

where $q(x)$, $x \varepsilon [a,b]$, and the solutions are in Λ and where we shall assume that $q(x)$ is continuous and hermitian, i.e. $q(x) = q(x)^*$ for all x in $[a,b]$. The

next theorem is basic for the study.

<u>Theorem 4.6</u>. Let $f(x)$ be an A-valued, continuous function on $[a,b]$. Then

$$(4.6) \qquad Y(x) = \psi(x)\int_a^x \alpha(t)f(t)dt + \phi(x)\int_x^b \beta(t)f(t)dt$$

is a particular solution of the differential equation

$$(4.7) \qquad y'' + (\lambda - q(x))y = f(x)$$

on $[a,b]$, where λ is a complex number, the functions ϕ, ψ are the solutions of

$$y'' + (\lambda - q(x))y = 0,$$

satisfying the initial conditions

$$\begin{cases} \phi(a) = \psi'(a) = \sin\mu \\ \phi'(a) = \psi(a) = \cos\mu, \end{cases}$$

$0 \leq \mu < \pi$, and α, β are the solutions of

$$y'' + y(\lambda - q(x)) = 0,$$

satisfying the initial conditions

$$\alpha(a) = \beta'(a) = \sin\mu$$

$$\alpha'(a) = \beta(a) = \cos\mu$$

$0 \leq \mu < \pi$.

<u>Proof</u>. Define

$$M(x) = \psi(x)\,\alpha(x) - \phi(x)\,\beta(x)$$

$$N(x) = \psi'(x)\,\alpha(x) - \phi'(x)\,\beta(x)$$

$$T(x) = \psi'(x)\,\alpha'(x) - \phi'(x)\,\beta'(x)$$

$$S(x) = \phi(x)\,\beta'(x) - \psi(x)\,\alpha'(x)$$

for all $x \in [a,b]$.

Then the functions $M(x)$, $N(x)$, $T(x)$ and $S(x)$ satisfy the first order linear system,

$$\begin{aligned} M'(x) &= N(x) - S(x) \\ N'(x) &= T(x) - (\lambda - q(x))M(x) \\ T'(x) &= (\lambda - q(x))S(x) - N(x)(\lambda - q(x)) \\ S'(x) &= M(x)(\lambda - q(x)) - T(x) \end{aligned} \tag{4.8}$$

and the initial conditions

$$
(4.9)\qquad
\begin{aligned}
M(a) &= -\cos\mu\,\sin\mu + \sin\mu\cos\mu = 0\\
N(a) &= \sin\mu\,\sin\mu + \cos\mu\cos\mu = e\\
T(a) &= \sin\mu\,\cos\mu - \cos\mu\sin\mu = 0\\
S(a) &= \sin\mu\,\sin\mu + \cos\mu\cos\mu = e.
\end{aligned}
$$

The functions $M(x) \equiv 0$, $N(x) \equiv e$, $T(x) \equiv 0$, and $S(x) \equiv e$, for all $x \in [a,b]$, satisfy the system (4.8) and the initial conditions (4.9) and so, by the existence and uniqueness theorem for linear, first order systems, it follows that for all $x \in [a,b]$,

$$
(4.10)\qquad
\begin{aligned}
\psi(x)\,\alpha(x) - \phi(x)\,\beta(x) &= 0\\
\psi'(x)\,\alpha(x) - \phi'(x)\,\beta(x) &= e\\
\psi'(x)\,\alpha'(x) - \phi'(x)\,\beta'(x) &= 0\\
\psi(x)\,\alpha'(x) - \phi(x)\,\beta'(x) &= -e.
\end{aligned}
$$

It is easy now to check that (4.6) is a particular solution of (4.7). Indeed, by differentiation and taking into account (4.10), we obtain

$$
Y'(x) = \psi'(x)\int_a^x \alpha(t)f(t)\,dt + \phi'(x)\int_x^b \beta(x)f(t)\,dt +
$$

$$
+ [\psi(x)\alpha(x) - \phi(x)\beta(x)]f(x) =
$$

$$
= \psi'(x)\int_a^x \alpha(t)f(t) + \phi'(x)\int_x^b \beta(t)f(t)\,dt
$$

so

$$Y''(x) = -(\lambda - q(x))\psi(x)\int_a^x \alpha(t)f(t)dt -$$

$$- (\lambda - q(x))\phi(x)\int_x^b \beta(t)f(t)dt + [\psi'(x)\alpha(x) - \phi'(x)\beta(x)]f(x) =$$

$$= - (\lambda - q(x))Y(x) + f(x).$$

We now obtain a useful property, for the further development of the theory.

