QUANTUM MECHANICAL PROBLEMS IN ONE-, TWO- AND THREE-DIMENSIONS

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SHUXI LI, B.Sc., M.Sc.

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AUTHOR: Shuxi Li, B.Sc. (Chongqing University, P.R. China)
M.Sc. (Chongqing University, P.R. China)

SUPERVISOR: Dr. Rajat K. Bhaduri

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ABSTRACT

The underlying theme of this thesis is to solve mathematical models related to physical problems in one, two and three spatial dimensions. In some cases, particularly in one spatial dimension, the equations could be solved exactly, while for models in higher dimensions, the solutions were approximate. Although the four separate chapters of the thesis are largely independent, quantum statistics played an important role in the models considered. In the following paragraphs, I briefly summarize the contents of the four chapters.

An exactly solvable field theoretical model in one spatial dimension and its classical bound state solution (including the zero mode) are presented in chapter one. The related bosonization and vacuum charge are discussed.

In chapter two, a many-anyon system in two dimensional space with a confining potential is solved using the Thomas-Fermi mean-field method. The ground-state energy and spatial distribution are obtained as functions of the statistical parameter.

In chapter three, Chern-Simions coupling to fermions in two dimensions is considered. A zero-mode soliton solution is obtained. Classical vortices in fluid mechanics are shown to be mathematically analogous to anyons. Some new results are derived using this analogy.

Finally, in three-dimensional space, the vacuum properties of the Nambu-Jona-Lasinio model are studied at both zero and finite temperatures. The vacuum condensation energy per unit volume is calculated and is identified as the MIT bag constant. Some other thermodynamic properties are also calculated, showing striking similarity to the BCS superconductor.

Abstract

A scaling law of the chiral condensation energy density in a nuclear medium is suggested.

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Chapter 1

An Exactly Solvable Model In 1+1 Dimensions

1.1 INTRODUCTION

Exactly solvable interacting models are of great importance in quantum field theories. Although the physics of the real world may not be described by exactly solvable models, such models are very useful. They can provide testing grounds for various proposed approximation schemes and give us insight into the physics of the system. In 3+1 dimensions, however, there are only a few models which are exactly solvable. Since the early 1970s, many studies have been devoted to 1+1 dimensional models. The study was pioneered by Dashen, Hasslacher and Neveu (Dashen *et al*, 1974), Scott, Chu and Mclaughlin (Scott *et al*, 1973), Goldstone and Jackiw (Goldstone and Jackiw, 1975), Chang, Ellis and Lee (Chang *et al*, 1975) and Coleman (Coleman, 1975). Exact solutions were worked out for various models (Dashen *et al*, 1974, Scott *et al*, 1973 and Chang *et al*, 1975), and second quantization was applied in some cases (Dashen *et al*, 1974, Goldstone and Jackiw, 1975). Many striking results were obtained from studying these 1+1 dimensional models. New concepts were introduced: Soliton; Vacuum Fermion charge fractionlization – a result from topological stable soliton (kink) polarizing the Dirac sea (Jackiw and Rebbi, 1976); Fermion-Boson duality – Fermion as a quantum soliton – the equivalence of quantum sine-Gordon (SG) equation and the massive Thirring (MT) model (Coleman, 1975, Mandelstam, 1975). These concepts and discoveries in quantum field theories have been of a great significance in connection to condensed matter physics (for a review, see Krive and Rozhavskii, 1987), and have led to the understanding of the possible mechanism of nonlinear conductivity of quasi-one-dimensional compounds (for example, TaS_3, $NbSe_3$, etc.) since then.

Since 1984, there has been much interest (Kahana *et al*, 1984, Mackenzie and Wilczek, 1984, Birse and Banerjee, 1985, Ripka and Kahana, 1985, Aitchison and Fraser, 1985 and Kahana and Ripka, 1984) in the study of a fermionic field coupled to a bosonic field, with the latter having solitonic solutions in three spatial dimensions. These, of course, are all solved numerically. The simplest approach to the problem is to treat the fermionic field classically in the background of a scalar soliton. On solving the resulting Dirac equation, one may study the behavior of the eigenstates and the Dirac sea under the influence of the solitonic coupling.

In this chapter, we study a model of a Fermion chirally coupled to a static background soliton in 1+1 dimensions. We solve analytically for bound state solutions of the fermionic field and soliton solutions of the background scalar field at the same time. Interestingly, for a special choice of an adjustable parameter in our model, the system reduces to the model studied by Chang, Ellis and Lee, which was proved to be equivalent to the MT model (Chang *et al*, 1975). For another choice of the parameter, the system reduces to the model studied by Mackenzie and Wilczek (Mackenzie and Wilczek, 1984) and Nogami and Beachey (Nogami and Beachey, 1986). The chapter is organized as follows: In section 2, the model Lagrangian and its solutions are presented. The relation to the MT model is discussed in section 3. In section 4, the expression of scalar field Lagrangian which supports the soliton in the chiral coupling is derived. The vacuum Fermion charge (VFC) is calculated in section 5. In section 6, the procedure of bosonization of our model is explained. The last section of this chapter summarizes the results.

1.2 AN EXACTLY SOLVABLE CLASS OF SOLITONS

In this chapter, we adopt the units of $\hbar = c = 1$, and work in the Minkowski space with the metric $diag\ (g_{\mu\nu}) = (1, -1)$. Consider a classical Lagrangian of a fermionic field ψ chirally coupled to a static topological soliton $\theta_c(x)$. The soliton has the property that it approaches different non zero constant values as $x \to \pm\infty$. The Lagrangian density is given by:

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}e^{i\gamma_5\theta_c(x)}\psi \ . \tag{1.2.1}$$

Applying the unitary chiral rotation

$$\psi(x) = e^{-i\theta_c(x)\gamma_5/2}q(x) \ , \tag{1.2.2}$$

and using the property that $\gamma^{\mu}\gamma^{5} = \epsilon^{\mu\nu}\gamma_{\nu}$ in 1+1 dimensions, we get the transformed Lagrangian from Eq.(1.2.1):

$$\mathcal{L}' = i\bar{q}\gamma^{\mu}\partial_{\mu}q - \frac{1}{2}\epsilon_{\mu\nu}\bar{q}\gamma^{\mu}q\partial^{\nu}\theta_c(x) - m\bar{q}q \ . \tag{1.2.3}$$

Namely, a system described by a chiral coupling Lagrangian (1.2.1) may be equivalently described by the Lagrangian (1.2.3) after applying a chiral rotation (1.2.2). For convenience, we will work with the transformed Lagrangian (1.2.3) in the rest of this chapter. The corresponding Dirac equation, keeping in mind that $\theta_c(x)$ is time-independent, is given by (with $\epsilon_{01} = -\epsilon_{10} = 1$)

$$i\frac{\partial q}{\partial t} = -i\gamma^5 \frac{\partial q}{\partial x} + \gamma^0 mq + \frac{1}{2}\frac{d\theta_c(x)}{dx} q \,. \tag{1.2.4}$$

In the following, we use the representation

$$\gamma^0 = \sigma_1 \,, \;\; \gamma^1 = i\sigma_3 \,, \;\; \gamma^5 = \gamma^0\gamma^1 = \sigma_2 \,, \tag{1.2.5}$$

where the σ_i's are the 2×2 Pauli matrixes given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \;\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \,, \;\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,.$$

For a stationary solution, $q(x,t) = q(x)e^{-iEt}$, with $q(x)$ a two by one vector. Denoting $q(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$, the eigen-equation can be written as (with $u' = du/dx$ etc.)

$$-v' + mv + V_D(x)u = Eu \,, \tag{1.2.6a}$$

$$u' + mu + V_D(x)v = Ev \,, \tag{1.2.6b}$$

where the Dirac potential is defined by

$$V_D(x) = \frac{1}{2}\frac{d\theta_c(x)}{dx} \,. \tag{1.2.7}$$

For a bound state solution, we make the transformations for u and v following from Chang, Ellis and Lee (Chang *et al*, 1975) that

$$u(x) = \eta(x)\cos\phi(x) \,, \tag{1.2.8a}$$

$$v(x) = \eta(x)\sin\phi(x) \,, \tag{1.2.8b}$$

where $\eta(x)$ and $\phi(x)$ are real functions. For a physically meaningful solution, we also demand that the solution be normalizable, i.e., that satisfy

$$\int_{-\infty}^{+\infty}(u^2+v^2)dx = \int_{-\infty}^{+\infty}\eta^2(x)\ \ dx = 1 \ .$$

Substituting the transformation (1.2.8) into Eq.(1.2.6) and simplifying yields the following coupled equations:

$$\eta' = -m\eta\cos 2\phi \tag{1.2.9}$$

$$\phi' = (V_D - E) + m\sin 2\phi \ . \tag{1.2.10}$$

These may be integrated exactly if $V_D(x)$ is chosen to be proportional to $\phi'(x)$. To be specific, we take the form

$$V_D(x) = (1+\frac{2}{\beta})\phi'(x)\ , \tag{1.2.11}$$

where β is an adjustable constant whose range of variation will be determined from the normalization condition. Substituting (1.2.11) into (1.2.10), we obtain

$$\frac{2}{\beta}\phi' = E - m\sin 2\phi \ . \tag{1.2.12}$$

In the following, we will solve the above equation and Eq.(1.2.9) for cases of the zero mode with $E = 0$ and the non-zero energy bound state with $0 < |E| < m$ respectively.

The Zero-Mode

In this case, the integration of Eq.(1.2.12) with $E = 0$ immediately gives (Bhaduri *et al*, 1989)

$$\phi_0(x) = \frac{1}{2}\arccos(\tanh\beta m(x-x_0))\ . \tag{1.2.13}$$

Without loss of generality, we assign x_0, the integration constant, to be zero in the subsequent equations.

Thus, from Eq.(1.2.11), we obtain the Dirac vector potential as

$$V_{0D}(x) = -m(\frac{\beta}{2}+1)\frac{1}{\cosh(\beta m x)} . \tag{1.2.14}$$

A little algebra, based on Eqs.(1.2.9) and (1.2.13), gives (see appendix A for detail)

$$\eta_0(x) = N\left(\frac{1}{\cosh(\beta m x)}\right)^{1/\beta} \tag{1.2.15}$$

where N is a normalization constant. Since $\cosh(\beta m x) \geq 1$, from the condition

$$\int_{-\infty}^{+\infty} \eta_0^2(x)\, dx = \int_{-\infty}^{+\infty} N^2(\cosh(\beta m x))^{-2/\beta} dx = 1 ,$$

the renormalizability of the solution requires that $\beta > 0$.

It is fairly easy to write the zero-mode Fermion eigen-state q_0 in the following form:

$$q_0 = N[\cosh(\beta m x)]^{-(\frac{1}{2}+\frac{1}{\beta})}\begin{pmatrix} e^{\beta m x/2} \\ e^{-\beta m x/2}\end{pmatrix}. \tag{1.2.16}$$

We have thus obtained the zero-mode solution for the Fermion in the associated Dirac potential. It is helpful to recall that the Dirac potential is defined in terms of the chiral soliton $\theta_c(x)$. Through the comparison of Eq.(1.2.7) with Eqs. (1.2.11) and (1.2.13), we immediately obtain the following expression for the soliton:

$$\theta_c(x) = (1+\frac{2}{\beta})\arccos[\tanh(\beta m x)] + \theta_0 , \tag{1.2.17}$$

where θ_0 is an integration constant and may be chosen such that $\theta_c(-\infty) = -\theta_c(+\infty)$ and is given by

$$\theta_0 = (1+\frac{2}{\beta})\frac{\pi}{2} .$$

Note that equation (1.2.17) may also be written in the equivalent form

$$\theta_c(x) = 2(1 + \frac{2}{\beta}) \arctan(e^{-\beta m x}) + \theta_0 \,. \tag{1.2.18}$$

This is the standard sine-Gordon like soliton. From the definition

$$Q = \frac{1}{2\pi}[\theta_c(\infty) - \theta_c(-\infty)] \,, \tag{1.2.19}$$

the topological charge of this soliton is found as

$$Q_0 = -\frac{1}{2}(1 + \frac{2}{\beta}) \,.$$

The Bound State of $|E| > 0$

For the case of $|E| > 0$, for simplicity, we assume $E > 0$ without loss of generality. The general solution of $\phi(x)$ may be obtained by integrating Eq.(1.2.12) directly, resulting in

$$\phi(x) = \arctan(\frac{1}{E}[m - k \tanh(\frac{k\beta x}{2})]) \,, \tag{1.2.20}$$

where $k = \sqrt{m^2 - E^2}$. Substituting $\phi(x)$ into Eq.(1.2.11), we find the expression for the Dirac vector potential $V_D(x)$:

$$V_D(x) = (\frac{\beta}{2} + 1) \frac{-k^2 E \cosh^{-2} \frac{k\beta x}{2}}{E^2 + (m - k \tanh \frac{k\beta x}{2})^2} \,, \tag{1.2.21}$$

where the eigen-energy E is to be determined by the normalization condition. Combining Eqs.(1.2.9) and (1.2.12), we find the function $\eta(x)$ to be (see appendix A for detail)

$$\begin{aligned} \eta(x) &= N(m \sin 2\phi - E)^{\frac{1}{\beta}} \\ &= N \left(\frac{k^2 E \cosh^{-2} \frac{k\beta x}{2}}{E^2 + (m - k \tanh \frac{k\beta x}{2})^2} \right)^{1/\beta} \,. \end{aligned} \tag{1.2.22}$$

The normalization condition

$$\int_{-\infty}^{+\infty} \eta^2(x)\, dx = \int_{-\infty}^{+\infty} N \left(\frac{k^2 E \cosh^{-2} \frac{k\beta x}{2}}{E^2 + (m - k \tanh \frac{k\beta x}{2})^2} \right)^{2/\beta} dx = 1 \qquad (1.2.23)$$

gives the relation between E and the normalization constant N, and $\beta > 0$ is always required to ensure the normalizability.

Finally, the Fermion eigenstate may be expressed as

$$\begin{aligned} q(x) &= \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \\ &= N \frac{(k^2 E \cosh^{-2} \frac{k\beta x}{2})^{1/\beta}}{[E^2 + (m - k \tanh \frac{k\beta x}{2})^2]^{1/\beta + 1/2}} \begin{pmatrix} E \\ m - k \tanh \frac{k\beta x}{2} \end{pmatrix} , \end{aligned} \qquad (1.2.24)$$

and the soliton is found to be

$$\begin{aligned} \theta_c(x) &= 2(1 + \frac{2}{\beta})\phi(x) + \theta_0 \\ &= 2(1 + \frac{2}{\beta}) \arctan[\frac{1}{E}(m - k \tanh \frac{k\beta x}{2})] + \theta_0 , \end{aligned} \qquad (1.2.25)$$

where θ_0 may be adjusted to satisfy the condition

$$\theta_c(-\infty) = -\theta_c(+\infty) .$$

Since

$$\theta_c(\infty) \to 2(1 + \frac{2}{\beta}) \arctan(\frac{m-k}{E}) + \theta_0 \ , \qquad (1.2.26a)$$

and

$$\theta_c(-\infty) \to 2(1 + \frac{2}{\beta}) \arctan(\frac{m+k}{E}) + \theta_0 \ , \qquad (1.2.26b)$$

the integral constant θ_0 is given by

$$\theta_0 = -(1 + \frac{2}{\beta})\frac{\pi}{2} ,$$

which is the same as that in the zero-mode. The topological charge of the soliton may then be calculated from Eq.(1.2.19) as:

$$Q = \frac{1}{\pi}(1+\frac{2}{\beta})(\arctan(\frac{m-k}{E}) - \arctan(\frac{m+k}{E}))$$
$$= -\frac{1}{\pi}(1+\frac{2}{\beta})\arctan\frac{k}{E}\,. \qquad (1.2.27)$$

Obviously, the absolute value of the topological charge is less than that of the zero-mode for a given β.

To find the eigen-energy explicitly, we need to carry out the integration of Eq.(1.2.23). We find that, only for some special values of β, which gives integer values for n with $n = \frac{2}{\beta} - 1 \geq 0$, the integral can be calculated exactly to get the energy expressions. Here we give the first few expressions of the energy for $n = 0,\ 1,\ 2$ and 3, i.e., $\beta = 2,\ 1,\ 2/3$, and 1/2: (detailed analytical calculations are given in appendix A)

$\beta = 2$,

$$E = m\cos\frac{1}{N^2}\,, \qquad (1.2.28)$$

$\beta = 1$,

$$k - E\arctan\left(\frac{k}{E}\right) = \frac{1}{2N^2}\,, \qquad (1.2.29)$$

$\beta = \frac{2}{3}$,

$$(m^2+2E^2)\arctan\frac{k}{E} - 3kE = \frac{2}{3N^2}\,, \qquad (1.2.30)$$

$\beta = \frac{1}{2}$,

$$\frac{k}{3E^4m^2}(3E^2-m^2)(m^2E^4+16m^6-E^6) + \frac{k}{2}(2m^2-3E^2)$$
$$-\frac{E}{2}(2E^2+3m^2)\arctan\left(\frac{k}{E}\right) = \frac{1}{4N^2}\,, \qquad (1.2.31)$$

where $k = \sqrt{m^2-E^2}$ and the normalization constant N may be considered as a free parameter.

