EXPLICIT BRAUER INDUCTION

AND

SHINTANI DESCENT

By

Brian McCudden B.A., M.A., M.Ed.

A thesis submitted to the School of Graduate Studies

in Partial Fulfillment of the requirements for the Degree

Doctor of Philosophy

McMaster University

July 1992.

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DOCTOR OF PHILOSOPHY

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: Explicit Brauer Induction and Shintani Descent

AUTHOR: Brian McCudden B.A. (York University)
M.A. (York University)
M.Ed.(University of Toronto)

SUPERVISOR: Professor V. Snaith

NUMBER OF PAGES: v, 122

ABSTRACT:

This paper demonstrates an application of Explicit Brauer Induction. Snaith [28], introduced this canonical form for Brauer's induction theorem. An overview of this development is given in chapter one with a new proof of Brauer's theorem.

In [29-35] Snaith applied Explicit Brauer Induction, primarily in the construction of invariants of representations of finite groups from invariants of one-dimensional characters. Following [28,29,30] Chapter two presents proof that the Grothendieck group $R_+(G,S^1)$ is a ring with properties of induction, restriction, inflation, and Frobenius reciprocity.

There exists a ring homomorphism

$$b : R_+(G,S^1) \longrightarrow R(G)$$

which was introduced as a footnote in [26,P.71]. We discuss the map T_G, [30 Pp.454-469], which is a section to b. Deligne [6, Pp.501-597] has devised generators for Ker b, in the case for G solvable. We examine the case $G = D_8$. Boltje [1,2] has devised another section to the map b, termed a_G. This development and examples are given in Chapter three.

In Chapter four, conjugacy classes and character values, for the matrix group $GL_2\mathbf{F}_q$ are reviewed for all irreducible representations including the cuspidal case [13,Pp.122].

The main result of this paper is contained in Chapter five. We develop the Explicit Brauer Induction formula for $a_G(\rho)$ where $G = GL_2\mathbf{F}_q$ and ρ is an irreducible representation of G. This development is used to describe Shintani descent between the irreducible representations of $GL_2\mathbf{F}_q{}^n$ and the irreducible representations of $GL_2\mathbf{F}_q$. The original derivation of Shintani descent [27, Pp.396-414], uses norms on character values. In the construction given here, the Shintani norm is not used, but rather, the correspondence is obtained by applying Hilbert Theorem 90 to the maximal one dimensional characters which appear in the expression for $a_G(\rho)$.

This thesis is dedicated to my wife Veronica and my sons James and Christopher with sincere thanks for their confidence in me and for their help and support.

I would like to thank Doctor Victor Snaith for his encouragement, patience and assistance in guiding my research.

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INTRODUCTION

This paper demonstrates several applications of Explicit Brauer Induction. Snaith,[28], introduced this canonical form for Brauer's Induction theorem as follows. For a finite group G, a finite dimensional representation is expressed as a linear combination of monomial represntations, for unitary, orthogonal (even dimensional) and symplectic representations. In these cases "monomial" means that the representation is of the type $\mathrm{Ind}_H^G(\phi : H \longrightarrow X_1)$ where $X_1 = U(1)$, $O(2)$ or $Sp(1)$, depending on the type of representation.

Let X_n be one of the classical groups (unitary, orthogonal or symplectic), $X_n = U(n)$, $O(2n)$ or $Sp(n)$. For $X_n = U(n)$ let Y_n denote the normalizer of the standard, diagonal, maximal torus. If $X_n = Sp(n)$, then Y_n will denote the normalizer of the diagonal subgroup, $X_1^n = Sp(1)^n$, and if $X_n = O(2n)$, then Y_n will denote the normalizer of the subgroup of diagonal 2 × 2 blocks (i.e., $O_2(\mathbb{R})^n = O(2)^n$ along the diagonal). In each case Y_n is given by the wreath product $Y_n = \Sigma_n \int X_1$. By using several special topological and algebraic properties of these groups, Snaith proved that for a representation, $\rho: G \longrightarrow X_n$, ρ may be given as the sum of monomial representations.

In Chapter one, Snaith's development of Explicit Brauer Induction is given along with a new proof of Brauer's Theorem reformulated as $\oplus_i R(H_i) \xrightarrow{\mathrm{Ind}_H^G} R(G)$ where H_i is a p-elementary subgroup of G, and R(G) is the complex representation ring of G. A key feature of Snaith's work is that the cannonical induction form may be used constructively. This is demonstrated for a new example $G = A_4$.

Let G be a finite group, then once a canonical form for Brauer induction has been obtained, a presentation for R(G) in terms of monomial representations can be made. When G is solvable this was done by Langlands and Deligne [6]. Serre [26, P.71] gives some indication and motivation for this presentation.

Snaith's development followed from a reformulation of Brauer's induction theorem. Let $R_+(G,S^1)$ be the free abelian group on the G - conjugacy classes $(H,\phi)^G = (G \supset H \xrightarrow{\phi} S^1)$ of linear characters of subgroups. Define $b:R_+(G,S^1) \longrightarrow R(G)$ by $b(G \supset H \xrightarrow{\phi} S^1) = \mathrm{Ind}_H^G(\phi)$. Brauer's Theorem states that b is onto.

Boltje [1] and Snaith [28], [29], [30] define an explicit Brauer induction formula as a section for the map, b. Given such a section, it may be used to obtain a presentation for R(G) in terms of generators $(G \supset H \xrightarrow{\phi} S^1)$. Snaith [28], [29] produced such a section T_G.

In Chapter two, we define a product on the free abelian group $R_+(G,S^1)$ given by:

$$(G \supset H \xrightarrow{\phi} S^1).\ (G \supset K \xrightarrow{\chi} S^1)$$
$$= \sum_{w \in K\backslash G/H} \supset (w^{-1} K w) \cap H \xrightarrow{w^*(\phi)\chi} S^1$$

With this product we then prove that $R_+(G,S^1)$ is a ring with the properties of induction, restriction and inflation. Further, we show that $R_+(G,S^1)$ has the Frobenius reciprocity property and hence that it is a Green Functor. Also in Chapter two, we give Deligne's development [6], of generators for the kernel of the map b. We compare the Snaith and Deligne constructions for the case of $G = D_8$.

The Snaith development [28], [29] of T_G is obtained by means of the algebraic topology of smooth group actions related to G - representations. Boltje [1],[3], has algebraically constructed another such section, a_G, where

$$a_G\colon R(G) \longrightarrow R_+(G,S^1).$$

We review the development of a_G in Chapter three, and give two examples for the cases $G = D_8$ and $G = PSL_2\mathbf{F}_5$. Boltje, Snaith and Symonds, [3], have derived both algebraic and topological formulae for a_G and T_G, and have constructed a formula that relates the two maps directly. This relation is also discussed in Chapter three.

The main result of this thesis is the use of Boltje's explicit Brauer induction formula to develop a correspondence between the irreducible representations of $GL_2\mathbf{F}_{q^d}$ with entries from $\mathbf{F}_{q^d}$ and the irreducible representations of $GL_2\mathbf{F}_q$. The original derivation of this correspondence was given by Shintani [27]. In Chapter four, the four conjugacy classes and character values for the matrix group $GL_2\mathbf{F}_q$ are reviewed for the four types of irreducible representations. This material is based on the development of the Weil representation [13, Pp.122-123] and material from Shintani, [27], Lang, [20] and Naimark and Stern [22].

The construction of the Weil representation, $r(\theta)$, is given by considering the complex vector space of functions $f:\mathbf{F}^*_{q^2} \longrightarrow C$, with some special properties, including the Fourier transform. The Weil representation, $r(\theta)$, is obtained from a character θ: $\mathbf{F}^*_{q^2} \longrightarrow \mathbb{C}^*$ which is distinct from its conjugate by Frobenius map, ie: for $\sigma \in G\ (\mathbf{F}_{q^2}/\mathbf{F}_q)$, and $z \in \mathbf{F}^*_{q^2}$ then $\theta(z) \neq \theta(\sigma(z))$. A defining feature of this type of representation is that it respects formulae given by the Bruhat decomposition for $GL_2\mathbf{F}_q$, [20], where the following matrices generate $GL_2\mathbf{F}_q$ -$\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}$, $\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, and where $\alpha,\ \delta \in \mathbf{F}^*_q$ and $u \in \mathbf{F}_q$. The other three irreducible representations are obtained by induction from characters of the Borel subgroup $B = \{\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}\}$ where $\alpha,\ \delta \in \mathbf{F}^*_q$ and $\beta \in \mathbf{F}_q$, [20].

In the first part of Chapter five, the calculation of some of the maximal pairs in the explicit Brauer induction formula $a_G(\rho)$ is given, where ρ is an irreducible representation of $G = GL_2\mathbf{F}_q$. The four maximal pairs obtained characterize the four types of irreducible representations which occur in the expansion of $a_G(\rho)$.

The Shintani Correspondence [27], for $GL_2\mathbf{F}_q$ is a one-one correspondence of the following form:

Let $F \in G(\mathbf{F}_{q^d}/\mathbf{F}_q)$ denote the Frobenius map then

$$\begin{array}{c}\{\text{Irreducible representations, } \nu\text{, of } GL_m\mathbf{F}_{q^d}\text{, fixed under } F\} \\ \Big\updownarrow \text{Sh} \\ \{\text{Irreducible representations, } SH(\nu)\text{, } \quad GL_m\mathbf{F}_q\}\end{array}$$

This correspondence is characterised by means of the Shintani norm. For $X \in GL_m\mathbf{F}_{q^d}$, define

$$N(X) = F^{n-1}(X)\ F^{n-2}(X)\ .....F(X)X.$$

By using the explicit Brauer induction formula $a_G(\rho)$, where ρ is an irreducible representation of $G = GL_2\mathbf{F}_{q^d}$, we describe a correspondence of the above type. In fact, this correspondence is the same as that of Shintani for the case of $GL_2\mathbf{F}_{q^d}$. The significant difference between Shintani's procedure and this one is in the application of Hilbert Theorem 90, [20], to the maximal one dimensional characters which appear in the formula $a_G(\rho)$. The Shintani norm is not used in this development.

CHAPTER 1

EXPLICIT BRAUER INDUCTION

This chapter is an overview of the development of Explicit Brauer Induction .

As expressed in [28],[29] for a finite group G, one often uses the Brauer - style induction theorem when studying representations (unitary, orthogonal or symplectic) of G. This approach is especially common in determining invariants in number theoretic contexts -eg: looking for invariants of representations of Galois groups.

The Brauer inductive theorem has a drawback in the sense that it is non- constructive; it is an existence theorem and cannot be used to produce explicit general formulae or to prove the existence of invariants of representations and to calculate them. Snaith developed a canonical form for the classical induction theorem, which he has termed "Explicit Brauer Induction". This theorem is constructive.

As an example, if $\rho: G \longrightarrow U(n)$ is a unitary representation of G, then Explicit Brauer Induction gives a canonical formula for ρ in the representation ring of G - in terms of monomial representations (those induced from one-dimensional characters of subgroups). Snaith has applied this technique to obtain formulae for arithmetical invariants of Galois representations, such as the Artin root number and the Deligne-Langlands local root number.

Once a canonical form for Brauer induction has been obtained, a presentation for R(G) in terms of monomial representations can be made. When G is a solvable group, this was done by Langlands and Deligne. Serre,[26, P.71] gives some indication and motivation of this problem as follows:

If N is the kernel of $\mathbb{Q} \times \mathrm{Ind}; \bigoplus_{H \in X} \mathbb{Q} \otimes R(H) \longrightarrow \mathbb{Q} \otimes R(G)$ where X is a family of subgroups stable under conjugation (e.g.: cyclic subgroups of G) and $G = \bigcup_i H_i$ where $H_i \in X$ then one can show that N is generated over $\mathbb{Q}$ by elements of the following types:

(i) For $H, H' \in X$, with $H' \subset H$, let $\chi' \in R(H')$ and $\chi = \mathrm{Ind}_{H'}^{H}(\chi) \in R(H)$ then $(\chi - \chi') \in N$.

(ii) For $H \in X$ (as above) and $x \in G$ set ${}^{x}H = xHx^{-1}$.
Let $\chi \in R$
(H) and ${}^{x}\chi \in R({}^{x}H)$
defined ${}^{x}\chi\,(xhx^{-1}) = \chi(h)$ for $h \in H$
then $(\chi - {}^{x}\chi) \in N$.

This gives a "presentation" for $\mathbb{Q} \times R(G)$ in terms of induced characters (H,χ). For G solvable this has been taken further by Langlands-Deligne to give such a presentation for R(G).

Later we will give a presentation for R(G) and return to the Deligne- Langlands presentation.

1.1 BRAUER INDUCTION THEOREM

Following Serre [26, Pp.68-79], Artin's Theorem states that, in R(G), the Grothendieck group of finite dimensional complex representations of G, every character ρ of G is a linear combination with rational coefficients of induced characters from cyclic subgroups. Brauer extended this result to maintain that each character ρ of G may be expressed as a linear combination with integar coefficients of characters induced from characters of elementary subgroups. Orthogonal and symplectic versions of Brauer's theorem may be found in [21, Pp. 66,73]

ie: $\rho: G \longrightarrow X_n$ where $X_n = U(n)$, $O(2n)$, $Sp(n)$ and $n \geq 1$.

Serre proved Brauer's Induction theorem as follows:

Consider the subgroup V_p (p-prime) of R(G) generated by characters induced from the p-elementary subgroups of G. A p-group is a group whose elements have order a power of p and a p-elementary group H, is a group which can be written C x P where C is cyclic, of order prime to p, and P is a p- group.

Serre proves that the index of V_p in R(G) is finite and prime to p, and then uses this result to show that the sum of the V_p, called V, equals R(G). The proof of the last assertion is outlined as follows: since the index of V in R(G) divides that of V_p (for $V_p \subset V$) so by the above result, this index is prime to p. Now since this is true for all primes p, the index of V in R(G) must be 1 and hence V = R(G).

Two other proofs of the Brauer Induction theorem are given in [34, P.499] and [38, P.168].

1.2 EXPLICIT BRAUER INDUCTION:

Let $R_F(G)$ denote the Grothendieck group of finite-dimensional F- representations of a finite group G, for a field F. Let X_n be any one of the following compact Lie groups: X_n = U(n), O(2n) or Sp(n) $n \geq 1$; ie: the unitary, orthogonal or symplectic groups respectively. Also suppose that $\rho : G \longrightarrow X_n$ is a unitary, orthogonal or symplectic representation of G.

If X_n is equal to U(n) or Sp(n) let Y_n denote the normaliser in X_n of the standard diagonal maximal "torus", X_1^n and if X_n = O(2n let Y_n denote the normaliser in X_n of the subgroup $O(2)^n$, given by th matrices which are the direct sum of an n-tuple of elements of O(2) [4, Chapter 11].

Explicit Brauer Induction as described by Snaith is a canonical form in which ρ is expressed, in $R_F(G)$, as a linear combination of representations induced from representations of subgroups of G into X_1. This type of induced representation is termed MONOMIAL.

In each case Y_n is the wreath product generated by Σ_n, the symmetric group, acting on X_1^n by the permutation action

1.2.1 $$Y_n = \Sigma_n \int X_1$$

Hence if $A \subset \mathbb{C}$ is any ring containing $^1/_{n!}$ - by the above choice of groups, the classifying spaces of X_n and Y_n have the same cohomology with coefficients in A. The inclusion of Y_n into X_n induces this cohomology isomorphism - so that the fibre X_n/Y_n has the A - cohomology of a point - and so:

1.2.2 $$H^*(X_n/Y_n;\ \mathbb{C}) \cong H^*(\text{point};\ \mathbb{C}) = \begin{cases} 0, & * \neq 0 \\ \mathbb{C}, & * = 0 \end{cases}$$

Snaith, [8, P.462], proves the above by using a calculation of Borel's. The inclusion of the diagonal torus T^n into $U(n)$ induces the cohomology isomorphism, which applied to this case via the Serre spectral sequence for the extension

$$T^n \rightarrowtail Y_n \twoheadrightarrow \Sigma_n$$

gives the required result.

A homomorphism of groups ϕ: $G \longrightarrow Y_n$ is called a monomial homomorphism. The equivalence relation imposed on monomial homomorphisms is induced by inner automorphisms of G and Y_n. Hence a monomial homomorphism is a homomorphism into a matrix group. It is not a representation until it is composed with the inclusion of Y_n into X_n. Let ρ: $G \longrightarrow Y_n$ be a monomial homomorphism, then for $i: H \hookrightarrow G$, $\mathrm{Res}^G_H(\rho)$ will denote the composition of ι and ρ. Since G is finite (or for G a compact Lie group with H of finite index in G) and T: $H \longrightarrow Y_n$ a monomial homomorphism then the induction construction [28] yields:

1.2.3 $$\mathrm{Ind}^G_H\ (T):\ G \longrightarrow Y_{nd}\ (d = [G:H])$$

A monomial representation means a representation of the form: $\mathrm{Ind}^G_H(\phi : H \longrightarrow X_1) \in R_F(G)$ with H of finite index in G (if G is a compact Lie group).

Monomial homomorphisms and representations are related in the following way:

1.2.4 **Lemma** Let $\rho : G \longrightarrow Y_n \subset X_n$ be a representation. It may be written as the direct sum of monomial representations. That is:

$$\rho = \sum_{i=1}^{t} \operatorname{Ind}_{H_i}^{G}(\rho_i) \in R_F(G). \quad \text{with } \rho_i : H_i \longrightarrow X_1$$

in which the set $\{\rho_i\}$ is well defined up to permutation and conjugation in G.

<u>PROOF</u> There is a canonical projection $\pi = Y_n \longrightarrow \Sigma_n$ and we have a split extension:

$$X_1^n \overset{j}{\rightarrowtail} \Sigma_n \int X_1 = Y_n \xrightarrow{\pi} \Sigma_n$$

If G does not act transitively through $\pi\rho$ on $\{1,2\ldots n\}$ then ρ gives one summand for each orbit. If the action of G is transitive, then if we set $H_1 = \{h \in G \mid \pi\rho(h)(1) = 1\}$ and write for $h \in H_1$, $\rho(h) = \sigma(h)\,[\rho_1(h), \rho_2(h)\ldots]$

(where $\sigma(h) \in \Sigma_n$, $\rho_i(h) \in X_1$) Since $\sigma(h)$ fixes 1 for all $h \in H$, then $\rho = \rho_1 : H \longrightarrow X_1$ is a homomorphism and hence $\rho = \operatorname{Ind}_H^G(\rho_1 : H \longrightarrow X_1)$ is a monomial homomorphism, and similarly for ρ_i $(1 \le i \le t)$ so that

1.2.5 $$\rho = \sum_{i=1}^{t} \operatorname{Ind}_{H_i}^{G}(\rho_i) \in R_F(G).$$

The association $(\rho_i : H_i \longrightarrow X_1) \longleftrightarrow (\rho : G \longrightarrow Y_n)$ defines a bijection between subhomorphisms defined on H_i and monomial homomorphisms ρ when $n = [G:H]$ and stab $(\pi.\rho_i)(1) = H_i$. Let $\{x_i ; 1 \le i \le n\}$ be a set of coset representatives for ${}^G/_H$. Given $g \in G$ define $\sigma(g) \in \Sigma_n$ by

$$\begin{cases} g x_i = x_{(g)(i)} h(i,g) \\ \text{and } h(i,g) \in H_i \end{cases} \quad \text{for } 1 \le i \le n$$

$\rho(g)$ is given by $\sigma(g)[\rho_i(h(1,g)), \rho_i(h(2,g)\ldots\rho_i(h(n,g))]$

By choosing different coset representatives say $\{x_i a_i ;\ 1 \le i \le n;\ a_i \in H_i\}$ the result is to conjugate

$\rho(g)$ above by $(\rho_1(a_1)^{-1}, \rho_1(a_2)^{-1}, \ldots \rho_1(a_n)^{-1})$

If we choose the stabiliser of $k (1 \leq k \leq n)$ instead of the stabilizer of 1 in $H_1 = \{h \in G \mid \pi\rho(h)(1) = 1\}$, this changes H_1 to pHp^{-1} for some $p \in G$ and changes the expression for $\rho(g)$ to its conjugate by p. That is,

$(G \supset H_1 \xrightarrow{\rho_1} X_1$ is mapped to $(G \supset K \xrightarrow{\chi} X_1)$ where these are equivalent in the sense that there exists $(g,y) \in G \times X_1$ such that $K = g\, H_1\, g^{-1}$ and $y(\rho_1(h))y^{-1} = \chi(ghg^{-1})$ for all $h \in H_1$. In addition ρ is composed with an inner automorphism of G and with a map induced by an inner automorphism of X_1.

Before expressing the Explicit Brauer Induction theorem we require some additional notation, [28, Pp.6-8]. Let $\rho: G \longrightarrow X_n$ be an F -representation where ρ is a unitary, orthogonal or symplectic representation of G. Consider the left action of G, by way of ρ on ${}^{X_n}/Y_n$.

Let M denote the orbit space $M = G \backslash ({}^{X_n}/Y_n)$
Let (J) denote the conjugacy class of a subgroup J, in G, and let $M_{(J)}$ be the set of those G- orbits of type (J), which is isomorphic to ${}^{G}/J$.

Set $\chi^{\#}_{(J)}$ as the Euler characteristic of $M_{(J)}$ with respect to compactly supported cohomology, $H^{*}(-; \mathbb{C})$. Alternatively, in terms of the usual Euler characteristic, χ, we have
1.2.6 $\chi^{\#}(X) = \chi(\bar{X}) - \chi(\bar{X} - X)$ where $\bar{X}$ denotes the closure of X.
Now, if, as in this case, X is a finitely triangulated space then $\chi(X)$ can be calculated combinatorially as the alternating sum over the dimension i, of the numbers of i- simplices of X.

If $g_J \in X_n$ is an element which lies above an orbit of type (J) in M, ie: the G- stabiliser of $g_J Y_n$ is conjugate to J, then denoting the stabliser of $g_J Y_n$ by $H(g_J, \rho)$ we have

$$H(g_J, \rho) = \rho^{-1}(g_J\, Y_n\, g_J^{-1}) = \mathrm{stab}_G\,(g_J\, Y_n)$$

Define $T_G(\rho) \in R_F(G)$ by

1.2.7 $$T_G(\rho) = \sum_{(J)} \chi^{\#}{}_{(J)} \ \mathrm{Ind}^G_{H(g_J,\rho)} \quad (g_J{}^{-1} \ \rho \ g_J) \in R_F(G)$$

In the above expression, (J) runs through the conjugacy classes of subgroups of G and $g_J \in X_n$ is any choice of element whose orbit is of type (J).

We express the Explicit Brauer Induction theorem as follows, [29, Pp. 284-286]:

Let ρ: $G \longrightarrow X_n$ be a representation; unitary, orthogonal or symplectic. By way of ρ, let G act on $X = X_n/Y_n$ by left translation with orbit space M.

1.2.8 Theorem: (i) In $R_F(G)$, $1 = \sum_{(J)} \chi^{\#}_{(J)} \ \mathrm{Ind}^G_J(1)$

where (J) runs over conjugacy classes of subgroups of G and $\chi^{\#}_{(J)}$ is the Euler characteristic..

(ii) In $R_F(G)$, $\rho = \sum_{(J)} \chi^{\#}_{(J)} \ \mathrm{Ind}^G_J(\mathrm{Res}^G_J(\rho)) = T_G(\rho)$

(iii) If $g_J \in X_n$ and $H(g_J,\rho) = \mathrm{stab}_G(g_J Y_n)$,

then $g_J^{-1} \ \rho(H(g_J,\rho) \ g_J \in Y_n$.

(And so by the previous Lemma 1.2.4

$\mathrm{Ind}^G_J(\mathrm{Res}^G_J(\rho))$ has a canonical form as a sum of monomial representations.)

<u>PROOF</u>: Part (iii) follows from the definitions

since $H(g_J, \rho) = \rho^{-1}(g_J Y_n \ g_J{}^{-1})$

$= \mathrm{stab}_G(g_J Y_n)$

$$\Rightarrow \quad g_J^{-1}\rho(H(g_J,\ \rho)g_J \in Y_n$$

Part (ii) follows from part (i) by application of the Frobenius reciprocity formula after multiplication by ρ.

$$\text{ie:} \quad \rho = \Sigma_{(J)} \chi^{\#}_{(J)} \operatorname{Ind}_J^G (1).\ \rho$$
$$= \Sigma_{(J)} \chi^{\#}_{(J)} \operatorname{Ind}_J^G (\operatorname{Res}_J^G (\rho)).$$

To prove part (i) we use the following result from [29, Pp.215-216]

Let ρ: $G \longrightarrow X$ be a unitary, orthogonal or symplectic representation and Y a closed subgroup of X. Then for any $g \in G$

1.2.9 Theorem $\qquad \chi(^X/Y) = {}_{\alpha}\Sigma_{(J)}\ \chi^{\#}(M_\alpha)\ |\ (^G/J)^g\ |$

$$= \Sigma_{(J)}\ \chi^{\#}(M_{(J)})\ |\ (^G/J)^g|$$

where $|(^G/J)\ ^g|$ is the number of cosets J such that $gzJ = zJ$.

The first sum is taken over the conjugacy classes (J) of subgroups J of G and over connected components M_α of orbit type subsets $M_{(J)} \subset G\backslash X/Y = M$.

The second sum is taken over orbit types, (J).

PROOF: The homogeneous space X_n/Y_n is a left G - space by means of the action

$$g(xY) = \rho(g)xY \qquad g \in G,\ x \in X_n$$

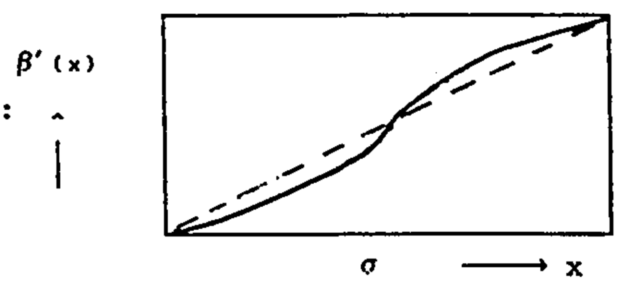
and this G - space admits a G - equivariant triangulation forming it into a finite G - simplicial complex in such a way that the orbit space, $M = G^{\backslash X_n}/Y_n$ inherits the G - quotient simplicial structure.

If $H \subset G$ then $M_{(H)}$ denotes the set of G - orbits in M which are G - isomorphic to $^G/H$. Since M is a compact stratified space (for $X_n = U(n)$, $SO(2n)$ or $Sp(n)$), it is possible to find a homotopy H′ from the identity map to a map β', defined β': $M \longrightarrow M = G^{\backslash X_n}/Y_n$ where H′ has only a finite number of isolated fixed points.

One may ensure that H' and β' are well balanced with respect to the orbit-type structure.

This can be done inductively by moving M very slightly within each simplex. One leaves all vertices fixed throughout the homotopy H'. On a one-simplex, σ, let H' be the homotopy which moves linearly within σ from the identity map to β'

ie:



Then, on each two simplex, one superimposes H' radially on top of the homotopy already given on the boundary - and proceed inductively, [29, Pp.281-282].

If $M = G\backslash X_n/Y_n$ has m - simplices $\{\Delta^m_i\}$, let $b_{m,i} \in \Delta^m_i$ denote the barycentre. Fix (f) will denote the set of fixed points of a self map f.

There exists a homotopy

1.2.10 $\quad H'(x,0) = x$ and $H'(x,1) = \beta'(x)$

which satisfies the following four properties:

(i) $\operatorname{Fix}(\beta') = \{b_{m,i}\}$ the set of barycentres of the triangulation.

(ii) H' preserves the triangulation of M and hence

$H'(M_{(H)},t) \subset M_{(H)}$ for each orbit type (H) and for all $t \in I = [0,1]$

(iii) The fixed point index of $\beta'|\Delta_i^m$ is denoted by $I_{\beta'|\Delta_i^m}(b_{m,i})$ and equals (-1^m) .

(iv) Let M_α be a connected component (ie:path component) of $M_{(H)}$. Denote by $\chi^\#(M_\alpha)$ the internal Euler characteristic of M_α

$$\chi^\#(M_\alpha) = \chi(M_\alpha) - \chi(M_\alpha - M_\alpha)$$

where χ is the usual Euler characteristic. With this notation,

1.2.11 $$\chi^\#(M_\alpha) = \sum_{b_{m,i} \in M_\alpha} I_{\beta'|\Delta_i^n}(b_{m,i})$$

The above formula is obtained by applying the Lefschetz fixed point theorem to

$$\beta' : M_\alpha \longrightarrow M_\alpha$$

Let π: ${}^{X}n/Y_n \longrightarrow M = G\backslash{}^{X}n/Y_n$ be the canonical projection and H', β' be as defined, then there exists a G - equivariant homotopy

1.2.12 $H: ({}^{X}n/Y_n) \times I \longrightarrow {}^{X}n/Y_n$ such that

(a) $H(xY_n, 0) = xY_n \quad \forall x \in X_n$

(b) $\pi(H(xY_n, t)) = H^1(\pi(xY_n),t)$ for all $x \in X_n$, $t \in I$

(c) If β: ${}^{X}n/Y_n \longrightarrow {}^{X}n/Y_n$ is defined by
$\beta(xY_n) = H(xY_n, 1)$
then Fix $(\beta) = \{\pi^{-1}(b_{m,i})\}$.

From [29, P214] one knows that $H^1 : MxI \longrightarrow M$ satisfying 1.2.10(ii) may be lifted to an equivariant homotopy starting at 1. For such a lifting, H, the map $\beta = (H,1)$: ${}^{X}n/Y_n \longrightarrow {}^{X}n/Y_n$, will satisfy 1.2.12(a) and (b) and Fix (β) will be a subset of

$$\{\pi^{-1}\ (\mathrm{Fix}\ (\beta'))\} = \{\ \pi^{-1}(b_{m,i})\}.$$

To make these two sets equal - ie; to satisfy (c), let us consider how H is constructed from H^1.

First, by taking a barycentric subdivision, the triangulation may be refined to ensure that Fix (β') consists of zero cells of the triangulation. Then H may be constructed by induction over the skeleton of the refined triangulation. Thus the homotopy

$$H\colon \{\pi^{-1}\ (b_{m,i})\} \times I \longrightarrow {}^{X}n/Y_n$$

may be defined to satisfy (c) and extended in such a way as to satisfy 1.2.12(a),(b). Now by 1.2.10 (ii), for each $b_{m,i}$, the path $H'(b_{m,i},t)$ does not leave the interior $\overset{o}{\Delta}{}^m_i$ of the simplex Δ^m_i of the original triangulation. Hence we may choose $(x_{m,i}Y_n)$ above $b_{m,i}$ and a slice, $S \subset {}^{X}n/Y_n$, through $(x_{m,i}Y)$

such that

(d) $\pi(S)$ is a neighbourhood of $b_{m,i}$ in Δ^m_i containing $H'(b_{m,i},t)$ for all $t \in I$ and

(e) If $V = \mathrm{stab}(x_m, iY_n)$ is the subgroup of G stabilising $(x_m, iY_n))$ then V fixes S

and $\psi : {}^{G}/V \times S \longrightarrow S$

$(g\ V, s) \longrightarrow gs$

defines an equivariant homeomorphism onto a neighbourhood of (x_m, iY_n).

Given 1.2.12(d) and (e) it is possible to choose a homeomorphism $\hat{\pi}\colon S \longrightarrow \pi(S)$. With $s \in S$, $g \in G$ and $t \in I$ define

1.2.13 $H(gs,\ t) = g\ (\hat{\pi}\ (H^1\ (\pi(s),\ t\ ))) = \psi\ (g\ V,\ \hat{\pi}\ (H^1(\pi(s),t)))$

Then H^1 is a well-defined, equivariant, lifted homotopy on a neighbourhood of each $\{b_{m,i}\}$. If $b_{m,i} = \pi(s)$

then $s = g(x_{m,i})Y$ and

$$\begin{aligned}\beta(s) &= H(s,1) = H(\psi(gV,\ x_{m,i}Y),1)\\ &= g\ (\hat{\pi}\ (\beta'(b_{m,i}))\\ &= g\ (x_{m,i})Y\\ &= s \text{ as needed.}\end{aligned}$$

With β, β' as defined we may consider the following commutative diagram:

1.2.14
$$\begin{tikzcd} {}^{X_n}/Y_n \arrow[r, "x\beta"] \arrow[d, "\pi"'] & {}^{X_n}/Y_n \arrow[d, "\pi"] \\ G\backslash X_n/Y_n \arrow[r, "\beta'"] & G\backslash X_n/Y_n \end{tikzcd}$$

Since X_n is connected $x\beta$ is homotopic to the identity map of ${}^{X_n}/Y_n$. Consequently its Lefschetz number is equal to the Euler Characteristic $\chi({}^{X_n}/Y_n)$.

that is $\chi({}^{X_n}/Y_n) = L(x\beta, {}^{X_n}/Y_n)$

1.2.15 $\quad = \sum_n (-1)^n(\text{Trace}:(x\beta)_*: Hn({}^{X_n}/Y_n;\ \mathbb{Q}) \longrightarrow Hn({}^{X_n}/Y_n;\mathbb{Q})$

However, by the Lefschetz fixed point theorem, we have

$$L(x\beta,\ {}^{X_n}/Y_n) = \sum_{s\in \text{Fix }(x\beta)} I_{x\beta}(s)$$

where $I_{x\beta}(s)$ is the local fixed point index at a point $s\in{}^{X_n}/Y_n$ such that $\beta(x) = s$. The only such points in Fix($x\beta$) lie above points $b_{m,i} \in$ Fix (β') and if J = stab ($b_{m,i}$), there are precisely $|({}^{G}/J)^x|$ points of Fix ($x\beta$) above $b_{m,i}$. This happens because β is the identity on $\pi^{-1}(b_{m,i})$ by 1.2.12(c).