<u>Theorem 4.7</u>. Let λ be a complex number such that either $Im\ \lambda \neq 0$, or $\lambda < \inf\{\sigma(q(x)): a \leq x \leq b\}$ when λ is real. If ϕ satisfies

$$(4.11) \qquad y'' + (\lambda - q(x))y = 0,$$

for all $x \in [a,b]$, and the initial conditions

$$\phi(a) = \sin\mu$$

$$\phi'(a) = \cos\mu, \qquad 0 \leq \mu \leq \pi/2,$$

then the element $\phi^{*\prime}(b)\phi(b)$ is invertible.

<u>Proof</u>. Let $\lambda = \omega + i\delta$. From (4.11) we obtain that

$$\phi^*(x)\phi''(x) + \phi^*(x)(\omega - q(x))\phi(x) + i\delta\phi^*(x)\phi(x) = 0.$$

Since

$$(\phi^*(x)\phi'(x))' = \phi'^*(x)\phi'(x) + \phi^*(x)\phi''(x),$$

then

$$(\phi^*(x)\phi'(x))' = \phi'^*(x)\phi'(x) - \phi^*(x)(\omega - q(x))\phi(x) -$$

$$- i\delta\phi^*(x)\phi(x)$$

and, by integration from a to b,

$$(4.12)\quad \phi^*(b)\phi'(b) = \phi^*(a)\phi'(a) + \int_a^b \phi'^*(x)\phi(x)dx -$$

$$- \int_a^b \phi^*(x)(\omega - q(x))\phi(x)dx - i\delta\int_a^b \phi^*(x)\phi(x)dx =$$

$$= \sin\mu\cos\mu + \int_a^b \phi'^*(x)\phi'(x)dx + \int_a^b \phi^*(x)(q(x) - \omega)\phi(x)dx -$$

$$- i\delta\int_a^b \phi^*(x)\phi(x)dx.$$

Since

$$Im(\phi^*(b)\phi'(b)) = -\delta\int_a^b \phi^*(x)\phi(x)dx$$

and the integral is a positive element of A, if follows from Proposition 4.4 that $\phi^*(b)\phi'(b)$ is invertible when $\delta = Im(\lambda) \neq 0$. If λ is real, i.e. $\lambda = \omega$, then from

$$\sigma(q(x) - \omega) = \sigma(q(x)) - \omega,$$

it follows that for all $x \in [a,b]$,

$$\sigma(q(x) - \omega) \geq \inf\{\sigma(q(x)) : x \in [a,b]\} - \omega > 0.$$

Thus the element $q(x) - \omega$ is strictly positive and hence by Proposition 4.2 it has a positive square root, $Q(x)$. Hence (4.12) becomes

$$\phi^*(b)\phi'(b) = \sin\mu\cos\mu + \int_a^b \phi'^*(x)\phi'(x)dx +$$

$$+ \int_a^b [Q(x)\phi(x)]^*[Q(x)\phi(x)]dx$$

and by Proposition 4.3 it follows that $\phi^*(b)\phi'(b)$ is

strictly positive and thus invertible.

Finally, if $\phi^*(b)\phi'(b)$ is invertible, so is $\phi'^*(b)\phi(b) = [\phi^*(b)\phi'(b)]^*$.

We can now prove the following theorem:

<u>Theorem 4.8</u>. Let λ be a complex number such that $Im\,\lambda \neq 0$ or, when λ is real, $\lambda < \inf\{\sigma(q(x) : x \in [a,b]\}$ and let ϕ,ψ be defined as in Theorem 4.6. Then the boundary value problem

$$y'' + (\lambda - q(x))y = f(x), \tag{4.13}$$

$$\begin{aligned} &y(a,\lambda)\cos\mu - y'(a,\lambda)\sin\mu = 0 \\ &\phi^{*\prime}(b,\lambda)y(b,\lambda) = 0, \qquad 0 \leq \mu \leq \pi/2, \end{aligned} \tag{4.14}$$

has a unique solution that can be written as

$$y(x,\lambda) = \int_a^b G(x,t,\lambda)f(t)dt,$$

where $G(x,t,\lambda)$, termed the Green's function of the problem (4.13), (4.14), is continuous at every point $a \leq x,t \leq b$.

<u>Proof</u>. The general solution of (4.13) is of the form

$$y(x,\lambda) = \psi(x)\int_a^b \alpha(t)f(t)dt + \phi(x)\int_a^b \beta(t)f(t)dt + \psi(x)A_1 + \phi(x)A_2$$

as follows from Theorem 4.6.