1.3 THE RELATION OF OUR MODEL TO THE MASSIVE THIRRING MODEL

It is interesting to note that, for the choice of $\beta = 2$, the nonzero energy solution of $E \neq 0$ takes the special form:

$$q(x) = \frac{NkE^{1/2}}{\cosh(kx)(E^2 + (m - k\tanh(kx))^2)} \begin{pmatrix} E \\ m - k\tanh(kx) \end{pmatrix} , \tag{1.3.1}$$

$$V_D(x) = \frac{-2\,k^2 E}{E^2\cosh^2(kx) + (m\cosh(kx) - k\sinh(kx))^2} , \tag{1.3.2}$$

$$\theta_c(x) = 4\arctan\frac{1}{E}(m - k\tanh(kx)) + \theta_0 , \tag{1.3.3}$$

with the energy

$$E = m\cos\frac{1}{N^2} . \tag{1.3.4}$$

On the other hand the zero-mode of $E = 0$, we have the simple form

$$q_0(x) = (\frac{m}{\pi})^{1/2}\frac{1}{\cosh(2mx)} \begin{pmatrix} e^{mx} \\ e^{-mx} \end{pmatrix} , \tag{1.3.5}$$

$$\theta_c(x) = 4\arctan(e^{-2mx}) + \theta_0 . \tag{1.3.6}$$

These are exact solutions for the massive Thirring model with the standard form of Lagrangian

$$\mathcal{L}_{MT} = \bar{q}i\gamma^\mu\partial_\mu q + \frac{\lambda^2}{2}(\bar{q}\gamma_\mu q)^2 - m\bar{q}q , \tag{1.3.7}$$

provided that the coupling constant λ satisfies the condition $\lambda^2 = \frac{2}{N^2}$. Obviously, $\lambda^2 = \pi$ gives the zero-mode. The equivalence of these two models may be understood by comparing the equation of motion generated from our Lagrangian Eq.(1.2.3) with that from the MT model Lagrangian Eq.(1.3.7). We verify that they are indeed equivalent to each other if we recognize the relation (Kogut and Susskind, 1975)

$$-\frac{1}{2}\epsilon_{\mu\nu}\partial^\nu\theta_c = \lambda^2(\bar{q}\gamma_\mu q) . \tag{1.3.8}$$

This may be easily checked with the solutions of Eqs.(1.3.1) to (1.3.6) (see appendix A for detail).

1.4 SCALAR FIELD LAGRANGIAN

From the static soliton solution to the scalar field of $E \neq 0$ (i.e. Eq.(1.2.25)), we can derive the scalar field potential from the well-known relation (Lee, 1981)

$$U(\theta_c) = \frac{1}{2}\left(\frac{d\theta_c}{dx}\right)^2 \tag{1.4.1}$$

Bearing in mind that the vector potential in the Dirac equation is also related to the soliton as seen from Eq.(1.2.7), namely, $V_D = \frac{1}{2}\frac{d\theta_c}{dx}$, we obtain,

$$U(\theta_c) = 2V_D^2 \ . \tag{1.4.2}$$

Note furthermore that $\theta_c = 2(1+\frac{2}{\beta})\phi$ + constant. We combine Eqs.(1.2.10) and (1.2.11), and obtain a simple relation between V_D and ϕ,

$$V_D = -(\frac{\beta}{2}+1)(m\sin(2\phi) - E)\ , \tag{1.4.3a}$$

leading to, a relation between U and ϕ:

$$U(\theta_c) = 2(\frac{\beta}{2}+1)^2(m\sin(2\phi) - E)^2\ . \tag{1.4.3}$$

The Lagrangian for the soliton is readily written down:

$$\begin{aligned}\mathcal{L}_{\theta_c} &= \frac{1}{2}\partial_\mu\theta_c\partial^\nu\theta_c - U(\theta_c) \\ &= \frac{1}{2}4(1+\frac{2}{\beta})^2\partial_\mu\phi\partial^\mu\phi - 2(\frac{\beta}{2}+1)^2(m\sin(2\phi) - E)^2.\end{aligned} \tag{1.4.4}$$

When divided by the overall factor $4(1+\frac{2}{\beta})^2$, Eq.(1.4.4) becomes

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\beta^2}{8}(m\sin(2\phi) - E)^2 \tag{1.4.5}$$

where β and m are parameters, and E is the eigen-energy of the Fermion. So, for $E = 0$, the Lagrangian becomes sine-Gordon model, which consistents with Eq.(1.2.18). The energy of the soliton ϕ may be calculated from the formula

$$E_\phi = \int_{-\infty}^{\infty} \left(\frac{1}{2} (\frac{d\phi}{dx})^2 + U(\phi) \right) dx \ , \tag{1.4.6}$$

which gives

$$E_\phi = \frac{\beta}{2}(k - E \arctan(\frac{k}{E})) \ . \tag{1.4.7}$$

Thus, the energy of the soliton θ_c is obtained:

$$\begin{aligned} E_{\theta_c} &= 4(1+\frac{2}{\beta})^2 E_\phi \\ &= 2\left(\beta(1+\frac{2}{\beta})^2 (k - E \arctan(\frac{k}{E}) \right) \ , \end{aligned} \tag{1.4.8}$$

where $k = \sqrt{m^2 - E^2}$. Because the soliton is coupled to a Fermion, the eigen-energy of the Fermion will affect the energy of the soliton. For example, as seen from the above result, for a Fermion with zero energy, the soliton reaches its maximum energy of $2\beta m(1+\frac{2}{\beta})^2$. On the other hand, when the Fermion bound state energy approaches its maximum value m, the energy of the background soliton approaches its minimum value, i.e., zero.

1.5 VACUUM FERMION CHARGE

The vacuum Fermion charge (VFC) of a system is defined to be the Fermion charge of the filled Dirac sea of the interacting system relative to the corresponding Fermion charge of a reference system. Conventionally the reference system is chosen to be the system defined by the free Hamiltonian H_0. The VFC may then be expressed as:

$$Q_f = \int_{-\infty}^{+\infty} (\rho^-(x) - \rho_0^-(x)) \, dx \ . \tag{1.5.1}$$

Here $\rho^-(x)$ is the negative energy state density with

$$\rho^-(x) = \sum_{i,\epsilon_i<0} \psi_i^\dagger(x)\psi_i(x) \qquad (1.5.2)$$

for the interacting system and $\rho_0(x)$ is that for the free system. Since the number of energy eigenstates is invariant, namely,

$$\int_{-\infty}^{+\infty} \rho(x)\, dx = \int_{-\infty}^{+\infty} \rho_0(x)\, dx, \qquad (1.5.3)$$

where

$$\rho(x) = \rho^+(x) + \rho^-(x)\,, \qquad (1.5.4)$$

with

$$\rho^+(x) = \sum_{i,\epsilon_i>0} \psi_i^\dagger(x)\psi_i(x)\,. \qquad (1.5.5)$$

and the free Dirac Hamiltonian is charge conjugate and parity reflection invariant, the density of the noninteracting system has the property that

$$\rho_0^+(x) = \rho_0^-(x)\,. \qquad (1.5.6)$$

The VFC Q_f of Eq.(1.5.1) may therefore be expressed in the form

$$Q_f = \frac{1}{2}\int_{-\infty}^{+\infty} (\rho^-(x) - \rho^+(x))\, dx\ . \qquad (1.5.7)$$

Thus, the VFC Q_f is a measure of the spectral asymmetry of the system. In this form, Q_f is no longer tied to the reference system, as is in the original definition Eq.(1.5.1).

It is proved by Ma, Nieh and Su (Ma *et al*, 1985) that, in the case of a vector coupling, the fractional part of Eq.(1.5.7) may be calculated from the vector potential in the Dirac equation, V_D. Apart from an integer contribution, the VFC is shown to be

$$Q_f = \frac{1}{\pi}\int_{-\infty}^{+\infty} V_D(x)\, dx\ . \qquad (1.5.8)$$

For the case of chiral coupling, on the other hand, the VFC is related to the chiral angle $\theta_c(x)$ due to the soliton effect, the relation is given by Goldstone and Wilczek (Goldstone and Wilczek, 1981)

$$Q_f = \frac{1}{2\pi}(\theta_c(+\infty) - \theta_c(-\infty)). \tag{1.5.9}$$

Eqs.(1.5.8) and (1.5.9) are consistent due to the relation $V_D = \frac{1}{2}\frac{d\theta_c}{dx}$. However, Eq.(1.5.8) is more general. Note that Eq.(1.5.8) is exactly the same as the topological charge of the background soliton. From our solution Eq.(1.2.25), we obtain

$$Q_f = -\frac{1}{\pi}(1 + \frac{2}{\beta})\arctan(\frac{k}{E}) \ . \tag{1.5.10}$$

Thus we have obtain a fractional VFC (generally other than $\frac{1}{2}$) contribution for the chiral coupling model. For $\beta = 2$, which corresponds to the MT model with $E = m\cos\frac{\lambda^2}{2}$ and $k = \sqrt{m^2 - E^2} = m\sin\frac{\lambda^2}{2}$, the VFC reduces to

$$Q = -\frac{2}{\pi}\arctan(\tan\frac{\lambda^2}{2}) = -\frac{\lambda^2}{\pi}, \tag{1.5.11}$$

where λ^2 is the coupling constant in the MT model. For a positive bound state, $0 \le \lambda^2 \le \pi$. The result shows that the zero-mode (i.e. $\lambda^2 = \pi$) in the MT model give rise to an integer other than $\frac{1}{2}$ fractional VFC, as is the case for scalar coupling (Jackiw and Rebbi, 1975). For non zero-mode, it gives a fractional value.

1.6 BOSONIZATION

It was proven perturbatively by Coleman in 1975 that the quantum SG model

$$\mathcal{L}_{SG} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{\alpha_0}{\beta_0^2}(1 - \cos(\beta_0\sigma)) \ , \tag{1.6.1}$$

is equivalent to the MT model

$$\mathcal{L}_{MT} = \bar{q} i \gamma^\mu \partial_\mu q + \frac{\lambda^2}{2} (\bar{q} \gamma_\mu q)^2 - m \bar{q} q \ , \tag{1.6.2}$$

provided that the coupling constants satisfy the relation (Coleman, 1975)

$$1 + \frac{\lambda^2}{\pi} = \frac{4\pi}{\beta_0^2} \ . \tag{1.6.3}$$

Later, Mandelstam (Mandelstam, 1975), Ha (Ha, 1984) and Noan (Noan, 1985) proved the same result in model independent ways — i.e. an operator solution of the field equation in one theory may be constructed by means of a nonlocal expression of the operator of another. Specifically, the Fermi fields F assume the form $F \sim e^B$, where B are the Bose operators, hence the anticommutation relations for the Fermi fields F appear if the commutator of the Bose operator is equal to a c-number $i\pi$: i.e,

$$F_1 F_2 = e^{B_1} e^{B_2} \equiv e^{B_2} e^{B_1} e^{[B_1, B_2]} = e^{B_2} e^{B_1} e^{i\pi} = -F_2 F_1. \tag{1.6.4}$$

More generally, the Bose-Fermi transformation is made possible through the following equivalences: (See, for example, Ha, 1984)

$$i\bar{q}\gamma^\mu \partial_\mu q \leftrightarrow \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma, \tag{1.6.5}$$

$$\bar{q} q \leftrightarrow -\frac{1}{\pi\alpha} \cos\left(\sqrt{4\pi}\sigma\right) , \tag{1.6.6}$$

$$i\bar{q}\gamma_5 q \leftrightarrow \frac{1}{\pi\alpha} \sin\left(\sqrt{4\pi}\sigma\right) , \tag{1.6.7}$$

$$\bar{q}\gamma_\mu q \leftrightarrow \frac{1}{\sqrt{\pi}\alpha} \epsilon^{\mu\nu} \partial_\nu \sigma \ , \tag{1.6.8}$$

$$\bar{q}\gamma^5 \gamma_\mu q \leftrightarrow \frac{1}{\sqrt{\pi}\alpha} \partial^\mu \sigma \ , \tag{1.6.9}$$

where $\frac{1}{\alpha}$ is an ultraviolet cutoff and α will be taken to zero at the end of any calculation involving a Fermi field.

Applying these transformations to the model we are considering, we obtain the bosonized version of our Lagrangian Eq.(1.2.3):

$$\mathcal{L}_B = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2\sqrt{\pi}\alpha}\partial_\mu\sigma\partial^\mu\theta_c + \frac{m}{\pi\alpha}\cos(\sqrt{4\pi}\sigma) \ , \tag{1.6.10}$$

where θ_c is the static background soliton given by Eq.(1.2.18) or Eq.(1.2.25), i.e.

$$\theta_c(x) = 2(1+\frac{2}{\beta})\arctan(e^{\beta m x}) + \theta_0 \ , \tag{1.6.11}$$

or,

$$\theta_c(x) = 2(1+\frac{2}{\beta})\arctan[\frac{1}{E}(m - k\tanh\frac{k\beta x}{2})] + \theta_0 \ , \tag{1.6.12}$$

where $k = \sqrt{m^2 - E^2}$, β and E may be considered as parameters. This bosonized Lagrangian should describe the same system as described by its Fermionic version. This is obvious for $\beta = 2$.

1.7 SUMMARY AND DISCUSSION

In this chapter on (1+1)-dimensional classical field theory, we demonstrated in detail that for a Fermion chirally coupled to a soliton of a certain form, there is always a bound state solution. We obtained the background soliton solution and the fermionic bound state simultaneously. We discussed the relationship of our model to the massive Thirring model. We derived explicitly the scalar field potential which supports the static background soliton. We also calculated the induced vacuum Fermion charge due to the polarization of the Dirac sea. Using Fermion-Boson transformation formulae, we obtained a bosonized version of our model.

It is interesting to note that the model we studied is a general one in the following sense. For a special choice of our parameter (i.e., $\beta = 2$), it reduces to the MT model and hence the chiral model studied by Chang, Ellis and Lee (Chang *et al*, 1975); for another choice of the parameter (i.e., $\beta \to \infty$), the model reduces to the "infinite thin soliton" model studied by Mackenzie and Wilczek (Mackenzie and Wilczek, 1984) and Nogami and Beachey (Nogami and Beachey, 1986). The former equivalence has been shown in section 3. The latter equivalence is explored in more detail here.

The model studied by Mackenzie and Wilczek and Nogami and Beachey may be effectively described by the Dirac potential $V_D = \theta\,\delta(x)$, where θ is a coupling constant. Therefore, the static soliton coupled to a Fermion is given by a step function $\theta_c(x) = \theta\frac{x}{|x|}$. The even-parity bound state energy is found to be $E = m\cos\theta$ for $0 \le \theta \le \pi$, and the vacuum Fermion charge is found to be (Bhaduri, 1987) $Q = -\frac{\theta}{\pi}$ for positive energy state $0 \le \theta \le \frac{\pi}{2}$.

In our model, when $\beta \to \infty$, the general solution of Eqs.(1.2.21) and (1.2.25) indeed gives a Delta function potential V_D and a step function soliton θ_c (see appendix A for detail), but the eigen-energy is not obtainable from the normalization condition of Eq.(1.2.23). However, we can solve the equation of motion directly by taking the limit $\beta \to \infty$ in Eq.(1.2.12) and obtain the relation:

$$E = m\sin(2\phi(x)) \ .$$

Since $\phi(x) \sim -\frac{1}{2}\theta_c(x)$, the only possible topological soliton solution may be obtained:

$$2\phi(x) = -\theta\frac{x}{|x|} + \frac{\pi}{2} \ .$$

It follows that

$$\theta_c(x) = -2\phi(x) + \theta_0 = \theta\frac{x}{|x|} \ ,$$

where we have chosen the integral constant θ_0 to be π. This yields a step function and its derivative is a delta function. The vector potential is obtained as

$$V_D(x) = \phi'(x) = -\theta\, \delta(x).$$

By taking $E = m\cos\theta$, $k = \sqrt{m^2 - E^2} = m\sin\theta$ and $\beta \to \infty$ in Eq.(1.5.8), we find the vacuum Fermion charge

$$Q_f = -\frac{1}{\pi}\arctan\frac{k}{E} = -\frac{\theta}{\pi}, \quad 0 \le \theta \le \frac{\pi}{2}.$$

The results are the same as those obtained by Mackenzie and Wilczek and Nogami and Beachey. For the zero-mode of $\theta = \frac{\pi}{2}$, the VFC is $\frac{1}{2}$ which is the same as that in the case of scalar coupling (Jackiw and Rebbi, 1976). Notice also that the VFC of Eq.(1.5.8), calculated from the formula $Q_f = \frac{1}{\pi}\int_{-\infty}^{\infty} V_D\, dx$, is valid only in the case of having a positive energy bound state. When the bound state energy becomes negative, the real vacuum charge should be understood as $1+Q_f$ because of the filling up of one more negative energy state in the Dirac sea. This situation is well discussed in the paper by Mackenzie and Wilczek (Mackenzie and Wilczek, 1984).