In addition, the local fixed point index $I_{(x\beta)}(s)$ at a point $s \in {}^{X_n}/Y_n$, such that $\pi(s) = b_{m,i}$ is equal to $I_{\beta'|\Delta_i^m}(b_{m,i})$

because π is locally trivial at s by 1.2.13.

Therefore we rewrite 1.2.15 as:

1.2.16

$$\chi({}^{X_n}/Y_n) = \sum_{s \in Fix(x\beta)} I_{x\beta)}(s) = \sum_{\alpha,(J)} \sum_{b_{m,i} \in M_\alpha} (I_{\beta'}|_{\Delta_i^m}(b_{m,i})\{|({}^G/J)^x|\}$$

where the sum is taken over the conjugacy classes, (J), of the subgroups J of G and over connected components, M_α, of $M_{(J)}$.

Substituting into the above expression the value of $\chi^\#(M_\alpha)$, 1.2.6 we have,.

1.2.17 ie: $\chi^\#(M_\alpha) = \chi(\bar{M}_\alpha) - \chi(\bar{M}_\alpha - M_\alpha) = \sum_{b_{m,i} \in M_\alpha} I_{\beta'}|_{\Delta_i^m}(b_{m,i})$

which gives $\chi({}^{X_n}/Y_n) = \sum_{\alpha,(J)} \chi^\#(M_\alpha)|({}^G/J)^x|$ as required. The second part of the theorem follows from the first since the Euler characteristic is additive:

ie: $$\sum_\alpha \chi^\#(M_\alpha) = \chi^\#(M_{(J)})$$

where M_α runs over the connected components of $M_{(J)}$.

Now from 1.2.2 we have:

$$H^*({}^{X_n}/Y_n;\, \mathbb{C}) \cong H^*(pt;\, \mathbb{C}) = \begin{cases} 0, & * \neq 0 \\ \mathbb{C}, & * = 0 \end{cases}$$

so that $\chi({}^{X_n}/Y_n) = 1$.

Thus substituting in the theorem just proved gives:

1.2.18 $$1 = \sum_{(J)} \chi^\#(M_{(J)})|({}^G/J)^x| = \sum_{(J)} \chi^\#_{(J)} \ \mathrm{Ind}_H^G(g_j,\rho)(1)$$

This proves part (i) of the theorem, 1.2.8. This proof can be shortened as follows for a finite G.

Triangulate ${}^{X_n}/Y_n$ so that it is a simplical complex on which $g \in G$ acts on a simplex σ by either:

1.2.19 (i) fixing every point

or (ii) mapping it homeomorphically onto a simplex $g(\sigma)$ with $\mathrm{int}(g(\sigma)) \cap \mathrm{int}(\sigma) = \phi$.

We call this a cellular action. Now $C_*\ ({}^{X_n}/Y_n : \mathbb{C})$, the simplical chains, are a sum of representations of the form

$$\{g(\sigma) \mid \dim(\sigma) = i,\ g \in {}^G/\mathrm{stab}\ (\sigma)\}$$

$$\cong \mathrm{Ind}_{\mathrm{stab}\ (\sigma)}^{G}\ (1)$$

Hence, in R(G)

1.2.21
$$1 = \sum_i (-1)^i\ H_i\ (X_{n/Y_n}\ ;\ \mathbb{C})$$

$$= \sum (-1)^i\ C_i\ ({}^{X_n}/Y_n;\ \mathbb{C})$$

$$= \sum_{(J)} \chi^{\#}_{(J)}\ \ \mathrm{Ind}_J^G\ (1)$$

where
$$\chi^{\#}_{(J)} = \sum_{\substack{\dim\ (\sigma) = i \\ (J) = \mathrm{stab}\ (\sigma)}} (-1)^i$$

$$= \sum_{\substack{\tau \text{ a simplex of} \\ \backslash X_n/Y_n \\ \text{whose orbit is } G/J}} (-1)^i$$

In using Explicit Brauer Induction to prove Brauer's Theorem in the classical case, we will need the following result.

1.2.22 LEMMA For G, a finite group, there exist M -subgroups H_i (every irreducible representation of H_i is monomial), such that

1.2.23 $$1 = \sum_i n_i \operatorname{Ind}_{H_i}^{G}(1)$$

Recall that $X_n = U(n)$ and $Y_n = \Sigma_n \int X_1$

PROOF This is proved by induction on the order of G. If G is an M-Group the result follows immediately.

If we have $\upsilon : G \longrightarrow X_n$, which is not monomial then application of 1.2.8(i) to υ, implies that no conjugate of υ is contained in Y_n and so $J \lneqq G$ in 1.2.8(i)

1.2.24 Brauer's Theorem The map $\lambda : \bigoplus_i R(H_i) \xrightarrow{\operatorname{Ind}_{H_i}^{G}} R(G)$ is surjective (H_i p- elementary).

PROOF: From 1.2.22 $1 = \sum_i n \operatorname{Ind}_{H_i}^{G}(1)$,
where H_i is an M-Group.
By Frobenius reciprocity for $x \in R(G)$

$$x = x.1 = \sum_i n_i \operatorname{Ind}_{H_i}^{G}(1)x$$
$$= \sum_i n_i \operatorname{Ind}_{H_i}^{G}(\operatorname{Res}_{H_i}^{G}(x))$$

and so since we are proceeding by induction on G we may assume that G is an M-group.

Consider A = Image (λ). Now by Frobenius Reciprocity A is an ideal in R(G).

In addition, if ρ is an irreducible representation with $\dim(\rho) \geq 2$, then since $\rho = \operatorname{Ind}_{H}^{G}(\phi)$ where $\phi : H \longrightarrow \mathbb{C}^*$ and $H \lneqq G$, then $\rho \in A$.

If G is not abelian, then there exists an irreducible ρ, with $\dim(\rho) \geq 2$, and if $\bar{\rho}$ is its dual then $\rho.\bar{\rho} \in A$. But if χ is a monomial representation of G, then

$$\langle \chi, \rho\, \bar{\rho} \rangle_G = \langle \chi\, \rho, \rho \rangle_G = \begin{cases} 1 \text{ if } \chi = 1 \\ 0 \text{ otherwise} \end{cases}$$

and so $\rho\, \bar{\rho} = 1 + \sum_i \rho_i$ where $\dim(\rho_i) \geq 2$

Hence $1 \in A$ and so $A = R(G)$.

It remains to consider the abelian case, where $G = G_1 \times G_2 \times \ldots \times G_r$

where G_i is a p_i- group such that $p_i \neq p_j$ for $i \neq j$

From Artin's theorem,

1.2.25 $\quad |G|.1 \in \sum_i \mathrm{Ind}_{H_i}^{G}(\phi)$ where $\phi : H_i \longrightarrow \mathbb{C}^*$

and H_i is cyclic.

We have for $\bar{G}_2 = G_2 \times G_3 \times \ldots \times G_r$

$$|G|/|G_1| .1 \in \sum_i (1_{G_1} \otimes \mathrm{Ind}_{H_i}^{\bar{G}_2}(\phi)) \in \otimes_i R(G_i) = R(G)$$

where H_i is cyclic

But $\quad |G|/|G_1| \cdot 1 \in \sum_i \mathrm{Ind}_{G_1 \times H_i}^{G}(\phi)$ for H_i cyclic [26,P.75],

where $G_1 \times H_i$ is p-elementary and these results are true for any G_i

and $\bar{G}_i = G_1 \times G_2 \times \ldots G_{i-1} \times G_{i+1} \ldots \times G_r$.

The HCF of the $|G|/|G_i|$ is one since each G_i is a p_i-group and $(|G_i|, |G_j|) = 1$.

Hence $1 \in \oplus_{i,j} R(G_i \times H_j)$

and so $\oplus_{i,j} R(G_i \times H_j) = R(G)$

1.3 We now give an example of an Explicit Brauer Induction application.

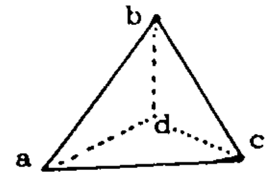
The following formula, [34, P.499], relates the Euler characteristic of the fixed point sets in

$$X = X_n/Y_n \text{ (ie: } \chi((X_n/Y_n)^H))$$

and the special Euler characteristics $\chi^{\#}_{(H)}$

1.3.1 $$\chi((X_n/Y_n)^H) = \chi^{\#}_{(H)} \; | N_G(H)/H \; | + \sum_{(J) \geq (H)} \chi^{\#}_{(J)} \; | \; (G/J)^H |$$

We may use this formula to evaluate the $\chi^{\#}_{(H)}$ for the specific case of $G = A_4$.



From [37], we have the following presentation:

1.3.2 $\quad A_4 = \langle 2,3,5:\ 2^2 = 3^2 = 5^3 = 1;$

$\quad 4 = 2.3 = 3.2,\ 2.3.5 = 5.2.$

$\quad 5.3 = 2.5\rangle$

A_4 is the group of even permutations on a set $\{a,b,c,d\}$; it is also isomorphic to the group of rotations in $\mathbb{R}^3$ which stabilize a regular tetrahedron with barycentre, the origin. Thus we can identify:

$$2 = (ab)(cd),\quad 3 = (ac)(bd)$$
$$4 = (ad)(bc),\quad 5 = (abc)$$

and $\{2,3,4\}$ correspond to the reflections of the tetrahedron through lines joining the midpoints of opposite sides. There are eight elements such as 5 which correspond to rotations of $\pm 120^0$ about a line joining a vertex and the barycentre.

Now $|A_4| = 12$ and since $\langle 5\rangle$ is not normal in A_4 then by Sylow theory, A_4 has four subgroups of order three and one subgroup of order four, and from the given presentation the conjugacy classes are four in number:

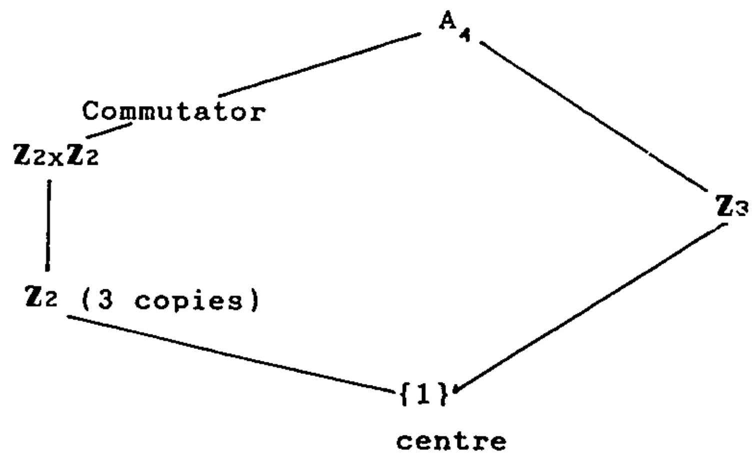
ie: 1, (2,3,4,) (5, 6=2.5, 7=3.5, 8=4.5)

(9=5, 10=2.9, 11=3.9, 12=4.9)

The last two classes are distinct because their images are distinct in $A_4/\langle 2,3\rangle \cong \mathbb{Z}_3$.

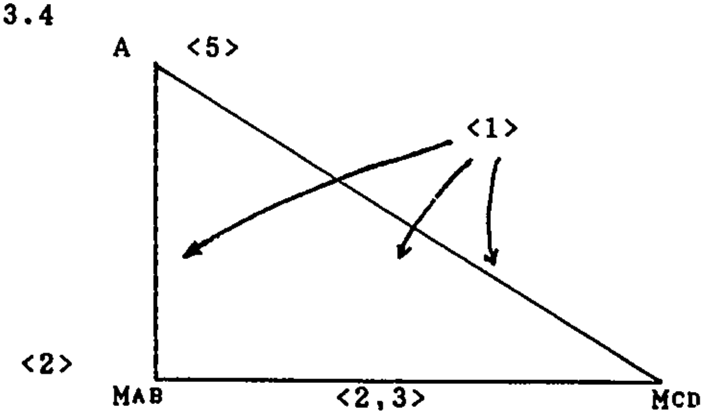
Hence we have the following lattice for A_4:

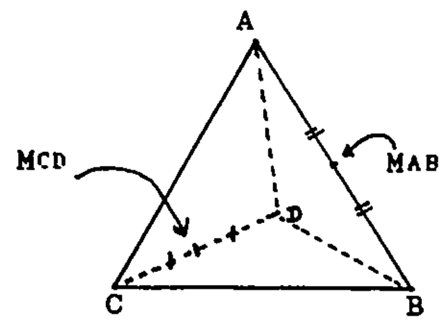
1.3.3



By adjusting the action of Σ_4 on $\mathbb{R}P^2$(the projectivised cube) as shown in [34, Pp,501-2], or by considering the action of A_4 on the tetrahedron, we obtain a fundamental domain as follows:

1.3.4





Hence we need only consider the subgroups <1>, <2>, <2,3> and <5> in evaluating 1.3.1.

From our earlier derivation of Explicit Brauer Induction 1.2.8(i) we demonstrate:

1.3.5
$$1 = \sum_{(H)} \chi^{\#}{}_{(H)} \ \operatorname{Ind}_{H}^{A_4} (1)$$

where by definition

$$\mathrm{Ind}_H^{A_4}(1) = |\text{(fixed points in } A_4/H)|$$

$$= |\{gH \mid xgH = gH \in A_4/H\}|$$

where x runs through the conjugacy classes of A_4.

We may use this definition of $\mathrm{Ind}_H^{A_4}(1)$ to verify the following table:

1.3.6

x (a representative of the class)	$\lvert(A_4/\langle 5\rangle)^x\rvert$	$\lvert(A_4/\langle 2,3\rangle)^x\rvert$	$\lvert(A_4/\langle 2\rangle)^x\rvert$	$\lvert(A_4/\langle 1\rangle)^x\rvert$
1	4	3	6	12
2	0	3	2	0
5	1	0	0	0
9	1	0	0	0

The top row is quite clear and since $\langle 2,3\rangle$ is normal in A_4, the third column may be verified. Since $A_4/\{5\} = \langle 5\rangle, 2\langle 5\rangle, 3\langle 5\rangle, 4\langle 5\rangle$ we see that 5 fixes $\langle 5\rangle$ as does $5^{-1} = 9$ which verifies column two. Now for column four we may calculate to verify the given results or one may observe that the inclusion of A_4 into Σ_4 maps $\langle 2\rangle$ into $\langle(ab),(cd)\rangle$, so that, as a space on which A_4 acts on the left,

$$A_4/\langle 2\rangle \xrightarrow{\quad\cong\quad} \Sigma_4/\{\langle(ab)(cd)\rangle, \langle(ac)(bd)\rangle\}$$

From [35, P.14] we have:

1.3.7 $\quad |(\Sigma_4/\langle(ab),(cd)\rangle^x| = \begin{cases} 2 \text{ if } x = 2 \\ 0 \text{ if } x = 5 \text{ or } 5^{-1} = 9 \end{cases}$

This gives the required results for column three.

Similarly, $\quad A_4/\langle 1\rangle \xrightarrow{\alpha} \Sigma_4/\langle(bd)\rangle$

so that from the same source, we have

1.3.8 $\quad |(A_4/\{1\})^x| = \begin{cases} 0 \text{ if } x = 2 \\ 0 \text{ if } x = 5 \text{ or } 5^{-1} = 9 \end{cases}$

From the formula for Explicit Brauer Induction 1.2.8(i) we have in $R(A_4)$ a relation of the form

$$1 = x_1 \operatorname{Ind}_{\langle 5\rangle}^{A_4}(1) + x_2 \operatorname{Ind}_{\langle 2,3\rangle}^{A_4}(1) + x_3 \operatorname{Ind}_{\langle 2\rangle}^{A_4}(1) + x_4 \operatorname{Ind}_{\langle 1\rangle}^{A_4}(1)$$

and by taking values from the table 1.3.6 we have:

1.3.9
(a) $1 = 4x_1 + 3x_2 + 6x_3 + 12x_4$
(b) $1 = 3x_2 + 2x_3$
(c) $1 = x_1$
(d) $1 = x_1$

Thus we must have

$$\begin{aligned} 1 &= x_1 \\ 1 &= 3x_2 + 2x_3 \\ -4 &= 4x_3 + 12x_4 \\ \text{or } -1 &= x_3 + 3x_4 \end{aligned}$$

We find the general relations of this kind by setting

$$\begin{aligned} x_3 &= 3k - 1 \\ x_2 &= \frac{1 - 6k + 2}{3} = 1 - 2k \\ x_4 &= \frac{-1 - 3k + 1}{3} = -k \end{aligned}$$

Hence we have:

1.3.10 $$1 = \mathrm{Ind}_{\langle 5\rangle}^{A_4}(1) + \mathrm{Ind}_{\langle 2,3\rangle}^{A_4}(1) - \mathrm{Ind}_{\langle 2\rangle}^{A_4}(1)$$

$$+ k\,[-2\,\mathrm{Ind}_{\langle 2,3\rangle}^{A_4}(1) + 3\,\mathrm{Ind}_{\langle 2\rangle}^{A_4}(1) - \mathrm{Ind}_{\langle 1\rangle}^{A_4}(1)]$$

Assume that A_4 acts via $\upsilon: A_4 \longrightarrow U(3)$ on $^{U(3)}/NT^3 = X$. From [37] there is a 3 dimensional representation of A_4 given by

$$2 \longrightarrow \begin{pmatrix} -1 & . & \\ . & 1 & \\ . & . & -1 \end{pmatrix} ; \quad 3 \longrightarrow \begin{pmatrix} . & -1 & 0 \\ . & . & 1 \\ -1 & . & . \end{pmatrix} ; \quad 5 \longrightarrow \begin{pmatrix} . & . & 1 \\ 1 & . & . \\ . & 1 & . \end{pmatrix}$$

Recalling the formula from 1.3.1 we have

$$\chi(X^H) = \chi^{\#}_{(H)} \left| {N_G(H)}/{H} \right| + \sum_{J>H} \chi^{\#}_{(J)} \left|({G}/{J})^H\right|$$

and from 1.3.4 we only need H = <1>, <2>, <2,3> and <5>.

For H = <5> we have $N_{A_4}(\langle 5\rangle)/_{\langle 5\rangle} = 1$ as $N_{A_4}(\langle 5\rangle) = \langle 5\rangle$

and we have $\chi^{\#}_{\langle 5\rangle} = 1$ from 1.3.8 and thus since we have no proper subgroup J>H, then the last term is zero, implying that $\chi(x^{(5)}) = 1$.

For H = <2,3>, since <2,3> is normal in A_4

$$|N_{A_4}(\langle 2,3\rangle)/_{\langle 2,3\rangle}| = {12}/{4} = 3$$

thus $\chi(X^{\langle 2,3\rangle}) = \chi^{\#}_{\langle 2,3\rangle}\cdot 3 + 0$ (as again there is no proper subgroup $J > \{\langle 2,3\rangle\}$ and the last term of the equation may be discounted).

Hence we have, by substituting for $\chi^{\#}_{\langle 2,3\rangle}$ from 1.3.9,

$$\chi(X^{<2,3>}) = (1 - 2k)3$$

For H = <2>, we have $\left|N_{A_4}(<2>)/<2>\right| = \left|\left\{x \in A_4 \mid x<2>x^{-1} = {<2>}/{<2>}\right\}\right|$

and we find $N_{A_4}<2> = <2,3> \Rightarrow \left|N_{A_4}(<2>)/<2>\right| = 2$

Thus $\chi(X^{<2>}) = \chi^{\#}_{<2>} \cdot 2 + \chi^{\#}_{<2,3>}\left|(A_4/<2,3>)^2\right|$

and from 1.3.6 by substituting for $\chi^{\#}_{<2,3>}$, $\chi^{\#}<2>$

we have $\chi(X^{<2>}) = (-1 + 3k) \cdot 2 + (1 - 2k) \cdot 3$

$$= -2 + 6k + 3 - 6k = 1$$

Lastly for H = <1> we have

1.3.11 $\quad \chi(X^{<1>}) = \chi^{\#}_{<1>}\left|N_{A_4}<1>/<1>\right| + \chi^{\#}_{<5>}\left|(A_4/<5>)^1\right|$

$$+ \chi^{\#}_{<2,3>}\left|A_4/<2,3>)^1\right| + \chi^{\#}_{<2>}\left|(A_4/<5>)^1\right|$$

and $\left|N_{A_4}(<1>)/_{<1>}\right| = 12$

so we have the following:

$\chi(X^{<1>}) = 4x_1 + 3x_2 + 6x_3 + 12x_4$

and by substituting for the x_i

thus $\chi(X^{(1)}) = 1$

1.3.12 $\quad \boxed{\begin{array}{l}\text{And so we have } \chi(X^{(5)}) = 1,\ \chi(X^{(2,3)}) = 3 - 6k \\ \chi(X^{(2)}) = 1 \text{ and } \chi(X^{(1)}) = 1 \text{ for some } k \in \mathbb{Z}\end{array}}$

We note that by taking k = 1 we obtain in $R(A_4)$, 1.3.13 $1 = \mathrm{Ind}_{\langle 5\rangle}^{A_4}(1) + 2\,\mathrm{Ind}_{\langle 2\rangle}^{A_4}(1) - \mathrm{Ind}_{\langle 2,3\rangle}^{A_4}(1) - \mathrm{Ind}_{\{1\}}^{A_4}(1)$

as seen from 1.3.4.

CHAPTER 2

THE RING $R_+(G,S^1)$

2.1 The applications of Explicit Brauer Induction that we will be considering are based on constructions of Snaith and Boltje.

Snaith [28]-[35] has defined constructions of invariants of representations of finite groups. We consider $R_+(G,S^1)$ as follows, [30]:

Definition: If G is a finite group and Π a compact Lie group then $R_+(G,\Pi)$ denotes the free abelian group on equivalence classes of subhomomorphisms

2.1.1 $(G \supset H \xrightarrow{\theta} \Pi)$

where two subhomomorphisms are equivalent, if and only if, they differ by the effect of inner automorphisms of G and of Π. That is, if H is fixed within its conjugacy class then the subhomomorphism given by 2.1.1 is equivalent to those obtained from it by any conjugation of elements of $N_G(H)$ and by elements of Π.

When $\Pi = \{1\}$ and G is finite then $R_+(G,\{1\})$ is the Burnside ring of G. When $\Pi = S^1$, the circle, and G is finite, then $R_+(G,S^1)$ is generated by one-dimensional characters of subgroups, up to conjugations within G.

ie: $G \supset H \xrightarrow{\theta} S^1$.

Snaith developed invariants for $\Pi = \Sigma_n \int S^1$ the n-fold wreath product, which is the subgroup that normalises the diagonal maximal torus $\left\{\begin{pmatrix} e^{i\theta} & & \\ & \ddots & \\ & & e^{i\theta}\end{pmatrix}\right\}$ of $U(n)$, the unitary group of nxn complex matrices. However for our purposes, we will only require the $\Pi = S^1$ case.

We will show that $R_+(G,S^1)$ is a ring as follows. Define a product on $R_+(G,S^1)$ by means of the formula:

$$\text{2.1.2}\qquad \left\{ \begin{aligned} &(G \supset H \xrightarrow{\theta} S^1).(G \supset K \xrightarrow{\psi} S^1) \\ &= \sum_{w \in K\backslash G/H} G \supset w^{-1}Kw \cap H \xrightarrow{w*(\theta).\psi} S^1 \end{aligned} \right.$$

where $w^*(\theta)h = \theta(w\ h\ w^{-1})$.

$R_+(G,S^1)$ has the following properties:

2.1.3 INDUCTION For $G \supset J$, we define

$$\mathrm{Ind}_J^G : R_+(J,S^1) \longrightarrow R_+(G, S^1).$$

by $\mathrm{Ind}_J^G (J \supset H \xrightarrow{\theta} S^1) = (G \supset H \xrightarrow{\theta} S^1)$.

2.1.4 INFLATION If $\lambda : W \twoheadrightarrow G$ is a surjection of groups, define $\mathrm{Inf}_G^W : R_+(G, S^1) \longrightarrow R_+(W, S^1)$

by $\mathrm{Inf}_G^W (G \supset H \xrightarrow{\theta} S^1) = (W \supset \lambda^{-1}(H) \xrightarrow{\theta\lambda} S^1)$.

2.1.5 RESTRICTION For $G \supset J$ we define

$$\mathrm{Res}_J^G : R_+(G, S^1) \longrightarrow R_+(J, S^1)$$

by $\mathrm{Res}_J^G (G \supset H \xrightarrow{\theta} S^1)$

$$= \sum_{x \in J\backslash G/H} \{J \supset J \cap (x\ H\ x^{-1}) \xrightarrow{(x^{-1})^*\theta} S^1\}$$

where $(x^{-1})^*(\theta)\ (j) = \theta(x^{-1}\ j\ x\ )$.

The product given in 2.1.2 is induced from the product of monomial representations of G; see [8,P.9, P.13]

2.1.6 ASSOCIATIVITY IN $R_+(G,S^1)$:

There is a bilinear pairing [3, P.507]

$$(-.-)\ R_+(G,\Sigma_n \int S^1) \times R_+(G,\Sigma_m \int S^1) \longrightarrow R_+(G{\times}G,\Sigma_{n+m} \int S^1)$$

given by $(G \supset U \xrightarrow{\upsilon} \Sigma_n \int S^1)\ .\ (G \supset V \xrightarrow{\eta} \Sigma_m \int S^1)$

$$= (G \times G \supset U \times V \xrightarrow{\upsilon \oplus \eta} \Sigma_{n+m} \int S^1)$$

From [30, Pp.457-8], application of the ρ- construction, 1.2.5, gives $R_+(G \times G, \Sigma_{n+m} \int S^1) \xrightarrow{\rho} R_+(G \times G, S^1)$

where $(G \times G \supset U \times V \xrightarrow{\upsilon \oplus \eta} \Sigma_{n+m} \int S^1) \longrightarrow \Sigma_{i,j} (G \times G \supset U_i \times V_j \xrightarrow{\upsilon_i \oplus \eta_j} S^1)$

However, in $R_+(G \times G, S^1)$

$$(G \times G \supset U \times G \xrightarrow{\text{Pro.j}} U \xrightarrow{\upsilon} S^1)[G \times G \supset G \times V]$$

$$= (G \times G \supset U \times V \xrightarrow{\text{Pro.j}} U \xrightarrow{\upsilon} S^1)$$

Hence, there is a map $(- * -)$

$$R_+(G,S^1) \times R_+(G,S^1) \longrightarrow R_+(G \times G, S^1)$$

where $(G \supset H \xrightarrow{\phi} S^1)*(G \supset K \xrightarrow{\chi} S^1) \longrightarrow (G \times G \supset H \times K \xrightarrow{\text{Pro.j}} H \xrightarrow{\phi} S^1$

Let $a = (G \supset H \xrightarrow{\phi} S^1)$, $b = (G \supset K \xrightarrow{\chi} S^1)$,
$c = (G \supset J \xrightarrow{\theta} S^1)$, then by extension $a*b*c \in R_+(G \times G \times G, S^1)$.

Consider the following maps:

2.1.7

$$\begin{tikzcd}
 & a * b * c \in R_+(GxGxG,S^1) \arrow[dl, "\Delta \times 1"'] \arrow[dr, "1 \times \Delta"] & \\
(ab) * c \in R_+(GxG,S^1) \arrow[dr, "\Delta"'] & & a * (bc) \in R_+(GxG,S^1) \arrow[dl, "\Delta"] \\
 & (ab)c \in R_+(G,S^1) \ni a(bc) &
\end{tikzcd}$$

where $\Delta \times 1$, $1 \times \Delta$ and Δ are given as below, and ab is the product given by 2.1.2.

Let $G \times G \times G \supset W \xrightarrow{\xi} S^1 \in R_+(G \times G \times G,\ S^1)$, then we have the following diagram from 2.1.7, where G is the diagonal subgroup of $G \times G$.

2.1.8

$$\begin{tikzcd}
 & (G \times G \times G \supset W \xrightarrow{\xi} S^1) \arrow[dl, "\Delta x 1"'] \arrow[dr, "1 x \Delta"] & \\
\displaystyle\sum_{\alpha \in \Delta G x G \backslash G x G x G / W} \Delta G x G \supset \Delta G x G \cap \alpha W \alpha^{-1} \xrightarrow{(\alpha^{-1})^* \xi} S^1 \arrow[d, "\Delta"] & & \displaystyle\sum_{z \in G x \Delta G \backslash G x G x G / W} G x \Delta G \supset G x \Delta G \cap z W z^{-1} \xrightarrow{(z^{-1})^* \xi} S^1 \arrow[d, "\Delta"] \\
\displaystyle\sum_{\substack{\alpha \in \Delta G x G \backslash G x G x G / W \\ \beta \in \Delta G \backslash \Delta G x G / \Delta G x G \cap \alpha W \alpha^{-1}}} \sum \Delta G \supset \Delta G \cap \beta(\Delta G x G)\beta^{-1} \cap \beta \alpha W \alpha^{-1} \beta^{-1} \xrightarrow{(\beta^{-1})^* (\alpha^{-1})^* \xi} S^1 & & \displaystyle\sum_{\substack{z \in G x \Delta G \backslash G x G x G / W \\ u \in \Delta G \backslash G x \Delta G / G x \Delta G \cap z W z^{-1}}} \sum \Delta G \supset \Delta G \cap u(G x \Delta G)u^{-1} \cap u z W z^{-1} u^{-1} \xrightarrow{(u^{-1})^* (z^{-1})^* \xi} S^1
\end{tikzcd}$$

where Δ is the restriction map as indicated.

Since $\mathrm{Stabiliser}_{\Delta G \times G}(\alpha W) = \Delta G \times G \cap \alpha W \alpha^{-1}$

and similarly $\mathrm{Stabiliser}_{G \times \Delta G}(z W) = G \times G \cap z W z^{-1}$

while $\mathrm{Stabiliser}_{\Delta G}(\beta \alpha W) = \Delta G \cap \beta (\Delta G \times G) \beta^{-1} \cap (\beta \alpha W \alpha^{-1} \beta^{-1})$

and $\mathrm{Stabiliser}_{\Delta G}(u z W) = \Delta G \cap u (G \times \Delta G) u^{-1} \cap (u z W z^{-1} u^{-1})$

then (α, β) corresponds to a ΔG-orbit of $\beta \alpha W$

and (z, u) corresponds to a ΔG-orbit of $u z W$

hence $g\ u\ z\ W$ is a ΔG-orbit of $[u\ z\ W]$ where $g \in \Delta G$
and if $g_1 \beta\ \alpha\ W$ is a ΔG-orbit of $[\beta\ \alpha\ W]$
we have $\beta\alpha = g_1^{-1}\ g\ u\ z\ w$

2.1.9 $\qquad = \hat{g}\ u\ z\ w$ where $\hat{g} \in \Delta\ G$ and $w \in W$

Hence the summations in 2.1.8 are breakdowns of the ΔG orbits and they correspond by 2.1.9, which proves associativity in $R_+(G,S^1)$

2.1.10 The identity for the product given by 2.1.2 is

$$(G \xrightarrow{Id} S^1)$$

ie: $(G \xrightarrow{Id} S^1).(H \xrightarrow{\phi} S^1) = \sum\limits_{s \in G/H} G \cap s^{-1}H\,s \xrightarrow{Id\phi(s-s^{-1})} S^1$

$$= (H \xrightarrow{\phi} S^1) = (H \xrightarrow{\phi} S^1).(G \xrightarrow{Id} S^1)$$

One also notes [34, P.514] that

$$G \xrightarrow{Id} S^1 \quad \in \quad \Omega(G) \in {}_+R\ (G,\hat{S}^1\ ))$$

where $\Omega(G)$ is the Burnside ring, whose basis consists of subhomomorphisms $(G \supset H \xrightarrow{Id} S^1)$

2.1.11 The distributive property is easily seen from the definition of 2.1.2. We have

$$(K \xrightarrow{\chi} S^1).\ [(H \xrightarrow{\phi 1} S^1) + (H \xrightarrow{\phi 2} S^1)]$$

$$= \sum_{S \in H\backslash G/K} s^{-1}H\ s \cap K \xrightarrow{\chi(\phi 1(s-s^{-1}))} S^1 \ + \sum_{S \in H\backslash G/K} s^{-1}H\ s \cap K \xrightarrow{\chi(\phi 2(s-\bar{s}^{1})\ )} S^1$$

$$= \sum_{S \in H\backslash G/K} s^{-1}H\ s \cap K \xrightarrow{\chi(\phi 1+\phi 2)(s-s^{-1})} S^1$$

$$= (K \xrightarrow{\chi} S^1).(H \xrightarrow{\phi 1} S^1) + (K \xrightarrow{\chi} S^1).(H \xrightarrow{\phi 2} S^1)$$

as required.

We will show that the ring $R_+(G,S^1)$ has the Frobenius reciprocity property:

2.1.12 $\quad \mathrm{Ind}\,{}_H^G\,(\mathrm{Res}\,{}_H^G(x).\ y) = x.\ \mathrm{Ind}\,{}_H^G(y)$

where $x = (G \supset J \xrightarrow{\phi} S^1)$, $y = (H \supset K \xrightarrow{\chi} S^1)$,

The product is that of 2.1.2 and the restriction map is defined as

$$\mathrm{Res}\,{}_H^G(x) = \sum_{z \in H\backslash G/J} H \supset (H \cap z\,J\,z^{-1}) \xrightarrow{(z^{-1})^*\phi} S^1$$

where $(z^{-1})^*\phi = \phi\,(z - z^{-1})$ and we denote

$$H \cap z\,J\,z^{-1} = H_z.$$

Then $\mathrm{Ind}\,{}_H^G\,(\mathrm{Res}\,{}_H^G\,(x).y)$

$$= \mathrm{Ind}\,{}_H^G\left[\left\{\sum_{z \in H\backslash G/J} H \supset H_z \xrightarrow{(z^{-1})^*\phi} S^1\right\}.(H \supset K \xrightarrow{\chi} S^1)\right]$$

$$= \mathrm{Ind}\,{}_H^G\left[\sum_{z \in H\backslash G/J}\ \sum_{\beta \in H_z\backslash H/K} H \supset (\beta^{-1}\,H_z\ \ \beta \cap K) \xrightarrow{\beta^*[(z^{-1})^*\phi]\chi} S^1\right.$$

2.1.13 $$= \sum_{z \in H\backslash G/J}\ \sum_{\beta \in H_z\backslash H/K} G \supset (\beta^{-1}\,H_z\ \beta \cap K) \xrightarrow{\beta^*[(z^{-1})^*\phi]\chi} S^1$$

However, $x.\ \mathrm{Ind}\,{}_H^G\,(y) = (G \supset J \xrightarrow{\phi} S^1).\ (G \supset K \xrightarrow{\chi} S^1)$

2.1.14 $$= \sum_{\alpha \in J\backslash G/K} G \supset \alpha^{-1}\,J\,\alpha \cap K \xrightarrow{\alpha^*(\phi).\chi} S^1$$

and we wish to show that these summations are the same.