In order to find the values of the constants A_1 and A_2, we make use of the boundary conditions (4.14). Since

$$y(a,\lambda) = \phi(a)\int_a^b \beta(t)f(t)dt + \psi(a)A_1 + \phi(a)A_2 =$$

$$= \sin\mu\int_a^b \beta(t)f(t)dt - A_1\cos\mu + A_2\sin\mu,$$

and

$$y'(a,\lambda) = \phi'(a)\int_a^b \beta(t)f(t)dt + \psi'(a)A_1 + \phi'(a)A_2 =$$

$$= \cos\mu\int_a^b \beta(t)f(t)dt + A_1\sin\mu + A_2\cos\mu,$$

it follows that

$$0 = y(a,\lambda)\cos\mu - y'(a,\lambda)\sin\mu = \sin\mu\cos\mu\int_a^b \beta(t)f(t)dt -$$

$$- A_1\cos^2\mu + A_2\sin\mu\cos\mu - \sin\mu\cos\mu\int_a^b \beta(t)f(t)dt -$$

$$- A_1\sin^2\mu - A_2\sin\mu\cos\mu = - A_1,$$

and hence $A_1 = 0$.

Also

$$0 = \phi'^*(b,\lambda)y(b,\lambda) = \phi^{*\prime}(b)\psi(b)\int_a^b \alpha(t)f(t)dt + \phi^{*\prime}(b)\phi(b)A_2,$$

and then Theorem 4.7 implies

$$A_2 = -[\phi'^*(b)\phi(b)]^{-1}\phi^{*\prime}(b)\psi(b)\int_a^b \alpha(t)f(t)dt.$$

Hence we can write

$$y(x,\lambda) = \int_a^b G(x,t,\lambda)f(t)dt, \tag{4.15}$$

where we define

$$G(x,t,\lambda) = \begin{cases} \psi(x)\alpha(t) - \phi(x)[\phi'^*(b)\phi(b)]^{-1}\phi'^*(b)\psi(b)\alpha(t), & t \le x, \\ \phi(x)\beta(t) - \phi(x)[\phi'^*(b)\phi(b)]^{-1}\phi'^*(b)\psi(b)\alpha(t), & t > x. \end{cases} \tag{4.16}$$

It remains to show that $G(x,t)$ is continuous on $[a,b] \times [a,b]$. Indeed at any point (x_0,t_0), $x_0 < t_0$, we have

$$\lim_{(x,t)\to(x_0,t_0)} G(x,t) =$$

$$= \lim_{(x,t)\to(x_0,t_0)} \{\phi(x)\beta(t) - \phi(x)[\phi'^*(b)\phi(b)]^{-1}\phi'^*(b)\psi(b)\alpha(t)\} =$$

$$= G(x_0,t_0).$$

since the functions $\phi(x)$, $\beta(t)$, $\psi(x)$ and $\alpha(t)$ are all

continuous, and the same argument applies at (x_0, t_0) if $x_0 > t_0$.

Suppose now that $x_0 = t_0$. Then $G(x,t)$ is continuous at (x_0, x_0) if and only if

$$\psi(x_o)\alpha(x_o) - \phi(x_o)[\phi'^*(b)\phi(b)]^{-1}\phi'^*(b)\psi(b)\alpha(x_o) =$$

$$= \phi(x_o)\beta(x_o) - \phi(x_o)[\phi'^*(b)\phi(b)]^{-1}\phi'^*(b)\psi(b)\alpha(x_o),$$

and this equality holds if and only if

$$\psi(x_o)\alpha(x_o) = \phi(x_o)\beta(x_o)$$

for all x_o on $[a,b]$. By (4.10) this is always the case and so $G(x,t)$ is everywhere continuous.

We now require the following theorem concerning the symmetry of the Green's function.

<u>Theorem 4.11</u>. Under the hypotheses of Theorems 4.6 and 4.8, when λ is real, the Green's function (4.16) is hermitian symmetric.

<u>Proof</u>. If λ is real, then $\phi^*(x) = \alpha(x)$ and $\psi(x) = \beta(x)$ on $[a,b]$. Thus

$$G(x,t) = \begin{cases} \psi(x)\phi^*(t) - \phi(x)[\phi'^*(b)\phi(b)]^{-1}\phi'^*(b)\psi(b)\phi^*(t) & \text{if } t \leq x \\ \phi(x)\psi^*(t) - \phi(x)[\phi'^*(b)\phi(b)]^{-1}\phi'^*(b)\psi(b)\phi^*(t) & \text{if } t > x. \end{cases}$$