Chapter 2

Anyon Gas In A Harmonic Oscillator Potential In 2+1 Dimensions

2.1 INTRODUCTION

Particles in two spatial dimensions may have arbitrary spin and, therefore, obey fractional statistics (Leinaas and Myrheim, 1977, Goldin *et al*, 1980 and 1981, Wilczek, 1982, Laughlin, 1983, Haldane, 1983, Halperin, 1984 and Fröhlich and Marchetti, 1991) as a result of having only a trivial rotational symmetry group. This may be explained in a simple example: Assume that there are two identical particles moving in space and that the wave function of the system is given by $\psi(1,2)$. Further, assume that the particles do not overlap each other. Then, for the exchange of particle 1 and 2, the resulting wave function acquires a phase, say, $e^{\pm i\gamma\pi}$. The phase factor contribution of the particles moving around the center point is different for clockwise ("+" sign), or anti-clockwise ("-" sign), if the particles are constrained on a plane, whereas there is no difference if they move in three dimensional space. As illustrated in Fig.2.1, in 3-space dimensions the path (*a*) is identical to the path (*b*) if we rotate the path around the axis which connects the two particles.

It then follows that $e^{-i\gamma\pi} = e^{+i\gamma\pi}$, and γ can take only odd or even integers. This gives Fermi or Bose statistics. However, there is no constraint on γ for the two space-dimensional case. Consequently, particles moving in two space dimensions may have arbitrary spin and multivalued wave function and therefore correspond to fractional statistics. Particles with fractional spins are called anyons. The statistics of these particles is expected to interpolate continuously between normal Boson and Fermion limits. Extensive studies have suggested that anyons may be responsible for the occurrence of the Fractional Quantum Hall Effect (FQHE) (Halperin, 1984, Arovas *et al*, 1984, Girvin and MacDonald, 1987 and Laughlin, 1990) and may have some connection to high temperature Superconductivity (Laughlin, 1988, Fetter *et al*, 1989 and Chen *et al*, 1989).

The quantum mechanics of anyons is complicated by the fact that their wave functions are multivalued. One can avoid this problem by treating anyons as Bosons or Fermions having single valued wave functions but interacting via a long-range Aharonov-Bohm type vector potential of appropriate strength. Starting with noninteracting Fermions, for example, this statistical interaction transmutes them to anyons and then to Bosons as the strength parameter varies from zero to one. Properties of noninteracting anyons in the above scheme have been studied extensively (For a recent review, see Mackenzie and Wilczek, 1988, Canright and Girvin, 1990 and Canright and

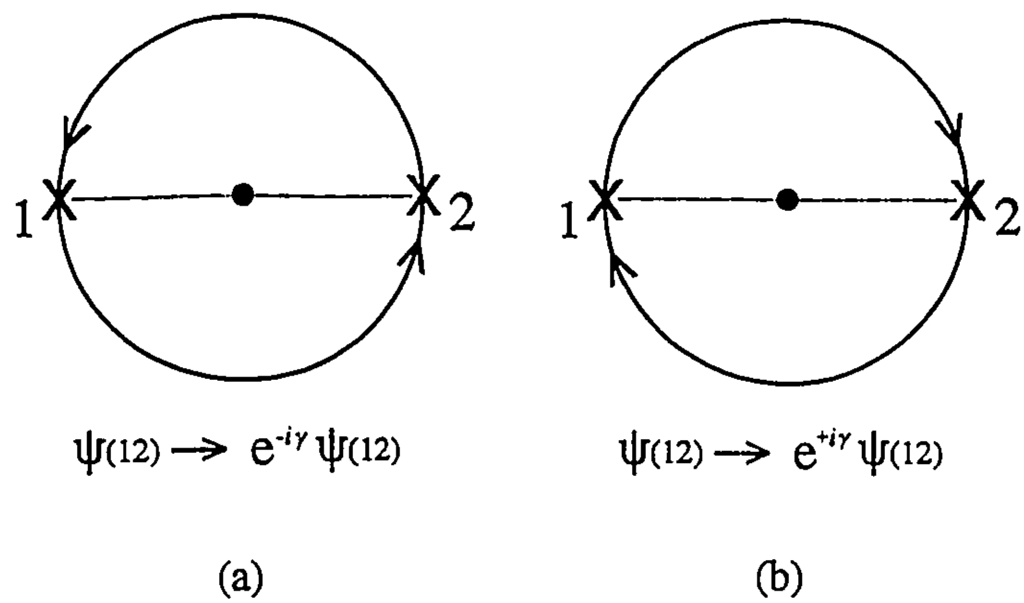


Fig.2.1: Schematical illustrations for anyons in two space dimensions.

Johnson, 1990). The interpolation of quantum states between fermionic and bosonic systems has been established for two (Leinaas and Myrheim, 1977) and three (Sporre *et al*, 1991 and Murthy *et al*, 1991) particles in a harmonic oscillator potential. Properties of an ideal gas of anyons in an external magnetic field have been studied in the Hartree-Fock mean-field (Hanna *et al*, 1989), Random-Phase-Approximation (Fetter *et al*, 1989) and in hydrodynamical (Wen and Zee, 1990) approaches. The behaviors of an anyon gas in other external potentials have also been studied by many groups. (see, for example, Suzuki *et al*, 1991, Dunne *et al*, 1991, Ma and Chang, 1991 and He *et al*, 1992).

In this chapter, we study a many-anyon system confined in a harmonic oscillator potential. We make the semi-classical Thomas-Fermi approximation to the interacting fermion system. We investigate how the system changes when the statistical interaction is switched on, and whether the system of Fermions (at $\alpha = 0$) interpolates to a system of Bosons (at $\alpha = 1$) for this case. The ground state Fermion energy, the angular momentum, and the spatial distributions of particle density are studied in the TF model. The chapter is organized as follows: In section 2, the formalism is outlined. In section 3, we review the Thomas-Fermi (TF) method. In section 4, we present the calculations for the anyon system. The results and conclusions are given in the last section.

2.2 THE FORMALISM

A very nice description of the concept of the anyon and its formalism may be found in the book by Wilczek (Wilczek, 1990) and, for example, in the papers by Hanna, Laughlin and Fetter (Hanna *et al*, 1989) and by Date, Krishna and Murthy (Date *et al*, 1992). Here we give a brief description of the anyon and the nonrelativistic formalism which we will use in the following discussions.

As we already noted in the introduction, "anyons" are particles obeying fractional statistics and are described by a multi-valued wave function. In particular, the multi-valued wave function Ψ is of the form

$$\Psi(\mathbf{r}_1, ..., \mathbf{r}_N, ...) = \prod_{i<j} e^{i\alpha\theta_{ij}} \psi(\mathbf{r}_1, ..., \mathbf{r}_N, ...) \,, \qquad (2.2.1)$$

where $\mathbf{r}_i$ is a coordinate representing the location of the ith particle in the x-y plane, α is an arbitrary real parameter (which serves as a coupling constant as we will see later), $\theta_{ij} = \arctan \frac{y_i - y_j}{x_i - x_j}$ is the azimuthal angle of $\mathbf{r}_{ij}$, and ψ is a single-valued Fermion (or Boson) wave function. For simplicity, we assume that the particles are spinless, and α varies between 0 and 1 , while $\hbar = c = e = 1$. Consider a nonrelativistical free anyon-gas obeying the Schrödinger equation

$$H\Psi(\mathbf{r}_1, ..., \mathbf{r}_N, ...) = E\Psi(\mathbf{r}_1, ..., \mathbf{r}_N, ...) \,, \qquad (2.2.2)$$

with H the free-particle Hamiltonian

$$H = \sum_{i=1} \frac{|\mathbf{p}_i|^2}{2m} \,. \qquad (2.2.3)$$

Note that $\mathbf{p} = -i\nabla$ and

$$\begin{aligned} \nabla_k \Psi(\mathbf{r}_1, ..., \mathbf{r}_N, ...) &= [\prod_{i<j} e^{i\alpha\theta_{ij}} \nabla_k + (\nabla_k \prod_{i<j} e^{i\alpha\theta_{ij}})] \psi(\mathbf{r}_1, ..., \mathbf{r}_N, ...) \\ &= \prod_{i<j} e^{i\alpha\theta_{ij}} (\nabla_k + [\nabla_k \ln(\prod_{i<j} e^{i\alpha\theta_{ij}})]) \psi(\mathbf{r}_1, ..., \mathbf{r}_N, ...) \,, \end{aligned} \qquad (2.2.4)$$

which can be rewritten as

$$\nabla_k \Psi(\mathbf{r}_1, ..., \mathbf{r}_N, ...) = \prod_{i<j} e^{i\alpha\theta_{ij}} [\nabla_k \psi(\mathbf{r}_1, ..., \mathbf{r}_N, ...) + i\,\alpha \sum_{j\neq k} \frac{\hat{\mathbf{z}} \times (\mathbf{r}_k - \mathbf{r}_j)}{|\mathbf{r}_k - \mathbf{r}_j|^2} \psi(\mathbf{r}_1, ..., \mathbf{r}_N, ...)] \,. \tag{2.2.5}$$

Therefore, we have

$$\nabla_k [\prod_{i<j} e^{i\alpha\theta_{ij}} \psi(\mathbf{r}_1, ..., \mathbf{r}_N, ...)] = \prod_{i<j} e^{i\alpha\theta_{ij}} D_k \psi(\mathbf{r}_1, ...\mathbf{r}_N, ...) \,, \tag{2.2.6}$$

where $D_k\psi = (\nabla_k + i\,\mathbf{A}_k)\psi$, and

$$\mathbf{A}_k = \alpha \sum_{j\neq k} \hat{\mathbf{z}} \times \frac{(\mathbf{r}_k - \mathbf{r}_j)}{|\mathbf{r}_k - \mathbf{r}_j|^2} \,. \tag{2.2.7}$$

Equation (2.2.2) may then be rewritten as

$$H'\psi = E\psi \,, \tag{2.2.8}$$

with an interacting Hamiltonian given by

$$H' = \sum_{i=1} \frac{1}{2m} (\mathbf{p}_i + \mathbf{A}_i)^2$$

where $\mathbf{A}_i$ defined by Eq.(2.2.7). It can be seen that a *free* anyon gas may be described as an *interacting* Fermion (or Boson with α replaced by $1 - \alpha$) gas with the "statistical" vector interactions. This vector interaction $\mathbf{A}$ induces a "fictitious" magnetic field B (along the $\hat{\mathbf{z}}$ direction) and a flux Φ according to $B(\mathbf{r}) = \nabla \times \mathbf{A} = 2\pi\alpha \sum_i \delta(\mathbf{r} - \mathbf{r}_i)$ and $\Phi = \int \nabla \times \mathbf{A}\, d\mathbf{r} = \oint \mathbf{A} \cdot d\mathbf{l} = N\alpha 2\pi$ respectively for N particles. Thus, fractional-statistics particles (namely, anyons) may be pictured as Fermions (or Bosons) acting as though each particle carried a solenoid with a fraction α of a flux quantum (Hanna *et al*, 1989).

To facilitate our discussion, we adopt the interacting picture to describe anyons and choose Fermions as the base particles interacting through the statistical interaction in the following calculations.

2.3 THE THOMAS-FERMI METHOD

Because of the complexity of the N-anyon problem, an appropriate approximation may be employed. In particular, we utilize the semi-classical Thomas-Fermi method which was originally developed for solving the self-consistent potential field in an atom (Thomas, 1926 and Fermi, 1928). The TF method for atoms in three space dimensions is outlined clearly by March (March, 1975). Following his approach, we generalize the method for an interacting many-Fermion system in two space dimensions (for a two space dimensional application for atoms, see the paper by Bhaduri *et al*, 1990).

Specifically, the ground state of the many-Fermion system may be obtained by letting Fermions fill the lowest energy states according to the Pauli principle up to a maximum energy, say, μ. This energy is called the Fermi energy, or chemical potential. For a given (effective) single-particle potential $V(\mathbf{r})$, the fastest moving Fermion, with total energy μ, may be described by the classical energy equation

$$\mu = \frac{p_f^2}{2m} + V(\mathbf{r})\,. \tag{2.3.1}$$

Here, m is the mass and p_f the maximum momentum of the Fermion (p_f is generally termed Fermi momentum). It may be shown that the Fermi energy of a stable system must be independent of r. This in turn implies that p_f must vary in space, since $V(\mathbf{r})$ depends on position. Therefore, we may rewrite Eq.(2.3.1) as

$$p_f^2(\mathbf{r}) = 2m\,[\mu - V(\mathbf{r})]\,. \tag{2.3.2}$$

Recall that the number of particles in the system is equal to the volume of the occupied phase space multiplied by spin degeneracy, namely, in two spatial dimensions,

$$N = (\text{spin deg.}) \times \frac{1}{(2\pi\hbar)^2} \int d^2r \int_{|p|<p_f} d^2p \,. \tag{2.3.3}$$

In particular, for spinless particles, *spin deg.* $= 1$. With the foregoing discussion in mind, replacing the mean particle density $\frac{N}{\int d^2r}$ by the local density $\rho(\mathbf{r})$, the relation to the Fermi momentum gives

$$\rho(\mathbf{r}) = \frac{1}{(2\pi\hbar)^2}\pi p_f^2(\mathbf{r}) \,. \tag{2.3.4}$$

(In the rest, we have taken the value of $\hbar = 1$.) From Eqs.(2.3.2) and (2.3.4), we obtain the TF relation between density $\rho(\mathbf{r})$ and potential energy $V(\mathbf{r})$:

$$\rho(\mathbf{r}) = \frac{m}{2\pi}(\mu - V(\mathbf{r})) \,. \tag{2.3.5}$$

The above equation is valid only in the classically allowed region, i.e., where $V(\mathbf{r}) \leq \mu$. The point $\mathbf{r}_0$, which is found from $V(\mathbf{r}_0) = \mu$, is called the classical "turning point". We define $\rho(\mathbf{r}) = 0$ in the classically forbidden region. The chemical potential in the Eq.(2.3.5) is to be determined from the normalization condition, that is

$$\int \rho(\mathbf{r}) \, d^2r = N \,, \tag{2.3.6}$$

where N is the particle number of the system. For a given potential $V(\mathbf{r})$, one can find $\rho(\mathbf{r})$ from Eq.(2.3.5). Note that when the potential $V(\mathbf{r})$ itself is a function of $\rho(\mathbf{r})$ (for example, in an atomic case, $V(\mathbf{r})$ obeys Poisson's equation $-\nabla^2 V = 4\pi e^2 \rho$, hence it depends on ρ.), self-consistency has to be invoked in solving for the density.

It is known that the TF method is suitable for calculating the ground state energy of the system. Therefore, it is desirable to find the expression for the ground state energy in terms of the particle density. To get the kinetic energy, note that its mean value for N-Fermion system in a volume $\mathcal{V}$ is given by

$$\bar{T} = N \int_0^{p_f} \frac{p^2}{2m} \frac{2\pi p dp}{\pi p_f^2} = \frac{1}{2} N \frac{p_f^2}{2m} . \tag{2.3.7}$$

The kinetic energy density is therefore

$$\mathcal{T} = \frac{N}{\mathcal{V}} \frac{p_f^2}{4m} . \tag{2.3.8}$$

Replacing $N/\mathcal{V}$ by the local density $\rho(\mathbf{r})$, and noting from Eq.(2.3.4) that p_f^2 is proportional to ρ, we get

$$T = \frac{\pi}{m} \int \rho^2(\mathbf{r}) d\mathbf{r} . \tag{2.3.9}$$

The potential energy is given by

$$V = \frac{1}{2} \int \int U(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}) \rho(\mathbf{r}') d\mathbf{r} d\mathbf{r}' + \int V(\mathbf{r}) \rho(\mathbf{r}) d\mathbf{r} , \tag{2.3.10}$$

where the factor of $\frac{1}{2}$ is put to avoid double counting for the pairwise two-body interacting potential $U(\mathbf{r}, \mathbf{r}')$, and $V(\mathbf{r})$ is a single particle potential. The final expression for the total energy of the system in two dimensional space is thus given by

$$E = \frac{\pi}{m} \int \rho^2(\mathbf{r})\, d\mathbf{r} + \frac{1}{2} \int \int U(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}) \rho(\mathbf{r}')\, d\mathbf{r} d\mathbf{r}' + \int V(\mathbf{r}) \rho(\mathbf{r}) d\mathbf{r} . \tag{2.3.11}$$

This completes the review of the TF method. In the following, we will use this method to find the spatial density distributions for the N-anyon system and calculate the ground state energy for the Fermions with the interaction coupling constant $0 \leq \alpha \leq 1$.