Consider the map:

2.1.15 $$\underset{(K \text{ orbits of } J\backslash G)}{J\backslash G/K} \xrightarrow{\qquad\quad \Pi \qquad\quad} \underset{(H \text{ orbits of } J\backslash G)}{J\backslash G/H}$$

The "fibre" $- \Pi^{-1}\,(J\,w\,H)$, is the orbit under H of JwK and

$(Jw)g = Jw$ if and only if $w\, g\, w^{-1} \in J$

ie: $g \in w^{-1}Jw$

Hence the H-orbit of Jz looks like $(H \cap w^{-1} J w)\backslash H$ and similarly the K orbit of Jz looks like $(K \cap w^{-1} J w)\backslash K$.

Thus $\Pi^{-1} (J w H) \cong (H \cap w^{-1} J w \backslash H / K)$ and this expression varies with z and shows that the two index sets

$$J\backslash G/K \quad \text{and} \quad \bigcup_{w \in J\backslash G/H} (H \cap w^{-1} J w) \backslash H/K$$

have the same number of elements.

Now define a map

$$\bigcup_{(w \in J\backslash G/H)} (H \cap w^{-1} J w)\backslash H/K \xrightarrow{\quad \mu \quad} J\backslash G/K$$

We do this by choosing representatives [w] for J\G/H so that for each w_i, $\beta \in H \cap w_i^{-1} J w_i\backslash H/K \xrightarrow{\mu} J w_i \beta K$

where $w_1, w_2, \ldots, w_t \in G$ represent J\G/K.

Since $w_s^{-1} = z_r$, then we may write 2.1.13 as

$$\sum_{\substack{\beta \in H \cap w^{-1} Jw\backslash H/K \\ w \in J\backslash G/H}} \sum G \supset \beta^{-1} (H \cap w^{-1} J w) \beta \cap K \xrightarrow{\chi\phi (\beta w \text{ —— } w^{-1} \beta^{-1})} S'$$

and by the above argument and the equality of index sets, 2.1.13 and 2.1.14 are equal.

We will now demonstrate that the following maps are ring homomorphisms.

2.1.16 $\quad b: R_+(G,S^1) \longrightarrow R(G)$ and

2.1.17 $\quad \mathrm{Res}_J^G : R_+(G,S^1) \longrightarrow R_+(J,S^1)$.

The map b is defined,[30, P.450], as

$$b\, (G \supset H \xrightarrow{\phi} S^1) = \mathrm{Ind}_H^G (H \xrightarrow{\phi} S^1 \longrightarrow U(1))$$

$$= \mathrm{Ind}\,_H^G\,(\phi) \in R(G)$$

Now $b[(G \supset H \xrightarrow{\phi} S^1).\ (G \supset K \xrightarrow{\chi} S^1)]$

$$= b\ [\sum_{w \in K\backslash G/H} w^{-1}\ K\ w \cap H \xrightarrow{w^*(\phi)\ \chi} S^1]$$

$$= \sum_{w \in K\backslash G/H} \mathrm{Ind}\,_{w^{-1}Kw\ \cap\ H}^G\ (w^*(\phi)\chi)$$

while $b[G \supset H \xrightarrow{\phi} S^1]\ .\ b[G \supset K \xrightarrow{\chi} S^1]$

$= \mathrm{Ind}\,_H^G\ (\phi)\ .\ \mathrm{Ind}\,_K^G\ (\chi)$ and by Frobenius reciprocity this becomes:

$$\mathrm{Ind}\,_H^G\ (\phi\ \mathrm{Res}\,_H^G\ (\mathrm{Ind}\,_K^G\ (\chi))$$

$$= \mathrm{Ind}\,_H^G\ (\phi \sum_{u \in H\backslash G/K} \mathrm{Ind}\,_{uKu^{-1} \cap\ H}^H\ (u^{-1})^*\chi$$

$$= \sum_{u \in H\backslash G/K} \mathrm{Ind}\,_{uKu^{-1} \cap\ H}^G\ ((u^{-1})^*(\phi)\ \chi) = \sum_{w \in K\backslash G/H} \mathrm{Ind}\,_{w^{-1}Kw\ \cap\ H}^G\ (w^*(\phi)\ \chi)$$

as required.

Snaith proves 2.1.17 as follows in [30, Pp.451-452]. The product of 2.1.2 may be rewritten in the following way: consider the bijection of double cosets given by:

2.1.18 $$\Delta G\backslash GxG/KxH \longleftrightarrow K\backslash G/H$$
$$(u,v) \longleftrightarrow w = u^{-1}v$$

where ΔG is the diagonal subgroup of GxG.

Then $$\sum_{w \in K\backslash G/H} G \supset w^{-1}\ K\ w \cap H \xrightarrow{w^*(\phi)\chi} S^1$$

$$= \sum_{(u,v) \in \Delta G\backslash GxG/KxH} (G \supset (u\ K\ u^{-1}) \cap (v\ H\ v^{-1}) \xrightarrow{(u^{-1})^*(\phi)(v^{-1})^*(\chi)} S^1)$$

by definition of $\mathrm{Res}\,_J^G$, 2.1.5, we have

$$\mathrm{Res}\,_J^G\ [(G \supset H \xrightarrow{\phi} S^1).(G \supset K \xrightarrow{\chi} S^1)] = \mathrm{Res}\,_J^G\ (x.y)$$

2.1.19

$$= \sum_{\substack{(u,v)\in\Delta G\backslash G\times G/K:H \\ z\in J\backslash G/(uKu^{-1})\cap(vHv^{-1})}} \sum (J \supset (zuKu^{-1}z^{-1}) \cap (zvHv^{-1}z^{-1}) \cap J \xrightarrow{(u^{-1}z^{-1})^*(\phi)(v^{-1}z^{-1})^*(x)} S^1$$

While $\mathrm{Res}_J^G(x)$. $\mathrm{Res}_J^G(y)$

$$= \left[\sum_{\alpha\in J\backslash G/K} J \supset J \cap \alpha K \alpha^{-1} \xrightarrow{(\alpha^{-1})^* y} S^1\right] \left[\sum_{\beta\in J\backslash G/H} J \supset J \cap \beta H \beta^{-1} \xrightarrow{(\beta^{-1})^*\phi} S^1\right]$$

2.1.20

$$= \sum_{\substack{(\alpha,\beta)\in J\times J\backslash G\times G/K\times H \\ (\sigma,\delta)\in\Delta J\backslash J\times J/[(J\cap(\alpha K\alpha^{-1}))\times(J\cap(\beta Hb^{-1}))]}} \sum J \supset \sigma\alpha K \alpha^{-1}\sigma^{-1} \cap \delta\beta H\beta^{-1}\delta^{-1} \cap J \xrightarrow{(\alpha^{-1}\sigma^{-1})^*(x)(\beta^{-1}\delta^{-1})^*(\phi)} S^1$$

The indexing sets in 2.1.19 and 2.1.20 cover the $\Delta(J)$-orbits of $(G \times G)/(K \times H)$.
In 2.1.19, the $\Delta(G)$-orbits are counted first and then each is broken into $\Delta(J)$-orbits, while in 2.1.20, the $J\times J$-orbits are counted first and these are then broken down in $\Delta(J)$-orbits. In 2.1.19, the general term is the $\Delta(J)$-orbit of (zu,zv), while the general term in 2.1.20 is the $\Delta(J)$-orbit of $(\sigma\alpha,\delta\beta)$, from which we see that the two sums are equal.

The bijection, given in 2.1.18, also shows that the product 2.1.2 is commutative.

We note from [38, Pp.159-164], that the properties of restriction and induction on the ring $R_+(G,S^1)$ combine to give a Mackey functor, and such a functor, with the Frobenius property, provides $R_+(G,S^1)$ with the structure of a Green functor. We may see this as follows:

From [38,P.164], we have the canonical pullback for $H<K<G$

$$\begin{tikzcd} G/H \times G/K \arrow[r, "P"] \arrow[d, "Q"'] & G/K \arrow[d, "q"] \\ G/H \arrow[r, "p"] & G/G \end{tikzcd}$$

The orbits A_i of $G/H \times G/K$ correspond to the double cosets $H\backslash G/K$. Let $P(i)$, $Q(i)$ be the restriction of P,Q to A_i. Then the definition of a Mackey functor requires that the diagram

$$\begin{tikzcd} M(G/H \times G/K) \arrow[r, "P_*"] & M(G/K) \\ M(G/H) \arrow[u, "Q^*"] \arrow[r, "p_*"] & M(G/G) \arrow[u, "q^*"'] \end{tikzcd}$$

is commutative, where p_*, P_* are induction maps and q^*, Q^* represent restriction maps.

Here we require that for $H \xrightarrow{\phi} S^1$

2.1.21 $$\mathrm{Res}^G_K \, \mathrm{Ind}^G_H (\phi) = \sum_{x \in H\backslash G/K} \mathrm{Ind}^K_{K \cap x^{-1}Hx} \, \mathrm{Res}^H_{H \cap xKx^{-1}} (x^{-1})^*(\phi)$$

and this double coset formula from [38 P.164], follows from the definitions 2.1.3, 2.1.5 and Frobenius reciprocity.

2.2 $\underline{\mathrm{Ker}\ b : R_+(G, S^1) \longrightarrow R(G)}$

In this section we will consider the above map from the perspective of Explicit Brauer Induction [29, Pp. 221-241] and in relation to [6, Pp. 507-519].

This consideration is motivated by a footnote from [26, P.71]. We recall from Chapter one that Serre makes the following observations.

2.2.1 If $N = \mathrm{Ker}\,(\mathbb{Q} \otimes \mathrm{Ind}: \bigoplus_{H \in X} \mathbb{Q} \otimes R\,(H) \longrightarrow \mathbb{Q} \otimes R\,(G)$, where X is a family of subgroups stable under conjugation and passage to subgroups. (eg: cyclic subgroups of G) and $G = \bigcup_i H_i$ where $H_i \in X$. Then N is generated over $\mathbb{Q}$ by elements of the following kinds:

(i) For $H, H' \in X$ and $H' \subset H$, let $\chi' \in R(H')$ and $\chi = \mathrm{Ind}_{H'}^{H}\,(\chi') \in R(H)$, then $\chi - \chi' \in N$

(ii) For $H \in X$, let $x \in G$ and $H^x = x\,H\,x^{-1}$. Then for $\chi \in R(H)$ and $\chi^x \in R(H^x)$ which is defined $\chi^x(x\,h\,x^{-1}) = \chi(h)$ for $h \in H$,

then $(\chi - \chi^x) \in N$.

These give a presentation for $\mathbb{Q} \otimes R(G)$ in terms of the induced characters. Deligne,[6], has extended this result for Ker b: $R_+(G,S^1) \longrightarrow R(G)$ in the case where G is solvable.

Deligne uses the notation $R_+(G)$ for $R_+(G,S^1)$ and the multiplication 2.1.2 is defined:

2.2.2 $$(H,\,\phi).(K,\psi) = \sum_{(x,y) \in G\times(G/H\times G/K)} (H^x \cap K^x, (\phi^x|\,H^x \cap K^y).(\chi^y|\,H^x \cap K^y))$$

where $(H,\phi) = (G \supset H \xrightarrow{\phi} S^1)$, $(K,\,\psi\,) = (G \supset K \xrightarrow{\chi} S^1)$, x,y are representatives of the orbits of G in G/H x G/K and $H^x = x\,H\,x^{-1}$, $\phi^x = \phi\,(x^{-1} - x)$

This is clear from the bijection of double cosets discussed in 2.1.18.

Induction, restriction and inflation and the map b in $R_+(G,S^1)$ are defined as in 2.1 of this paper. The homomorphism, b, is surjective by Brauer's Theorem. Every element of $R_+(G,S^1)$

is a character and by Brauer's Theorem every character of G can be written as a linear combination with integer coefficients induced from characters of elementary subgroups.

Deligne then proceeds as follows:

An element R of $\mathrm{Ker}\ (R_+(G, S^1) \longrightarrow R(G))$ is said to be of Type I, II or III if there exists a quotient $A = G/B$, of a subgroup $B \subset G$ such that R is obtained from an element S of one of the following types related to

$\mathrm{Ker}\ (R_+(A,\ S^1) \longrightarrow R(A))$

(i) from inflation from A to G

(ii) torsion by a character of B.

where torsion is defined by Deligne as twisting by χ,

or multiplication by (G,χ), denoted by $.\chi$ and so

$(H,\phi)\ .\chi = (H,\ \phi.\ \chi|_H)$ where $[(H,\ \phi) = G \supset H \xrightarrow{\phi} S^1]$

(iii) induction from B to G.

2.2.3

<u>Type I</u>: $A \cong \mathbb{Z}/p$ $\quad S = (e\ [1]) - \sum_{\chi \in A^*} (A,\ \chi)$, where p is a prime

2.2.4

<u>Type II</u>: A is a central extension of $(\mathbb{Z}/k)^2$ by an abelian group Z, H_i $(i = 1,2)$ is the reciprocal image in A of second order factors Z/k and χ_i is a character of H_i. One assumes that

$\chi_1 \big|_Z = \chi_2 \big|_Z$ and that the character of Z is nontrivial on a commutator.

We have

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & H_1 \arrow[d] \\ H_2 \arrow[r, hook] & A \end{tikzcd}$$

The relation for S is given by $S = (H_1,\chi_1) - (H_2,\ \chi_2)$

2.2.5

Type III: A is a semi-direct product H.C; χ is a character of H and C is minimal among the non-trivial commutative subgroups of A. For every character μ of C, A_μ is the fixed point set of μ and $\{\chi, \mu\}$ is the character of A_μ which extends to $\chi|\ H \cap A_\mu$ and μ.

S expresses the following relation where μ runs over a set of orbit representatives of A given from C^* by conjugation.

$$\mathrm{Ind}_H^A(\chi) = \sum_{\mu \in C^*/A} \mathrm{Ind}_{A_\mu}^A(\{\chi, \mu\})$$

These relations are special cases of a result of Langlands.

2.2.6 Theorem (Langlands) [6, P.513]

Suppose that G = H.C, with C commutative, and let χ be a character of H. For all $\mu \in C^*$, let $G_\mu \subset G$ be the fixed point set of $\mu \in G$ acting on C^* by conjugation. If $\chi|\ H \cap C = \mu|\ H \cap C$, let $\{\chi, \mu\}$ be the character of $G_\mu = (H \cap G_\mu).C$, which extends from $\chi|H \cap G_\mu$ and μ.

For μ a representative of a set of G-conjugacy classes of characters of C such that $\chi \mid H \cap C = \mu \mid H \cap C$,

$$\text{then } \mathrm{Ind}_H^G(\chi) = \sum_{\substack{\chi|H \cap C = \mu|H \cap C \\ \mu \in C^*/G}} \mathrm{Ind}_{G_\mu}^G(\{\chi, \mu\})$$

For Type I: $H = \{e\}$, $C = A$

For Type II: $H = H_1$, $C = H_2$

For Type III: H and C are as expressed.

Every relation induced by induction, inflation or torsion on one of Type I, II or III is of the same type.

Deligne proves his main theorem by adapting the following theorem:

2.2.7 If G is nilpotent, then

$$\mathrm{Ker}\ b: R_+(G) = R_+(G,S^1) \longrightarrow R(G)$$

is generated (as a $\mathbb{Z}$-module) by relations of Type I and II.

Lemma: If G is commutative, Ker (b) is generated by relations of Type I.

One must show that for every subgroup H of G and every $\chi \in H^*$ the relation

$$\mathrm{Ind}\,_H^G\,(\chi) = \sum_{\substack{\phi \in G^* \\ \phi|H=\chi}} [\phi]$$

is a consequence (ie: a linear combination) of relations of Type I. This is so by the following:

For $H' \subset H'' \subset G$ with $[H'': H']$ prime and for χ a character of H'(always induced by a character of G)

$$\mathrm{Ind}\,_{H'}^G\,(\chi) = \sum_{\phi|H'=\chi} \mathrm{Ind}\,_{H''}^G\,(\phi)$$

and this follows by applying the Jordan-Hölder theorem to G/H. Deligne then goes on to repeat a result of Langlands (unpublished).

2.2.8 Theorem.

If G is solvable then the kernel of b: $R_+(G,S^1) \longrightarrow R(G)$ is generated by relations of the Types I, II and III respectively.

2.2.9 Lemma If G is solvable, the $\mathbb{Z}$- submodule N(G) of $R_+(G,S^1)$ which is generated by relations of the Type I, II and III is an ideal. We will return to this development later.

First we review Snaith's development of generators for Ker(b).

In [29, P.221-241], and [30,Pp.447-469] Snaith has developed a presentation for R(G), the additive group of unitary representations for the finite group G.

The key part of this is the description of maps τ_G, ρ_G and $T_G = \rho_G.\tau_G$, for which

$b\pi(\tau_G\ (\upsilon)) = \upsilon \in R(G)$ for υ: $G \longrightarrow U(n)$, a unitary representation of G, and $b.(T_G(\upsilon) = \upsilon \in R(G)$.

2.2.10 where b_n: $R_+(G,\Sigma_n \int S^1) \longrightarrow R(G)$,
and $\Sigma_n \int S^1$ is the n-fold wreath product, which is the subgroup which normalises the diagonal, maximal torus of the unitary group U(n) of n x n complex matrices.

Then b_n is given by,

$$b_n\ (G \supset H \xrightarrow{\phi} \Sigma_n \int S^1) = \mathrm{Ind}_H^G\ [H \xrightarrow{\phi} \Sigma_n \int S^1 \longrightarrow U(n)]$$

If $X = U(n)/(\Sigma_n \int S^1)$ and G acts on X (from the left) by left translation via $\upsilon : G \longrightarrow U(n)$, then for $M = G\backslash X$ and for $M(H)$ the orbit-stratum of M of the form G/H, let $\{M_\sigma\}$ be the set of connected components of the orbit strata $M_{(H)}$ and $M_{(H)}$ depends only on the conjugacy class (H). For each M_σ, choose $g_\sigma \in U(n)$ such that the orbit of g_σ is M_σ. Set

$$H_\sigma\ (\upsilon) = \upsilon^{-1}\ [g_\sigma\ (\Sigma_n \int S^1)\ g_\sigma^{-1}\ ] = \mathrm{Stab}_G(g_\sigma\ [\Sigma_n \int S^1])$$

and for $\chi_\sigma^{\#}$ the Euler characteristic of M_σ,

2.2.11

$$\text{define } \tau_G(\upsilon) = \sum_\sigma \chi_\sigma^{\#}\ (G \supset H_\sigma(\upsilon) \xrightarrow{g_\sigma^{-1}\upsilon g_\sigma} \Sigma n \int S^1) \in R_+(G,\ \Sigma \int S^1).$$

In [9, Pp. 223-230], Snaith proves that $\tau_G(\upsilon)$ is well defined, depends only on the image of υ in R(G) and that $b_n(\tau_G(\upsilon)) = \upsilon$. He also proves that τ_G is natural with respect to restriction. Snaith continues to describe the map ρ_G, a monomial homomorphism into $\Sigma_n \int S^1$, of the form $\rho_G : R_+(G,\Sigma_n \int S^1) \longrightarrow R_+(G,\ S^1)$

2.2.12 where $\rho_G\ (G \supset H \xrightarrow{\phi} \Sigma_n \int S^1) = \sum_\sigma \mathrm{Ind}_{H_\sigma}^G\ (\rho_\sigma)$

where $(\rho_\sigma: H_\sigma \longrightarrow S^1)$ is obtained as follows.

Consider $H \xrightarrow{\phi} \Sigma_n \int S^1 \longrightarrow (\Sigma_n \int S^1) / (S^1)^n \cong \Sigma_n$

and the action of H on the set $\{1, 2, \ldots, n\}$. From each H-orbit, choose a representative, σ and let $H_\sigma = \mathrm{Stab}_H \sigma$).

$\mathrm{Res}_{H_\sigma}^H(\phi)$ has a homomorphism $\phi_\sigma: H_\sigma \longrightarrow S^1$ as its (σ,σ)-th matrix entry.

Then ρ_G is a well defined homomorphism which commutes with induction, inflation and restriction. In addition for

$x \in R_+(G,\Sigma_n \int S^1)$, we have $b_n(x) = b \,.\, (\rho_G(x)) \in R(G)$

and we have a well defined homomorphism $T_G = \rho_G(\tau_G)$ with the following properties.

2.2.13 <u>Theorem:</u> (i) $T_G(\upsilon) \in R_+(G,S^1)$ is well-defined depending only on the class of $\upsilon \in R(G)$ and $b(T_G(\upsilon))=\upsilon \in R(G)$.

(ii) $T_G(\upsilon)$ is natural with respect to induction, inflation and restriction.

and (iii) For $\upsilon: G \longrightarrow U(n)$ and $\mu : G \longrightarrow U(m)$

then $T_G(\upsilon \oplus \mu) = T_G(\upsilon)\ \varepsilon\ \tau_G(\mu) + \varepsilon\ \tau_G(\upsilon)\ T_G(\mu)$

where $\varepsilon: R_+(G,S^1) \longrightarrow R_+(G, \{1\})$ is given by $\varepsilon(G \supset H \xrightarrow{\phi} S^1) = (G \supset H \longrightarrow \{1\})$

We will now consider ker b: $R_+(D_8,S^1) \longrightarrow R(D_8)$. We will follow Snaith's [29, Pp.231-232,239-240] calculations and use Deligne's technique. These two strategies will produce two different sets of generators.

Let $F \leq R_+(G,S^1)$ be the subgroup defined by the following generators

2.2.14 (i) For $\upsilon: G \supset H \xrightarrow{\phi} S^1$, $(T_G(\upsilon) - \upsilon) \in F$

(ii) For $\upsilon: G \supset H \xrightarrow{\phi} S^1$ and $\mu: G \supset J \xrightarrow{\chi} S^1$,

then $[T_G(b(\mu) \oplus b(\upsilon)) - T_G(b(\upsilon) - T_G(b(\mu))] \in F$

and F is the kernel of b.

Consider the nilpotent group, D_8. We will compare Deligne and Snaith relations for the kernel of b.

$$D_8 = \{x,y \mid x^4 = y^2 = 1 \quad x\ y\ x = y\}$$

and subgroups are given by

D_8 $\langle x\rangle$, $\langle x^2\rangle$, $\langle x^4\rangle = 1$, $\langle x^2,y\rangle$, $\langle x^2,xy\rangle$ and $\langle y\rangle$.

From [26, P37], we have the following character table for D_8:

	1	x and x^3	x^2	y and x^2y	xy and x^3y
1	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	-1	1	1	-1
χ_3	1	-1	1	-1	1
ρ	2	0	-2	0	0

where 1, χ_1, χ_2, χ_3 are the characters of the four representations of degree 1 and ρ is the character of the single irreducible representation of degree 2.

Following Snaith [29, Pp.204-206], [30, Pp.466-469], we first set up an expression for the kernel of

$$b: R_+(D_8, S^1) \longrightarrow R(D_8)$$

and relate this expression to Deligne relations.

For the subhomomorphism

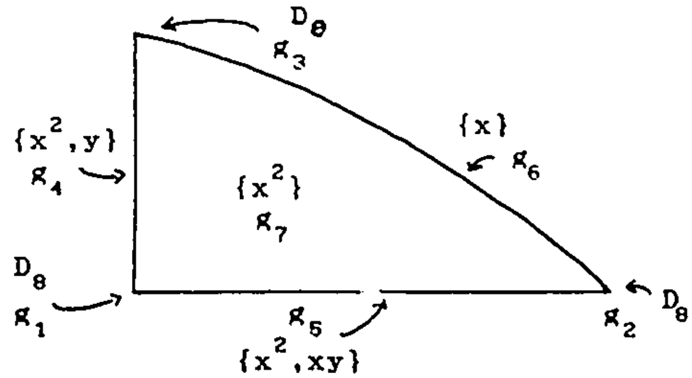
$$D_8 \supset \langle x \rangle \xrightarrow{\upsilon} S^1 \in R_+(D_8, S^1)$$

given by $\upsilon(x) = i$ $(i^2=1)$, we set

$$\underline{\upsilon} = b(\upsilon): D_8 \longrightarrow U(2)$$

with $\underline{\upsilon}(x) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $\underline{\upsilon}(y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The orbit space of the D_8 -action is given by a quadrant of a disc with the orbit types $\{M_\alpha\}$ and representatives $g_\alpha \in U(2)$, where the orbit of g_α lies within M_α. This orbit space is represented by the following diagram:



$g_1 = [\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}]$, the other $\{g_i\}$ are given by Table B, [29, P.190] and [29, P.204]. An expression for($T_{D_8}(b(\upsilon)) - \upsilon$) where

$$\upsilon = \sum_j \chi_j^{\#} \operatorname{Ind}^{G}_{H(g_j,\upsilon)} \operatorname{Res}^{G}_{H(g_j,\ \upsilon)}(g_j^{-1}\ \upsilon\ g_j)$$

is obtained as follows:

2.2.15 $(D_8 \supset \langle x\rangle \xrightarrow{\upsilon} \mu_4) + \rho(g_2^{-1}\ \underline{\upsilon}\ g_2 : D_8 \longrightarrow \Sigma_2 \int S^1$

$+ \rho(g_3^{-1}\ \underline{\upsilon}\ g_3 : D_8 \longrightarrow \Sigma_2 \int S^1) - (D_8 \supset \langle x^2, y\rangle \xrightarrow{\rho(g_4^{-1}\underline{\upsilon}g_4)} \Sigma_2 \int S^1$

$-(D_8 \supset \langle x^2, xy\rangle \xrightarrow{\rho(g_5^{-1}\underline{\upsilon}g_5)} \Sigma_2 \int S^1) - (D_8 \supset \langle x\rangle \xrightarrow{\rho(g_6^{-1}\underline{\upsilon}g_6)} \Sigma_2 \int S^1)$

$-(D_8 \supset \langle x^2\rangle \xrightarrow{\rho(\underline{\upsilon})} \Sigma_2 \int S^1) - (D_8 \supset \langle x\rangle \longrightarrow \mu_4)$

where $\underline{\upsilon} = \operatorname{Ind}^{D_8}_{\langle x\rangle}(\upsilon) = b(\upsilon)$ and $\mu_4 = \langle i\rangle$.

Under the map b,[29,P.206], this expression is zero in $R(D_8)$.

2.2.16 ie: $\mathrm{Ind}_{\langle x^2\rangle}^{D_8}(\rho(\underline{\upsilon})) + \rho(g_2^{-1}\underline{\upsilon}\, g_2) + \rho(g_3^{-1}\underline{\upsilon}\, g_3)$

$$-\mathrm{Ind}_{\langle x^2,y\rangle}^{D_8}(\rho(g_4^{-1}\,\underline{\upsilon} g_4) - \mathrm{Ind}_{\langle x^2,xy\rangle}^{D_8}\rho(g_5^{-1}\underline{\upsilon}\, g_5) - \mathrm{Ind}_{\langle x\rangle}^{D_8}\rho(g_6^{-1}\underline{\upsilon} g_6) = 0$$

$\mathrm{Ind}_{\langle x^2\rangle}^{D_8}(\rho(\underline{\upsilon}))$ is induced from $\rho(\underline{v})|\langle x^2\rangle = 2L$
where L is the non-trivial, one dimensional representation, given by $L(x\ ) = -1$. Also $\mathrm{Ind}_{\langle x^2,y\rangle}^{D_8}(\rho(g_4^{-1}\ \underline{\upsilon}\ g_4)$ is induced from

$$(\rho(g_4^{-1}\ \underline{\upsilon}\ g_4) \mid \langle x^2,y\rangle) \cong \mathrm{Ind}_{\langle x^2\rangle}^{\langle x^2,y\rangle}\ (L)$$

and $\mathrm{Ind}_{\langle x^2,xy\rangle}^{D_8}(\rho(g_5^{-1}\ \underline{\upsilon}\ g_5)$ is induced from

$$(\rho(g_5^{-1}\ \underline{\upsilon}\ g_5) \mid \langle x^2,xy\rangle) \cong \mathrm{Ind}_{\langle x^2\rangle}^{\langle x^2,xy\rangle}\ (L),$$

with the result that the first, fourth and fifth terms in 2.2.16 cancel.

In addition, $\rho(g_2^{-1}\underline{\upsilon} g_2)$, for a representative $g_2 = \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$
(taken from [29, P.190, Table B])
is given by $D_8 \supset \langle x^2,y\rangle \xrightarrow{\varepsilon_2} \{\mp 1\}$
where $\varepsilon_2(x^2) = -1$ and $\varepsilon_2(y) = 1$.
Further, $\rho(g_3^{-1}\underline{\upsilon}\ g_3)$, for the representative $g_3 = \frac{1}{2}\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}$
is given by $D_8 \supset \langle x^2,xy\rangle \xrightarrow{\varepsilon_3} \{\mp 1\}$
where $\varepsilon_3(x^2) = -1$, $\varepsilon_3(xy) = 1$, and
$\rho(g_6^{-1}\ \underline{\upsilon}\ g_6)$ for $g_6 = \frac{1}{2}\begin{bmatrix}1+i & -2\\ 2 & 1-i\end{bmatrix}$ is given by
$D8 \supset \langle x^2\rangle \xrightarrow{\varepsilon} \{\mp 1\}$ where $\varepsilon(x^2) = -1$.

Hence terms two, three and six of 2.2.16 become

$$\mathrm{Ind}\ {}^{D_8}_{\langle x^2,y\rangle}(\epsilon_2) + \mathrm{Ind}\ {}^{D_8}_{\langle x^2,xy\rangle}(\epsilon_3) - \mathrm{Ind}\ {}^{D_8}_{\langle x\rangle}(\epsilon).$$

Now $\mathrm{Ind}\ {}^{D_8}_{\langle x^2,y\rangle}(\epsilon_2|\langle x^2\rangle) = \mathrm{Ind}\ {}^{D_8}_{\langle x^2,xy\rangle}(\epsilon_3|\langle x^2\rangle) = \rho$, the unique irreducible two-dimensional representation of D_8.

But ρ is isomorphic to

$$\mathrm{Ind}\ {}^{D_8}_{\langle x\rangle}(\alpha) = \mathrm{Ind}\ {}^{D_8}_{\langle x\rangle}(\alpha^3)$$

where $\alpha:\langle x\rangle \hookrightarrow S^1$ is an inclusion, also

$$\mathrm{Ind}\ {}^{D_8}_{\langle x\rangle}(\epsilon)|\langle x\rangle \cong \alpha + \alpha^3$$

Thus the second, third and sixth terms in 2.2.16 cancel.

We now look at 2.2.16 using Deligne Relations, [6, Pp.513-517]. In the proof of 2.2.8 we recall that if we have a relation

$$\Sigma m_i\ \mathrm{Ind}\ {}^{G}_{H_i}(\chi_i) = 0 \text{ where } H_i \supset Z(G)$$

then this relation is induced by relations for H_iC, G, H_i and χ_i given by

2.2.17

$$\mathrm{Ind}\ {}^{H_iC}_{H_i}(\chi_i) = \sum_{\substack{\chi_i|H_i\cap C=\mu|H_i\cap C \\ \mu\in C^*/H_i}} \mathrm{Ind}\ {}^{H_iC}_{(H_iC)\mu}(\{\chi_i,\mu\})$$

where C is a normal commutative subgroup of G containing $Z(G) = Z$ and such that $[C:Z] = p$ (a prime) and its image is central in G/Z; $(H_iC)_\mu$ is the stabilizer of μ in H_iC and $(\{\chi_i,\mu\})$ is the character of $(H_iC)\mu$ which extends from $\chi_i|H_i\cap(H_iC)_\mu$ and μ.

In this case, for $G = D_8$, we examine 2.2.16 which has $H_i \supset Z = \langle x^2\rangle$ for all H_i and $C \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ which can be either $\langle x^2,y\rangle$ or $\langle x^2,xy\rangle$.

Set $C = \langle x^2,y\rangle$ then $C^* = \{1, \epsilon_2, \xi, \eta\}$

where $\epsilon_2(x^2) = -1$, $\epsilon_2(y) = 1$, $\xi(x^2) = 1$, $\xi(y) = -1$

and $\eta(x^2) = -1$, $\eta(y) = -1$.

Applying 2.2.17 to the terms of 2.2.16 gives the following:

<u>Term 1</u> $\quad \mathrm{Ind}_{\langle x^2\rangle}^{D_8} \rho(\underline{\upsilon})$ is induced from

$$\mathrm{Ind}_{\langle x^2\rangle}^{\langle x^2\rangle.\langle x^2,y\rangle} (\rho(\underline{\upsilon})) = \mathrm{Ind}_{\langle x^2\rangle}^{C} (\rho(\underline{\upsilon}))$$

$$= \sum_{\substack{\rho(\underline{\upsilon})|\langle x^2\rangle=\mu|\langle x^2\rangle \\ \mu\in C^*/\langle x^2\rangle}} \mathrm{Ind}_{(C)_\mu}^{C} (\{\rho(\underline{\upsilon}),\mu\})$$

But $C_\mu = C$ and $[\rho(\underline{\upsilon}) \mid \langle x^2\rangle](x^2) = -I_2$

so that $(\{\rho(\underline{\upsilon}),\ \mu\})$ can be ϵ_2 or η.