Hence $G(x,t)$ is hermitian symmetric if and only if

$$G^*(t,x) = \begin{cases} \psi(x)\phi^*(t) - \phi(x)\psi^*(b)\phi'(b)[\phi^*(b)\phi'(b)]^{-1}\phi^*(t) & \text{if } t \leq x \\ \phi(x)\psi^*(t) - \phi(x)\psi^*(b)\phi'(b)[\phi^*(b)\phi'(b)]^{-1}\phi^*(t) & \text{if } t > x, \end{cases}$$

is equal to $G(x,t)$, and this is clearly so if

$$\phi(x)[\phi'^*(b)\phi(b)]^{-1}\phi'^*(b)\psi(b)\phi^*(t) =$$

$$\phi(x)\psi^*(b)\phi'(b)[\phi^*(b)\phi'(b)]^{-1}\phi^*(t),$$

i.e. if

$$\phi^*(b)\psi(b)[\phi^*(b)\phi'(b)] = [\phi'^*(b)\phi(b)]\psi^*(b)\phi'(b) .$$

But this last equality holds because of (4.10). Consequently, $G(x,t,\lambda)$ is real symmetric.

We can state now an expansion theorem for Λ-valued functions.

<u>Theorem 4.12</u>. Let Λ be an L^2-separable $\mathcal{H}$-algebra,

$[a,b]$ a finite interval, $q(x)$ an A-valued, continuous function on $[a,b]$ such that

$$\inf \{\sigma(q(x)) : x \in [a,b]\}$$

is finite, and ϕ as in Theorem 4.6, for $0 \leq \mu \leq \pi/2$.

Then the problem

$$y'' + (\lambda - q(x))y = 0,$$

$$y(a,\lambda) \cos \mu - y'(a,\lambda) \sin \mu = 0, \quad 0 \leq \mu \leq \pi/2$$
$$\phi'^*(b,\lambda)y(b,\lambda) = 0,$$

has a solution only for a countable number of real values of : $\lambda_1, \ldots$. Also if Z_i is the solution corresponding to λ_i, then for any A-valued, twice continuously differentiable function $g(x)$ on $[a,b]$ satisfying the boundary conditions

$$(4.17) \qquad \begin{aligned} 0 &= g(a) \cos \mu - g'(a) \sin \mu, \quad 0 \leq \mu \leq \pi/2 \\ 0 &= g(b), \end{aligned}$$

we have

$$g(x) = \sum_i (g, Z_i)\, Z_i(x).$$

<u>Proof</u>. Let λ_o be a fixed real number such that

$$\lambda_o < \inf\{\sigma(q(x)) : x \in [a,b]\} .$$

By Theorem 4.8, the solution to the problem

$$(4.18)\qquad \begin{aligned} &y'' + (\lambda_o - q(x))y = f(x) \\ &y(a,\lambda_o)\cos\mu - y'(a,\lambda_o)\sin\mu = 0, \quad 0 \le \mu \le \pi/2 \\ &\phi'^*(b,\lambda_o)y(b,\lambda_o) = 0, \end{aligned}$$

where $f \in C(a,b;\Lambda)$, is given by

$$Kf(x) = \int_a^b G(x,t,\lambda_o)f(t)dt,$$

with a kernel $G(x,t,\lambda_o)$ that is continuous and hermitian symmetric.

Consequently, the operator K is self-adjoint and compact on $L^2(a,b;\Lambda)$. Thus it has a countable orthonormal set of eigenfunctions $Z_1, \ldots$, corresponding to its non-zero eigenvalues $\lambda_1, \ldots$, such that for all $f \in C(a,b;\Lambda)$ we have

$$(4.19)\qquad Kf(x) = \sum_i (Kf,Z_i)Z_i(x)$$

The eigenfunctions Z corresponding to non-zero

eigenvalues λ^{-1} are given by the integral equation

$$\lambda \int_a^b G(x,t,\lambda_o) Z(t)\,dt = Z(x). \tag{4.20}$$

Hence they satisfy the problem

$$\begin{aligned} Z'' + (\nu - q(x))Z &= 0, \quad \nu = \lambda_o - \lambda \\ Z(a,\nu)\cos\mu - Z'(a,\nu)\sin\mu &= 0, \quad 0 \le \mu \le \pi/2 \\ \phi^*(b,\nu)\, Z(b,\nu) &= 0. \end{aligned} \tag{4.21}$$

On the other hand any non-trivial solution Z of the problem (4.21) satisfies (4.20). Therefore Z is an eigenfunction of the operator K corresponding to a non-zero eigenvalue $(\lambda_o - \nu)^{-1}$.