2.4 THE TF APPROACH FOR A MANY-ANYON SYSTEM IN A HARMONIC OSCILLATOR POTENTIAL

In this section, we study a finite system of N anyons which are confined in an external harmonic oscillator potential. As stated in the previous sections, we treat the anyons as Fermions but interacting through a statistical vector interaction. We first derive the effective single particle Hamiltonian for the system and then apply the TF method to solve the particle density. Note that due to the special nature of the vector interaction, the TF relation is modified.

The effective single particle Hamiltonian of the many-anyon system

We start with a many-body Hamiltonian of N-Fermions confined in a harmonic oscillator one-body potential

$$V(r) = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}m\omega^2(x^2+y^2) ,$$

and interacting through a statistical potential $\mathbf{A}(\mathbf{r})$:

$$H = \sum_{i=1}^{N} \frac{(\mathbf{p}_i + \mathbf{A}_i)^2}{2m} + \sum_{i=1}^{N} V(r_i) \quad , \tag{2.4.1}$$

where

$$\mathbf{A}_i = \sum_{j(\neq i)=1}^{N} \mathbf{A}_{ij} \, , \quad \mathbf{A}_{ij} = \alpha \frac{\hbar c}{e} \frac{\hat{\mathbf{z}} \times (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} \quad , \tag{2.4.2}$$

are the statistical interacting vector potentials. The notation is essentially the same as that in the paper by Hanna *et al.* (Hanna *et al.*, 1989). For convenience, we set $\hbar c/e = 1$. Now define an auxiliary scalar function (Li and Bhaduri, 1991)

$$\phi(\mathbf{r}_i) = \sum_{j(\neq i)=1}^{N} \ln|\mathbf{r}_i - \mathbf{r}_j| \, . \tag{2.4.3}$$

The components of **A** may be expressed as

$$A_a(\mathbf{r}_i) = -\alpha\epsilon_{ab}\partial^b\phi(\mathbf{r}_i) \ , \tag{2.4.4}$$

where the indices i, j refer to particles, and a, b to the components (x, y), and $\epsilon_{xy} = -\epsilon_{yx} = 1$. To obtain the mean-field Hamiltonian, we define a single-particle density $\rho(\mathbf{r})$ with normalization $\int \rho(\mathbf{r})\, d^2r = N$, and make the mean-field approximation to $\phi(\mathbf{r})$ from Eq.(2.4.3), i.e.,

$$\phi(\mathbf{r}) = \int \rho(\mathbf{r}')\ \ln|\mathbf{r}-\mathbf{r}'|\ d^2r' \ . \tag{2.4.5}$$

Notice that the magnetic field in two space dimensions is defined by

$$B(\mathbf{r}) = (\nabla \times \mathbf{A}(\mathbf{r}))_z \ ,$$

and from Eqs.(2.4.4) and (2.4.5) the mean magnetic field is given by

$$B(\mathbf{r}) = 2\pi\alpha\rho(\mathbf{r}) \ .$$

This proportionality of the mean magnetic field to the spatial density is a basic property in Chern-Simons theory (Chen *et al*, 1989 and Jackiw and Pi, 1990). Assuming a cylindrically symmetric density distribution $\rho(r)$ for the fermionic $L = 0$ system, the angular integration in Eq.(2.4.5) can be easily performed, yielding

$$\phi(r) = 2\pi[\,\ln r \int_0^r \rho(r')\, r'dr' + \int_r^\infty \ln r' \rho(r')\, r'dr'] \ . \tag{2.4.6}$$

It follows that $d\phi/dr = N(r)/r$, where

$$N(r) = 2\pi \int_0^r \rho(r')\, r'dr' \tag{2.4.7}$$

is just the particle number in an area within radius r. In term of $N(r)$, we have therefore

$$\frac{\partial\phi}{\partial x} = \frac{d\phi}{dr}\frac{\partial r}{\partial x} = \frac{x}{r^2}N(r) \, ,$$
$$\frac{\partial\phi}{\partial y} = \frac{d\phi}{dr}\frac{\partial r}{\partial y} = \frac{y}{r^2}N(r) \, .$$

The mean vector potential Eq.(2.4.4) may be rewritten in a short form due to the above relations:

$$\mathbf{A}(\mathbf{r}) = \frac{\alpha N(r)}{r^2}\hat{\mathbf{z}} \times \mathbf{r} \, .$$

Note that

$$\nabla \cdot \mathbf{A} = 0$$

and

$$\begin{aligned} \mathbf{A} \cdot \mathbf{p} &= \frac{\alpha N(r)}{r^2}(\hat{\mathbf{z}} \times \mathbf{r}) \cdot \mathbf{p} \\ &= \frac{\alpha N(r)}{r^2}(\mathbf{r} \times \mathbf{p})_z \\ &= \frac{\alpha N(r)}{r^2} l \, , \end{aligned}$$

where $l = xp_y - yp_x$ is the single-particle angular momentum operator. The mean potential is therefore angular momentum dependent. The effective Hamiltonian of a single particle with angular momentum l is then given by

$$\begin{aligned} h &= \frac{1}{2m}(\mathbf{p} + \mathbf{A})^2 + \frac{1}{2}m\omega^2 r^2 \\ &= \frac{\mathbf{p}^2}{2m} + \frac{\alpha^2}{2mr^2}N^2(r) + \frac{\alpha l}{mr^2}N(r) + \frac{1}{2}m\omega^2 r^2 \, . \end{aligned}$$

Note that the single-particle angular momentum is not quantized in the semi-classical Thomas-Fermi method, because only the particle density, not the wave function, is considered. The term linear in l in the above equation breaks time-reversal symmetry at the mean field level, consequently the spatial density for a partial wave, $\rho_l(r)$, is different for positive or negative l. Writing $\mathbf{p}^2 = p_r^2 + l^2/r^2$, we obtain (Li *et al*, 1992)

$$h = \frac{p_r^2}{2m} + \frac{(l + \alpha N(r))^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2 \ . \tag{2.4.8}$$

The classical energy equation for the most energetic particle with partial wave Fermi momentum $p_f^{(l)}$ and chemical potential μ for this Hamiltonian is therefore given by

$$(p_f^{(l)})^2 = 2m(\mu - m^2\omega^2 r^2 - \frac{(l + \alpha N(r))^2}{r^2}) \ . \tag{2.4.9}$$

Our task will be to find the relation between $p_f^{(l)}$ and the local particle density $\rho(r)$.

The extended Thomas-Fermi relation

Recall that from phase space considerations, we have

$$N = \int d^2r \frac{1}{(2\pi\hbar)^2} \int_{|p|<p_f} d^2p \ .$$

Note also that $rd^2p = rdp_r dp_\theta = dp_r dl$, where $dl = rdp_\theta$, it follows that

$$N = \int 2\pi dr \frac{1}{(2\pi\hbar)^2} \int_{p_r<p_f^{(l)}} dp_r dl = \int N_l \, dl \ ,$$

where we have defined

$$N_l = \frac{1}{2\pi\hbar^2} \int dr \int_{p_r<p_f^{(l)}} dp_r = \frac{1}{\pi\hbar^2} \int dr p_f^{(l)} \ .$$

Therefore, the mean partial wave density may be identified as

$$\rho_l = \frac{N_l}{\int dr} \ , \tag{2.4.10}$$

and the desired local relation is obtained (recall that $\hbar = 1$):

$$\pi\rho_l(r) = p_f^{(l)}(r) . \tag{2.4.11}$$

It is straightforward to derive the relation between the total density $\rho(r)$ and the partial wave density $\rho_l(r)$. Namely, from the normalization condition:

$$N = \int \rho(r)\, 2\pi r dr = \int N_l\, dl = \int \rho_l(r)\, dl dr ,$$

we find the relation

$$\rho(r) = \frac{1}{2\pi r} \int \rho_l(r)\, dl . \tag{2.4.12}$$

Substituting for $p_f^{(l)}$ in Eq.(2.4.9), we obtain the partial wave TF relation for $\rho_l(r)$

$$\pi\rho_l(r) = [\, 2m\mu - (m\omega r)^2 - \frac{(l + \alpha N(r))^2}{r^2}]^{1/2} . \tag{2.4.13}$$

At this point, it is more convenient to use the following dimensionless quantities

$$x = \sqrt{m\omega} r\ , \ \ \rho_l(x) = \sqrt{m\omega}\rho_l(r)\ , \ \ \rho(x) = (m\omega)^{-1}\rho(r)\ , \ \ N(x) = N(r)\ , \ \ \mu' = 2\mu/\omega$$

to simplify the notation. Furthermore, substituting $\rho_l(x)$ from Eq.(2.4.13) into Eq.(2.4.12), we get

$$\rho(x) = \frac{1}{2\pi^2 x} \int_{-l_{max}}^{l_{max}} (\mu' - x^2 - \frac{(l + \alpha N(x))^2}{x^2})^{1/2} dl . \tag{2.4.14}$$

At $l = l_{max}$, the above integrand goes to zero ($0 \leq \alpha \leq 1$). The integration is easily carried out to yield

$$\rho(x) = \frac{1}{4\pi^2 x^2} \left(x^2(\mu' - x^2)[\frac{\pi}{2} + \arcsin(1 - \frac{2\alpha N(x)}{x\sqrt{\mu' - x^2}})] \right.$$
$$\left. + 2[x\sqrt{\mu' - x^2} - 2\alpha N(x)][\alpha N(x)(x\sqrt{\mu' - x^2} - \alpha N(x))]^{\frac{1}{2}} \right) . \tag{2.4.15}$$

We finally obtain the extended TF relation. The above density is valid only within the classically allowed region. When $x > x_0$, the density is defined as zero. The chemical potential μ' and the "turning point" x_0 may be solved from the relations

$$x_0\sqrt{\mu' - x_0^2} = \alpha N \ , \tag{2.4.16}$$

and

$$\int_0^{x_0} \rho(x) 2\pi x dx = N \ , \tag{2.4.17}$$

where $N = N(x_0)$ is the total particle number in the potential well.

Once the spatial density $\rho(x)$ is obtained, the ground state energy of the interacting N-Fermion system with $L = 0$ is readily calculated. In appendix B, we give the detailed derivation of the relation between total energy and spatial density in two spatial dimensions. For simplicity, we do not consider the exchange effect. Due to the long-range property of the potential, we expect that the exchange effect may not be very important, although this should be verified. The resulting relation is given by

$$E = \pi \int \rho^2(x) d^2x + \int V(x)\rho(x)\, d^2x + \frac{1}{2}\int U(x)\rho(x) d^2x + \Delta E \ , \tag{2.4.18}$$

where E is in units of ω,

$$V(x) = \frac{1}{2}x^2 = V(r)/\omega \ ;$$

$$U(x) = \frac{\alpha^2 N^2(x)}{2x^2} = U(r)/\omega$$

and ΔE the contribution from the linear l-dependent term in Eq.(2.4.8). This is given by

$$\begin{aligned} \Delta E &= \frac{1}{2}\int dx \int_{-l_{max}}^{l_{max}} dl \ \frac{\alpha l N(x)}{x^2} \rho_l(x) \qquad (2.4.19) \\ &= \frac{1}{2}\int dx \frac{\alpha N(x)}{\pi x^3} \left(\frac{8}{3}[\alpha N(x)(x\sqrt{\mu' - x^2} - \alpha N(x))]^{3/2} - \alpha N(x) 2\pi^2 x^2 \rho(x) \right) . \end{aligned}$$

The multiplicative factor of $\frac{1}{2}$ in the integral involving $U(x)$ in Eq.(2.4.18) and in ΔE above has been put in to avoid double counting, since these terms were derived from a mean-field approximation to the pair potential (see Eqs.(2.4.4) and (2.4.5)). Note, from Eq.(2.4.9), that $(l+\alpha N(x))$ may be considered to be the effective angular momentum of an anyon (without statistical interaction) moving in the original harmonic potential. (Of course, the total angular momentum of Fermions in the system is always conserved and is zero in the ground state.) So the total angular momentum of the "anyons" in the system is computed as follows:

$$\begin{aligned} < L_{eff} > &=< l+\alpha N(r) >= \int dr \int (l+\alpha N(r))\rho_l(r)dl \\ &= \int dr \frac{8}{3}[\alpha N(r)(r\sqrt{\mu' - r^2} - \alpha N(r))]^{3/2} . \end{aligned} \tag{2.4.20}$$

2.5 A TOY TEST MODEL

Before carrying out the numerical computation for the energy and density, we intend to give an idea of how accurate this extended TF approximation is. To this end, we apply the method to a simple solvable model—a model of N anyons constrained to move on a circle. Consider N Fermions with an interaction $\dot{\theta}_{ij}$ in the Lagrangian in 2+1 dimensions:

$$\mathcal{L} = \frac{1}{2}\sum_i(\dot{r}_i^2 + r_i^2\dot{\theta}_i^2) + \alpha\sum_{i<j}\dot{\theta}_{ij} , \tag{2.5.1}$$

where $0 \leq \alpha \leq 1$ is the interaction coupling strength. The particle mass is taken to be unity for simplicity and $\theta_{ij} = \arctan\frac{y_i - y_j}{x_i - x_j}$ is the azimuthal angle of $\mathbf{r}_i - \mathbf{r}_j$. The corresponding Hamiltonian is given by

$$\mathcal{H} = \sum_i \frac{1}{2}(\mathbf{p}_i + \mathbf{A}_i)^2 , \tag{2.5.2}$$

with an interacting vector potential

$$\mathbf{A}_i = \alpha \sum_{j(\neq i)} \frac{\hat{z} \times (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} . \tag{2.5.3}$$

This is the same Hamiltonian as that considered in section 4. The Lagrangian Eq.(2.5.1) or the Hamiltonian (2.5.2) in fact describes an anyon gas in 2+1 dimensions. When we constrain these particles to move on a circle, the particles are no longer anyons but normal particles moving in an effectively one dimensional space. We can easily show this by expressing $\dot{\theta}_{ij}$ in terms of $\dot{\theta}_i$ and $\dot{\theta}_j$ by an algebraic method (see appendix B for details):

$$\dot{\theta}_{ij} = \frac{1}{2}(\dot{\theta}_i + \dot{\theta}_j) . \tag{2.5.4}$$

Here $\theta_i = \arctan \frac{y_i}{x_i}$ is the azimuthal angle of the i-th particle.

Since

$$\begin{aligned}\sum_{i<j}(\dot{\theta}_i + \dot{\theta}_j) &= \frac{1}{2}\sum_{i\neq j}(\dot{\theta}_i + \dot{\theta}_j) \\ &= \frac{1}{2}\sum_{i,j}(\dot{\theta}_i + \dot{\theta}_j) - \frac{1}{2}\sum_{i=j} 2\dot{\theta}_i \\ &= \sum_{i,j}\dot{\theta}_i - \sum_{i}\dot{\theta}_i \\ &= (N-1)\sum_{i}\dot{\theta}_i ,\end{aligned}$$

the Lagrangian Eq.(2.5.1) is thus simplified as

$$\mathcal{L} = \frac{1}{2}\sum_{i}\dot{\theta}_i^2 + \frac{\alpha(N-1)}{2}\sum_{i}\dot{\theta}_i , \tag{2.5.5}$$

where $r_i = r_j = 1$ is used. It is clear that the particles are decoupled. The system becomes free. However, we will show in next section that the system may also be described by a Hamiltonian with an interacting vector potential.