Hence term one becomes $\epsilon_2 + \eta$.

<u>Terms 2 and 3</u> give the same result as follows

$$\rho(g_2^{-1}\ \underline{\upsilon}\ g_2) = \sum_{\substack{g_2^{-1}\underline{\upsilon}\ g_2|\langle x^2\rangle=\mu|\langle x^2\rangle \\ \mu\in C^*/D_8}} \mathrm{Ind}_{(D_8)_\mu}^{D_8}(\{\rho(g_2^{-1}\ \underline{\upsilon}\ g_2),\mu\})$$

Since $(D_8)_\mu = C$ and $[\rho(g_2^{-1}\ \underline{\upsilon}\ g_2) \mid \langle x^2\rangle]\ (x^2) = -I_2$.

Then $(\{\rho(g_2^{-1}\ \underline{\upsilon}\ g_2),\mu\})$ can be ϵ_2 or η.

So that terms two or three are $\mathrm{Ind}_C^{D_8}(\epsilon_2) + \mathrm{Ind}_C^{D_8}(\eta)$.

<u>Term 4</u> $\quad$ Here $H = C$ so that

$$\mathrm{Ind}_C^{HC} (\rho(g_4^{-1}\ \underline{\upsilon}\ g_4) = \sum_{\substack{\rho(g_4^{-1}\underline{\upsilon}g_4)|C \\ \mu\in C^*}} \mathrm{Ind}_{C_\mu}^{C} (\{\rho(g_4^{-1}\ \underline{\upsilon}\ g_4),\mu\})$$

$$= \epsilon_2 + \eta$$

<u>Term 5</u> In this case $H = \langle x^2, xy\rangle$, so that the expression

$$\mathrm{Ind}_{\langle x^2,xy\rangle}^{D_8} (\rho(g_5^{-1}\ \underline{\upsilon}\ g_5) = \sum_{\substack{\rho(g_5^{-1}\underline{\upsilon}\ g_5)|\langle x^2\rangle \\ \mu\in C^*/\langle x^2,xy\rangle}} \mathrm{Ind}_{(D_8)_\mu}^{D_8} \{\rho(g_5^{-1}\ \underline{\upsilon}\ g_5),\ \mu\})$$

$$= \mathrm{Ind}_C^{D_8} (\varepsilon_2) + \mathrm{Ind}_C^{D_8} (\eta)$$

<u>Term 6</u> Here $H = \langle x\rangle$, hence the expression

$$-\quad \mathrm{Ind}_{\langle x\rangle}^{D_8} (\rho(g_6^{-1}\ \underline{\upsilon}\ g_6) = \sum_{\substack{\rho(g_6^{-1}\underline{\upsilon}\ g_6)|\langle x^2\rangle \\ \mu\in C^*/\langle x\rangle}} \mathrm{Ind}_{(D_8)_\mu}^{D_8} \{\rho(g_6^{-1}\ \underline{\upsilon}\ g_6),\ \mu\})$$

$$= \mathrm{Ind}_C^{D_8} (\varepsilon_2) + \mathrm{Ind}_C^{D_8} (\eta)$$

Hence terms one and four cancel, as do two and three with five and six. In this case it is clear that Ker b is generated by Deligne Type II relations or $[T_{D_8} (b(\upsilon)) - \upsilon]$ elements.

CHAPTER 3

THE EXPLICIT INDUCTION HOMOMORPHISM - a_G

In [1 ,§1], Boltje considers $R_+(G,S^1)$ from another point of view. It is the Grothendieck ring of the category of monomial representations of G. The objects of this category are finite dimensional $\mathbb{C}$-vector spaces, V, together with a decomposition $V = V_1 \oplus \dots \oplus V_n$ into one-dimensional subspaces $V_1,\dots V_n$, called the lines of V and a $\mathbb{C}$-linear, left G-action on V, which permutes the lines of V.

Morphisms between two monomial representations

$V = V_1 \oplus \dots \oplus V_n$ and $W = W_1 \oplus \dots \oplus W_m$,

are linear maps, $f\colon V \longrightarrow W$

such that for each $1 \le i \le n$ there exists a $1 \le j \le m$

such that $f(V_i) \subset W_j$.

$V = V_1 \oplus \dots V_n$ is irreducible if G permutes the lines, $\{V_i\}$, transitively. The G-orbits of the lines provide the only decomposition of V into a sum of simple (irreducible) monomial representations. The forgetful functor F from the category of monomial representations of G to the category of $\mathbb{C}G$-modules, only forgets the decomposition into the lines and commutes with the morphisms $\oplus$, $\otimes$, $\overset{*}{-}$ (dual), det, restriction and induction.

Boltje proves that $R_+(G,S^1)$ is a ring by considering it as the free abelian group with basis M_G/G (where M_G is the set of all pairs (H,ϕ) with $H \le G$ and $(G \supset H \xrightarrow{\phi} S^1)$, which is the set of G-orbits. The orbit containing (H,ϕ) is given by $\overline{(H,\phi)}^G$.

The definitions $(H,\phi) \le (H^1,\phi^1) \Longleftrightarrow H \le H^1$ and $\phi = \phi^1|_H$

and $\overline{(H,\phi)}^G \le \overline{(H^1,\phi^1)}^G \Longleftrightarrow (H,\phi) \le {}^S(H^1,\phi^1)$ for some $s \in G$

where ${}^S(H^1,\phi^1) = (s^{-1}\, H^1\, s,\ \phi^1\, (s^{-1} - s)$

give M_G and M_G/G the structure of a partially ordered set.

To each pair $(H,\phi) \in M_G$, associate the irreducible monomial representation

$$M_G(H,\phi) = \mathrm{Ind}_H^G\, \mathbb{C}_\phi$$

where $\mathbb{C}_\phi$ is the one dimensional representation of H corresponding to ϕ. The underlying $\mathbb{C}G$-module of $M_G(H,\phi)$ is $\mathrm{Ind}_H^G\, \mathbb{C}_\phi$.

$M_G(H,\phi)$ is simple, since the action of G on the lines $g \otimes_{\mathbb{C}H} \mathbb{C}\phi$, where $g \in {}^G/_H$, is the same as the action of G on ${}^G/_H$, which is transitive.

Each irreducible monomial representation $V = V_1 \oplus \ldots \oplus V_n$ is isomorphic to some $M_G(H,\phi)$. For $H = \mathrm{stab}\,(V_1)$ and ϕ, the character of the $\mathbb{C}H$-module V_1, and for each i, choose $g_i \in G$, such that $g_i V_1 = V_i$, then $g_i \in G/H$. Let $0 \neq v \in V_1$, then $v_i = g_i v \in V_i$ and $v_i \longleftrightarrow g_i \otimes_{\mathbb{C}H} 1$ extends to an isomorphism between V and $M_G(H,\phi)$.

In Boltje's notation the multiplication defined in 2.1.2 is given by

$$\text{3.1.1} \qquad \overline{(H,\phi)}^G \cdot \overline{(K,\chi)}^G = \sum_{g \in H\backslash G/K} \overline{(H \cap {}^gK,\ \phi.{}^g\chi)}^G$$

Since G acts trivially on the elements (G,ϕ), $\phi \in \hat{G}$, then $R_+(G,S^1)$ contains the group ring, $\mathbb{Z}(\hat{G})$ as a subring.
Hence $R_+(G,S^1)$ is a $\mathbb{Z}(\hat{G})$-module and there is a decomposition:

$$\text{3.1.2} \qquad R_+(G,S^1) = \mathbb{Z}(\hat{G}) \oplus \sum_{\substack{(H,\phi) \\ H \lneq G}} \mathbb{Z}\langle (H,\phi) \rangle$$

as $\mathbb{Z}(\hat{G})$-modules. Projecting onto $\mathbb{Z}(\hat{G})$ gives the homomorphism:

$$\Pi_G : R_+(G,S^1) \longrightarrow \mathbb{Z}(\hat{G})$$

$$\overline{(H,\phi)}^G \longrightarrow \begin{cases} \phi, & \text{if } H = G \\ 0, & \text{if } H < G \end{cases}$$

and this is a $\mathbb{Z}(\hat{G})$-algebra homomorphism [1 §1.9].

For restriction defined as in 2.1.5, there is a ring homomorphism

$$\rho_G : R_+(G,S^1) \longrightarrow \mathbb{Z}(\hat{H}) \text{ where } H \leq G$$

$$\text{given by } x \longrightarrow \Pi_H\,(\mathrm{Res}_H^G(x))$$

where G acts on the ring $\prod_{H\leq G} \mathbb{Z}(\hat{H})$ by conjugation

and the image of ρ_G is contained in $(\prod_{H\leq G} \mathbb{Z}(\hat{H}))^G$,

as the subring of G-invariant elements of $(\prod_{H\leq G} \mathbb{Z}(\hat{H}))^G$.

Boltje shows that

3.1.3 $\quad \rho_G : R_+(G,S^1) \longrightarrow (\prod_{H\leq G} \mathbb{Z}(\hat{H}))^G$.

is an injective ring homomorphism with finite cokernel,[1, § 1.20,1.21]. The importance of this map is the fact that it is an isomorphism when tensored with the rational numbers.

$$\text{ie:}\quad \rho_G : \mathbb{Q} \otimes (R_+(G,S^1) \xrightarrow{\cong} (\prod_{H\leq G} \mathbb{Q}(\hat{H}))^G.$$

The rings $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$ and $\mathbb{Q} \otimes_{\mathbb{Z}} R_+(G,S^1)$ are $\mathbb{Q}$-vector spaces with bases Irr G (the set of irreducible, complex representations of G) and $\mathcal{M}G/G$ respectively.

Boltje assumes that for each finite G, there exists a map:

3.1.4 $\quad a_G : \mathbb{Q} \otimes_Z R(G) \longrightarrow \mathbb{Q} \otimes_Z R_+(G,S^1)$

given by $\chi \longrightarrow \sum_{\overline{(H,\phi)}^G \in \mathcal{M}G/G} \alpha_{\overline{(H,\phi)}^G}(\chi)\ (H,\phi)^G$ where $\alpha_{(H,\alpha)^G}(\chi) \in \mathbb{Q}$.

Additionally one assumes that a_G commutes with restriction to subgroups in $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$ and $\mathbb{Q} \otimes R_+(G,S^1)$

Boltje shows by induction that the coefficients $\alpha_{(H,\phi)^G}(\chi)$ are determined by the coefficients $\alpha_{\overline{(H,\phi)}^H}(\chi|_H)$ so that:

3.1.5

$$\alpha_{\overline{(H,\phi)}^H}(\chi|_H))_{(H,\phi)^G \in \mathcal{M}_G/G} = \Gamma(G) \ .\ (\alpha_{(H,\phi)^G}(\chi))_{(H,\phi)^G \in \mathcal{M}_G/G}(\chi)$$

where $\Gamma(G)$ is an invertible matrix over $\mathbb{Q}$ for the bilinear form:

$[(H,\phi)^G,(H^1,\phi^1)^G]$ and is an upper triangular matrix with $(N_G(H,\phi)\ ;\ H)$ as its diagonal entries.

Here $(H,\phi)^G \le (H^1,\phi^1)^G$, and both are elements of M_G/G. We need an expression for the left side of 3.1.5 such that it is G-invariant under conjugation and additive in χ. Suppose $a_G(\phi) = (G,\phi)^G$ for all $\phi \in \hat{G}$ and suppose that

3.1.6 $\quad \alpha_{(G,\phi)^G}(\chi) = (\phi,\chi)$ the Schur Inner product given by

$$\sum_{g \in G} \phi(g) \cdot \chi(g)$$

Corresponding to the map ρ_G, 3.1.3, there is a similar decomposition of $\mathbb{Z}[\hat{G}]$ modules and a projection (P_G) for $R(G)$.

3.1.7 $$R(G) = \mathbb{Z}[\hat{G}] \oplus \Big(\sum_{\substack{\chi \in Irr(G) \\ \dim(\chi) > 1}} \mathbb{Z}(\chi) \Big)$$

3.1.8 $$P_G : R(G) \longrightarrow \mathbb{Z}(\hat{G})$$

3.1.9 $$\text{given by } P_G(\chi) = \sum_{\phi \in \hat{G}} (\phi,\chi) \cdot \phi$$

and from 3.1.6 and 3.1.7, P_G is the identity map on linear irreducible characters of G and is zero on the other irreducibles. The expression, 3.1.9, is equivalent to the commutativity of the diagram:

3.1.10
$$\begin{tikzcd} \mathbb{Q} \otimes R(G) \arrow[r, "a_G"] \arrow[dr, "P_G"'] & \mathbb{Q} \otimes R_+(G,S^1) \arrow[d, "\Pi_G"] \\ & \mathbb{Q}\hat{G} \end{tikzcd}$$

Hence for fixed G, if a_G is compatible with restrictions to all subgroups $H \le G$ and 3.1.10 commutes for all $H \le G$, then the following equation gives values for $(\alpha_{(H,\phi)^H}(\chi|_H))$.

$$3.1.11(a) \qquad (\alpha_{(H,\phi)^H}(\chi|_H) = [(\phi,\chi|_H)]_{(H,\phi)^G \in \mathcal{M}_G/G}$$

$$= \Gamma(G).\ (\alpha_{(H,\phi)^G}(\chi))_{(H,\phi)^G \in \mathcal{M}_G/G}$$

$$\text{(or) } 3.1.11(b) \qquad (\phi,\chi|_H) = \sum_{\overline{(H,\phi)}^G \leq \overline{(H^1,\phi^1)} \in \mathcal{M}_G/G} \gamma^G_{(H,\phi),(H^1,\phi^1)}\ \alpha_{(H,\phi)^G}(\chi)$$

$$\text{where } \gamma^G_{(H,\phi)\ (H^1,\phi^1)} = |\{s \in H\backslash G/H^1 \mid (H,\phi) \leq {}^{s}(H^1,\phi^1)\}|$$

This implies that there is at most one family of maps

$a_G: \mathbb{Q} \otimes_Z R(G) \longrightarrow \mathbb{Q} \otimes_Z R_+(G,S^1)$

subject to the following conditions:

The following diagram is commutative

(i)

$$3.1.12 \qquad \begin{tikzcd} \mathbb{Q}\otimes_Z R(G) \arrow[r, "a_G"] \arrow[d, "Res^G_H"'] & \mathbb{Q}\otimes_Z R_+(G,S^1) \arrow[d, "Res^G_H"] \\ \mathbb{Q}\otimes_Z R(H) \arrow[r, "a_H"] & \otimes_Z R_+(H,S^1) \end{tikzcd}$$

and (ii) the diagram, 3.1.10, is commutative for all G.

3.1.13 a_G is defined as follows: For a finite group G, the map

$a_G: \mathbb{Q} \otimes_Z R(G) \longrightarrow \mathbb{Q} \otimes_Z R_+(G,S^1)$

is given by

$$a_G(\chi) = \sum_{(H,\phi)^G \in \mathcal{M}_G/G} \alpha_{(H,\phi)^G}(\chi)\ (H,\phi)^G, \text{ for } \chi \in \mathbb{Q} \otimes_{\mathbb{Z}} R_+(G,S^1)$$

where all the coefficients $\alpha_{(H,\phi)^G}(\chi) \in \mathbb{Q}$ are given as the unique solutions of 3.1.11.(b). Boltje then uses the adjoint map, as follows, to show that the coefficients in $a_G(\chi)$, that is $\alpha_{(\overline{H,\phi})^G}(\chi) \in \mathbb{Q}$, are in fact integral.

3.1.14 Proposition:

a_G is the right adjoint map of b with respect to the bilinear forms $[-,-]$ and $(-,-)$.

ie: $[x, a_G(\chi)] = (b(x), \chi)$

where $b: R_+(G,S^1) \longrightarrow R(G)$ given by $(H,\phi)^G \longrightarrow \mathrm{Ind}_H^G(\phi)$

$\chi \in \mathbb{Q} \otimes_Z R(G)$, and $x \in \mathbb{Q} \otimes_Z R_+(G,S^1)$

and where $[-,-]$ is defined on $R_+(G,S^1)$

as $[(H,\phi)^G, (H^1,\phi^1)^G] = \gamma^G_{(H,\phi)(H^1,\phi^1)}$

$= |\{s \in H\backslash G/H^1 \mid (H,\phi) \le {}^S(H^1,\phi^1)\}|$

The bilinear form $(-,-)$ is defined on R(G) as the scalar product

$$(\alpha,\beta) = \frac{1}{|G|}\sum_{g\in G}\alpha(g)\,\beta(g) \quad \text{where } \alpha,\ \beta \in R(G)$$

This proposition is proved by expanding

$[x, a_G(\chi)]$ for $x = (H,\phi)^G$ and using 3.1.11(b).

3.1.14 can be used as a definition for a_G, since $[-,-]$ is not degenerate on $\mathbb{Q} \otimes_Z R(G)$.

3.1.15 The third definition of $a_G(\chi)$ follows from the map r_G defined as:

$$r_G = (P_H \,\mathrm{Res}_H^G)_{H\le G} : \mathbb{Q} \otimes_{\mathbf{Z}} R(G) \longrightarrow (\prod_{H\le G} \mathbb{Q}\,[\hat{H}])^G$$

$$\text{where } r_G(\chi) = \Big(\sum_{\phi\in\hat{H}} (\phi,\ \chi|_H)\ \phi\Big)_{H\le G}$$

The map r_G is well defined since $({}^S\phi,\ \chi|_{{}^sH}) = (\phi,\chi|_H)$ for all $s \in G$.

It is injective, since by definition P_H is the identity on cyclic groups and by Artin's theorem each character $\chi \in R(G)$ is given by its restriction to cyclic groups.

3.1.16 Proposition:

The following diagram commutes for each G.

$$\begin{tikzcd} \mathbb{Q}\otimes_Z R(G) \arrow[r, "a_G"] \arrow[dr, "r_G"'] & \mathbb{Q}\otimes_Z R_+(G,S^1) \arrow[d, "\rho_G"] \\ & (\prod_{H\leq G} \mathbb{Q})^{\wedge G} \end{tikzcd}$$

Proof: For $\chi \in \mathbb{Q} \otimes_Z R(G)$

$$r_G(x) = \sum_{\phi\in \hat{H}} (\phi, \chi|_H)$$

now $$\rho_G\, a_G\,(\chi) = \rho_G \left[\sum_{(H,\phi)^G \in M_G/G} \alpha_{(H,\phi)^G}(\chi)\;(H,\phi)^G\right]$$

$$= \sum_{(H,\phi)^G \in M_G/G} \alpha_{(H,\phi)^G}(\chi) \left(\sum_{\phi\in\hat{H}} \gamma^G_{(H,\phi),\,(H^1,\phi^1)}\right)_{H\leq G}$$

which is equivalent to 3.1.11(b).

We have seen, 3.1.12, that a_G represents at most one family of maps subject to the conditions 3.1.12(i) and (ii).

3.1.17 Theorem

The family of maps $a_G : \mathbb{Q} \otimes_Z R(G)\ \mathbb{Q} \otimes_Z R_+(G,S^1)$ exists and is unique.

Proof: a_G is unique in the sense that it satisfies the requirement that 3.1.10 is commutative. This follows from 3.1.11(b) by taking the case H = G

then $(\phi,\chi) = \gamma^G_{(G,\phi),(G,\phi)}\ \alpha_{(G,\phi)^G}(\chi)$

and $\gamma^G_{(G,\phi),(G,\phi)} = (N_G(G,\phi):G) = 1$

It remains to check that a_G satisfies 3.1.12(i) ie: that a_G commutes with restrictions to subgroups.

Let $H \leq G$ and $\chi \in \mathbb{Q} \otimes_Z R(G)$, then for all $x \in \mathbb{Q} \otimes_Z R_+(G,S^1)$:

$[x, \mathrm{Res}\,^G_H(a_G(\chi))]_H = [\mathrm{Ind}\,^G_H\,(x),\ a_G(\chi)]_G$

by the Frobenius reciprocity property in $R_+(G,S^1)$

$= (b(\mathrm{Ind}_H^G(x)), \chi)_G$ from 3.1.14

$= (\mathrm{Ind}_H^G(b|_H(x)), \chi)_G$

$= (b|_H(x), \chi|_H)_H$

$= [x, a_H(\chi|_H)]_H$ by 3.1.14

and this implies that $\mathrm{Res}_H^G(a_G(\chi)) = \alpha_{(H,\phi)^G}(\chi|_H)$, since the bilinear form $[-, -]$ is $R_+(G,S^1)$ is non-degenerate. ($[-, -]$ is an upper triangular matrix with $(N_G(H):H)$ as its diagonal entries)

Following [3, P.13] and [1, P.40], we summarize the properties of a_G in the the following result.

3.1.18 Theorem

(a) There is one and only one family of maps

$a_G : R(G) \longrightarrow R_+(G,S^1)$, satisfying conditions

3.1.12 (i) and (ii)

(b) For all G, $b.a_G = \mathrm{Id}: R(G) \longrightarrow R(G)$

(c) For all $\phi \in \hat{G}$, $a_G(\phi) = (\phi : G \longrightarrow S^1)$

(d) The homomorphism, a_G, is a $\mathbb{Z}[\hat{G}]$- module homomorphism.

(e) If S_G is the partially ordered set of subgroups of G, then for $K, H \in S_G$, define the Möbius function of S_G by the formula

$$\mu_{K,H} = \sum (-1)^i \, \# \{i\text{-chains } K = C_0 < C_1 \ldots < C_i = H \text{ in } S_G\}$$

We have, for $\chi \in R(G)$, the following explicit formula:

$$a_G(\chi) = |G|^{-1} \sum_{K \le H \in S_G} |K| \, \mu_{K,H} \, \mathrm{Ind}_K^G (\mathrm{Res}_K^H (P_H \, \mathrm{Res}_H^G(\chi)))$$

where $\mathrm{Ind}_K^G((\upsilon : J \longrightarrow S^1)) = (\upsilon : J \longrightarrow S^1) \in R_+(G,S^1)$

or $$a_G(\chi) = |G|^{-1} \sum_{(K,\psi) \le (H,\phi) \in M_G} |K| \, \mu^{M_G}_{(K,\psi),(H,\phi)} \, (\phi, \chi|_H) \, (K, \psi)^G$$

....

and $\mu^{\mathcal{M}_G}_{(K,\chi),(H,\phi)}$ is the Möbius function on $\mathcal{M}_G$ given by $\sum_{i\geq 0}(-1)^i_{\#}$ {i-chains from $(K,\psi$ to (H,ϕ) in $\mathcal{M}_G$}

(f) For all $\chi \in R(G)$, if $(\phi,\chi|_H) = 0$, or there exists some $(K,\psi) > (H,\phi)$ in $\mathcal{M}_G$ with $(\psi,\chi|_K) = (\phi,\chi|_H)$ then the coefficient of $(H,\phi)^G$ in $a_G(\chi)$ is zero.

3.2 EXAMPLES OF $a_G(\chi)$ FOR $G = D_8$, $PSLF_5$.

From [26, P.37] we have the following character table for $G= D_8 = \langle x,y | x^4 = y^2= 1,\ xyx = y\rangle$

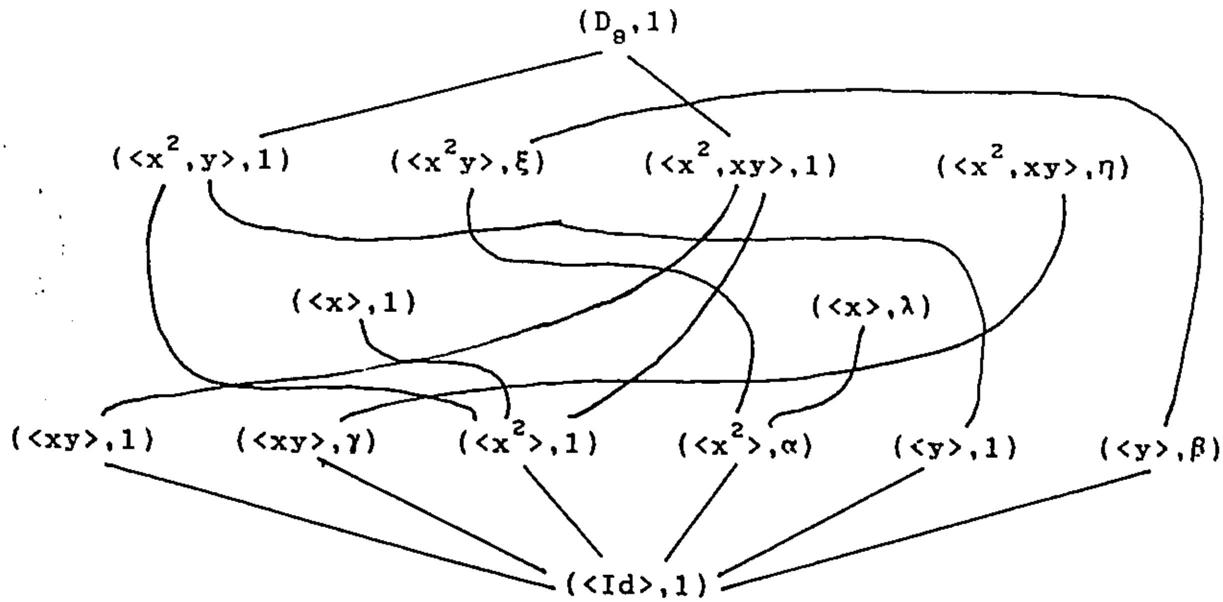
3.2.1

	1	x	x^2	x^3	y	xy	x^2y	x^3y
1	1	1	1	1	1	1	1	1
χ_1	1	1	1	1	-1	-1	-1	-1
χ_2	1	-1	1	-1	1	-1	1	-1
χ_3	1	-1	1	-1	-1	1	-1	1
υ	2	$i+i^{-1}$ $=0$	-2	i^3+i^{-3} $=0$	0	0	0	0

where $i = \sqrt{-1}$, the χ_i are linear characters and υ is a character of degree 2.

We have the following poset diagram for the pairs $(H,\phi) \in \mathcal{M}_G$

3.2.2



where $\alpha : \langle x^2\rangle \longrightarrow \mathbb{C}^*$ is given by $x^2 \longrightarrow -1$

$\beta : \langle y\rangle \longrightarrow \mathbb{C}^*$ is given by $y \longrightarrow -1$

$\gamma : \langle xy\rangle \longrightarrow \mathbb{C}^*$ is given by $xy \longrightarrow -1$

$\lambda : \langle x\rangle \longrightarrow \mathbb{C}^*$ is given by $x \longrightarrow i$

$\eta : \langle x^2,xy\rangle \longrightarrow \mathbb{C}^*$ is given by $x^2 \longrightarrow -1, xy \longrightarrow 1$

and $\xi : \langle x^2,y\rangle \longrightarrow \mathbb{C}^*$ is given by $x^2 \longrightarrow -1, y \longrightarrow 1$

We will evaluate $a_G(\upsilon)$ for the nonlinear character υ.

From 3.1.11(b) we have the following expression:

3.2.3 $$a_{D8}(\upsilon) = a\overline{(D_8,1)}^{D_8} + b\overline{(\langle x^2,y\rangle,1)}^{D_8} + c\overline{(\langle x^2,y\rangle,\xi\rangle}^{D_8} + d\overline{(\langle x^2,xy\rangle,1)}^{D_8}$$

$$+ e\overline{(\langle x^2,xy\rangle,\eta)}^{D_8} + f\overline{(\langle x\rangle,1)}^{D_8} + g\overline{(\langle x\rangle\lambda)}^{D_8} + h\overline{(\langle xy\rangle,1)}^{D_8}$$

$$+ j\overline{(\langle xy\rangle,\sigma)}^{D_8} + k\overline{(\langle x^2\rangle,1)}^{D_8} + l\overline{(\langle x^2\rangle,\alpha)}^{D_8} + m\overline{(\langle y\rangle,1)}^{D_8}$$

$$+ \overline{n(\langle y\rangle,\beta)}^{D_8} + \overline{p(\langle Id\rangle,1)}^{D_8}$$

We use some results from [1, P.50]

3.2.4 (i) For each character χ of G, and each $(H,\phi) \in \mathcal{M}_G$

if $(\phi,\chi|_H) = 0$ then $\alpha_{(H,\phi)^G}(\chi) = 0$.

(ii) For each character χ of G with $(H,\phi) < (H^1,\phi^1)$ and

$(\phi^1,\chi|_{H^1}) = (\phi,\chi|_H)$ then $\alpha_{(H,\phi)^G}(\chi) = 0$.

(iii) For $\chi \in R(G)$ with (H,ϕ) maximal and $(\phi,\chi|_H) \neq 0$,

then $\alpha_{(H,\phi)^G}(\chi) = \dfrac{(\phi,\chi|H)}{(N_G(H,\phi):H)}$

(iv) For all $\chi \in R(G)$ and all $(H,\phi) \in \mathcal{M}_G$,

if $Z(G) \leq H$ then $\alpha_{(H,\phi)^G}(\chi) = 0$.

We note that by property 3.2.4(i) the following coefficients in 3.2.3 are zero (a,b,d,f,k). Since $Z(D_8) = \langle x^2\rangle$, then by (iv) (h,j,m,n,p) are zero. This reduces the expression to

3.2.5 $$a_G(\upsilon) = \overline{c(\langle x^2y\rangle,\xi)}^{D_8} + \overline{e(\langle x^2,xy\rangle,\eta)}^{D_8}$$
$$+ \overline{g(\langle x\rangle,\lambda)}^{D_8} + \overline{l(\langle x^2\rangle,\alpha)}^{D_8}$$

We note that $(\langle x^2,y\rangle,\xi)$, $(\langle x^2,xy,\eta)$ and $(\langle x\rangle,\lambda)$ are maximal pairs in 3.2.2. Hence to find c, e, and g we use 3.2.4(iii).

To illustrate, we find c. The Schur inner product given by,

$$(\xi,\upsilon|_{\langle x^2,y\rangle}) = \frac{1}{4}\sum_{g\in\langle x^2,y\rangle} \xi(g)\ \upsilon|_{\langle x^2,y\rangle}(g)$$
$$= \frac{1}{4}\ [2 + 0 + 0 + (-1)\ (-2)] = 1$$

while $(N_{D_8}(\langle x^2,y\rangle,\xi) : \langle x^2,y\rangle) = 1$ and so $c = 1$.

Similarly e and g have value 1.

To calculate l we must use the formula given in 3.1.11(b). Here $(\alpha,\upsilon|_{\langle x^2\rangle}) = 2$ and we have

3.2.6 $$2 = \frac{|N_{D_8}(\langle x^2\rangle,\alpha)|}{|\langle x^2\rangle|}(1) + \gamma^{D_8}_{(\langle x^2\rangle,\alpha),\ (\langle x\rangle,\sigma)}(g)$$

$$+\gamma^{D_8}_{(\langle x^2\rangle,\alpha),\ (\langle x^2,y\rangle,\xi\rangle}(c) + \gamma^{D_8}_{(\langle x^2\rangle,\alpha),\ (\langle x^2,xy\rangle,\eta)}(e),$$

using the formula for $\gamma^{G}_{(H,\chi),(H^1,\phi^1)}$ given by 3.1.11(b), we find that

$$\gamma^{D_8}_{(\langle x^2\rangle,\ \alpha),\ (\langle x\rangle,\ \sigma)} = 2$$

$$= \gamma^{D_8}_{(\langle x^2\rangle,\ \alpha),\ (\langle x^2,y\rangle,\xi)} = \gamma^{D_8}_{(\langle x^2\rangle,\alpha),\ (\langle x^2,xy\rangle,\eta)}$$

and since $Z(D_8) = \langle x^2\rangle$ substitution in 3.2.6 gives $l = -1$ and we rewrite 3.2.5 as follows:

3.2.7 $$a_{D_8}(\upsilon) = \overline{(\langle x^2,y),\xi)}^{D_8} + \overline{(\langle x^2,xy\rangle,\eta)}^{D_8} + \overline{(\langle x\rangle,\lambda)}^{D_8} - \overline{(\langle x^2\rangle,\alpha)}^{D_8}$$

We also note that $b(a_{D_8}(\upsilon) =$

$$\mathrm{Ind}^{D_8}_{\langle x^2,y\rangle}(\xi) + \mathrm{Ind}^{D_8}_{\langle x^2,xy\rangle}(\eta) + \mathrm{Ind}^{D_8}_{\langle x\rangle}(\lambda) - \mathrm{Ind}^{D_8}_{\langle x^2\rangle}(\alpha)$$

$= b.T_{D_8}(\upsilon)$ from 2.2.12 and the subsequent calculation for D_8.

In fact, $a_G(\upsilon)$ and $T_G(\upsilon)$ are related and coincide when $\dim(\upsilon) \le 2$ as is the case for D_8. The relation between these two maps is shown in [3, §3,§4], to be

3.2.8 $$T_G(\upsilon) = a_G(\upsilon).\hat{\varepsilon}_G(\upsilon) \text{ where}$$

$\hat{\varepsilon}_G$ is given by $\varepsilon_G\,\tau_G$ and where for $\Omega(G)$, the Burnside ring, and $\Pi_n = \Sigma_n \int S^1$,

ϵ_G: $R_+(G,\Pi_n) \longrightarrow \Omega(G)$ is given by

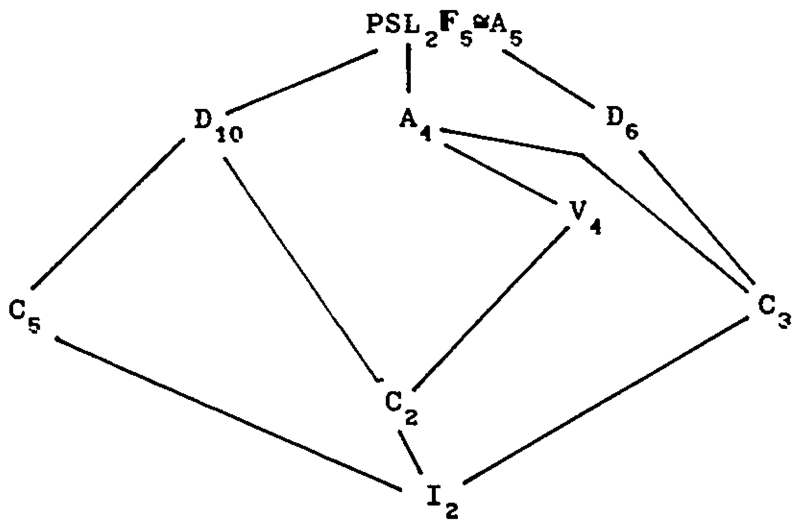
$$(H \xrightarrow{\upsilon} \Pi_n)_G \longrightarrow (H)_G \text{ (the conjugacy class of H in G).}$$

and τ_G is defined as in 2.2.11.