Finally let $g(x)$ be twice continuously differentiable on a,b, and satisfy the boundary conditions,

$$\begin{aligned} g(a)\cos\mu - g'(a)\sin\mu &= 0, \quad 0 \le \mu \le \pi/2 \\ g(b) &= 0. \end{aligned}$$

Then $g(x)$ is a solution of the problem (4.18) if $f(x)$ is taken as

$$f(x) = g''(x) + (\lambda_o - q(x))g(x).$$

Hence $g(x)$ is given by

$$g(x) = Kf(x) = \int_a^b G(x,t,\lambda_o) f(t)\,dt,$$

and so by (4.19)

$$g(x) = \sum_i (g,Z_i)\, Z_i(x).$$

§ 4.4. <u>An extension of two of Weyl's theorems.</u>

We present here two theorems concerning integrable square solutions on (o,∞).

<u>Theorem 4.13</u>. Let ϕ and ψ be two linearly independent solutions of

$$Ly \equiv -y'' + q(x)y = \lambda_o y,$$

where λ_o is a fixed complex number and ϕ, ψ belong to $L^2(0,\infty;A)$, i.e.

$$|||\phi|||_\pi^2 = \int_0^\infty \pi(\phi^*(t)\phi(t))\,dt < \infty, \quad |||\psi|||_\pi^2 = \int_0^\infty \pi(\psi^*(t)\psi(t)\,dt < \infty.$$

Then any solution of

$$Ly = \lambda y$$

for an arbitrary complex number λ, belongs to $L^2(0,\infty;A)$.

<u>Proof</u>. Let α and β be the associated functions of ϕ and ψ, respectively, i.e.

$$\alpha'' + \alpha(\lambda_o - q(x)) = 0, \quad \beta'' + \beta(\lambda_o - q(x)) = 0,$$

and

$$\alpha(0) = \phi(0), \qquad \beta(0) = \psi(0)$$
$$\alpha'(0) = \phi'(0), \qquad \beta'(0) = \psi'(0).$$

Let y be a solution of

$$y'' + (\lambda - q(x))y = 0.$$

This we can write as

$$Ly = \lambda_o y + (\lambda - \lambda_o)y,$$

and we can apply the method of variation of parameters to obtain

$$y(x) = \phi(x)c_1 + \psi(x)c_2 + (\lambda_o - \lambda)\int_p^x [\psi(x)\alpha(t) - \phi(x)\beta(t)]y(t)dt,$$

where p is an arbitrary constant.

We define

$$|||y|||_p(x) = \{\int_p^x \|y(t)\|^2 dt\}^{\frac{1}{2}}$$

(4.22)

$$|||y|||_p(\infty) = |||y|||_p.$$

(4.23) $$\|\int_p^x [\psi(x)\alpha(t) - \phi(x)\beta(t)]y(t)dt\| \leq$$

$$\leq \int_p^x \{\|\psi(x)\|\cdot\|\alpha(t)\| + \|\phi(x)\|\cdot\|\beta(t)\|\}\|y(t)\|dt \leq$$

$$\leq [\|\psi(x)\|\{\int_p^x \|\alpha(t)\|^2 dt\}^{\frac{1}{2}} + \|\phi(x)\|\{\int_p^x \|\beta(t)\|^2 dt\}^{\frac{1}{2}}]|||y|||_p(x),$$

the last inequality following by Schwarz's inequality.

Let $p \leq x \leq u$. Then $|||y|||(x) \leq |||y|||(u)$, and since

$$|||\psi|||_o^2 = \int_0^\infty \|\psi(t)\|^2 dt \leq M^2 < \infty,$$

$$|||\phi|||_o^2 = \int_0^\infty \|\phi(t)\|^2 dt \leq M^2 < \infty,$$

by hypothesis, (4.23) implies that

$$\left\| \int_p^x [\psi(x)\alpha(t) - \phi(x)\beta(t)]y(t)dt \right\| \leq$$

$$\leq M\,[\,\|\psi(x)\| \;\; |||y|||_p(x) + \|\phi(x)\| \;\; |||y|||_p(x)\,] \leq$$

$$\leq M\;|||y|||_p(u)\,[\;\|\psi(x)\| + \|\phi(x)\|\;].$$

Therefore

$$|||y|||_p(u) = \{\int_p^u \|y(x)\|^2\,dx\}^{\frac{1}{2}} \leq \|c_1\|^2\;|||\psi|||_p +$$

$$+ \|c_2\| \;\; |||\phi|||_p + |\lambda_o - \lambda|\,\{\int_p^u [M|||y|||_p(u)\;\|\psi(x)\|\,]^2\,dx\}^{\frac{1}{2}} +$$

$$+ |\lambda_o - \lambda|\;\{\int_p^u [M|||y|||_p(u)\;\|\phi(x)\|\,]^2 dx\}^{\frac{1}{2}} \leq$$