The Hamiltonian derived from above simplified Lagrangian is easily found

$$\mathcal{H} = \frac{1}{2}\sum_i (p_{\theta i} - \frac{\alpha}{2}(N-1))^2 , \tag{2.5.6}$$

where $p_{\theta i} = -i\frac{d}{d\theta_i}$. This Hamiltonian is trivial, and its single particle spectrum ϵ is given by

$$\epsilon_l = \frac{1}{2}(l - \frac{\alpha}{2}(N-1))^2 , \tag{2.5.7}$$

where l is the single particle angular momentum quantum number and must be taken as an integer. For a system with total angular momentum zero, the ground state is given by

$$E_{exa} = \sum_{l=-l_{max}}^{l=l_{max}} \frac{1}{2}(l - \frac{1}{2}\alpha(N-1))^2 , \tag{2.5.8}$$

where $l_{max} = \frac{N-1}{2}$ for an odd number of particles and $l_{max} = \frac{N}{2}$ for an even number of particles. Carrying out the summation over l, we obtain

$$E_{exa} = \begin{cases} \frac{N(N^2-1)}{24} + \frac{\alpha^2 N(N-1)^2}{8}, & \text{if } N = \text{odd;} \\ \frac{N(N+1)(N+2)}{24} + \frac{\alpha^2 N(N-1)^2}{8}, & \text{if } N = \text{even.} \end{cases} \tag{2.5.9}$$

For large N, E_{exa} may be approximated as

$$E_{exa} \simeq (1 + 3\alpha^2)\frac{N^3}{24} . \tag{2.5.10}$$

Interacting Hamiltonian

Starting from the Hamiltonian Eq.(2.5.2), with the constraint of $r_i = r_j = 1$, we obtain

$$\mathcal{H} = \sum_i \frac{1}{2}(\mathbf{p}_i + \mathbf{A}_i)^2 , \tag{2.5.11}$$

with

$$\mathbf{p}_i = p_{\theta i}\hat{\mathbf{e}}_\theta , \quad \mathbf{A}_i = \sum_{j(\neq i)} \alpha\hat{z} \times \hat{\mathbf{r}}_{ij} \frac{1}{2\sin\frac{\theta_i - \theta_j}{2}} , \tag{2.5.12}$$

where variables with $\hat{}$ are unit vectors indicating directions and $\hat{\mathbf{r}}_{ij} = \hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j$ represents the direction of the relative coordinates of particles i and j.

It is interesting to note that this simple (noninteracting) system may also be described as an *interacting* system in the above formalism. This equivalence enables us to test the accuracy of the Thomas-Fermi approximation for a system with interacting vector potential.

Now we are ready to deal with the interacting vector potential. By mean-field approximation, the summation in Eq.(2.5.12) may be replaced by a space integral weighted by the particle density ρ, we then have

$$\mathbf{A} = \alpha\hat{z} \times \int_0^{2\pi} \rho(\theta')\frac{\hat{\mathbf{r}} - \hat{\mathbf{r}}'}{-2\sin\frac{\theta'}{2}}d\theta' \, . \tag{2.5.13}$$

Under the assumption of particle density being independent of the angular variable as in section 4, the particle density becomes constant. Notice that the unit vector $\hat{\mathbf{r}}'$ may be decomposed along $\hat{\mathbf{r}}$ and $\hat{\mathbf{e}}_\theta$ directions as $\hat{\mathbf{r}}' = \cos\theta'\hat{\mathbf{r}} + \sin\theta'\hat{\mathbf{e}}_\theta$, where θ' is the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ and the unit vector $\hat{\mathbf{e}}_\theta = \hat{z} \times \hat{\mathbf{r}}$, which is perpendicular to $\hat{\mathbf{r}}$. Hence the integration with respect to θ' in Eq.(2.5.13) may easily be carried out, and the result gives

$$\mathbf{A} = -4\alpha\rho\hat{\mathbf{e}}_\theta \, . \tag{2.5.14}$$

Because ρ is a constant and equal to $N/(2\pi)$ from the normalization relation, the above equation gives $\mathbf{A} = -2\alpha N/\pi\mathbf{e}_\theta$. This in turn implies that the effective single particle Hamiltonian is of the form

$$h_{eff} = \frac{1}{2}(p_\theta - \frac{2\alpha N}{\pi})^2 \, . \tag{2.5.15}$$

Comparing this effective Hamiltonian h_{eff} with the exact Hamiltonian h in Eq.(2.5.6), we notice the difference between $A = -\frac{\alpha(N-1)}{2}$ and $A_{eff} = -\frac{2\alpha N}{\pi}$.

The classical energy equation for the fastest moving particle may be written as

$$\frac{1}{2}(p_{\theta f} - \frac{2\alpha N}{\pi})^2 = \epsilon_f \,, \tag{2.5.16}$$

where ϵ_f is the Fermi energy (or chemical potential) of the system and $p_{\theta f}$ is the Fermi momentum. The above equation shows that the Fermi momentum must be independent of the angular variable θ, since the Fermi energy is a constant. On the other hand, the particle density is related to the Fermi momentum through the phase space consideration (for a zero total angular momentum system):

$$N = \int_0^{2\pi} d\theta \int_{|p_\theta|<p_{\theta f}} \frac{dp_\theta}{2\pi} = 2p_{\theta f} \,. \tag{2.5.17}$$

The system has only one degree of freedom, θ, therefore the density of particles found from Eq.(2.5.17) should also be a constant:

$$\rho = \frac{N}{\int d\theta} = \frac{p_{\theta f}}{\pi} \,. \tag{2.5.18}$$

Note that in the Thomas-Fermi approximation, the expression of the kinetic energy of a system is dimension dependent. In this effectively one spatial dimensional case, the average kinetic energy of the system is given by

$$\bar{T} = \frac{N}{2p_{\theta f}} \int_{-p_{\theta f}}^{p_{\theta f}} \frac{p_\theta^2}{2} dp_\theta = N\frac{p_{\theta f}^2}{6} \,, \tag{2.5.19}$$

and the mean contribution from the linear p_θ dependent term is zero. Moreover, from Eq.(2.5.18), replacing $N/2\pi$ or $p_{\theta f}/\pi$ by the local density ρ (which is a constant in this particular case) in the above relation, the kinetic energy density is then proportional to ρ^3. The final result of the ground state energy for this system calculated from the effective Hamiltonian is thus given by (by means of the equation (2.3.10) for the potential energy)

$$E_{eff}^{TF} = \frac{\pi^2}{6}\int \rho^3 d\theta + \frac{1}{2}(\frac{2\alpha N}{\pi})^2 \int \rho \, d\theta$$
$$= \frac{N^3}{24} + \frac{\alpha^2 N^3}{\pi^2} . \qquad (2.5.20)$$

The large N limit gives

$$E_{eff}^{TF} \simeq (1 + \frac{24\alpha^2}{\pi^2})\frac{N^3}{24} . \qquad (2.5.21)$$

Discussion for the toy model

We have computed the ground state energy for N particles moving on a unit circle with total angular momentum zero exactly and by TF method. It is interesting to compare the results, i.e, Eqs.(2.5.9) and (2.5.20). For large number of particles, the ratio of $R(\alpha) = E_{eff}^{TF}/E_{exa}$ may be computed from Eqs.(2.5.10) and (2.5.21):

$$R(\alpha) = \frac{8(\pi^2 + 24\alpha^2)}{\pi^2(8 + 24\alpha^2)} . \qquad (2.5.22)$$

The accuracy obviously depends on the interaction coupling strength α. It is found quantitatively that for $1 \geq \alpha \geq 0$,

$$0.86 \leq R(\alpha) \leq 1 . \qquad (2.5.23)$$

This shows that the accuracy of applying the extended TF approximation to this system with vector interaction is better than 86%. This result is reasonable. (Of course it is not as good as the result of applying the TF method directly to the exact Hamiltonian (i.e., Eq.(2.5.6)). In that case, the ground state energy is obtained as

$$E_{TF} = \frac{N^3}{24} + \frac{\alpha^2 N(N-1)^2}{8} . \qquad (2.5.24)$$

and for large N,

$$E_{TF} \simeq (1+3\alpha^2)\frac{N^3}{24}\,, \tag{2.5.25}$$

the result is exactly the same as the large N limit for the exact solution.) It is understandable that the TF method gives a better result when being applied to a noninteracting system than that when being applied to an interacting system. In the interacting potential picture, the accuracy of 86% is fairly good considering the simplicity of this model (since there is no r dependence in this model).

In conclusion, the extended TF approximation is valid in calculating a many-body system with interacting *vector* potentials.

2.6 THE NUMERICAL METHOD AND RESULTS

With confidence in mind, we continue the numerical calculations for the ground state energy of the Fermions and the effective angular momentum of the anyons as functions of the interaction strength α. To solve equation (2.4.17) with the normalization condition (2.4.19), we choose to use a general-purpose computer program Colsys written by Ascher, Christiansean and Russell (Ascher *et al*, 1979). The program is designed for solving up to 20 coupled ordinary differential equations of up to fourth order with boundary conditions. We therefore need to convert our problem to the standard form — two coupled nonlinear differential equations of first order (or one nonlinear second order differential equation) with a two-point boundary value condition.

From Eq.(2.4.6), we have

$$\frac{dN(x)}{dx} = \pi x \rho(x)\,. \tag{2.6.1}$$

Substituting the above in Eq.(2.4.6), we obtain one nonlinear differential equation

$$\frac{dN(x)}{dx} = \frac{1}{2\pi x}\left(x^2(\mu' - x^2)[\frac{\pi}{2} + \arcsin(1 - \frac{2\alpha N(x)}{x\sqrt{\mu' - x^2}})] + 2[x\sqrt{\mu' - x^2} - 2\alpha N(x)][\alpha N(x)(x\sqrt{\mu' - x^2} - \alpha N(x))]^{\frac{1}{2}}\right), \quad (2.6.2)$$

with the boundary $N(x_0) = N$, and $N(0) = 0$. Since the boundary point x_0 is also unknown and should be solved for self-consistently, a further consideration has to be made. To eliminate the unknown quantity x_0 from the boundary condition, we introduce another variable t and define $x = x_0 t$ with t varying between the known range 0 and 1, and the unknown quantity x_0 satisfies the differential equation

$$\frac{dx_0}{dt} = 0 \quad (2.6.3)$$

Assuming $N(x) = N(t)$, and from Eq.(2.4.16), $\mu' = x_0^2(1 + \frac{\alpha^2 N^2}{x_0^4})$. we then have the following coupled equations

$$\frac{dN(t)}{dt} = \frac{x_0^4}{2\pi t}\left(t^2[(1 + \frac{\alpha^2 N^2}{x_0^4}) - t^2][\frac{\pi}{2} + \arcsin(1 - \frac{2\alpha N(t)/x_0^2}{t\sqrt{(1 + \frac{\alpha^2 N^2}{x_0^4}) - t^2}})] + 2[t\sqrt{(1 + \frac{\alpha^2 N^2}{x_0^4}) - t^2} - \frac{2\alpha N(t)}{x_0^2}][\alpha N(t)(t\sqrt{(1 + \frac{\alpha^2 N^2}{x_0^4}) - t^2} - \frac{\alpha N(t)}{x_0^2})]^{1/2}\right), \quad (2.6.4)$$

and

$$\frac{dx_0}{dt} = 0 , \quad (2.6.5)$$

with the known boundary conditions $N(0) = 0$, and $N(1) = N$. In this way we have converted the integral differential equations to coupled nonlinear ordinary differential equations with two-point boundary condition. This is now a well-defined two-point boundary value problem and can be solved easily by the computer code Colsys.

The numerical calculation is done on a SUN workstation. The numerical results (Li *et al*, 1992) of $x\rho(x)$ from solving Eq.(2.4.17) are displayed in Fig.2.2 for $\alpha = 0$, 0.5 and 1 for $N = 10$ and $N = 190$ two cases. The density profiles are very peaked at the origin as $\alpha \to 1$, consequently $x\rho(x)$ tends to be flat, especially for large N. For $\alpha = 0$, the quantum fermionic ground state is an $L = 0$ noninteracting "closed shell" nondegenerate system with the lowest states in the harmonic oscillator potential well filled according to the Pauli principle. With a non zero α, we have an interacting Fermi system with the total angular momentum L conserved, and the ground state is still nondegenerate. It is the density (and energy) of this ground state that we are calculating in the TF approximation.

Alternatively, this fermionic ground state may also be viewed as a noninteracting many-anyon state confined in the external oscillator potential. At $\alpha = 1$, this anyon-state becomes an excited noninteracting bosonic state. Note that the interacting fermionic wave function (at $\alpha = 1$) in the noninteracting picture has to be multiplied, in a quantum calculation, with an overall phase factor $\prod_{j<k} e^{\pm i\theta_{jk}}$ to convert it to a symmetric Bose state. Therefore a total angular momentum of $\pm\frac{N(N-1)}{2}\hbar$ units is imparted to the system. In our TF calculation, the effective angular momentum of Bosons for $\alpha = 1$ is calculated for various particle numbers as listed in Table (II.1). The results show that the fermionic system indeed becomes a bosonic one (when $\alpha = 1$) and acquires approximately $\frac{N(N-1)}{2}\hbar$ of angular momentum. The least energy of the excited noninteracting bosonic state, compatible with this angular momentum, is $\frac{N(N+1)}{2}\hbar\omega$. For large N, there may be many states with

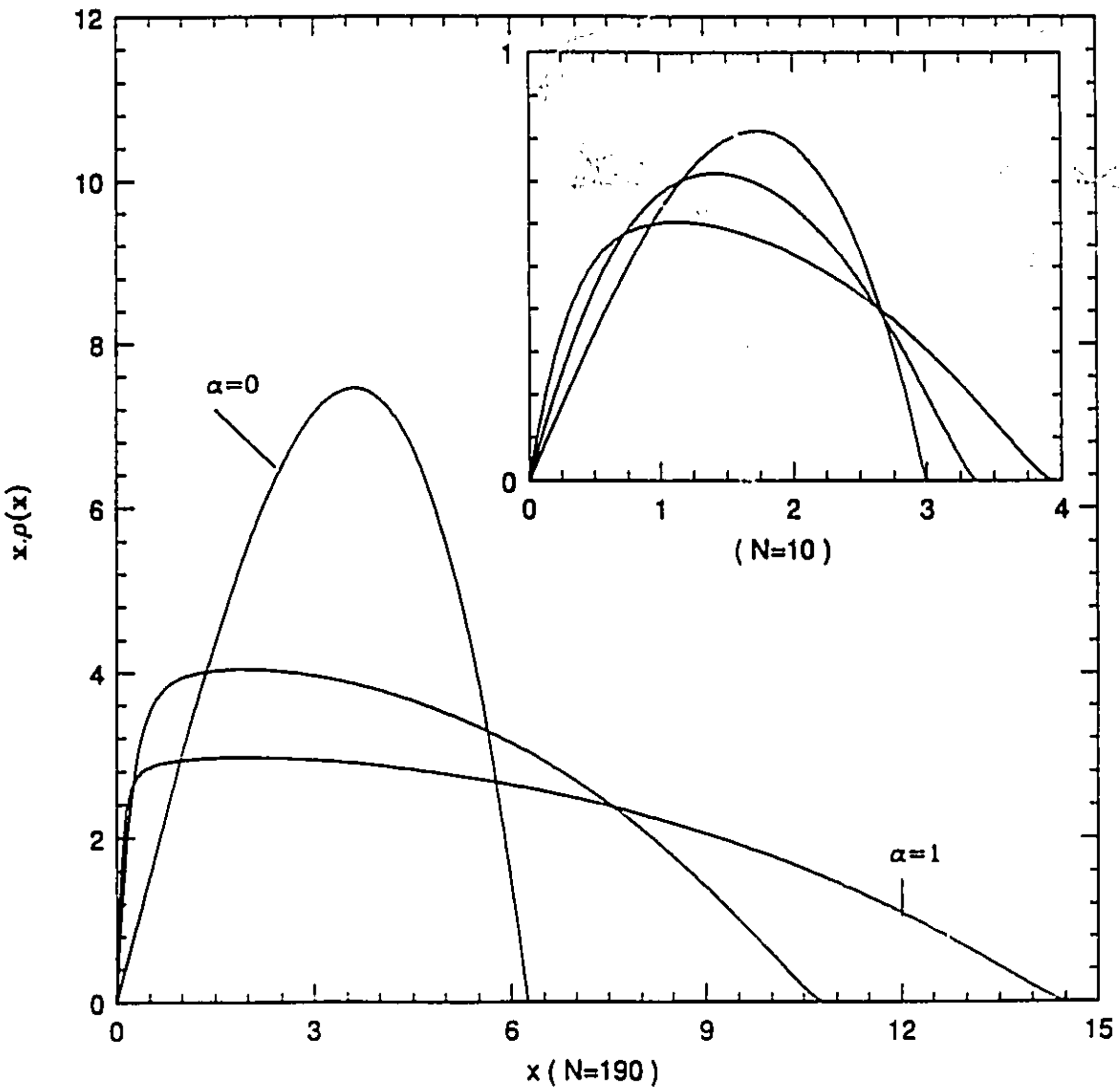


Fig.2.2: The spatial distributions of $x\rho(x)$ versus x (x is dimensionless variable) for particle number $N = 10$ (insert) and $N = 190$ are shown. The curve with the smallest intercept on the x-axis corresponds to $\alpha = 1$. Shown here are the three different curves with $\alpha = 0.0$. 0.5. and 1.0

the same angular momentum at this energy (i.e., the state is highly degenerated). The statistical interaction, however, drives each fermionic state to a unique combination of these degenerated bosonic states, even when this degeneracy is present. This has been explicitly demonstrated in the numerical quantum calculations of 3-anyon system (Sporre *et al*, 1991 and Murthy *et al*, 1991). Our calculated TF density should therefore be the same as that of the (noninteracting) interpolated bosonic state. For $N = 3$, in the quantum calculation, this bosonic state is a unique symmetrized combination of the configurations ($1s$, $1p$, $1d$), ($(1s)^2$, $1f$) and $(1p)^3$, with the center-of-mass in the ground state. The exact (normalized) wave function for three excited bosons (i.e. corresponding to $\alpha = 1$ in our case) with center of mass in the ground state, the relative energy being $5\hbar\omega$ and angular momentum $3\hbar$ is given by (Murthy *et al*, 1991)

$$|\psi>=\sqrt{\frac{1}{8\pi^3}}(3\rho_+^2 - \lambda_+^2)\lambda_+ e^{-\frac{\rho^2+\lambda^2}{2}} , \qquad (2.6.6)$$

where the relative coordinates λ and ρ are defined as

$$\lambda = \frac{1}{\sqrt{6}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3) , \qquad (2.6.7)$$

and

$$\rho = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2) . \qquad (2.6.8)$$