We will now consider $a_G(G)$ for $G = PSL_2\mathbf{F}_5$.

From [7, P.286,§262] we can form the following poset table for $PSL_2\mathbf{F}_5$ and the related information on conjugacy classes.

3.2.9

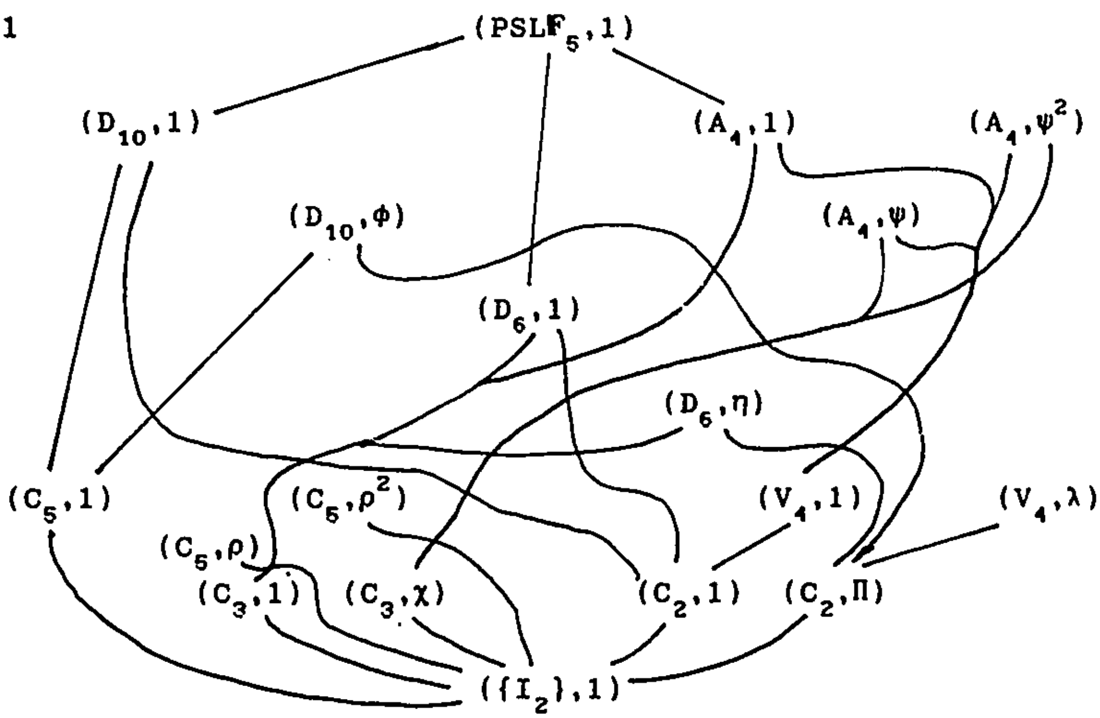


3.2.10

SUBGROUP	A PRESENTATION	NUMBER OF CONJUGATES OF EACH TYPE
C_2	$\{\langle \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \rangle \mid z \in \mathbf{F}_5\}$	15
C_3	$\{\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \rangle\}$	10
C_5	$\{\langle \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \rangle \mid \alpha \in \mathbf{F}_5\} \cong \mathbf{F}_5^+$	6
V_4	$\{\langle \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle \mid z \in \mathbf{F}_5$	5
D_6	$\{\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \rangle\}$	10
D_{10}	$\{\langle \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \rangle\} \mid \alpha \in \mathbf{F}_5\}$	6
A_4	$\{\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \rangle\}$	5

Before we can develop expressions for a_G where $G = PSL_2\mathbf{F}_5$, we need some preliminaries. First, the poset table for M_G, is as follows:

3.2.11



We note that for $\zeta: A_4 \longrightarrow A_4^{ab} \cong \mathbf{Z}/3$, then since $N_G(A_4) = A_4$,

$\zeta = 1, \psi$ or ψ^2 where $\psi : \mathbb{Z}/3 \hookrightarrow S^1$. Also $D_{10} \longrightarrow D_{10}^{ab} \cong \mathbb{Z}/2 \xrightarrow{\zeta} \{\pm 1\} \subset S^1$ gives $\zeta = 1$ or ϕ where $\phi(x) = -1$ for $x \in \mathbb{Z}/2$ and similarly for D_6 we have $\eta : \mathbb{Z}/2 \longrightarrow \{\pm 1\}$.

In the V_4 case we have $1, \lambda_1, \lambda_2, \lambda_1\lambda_2 : V_4 \longrightarrow \{\pm 1\}$ and $(V_4, \lambda_1) \sim (V_4, \lambda_2) \sim (V_4, \lambda_1\lambda_2) = (V_4, \lambda)$ where the conjugation is given by the elements $t \in A_4$ (where $t^3 = 1$).

Since $N_G(C_5) = D_{10}$ then $\lambda : C_5 \hookrightarrow S^1$ gives $\rho = 1, \rho, \rho^2$ where $(C_5, \rho) \sim (C_5, \rho^4)$; $(C_5, \rho^2) \sim (C_5, \rho^3)$ by conjugation in D_{10} and $\rho(x) = \exp(2\pi i/5)$ for $x \in C_5$.

Similarly for C_3, $N_G(C_3) = D_6$ and so $\chi : C_3 \hookrightarrow S^1$ gives $\chi = 1, \chi$ where $(C_3, \chi) \sim (C_3, \chi^2)$ and $\chi(x) = \exp(2\pi i/3)$ for $x \in C_3$.

From [1, P.52 or 14] we have the following : $A_5 \cong PSL_2\mathbf{F}_5$ has five characters of order, 1,3,3,5 and 5. We note,[14, Pp.132,133] that the conjugacy classes and character table for A_5 may be tabulated as follows:

CONJUGACY CLASSES FOR A_5

i	NUMBER IN CLASS	DESCRIPTION OF CLASS (C_i)
1	1	{Id}
2	15	elements (12)(34) of order 4
3	20	elements (124) of order 3
4	12	conjugates of (12345)
5	12	conjugates of (12345)

<u>CHARACTER TABLE FOR A_5</u>

	C_1	C_2	C_3	C_4	C_5
h_i	1	15	20	12	12
1	1	1	1	1	1
$\upsilon_{3,1}$	3	-1	0	α_1	α_2
$\upsilon_{3,2}$	3	-1	0	α_2	α_1
υ_4	4	0	1	-1	-1
υ_5	5	1	-1	0	0

where $\alpha_1 = \frac{1 + \sqrt{5}}{2}$ and $\sigma_2 = \frac{1 - \sqrt{5}}{2}$

Boltje [1], uses this information to calculate $a_G(\upsilon)$ for $\upsilon \in R(A_5)$, but we will demonstrate an alternate calculation for $a_G(\upsilon_4)$.

3.2.12

$$\begin{aligned} a_G(\upsilon 4) = {} & a(\overline{PSL_2F_5, 1})^G + b(\overline{A_4,\ 1})^G + c(\overline{A_4,\ \phi})^G \\ & + d(\overline{A_4, \phi^2})^G + e(\overline{D_{10}, 1})^G + f(\overline{D_{10}, \phi})^G \\ & + g(\overline{D_6,\ 1})^G + h(\overline{D_6,\ \eta})^G + i(\overline{C_6,\ 1})^G \\ & + j(\overline{C_5,\ \rho})^G + k(\overline{C_5, \rho^2})^G + l(\overline{V_4,\ 1})^G \\ & + m(\overline{V_4,\ \lambda})^G + n(\overline{C_3,\ 1})^G + \hat{o}(\overline{C_3,\ \chi})^G \\ & + p(\overline{C_2,\ 1})^G + q(\overline{C_2,\ \Pi})^G + r(\overline{[I], 1})^G \end{aligned}$$

We will develop the notion of Weil (Cuspidal) representations in

Chapter 4. We can apply Snaith's unpublished work on the characters of such representations since they are of order (q - 1) for $GL_2\mathbf{F}_q$ and irreducible, as is υ_4.

We may use the properties 3.2.4 to calculate these coefficients in conjunction with the character tables from [26, Pp.39-42].

We will use 3.2.4 (i) and (ii) to calculate some coefficients in 3.2.12.

For $(A_4,\phi)^G$, we have $(\phi,\upsilon_4|_{A_4}) = \frac{1}{12}\sum_{g\in A_4}(\phi(g)\ .\ \overline{\upsilon_4|_{A_4}(g)}$,

consider the presentation for A_4 given by

$\{\langle x\rangle, \langle y\rangle, \langle z\rangle \mid x^2 = y^2 = z^3 = 1, xy = yz,\ zx = xyz,\ zy = xz\}$

and the character values given by,

$$\text{and}\quad \left\{\begin{matrix} x \\ y \\ xy \\ 1 \end{matrix}\right. \longrightarrow 1 \quad \left\{\begin{matrix} zx \\ zy \\ zxy \\ z \end{matrix}\right. \longrightarrow \omega \ \text{and} \quad \left\{\begin{matrix} z^2x \\ z^2y \\ z^2xy \\ z^2 \end{matrix}\right. \longrightarrow \omega^2 \quad \text{where } \omega=\exp(2\Pi i/3)$$

The conjugacy class of a matrix $X\in GL_2\mathbf{F}_q$, where $q = p^s$ and p is prime, is determined by its minimal polynomial. Hence those elements in A_4 of order three ie: those matrices represented by zx, zy, zxy, z^2x, z^2y, z^2xy, z, z^2 have minimal polynomial $t^2 + t + 1 = 0$. In an unpublished work Snaith has developed a table as follows for the character values of Weil representations.

3.2.13 <u>Conjugacy Classes and Character Values for $G = GL_2\mathbf{F}_q$</u>

Type	Minimal Polynomial	Conjugacy Class Representation	Character Value
I	$(t-\alpha)(t-\beta)$ $\alpha,\beta \in \mathbf{F}_q^*$	$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$	0
II	$(t-\alpha)^2$ $\alpha \in \mathbf{F}_q^*$	$\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$	$-\theta(\alpha)$
III	$(t-\alpha)$ $\alpha \in \mathbf{F}_q^*$	$\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$	$(q-1)\theta(\alpha)$
IV	$t^2-(x+F(x))t+xF(x)$ $F(x)\neq x \in \mathbf{F}_{q^2}^*$	$\begin{pmatrix}0 & -xF(x)\\ 1 & (x+F(x))\end{pmatrix}$	$-\{\theta(x)+\theta(F(x))\}$

where $\theta : \mathbf{F}_q^{*2} \longrightarrow \mathbb{C}^*$ and $F(x) = x^q$ is the Frobenius map

Here we find the character value of $\upsilon_4|_{A_4}(g)$ from case (iv), is given by

$$-(\theta(\upsilon) + \theta(F(\upsilon)) \text{ where}$$

$$\theta:\mathbf{F}_5^*2 \longrightarrow \mathbb{C}^*,\ v \in \mathbf{F}_q^* \text{ and } v \neq F(v).$$

3.2.14

$$\text{Thus } (\phi,\upsilon_4|_{A_4}) = \tfrac{1}{12}\{[4(4(1)) - 4\omega\overline{[(\theta(v) +\theta(F(v))]}$$

$$- 4\omega^2[\overline{\theta(v) + \theta(F(v)}]\}$$

$$= \tfrac{1}{3}\ [4(1 + \overline{\theta(v) + \theta(F(v)})]$$

where $\omega = \exp 2\pi i/_3$

3.2.15 Similarly $(\chi,\upsilon_4|_{C_3}) = (\phi,\upsilon_4|_{A_4}) = (\phi^2,\ \upsilon_4|_{A_4})$

and from [1, P.50(ii)], this implies that

$c = d = 0$

Similarly evaluations of Schur inner products will show that $a = e = f = l = i = 0$.

By using 3.2.4 (iii), we may calculate the coefficients b, g, h, j, k, m amd $\hat{o}$, since the corresponding pairs are maximal in M_G.

Let us consider $(A_4, 1)$, then

3.2.16 $$b = \frac{(1, \upsilon|_{A_4})}{N_G(A_4, 1):A_4} = 1$$

and similarly $g = h = j = k = m = \hat{o} = 1$.
With these values we may rewrite 3.2.12 as

$$\begin{aligned} a_G(\upsilon_4) = {} & (\overline{A_4, 1})^G + (\overline{D_6, 1})^G + (\overline{D_6, \eta})^G \\ & + (\overline{C_5, \rho})^G + (\overline{C_5, \rho^2})^G + (\overline{V_4, \lambda})^G \\ & + n(\overline{C_3, 1})^G + (\overline{C_3, \chi})^G + p(\overline{C_2, 1})^G \\ & + q(\overline{C_2, \Pi})^G + r(\overline{[I_2], 1})^G \end{aligned}$$

The remaining coefficients may be calculated by applying 3.1.11(b). Let us consider the cases $(C_3, 1) \le (H^1, \phi^1)$ and $(C_2, 1) \le (H^1, \phi^1)$. These will give us equations to determine n and p.

3.2.17 $$(1, \upsilon_4|_{C_3}) = \frac{1}{3} \sum_{g \in C_3} \{1(g)\ \upsilon_4|_{C_3}(g)\}$$

and once again we use 3.2.13 to obtain the characters of elements of $C_3 \cong \{<\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}>\}$

ie: $(1, \upsilon_4|_{C_3}) = \frac{1}{3}\{1(4) + 1(-\omega-\omega^2) + 1(-\omega-\omega^2)\}$ where $\omega = \exp(2\pi i/3)$

$$= \frac{1}{3}\{4 - 2(\omega + \omega^2)\}$$

$$= 2$$

hence we have

3.2.18 $$2 = \gamma^G_{(C_3,1),(C_3,1)}(n) + \gamma^G_{(C_3,1),(D_6,1)}(g) + \gamma^G_{(C_3,1),(D_6,\eta)}(h) + \gamma^G_{(C_3,1),(A_4,1)}(b)$$

and $$\gamma^G_{(H,\phi),(H^1,\phi^1)} = |\{s \in H\backslash^G/H^1 \mid (H,\phi) \le {}^s(H^1,\phi^1)\}|$$

hence $$\gamma_{(C_3,1),(C_3,1)} = |N_G(C_3,1)|/|C_3| = 2$$

since $N_G(C_3,1) = D_6$

while $\gamma^G_{(C_3,1),(D_6,1)} = 1 = \gamma^G_{(C_3,1),(D_6,\eta)}$

and $\gamma^G_{(C_3,1),(A_4,1)} = 2$, for we have $s_1 \in D_6$ and $s_2 \in A_4$ since A_4 has no subgroup of order six. Substituting for the above and known coefficients gives $n = -1$.

Similarly for $(C_2,1)$ we have

3.2.19 $$(1,\upsilon_4|_{C_2}) = \frac{|N_G(C_2,1)|}{|C_2|}(p) + \gamma^G_{(C_2,1),(V_4,1)}.(1) + \gamma^G_{(C_2,1),(V_4,1)}(m)$$

$$+ \gamma^G_{(C_2,1)(D_6,1)}(g) + \gamma^G_{(C_2,1),(D_{10},1)}(e) + \gamma^G_{(C_2,1),(A_4,1)}(b)$$

and $l = e = 0$. Again from 3.2.1 we have

$$(1,\upsilon_4|_{C_2}) = \tfrac{1}{2}\{1.4 + 1(0)\} = 2,$$

$\gamma^G_{(C_2,1),(V_4,\lambda)} = 1$, since $N(C_2) = V_4$, and we have one case for $s \in V_4$.

$\gamma^G_{(C_2,1),(A_4,1)} = 1$ for the case of $s = A_4$, since $N_G(A_4) = A_4$

however, $\gamma^G_{(C_2,1),(D_6,1)} = 2$ as we have two possible cases, $s_1 \in V_4$ or $s_2 \in D_6$.

Hence 3.2.19 becomes:

$$2 = 2p + 1 + 2 + 1 \text{ which gives } p = -1.$$

We have yet to calculate q and r. To do so we make use of two additional expressions from [1, P.50, 4(vi) and 4(vii)].

3.2.20 (a) For all $\chi \in R(G)$, $\sum_{(H,\phi)^G \in M_G/_G} (G:H)\, \alpha_{(H,\phi)^G}(\chi) = \chi(1)$

(b) For all $\chi \in R(G)$, $\sum_{(H,\phi)^G \in \mathcal{M}_G/_G} \alpha_{\overline{(H,\phi)}^G}(\chi) = \chi(1)$

and from 3.2.13 we have

$$\upsilon(1) = \upsilon(I_2) = 4(\theta(1)) = 4 \text{ where } \theta : \mathbb{F}_{5^2}^* \longrightarrow \mathbb{C}^*$$

So from 3.2.20(b) we have

$$q + r = -1$$

while from 3.2.20(a) we obtain

$q + 2r = -1$ which gives $q = -1$ and $r = 0$.

It follows that:

$$\begin{aligned} a_G(\upsilon_4) = {} & \overline{(A_4, 1)}^G + \overline{(D_6, 1)}^G + \overline{(D_6, \eta)}^G \\ & + \overline{(C_5, \rho)}^G + \overline{(C_5, \rho^2)}^G + \overline{(V_4, \lambda)}^G \\ & - \overline{(C_3, 1)}^G + \overline{(C_3, \chi)}^G - \overline{(C_2, 1)}^G \\ & - \overline{(C_2, \Pi)}^G. \end{aligned}$$

CHAPTER 4

THE IRREDUCIBLE REPRESENTATIONS OF $GL_2\mathbf{F}_q$

Following [7, 22, 27, and 36] we review a construction of the irreducible representations of $GL_2\mathbf{F}_q$. We begin with the cuspidal or Weil representations.

4.1 *CUSPIDAL OR WEIL REPRESENTATIONS:*

Let $\mathbf{F}_q$ denote the finite field of order q, then $\mathbf{F}_q^{\,2}/\mathbf{F}_q$ is a quadratic extension. We define the irreducible, cuspidal representations

4.1.1 $$r(\theta) : GL_2\mathbf{F}_q \longrightarrow GL_{q-1}(\mathbb{C})$$

which are associated to a character

4.1.2 $$\theta : \mathbf{F}_q^{*2} \longrightarrow \mathbb{C}^*$$

These representations are due to A.Weil. The construction, in greater generality is described in [13, P.122], where the Weil representation is described in the case where $\mathbf{F}_q$ is replaced by a local field. This construction is a modification of Weil's.

4.1.3 Before giving the formula for $r(\theta)$ above, we recall the manner in which GL_2F is generated, when F is any field.

Define the following subgroups:

$$B = \left\{ X \in GL_2F \mid X = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \right\}$$

where $B \leq GL_2F$ is called the *BOREL Subgroup*;

$$U = \left\{ Y \in B \mid Y = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right\}$$

where $U \leq B$ is called the *unitriangular subgroup*.

Set $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2F$

The BRUHAT DECOMPOSITION OF GL_2F has the form:

4.1.3(a) $\quad GL_2F = B \cup B\,w\,U$

This is straightforward for 2x2 matrices and is described in [20, Pp. 472-476].

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2F$ with $c \neq 0$

then A may be written uniquely:

4.1.3(b) $\quad A = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} w \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$

where $\delta = -c$, $u = dc^{-1}$, $\beta = -a$ and $\alpha = -\det(A)\,c^{-1}$

4.1.4 Theorem If F is any field, then GL_2F is generated by matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}, \quad \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \text{ and } w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ where } \alpha, \delta \in F^* \text{ and } u \in F,$$

which are subject to the following relations:

(i) $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \delta^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha u \delta^{-1} \\ 0 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}$

(iii) $w \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} w^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$

(iv) $w \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} w = \begin{pmatrix} -u^{-1} & 0 \\ 0 & -u \end{pmatrix} \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & -u^{-1} \\ 0 & 1 \end{pmatrix}$

(v) $w^4 = 1$

Proof: See [22,Pp.60-131], and [20,Pp.472-476].

Folowing is a summary of the derivation of $r(\theta)$ taken from an unpublished manuscript of V. Snaith.

Given $\theta : \mathbf{F}_q^{*2} \longrightarrow \mathbb{C}^*$ as in 4.1.2, denote by H the $\mathbb{C}$ vector space given by:

$$4.1.5 \quad H = \left\{ f\colon \mathbf{F}_q^{*2} \longrightarrow \mathbb{C} \mid f(t^{-1}x) = \theta(t)\, f(x) \text{ if } N(t) = 1 \right\}$$

where $N = N_{\mathbf{F}_{q^2}/\mathbf{F}_q} : \mathbf{F}_q^{*2} \longrightarrow \mathbf{F}_q^*$ is the surjective norm given by $N(t) = t\sigma(t)$ for $t \in \mathbf{F}_q^{*2}$ and $\sigma(t)$ is the Frobenius map $\sigma(t) = t^q$ and $\sigma \in \mathrm{Gal}\,(\mathbf{F}_{q^2}/\mathbf{F}_q)$.

Since $N_{\mathbf{F}_{q^2}/\mathbf{F}_q}$ is surjective then $\dim_{\mathbb{C}} H = (q - 1) = |\mathbf{F}_q^*|$

We will require the following result:

4.1.6 Lemma

Let $\chi\colon G \longrightarrow \mathbb{C}^*$ be a non-trivial homomorphism. If G is a finite abelian group, then $\sum_{g\in G} \chi(g) = 0$

Proof: $$\sum_{g\in G}\chi(g) = |\mathrm{Ker}\ \chi| \sum_{j=0}^{n-1} \chi(g)^j$$

$$= |\mathrm{Ker}\ \chi| . 0$$

when g Ker χ generates $G/_{\mathrm{Ker}\ \chi} \cong \mathbb{Z}/_n$.

4.1.7 Corollary If θ, in 4.1.2 is non-trivial and for $f\in H$, the function space of 4.1.5, then $\sum_{v\in\mathbf{F}_q^{*2}} f(v) = 0$.

Proof: Let $x_1, x_2, \ldots\ldots, x_{q-1} \in \mathbf{F}_q^{*2}$ be a set of coset representatives for $\mathbf{F}_q^{*2}/_{\mathrm{Ker}(N)} \cong \mathbf{F}_q^*$

Then the above sum may be rewritten as:

$$\sum_{v\in\mathbf{F}_q^{*2}} f(v) = \sum_{i=1}^{q-1} \sum_{t\in \mathrm{Ker}(N)} f(t\, x_i)$$

$$= \sum_{i=1}^{q-1} \left[\sum_{t \in Ker(N)} \theta(t^{-}) \right] f(x_i)$$

$= 0$ by Lemma 4.1.6.

4.1.8 The Fourier Transform on *H*.

If $f \in H$, define the *Fourier transform* $\hat{f}$, of f by the formula

$$\hat{f}\,(z) = -(q^{-1}) \sum_{y \in \mathbf{F}_q^{*2}} f(y)\, \omega_{\mathbf{F}_q{}^2}\,(y\,\sigma\,(z) + z\,\sigma\,(y))$$

where $z \in \mathbf{F}_q^{*2}$ and $\omega_{\mathbf{F}_q{}^2}$ is the non-trivial additive character given by $\omega_{\mathbf{F}_q{}^2} : \mathbf{F}_q{}^2 \longrightarrow \mathbb{C}^*$

such that $\omega_{\mathbf{F}_q{}^2}(y) = \exp\,(2\pi i\,(\mathrm{Trace}_{\mathbf{F}_q{}^2/\mathbf{F}_p}\,(y))\,p^{-1})$

where $q = p^s$, for p a prime.

We will also need the following result:

4.1.9 Lemma The map which assigns to $f \in H$, the Fourier transform, $\hat{f}$, defines a $\mathbb{C}$ - linear endomorphism of H, of order four.

Proof: Let $z \in \mathbf{F}_q^{*2}$ and $t \in \mathrm{Ker}\,(N)$, then by 4.1.8,

$$\hat{f}\,(tz) = -(q^{-1}) \sum_{y \in \mathbf{F}_q^{*2}} f(y)\, \omega_{\mathbf{F}_q{}^2}\,(y\sigma(tz) + tz\sigma(y))$$

$$= -(q^{-1}) \sum_{y \in \mathbf{F}_q^{*2}} f\,(y)\, \omega_{\mathbf{F}_q{}^2}\,(yt^{-1}\sigma\,(z) + z\sigma(t^{-1}y))$$

since $\sigma(t) = t^{-1}$

$$= -(q^{-1}) \sum_{v \in \mathbf{F}_q^{*2}} f\,(tv)\ \omega_{\mathbf{F}_q^2}\ (v\sigma(z) + z\sigma(v))$$

where $yt^{-1} = v$

$$= -(q^{-1}) \sum_{v \in \mathbf{F}_q^{*2}} \theta\,(t^{-1})\ f(v)\ \omega_{\mathbf{F}_q^2}\ (v\sigma(z) + z\sigma(v))$$

$$= \theta(t^{-1})\hat{f}(z)$$

Therefore the Fourier transform, which is $\mathbb{C}$ - linear, gives an endomorphism of H. To obtain the order of the endomorphism we will show that

4.1.10 $\quad \hat{\hat{f}}(z) = f(-z)$ for all $z \in \mathbf{F}_q^{*2}$, $\quad f \in H$.

By 4.1.8:

$$\hat{\hat{f}}(z) = -(q^{-1}) \sum_{y \in \mathbf{F}_q^{*2}} \hat{f}(y)\ \omega_{\mathbf{F}_q^2}\ (y\sigma(z) + z\sigma(y))$$

$$= (q^{-2}) \sum_{y,v \in \mathbf{F}_q^{*2}} f(v)\omega_{\mathbf{F}_q^2}\ (v\sigma(y)+y\sigma(v))\ \omega_{\mathbf{F}_q^2}\ (y\sigma(z)+z(\sigma(y))$$

$$= (q^{-2}) \sum_{y,v \in \mathbf{F}_q^{*2}} f(v)\ \omega_{\mathbf{F}_q^2}\ (y\sigma(z + v) + (z + v)\sigma(y))$$

Now when $z + v \neq 0$, then by 4.1.6

$$\sum_{y \in \mathbf{F}_q^{*2}} \omega_{\mathbf{F}_q^2}\ (y\sigma(z + v) + (z + v)\sigma(y)) = -1 \quad \text{and when}$$

$$z + v = 0 \text{ then} \sum_{y \in \mathbf{F}_q^{*2}} \omega_{\mathbf{F}_q^2}\ (0) = (q^2 - 1)$$

And so the expression for $\hat{\hat{f}}(z)$ can be reduced to

$$(q^{-2})\left[(q^2-1)\ f(-z) - \sum_{\substack{v\in \mathbf{F}_q^{*2} \\ v\neq -z}} f(v)\right]$$

and by 4.1.7 since $\sum_{v\in \mathbf{F}_q^{*2}} f(v) = 0$

then $\sum_{\substack{v\in \mathbf{F}_q^{*2} \\ v\neq -z}} f(v) = 0 - f(-z)$ from 4.1.7,

and so the expression becomes:

$$(q^{-2})\ [(q^2-1)\ f(-z) - (\ -f(-z\ ))] = f(\ -z)$$

Thus the endomorphism given by the Fourier transform has order four.

4.2.1 The Weil representation : Definition

Let θ: $\mathbf{F}_q^{*2} \longrightarrow \mathbb{C}^*$ be a non-trivial character as in 4.1.2. Define a representation

$$r(\theta) : GL_2\mathbf{F}_q \longrightarrow \mathrm{Aut}_{\mathbb{C}}\ (H) = GL(H)$$

by the following formulae $(f \in H,\ x \in \mathbf{F}_q^{*2})$

(i) $[r(\theta)]\ (w)\ f(x) = \hat{f}(x)$ where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(ii) $[r(\theta)\ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\ f]\ (x) = \omega_{\mathbf{F}_q}(u\ N_{\mathbf{F}_q{}^2/\mathbf{F}_q}\ (x))\ f(x)$

where $u \in \mathbf{F}_q$

(iii) $[r(\theta)\ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\ f\ ]\ (x) = \theta(\beta)\ f(\beta x)$

where $\alpha \in \mathbf{F}_q^*$, $\beta \in \mathbf{F}_q^{*2}$ and $N_{\mathbf{F}_q{}^2/\mathbf{F}_q}\ (\beta) = \alpha$

4.2.2 Lemma The expressions defined in 4.2.1. (ii) and (iii) give well defined functions lying in H.

Proof:

For (ii), suppose that $t \in \mathrm{Ker}(N_{\mathbf{F}_q{}^2/\mathbf{F}_q})$ then

$$\omega_{\mathbf{F}_q}(u\,N(tx))\,f(tx) \quad \text{where } N = N_{\mathbf{F}_q{}^2/\mathbf{F}_q}$$

$$= \omega_{\mathbf{F}_q}(u\,N(x)\,\theta(t^{-1})\,f(x) \quad \text{by } 4.1.5$$

and so $r(\theta)\,[\begin{smallmatrix}1 & u\\ 0 & 1\end{smallmatrix}]\,f \in H$.

If, in (iii) β is replaced by βt, then

$\theta(\beta t)\,f(t\beta x) = \theta(\beta)\,\theta(t)\,\theta(t^{-1})\,f(\beta x)$ by 4.1.5

$= \theta(\beta)\,f(\beta x)$

and so (iii) is well defined.

Further, $\theta(\beta)\,f(\beta t x) = \theta(\beta)\,f(\beta x)\,\theta\,(t^{-1})$

and so $r(\theta)\,(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix})\,f \in H$.

Before we can check that the formulae for $r(\theta)$, defined in 4.2.1, respect the relations 4.1.4 (i) - (v), we must calculate the value of $r(\theta)\,(\begin{smallmatrix}1 & 0\\ 0 & \delta\end{smallmatrix})$ which is implied by 4.1.4 (iii) and 4.2.1. To obtain this result, the following known fact is needed.

4.2.3 $$\hat{h}(x) = \hat{f}(x\sigma(b)^{-1})$$

for $f \in H$, $x, \in \mathbf{F}_q^{*2}$ and $h(x) = f(bx)$

4.2.4 Lemma

If $f \in H$ and $x \in \mathbf{F}_q^{*2}$ then

$[r(\theta)\,(\begin{smallmatrix}1 & 0\\ 0 & \delta\end{smallmatrix})\,(x) = \theta(\beta)\,f(x\sigma(\beta)^{-1})$

where $N(\beta) = \delta$.

By 4.1.9, $r(\theta)\,(w^{-1})\,f = \hat{h}$ where $h(z) = f(-z)$.

Therefore $[r(\theta)\,(\begin{smallmatrix}1 & 0\\ 0 & \delta\end{smallmatrix})\,f]\,(x) = [r(\theta)\,w\,(\begin{smallmatrix}\delta & 0\\ 0 & 1\end{smallmatrix})\,w^{-1}f]\,(x)$.

$$= r(\theta)\,(w)\,\{z \longrightarrow \theta(\beta)\,\hat{h}(\beta z)\}\,(x)$$

$$= \theta(\beta)\,\hat{\hat{h}}(x\sigma(\beta)^{-1}) \text{ by 4.2.3 and definition 4.2.1 (i)}$$

$$= \theta(\beta)\,h(-x\sigma(\beta)^{-1}) \text{ by 4.1.10}$$

$$= \theta(\beta)\,f(x\sigma(\beta)^{-1}) \quad \text{as required.}$$

4.2.5 Lemma The formulae expressed in 4.2.1 respect the relations of 4.1.4 (ii), (iii) and (v).

Proof:

(a) Since $\omega_{\mathbf{F}_q}(u_1 + u_2)\, N(x) = \omega_{\mathbf{F}_q}(u_1 N(x))\, \omega_{\mathbf{F}_q}(u_2 N(x))$, where $N = N_{\mathbf{F}_{q^2}/\mathbf{F}_q}$ it is clear that 4.2.1 (ii) is linear in $u \in \mathbf{F}_q$. Therefore $r(\theta)$ respects the relation 4.1.4(ii).

(b) Relation (iii) follows from 4.2.4.

(c) Statements 4.2.1 (i) and 4.1.9 ensure that $r(\theta)$ respects relation (v).