$$\leq (\|c_1\| + \|c_2\|)M + |\lambda_o - \lambda|M|||y|||_p(u)\{\int_p^u \|\psi(x)\|^2\,dx\}^{\frac{1}{2}} +$$

$$+ |\lambda_o - \lambda|M|||y|||_p(u)\{\int_p^u \|\phi(x)\|^2\,dx\}^{\frac{1}{2}} \leq$$

$$\leq \; (\|c_1\| + \|c_2\|)M + 2|\lambda_o - \lambda|M^2 |||y|||_p(u).$$

If we take p large enough so that $|\lambda_o - \lambda| M^2 < \frac{1}{4}$, we have

$$|||y|||_p(u) < 2(\|c_1\| + \|c_2\|)M.$$

Since the right-hand side is independent of u, we have

$$|||y|||_p \leq 2(\|c_1\| + \|c_2\|)M.$$

Consequently, $|||y|||_o < \infty$ and, by the continuity of π we have $|||y|||_\pi < \infty$.

<u>Theorem 4.14</u>. (Nested Circles Theorem). Let $q(x)$ be an A-valued, continuous, hermitian function on $[0,\infty)$ and let ϕ and ψ be the solutions of the differential equation

$$(4.23) \qquad y'' + (\lambda - q(x))y = 0, \qquad Im\,\lambda > 0,$$

determined by the initial conditions

$$\phi(0,\lambda) = e, \qquad \psi(0,\lambda) = 0,$$
$$\phi'(0,\lambda) = 0, \qquad \psi'(0,\lambda) = e,$$

and assume that

$$\lim_{b\to\infty} \int_0^b \pi(\psi^*(t,i)\psi(t,i))dt = \infty ,$$

i.e. $\psi(t,i) \notin L^2(0,\infty;A)$.

Then there exists a unique linearly independent solution $\chi(t,\lambda)$ of (4.24) for each λ, $Im\,\lambda > 0$, such that $\chi \in L^2(0;\infty;A)$.

<u>Proof</u>. Let χ be a solution of (4.24), $\chi \neq \psi$. It can be written as

$$\chi(t,\lambda) = \phi(t,\lambda) + \psi(t,\lambda)m(\lambda)$$

for some $m(\lambda) \in A$.

We shall make use of the following well-known Lagrange's identity:

$$(4.25) \qquad \int_0^b [z^*Ly - (Lz)^*y]dt = [y,z](b) - [y,z](0),$$

where

$$Ly \equiv -y'' + q(x)y$$

and

(4.26) $$[y,z](t) = z^{*\prime}(t)y(t) - z^{*}(t)y'(t) .$$

For every fixed $b > 0$, let us consider the set C_b of points m such that

(4.27) $$\pi[\chi,\chi](b) = 0.$$

We can give two explicit analytic definitions of C_b as follows.

From (4.25) applied to the solution χ, we obtain

$$\pi[\chi,\chi](b) = \pi[\chi,\chi](0) + \int_0^b \pi[\chi^* L\chi - (L\chi)^*\chi]dt ,$$

which becomes

$$\pi[\chi,\chi](b) = -2i\,\pi(Im\ m) + (\lambda - \bar{\lambda})\int_0^b \pi(\chi^*\chi)dt,$$

since $L\chi = \lambda\chi$.

Hence

(4.28) $$C_b = \{ m : \pi(Im\ m) = Im\ \lambda \int_0^b \pi(\chi^*(t)\chi(t))dt \} .$$

The second expression for C_b needs more elaboration. Indeed we have that

(4.29) $$0 = \pi[\chi,\chi](b) = \pi\{\chi'^*(b)\chi(b) - \chi^*(b)\chi'(b)\} =$$

$$= \pi \{ [\phi'^*(b) + m^*\psi'^*(b)][\phi(b) + \psi(b)m] -$$

$$- [\phi^*(b) + m^*\psi^*(b)] [\phi'(b) + \psi'(b)m]\} .$$

To simplify the notation we set

$$[\psi,\psi](b) = \psi'^*(b)\psi(b) - \psi^*(b)\psi'(b) = iR$$

$$[\psi,\phi](b) = \phi'^*(b)\psi(b) - \phi^*(b)\psi'(b) = S$$

$$[\phi,\phi](b) = \phi'^*(b)\phi(b) - \phi^*(b)\phi'(b) = iT.$$

It is clear that R and T are positive elements of the algebra A. Indeed from (4.25) applied to ψ we have