λ_+ and ρ_+ are the complex expressions of λ and ρ. Defining the center of mass coordinate as

$$\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 , \qquad (2.6.9)$$

we have the relation

$$\rho^2 + \lambda^2 = r_1^2 + r_2^2 + r_3^2 - 3R^2 . \qquad (2.6.10)$$

The symmetrical wave function (2.6.6) may be decomposed as the sum of wave functions $|\psi_1>$, $|\psi_2>$ and $|\psi_3>$, which correspond to states $(1s,\ 1p,\ 1d)$, $((1s)^2,\ 1f)$ and $(1p)^3$, which are defined as

$$|\psi_1>=\sqrt{\frac{1}{4\pi^3}}[z_1(z_2^2+z_3^2)+z_2(z_3^2+z_1^2)+z_3(z_1^2+z_2^2)]e^{-\frac{|z_1|^2+|z_2|^2+|z_3|^2}{2}}\,, \quad (2.6.11)$$

$$|\psi_2>=\sqrt{\frac{1}{6\pi^3}}(z_1^3+z_2^3+z_3^3)e^{-\frac{|z_1|^2+|z_2|^2+|z_3|^2}{2}}\,, \quad (2.6.12)$$

and

$$|\psi_3>=\sqrt{\frac{3}{\pi^3}}(z_1z_2z_3)e^{-\frac{|z_1|^2+|z_2|^2+|z_3|^2}{2}}\,, \quad (2.6.13)$$

respectively with $z_i = x_i + iy_i$ the complex coordinate for the i-th particle. A little algebra gives

$$|\psi>=-\frac{1}{\sqrt{3}}|\psi_1>+\frac{\sqrt{2}}{3}|\psi_2>+\frac{2}{3}|\psi_3>\,, \quad (2.6.14)$$

This implies that the probability of finding the particles in state 1 is 33%, in state 2 is 22% and in state 3, 44%. On the contrary, in our numerical TF calculation of density distribution, there is a huge peak in the origin, indicating that particles are mostly in state 2. Thus, the density distribution for the 3 particle case is not reliable. This may be due to that we have no control for the center of mass in the TF method. However, we expect that, for large N, the center of mass remains in the ground state, and the accuracy of the density distribution for large N should improve.

In Table (II.1), we also display the calculated TF ground state energy of the Fermion system with zero total angular momentum (from Eq.(2.4.18)). Two cases of noninteracting ($\alpha = 0$) and fully interacting ($\alpha = 1$) are included for particle numbers $N = 3,\ 10,\ 45,\ 105,\ 153$ and 190. The exact results for the noninteracting N-Fermion ground state in a harmonic potential are

	$\alpha = 0$			$\alpha = 1$		
N	E_{exact}	E_{TF}	$E_{<}$	E_{TF}	$< L >$	$< L_{eff} >$
3	5	4.89	6	5.82	3	3.96
10	30	29.81	55	50.13	45	47.01
45	285	284.60	1035	973.17	990	989.47
105	1015	1014.40	5565	5451.23	5460	5440.72
153	1785	1784.22	11781	11743.52	11618	11585.70
190	2470	2469.19	18145	18258.27	17955	17891.28

Table (II.I) : The results of Thomas-Fermi calculations for the energy and the effective angular momentum of N-Fermion systems. These are compared with the exact energy E_{exact} at $\alpha = 0$. and with $E_{<}$ and $< L >$ which are the lower bound of energy and compatible angular momentum of the corresponding bosonic state at $\alpha = 1$. The results for energy are in units of $\hbar\omega$ and for the effective angular momentum are in units of $\hbar$.

known, and are tabulated for comparison. It is interesting that in two spatial dimensions, the TF calculation for energy is fairly accurate even for $N = 3$, and is nearly exact (for $\alpha = 1$) for very large N. The variation in the energy (for a given N) as a function of α is also shown in Fig.2.3 for $N = 3$ and $N = 190$. For $N = 3$ the numerical quantum result (Murthy *et al*, 1991, and Sporre *et al*, 1991) is also shown for comparison.

For $N = 190$, the near-linear behavior of the interpolating curve for $\alpha > 0.2$, and also the flatness of the curve near $\alpha = 0$ (for all N) can be clearly seen. The latter feature has been noted in perturbative quantum calculations by other groups recently (Khare and McCabe, 1991 and Sen, 1991), and supports the validity of the TF results. In Fig.2.4, we plot the computed angular momentum $< L_{eff} >$ of anyons from Eq.(2.4.20) for $N = 45$ as a function of α. The interpolation between Fermion (for $\alpha = 0$) and Boson (for $\alpha = 1$) is near-linear. Finally, in Fig.2.5, we give the schematic picture of the change from a unique Fermion state to a Bosonic state which is a unique combination of the degenerated states.

2.7 DISCUSSION AND CONCLUDING REMARKS

In this chapter, we have carried out calculations for the many-anyon system in a harmonic oscillator potential in the TF approximation. The results show some interesting properties and may help us understand more deeply the behavior of the many-anyon system. The accuracy of this approximation is tested in a simple solvable vector interacting model. The result shows a fair agreement to the exact solution. Before concluding, it is interesting to note that our effective single particle Hamiltonian Eq.(2.4.9) (apart from the harmonic potential) is essentially the same as the one given by

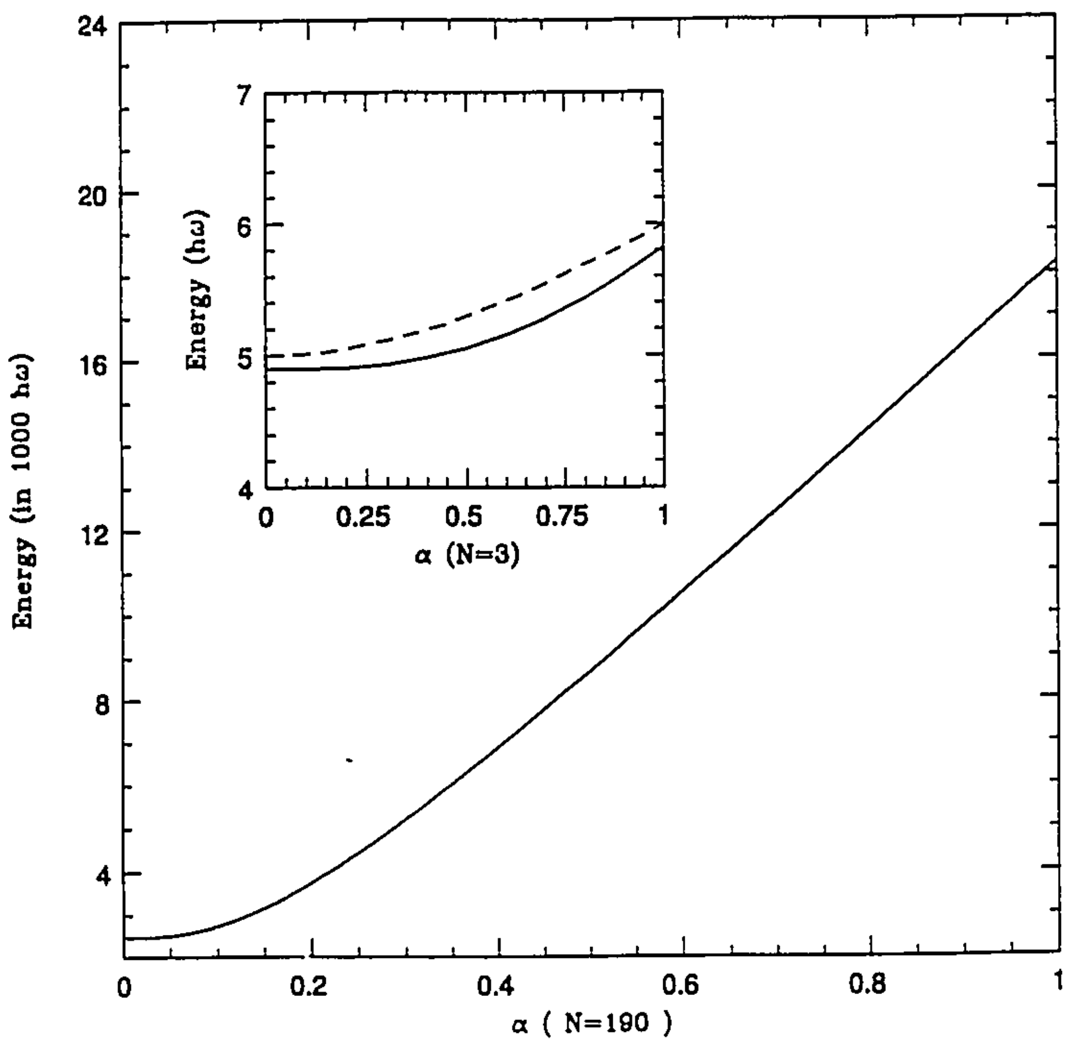


Fig.2.3: The TF energies (E's) for $N = 3$ (insert) and $N = 190$ are shown as functions of the coupling strength α. For the case of three particles, the variation in E is compared with the quantum result (dashed line in the insert) obtained from Murthy *et al.* 1991 and Sporre *et al.* 1991.

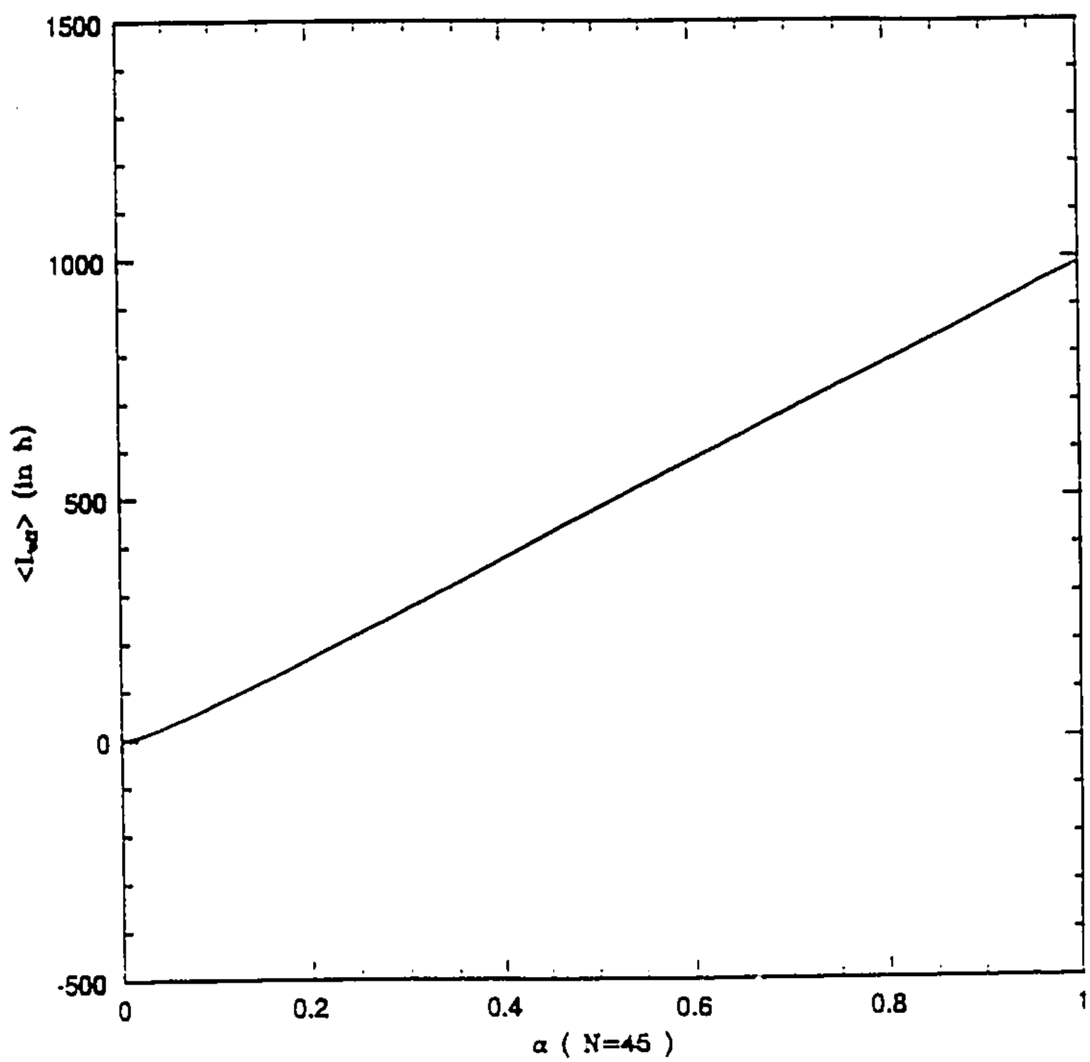
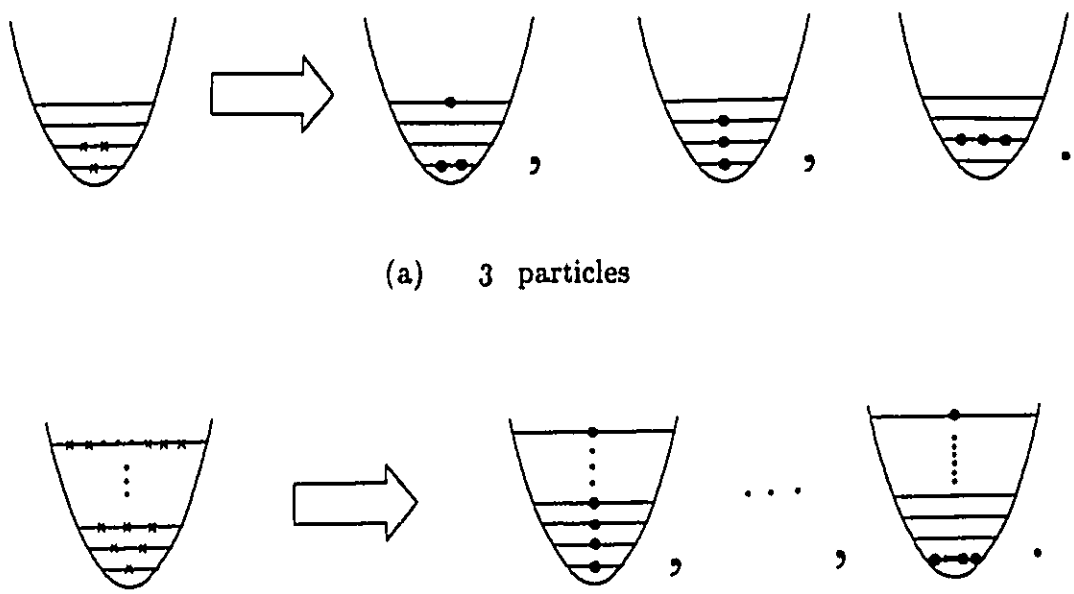


Fig.2.4: The total effective angular momentum of anyons as a function of the coupling strength α for particle number $N = 45$.



(a) 3 particles

(b) N particles

$\alpha = 0$ $\alpha = 1$

Fig.2.5: Schematic illustration of the transition from fermionic to bosonic system due to the statistical interaction. The Fermions are shown by × and Bosons by •. (a): In the case of three particles, there are three possible excited bosonic configurations which are degenerate with the right energy and angular momentum, but only a unique combination of these three states corresponds to the statistical interaction at $\alpha = 1$. (As shown in the text.) (b): For a large number of N particles, the number of the bosonic degenerate configurations is large. Shown are the only two extreme cases i.e., stretched configuration with one particle in each state up to the N-th energy level, and condensed one (far right) with $N-1$ particles in the ground state and one particle in the highest energy level to account for the angular momentum.