4.2.6 Lemma The definition of $r(\theta)$ in 4.2.1 respects the relation 4.1.4 (i).

Proof Let $N = N_{\mathbf{F}_{q^2}/\mathbf{F}_q}$ and $r(\theta)$ be r, and suppose that $N(a) = \alpha$ and $N(d) = \delta$. Then from 4.2.1 and 4.2.4 we have the following

$$[r \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} r \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \delta^{-1} \end{pmatrix} f]\,(x)$$

$$= [r \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \left\{ z \longrightarrow \theta(a^{-1})\, \theta(d^{-1})\, f(a^{-1} z\, \sigma(d)) \right\} (x)$$

by 4.2.1(iii) and 4.2.4.

$$= [r \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \left\{ z \longrightarrow \omega_{\mathbf{F}_q}(u\, N(z))\, \theta(a^{-1} d^{-1})\, f(a^{-1} z \sigma(d)) \right\} (x)$$

by 4.2.1 (ii)

$$= \theta(ad)\, \omega_{\mathbf{F}_q}(u\, N(ax\sigma(d)^{-1}))\, \theta(a^{-1} d^{-1})\, f(x)$$

by 4.2.1(iii) and 4.2.4

$$= \omega_{\mathbf{F}_q}(u\, \alpha\, \delta^{-1}\, N(x))\, f(x) \text{ since } N(\sigma(d)) = N(d) = \delta$$

$$= [\, r(\theta) \begin{pmatrix} 1 & \alpha u \delta^{-1} \\ 0 & 1 \end{pmatrix} f(x) \text{ as required.}$$

4.2.7 Lemma The definition of $r(\theta)$ in 4.2.1 respects the relation 4.1.4 (iv).

Proof: As above, let $N = N_{\mathbf{F}_{q^2}/\mathbf{F}_q}$ and r represent $r(\theta)$, then we will show that:

$$r\ (w)\ r\ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\ r\ (w)\ f(x)$$

4.2.8(a) $$= -\ (q^{-1}) \sum_{v \in \mathbf{F}_q^{*2}} f(v)\ \omega_{\mathbf{F}_q}\ (-N\ (x + v)\ u^{-1})$$

4.2.8(b) $$= (r\ \begin{pmatrix} -u^{-1} & 0 \\ 0 & u \end{pmatrix})\ r\ \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}\ r\ (w)\ r\ \begin{pmatrix} 1 & -u^{-1} \\ 0 & 1 \end{pmatrix}\ f(x)$$

Let $b \in \mathbf{F}_q^{*2}$ such that $N(b) = -u$, then

$$[r\ (w)\ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\ r\ (w)\ f\ ]\ (x)$$

$$= r\ (w)\ \{z \longrightarrow \omega_{\mathbf{F}_q}\ (u\ N\ (z)\ \hat{f}(z))\}\ (x) \quad \text{by 4.2.1}$$

$$= -(q^{-1}) \sum_{y \in \mathbf{F}_q^{*2}} \omega_{\mathbf{F}_q}\ (u\ N\ (y)\ \hat{f}(y)\ \omega_{\mathbf{F}_q}(y\sigma(x) + x\sigma(y))$$
by 4.1.8 and 4.2.1(i)

$$= (q^{-2}) \sum_{y,v \in \mathbf{F}_q^{*2}} \omega_{\mathbf{F}_q}(uy\sigma(y)\ \omega_{\mathbf{F}_q}(y\sigma(x)+x\sigma(y))\omega_{\mathbf{F}_q}(y\sigma(v)+v\sigma(y))f(v)$$
by 4.1.8

$$= (q^{-2}) \sum_{y,v \in \mathbf{F}_q^{*2}} \omega_{\mathbf{F}_q}(uy\sigma(y) + \sigma(y)(x+v) + y\sigma(x+v))\ f(v)$$

In the above expression, if $N(z) = u$ then by substituting $a = x + v$ we obtain

$$\sum_{y \in \mathbf{F}_q^{*2}} \omega_{\mathbf{F}_q}(N\ (zy + a\sigma(z^{-1}))\ \omega_{\mathbf{F}_q}\ (-N\ (a)\ u^{-1})$$

This follows from

$$N\ (zy + a\sigma(z^{-1})) = (zy + a\sigma(z^{-1})\ (\sigma(z)\ \sigma(y) + \sigma(a)\ z^{-1})$$

$$= N(zy) + y\sigma(a) + a\sigma(y) + N(a)\ N\ (\sigma(z^{-1}))$$

$$= u\ y\ \sigma(y) + y\ \sigma(a) + a\ \sigma(y) + N\ (a)\ u^{-1}$$

And the above sum is equal to

$$-1 + \left\{ \sum_{y \in \mathbf{F}_q^{*2}} \omega_{\mathbf{F}_q} N(\ yz + a\sigma(z^{-1}))\ \omega_{\mathbf{F}_q}(\ -N\ (a)\ u^{-1}) \right\}$$

$$= -1 + \left\{ \sum_{s \in \mathbf{F}_q^2} \omega_{\mathbf{F}_q}(N(s)) \right\} \omega_{\mathbf{F}_q}(-N(a)u^{-1}) \Big\} \text{ since } z, a\sigma(z^{-1}) \in \mathbf{F}_q{}^2$$

$$= -1 + \omega_{\mathbf{F}_q}(-N(a)u^{-1}) \{\omega_{\mathbf{F}_q}(0) + (q+1) \sum_{t \in \mathbf{F}_q^*} \omega_{\mathbf{F}_q}(t)\}$$

since $N(s)$ maps $\mathbf{F}_q^{*2}$ onto $\mathbf{F}_q^*$

$$= -1 + \omega_{\mathbf{F}_q} (-N(a)\ u^{-1}) \ \{ 1 + (q+1)\ (-1)\} \text{ by } 4.1.6$$

$$= -1 - q\, \omega_{\mathbf{F}_q} (-N\ (a)\ u^{-1})$$

Substituting this formula into the expression for

$[r\ (w)\ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\ r\ (w)\ f]\ (x)$ gives

$$-(q^{-2}) \left\{ \sum_{v \in \mathbf{F}_q^{*2}} f(v) \right\} - (q^{-1}) \left\{ \sum_{v \in \mathbf{F}_q^{*2}} f(v)\ \omega_{\mathbf{F}_q} (-N\ (x+v)\ u^{-1}) \right\}$$

$$= -(q^{-1}) \sum_{v \in \mathbf{F}_q^{*2}} f(v).\ \omega_{\mathbf{F}_q} (-N(x+v)\ u^{-1}) \quad \text{by } 4.1.7$$

This proves 4.2.8(a).

To establish 4.2.8(b) we note that the expression

$$[r\ \begin{pmatrix} -u^{-1} & 0 \\ 0 & -u \end{pmatrix}\ r\ \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}\ r\ (w)\ r\ \begin{pmatrix} 1 & -u^{-1} \\ 0 & 1 \end{pmatrix}\ f\ ]\ (x)$$

$$= [r\ \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}\ r\ (w)\ r\ \begin{pmatrix} 1 & -u^{-1} \\ 0 & 1 \end{pmatrix}\ f(-x\ u^{-1})$$

from 4.2.1 (iii) and 4.2.4 and where $N(b) = -u$

$$= \omega_{\mathbf{F}_q}[-u\ N\ (-xu^{-1})]\{r(w)(z \longrightarrow \omega_{\mathbf{F}_q}(-(N)z)\ u^{-1})\ f(z))\}(-xu^{-1})$$

by 4.2.1 (ii)

$$= -(q^{-1}) \sum_{y \in \mathbf{F}_q^{*2}} f(y)\ \omega_{\mathbf{F}_q}(-x\sigma(x)u^{-1} - x\sigma(y)u^{-1}\sigma(x)yu^{-1} - y\sigma(y)u^{-1})$$

$$= -(q^{-1}) \sum_{y \in \mathbf{F}_q^{*2}} f(y)\ \omega_{\mathbf{F}_q}[-N\ (x + y)\ u^{-1}]$$

which establishes 4.2.8(b)

4.2.9 Theorem

The formulae of 4.2.1 extend to define a unique representation

$$r(\theta) : GL_2\mathbb{F}_q \longrightarrow \mathrm{Aut}_{\mathbb{C}}(H) \cong GL_{q-1}\mathbb{C}.$$

Proof: This theorem follows from 4.1.3 and 4.1.10-4.2.7.
From 4.2.2, 4.2.4 it follows that the formulae do define a unique automorphism of H, while 4.2.5 - 4.2.7 verify that the formulae respect the relations (i) - (v) of 4.1.4.

4.2.10 Definition: A representation

$$\rho: GL_m\mathbb{F}_q \longrightarrow GL_e(\mathbb{C})$$

is <u>cuspidal</u> if, for each partition

$$M = (m_1, m_2, \ldots..m_r) \quad (m_i \geq 0, \sum_{i=1}^{r} m_i = m)$$

$$\mathrm{Res}_{U_M}^{GL_m\mathbb{F}q}(\rho): U_M \longrightarrow GL_e(\mathbb{C})$$

has no non-zero fixed vectors. In this case U_M denotes the sub-group of $GL_m\mathbb{F}_q$ consisting of unipotent matrices of the form

$$\begin{bmatrix} I_{m_2} & A_1 & A_2\ldots.. & \\ 0 & I_{m_2} & B_1\ldots & \\ 0 & 0 & & \\ 0 & 0 & I_{m_r} & \end{bmatrix}$$

4.2.11 Theorem The Weil representation, $r(\theta)$, is cuspidal.

Proof: The only non-trivial partition of 2 is $M = (1,1)$ in which case U_m is the unitriangular group $U = \{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\}$ of 4.1.2.

From 4.2.1 (ii), $f \in H$ is fixed by U if and only if

$$\omega_{\mathbf{F}_q}(u\,N(x)) = 1 \quad \text{for all } x \in \mathbf{F}_q^{*2}$$

for which $f(x) \neq 0$ and for all $u \in \mathbf{F}_q$. But $\omega_{\mathbf{F}_q}$ is non-trivial, and so the only possible way that this may occur is for $f(x)$ to be identically zero.

4.3.1 *Character values of $r(\theta)$*

We proceed as follows,

Definition: $\begin{cases} \chi_\theta : \{\text{conjugacy classes of } GL_2\mathbf{F}_q\} \longrightarrow \mathbb{C} & \text{where} \\ \chi_\theta(g) = \text{Trace}\,(r(\theta)(g)) \end{cases}$

In the following computations, it is convenient to choose a basis for H, the representation space of 4.1.5.

To do so, let $v_1, v_2, \dots v_{q-1} \in \mathbf{F}_q^{*2}$ be a set such that

$$\left\{ N_{\mathbf{F}_q{}^2/\mathbf{F}_q}(v_j);\ 1 \le j \le q-1 \right\} = \mathbf{F}_q^{*}.$$

Hence the $\{v_i\}$ are a complete set of coset representations for $\mathbf{F}_q^{*2}/\text{Ker}\,(N_{\mathbf{F}_q{}^2/\mathbf{F}_q})$.

Set $f_i \in H$ ($i = 1,2,\dots q-1$) equal to the function in H which satisfies

4.3.2 $$f_i(v_j) = \left\{ \begin{matrix} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{matrix} \right\}$$

The values of f_i on $\mathbf{F}_q^{*2}$ are uniquely determined by 4.1.5, so that $\{f_1, f_2, \dots f_{q-1}\}$ is a basis for H.

If $g \in GL_2\mathbf{F}_q$ and $r(\theta)(g)f_i = \sum_{j=1}^{q-1} a_{ij} f_i$

then we have:

4.3.3 $$\begin{cases} a_{ij} = (r(\theta)(g)\,f_i)(v_j) \quad \text{and} \\ \chi_\theta(g) = \sum_{i=1}^{q-1} a_{ij} \end{cases}$$

4.3.4 *Conjugacy Classes in $GL_2\mathbf{F}_q$*

We continue to follow Snaith's unpublished manuscript and [21, Pp .60-131]. The conjugacy classes of $GL_2\mathbf{F}_q$, are based on a determination of the minimal polynomial of $g \in GL_2\mathbf{F}_q$. Since the minimal polynomial must have degree 1 or 2, the representatives of each class are tabulated below, together with their minimal polynomials:

4.3.5

	minimal polynomial	conjugacy class representative
(I)	$(t-\alpha)(t-\beta)$ $\alpha,\beta \in \mathbf{F}_q^*$ are distinct	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$
(II)	$(t-\alpha)^2$ $\alpha \in \mathbf{F}_q^*$	$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$
(III)	$(t-\alpha)$ $\alpha \in \mathbf{F}_q^*$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$
(IV)	$t^2-(x+\sigma(x))t+x\sigma(x)$ $x \in \mathbf{F}_q^{*2}, \sigma(x)=x^q$ and $x \neq \sigma(x)$	$\begin{pmatrix} 0 & -x\sigma(x) \\ 1 & x+\sigma(x) \end{pmatrix}$

Conjugacy classes in GL_2F

The calculations which follow will establish the character values for $r(\theta)$ in the above four cases.

4.3.6 Lemma Let $g = \begin{pmatrix} 0 & -x\sigma(x) \\ 1 & x+\sigma(x) \end{pmatrix} \in GL_2\mathbf{F}_q$ denote the conjugacy class representative of type IV in 4.3.5. Then, for $f \in H$,

$$r(\theta)(g)f)(z)=(-q)^{-1}\theta(-x)\sum_{y\in\mathbf{F}_q^{*2}} f(y)\omega_{\mathbf{F}_q}\{(x+\sigma(x))N(y)+y\sigma(xz)+xz\sigma(y)\}$$

where $z, x \in \mathbf{F}_q^{*2}$ and $N = N_{\mathbf{F}_q{}^2/\mathbf{F}_q}$ is the norm map.

Proof Let $t = x + \sigma(x)$, $d = x\sigma(x) = N(x)$ and let $a \in \mathbf{F}_q^{*2}$

such that $N(a) = -1$. Let $r(\theta)$ be represented by r. First, note that for $x \neq \sigma(x)$, from 4.1.3(b)

4.3.7 $$\begin{pmatrix} 0 & -N(x) \\ 1 & x+\sigma(x) \end{pmatrix} = \begin{pmatrix} -x\sigma(x) & 0 \\ 0 & -1 \end{pmatrix} w \begin{pmatrix} 1 & x+\sigma(x) \\ 0 & 1 \end{pmatrix}$$

where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Then by 4.2.1, and 4.2.4 and 4.3.7 we have

$$(r(g)f)(z) = [\, r\begin{pmatrix} -d & 0 \\ 0 & -1 \end{pmatrix} r(w)\, r\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} f]\,(z)$$

$$= r\begin{pmatrix} -d & 0 \\ 0 & -1 \end{pmatrix} r(w)\, \{ s \longrightarrow \omega_{\mathbf{F}_q}(tN(s))\, f(s)\}\,(z)$$

$$= (-q^{-1})\theta(ax)\,\theta(a) \left\{ \sum_{y \in \mathbf{F}_q^{*2}} f(y)\, \omega_{\mathbf{F}_q}[tN(y) + y\sigma(axz\sigma(a)^{-1})] \right\}$$

$$= (-q^{-1})\theta(x) \left\{ \sum_{y \in \mathbf{F}_q^{*2}} f(ya^{-2})\omega_{\mathbf{F}_q}[tN(ya^{-2}) - ya^{-2}\sigma(xz) - \sigma(ya^{-2})xz] \right\}$$

since $a\sigma(a) = -1$ and $N(a^{-2}) = 1$

$$= (-q^{-1})\theta(x)\theta(-1) \left\{ \sum_{v \in \mathbf{F}_q^{*2}} f(v)[tN(v) + v\sigma(xz) + xz\sigma(v)] \right\}$$

by setting $v = -ya^{-2}$. This is the required result.

4.3.8 Lemma With the notation of 4.3.1 and 4.3.6,

$$\chi_\theta(g) = \text{Trace}\,(r(\theta)\,[\begin{pmatrix} 0 & -d \\ 1 & t \end{pmatrix}]$$
$$= -\{\theta(x) + \theta(\sigma(x))\}.$$

Proof With the notation of 4.3.6, and from 4.3.3

$$\chi_\theta(g) = \sum_{i=1}^{q-1} (r(\theta)(g)\, f_i)\,(v_i)$$

$$=(-q^{-1})\theta(-x)\sum_{i=1}^{q-1}\sum_{y\in\mathbf{F}_q^{*2}} f_i(y)\,\omega_{\mathbf{F}_q}\{tN(y)+y\sigma(x(v_i))+x(v_i)(\sigma(y)\}$$

$$=(-q^{-1})\theta(-x)\sum_{i=1}^{q-1}\sum_{u\in Ker(N)} f_i(uv_i)\omega_{\mathbf{F}_q}\{tN(v_i)+u\sigma(x)N(v_i)+\sigma(u)(x)N(v_i)\}$$

by setting $y = uv_i$, using 4.3.1, 4.3.2 and since $f_i(uv_j) = 0$ if $j\neq i$ and $N(u) = 1$,

$$=(-q^{-1})\theta(-x)\sum_{i=1}^{q-1}\sum_{u\in Ker(N)} \theta(u)^{-1}\omega_{\mathbf{F}_q}\{N(v_i)[t+u\sigma(x)+x\sigma(u)]\}$$

If the expression $t + u\sigma(x) + x\sigma(u)$ is non-zero then

$$\sum_{i=1}^{q-1}\omega_{\mathbf{F}_q}\{N(v_i)[t+u\sigma(x)+x\sigma(u)]\} = -1$$

from 4.3.1 and 4.1.6.

This will be true for all values of $u\in Ker(N)$ with the exceptions $u = -1$ and $u = -x\sigma(x)^{-1}$, when

$$t + u\sigma(x) + \sigma(u)x = x + \sigma(x) + u\sigma(x) + \sigma(u)x = 0.$$

Hence, $-(-q^{-1})\theta(-x)\sum_{i=1}^{q-1}\sum_{u\in Ker(N)} \theta(u)^{-1}\omega_{\mathbf{F}_q}[t+u\sigma(x)+x\sigma(x)]$

$$= -(-q)^{-1}\theta(-x)\left\{(-1)\left[\sum_{\substack{u\in Ker(N)\\ u\neq -1,\,-x\sigma(x)^{-1}}}\theta(u)^{-1}\right]+(q-1)\left[\theta(-1)^{-1}+\theta(-x\sigma(x)^{-1})^{-1}\right]\right\}$$

$$= -(q^{-1})\theta(-x)\{\theta(-1)^{-1}+\theta(-x\sigma(x)^{-1})^{-1}\,[1+q-1]\}\quad\text{by 4.1.6.}$$

Hence $\quad\chi_\theta(g) = q(-q^{-1})\theta(-x)\{\theta(-1)+\theta(\sigma(x)x^{-1}\}$
$\qquad\qquad = -\{\theta(x) + \theta(\sigma(x))\}.$

4.3.9 Lemma

If $g = \begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix} \in GL_2\mathbf{F}_q$ as in 4.3.5, cases I or III, then

$$\chi_\theta\ (g) = \begin{cases} (q-1)\theta(\alpha) & \text{if } \alpha = \delta \\ 0 & \text{if } \alpha \neq \delta \end{cases}$$

Proof:

Let $a,b \in \mathbf{F}_q^{*2}$ be such that $N(a) = \alpha$, $N(b) = \delta$, where N is the norm $N_{\mathbf{F}_q{}^2/\mathbf{F}_q}$.

By 4.2.1 and 4.3.3 we have

$$\chi_\theta(g) = \sum_{i=1}^{q-1} (r(\theta)\ (g)\ f_i\ (v_i)$$

$$= \sum_{i=1}^{q-1} \theta(a)\theta(b)\ f_i(av_i\ \sigma(b)^{-1})$$

When $\alpha \neq \delta$ then $N(a\sigma(b^{-1}) \neq 1$ and $f_i(av_i\sigma(b)^{-1}) = 0$ for all i

But if $\alpha = \delta$ (and $a = b$) then we have

$$\chi_\theta(g) = \sum_{i=1}^{q-1} \theta(a)\theta(b)\theta(a^{-1})\theta(\sigma(b))$$

$$= (q-1)\theta(b\sigma(b)) = (q-1)\theta(\delta) \text{ as required.}$$

4.3.10 Lemma

If $g = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \in GL_2\mathbf{F}_q$ as in 4.3.5, Case II,

then $\chi_\theta(g) = -\theta(\alpha)$.

Proof Let $N_{\mathbf{F}_q{}^2/\mathbf{F}_q}\ (a) = N(a) = \alpha$. By 4.2.1 and 4.3.3

$$\chi_\theta(g) = \chi_\theta\ [(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix})\ (\begin{smallmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{smallmatrix})]$$

$$= \sum_{i=1}^{q-1} \theta(a^2)\ [r(\theta)\ (\begin{smallmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{smallmatrix})\ f_i]\ [av_i\sigma(a)^{-1}]$$

$$= \sum_{i=1}^{q-1} \theta(a^2)\ \omega_{\mathbf{F}_q}[a^{-1}\ N\ (v_i a\sigma(a)^{-1}]\ f_i]\ [av_i\sigma(a)^{-1}]$$

$$= \sum_{i=1}^{q-1} \theta(\alpha)\ \omega_{\mathbf{F}_q}[a^{-1}\ N\ (v_i)] \quad \text{since } N(a\sigma(a)^{-1}) = 1$$

Hence $\chi_\theta(g) = -\theta(\alpha)$ since $\sum_{i=1}^{q-1} \omega_{\mathbf{F}_q}[a^{-1} N(v_i)] = 1$ by 4.1.6.

Using the results of 4.3.8 - 4.3.10 gives the following table of character values for the Weil Representation $r(\theta)$.

4.3.11

CONJUGACY CLASS REPRESENTATIVE	NUMBERS IN CONJUGACY CLASS	CHARACTER VALUE $\chi_\theta(X)$
(I) $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$	$q(q+1)$	0
(II) $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$	q^2-1	$-\theta(\alpha)$
(III) $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	1	$(q-1)\theta(\alpha)$
(IV) $\begin{pmatrix} 0 & -x\sigma(x) \\ 1 & x+\sigma(x) \end{pmatrix}$	$q^2 - q$	$-(\theta(x)+\theta(\sigma(x)))$ where σ is the Frobenius map, $\sigma(x)=x^q$

The numbers of elements in each class come from [7], [22], or [36].

4.3.12 Theorem

The formulae of 4.2.1 characterize a unique, well defined $(q-1)$ dimensional irreducible representation, $r(\theta)$ of $GL_2\mathbf{F}_q$.

Proof From 4.1.5, 4.2.2 and 4.2.9 we know that $r(\theta)$ is a unique, well defined $(q-1)$ dimensional representation. It remains to show that $r(\theta)$ is irreducible. This is equivalent to demonstrating that the Schur inner product $\langle r(\theta), r(\theta)\rangle$ is equal to 1. By definition of the Schur inner product we have

$$\langle r(\theta), r(\theta)\rangle = \frac{1}{|GL_2\mathbf{F}_q|} \sum_{g\in GL_2\mathbf{F}_q} \chi_\theta(g)\, \overline{\chi_\theta(g)}$$

$$= \sum_{g \in GL_2\mathbf{F}_q} | \chi_\theta (g) |^2$$

$$= 0+(q^2-1) \sum_{\alpha \in \mathbf{F}_q^*} (-\theta(\alpha))(-\theta(\alpha))) + \sum_{\alpha \in \mathbf{F}_q^{*2}} (q-1)\theta(\alpha) \ . \ (q-1)\theta(\alpha)$$

$$+(q^2-q) \sum_{\substack{\text{paire } (x,\sigma(x)) \\ \text{where } x \neq \sigma(x)}} -\{\theta(x) + \theta(\sigma(x)\}\{-\overline{(\theta(x)+\theta(\sigma(x))}\}$$

We obtain this result by considering the four cases of 4.3.11.

First, $(\theta(x) + \theta(\sigma(x)) \ (\overline{\theta(x) + \theta \ (\sigma(x)})$

$= (\theta(x) + \theta(x^q)) \ (\theta(x^{-1}) + \theta(x^{-q}) = 2 + \theta(x^{q-1}) + \theta(x^{1-q})$

Then by substituting we have

$\langle r(\theta), r(\theta)\rangle \ |GL_2\mathbf{F}_q|$

$= 0 +(q^2-1)(q-1) + (q- 1)^2(q-1)$

$$+(q^2 - q) \sum_{\substack{\text{paire } (x,\sigma(x)) \\ x \neq \sigma(x)}} \{2 + \theta(x^{q-1}) + \theta(x^{1-q})\}$$

$= \ (q^2-1)(q-1)+(q-1)^3+2(q^2-q)^2+(q^2-q)[(q+1)(1-q)]$

$= (q-1)^2 \ [(q+1) + (q-1) + 2q^2 - q(q+1)]$

$= (q-1)^2 \ [q^2+q] = (q^2-1) \ (q^2-q)$

and since $|GL_2\mathbf{F}_q| = (q^2-1) \ (q^2-q)$, this implies that $\langle r(\theta), r(\theta)\rangle = 1$ ie: $r(\theta)$ is irreducible.

4.3.13 We will need the following characterisation for $r(\theta)$ in Chapter 5, for Explicit Brauer Induction in $GL_2\mathbf{F}_q$.

Consider the abelian subgroup H of B given by

$$\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} | \alpha \in \mathbf{F}_q^*, \beta \in \mathbf{F}_q\}.$$

There is an isomorphism:

$\gamma : H \xrightarrow{\cong} \mathbf{F}_q^* \times \mathbf{F}_q$ given by

4.3.14 $\quad \gamma\{\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix}\} = (\alpha, \beta\alpha^{-1})$.

We define the additive character

$\omega = \omega_{\mathbf{F}_q} : \mathbf{F}_q \longrightarrow \mathbb{C}^*$ by the formula

4.3.15 $\quad \omega(x) = \exp(2\pi i(\mathrm{Trace}_{\mathbf{F}_q/\mathbf{F}_p}(x))/p)$. Where $q = p^s$ for some s and define a character:

$\theta : \mathbf{F}_q^{*2} \longrightarrow \mathbb{C}^*$ such that for $y \in \mathbf{F}_q^{*2}$,

$\theta(y) \neq \theta(\sigma(y))$ where σ is the Frobenius map.

We denote a one dimensional representation of H, $\theta \otimes \omega$ by the formula,

4.3.16 $\quad \theta \otimes \omega = (\theta \otimes \omega)\gamma : H \longrightarrow \mathbb{C}^*$ given by

$$(\theta \otimes \omega)\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix} = \theta(\alpha)\omega(\beta\alpha^{-1})$$

By induction, we obtain a(q-1) dimensional representation of B, ie: $\mathrm{Ind}_H^B(\theta \otimes \omega)$

Finally we note that

4.3.17 Theorem: $\mathrm{Res}_B^G(r(\theta)) = \mathrm{Ind}_H^B(\theta \otimes \omega)$

Proof: Let $g = \begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix} \in B$, then check the character value of $\mathrm{Ind}_H^B(\theta \otimes \omega)$ at g, and compare the result with the corresponding value of $r(\theta)$ from 4.3.11.

Since H B,

$$\mathrm{Ind}_H^B(\theta \otimes \omega)(g) = \sum_{\substack{u=\begin{pmatrix}z & 0\\ 0 & 1\end{pmatrix}\\ z^{-1}(g)z\in H}} (\theta \otimes \omega)(u^{-1}gu)$$ since u gives a set

of coset representations for $B/_H$.

Since $z \in \mathbf{F}_q^*$, this expression is zero with the exception of $\alpha = \delta$ in g when it becomes

$$\mathrm{Ind}_H^B(\theta \otimes \omega)(g) = \sum_{z\in\mathbf{F}_q^*} (\theta \otimes \omega)\begin{pmatrix}\alpha z^{-1} & \beta\\ 0 & \alpha\end{pmatrix}$$

$$= \sum_{z\in\mathbf{F}_q^*} (\theta(\alpha)\omega(z^{-1}\beta)$$

By 4.1.6, $\sum_{z \in \mathbf{F}_q^*} \omega(z^{-1}\beta) = \begin{cases} -1 \text{ if } \beta \neq 0 \\ (q-1) \text{ if } \beta = 0 \end{cases}$

and these results concur with the character values for types II and III in 4.3.1 as required.

4.4 *ADDITIONAL REPRESENTATIONS OF $GL_2\mathbf{F}_q$*

To begin, we recall some notation and theory from Shintani [27,Pp.396-414].

For a pair (χ_1,χ_2) of characters of $\mathbf{F}_q^*$, denote the representation of $GL_2\mathbf{F}_q$ induced from the one dimensional character

$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \longrightarrow \chi_1(\alpha)\chi_2(\delta)$ of B (the Borel subgroup) by $R_{(\chi_1,\chi_2)}$

The representation space $V_{(\chi_1,\chi_2)}$ is the space of complex valued functions on $GL_2\mathbf{F}_q$ satisfying

$$f\left\{\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} x\right\} = \chi_1(\alpha)\chi_2(\delta)f(x) \quad \ldots \forall \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in B.$$

The representation $R_{(\chi_1,\chi_2)}$ is given by

$[R_{(\chi_1,\chi_2)}(g)f]x = f(xg)$ and we have the following facts:

4.4.1 $\quad R_{(\chi_1,\chi_2)}$ is irreducible $< \quad >$ $\chi_1 \neq \chi_2$

4.4.2 $\quad$ Two representations $R_{(\chi_1,\chi_2)}$, $R_{(\chi_1',\chi_2')}$ are equivalent

$< \quad > (\chi_1,\chi_2) = (\chi_1',\chi_2')$ <u>OR</u> $(\chi_1,\chi_2) = (\chi_2',\chi_1')$

4.4.3 $\quad$ Every irreducible representation of $GL_2\mathbf{F}_q$ is equivalent to some representation of $GL_2\mathbf{F}_q$ given by one of:

(a) L_χ: $L_\chi(g) = \chi(\det(g))$, where $\chi : \mathbf{F}_q^* \longrightarrow \mathbb{C}^*$ is of dimension one.

(b) Some subrepresentation S_χ of $R_{(\chi,\chi)}$ given on a representation space

$$V^{1}_{(\chi,\chi)} = \left\{ f \in V_{(\chi,\chi)},\ \sum_{x\in GL_2\mathbf{F}_q} \chi^{-1}(\det x) f(x) = 0 \right\}$$

and of dimension q

(c) $R_{(\chi_1,\chi_2)}$ as in 4.4.1 of dimension (q+1)

and (d) Cuspidal representations, of dimension (q-1)

Further from [27,§4.3⁰] , if m is the degree of $\mathbf{F}^{*}_{q^m}$, a finite extension of a finite field $\mathbf{F}^{*}_q$, then for <u>m odd</u> $R_{(\chi_1,\chi_2)}$ is Frobenius invariant <u>if and only if</u> both χ_1, χ_2 are Frobenius invariant.

For m even, $R_{(\chi_1,\chi_2)}$ is Frobenius invariant if

$(\chi_1^{\sigma},\chi_2^{\sigma}) = (\chi_2,\chi_1)$ where $[\chi_1^{\sigma}=\chi_1 \circ \sigma]$, σ is the Frobenius map, and where $\chi_1 \neq \chi_2$ and χ_1,χ_2 are <u>not</u> Frobenius invariant.

We have calculated the character values for the cuspidal representation (the $r(\theta)$ case). We will now calculate the character values for $R_{(\chi_1,\chi_2)}$, L_χ, and S_χ.

As in 4.3.11, we look at the 4 conjugacy classes of $GL_2\mathbf{F}_q$, given by the analysis of the minimal polynomials. We then apply the standard formula,[27,P.396]:

4.4.4

$$\mathrm{Ind}_B^{GL_2\mathbf{F}_q}(\chi_1 \otimes \chi_2)(g) = \frac{1}{|B|} \sum_{\substack{z\in GL_2\mathbf{F}_q \\ z^{-1}gz\in B}} (\chi_1 \otimes \chi_2)(z^{-1}gz) \text{ for } g \in GL_2\mathbf{F}_q$$

We begin with the definition $L_\chi\colon g \longrightarrow \chi(\det g)$.

To examine S_χ we observe that

4.4.5 $\quad R_{(\chi,\chi)} = \mathrm{Ind}_B^{GL_2\mathbf{F}_q}(\chi \otimes \chi)\colon B \longrightarrow \mathbb{C}^*$

where $\chi: \mathbf{F}_q^* \longrightarrow \mathbb{C}^*$

also $X = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \longrightarrow \chi(a)\ \chi(c) = \chi(\det(X))$

Choose $g_1, \ldots, g_s$ as coset representatives for $GL_2\mathbf{F}_{q/B}$ and set

$$\left\{ z = \sum g_i \otimes f(g_i);\ f(g_i) \in \mathbb{C} \right\}$$

We recall [27,Pp.396-7] that the representation space for $R_{(\chi_1,\chi_2)}$.

is given by $V_{(\chi_1,\chi_2)} = \left\{ \begin{array}{l} f\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} x\} = \chi_1(a)\chi_2(c) f(x) \\ \forall \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B \text{ and } x \in GL_2\mathbf{F}_q) \end{array} \right\}$

Consider the subspace of $V_{(\chi,\chi)}$, given by

4.4.6 $$V'_{(\chi,\chi)} = \left\{ \phi \in V_{(\chi,\chi)};\ \sum_{x \in GL_2\mathbf{F}_q} (\chi(\det x))^{-1} f(x) = 0 \right\}$$

where $f \longleftrightarrow \sum g \underset{B}{\otimes} f(g)$

If we let:

4.4.7 $$\psi:\ \mathbb{C}[GL_2\mathbf{F}_q] \underset{\mathbb{C}B}{\otimes} \mathbb{C}_{(\chi,\chi)} \longrightarrow \mathbb{C}$$

be a $\mathbb{C}$ linear well defined map where the action on $\mathbb{C}$ is given by

4.4.8
$$\begin{array}{ccc} g \otimes f(g) & \longrightarrow & \chi\det(g)) f(g) \\ \Big\downarrow h & & \Big\downarrow \\ hg \otimes f(g) & \longrightarrow & \chi(\det(hg)) f(g) \\ & & \shortparallel \\ & & \chi(\det(h))(\chi(\det g) f(g) \end{array}$$

and

$$g\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \otimes \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} f(g) \longrightarrow \begin{array}{l} \chi((\det g)ac)[\chi(ac)^{-1} f(g)] \\ \qquad \shortparallel \\ \chi(\det(g))\ f(g) \end{array}$$

Then $V'(\chi,\chi)$ corresponds to $\ker(\psi)$ where

$\ker(\psi) = S_\chi \subset R_{(\chi,\chi)} = S_\chi \oplus \mathbb{C}_{L_\chi}$

and where $g \in GL_2\mathbf{F}_q$ acts as $\chi(\det(g))$

4.4.9 so that $R_{(\chi,\chi)} = L_\chi \oplus S_\chi$.