$$[\psi,\psi](b) = [\psi,\psi](0) + \int_0^b (\lambda - \bar{\lambda})\psi^*(t)\psi(t)dt$$

and since $[\psi,\psi](0) = 0$ we obtain

$$R = 2\, Im\, \lambda \int_0^b \psi^*(t)\psi(t)dt.$$

Similarly

$$T = 2\, Im\, \lambda \int^b \phi^*(t)\phi(t)dt.$$

Since $\psi(t)$ and $\phi(t)$ can only have isolated

zeros, it follows that R and T are strictly positive and hence (4.29) can be written as

$$0 = \pi(im^*Rm - m^*S^* + Sm + iT) .$$

Thus, with the substitutions

$$R^{\frac{1}{2}}m = Z, \qquad iS = W, \qquad P = R^{-\frac{1}{2}}W^*,$$

we obtain as the equation for C_b,

$$(4.30) \qquad C_b = \{Z : \pi((Z - P)^*(Z - P)) = r_b^2\} ,$$

where

$$(4.31) \qquad r_b^2 = \pi(W(b)R^{-1}(b)W^*(b) - T(b)) .$$

Equation (4.30) suggest that we call C_b a circle. Then the continuity of π and equations (4.28) and (4.30) give

$$(4.32) \quad \textit{Interior-}C_b = \{m : \pi(Im\, m) > Im\, \lambda \int_0^b \pi(\chi^*(t)\chi(t))dt\},$$

and

$$(4.33) \quad \textit{Interior}\ C_b = \{Z : \pi((Z - P)^*(Z - P)) < r_b^2,\ m = R^{-\frac{1}{2}}Z\}.$$

Let c be an arbitrary real number such that $0 < b < c$, and let $m \in C_c$. Then

$$\pi(Im\, m) = Im\, \lambda \int_0^c \pi(\chi^*(t)\chi(t))dt > Im\, \lambda \int_0^b \pi(\chi^*(t)\chi(t)dt ,$$

and (4.32) implies that $m \in Interior\, C_b$. This means that C_c is contained in C_b and so $r_c^2 < r_b^2$.

Any sequence $b_1, \ldots, b_n, \ldots$ of increasing positive numbers approaching infinity defines a sequence $r_{b_1}^2, \ldots, r_{b_n}^2, \ldots$ of strictly decreasing positive numbers bounded below by 0. Hence $\lim_{b \to \infty} r_b = r_\infty$ exists.

Let

$$m_\infty \in C_\infty = \{Z : \pi((Z - P)^*(Z - P)) = r_\infty^2, m = R^{-\frac{1}{2}}Z\}.$$

We have that $m_\infty \in Interior\, C_b$, for all $b > 0$, and therefore

$$\pi(Im\, m_\infty) > Im\, \lambda \int_0^b \pi\{[\phi^*(t,\lambda)+m_\infty^*(\lambda)\psi^*(t,\lambda)][\phi(t,\lambda) + \psi(t,\lambda)m_\infty(\lambda)]\}dt.$$

We can take limits as $b \to \infty$ to obtain

$$(4.34) \qquad \int_0^\infty \pi\{[\phi^* + m_\infty^* \psi^*][\phi + \psi m_\infty]\}dt \leq \frac{\pi(Im\, m_\infty)}{Im\, \lambda} < \infty .$$

Consequently $\chi = \phi(t,\lambda) + \psi^*(t,\lambda)m_\infty$ belongs to $L^2(0,\infty;A)$, for every m_∞ in C_∞.

But the previous theorem states that there is at most one linearly independent solution χ in $L^2(0,\infty;A)$ for each value of λ. Hence $r_\infty = 0$ and

$$(4.35) \qquad m_\infty = \lim_{b\to\infty} R^{-1}(b,\lambda)W^*(b,\lambda) =$$

$$= \lim_{b\to\infty} [\psi,\psi]^{-1}(b)[\phi,\psi](b).$$

§ 4.5. <u>An approximation theorem for functions on $(-\infty,\infty)$.</u> In this section we assume that A is a countably compact, separable H-algebra, with the property that there exists a basis $\{e_n\}$ such that for every n there exists m, $m \geq n$, for which $e_1, \ldots, e_m$ is a basis for the subalgebra A_m generated by $e_1, \ldots, e_n$. (a,b) will represent a finite or infinite interval.

<u>Theorem 4.15</u>. Let A be an algebra with the above properties and let

$$q(x) = \sum_{j=1}^{n} q_j(x)e_j$$

where $q_j(x) = (q(x),e_j)$ are assumed complex-valued and continuous, for all $1 \leq j \leq n$ (n fixed), on (a,b), and such that there exists $\lambda_0 < \{\sigma(q(x) : x \in (a,b)\}$.