Jackiw and Redlich (c.f. Eq.(14b) of paper by Jackiw and Redlich, 1983)), which is derived for a single particle in a flux field generated by a solenoid. In their Eq.(14b), the single particle Hamiltonian may be extracted as

$$h = -\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{[\hbar(-i\frac{\partial}{\partial\phi}) - (\frac{e}{2\pi c})\Phi(r)]^2}{2mr^2} ,$$

where $\Phi(r)$ is the flux enclosed in the region of πr^2. Hence, the effective angular momentum of the cyon (i.e. anyon (Wilczek, 1982 and Goldhaber and Mackenzie, 1988)) is obtained:

$$L = l - \hat{\Phi}(r) ,$$

where l is an eigen value of $-i\frac{\partial}{\partial\phi}$, and is an integer in the quantum mechanic treatment, $\hat{\Phi}(r) = \Phi(r)/\Phi_0$, and $\Phi_0 = \frac{2\pi c}{e}$ is the magnetic flux quantum. In our formalism, the flux field is generated by the N anyons, thus, we would expect that

$$-\alpha N(r) \sim \Phi(r)/\Phi_0 .$$

The exact relation may be derived from the Chern-Simons relation of $B(r) = 2\pi\alpha\rho(r)$. We obtain

$$N(r) = \int_0^r \rho(r')2\pi r'dr' = \frac{1}{2\pi\alpha}B(r')2\pi r'dr' = \frac{1}{2\pi\alpha}\Phi(r) = -\frac{1}{\alpha}\hat{\Phi}(r) ,$$

where $\Phi_0 = -2\pi$, consistent with our notation. Thus the equivalence is exact. As pointed out by Wilczek in 1982, the total angular momentum of an anyon consists of two parts, one is from the contribution of a normal particle (i.e. Fermion or Boson according to which basis one is using), and the other is from that of the magnetic flux. The angular momentum of a normal particle is still conserved, but the agular momentum of an anyon may change with the change of the magnetic flux number. In our case, the magnetic flux is generated by a statistical interaction, and therefore the total angular momentum of an anyon changes with the statistical interaction strength α.

We conclude this chapter with some comments on the accuracy of the TF method. First, in our calculation, we did not include the exchange effect. Second, we notice that the statistical interaction, being a function of relative coordinates only, cannot excite the center-of-mass of the system. This constraint is not applicable to the TF approximation, and introduces errors for the particle density distribution functions in the method, especially for small particle number. The calculated ground state energy of the zero angular momentum fermionic state for $N = 3$ is still surprisingly good, but the effective angular momentum of anyons and the particle density distribution are not accurate. The accuracy should improve with large N. Our numerical results show two curious aspects. As already mentioned, the density is very peaked at the origin as $\alpha \to 1$. A bosonic state with a large fraction of particles in the s-state will also have this peaked behavior near the origin. This suggests that the bosonic state reached through the statistical interaction has this character, although caution is advocated in view of the approximations used. Finally, the nonlinear behavior of the energy for small α and near-linear behavior for $\alpha > 0.2$ for large N is somewhat unexpected, but the linear behavior of the angular momentum against α is in agreement with the argument we made before.

Chapter 3

Solitons, Vortices and Quantum Anyons

3.1 INTRODUCTION

From previous chapter, we know that an anyon may be pictured as a "charged" point particle moving, say, in the x-y plane, with a delta function "magnetic" field attached to its site carrying an associated fractional flux. This magnetic field will cause nonclassical forces away from its site, and may have significant effect quantum mechanically. In a sense this singular magnetic field at the site of an anyon is akin to a point vortex in a two dimensional fluid with singular vorticity. Indeed, in the Abelian Higgs model, when the matter field is gauged by a Chern-Simons field, quantum vortex solutions have been obtained (Paul and Khare, 1986, Hong *et al*, 1990 and Jackiw and Weinberg, 1990) that are fractionally charged, carrying an integer number of flux quanta. Such vortices are known to obey fractional statistics

(Fröhlich and Marchetti, 1989). The quantum superfluidity of a fractional statistics fluid has been examined in this connection by many groups in the last decade (Marsden and Weinstein, 1983, Chiao *et al*, 1985, Girvin, 1987 Goldin *et al*, 1987 and Wen and Zee, 1990). The excitation of anyons is intrinsically a quantum phenomenon, and these collective excitations may rightly be regarded as vortices in the above sense (Stone, 1990 and Fradkin, 1989). On the other hand, since 1990, solitonic solutions to the Chern-Simons gauge coupling have been discovered, which might be helpful for a further understanding of anyons. The solitons were first found in relativistic scalar theories (Jackiw and Weinberg, 1990 and Hong *et al*, 1990) and then found in nonrelativistic field theory (Jackiw and Pi, 1990, 1991). In both cases, static self-dual solitonic solutions were obtained for a charged self-interacting matter field coupled to the Chern-Simons gauge potentials. The matter field density in the Schrödinger equation is found to obey the Liouville equation.

In this chapter, however, we show that the self-dual solitonic solutions may also be obtained from a free Dirac field with the same Chern-Simons gauge potential. We also demonstrate that the classical equations of fluid mechanics have a close correspondence with the equations used in anyon physics. For example, the singular statistical vector potential of anyons is noted to be of the same form as the velocity vector of point vortices that arise from the solutions of the Eular equation. The anyonic Lagrangian used in the nonrelativistic theory is intimately connected (Arovas *et al*, 1985) with a nondynamical Chern-Simons gauge field. We then examine the analogy between *quantum* anyons and *classical* planar vortices of fluid mechanics. By using the analogy with classical fluid mechanics, we obtain identical solutions of the Liouville equation for fluid vortices and anyons. Furthermore, using the analogy with fluid mechanics, we are able to derive the energy-density functional of an anyonic system that yields the Liouville equation. Finally, by this analogy, we may also obtain an anyonic current that is proportional to the vector potential, which implies that the mass of the vector potential (or the "magnetic" field) is generated dynamically (Bhaduri, 1988).

The chapter is organized as follows: In section 2, we study a matter field gauged with the "Chern-Simons" potential. We show that it can support soliton solutions which are the same as those in the case of Schrödinger field. We demonstrate that the classical zero-mode solutions of the gauged Dirac equation satisfies the Liouville equation, which describes n solitons (Jackiw and Pi, 1990, 1991 and Li and Bhaduri, 1991) and may also be pictured as $2n$ vortices. In section 3, we turn to a totally different field: we study classical vortices in fluid mechanics and find similar solutions to the quantum system. In section 4, the analogy between quantum anyons and classical planar vortices of fluid mechanics is pointed out. With the analogy, we obtain another kind of velocity field for the fluid vortices, and derive an energy density functional for the anyons where the particle density satisfying the same Liouville equation (Bhaduri and Li, 1992). Conclusions and remarks are given in the last section.

3.2 PLANAR SOLITONS OF THE GAUGED DIRAC EQUATION

Jackiw and Pi (Jackiw and Pi, 1990, 1991) have obtained the static self-dual solitonic solutions of a nonrelativistic nonlinear self-interacting matter field coupled to a Chern-Simons gauge potential. The density distributions of these zero-energy solutions were shown to obey the Liouville equation. On the other hand, Mackenzie and Matheson have demonstrated perturbatively that there exist n modulo 2 zero modes in an Abelian Higgs field coupled to charged fermions plus the Chern-Simons term (Mackenzie and Matheson, 1991). We want to show that there exist infinitely many zero-mode solutions which are solitons even for the free Dirac field in the Abelian Chern-Simons gauge without Higgs field, and that the Fermion density obeys the same Liouville equation.

In the following, we use the units of $\hbar = c = 1$, and relativistic notation with the metric diag($g_{\mu\nu}$)=diag(1, −1, −1). In our notation, Greek letters represent time and space, and Roman letters represent space only. The Lagrangian density we are considering may be written as

$$\mathcal{L} = \bar{\psi}\gamma^{\mu}(i\partial_{\mu} + eA_{\mu})\psi + \frac{k}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_{\lambda} , \tag{3.2.1}$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the Abelian Chern-Simons (CS) field strength. As demonstrated by Arovas *et al*, the CS field has the effect of changing the statistics, with the induced spin of $s = \frac{e^2}{4\pi k}$ (Arovas *et al*, 1985). The Lagrangian (3.2.1) therefore describes an anyon field. The equations of motion obtained from the variation of the action with respect to gauge field read

$$B = \epsilon^{ij}\partial_i A_j = -\frac{e}{k}J^0 , \tag{3.2.2}$$

$$E^i = F^{0i} = \frac{e}{k}\epsilon^{ij}J_j , \tag{3.2.3}$$

where $J^\mu = (J^0, \mathbf{J})$ is the gauge invariant conserved four-vector current with $J^0 = \psi^\dagger\psi = \rho$ the field density and $\mathbf{J} = \psi^\dagger\alpha\psi$ the vector current. The latter is zero for the eigenstates under consideration. Eqs.(3.2.2) and (3.2.3) show that the "electromagnetic" fields E and B are in fact not independent quantities, and are determined entirely by the matter field ψ. The solutions of the potential in Eqs.(3.2.2) and (3.2.3) are formally given by

$$A^i = \partial^i\omega(\mathbf{r}) + \epsilon^{ij}\int \partial_j G(\mathbf{r}-\mathbf{r}')J^0(\mathbf{r}')\, d^2r' , \qquad (3.2.4)$$

and

$$A^0 = \partial^0\omega(\mathbf{r}) - \epsilon^{ij}\int \partial_i G(\mathbf{r}-\mathbf{r}')J_j(\mathbf{r}')\, d^2r' , \qquad (3.2.5)$$

where $G(\mathbf{r}-\mathbf{r}')$ is the Green's function of the operator ∇^2: $\nabla^2 G(\mathbf{r}) = \delta(\mathbf{r})$, which, in two dimensions, is given by $G(\mathbf{r}) = \frac{1}{2\pi}\ln|\mathbf{r}|$ and $\omega(\mathbf{r})$ is an arbitrary scalar function.

To derive the Hamiltonian density of the system, note that the canonical momenta of the system may be obtained from

$$\Pi_\psi = \frac{\delta\mathcal{L}}{\delta\dot\psi} = \bar\psi i\gamma_0 , \quad \Pi_{A_\mu} = \frac{\delta\mathcal{L}}{\delta\dot A_\mu} = \frac{k}{2}\epsilon^{0\mu\lambda}A_\lambda . \qquad (3.2.6)$$

The energy density of the system is calculated by substituting the above equations in the relation

$$\mathcal{H} = \Pi_\psi\dot\psi + \Pi_{A_\mu}\dot A_\mu - \mathcal{L}$$

and we obtain

$$\begin{aligned}\mathcal{H} &= \bar\psi i\gamma^0\partial_0\psi + \frac{k}{2}\epsilon^{0\mu\lambda}A_\lambda\partial_0 A_\mu - \bar\psi i\gamma^0\partial_0\psi - \bar\psi e\gamma^0 A_0\psi \\ &\quad - \bar\psi(i\gamma^i\partial_i + e\gamma^i A_i)\psi - \frac{k}{2}\epsilon^{\mu\nu\lambda}\partial_\mu A_\nu A_\lambda \\ &= \frac{k}{2}\epsilon^{0\mu\lambda}\partial_0 A_\mu A_\lambda - eA_0\rho - \frac{k}{2}\epsilon^{\mu\nu\lambda}\partial_\mu A_\nu A_\lambda + \psi^\dagger H_D\psi ,\end{aligned} \qquad (3.2.7)$$

with H_D, the Dirac Hamiltonian, given by

$$H_D = \alpha \cdot (-i\nabla - e\mathbf{A}) , \tag{3.2.8}$$

where $\alpha = \gamma_0\gamma$, $\beta = \gamma_0$, and the second term in Eq.(3.2.7) can be rewritten as

$$\begin{aligned} -e\rho A_0 &= k\epsilon^{0ij} A_0 \partial_i A_j \\ &= \frac{k}{2}(\epsilon^{ij0} A_0 \partial_i A_j + \epsilon^{i0j} A_j \partial_i A_0) = \frac{k}{2}\epsilon^{i\mu\lambda}\partial_i A_\mu A_\lambda , \end{aligned}$$

where the CS relation Eq.(3.2.2) is used. This term, together with the first term in the Hamiltonian density cancels exactly the third term in Eq.(3.2.7). The resulting Hamiltonian density takes the simple form

$$\mathcal{H} = \psi^\dagger H_D \psi , \tag{3.2.9}$$

and formally there is no contribution from the CS term.

For a static solution, $\psi(\mathbf{r}, t) = \psi(\mathbf{r})e^{-i\epsilon t}$, the equation of motion for the Fermion field reads

$$\alpha \cdot (-i\nabla - e\mathbf{A})\psi = (\epsilon - eA_0)\psi , \tag{3.2.10}$$

and the right hand side of above equation is zero for the zero-mode solution where A_0 is zero due to vanishing of the current in this case. Note that the Dirac Hamiltonian itself depends on the field density through the CS gauge potential $\mathbf{A}$ and hence is nonlinear, although it takes the same form as the one with "normal" gauge potential. It also obeys the following relation formally

$$\{H_D, [H_D, \beta]\} = [H_D^2, \beta] = 0 . \tag{3.2.11}$$

Consequently, all the stationary states ψ_ϵ and $\psi_{-\epsilon} = [H_D, \beta]\psi_\epsilon$ are symmetrically placed about $\epsilon = 0$. The zero-energy states, denoted by $\psi_\pm$, may be written as $\psi_\pm = \psi_0\chi_\pm$, where $\chi_\pm$ are the spin up or down Pauli spinors, and ψ_0 is a scalar function. The Hamiltonian (3.2.10) may be expressed as a matrix of the form (Aharonov and Casher, 1987):

$$H_D = \begin{pmatrix} 0 & -iD^- \\ -iD^+ & 0 \end{pmatrix} \tag{3.2.12}$$

in a representation $\alpha_1 = \sigma_1$, $\alpha_2 = \sigma_2$, $\beta = \sigma_3$, and $D^\pm = D^1 \pm iD^2$, $\mathbf{D} = (\nabla - ie\mathbf{A})$. It then follows from Eq.(3.2.10) that

$$\begin{pmatrix} 0 & -iD^- \\ -iD^+ & 0 \end{pmatrix} \psi_0\chi_\pm = 0 \, , \tag{3.2.13}$$

this yields

$$D^-\psi_0 = 0 \, , \quad or \quad D^+\psi_0 = 0 \, , \tag{3.2.14}$$

but not both, since $[D^-, D^+] = 2eB \neq 0$. These, of course, are well-known properties of the gauged Dirac equation. If we define $A^i = \partial^i\omega + \epsilon^{ij}\partial_j\phi$, where $\omega(\mathbf{r})$ is an arbitrary scalar function, then $\phi(\mathbf{r})$ satisfies the equation

$$\nabla^2\phi(\mathbf{r}) = -B = \frac{e}{k}\rho(\mathbf{r}) \, . \tag{3.2.15}$$

This in turn implies from the formal solution Eq.(3.2.4), that

$$\phi(\mathbf{r}) = \frac{e}{2\pi k}\int \ln|\mathbf{r} - \mathbf{r}'| \, \rho(\mathbf{r}') \, d\mathbf{r}' \, , \tag{3.2.16}$$

and $\phi(\mathbf{r})$ determines $\mathbf{A}$. From the Eq.(3.2.14), or

$$D_1\psi_0 = \mp iD_2\psi_0 \, , \tag{3.2.17}$$

the self-dual character of this equation was recognized by Jackiw and Pi (Jackiw and Pi, 1990, 1991) when it is written as

$$\mathbf{D}\psi_0 = \mp i\mathbf{D} \times \psi_0 \, . \tag{3.2.17a}$$

Following Jackiw and Pi, we assume the same form of the wave function for the zero-mode:

$$\psi_0(\mathbf{r}) = \sqrt{\rho(\mathbf{r})}e^{ie\omega} , \tag{3.2.18}$$

where $\rho(\mathbf{r}) = \psi_0^\dagger\psi_0$ is the density of the field and $\omega(\mathbf{r})$ is an arbitrary scalar function. From Eq.(3.2.17), the formal solution of the vector potential $\mathbf{A}(\mathbf{r})$ is obtained (see appendix C for detail):

$$\mathbf{A}(\mathbf{r}) = \nabla\omega(\mathbf{r}) \mp \frac{1}{2e}\nabla \times \ln\rho(\mathbf{r}) . \tag{3.2.19}$$

Comparing with the definition of $\phi(\mathbf{r})$ and Eq.(3.2.15), we obtain immediately the nonlinear Liouville equation for $\rho(\mathbf{r})$:

$$\nabla^2 \ln\rho(\mathbf{r}) = \mp\frac{2e^2}{k}\rho(\mathbf{r}) . \tag{3.2.20}$$