We now calculate the values of characters for $R_{(\chi_1,\chi_2)}$ and use these values to build a table for $R_{(\chi,\chi)}$, S_χ and L_χ.

For the first conjugacy class representative

$$g = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},\ \alpha \neq \beta \in \mathbf{F}_q^*$$

$$\text{Then } \mathrm{Ind}_B^{GL_2\mathbf{F}_q} (\chi_1 \otimes \chi_2) = \frac{1}{|B|} \sum_{\substack{z \in GL_2\mathbf{F}_q \\ z^{-1}gz \in B}} (\chi_1 \otimes \chi_2)(z^{-1}gz)$$

Let $z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$z^{-1}gz = \frac{1}{\det z} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, d \in \mathbf{F}_q^* \text{ and } b, c \in \mathbf{F}_q$$

$$= \frac{1}{\det z} \begin{pmatrix} d\alpha & -b\beta \\ -c\alpha & a\beta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \frac{1}{\det z} \begin{pmatrix} ad\alpha - bc\beta & bd\alpha - bd\beta \\ -ac\alpha + ac\beta & -bc\alpha + ad\beta \end{pmatrix}.$$ So if $z^{-1}gz \in B$

then $ac\alpha = ac\beta \implies a$ or $c = o$

4.4.10 (i) If $c = 0$ then $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)(z^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} z = \chi_1(\alpha)\chi_2(\beta)$

and then since $z \in B$, we have

$$\frac{1}{|B|} \sum_{z \in B} \chi_1 \otimes \chi_2\ (z^{-1})\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} z = \chi_1(\alpha)\ \chi_2(\beta)$$

(ii) If $a = 0$ then we can consider

$$z = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z'$$

$$\frac{1}{|B|} \sum_{\substack{z=\binom{0\ b}{c\ a} \\ z^{-1}gz\in B}} \chi_1 \otimes \chi_2 \, (z^{-1}gz) = \frac{1}{|B|} \sum_{z'\in B} (\chi_1 \otimes \chi_2)[(z')^{-1}\begin{pmatrix}\beta & 0\\ 0 & \alpha\end{pmatrix} z']$$

$$= \chi_1(\beta)\chi_2(\alpha)$$

Thus if $\alpha \neq \beta$

4.4.11 $$R_{(\chi_1,\chi_2)} \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} = \chi_1(\alpha)\chi_2(\beta) + \chi_1(\beta)\chi_2(\alpha).$$

Since $R_{(\chi,\chi)} = S_\chi \oplus L_\chi$ we have

$$S_\chi\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} = R_{(\chi,\chi)}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} - L_\chi\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$$

where $\alpha \neq \beta$

4.4.12 $$S_\chi\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} = 2\chi(\alpha)\chi(\beta) - \chi(\alpha)\chi(\beta)$$
$$=\chi(\alpha)\chi(\beta)$$

We consider the second conjugacy class representative given by $g = \begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$ calculating $R_{(\chi_1,\chi_2)}$ as before.

We find that $z^{-1}gz \in B \Rightarrow c = 0$

so that the expression for $\mathrm{Ind}_B^{GL_2\mathbf{F}_q}(\chi_1 \otimes \chi_2)$ becomes $\chi_1(\alpha)\chi_2(\alpha)$

Thus $R_{(\chi,\chi)}\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix} = \chi(\alpha)^2$

which implies $S_\chi\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix} = R_{(\chi,\chi)}\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix} - L_\chi\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$

4.4.13 or $S_\chi(g) = \chi(\alpha)^2 - \chi(\alpha)^2 = 0$

Given the third conjugacy class representative $g = \begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$ we have, since $g \in Z(GL_2\mathbf{F}_q)$, a value for every $z \in GL_2\mathbf{F}_q$.

ie: $$R_{(\chi_1,\chi_2)}\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix} = \frac{|GL_2\mathbf{F}_q|}{|B|}\chi_1(\alpha)\chi_2(\alpha)$$
$$= (q+1)\,\chi_1(\alpha)\chi_2(\alpha)$$

4.4.14 Therefore $S_\chi\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix} = (q+1)\chi(\alpha)^2 - \chi(\alpha)^2$

Finally we consider the fourth conjugacy class representative given by

$$g = \begin{pmatrix}0 & -N(x)\\ 1 & Tr(x)\end{pmatrix},\ x \in \mathbf{F}_q^{*2}$$

where $N(x) = x\sigma(x)$ and where $\{N_{\mathbf{F}_q^2/\mathbf{F}_q}(x)\} \in \mathbf{F}_q^*$,

$Tr(x) = x + \sigma(x)$, and σ is the Frobenius map.

Now for $z = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $z^{-1}(g)\ z \in B \Rightarrow a = c = 0$

so that $R_{(\chi_1,\chi_2)}(g) = 0$

Alternately, the eigenvalues of g are in $(\mathbf{F}_q^2 - \mathbf{F}_q)$

(ie: examine $t^2 - t(Tr(x)) + N(x) = 0$)

while those of $z^{-1}gz \in B$ are in $\mathbf{F}_q^2$.

Then, since $L_\chi(g) = \chi(\det g)$

$= \chi(N_{\mathbf{F}_q^2/\mathbf{F}_q}(x)) \neq 0$

4.4.15 $S_\chi(g) = 0 - \chi(N_{\mathbf{F}_q^2/\mathbf{F}_q}(x))$

We note that the list of irreducible characters of $GL_2\mathbf{F}_q$ given by L_χ, S_χ, $R_{(\chi_1,\chi_2)}$ and $r(\theta)$ is complete:

We have the following table:

4.4.16

TYPE OF IRREDUCIBLE REPRESENTATION	QUANTITY	DIMENSION
L_χ	$(q-1)$	1
S_χ	$(q-1)$	q
$R_{(\chi_1,\chi_2)}$	$(q-1)(q-2)/2$	$(q+1)$
$r(\theta)$	$(q^2-q)/2$	$(q-1)$

Then by [20, P.648], we have the following summation:

$$|GL_2\mathbb{F}_q| = (q-1)^2(q^2+q)$$

$$= (q-1)(1^2) + (q-1)q^2 + (q-1)(q-2)/2\ (q+1)^2$$

$$+ (q^2-q)/2\ (q-1)^2$$

We can summarize the results of 4.4 in the following table:

4.4.17 CHARACTER VALUES FOR THE FOUR CONJUGACY CLASSES OF $GL_2\mathbb{F}_q$ FOR $r(\theta), R_{(\chi_1,\chi_2)}, L_\chi, S_\chi$

REPN.	CLASS 1 $g = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$ $\alpha \neq \beta$	CLASS 2 $g = \begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$	CLASS 3 $g = \begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$	CLASS 4 $g = \begin{pmatrix}0 & -N(x)\\ 1 & T(x)\end{pmatrix}$ $x \in \mathbb{F}_q^{*2}$
$r(\theta)$	0	$-\theta(\alpha)$	$(q-1)\theta(\alpha)$	$-\{\theta(x)+\theta(\sigma(x))\}$
$R_{(\chi_1,\chi_2)}$	$\chi_1(\alpha)\chi_2(\beta)$ $+\chi_1(\beta)\chi_2(\alpha)$	$\chi_1(\alpha)\chi_2(\alpha)$	$(q+1)\chi_1(\alpha)\chi_2(\alpha$	0
S_χ	$\chi(\alpha)\chi(\beta)$	0	$q\chi(\alpha)^2$	$-\chi(N_{\mathbb{F}_q{}^2/\mathbb{F}_q}(x))$
L_χ	$\chi(\alpha)\chi(\beta)$	$\chi(\alpha)^2$	$\chi(\alpha)^2$	$\chi(N_{\mathbb{F}_q{}^2/\mathbb{F}_q}(x))$

CHAPTER 5

$a_G(\rho)$ WHERE ρ IS AN IRREDUCIBLE REPRESENTATION OF $G = GL_2\mathbf{F}_q$

5.1 MAXIMAL PAIRS IN $\mathcal{M}_G$

We recall from 3.1, [1,§1.26] and [1,§4(v)] that if $\mathcal{M}_G$ ($G = GL_2\mathbf{F}_q$) represents the poset of pairs (K,ϕ) with $K \leq G$ and $\phi : K \longrightarrow \mathbb{C}^*$, then from,[1,§ 4(v)], if $\chi \in R(G)$ and (K,ϕ) is maximal in $\mathcal{M}_G$, the coefficient of $(K,\phi)^G$ in $a_G(\chi)$ is given by:

$$\langle\phi,\chi|_K\rangle \;.\; [N_G(K,\ \phi):KH]^{-1}$$

ie: the multiplicity of (K,ϕ) in $a_G(\chi)$

5.1.1 and $N_G(K,\phi) = \{X \in G \mid (X\ K\ X^{-1},(X^{-1})^*(\phi)) = (K,\phi)\}$

We will evaluate some of the maximal pairs $(H,\phi)^G$ for $a_G(\rho)$, where ρ runs through the irreducible representations of $G = GL_2\mathbf{F}_q$.

First we consider [27, P410 §4], the one dimensional character L_χ where $\chi:\mathbf{F}_q^* \longrightarrow \mathbb{C}^*$. Since any one dimensional character of G is equivalent to some $L_\chi = \chi.(\det -)$, we then have a maximal pair $(G,\chi.\det)$ ie: TYPE A.

Second, we consider (H,ϕ) where $\phi:H \longrightarrow \mathbb{C}^*$.
From 4.3.14 we recall the isomorphism,

5.1.2 $\gamma : H = \{\begin{pmatrix}\alpha & \beta\\ \sigma & \alpha\end{pmatrix}\} \cong \mathbf{F}_q^* \times \mathbf{F}_q\}$ given by $\gamma\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix} = (\alpha,\beta\alpha^{-1})$.

We define a character $(\tau \otimes \psi_{\mathbf{F}_q})\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix} = (\tau(\alpha).\psi_{\mathbf{F}_q}(\beta\alpha^{-1}))$ where $\tau:\mathbf{F}_q^* \longrightarrow \mathbb{C}^*$ is nontrivial and $\psi_{\mathbf{F}_q}:\mathbf{F}_q \longrightarrow \mathbb{C}^*$ is any homomorphism.

We consider the pair $(H,\tau \otimes \psi)$. First we note that B, the Borel subgroup, is maximal in $G = GL_2\mathbf{F}_q$. In addition, for any proper subgroup K of G, where $H \le K$, then $K \subset B = N_G(H)$.
Further we have

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} x^{-1} & -x^{-1}yz^{-1} \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & x\beta z^{-1} \\ 0 & \alpha \end{pmatrix}$$

Hence conjugation in B multiplies the term β by xz^{-1}. Since $\psi(\beta)$ and $\psi(xz^{-1}\beta)$ are distinct except when $x = z$, then $N_G(H,\tau \otimes \psi) \le H$ hence the pair $(H,\tau \otimes \psi)$ is maximal with normaliser H. This is TYPE B.

Third, we consider $(B,\lambda_1 \otimes \lambda_2)$ in the sense of 4.4.1 and 4.4.4 where λ_1, λ_2 are distinct characters of $\mathbf{F}_q^*$.

From $\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} z & -x \\ -y & w \end{pmatrix}\frac{1}{D}$ where $D = \det\begin{pmatrix} w & x \\ y & z \end{pmatrix}$

$$= \begin{pmatrix} wa & wb+xc \\ ya & yb+zc \end{pmatrix}\begin{pmatrix} z & -x \\ -y & w \end{pmatrix}\frac{1}{D}$$

$$= \begin{pmatrix} [waz - wby - xcy] & [-wax+w^2b+wx] \\ [yaz - y^2b - yzc] & [-yxa+wyb+wzc] \end{pmatrix}\frac{1}{D}$$

we have $N_G(B) = B$

also $\begin{pmatrix} x & y \\ o & z \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} x^{-1} & -yx^{-1}z^{-1} \\ 0 & z^{-1} \end{pmatrix} = X\ A\ X^{-1}$

$$= \begin{pmatrix} x\alpha & x\beta+y\sigma \\ 0 & z\sigma \end{pmatrix}\begin{pmatrix} x^{-1} & -yx^{-1}z^{-1} \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & -\alpha y+x\beta z^{-1}+y\sigma z^{-1} \\ 0 & \sigma \end{pmatrix}$$

Thus from the definition $(\lambda_1 \otimes \lambda_2)\begin{pmatrix} \alpha & \beta \\ 0 & \sigma \end{pmatrix} = \lambda_1(\alpha)\lambda_2(\sigma)$

we have $(\lambda_1 \otimes \lambda_2)(X\ A\ X^{-1}) = \lambda_1(\alpha)\lambda_2(\sigma)$

and so $N_G(B,\lambda_1 \otimes \lambda_2) \le B$

ie: $(B,\lambda_1 \otimes \lambda_2)$ is maximal ie:TYPE C.

We now examine the fourth type of maximal pair $(\mathbf{F}_q^{*2},\rho)$, where ρ and $\sigma(\rho)$ are distinct. From an unpublished paper of Snaith's we have the following:

First consider a generator $x \in \mathbf{F}_q^{*2}$. With respect to the $\mathbf{F}_q$ - basis of $\mathbf{F}_q{}^2$ given by $\{1,x\}$, there is an embedding

$$i: \mathbf{F}_q^{*2} \longrightarrow GL_2\mathbf{F}_q = G \text{ where}$$

5.1.3 $i(x) = \begin{pmatrix} 0 & -x\sigma(x) \\ 1 & x+\sigma(x) \end{pmatrix}$ and where σ is the Frobenius map,

ie: $\sigma \in \mathrm{Gal}(\mathbf{F}_q{}^2/\mathbf{F}_q)$

This matrix, representing the fourth type of conjugacy class in $GL_2\mathbf{F}_q$, 4.4.17 generates a cyclic subgroup isomorphic to $\mathbf{F}_q^{*2}$. Also the Frobenius map, σ, corresponds to conjugation by the matrix

$$f = \begin{pmatrix} 1 & x+\sigma(x) \\ 0 & -1 \end{pmatrix}.$$

Multiplication by x, with respect to the $\mathbf{F}_q$ - basis $\{1,x\}$ is given by the matrix $i(x)$.

Hence $f^{-1}(i(x))f = \begin{pmatrix} 1 & (x+\sigma(x)) \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & (-x\sigma(x)) \\ 1 & (x+\sigma(x)) \end{pmatrix} \begin{pmatrix} 1 & (x+\sigma(x)) \\ 0 & -1 \end{pmatrix}$

5.1.4 $= \begin{pmatrix} (x+\sigma(x)) & [x+\sigma(x)]^2 + [-x\sigma(x)] \\ -1 & -x-\sigma(x) \end{pmatrix} \begin{pmatrix} 1 & (x+\sigma(x)) \\ 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} (x+\sigma(x)) & x\sigma(x) \\ -1 & 0 \end{pmatrix}$$

and $(1,x)\begin{pmatrix} x+\sigma(x) & x\sigma(x) \\ -1 & 0 \end{pmatrix} = (\sigma(x), x\sigma(x))$

ie: this matrix represents multiplication by $\sigma(x)$ with respect to the $\mathbf{F}_q$ - basis $(1,x)$.

5.1.5 If we set $J = \langle x,\sigma\rangle \subset G$,

then since $\det i(x) = N_{\mathbf{F}_q{}^2/\mathbf{F}_q}(x)$ we have

5.1.6 $\mathbf{F}_q^{*2} \cap SL_2\mathbf{F}_q = \langle x^{q-1}\rangle \cong \mathbf{Z}/(q+1)$

In addition, if q is odd then

$\det(x^{\frac{q-1}{2}}) = -1$ so that

5.1.7 $\sigma(x^{\frac{q-1}{2}}) = \sigma' \in SL_2\mathbf{F}_q$

Also $\sigma'(x^{q-1})(\sigma')^{-1} = \sigma(x)^{q-1} = x^{q^2-q} = (x^{q-1})^{-1}$

When q is even, set $\sigma' = \sigma$,
so that for all q,

5.1.8 $\quad K = J \cap SL_2\mathbf{F}_q$

$\quad = \langle x^{q-1}, \sigma' \rangle \cong D_{2(q+1)}$

5.1.9: PROPOSITION

(i) K is maximal in $SL_2\mathbf{F}_q$

(ii) J is maximal in $GL_2\mathbf{F}_q$

(iii) $J = N_G(\mathbf{F}_q^{*2})$, the normaliser of $\mathbf{F}_q^{*2}$ in $GL_2\mathbf{F}_q$

PROOF: Consider the canonical map,

$$\pi : SL_2\mathbf{F}_q \longrightarrow PSL_2\mathbf{F}_q$$

which is an isomorphism for q even and a (central) double covering for q odd. The image, $\pi(K)$, is isomorphic to a dihedral group ($D_{(q+1)}$ if q is odd and $D_{2(q+1)}$ if q is even).

Now recall the results of [7,P.286,§262] concerning the maximal subgroups of $PSL_2\mathbf{F}_q$. Dickson shows that any maximal subgroup of $PSL_2\mathbf{F}_q$ is conjugate to one of the following:-

(a) $\{\pi(\begin{smallmatrix}\alpha & u\\ 0 & \alpha^{-1}\end{smallmatrix})\}$,

(b) $\pi(K)$,

(c) $PSL_2\mathbf{F}_t$ for suitable $\mathbf{F}_t$ contained in $\mathbf{F}_q$,

or (d) exceptional subgroups of octahedral, tetrahedral or icosahedral type.

Hence $\pi(K)$ is maximal and it follows that K is maximal since $\mathrm{Ker}(\pi)$ lies within K.

If $K \subset H \subset GL_2\mathbf{F}_q$ we may consider the following commutative diagram:

$$\begin{tikzcd}
K \arrow[r, tail] \arrow[d] & J \arrow[r, two heads, "\det"] \arrow[d] & \mathbf{F}_q^* \arrow[d, "1"] \\
SL_2\mathbf{F}_q \cap H \arrow[r, tail] \arrow[d] & H \arrow[r, two heads, "\det"] \arrow[d] & \mathbf{F}_q^* \arrow[d, "1"] \\
SL_2\mathbf{F}_q \arrow[r, tail] & GL_2\mathbf{F}_q \arrow[r, "\det"] & \mathbf{F}_q^*
\end{tikzcd}$$

Since $\det(J) = \mathbf{F}_q^*$ we see that if $SL_2\mathbf{F}_q \cap H = SL_2\mathbf{F}_q$ then $H = GL_2\mathbf{F}_q$ while $SL_2\mathbf{F}_q \cap H = K$ implies that $J = H$.

It remains to prove part (iii).

5.1.10 We wish to show that there is an isomorphism

$$N_G(\mathbf{F}_q^{*2}) / \mathbf{F}_q^{*2} \cong \mathrm{Gal}\ (\mathbf{F}_q^2 / \mathbf{F}_q) = \langle\sigma\rangle.$$

Clearly we have a map from Gal $(\mathbf{F}_q{}^2/\mathbf{F}_q)$ to the normaliser of $\mathbf{F}_q^{*2}$. Conversely, if X normalises $\mathbf{F}_q^{*2}$ then X induces an $\mathbf{F}_q$ - endomorphism of the subalgebra of $M_2\mathbf{F}_q$ (the 2x2 matrices with entries in $\mathbf{F}_q$) which is generated by $\mathbf{F}_q^{*2}$. However this subalgebra is isomorphic to $\mathbf{F}_q{}^2$. Hence X determines an element of $\mathrm{Gal}(\mathbf{F}_q{}^2/\mathbf{F}_q)$. These two maps are inverse isomorphisms of the type required in 5.1.10.

Thus we have $N_G(\mathbf{F}_q^{*2}) = \langle\mathbf{F}_q^{*2},\sigma\rangle$ and we have noted that conjugation by σ induces the Frobenius map on $\mathbf{F}_q^{*2}$. Hence $f \in N_G(\mathbf{F}_q^{*2},\rho)$ and so $N_G(\mathbf{F}_q^{*2},\rho) = \mathbf{F}_q^{*2}$.

5.2 EXPLICIT BRAUER INDUCTION ON $GL_2\mathbf{F}_q$

We will now look at Explicit Brauer Induction on $GL_2\mathbf{F}_q = G$ using the formula $a_G(\rho)$ where ρ is an irreducible representation of $G = GL_2\mathbf{F}_q$ ie: $\rho:GL_2\mathbf{F}_q \longrightarrow GL(V)$.

5.2.1 Let $a_G(\rho) = \sum_r a_r(\overline{H,\tau \otimes \psi})^G$

$$+ \sum_s b_s(\overline{B,\lambda_1 \otimes \lambda_2})^G$$

$$+ \sum_t c_t(\overline{\mathbf{F}_q^{*2},\rho})^G$$

$$+ \sum_u d_u(\overline{G,\chi_u.\det})^G$$

$$+ \ . \ . \ .$$

where a_r, b_s, c_t and d_u represent the coefficients of the maximal pairs of the four TYPES A - D given in 5.1. The ellipsis denote the sum of all terms of other kinds. We will now calculate the values of these coefficients for each of the four irreducible representations, L_χ, S_χ, $R_{(\chi_1,\chi_2)}$ and $r(\theta)$.

5.2.2 $a_G(L_\chi) = (G,\chi.\ \det)^G$ by [1,P.25, §2.17].

5.2.3 First we will calculate the values of the coefficients for $R_{(\chi_1,\chi_2)}$ and then use $R_{(\chi,\chi)}$ to compute those for S_χ. Now from [1,§4(v)] we know that the coefficient of the maximal pairs of TYPES A - D is given by the Schur inner product, since the expression $[N_G(H,\phi):H]$ is 1 for TYPES A - D.

First we examine TYPE B:

the coefficient of $(\overline{H,\tau \otimes \psi})^G$ is given by $\langle \tau \otimes \psi, \mathrm{Res}_H^G(R_{(\chi_1,\chi_2)})\rangle$

$= \langle \tau \otimes \psi, \mathrm{Res}_H^G \mathrm{Ind}_B^G (\chi_1 \otimes \chi_2)\rangle$ from 4.4.4

5.2.4 $=\langle \tau \otimes \psi, \sum_{z \in H\backslash G/B} \mathrm{Ind}_{H \cap zBz^{-1}}^H (\chi_1 \otimes \chi_2)\rangle$ by [26,P.56].

From the Bruhat decomposition $G = B \cup B\,w\,U$ where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $U = \{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbf{F}_q\}$ we have two cases:

we find that B gives $z = 1$

and from B w U we obtain $H\backslash Bw$

and since $\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$ is a coset representative for B\H

we have $\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}0 & \alpha\\ -1 & 0\end{pmatrix} = x$.

Hence 5.2.4 becomes

$$\langle \tau \otimes \psi, \{(\chi_1 \otimes \chi_2) + \operatorname{Ind}{}^{H}_{H\cap xBx^{-1}} (\chi_1 \otimes \chi_2)\}\rangle$$

$$= \langle \tau \otimes \psi, \{(\chi_1 \otimes \chi_2) + \operatorname{Ind}{}^{H}_{Z(G)} (\chi_1 \otimes \chi_2)\}\rangle$$

where $Z(G) \cong \{\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix} | \alpha \in \mathbf{F}_q^*\} \cong \mathbf{F}_q^*$

and by Frobenius reprocity this becomes

5.2.5 $\langle \tau \otimes \psi, \{\chi_1 \chi_2 \otimes 1 + (\chi_1 \chi_2 \otimes \operatorname{Ind}{}^{H}_{\mathbf{F}_q^*} (1)\}\rangle$

$$= \begin{cases} 1 \text{ if } \tau = \chi_1 \chi_2 \\ 0 \text{ otherwise} \end{cases}$$

Next we consider the Schur inner product for TYPE C:

ie: $(B, \lambda_1 \otimes \lambda_2)$.

This is given by

5.2.6 $\langle \lambda_1 \otimes \lambda_2, \operatorname{Res}{}^{G}_{B} (\operatorname{Ind}{}^{G}_{B} ((\chi_1 \otimes \chi_2))\rangle$

$$= \langle \operatorname{Ind}{}^{G}_{B} (\lambda_1 \otimes \lambda_2), \operatorname{Ind}{}^{G}_{B}(\chi_1 \otimes \chi_2)\rangle$$

and from [27, P.411, §4.1], $\operatorname{Ind}{}^{G}_{B}(\chi_1 \otimes \chi_2)$ is irreducible if and only if $\chi_1 \neq \chi_2$ and so the value of 5.2.6 is given by:

$$\begin{cases} 1 \text{ if } [\lambda_1, \lambda_2] = [\chi_1, \chi_2] \text{ or } [\chi_2, \chi_1] \text{ from 4.4.1, 4.4.2} \\ 0 \text{ otherwise} \end{cases}$$

We now examine the Schur inner product for TYPE D; ie: $(\overline{\mathbf{F}_q^{*2}}, \rho)$.

As above this is given by:

5.2.7 $\langle \rho, \operatorname{Res}{}^{G}_{\mathbf{F}_q^{*2}} (\operatorname{Ind}{}^{G}_{B}(\chi_1 \otimes \chi_2))\rangle$

$$= \langle \rho, \sum_{z \in \mathbf{F}_q^{*2} \backslash G/B} \operatorname{Ind}{}^{\mathbf{F}_q^{*2}}_{\mathbf{F}_q^{*2} \cap zBz^{-1}} (\chi_1 \otimes \chi_2\rangle$$

Again, as in 5.2.4, we use the Bruhat decomposition and obtain $z = 1$ from B.
We examine $\mathbf{F}_q^{*2} \backslash B\, w\, U/B$, or $\mathbf{F}_q^{*2} \backslash B\, w$;
since $\mathbf{F}_q^{*2} \backslash B$ can be represented by $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$
then we have $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\beta & \alpha \\ -1 & 0 \end{pmatrix} = y$.
Then 5.2.7 becomes:

$$\langle \rho, \{ \mathrm{Ind}_{Z(G)}^{\mathbf{F}_q^{*2}} (\chi_1 \otimes \chi_2) + \mathrm{Ind}_{\mathbf{F}_q^{*2} \cap\, y\, B\, y^{-1}}^{\mathbf{F}_q^{*2}} (\chi_1 \otimes \chi_2) \rangle$$

To find $\mathbf{F}_q^{*2} \cap\, y\, B\, y^{-1}$ we recall that $\mathbf{F}_q^{*2}$ is a cyclic group generated by the matrix $\begin{pmatrix} 0 & -x\sigma(x) \\ 1 & x+\sigma(x) \end{pmatrix}$ where $\sigma \in \mathrm{Gal}\,({}^{\mathbf{F}_q}/\mathbf{F}_q)$, is the Frobenius map.
Then examination of $y\, B\, y^{-1}$ gives $\mathbf{F}_q^{*2} \cap\, y\, B\, y^{-1} \cong \phi$, the null set.
So 5.2.7 becomes $\langle \rho,\ \mathrm{Ind}_{\mathbf{F}_q^{*}}^{\mathbf{F}_q^{*2}} (\chi_1 \otimes \chi_2) \rangle$
and by Frobenius reciprocity this is $\langle \mathrm{Res}_{\mathbf{F}_q^{*}}^{\mathbf{F}_q^{*2}} (\rho),\ \chi_1\, \chi_2 \rangle$

which has value $\begin{cases} 1 \text{ for } \mathrm{Res}_{\mathbf{F}_q^{*}}^{\mathbf{F}_q^{*2}} (\rho) = \chi_1\, \chi_2 \\ 0 \text{ otherwise} \end{cases}$

Finally for $R_{(\chi_1, \chi_2)}$ we clearly see that there are no terms of TYPE A and so we have the expression:

$$5.2.8 \quad a_G(R_{(\chi_1,\chi_2)}) = \sum_{\psi \neq 1} (\overline{H,\ \chi_1\, \chi_2 \otimes \psi})^G + (\overline{B,\ \chi_1 \otimes \chi_2})^G$$
$$+ (\overline{B,\ \chi_2 \otimes \chi_1})^G + \overline{(\mathbf{F}_q^{*2},\ \mathrm{Ind}_{\mathbf{F}_q^{*}}^{\mathbf{F}_q^{*2}} (\chi_1 \otimes \chi_2))}^G + \ldots$$

Next we will consider the representation $R_{(\chi,\chi)}$ to obtain the coefficients of the maximal pairs for TYPES A - D for S_χ.
If $\chi_1 = \chi_2$ in the calculations given in 5.2.5, 5.2.6 and 5.2.7 we have the following:

5.2.9 (i) $\langle \tau \otimes \psi, (\chi^2 \otimes 1 + (\chi^2 \otimes \mathrm{Ind}\,_1^{F_q}\,(1))\rangle$

$$= \begin{cases} 1 \text{ if } \tau = \chi^2 \\ 0 \text{ otherwise} \end{cases}$$

(ii) $\langle \mathrm{Ind}\,_B^G(\lambda_1 \otimes \lambda_2, \mathrm{Ind}\,_B^G(\chi \otimes \chi)\rangle$

$$= \begin{cases} 1 \text{ if } [\lambda_1, \lambda_2] = [\chi, \chi] \\ 0 \text{ otherwise} \end{cases}$$

(iii) $\langle \rho, \mathrm{Ind}\,_{F_q^*}^{F_q^{*2}}\,(\chi \otimes \chi)\rangle$

$$= \langle \mathrm{Res}\,_{F_q^*}^{F_q^{*2}}\,(\rho), \chi^2\rangle$$

As noted in 4.4.9 we have $R_{(\chi,\chi)} = L_{(\chi)} \oplus S_{(\chi)}$

and so $a_G(R_{(\chi,\chi)}) = a_G(L_\chi) + a_G(S_\chi)$

ie: $a_G(R_{(\chi,\chi)}) = a_G(\overline{G, \chi.(\det -)})^G + a_G(S_\chi)$

hence we have:

5.2.10 $$a_G(S_\chi) = \sum_{\psi \neq 1} \overline{(H, \chi^2 \otimes \psi)}^G + \overline{(B, \chi \otimes \chi)}^G + \overline{(F_q^{*2}, \mathrm{Ind}\,_{F_q^{*2}}^{F_q^{*2}}\,(\chi \otimes \chi))}^G + \ . \ . \ .$$

Last we will calculate the coefficients of the maximal pairs, TYPES A - D, for the cuspidal representation $r(\theta)$. We recall from 4.4.16 that $r(\theta)$ is a (q-1) dimensional, irreducible representation of G and hence there are no terms of TYPE A.

Let us consider terms of TYPE B. As before, in 5.2.4, we evaluate the Schur inner Product, which for $r(\theta)$ gives:

5.2.11 $\langle \tau \otimes \psi, \mathrm{Res}\,_H^G\,(r(\theta))\rangle$

$= \langle \tau \otimes \psi, \mathrm{Res}\,_H^B\,(\mathrm{Ind}\,_H^B\,(\theta \otimes \omega)\rangle$ from 4.3.17

$$= \sum_{z \in B/H} \langle \tau \otimes \psi, \mathrm{Ind}\,_{H \cap zBz^{-1}}^H (z^{-1})^*(\theta \otimes \omega)\rangle$$

and since matrices $\{\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\}$ $x \in \mathbf{F}_q^*$ form a set of coset representations for B/H, this expression becomes

5.2.12 $\sum_{z \in B/H} \langle \tau \otimes \psi, (z^{-1})^*(\theta \otimes \omega)\rangle$

and since $\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & x^{-1}\beta \\ 0 & \alpha \end{pmatrix}$

where α, $x \in \mathbf{F}_q^*$, $\beta \in \mathbf{F}_q$

and from 4.3.14, $\begin{pmatrix} \alpha & x^{-1}\beta \\ 0 & \alpha \end{pmatrix} \longrightarrow (\alpha, x^{-1}\beta\alpha^{-1})$

then 5.2.12 can be written:

$$\sum_{x \in \mathbf{F}_q^*} \langle \tau \otimes \psi, \mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}} (\theta) \otimes \omega(x^{-1}.-)\rangle$$

$$= \begin{cases} 1 \text{ if } \mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}} (\theta) \otimes \omega_{\mathbf{F}_q}(x^{-1}. -) = \tau \otimes \psi \\ 0 \text{ otherwise} \end{cases}$$

Let us consider the terms of TYPE C. The Schur inner product for this case gives

$$\langle \lambda_1 \otimes \lambda_2, \mathrm{Ind}_H^B(\theta \otimes \omega)\rangle$$
$$= \langle \mathrm{Res}_H^B (\lambda_1 \otimes \lambda_2), (\theta \otimes \omega)\rangle$$
$$= \langle \lambda_1 \lambda_2 \otimes 1, \theta \otimes \omega\rangle = 0$$

and so there are no terms of TYPE C.

To find the coefficient of a term of TYPE D in $a_G(r(\theta))$, we examine the Schur inner product

$$\langle \rho, \mathrm{Res}_{\mathbf{F}_q^{*2}}^{G} (r(\theta))\rangle$$

and this is given by $\dfrac{1}{q^2-1} \sum_{g \in \mathbf{F}_q^{*2}} \overline{\rho(g)}.\ r(\theta)\ (g)$

We recall that the four conjugacy classes of $G = GL_2\mathbf{F}_q$ have representatives, which are given as follows:

$g_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$; $g_2 = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$; $g_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$; $g_4 = \begin{pmatrix} 0 & -N(x) \\ 1 & Tr(x) \end{pmatrix}$

where $\alpha, \beta \in \mathbf{F}_q^*$; $x \in \mathbf{F}_q^{*2}$, $N(x) = x\sigma(x)$

and $Tr(x) = x + \sigma(x)$ where $\sigma(x)$ is the Frobenius map.