Let ϕ and ψ be the solutions of

$$y'' + (\lambda - q(x))y = 0,$$

satisfying the initial conditions

$$\phi(t_o,\lambda) = \psi'(t_o,\lambda) = e$$

$$\phi'(t_o,\lambda) = \psi(t_o,\lambda) = 0,$$

where $t_o \in (a,b)$.

Then there exists a complex matrix-valued $\sigma_m = \{\sigma_{mij}\}$ such that

$$\phi_j(\lambda) = \int_a^b \pi(e_j^*\psi^*(t,\lambda)f(t))dt,$$

$1 \leq j \leq m$, $m = m(n)$, and

$$f(x) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{m} \phi_i(\lambda)\psi(x,\lambda)e_j d\sigma_{mij}(\lambda)$$

define mutually inverse isometric mappings from $L^2(a,b;A_m)$ to $L^2_{\sigma_m}$ where A_m is the subalgebra generated by $e_1,\ldots,e_n$, and with basis $e_1, \ldots, e_m$.

<u>Proof</u>. We can consider the differential equation

$$y'' + (\lambda_o - q(x))y = f(x)$$

where $f(x)$ is A_m-valued and of compact support $[\alpha,\beta]$.

Let $\{Z_{ms}\}$, $s = 1, 2, \ldots$ be a complete orthonormal set of functions in $L^2(a,b;A_m)$ as in the proof of Theorem 4.12. It is clear that for each positive integer s,

$$Z_{m,s}(x) = \psi(x)C_s,$$

for some constant C_s in A_m. If we put

$$C_s = \sum_{j=1}^{m} \alpha_{js} e_j,$$

we have that

$$\int_\alpha^\beta \pi(f^*(t)f(t))dt = \sum_{s=1}^{\infty} \left| \int_\alpha^\beta \pi(Z^*_{ms}(t)f(t))dt \right|^2 =$$

$$= \sum_{s=1}^{\infty} \left(\sum_{s=1}^{m} \alpha_{js} \int_\alpha^\beta \pi(e_j^* \psi^*(t)f(t)dt) \left(\overline{\sum_{k=1}^{m} \bar{\alpha}_{ks} \int_\alpha^\beta \pi(e_k^* \psi^*(t)f(t))dt} \right) \right) =$$

$$= \int_{-\infty}^{\infty} \sum_{j,k=1}^{m} \phi_j(\lambda)\overline{\phi_k(\lambda)} d\sigma_{mjk}(\lambda;\alpha,\beta)$$

where

$$\phi_j(\lambda) = \int_\alpha^\beta \pi(e_j^* \psi^*(t,\lambda)f(t))dt$$

and

$$\sigma_{mjk}(\lambda;\alpha,\beta) = \begin{cases} -\sum\limits_{o\geq\lambda_p>\lambda} \alpha_{jp}\bar{\alpha}_{kp}, & \text{if } \lambda < 0 \\ \sum\limits_{o>\lambda_p\geq\lambda} \alpha_{jp}\bar{\alpha}_{kp}, & \text{if } \lambda > 0. \end{cases}$$

The remaining part of the proof follows as in Theorem 3.1. The functions to use in the proof of a lemma analogous to Lemma 3.1 are defined now as follows: $v_k(x,h) = v(x,h)e_k$, where $v(x,h) > 0$ for $x \in (t_o, t_o+h)$, $v(x,h) = 0$ otherwise, and

$$\int_0^h v(t,h)dt = 1.$$

<u>Corollary 4.16</u>. Let $f(x)$ be given by

$$f(x) = \sum_{j=1}^{\infty} f_j(x)e_j.$$

Then $f(x)$ can be written as

$$(4.36) \qquad f(x) = \lim_{m\to\infty} \int_{-\infty}^{\infty} \sum_{i,j=1} \phi_{im}(\lambda)\psi(x,\lambda)e_j d\sigma_{mij}(\lambda),$$

where

$$(4.37) \qquad \phi_{im}(\lambda) = \int_a^b \pi(e_i^*\psi^*(t,\lambda)F_m(t))dt.$$

<u>Proof</u>. By defining $F_m(x)$ as

$$F_m(x) = \sum_{j=1}^{m} f_j(x) e_j,$$

we can apply the theorem to F_m to obtain (4.37) and

$$F_m(x) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{m} \phi_{im}(\lambda)\psi(x,\lambda) e_j d\sigma_{mij}(\lambda).$$

Since $f(x) = \lim_{m \to \infty} F_m(x)$ in the $L^2(a,b;A)$-norm, (4.36) follows.

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