In addition $\mathbf{F}_q^* \cong \{(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix})\}$, while $\mathbf{F}_q^{*2} \cong \{(\begin{smallmatrix}0 & -N(x)\\ 1 & Tr(x)\end{smallmatrix})\}$

Hence we have $\langle\rho, \operatorname{Res}^{G}_{\mathbf{F}_q^{*2}} (r(\theta))\rangle$

$$= \frac{1}{(q^2-1)}\left\{\sum_{\alpha\in\mathbf{F}_q^*} \overline{\rho(\alpha)}(q-1)\ \theta(\alpha) - \sum_{x\in\mathbf{F}_q^{*2}-\mathbf{F}_q^*} \overline{\rho(x)}\ [\theta(x) + \theta(\sigma(x))]\right\}$$

from 4.3.11 this expression becomes,

$$\frac{(q+1)}{(q^2-1)} \sum_{\alpha\in\mathbf{F}_q^*} \overline{\rho(\alpha)}.\ \theta(\alpha) - \langle\rho,\theta\rangle - \langle\rho,\ \theta(\sigma(-))\rangle$$

$$= \langle \operatorname{Res}^{\mathbf{F}_q^{*2}}_{\mathbf{F}_q^*}(\rho),\ \operatorname{Res}^{\mathbf{F}_q^{*2}}_{\mathbf{F}_q^*} (\theta)\rangle - \langle\rho,\theta\rangle - \langle\rho,\theta(\sigma(-))\rangle$$

$$= \begin{cases} 1 \text{ for } \operatorname{Res}^{\mathbf{F}_q^{*2}}_{\mathbf{F}_q^*}(\rho) = \operatorname{Res}^{\mathbf{F}_q^{*2}}_{\mathbf{F}_q^*}(\theta) \text{ and } \rho \neq \theta,\ \theta(\alpha(-1)) \\ 0 \text{ otherwise.} \end{cases}$$

This information may be summarized as follows:

5.2.13 THEOREM: The coefficients of the maximal pairs of TYPES A - D for the four irreducible representations $v = \{L_\chi,\ S_\chi,\ R_{(\chi_1,\chi_2)}, r(\theta)\}$ in the Explicit Brauer Induction formula $a_G(v)$ are given by:

(i) $a_G(L_\chi) = \overline{(G,\chi\ .\ \det)}^G$ where $G = GL_2\mathbf{F}_q$

(ii) $a_G(S_\chi) = \sum_{1\neq\psi} \overline{(H,\chi^2 \otimes \psi)}^G$

$$+ B,\overline{(\chi \otimes \chi)}^G + (\mathbf{F}_q^{*2},\ \operatorname{Ind}^{\mathbf{F}_q^{*2}}_{\mathbf{F}_q^*} (\chi \otimes \chi)) + \ldots$$

$$\text{(iii)}\quad a_G(R_{(\chi_1,\chi_2)}) = \sum_{\gamma\neq\chi} (\overline{H,(\chi_1\ \chi_2)\otimes\psi})^G$$

$$+ (\overline{B,\chi_1\otimes\chi_2})^G + (\overline{B,\chi_2\otimes\chi_1})^G$$

$$+ (\overline{\mathbf{F}_q^{*2}\ \mathrm{Ind}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}}(\chi_1\otimes\chi_2)})^G + \ldots$$

$$\text{(iv)}\quad a_G(r(\theta)) = \sum_{\alpha\in\mathbf{F}_q^*} (\overline{H,\ \mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}}(\theta)\otimes\omega(\alpha^{-1}.\ -\ )})^G$$

$$+ \sum_{\rho\neq\theta,\theta(\sigma(-));\ \mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}}(\rho) = \mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}}(\theta)} (\overline{\mathbf{F}_q^{*2},\ \rho})^G$$

$$+ \ldots$$

If we chart the information given in THEOREM 5.2.13, it is evident that the value of the coefficients of the maximal pairs pick out the irreducible representations, φ, in the Explicit Brauer Induction formula, $a_G(\varphi)\in R_+(GL_2\mathbf{F}_q)$.

5.2.14

COEFFICIENTS OF MAXIMAL PAIRS IN $a_G(\varphi)$				
IRR. REPN.	$(\overline{G,\chi\cdot\det})^G$	$(\overline{H,\tau\otimes\psi})^G$	$(\overline{B,\ \lambda_1\otimes\lambda_2})^G$	$(\overline{\mathbf{F}_q^{*2},\rho})^G$
L_χ	1	0	0	0
S_χ	0	$\begin{cases} 1 & \text{for } \phi=\chi^2 \text{ and } \psi\neq 1 \\ 0 & \text{otherwise}\end{cases}$	$\begin{cases} 1 & \text{for } \lambda_1=\lambda_2=\chi \\ 0 & \text{otherwise}\end{cases}$	$\begin{cases} 1 & \text{for } \mathrm{Res}_{\mathbf{F}_q}^{\mathbf{F}_q^{*2}}(\rho)=\chi^2 \\ 0 & \text{otherwise}\end{cases}$
$R_{(\chi_1,\chi_2)}$	0	$\begin{cases} 1 & \text{for } \phi=\chi_1\chi_2 \text{ and } \psi\neq 1 \\ 0 & \text{otherwise}\end{cases}$	$\begin{cases} 1 & \text{for } \lambda_1=\chi_1,\lambda_2=\chi_2 \text{ or } \lambda_1=\chi_2,\lambda_2=\chi_1 \\ 0 & \text{otherwise}\end{cases}$	$\begin{cases} 1 & \text{for } \mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}}(\rho)=\chi_1\ \chi_2 \\ 0 & \text{otherwise}\end{cases}$
$r(\theta)$	0	$\begin{cases} 1 & \text{for } \mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}}(\theta)=\phi \text{ and } \chi=\omega(x^{-1}\cdot -) \text{ for } x\in\mathbf{F}_q^* \\ 0 & \text{otherwise}\end{cases}$	0	$\begin{cases} 1 & \text{for } \mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}}(\rho)=\mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}}(\theta) \text{ and } \rho\neq\theta,\theta(\sigma(-)) \\ & \text{otherwise}\end{cases}$

The lack of TYPE C terms picks out the $r(\theta)$ case. The $(\overline{H,\tau\otimes\psi})^G$ terms indicate the value over which the summation of terms of type, $(\overline{\mathbf{F}_q^{*2},\rho})^G$, are taken. Note that the characters θ, and $\theta(\sigma^*(-))$ on $\mathbf{F}_q^{*2}$ are the only ones that do not appear in the sum. We also note that $r(\theta) = r(\theta)\sigma(-))$, which can be seen by conjugating with the Frobenius matrix $\sigma = \begin{pmatrix}1 & x+\sigma(x)\\ 0 & -1\end{pmatrix}$where $x\in\mathbf{F}_q^{*2}$, on the four conjugacy class representatives of 4.3.11 and observing that the resulting matrices are in the same classes and thus have the same character values for $r(\theta)$.

5.3 THE SHINTANI CORRESPONDENCE AND EXPLICIT BRAUER INDUCTION

In [27], Shintani devised a one-one correspondence between the irreducible representations of $GL_m\mathbf{F}_{q^d}$ and $GL_m\mathbf{F}_q$.

If R: $GL_m\mathbf{F}_q{}^d \longrightarrow GL_n\mathbb{C}$ is Frobenius (F) invariant, ie: R(F(-)) is equivalent to R, then there exists a linear transformation I_F of the representation space of R such that

5.3.1 $\quad R(g)\ I_F = I_F\ R(F(g)) \quad (\forall g \in GL_m\mathbf{F}_q{}^d)$ and $F \in G(\mathbf{F}_q{}^d/\mathbf{F}_q)$

If χ is a class function on $GL_m\mathbf{F}_q{}^d$ as follows:

$$\chi(x) = \begin{cases} \chi(x^1) \text{ if } \exists \text{ an } x^1 \in GL_m\mathbf{F}_q \\ \text{where } x^1 \approx x \text{ and } x \in GL_m\mathbf{F}_q{}^d \\ 0 \text{ otherwise} \end{cases}$$

Then with the above notation Shintani stated that: For a suitable normalization of I_F, there exists an irreducible character χ_R of $GL_m\mathbf{F}_q$, which satisfies:

$$\text{Trace } I_F(R(g)) = \chi_R(\text{Norm}_{\mathbf{F}_q{}^d/\mathbf{F}_q}(g))$$

for every $g \in GL_m\mathbf{F}_q{}^d$ and where Norm $_{\mathbf{F}_q{}^d/\mathbf{F}_q}(g)$, *the Shintani norm*, is defined as $\quad F^{d-1}(g) . F^{d-2}(g) . . . . F(g) . (g)$.

Let $N(g) = \text{Norm}_{\mathbf{F}_q{}^d/\mathbf{F}_q}(g) \in GL_m\mathbf{F}_q{}^d$, then two elements of $GL_m\mathbf{F}_q$ are conjugate if and only if they are conjugate in $GL_m\mathbf{F}_q{}^d$. Therefore the conjugacy class of Norm $_{\mathbf{F}_q{}^d/\mathbf{F}_q}(g)$ contains a unique conjugacy class of $GL_m\mathbf{F}_q$, for F acts naturally on $GL_m\mathbf{F}_q^d$ with the fixed point set $GL_m\mathbf{F}_q$, and this unique class depends only on the conjugacy class of g.

Further, the map $R \longrightarrow \chi_R$ is a bijection from the set of equivalence classes of Frobenius invariant, irreducible representations of $GL_m\mathbf{F}_q{}^d$ onto the similar set for $GL_m\mathbf{F}_q$.

As described by Kawanaka, [18,Pp. 425-450], this correspondence is called a *Shintani descent, lifting (or base change)*. In his paper, Shintani details results of his correspondence for GL_2 of a local field, F.

Consider the following notation;

5.3.2 $\quad$ If ρ: $GL_2\mathbf{F}_q{}^d \longrightarrow GL_n\mathbb{C}$ is a Frobenius invariant, irreducible representation and $Sh(\rho): GL_2\mathbf{F}_q \longrightarrow GL_n\mathbb{C}$ then, following Shintani, [27,P.400], we have

Trace $\rho(x)$ = Trace $Sh(\rho)$ $(N(x))$ for every $x \in GL_2\mathbf{F}_{q^d}$.

Consider the exact sequence

$$\mathbf{F}_{q^d}^{*} \xrightarrow{F/1} \mathbf{F}_{q^d}^{*} \xrightarrow{N_{\mathbf{F}_{q^d}/\mathbf{F}_q}} \mathbf{F}_q^{*} \longrightarrow \{1\}$$

which follows from Hilbert Theorem 90,[20, P.335].

Then we have for $\upsilon : \mathbf{F}_{q^d}^{*} \longrightarrow \mathbb{C}^{*}$ with $\upsilon = \mathbf{F}^{*}(\upsilon)$ from Shintani [27,P. 396], the following commutative diagram where $Sh(\upsilon) : \mathbf{F}_q^{*} \longrightarrow \mathbb{C}^{*}$ is unique

$$\begin{tikzcd}
\mathbf{F}_{q^d}^{*} \arrow[r, "F/1"] & \mathbf{F}_{q^d}^{*} \arrow[r, "N_{\mathbf{F}_{q^d}/\mathbf{F}_q}"] \arrow[dr, "\upsilon"'] & \mathbf{F}_q^{*} \arrow[r] \arrow[d, "Sh(\upsilon)"] & \{1\} \\
& & \mathbb{C}^{*} &
\end{tikzcd}$$

We wish to construct a similar correspondence based on Explicit Brauer Induction. This will be done by applying Hilbert Theorem 90 to the maximal characters in the Explicit Brauer Induction formula for $a_G(\rho)$, where $G = GL_2\mathbf{F}_{q^d}$ and ρ is an irreducible representation of G. We note that this construction is distinct from that of Shintani, and makes no use of the Shintani norm.

First we note that Shintani descent applies to irreducible representations, ρ, which are Frobenius invariant.

So for $F \in G(\mathbf{F}_{q^d}/\mathbf{F}_q)$ we have:

$$\rho = \rho(F(-)) = F^{*}(\rho).$$

and for $GL_2\mathbf{F}_{q^d} = G$

$$a_G(\rho) = a_G(F^{*}(\rho)) = F^{*}(a_G(\rho))$$

where $F^{*}: R_{+}(GL_2\mathbf{F}_{q^d}) \longrightarrow R_{+}(GL_2\mathbf{F}_{q^d})$

is given by $F^{*}(J,\phi)^G = (F(J),\ \phi(F^{-1} -))^G$

Since $F^*(\rho)$ is irreducible, the maximal terms of TYPES A - D in $a_G(F^*(\rho))$ can be obtained by applying F^* to the maximal terms of TYPES A - D in $a_G(\rho)$.

We will develop this correspondence, termed Υ, as follows:

Case 1 : For $\rho = L_\chi$, where $L_\chi : GL_2\mathbf{F}_q^d \xrightarrow{\det} \mathbf{F}_q^{*d} \xrightarrow{\chi} \mathbb{C}^*$, from Shintani, [27, P.411], we have:

$F^*(L_\chi) = L_\chi$ if and only if $F^*(\chi) = \chi$.

This follows from Hilbert Theorem 90, which gives the following commutative diagram:

5.3.3
$$\begin{tikzcd} \mathbf{F}_q^{*d} \arrow[rr, "\chi"] \arrow[dr, "N=N_{\mathbf{F}_q{}^d/\mathbf{F}_q}"'] & & \mathbb{C}^* \\ & \mathbf{F}_q^* \arrow[ur, "\exists\ \bar{\chi}"'] & \end{tikzcd}$$

In this case we set:

$\Upsilon(L_\chi) = L_{\bar{\chi}}$ with

$L_\chi(g) = L_{\bar{\chi}}(N(g))$

ie: $\chi(\det g) = \bar{\chi}(N(\det(g))$

Case 2 For $\rho = R_{(\chi_1,\chi_2)}$ where $R_{(\chi_1,\chi_2)} = \mathrm{Ind}_B^G((\chi_1 \otimes \chi_2)$

and $\chi_1, \chi_2 : \mathbf{F}_q^{*d} \longrightarrow \mathbb{C}^*$

Then $F^*(a_G(\rho)) = F^*\Big\{ \sum_{\psi \neq 1} (\overline{H.\ \chi_1\ \chi_2 \otimes \psi})^G$

$+ (\overline{B,\ \chi_1 \otimes \chi_2})^G + (\overline{B,\ \chi_2 \otimes \chi_1})^G + \ldots \Big\}$

$$= \sum_{\psi \neq 1} \overline{(H, F^*(\chi_1\ \chi_2) \otimes \psi)}^G + \overline{(B,\ F^*(\chi_1) \otimes F^*(\chi_2))}^G$$

$$+ \overline{(B, F^*(\chi_2) \otimes F^*(\chi_1))}^G + \ldots$$

and from Shintani [27,P.412] we have two possible cases:

(i) $\chi_1 = F^*(\chi_1)$ and $\chi_2 = F^*(\chi_2)$

or (ii) $\chi_1 = F^*(\chi_2)$ and $\chi_2 = F^*(\chi_1)$

with $\chi_1\ \chi_2 = F^*(\chi_1\ \chi_2)$.

For the first case, we again obtain the following commutative diagram:

5.3.4
$$\begin{tikzcd} \mathbf{F}_q^{*d} \arrow[rr, "\chi_i"] \arrow[dr, "N=\mathrm{Norm}\ \mathbf{F}_q{}^d/\mathbf{F}_q"'] & & \mathbb{C}^* \\ & \mathbf{F}_q^* \arrow[ur, "\exists\ \bar{\chi}_i \text{ for } i=1,2 \text{ and } \bar{\chi}_i : \mathbf{F}_q^* \longrightarrow \mathbb{C}^*"'] & \end{tikzcd}$$

and so we may set

$T(R(\chi_1,\ \chi_2) = R(\bar{\chi}_1,\ \bar{\chi}_2)$

In case (ii) consider the surjective homomorphism

$$\lambda : G(\mathbf{F}_q{}^d/\mathbf{F}_q) \cong \mathbf{Z}/_d \longrightarrow \{\pm 1\}$$

which is defined by $\lambda(g) = (-1)^{i-1}$ if $g(\chi_1) = \chi_i$.
This definition demands that d be even and we see that
$\mathrm{Ker}(\lambda) = G(\mathbf{F}_q{}^d/\mathbf{F}_{q^2})$. Consider $G(\mathbf{F}_q{}^d/\mathbf{F}_q) \longrightarrow\!\!\!\!\!\rightarrow G(\mathbf{F}_q{}^2/\mathbf{F}_q)$ given by

$$F \longrightarrow F^{d/2}$$

where $F^*(\chi_1) = \chi_2$ and $F^*(\chi_2) = \chi_1$

so that $(F^2)^*(\chi_i) = \chi_i$ and so

$(F^2)^*$ generates $G(\mathbf{F}_{q^d}/\mathbf{F}_{q^2})$

Thus we may apply Hilbert Theorem 90 to obtain the following commutative diagram:

5.3.5
$$\begin{tikzcd}
\mathbf{F}_{q}^{*d} \arrow[rr] \arrow[dr, "\mathrm{Norm}(\mathbf{F}_{q^d}/\mathbf{F}_{q^2})"'] & & \mathbb{C}^* \\
& \mathbf{F}_{q}^{*2} \arrow[ur, "\bar{\chi}_1 \text{ where } \bar{\chi}_1 \text{ is unique}"'] &
\end{tikzcd}$$

Further, if the Frobenius transformation, σ , generates $G(\mathbf{F}_{q^2}/\mathbf{F}_q)$ and if $\bar{\chi}_2 = \bar{\chi}_1(\sigma(-))$ for $x \in \mathbf{F}_q^{*2}$

then $\sigma^*(\bar{\chi}_2) = \sigma^*(\sigma^*((\bar{\chi}_1))$

and so $\sigma^*(\bar{\chi}_2) = \bar{\chi}_1$

also for $z \in \mathbf{F}_q^{*d}$, we have

$\bar{\chi}_2(N(z)) = \sigma^*(\chi_1(N(z))$

$= \sigma^*(\chi_1(z))$

from Hilbert Theorem 90 as shown in the diagram above, and so

$$\bar{\chi}_2(N(z)) = \sigma^*(\chi_1(z)) = \chi_2(z).$$

So in case(ii) since $F^*(\chi_1\ \chi_2) = \chi_1\ \chi_2$,

there exists a unique $\bar{\chi}_{1,2} : \mathbf{F}_q^* \longrightarrow \mathbb{C}^*$

such that $\chi_1(z)\ \chi_2(z) = \bar{\chi}_{1,2}(N_{\mathbf{F}_{q^d}/\mathbf{F}_q}(z))$

Indeed we will show that:

$$\mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}}(\bar{\chi}_1) = \bar{\chi}_{1,2} = \mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}}(\bar{\chi}_2)$$

If we consider the following diagram:

5.3.6

$$\begin{tikzcd}
r \in \mathbf{F}_q^{*d} \arrow[rrr, "\chi_1\,\chi_2"] \arrow[dr, mapsto] \arrow[dr, shift left] & & & \mathbb{C}^* \\
& v \in \mathbf{F}_q^{*2} \arrow[dr, mapsto] \arrow[dr, shift left] & & \\
& & w \in \mathbf{F}_q^* \arrow[uur, "\bar{\chi}_{1,2}"'] &
\end{tikzcd}$$

where $N_{\mathbf{F}_{q^d}/\mathbf{F}_{q^2}}(r) = v = N(r)$ and $N_{\mathbf{F}_{q^2}/\mathbf{F}_q}(v) = w$

Then $\bar{\chi}_1(w) = \bar{\chi}_1(N_{\mathbf{F}_{q^2}/\mathbf{F}_q}(v))$

$$\begin{aligned}
&= \bar{\chi}_1(v\sigma(v)) \\
&= \bar{\chi}_1(v)\ \bar{\chi}_1(\sigma(v)) \\
&= \bar{\chi}_1(v)\ \bar{\chi}_2(v) \\
&= \bar{\chi}_1(N(r)\ \bar{\chi}_2(N(r)) \\
&= \chi_1(r)\ \chi_2(r) = \bar{\chi}_{1,2}(N_{\mathbf{F}_{q^d}/\mathbf{F}_q}(r)) \\
&= \bar{\chi}_{1,2}(w)
\end{aligned}$$

and similarly $\bar{\chi}_2(w) = \bar{\chi}_{1,2}(w)$.

Examination of these characters, and, since $\bar{\chi}_i \neq \sigma^*(\bar{\chi}_i)$ implies that $\Upsilon(R_{(\chi_1,\chi_2)})$ is cuspidal, [27,P.411], which leads to the following expression in $R_+(GL_2\mathbf{F}_q, S^1)$:

$$\sum_{\alpha\in\mathbf{F}_q^*} \overline{(H,\ \bar{\chi}_{1,2} \otimes \omega_{\mathbf{F}_q^*} \cdot (\alpha^{-1} \cdot -))}^{G}$$

$$+\sum_{\rho\in\{\bar{\chi}_1,\bar{\chi}_2\},\ \mathrm{Res}^{\mathbf{F}_q^{*2}}_{\mathbf{F}_q^*}(\rho) = \bar{\chi}_{1,2}} \overline{(\mathbf{F}_q^{*2},\ \rho)}^{G}$$

$+ \ . \ . \ .$ (where $G = GL_2\mathbf{F}_q$)

This is the expression in maximal terms of TYPE A - D for $a_G(r(\bar{\chi}_1))$. Thus in case (ii), set $\Upsilon(R_{(\chi_1,\chi_2)}) = r(\bar{\chi}_1) = r(\bar{\chi}_2)$.

CASE 3: As in case 2(i), if $S_\chi = F^*(S_\chi)$, then it follows that $\chi = F^*(\chi)$ Shintani [27,P.411]. By Hilbert Theorem 90 we have the following commutative diagram:

5.3.7
$$\begin{tikzcd}
\mathbb{F}_q^{*d} \arrow[rr, "\chi"] \arrow[dr, "N=\text{Norm } \mathbb{F}_{q^d}/\mathbb{F}_q"'] & & \mathbb{C}^* \\
& \mathbb{F}_q^* \arrow[ur, "\bar{\chi}"'] &
\end{tikzcd}$$

where $\bar{\chi}:\mathbb{F}_q^* \longrightarrow \mathbb{C}^*$ is unique

and so in this case set $\Upsilon(S_\chi) = S_{\bar{\chi}}$.

CASE 4: In the cuspidal, $r(\theta)$ case, consider if $r(\theta) = F^*(r(\theta))$, then we have from Theorem 5.2.13

$$a_G(r(\theta)) = a_G(F^*(r(\theta))) = \sum_{\alpha\in\mathbb{F}_q^{*d}} \overline{\left(H, \operatorname{Res}^{\mathbb{F}_q^{*2d}}_{\mathbb{F}_q^{*d}} \left(F^*(\theta) \otimes \omega_{\mathbb{F}_{q^d}}(\alpha^{-1}\cdot\ -\ )\right)\right)^G}$$

5.3.8
$$+ \sum_{\rho\neq\theta,\sigma^*(\theta);\ \operatorname{Res}^{\mathbb{F}_q^{*2d}}_{\mathbb{F}_q^{*d}}(\rho) = \operatorname{Res}^{\mathbb{F}_q^{*2d}}_{\mathbb{F}_q^{*d}}(\theta)} \overline{(\mathbb{F}_q^{*2d},\ F^*(\rho))}$$

$$+ \ . \ . \ . \qquad (\text{where } G = GL_2\mathbb{F}_{q^d})$$

and also since $a_G(r(\theta)) = \sum_{\alpha\in \mathbf{F}_q{}^d} \overline{(H,\ \mathrm{Res}^{\mathbf{F}_q^{*2d}}_{\mathbf{F}_q^{*d}} (\theta \otimes \omega_{\mathbf{F}_q^d} (\alpha^{-1} \cdot -))^G}$

$$+ \sum_{\rho\neq\theta,\sigma^*(\theta)\ ;\ \mathrm{Res}^{\mathbf{F}_q^{*2d}}_{\mathbf{F}_q^{*d}}(\rho) = \mathrm{Res}^{\mathbf{F}_q^{*2d}}_{\mathbf{F}_q^{*d}}(\theta)} (\overline{F,\rho})^G + \ldots$$

Hence we have the following two implications:

$$\mathrm{Res}^{\mathbf{F}_q^{*2d}}_{\mathbf{F}_q^{*d}} (F^*(\theta)) = \mathrm{Res}^{\mathbf{F}_q^{*2d}}_{\mathbf{F}_q^{*d}} (\theta)$$

This is similar to case (ii) $R_{(\chi_1,\chi_2)}$ discussed earlier, so we consider the following diagram:

5.3.9
$$\begin{tikzcd}
\mathbf{F}_q^{*d} \arrow[rr, "\theta"] \arrow[dr, "N=\mathrm{Norm}(\mathbf{F}_q{}^d/\mathbf{F}_q)"'] & & \mathbb{C}^* \\
& \mathbf{F}_q^* \arrow[ur, "\bar{\theta}"'] &
\end{tikzcd}$$

where $\bar{\theta}:\mathbf{F}_q^* \longrightarrow \mathbb{C}^*$ is unique and $\theta(z)=\bar{\theta}(N(z))$ for $z\in\mathbf{F}_q^{*d}$

The second term in 5.3.8 implies that F^* must permute the elements of the set $[\{\theta,\sigma^*(\theta)\},\ \mathrm{Res}^{\mathbf{F}_q^{*2d}}_{\mathbf{F}_q^{*d}} (\rho) = \mathrm{Res}^{\mathbf{F}_q^{*2d}}_{\mathbf{F}_q^{*d}} (\theta)]$

since $F^*(\rho) = \rho$ for all but these cases.

This in turn, implies that $F^*(\theta) = \sigma^*(\theta)$.
Since $F \in G(\mathbf{F}_q{}^{2d}/\mathbf{F}_q) \cong \mathbf{Z}/_{2d}$ and $\sigma = F^d$, then

$(F^{d-1})^*(\theta) = \theta$ for

$(F^*)(F^{d-1})^*(\theta) = (F^d)^*\theta = \sigma^*(\theta) = F^*(\theta)$

and since θ is not Galois invariant in the cuspidal case, that is,

$\theta \neq \sigma^*(\theta)$, then the subgroup generated by F^{d-1} must be a proper subgroup of $\langle F \rangle$.

Now the order of $\langle F \rangle$ is $2d$ and we consider the HCF of $2d$ and $(d-1)$. If $k | d-1$, then $k | 2d-2$ and if $k | 2d$, then $k = 1$ or 2.

But $k = 1$ implies that $\langle F^{d-1} \rangle = \langle F \rangle$, hence HCF$(d - 1, 2d)$ is 2 and so d is odd.

This implies that:

5.3.10 $\quad \langle F^{d-1} \rangle \cong \mathbb{Z}/_d = G(\mathbb{F}_q{}^{2d}/_{\mathbb{F}_q{}^2})$

and by Hilbert Theorem 90 we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{F}_q^{*2d} \arrow[rr, "\theta"] \arrow[dr, "N=N_{\mathbb{F}_q{}^{2d}/\mathbb{F}_q{}^2}"'] & & \mathbb{C}^* \\
& \mathbb{F}_q^{*2} \arrow[ur, "\tilde{\theta}"'] &
\end{tikzcd}$$

where $\tilde{\theta}: \mathbb{F}_q^{*2} \longrightarrow \mathbb{C}^*$ is unique and $\theta(z) = \tilde{\theta}(N_{\mathbb{F}_q{}^{2d}/\mathbb{F}_q{}^2}(z))$ for all $z \in \mathbb{F}_q^{*2d}$

We now demonstrate that $\tilde{\theta}$ restricts to $\tilde{\theta}$ on $\mathbb{F}_q^*$.
If $x \in \mathbb{F}_q^*$ and $y \in \mathbb{F}_q^{*d}$ such that $N_{\mathbb{F}_q{}^d/\mathbb{F}_q}(y) = x$, then

$$\operatorname{Res}^{\mathbb{F}_q^{*2}}_{\mathbb{F}_q^*}(\tilde{\theta}\ (x)) = \operatorname{Res}^{\mathbb{F}_q^{*2}}_{\mathbb{F}_q^*}(\tilde{\theta}\ (N_{\mathbb{F}_q{}^d/\mathbb{F}_q}(y)))$$

$$= \tilde{\theta}(N_{\mathbb{F}_q{}^{2d}/\mathbb{F}_q{}^2}(y)) \text{ since } \mathbb{Z}/_d \cong \langle F^{d-1} \rangle = G(\mathbb{F}_q{}^{2d}/_{\mathbb{F}_q{}^2})$$

$$\cong G(\mathbb{F}_q{}^d/_{\mathbb{F}_q})$$

$$= \mathrm{Res}_{\mathbf{F}_q^{*d}}^{\mathbf{F}_q^{*2d}} (\theta(y)) \text{ since } (\theta(y) = \tilde{\theta}(N_{\mathbf{F}_q^{2d}/\mathbf{F}_q^{d}}(y) \text{ for } y \in \mathbf{F}_q^{*d}$$

$$= \bar{\theta}(N_{\mathbf{F}_q^{d}/\mathbf{F}_q}(y)) \text{ from 5.3.9}$$

$$= \bar{\theta}(x)$$

The above examination of characters θ, $\bar{\theta}$ and $\tilde{\theta}$ suggests the formation:

$$\sum_{\alpha \in \mathbf{F}_q^*} \overline{(H, \bar{\theta} \otimes \omega_{\mathbf{F}_q} (\alpha, -)}^{G}$$

$$+ \sum_{\rho \neq \tilde{\theta}, \sigma^*(\tilde{\theta});\ \ \mathrm{Res}_{\mathbf{F}_q^*}^{\mathbf{F}_q^{*2}} (\rho) = \bar{\theta}} \overline{(\mathbf{F}_q^{*2}, \rho)}^{G} + \ldots$$

This is the expression for $a_{GL_2\mathbf{F}_q}(r(\tilde{\theta}))$ in the sense of THEOREM 5.2.13 section (iv).

Hence we set

5.3.12 $\qquad \Upsilon(r(\theta)) = r(\tilde{\theta})$.

It is clear in all the above cases that the correspondence given by Υ is one-one, since $\bar{\chi}, (\bar{\chi}_1, \bar{\chi}_2)$ and $\tilde{\theta}$ are unique characters obtained from the corresponding character in $GL_2\mathbf{F}_q{}^d$.

We may further clarify that this correspondence is in fact coincident with Shintani descent. As the Explicit Brauer Induction construction for a correspondence of the Shintani type does not use the Shintani norm, 5.3.1, on character values of $GL_2\mathbf{F}_q{}^d$, it is to be hoped that this technique promises to generalize to an intrinsic description of Sh(-) for $GL_m F$, for F a local field.

5.3.13 Theorem The correspondence given by $\rho \longleftrightarrow \Upsilon(\rho)$ is 1 - 1 where

(i) ρ is an irreducible representation of $GL_2\mathbf{F}_q{}^d$, which is fixed by $F \in G(\mathbf{F}_q^d / \mathbf{F}_q)$ and

(ii) $\Upsilon(\rho)$ is an irreducible representation of $GL_2\mathbf{F}_q$ coinciding with the Shintani correspondence [27,P.410 - 414].

Proof; The values of $\Upsilon(\rho)$ and $Sh(\rho)$ coincide on the character table for the four conjugacy class representations given by 4.3.11. We give a table for the conjugacy class

$$\text{Type I: } g = \{\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix},\ \alpha,\ \beta \in \mathbf{F}_q^*\}$$

	L_χ	$R_{(\chi_1,\chi_2)}$	S_χ	$r(\theta)$
Character values	$\chi(\alpha\beta)$	$\chi_1(\alpha)\chi_2(\beta)$ $+\chi_1(\beta)\chi_2(\alpha)$	$\chi(\alpha\beta)$	0

	$L_{\bar\chi}$	$R_{(\bar\chi_1,\bar\chi_2)}$	$S_{\bar\chi}$	$r(\tilde\theta)$
Character values	$\bar\chi(N(\alpha\beta)$ $= \chi(\alpha\beta)$	$\bar\chi_1 N(\alpha)\bar\chi_2 N(\beta)$ $+\bar\chi_1 N(\beta)\bar\chi_1(N(\alpha))$ $=\chi_1(\alpha)\chi_2(\beta)$ $+\chi_1(\beta)\chi_2(\alpha)$	$\bar\chi(N(\alpha\beta))$ $= \chi(\alpha\beta)$	0

The equality of character values on $\Upsilon(L_\chi)$ and $Sh(L_\chi)N(-)$follows from the definitions of $\Upsilon(L_\chi)$ and the Shintani descent definition 5.3.2:

$$Sh(\rho)\ (N(x)) = \rho(x) \text{ for } x \in \mathbf{F}_q^{*d} \text{ and } \rho:\mathbf{F}_q^{*d} \longrightarrow \mathbb{C}^* \text{ where}$$
$$Sh(\rho) : \mathbf{F}_q^* \longrightarrow \mathbb{C}^* \text{ and } \rho \text{ is Frobenius invariant.}$$

The same approach indicates identical character values for types II - IV.

We note that for Type IV, the $r(\theta)$ case is as follows:

$Sh(r(\theta)(N(y)) = r(\theta)(y)$ for $y \in \mathbf{F}_q^{*d}$ with character value $-\{\theta(x) + \theta(\sigma(x))\}$ where $x \in \mathbf{F}_q^{*2d}$ and the corresponding character value for $r(\tilde{\theta})$ is $-\{\tilde{\theta}(\tilde{x}) + \tilde{\theta}(\sigma(\tilde{x}))\}$ where $\tilde{x} \in \mathbf{F}_q^{*2}$.

Since $N_{\mathbf{F}_q{}^{2d}/\mathbf{F}_q{}^{d}}$ is surjective we have

$(\tilde{\theta})(\tilde{x}) = \tilde{\theta}(N_{\mathbf{F}_q{}^{2d}/\mathbf{F}_q{}^{d}}(x)) = \theta(x)$ and $\sigma \in G(\mathbf{F}_q{}^{2d}/\mathbf{F}_q{}^{d}) \cong (G(\mathbf{F}_q{}^{2}/\mathbf{F}_q)$

This completes the proof of Theorem 5.3.13.

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