The solution to the above equation is known. As argued by Jackiw and Pi, for a physically meaningful density solution, the coefficient of the right hand side in the above equation must be negative. For simplicity, we consider only the cylindrically symmetric density solution, which is given by

$$\rho(r) = \frac{4n^2}{\alpha r^2}\left[\left(\frac{r_0}{r}\right)^n + \left(\frac{r}{r_0}\right)^n\right]^{-2} , \tag{3.2.21a}$$

or

$$\psi_0(\mathbf{r}) = e^{ie\omega}\frac{2n}{\sqrt{\alpha}r}\left[\left(\frac{r_0}{r}\right)^n + \left(\frac{r}{r_0}\right)^n\right]^{-1} . \tag{3.2.21b}$$

Consequently, the vector potential takes form

$$\begin{aligned} A^i(r) &= \partial^i\omega \pm \frac{1}{e}\epsilon^{ij}\frac{\hat{r}^j}{r}\left(n - 1 - \frac{2n}{1+(r_0/r)^{2n}}\right) \\ &= \mp\frac{1}{e}\epsilon^{ij}\frac{\hat{r}^j}{r}\frac{2|n|}{1+(r_0/r)^{2|n|}} . \end{aligned} \tag{3.2.22}$$

In the above equations, $\alpha = \frac{e^2}{|k|}$, $|n| \geq 1$ with n an integer, r_0 is an unphysical free parameter, $\omega = \pm\frac{1}{e}(|n| - 1)\theta(\mathbf{r})$ and $\theta(\mathbf{r}) = \arctan(\frac{y}{x})$ is the azimuthal angle. The choice of ω guarantees that the gauge field $\mathbf{A}$ is finite at the origin. The requirement of integer n is to ensure the singlevaluedness of the wave function. The charge of the field may be easily calculated (see appendix C)

$$Q = e \int \rho(r)\, 2\pi r dr = \frac{4\pi |n| e}{\alpha} = \frac{4\pi |k||n|}{e}\,, \tag{3.2.23}$$

which may be fractional in general. The "magnetic" flux is then given by:

$$\Phi = -\frac{k}{|k|}\frac{4\pi}{e}|n| = \pm 2|n|\Phi_0\,, \tag{3.2.24}$$

where $\Phi_0 = \frac{2\pi}{e}$ is the flux quantum. This shows that the flux number is an even integer. The solution describes n zero-energy solitons, with each soliton carrying two units of flux and fractional charge. Consequently, the zero-mode anyon field is constructed by n localized solitons with each one carrying two units of flux lines and fractional number of charges. Note that the charge we obtained (cf. Eq.(3.2.23)) is the same as that in the CS Abelian Higgs vortex model with the vortex number of $2n$ (Mackenzie and Matheson, 1991). Therefore, the anyon field may also be pictured as $2n$-vortices in a plane.

We should mention at this point that, in his earlier paper (Jackiw, 1984), Jackiw considered a similar system and no soliton solutions were obtained. The reason is that he did not solve the magnetic field self-consistently, but considered it as an external field (For example, he assumed a constant magnetic field configuration). In our case, however, we do not assume any external magnetic field, but determine it self- consistently from the solution of the matter field. We can easily show that the Liouville equation (3.2.20) does not permit a constant solution, and hence from Eq.(3.2.2) a constant magnetic field configuration is not possible. The situation is very different from that studied by Jackiw.

3.3 CLASSICAL VORTICES IN FLUID MECHANICS

Review Of Basic Equations

Consider a classical incompressible fluid whose motion is confined in the x-y plane. In the absence of viscosity and external forces, the Navier-Stokes equation reduces to Euler's equation (See, for example, the text book by Rutherford, 1959),

$$\frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho_f}\nabla p\,. \tag{3.3.1}$$

Here, $\mathbf{u}(x, y, t)$ is the fluid velocity field, p is the pressure and ρ_f is the constant incompressible fluid density. The continuity equation for the incompressible fluid is given by (Rutherford, 1959)

$$\nabla\cdot\mathbf{u} = 0\,. \tag{3.3.2}$$

(Note that this is the same as the gauge condition in the quantum mechanics.) Because of the above property, u may be expressed as the curl of an arbitrary scalar function, say $\psi(x, y)$, which is called the stream function in fluid dynamics. (Note that in two spatial dimensions, the curl of a scalar S is a vector $\mathbf{V}$, and the curl of a vector is a scalar. These are realized by the component formulae $V^i = \epsilon^{ij}\partial_j S$ and $S = \epsilon^{ij}\partial_i V_j$. A scalar may be considered as a vector with only $\hat{z}$ component.)

$$\mathbf{u} = \nabla\times\psi\,. \tag{3.3.3}$$

The vorticity ξ of a vortex is generally defined as

$$\xi = \nabla\times\mathbf{u}\,. \tag{3.3.4}$$

This is a scalar. Substituting in Eq.(3.3.4) from Eq.(3.3.3), and noticing that $\nabla \cdot \psi$ in the x-y plane is zero, the vorticity ξ may then be expressed as

$$\xi = \nabla \times (\nabla \times \psi) = -\nabla^2 \psi \,. \tag{3.3.5}$$

To obtain the equation of motion for ξ, the curl of both sides of Eq.(3.3.1) is taken:

$$\frac{\partial \xi}{\partial t} + \nabla \times [(\mathbf{u} \cdot \nabla)\mathbf{u}] = 0 \,. \tag{3.3.6}$$

The second term in the above equation has a nonzero component only along the z direction. A little algebra immediately yields (proofs given in appendix C)

$$\nabla \times [(\mathbf{u} \cdot \nabla)\mathbf{u}] = -\nabla \times (\mathbf{u} \times \xi) = \nabla \cdot (\xi \mathbf{u}) \,. \tag{3.3.7}$$

With the use of above identities, Eq.(3.3.6) reduces to

$$\frac{\partial \xi}{\partial t} + \nabla \cdot (\xi \mathbf{u}) = 0 \,. \tag{3.3.8}$$

When ξ is not explicitly time dependent, the equation for steady-state motion is given by (see for example, the text book by Milne-Thompson, 1955)

$$\nabla \cdot (\xi \mathbf{u}) = 0 \,. \tag{3.3.9}$$

Since $\nabla \cdot \mathbf{u} = 0$, Eq.(3.3.9) is equivalent to $\mathbf{u} \cdot \nabla \xi = 0$. From Eq.(3.3.3), this reduces to

$$\frac{\partial \psi}{\partial y}\frac{\partial \xi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \xi}{\partial y} = 0 \,. \tag{3.3.10}$$

A general solution for this equation may be expressed as

$$\xi = f(\psi) \,, \tag{3.3.11}$$

where $f(\psi)$ can be any function of ψ. This shows that for a nonviscous fluid in steady motion, the vorticity is a constant along a streamline (Rutherford, 1959).

Point Vortices

We now consider a class of solution of Eq.(3.3.11) satisfying

$$\xi = -\nabla^2 \psi ,$$

with ψ given by

$$\psi(\mathbf{r}) = -\frac{1}{2\pi}\sum_{j=1}^{N} c_j \ln(|\mathbf{r}-\mathbf{r}_j|/r_0) , \tag{3.3.12}$$

here c_j's are the "strengths" of the vortices, and r_0 is an arbitrary length parameter. The vorticity is zero everywhere except at certain isolated points $\mathbf{r} = \mathbf{r}_j$, which is given by the sum of the delta functions:

$$\xi(\mathbf{r}) = \sum_{j=1}^{N} c_j \delta(\mathbf{r}-\mathbf{r}_j) . \tag{3.3.13}$$

Further, the velocity vector of the ith point vortex moving in the field of the remaining $N-1$ point vortices is given by

$$u_x(\mathbf{r}_i) = \frac{\partial\psi}{\partial y_i} = -\frac{1}{2\pi}\sum_{j(\neq i)=1}^{N} c_j \frac{(y_i - y_j)}{r_{ij}^2} , \tag{3.3.14a}$$

$$u_y(\mathbf{r}_i) = -\frac{\partial\psi}{\partial x_i} = \frac{1}{2\pi}\sum_{j(\neq i)=1}^{N} c_j \frac{(x_i - x_j)}{r_{ij}^2} . \tag{3.3.14b}$$

For the special case of an isolated vortex at the origin,

$$\psi = \frac{c_0}{2\pi}\ln r , \quad u_r = 0, \quad u_\theta = \frac{c_0}{2\pi r} . \tag{3.3.15}$$

The streamlines are circles of constant r with a singularity at the origin. The "circulation" around any circle is a constant:

$$\oint \mathbf{u}\cdot d\mathbf{r} = \int_0^{2\pi} u_\theta r \, d\theta = c_0 . \tag{3.3.16}$$

The vortex is like a whirlpool, since the tangential velocity of a fluid point on the circle around the origin increases inversely to the radius.

The Liouville Equation For The Vorticity

We now consider a different class of vortices by choosing

$$\xi = f(\psi) = e^{(\frac{2}{a}\psi)}\,, \quad a > 0\,. \tag{3.3.17}$$

Here a is an arbitrary constant. This choice for $f(\psi)$ is of special interest because it leads to a Liouville equation for ξ whose solutions are known. These solutions have been studied by Jackiw and Pi (Jackiw and Pi, 1990, 1991) in connection with the Chern-Simons theory. Note that the choice of Eq.(3.3.17) automatically satisfies Eq.(3.3.10), and combining with Eq.(3.3.5) yields the Liouville equation:

$$\xi = -\nabla^2\psi = -\frac{a}{2}\nabla^2 \ln\xi\,. \tag{3.3.18}$$

The rotationally symmetric solutions were given by (Jackiw and Pi, 1990, 1991)

$$\xi(r) = \frac{4am^2}{r^2}\left[\left(\frac{r}{r_0}\right)^m + \left(\frac{r_0}{r}\right)^m\right]^{-2}\,, \quad m \neq 0\,, \tag{3.3.19}$$

where r_0 is an arbitrary scaling parameter, and m is a constant. Notice that in classical fluid dynamics, the constant m does not have to be an integer, as in the case of quantum mechanics where one must assure the singlevaluedness of wave function. The velocity field is obtained from the relation

$$\mathbf{u} = \frac{a}{2}\nabla \times \ln\xi\,. \tag{3.3.20}$$

The circulation in this case is also a constant and is given by

$$\oint \mathbf{u}\cdot d\mathbf{r} = \int_{s'} (\nabla \times \mathbf{u})\cdot ds = \int \xi\, 2\pi r dr = 4\pi|m|a\,. \tag{3.3.21}$$

Note that for large r, $\xi(r)$ of Eq.(3.3.19) falls off as $r^{-2(|m|+1)}$. The stream function for large r again goes as $\ln r$, with $u_r = 0$, $u_\theta \propto r^{-1}$. Consequently, as for an isolated vortex, the kinetic energy $\frac{1}{2}\rho_f \int u^2 d^2r$ also increases logarithmically with the radius of the vortex.

3.4 QUANTUM ANYONS AND CLASSICAL VORTICES: AN ANALOGY

The Statistical Vector Potential And The Mean Magnetic Field

Fractional statistics particles (i.e., anyons) in two spatial dimensions may be mimicked by normal particles (i.e., Bosons or Fermions) interacting with a statistical vector potential **A**. The "magnetic" field generated from this potential is given by $B = \nabla \times \mathbf{A}$. In the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$. Therefore, the vector potential may always be expressed as a curl of an auxiliary scalar function, say, ϕ, i.e., $\mathbf{A} = \nabla \times \phi$ (Li and Bhaduri, 1991). We notice that the formalism used in anyon physics is essentially the same as that used in two-dimensional classical fluid mechanics, as we have seen from the previous section. We find the following one to one correspondences: velocity field in the fluid mechanics and the statistical vector potential, i.e. $\mathbf{u} \rightarrow \mathbf{A}$; the stream function and the auxiliary function, i.e., $\psi \rightarrow \phi$; the vorticity of the classical vortex and the magnetic field of the anyon, i.e., $\xi \rightarrow B$ and finally, the circulation of the vortex and the flux of the magnetic field. In Table 3.I, we summarize those formulae used in both two dimensional classical fluid mechanics and anyon physics. Note that the statistical interaction experienced by a point particle at $\mathbf{r} = \mathbf{r}_i$ due to its interaction with the remaining $N-1$ particles is (Hanna, 1989) (in the bosonic basis)

$$\mathbf{A}(\mathbf{r}_i) = \sum_{j(\neq i)=1}^{N} \mathbf{A}(\mathbf{r}_i, \mathbf{r}_j) = \alpha \frac{\hbar c}{e} \hat{\mathbf{z}} \times \sum_{j \neq i} \frac{(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} . \tag{3.4.1}$$

Here, $\hat{\mathbf{z}}$ is a unit vector in the z-direction, and α is a parameter that is zero for noninteracting bosons, and may be varied continuously to up $\alpha = 1$. Note that Eq.(3.4.1) is identical to the point vortices vector field of fluid mechanics of

Eqs.(3.3.14a) and (3.3.14b) with the correspondance $\mathbf{u} \to \mathbf{A}$, and $\frac{c_j}{2\pi} \to \alpha(\frac{hc}{e})$. For the description of the anyon gas, the Hamiltonian is written as

$$H = \frac{1}{2m}\sum_i (\mathbf{p}_i + \frac{e}{c}\mathbf{A}(\mathbf{r}_i))^2 . \qquad (3.4.2)$$

In other words, each Boson has a "charge" e, and is interacting with the others through the vector coupling. The Boson may be thought to carry a magnetic solenoid containing a fraction α of a flux quantum. From Table III.I, the magnitude of the magnetic field B (along the $\hat{z}$ direction) corresponds to the vorticity ξ. From this analogy, using Eq.(3.3.12), and replacing $c_j \to 2\pi\alpha(\frac{hc}{e})$, we obtain

$$B(\mathbf{r}) = (\frac{hc}{e})\alpha \sum_{j=1}^{N} \delta(\mathbf{r} - \mathbf{r}_j)$$

or

$$B(\mathbf{r}) = (\frac{hc}{e})\alpha n(\mathbf{r}) , \qquad (3.4.3)$$

where $n(\mathbf{r}) = \sum_j \delta(\mathbf{r} - \mathbf{r}_j)$ is the density of bosons carrying the flux lines. This density $n(\mathbf{r})$ has *no* correspondence with the constant fluid density ρ_f of the earlier section. It is proportional to the mean magnetic field when it is gauged with a Chern-Simons vector field, but here this relationship is obtained purely from analogy with classical fluid mechanics.

The Liouville Equation

We next consider the fluid Eq.(3.3.18) for vorticity, and replace $\xi \to B$ as before. At this point it is better to be cautioned that dimensionally ξ and B are not the same. It is easily checked that the vorticity ξ has a dimension of T^{-1}, whereas the field B is of dimension $M^{1/2}T^{-1}$. Therefore, when ξ is "replaced" by B in Eq.(3.3.18), the constant "a" appearing on the right-hand

Fluid Mechanics	Anyon Physics
Velocity Field $\mathbf{u}$	Vector Potential $\mathbf{A}$
$\nabla \cdot \mathbf{u} = 0$	$\nabla \cdot \mathbf{A} = 0$
Stream function ψ	Auxiliary Function ϕ
$\mathbf{u} = \nabla \times \psi$	$\mathbf{A} = \nabla \times \phi$
Vorticity ξ	Magnetic Field B
$\xi = \nabla \times \mathbf{u}$	$B = \nabla \times \mathbf{A}$
$= -\nabla^2 \psi$	$= -\nabla^2 \phi$
$\nabla \cdot (\xi \mathbf{u}) = 0$	$\nabla \cdot (B\mathbf{A}) = 0$
Point Vortices	Anyons
$\xi(\mathbf{r}) = \sum_j^N c_j \delta(\mathbf{r} - \mathbf{r}_j)$	$B(\mathbf{r}) = \frac{hc}{e} \alpha n(\mathbf{r})$,
	$n(\mathbf{r}) = \sum_j^N \delta(\mathbf{r} - \mathbf{r}_j)$
$\mathbf{u}(\mathbf{r}_i) = \frac{1}{2\pi} \sum_{j(\neq i)} c_j \frac{\hat{z}\times(\mathbf{r}_i - \mathbf{r}_j)}{\lvert\mathbf{r}_i - \mathbf{r}_j\rvert^2}$	$\mathbf{A}(\mathbf{r}_i) = \alpha \frac{\hbar c}{e} \sum_{j(\neq i)} \frac{\hat{z}\times(\mathbf{r}_i - \mathbf{r}_j)}{\lvert\mathbf{r}_i - \mathbf{r}_j\rvert^2}$
Localized vortices	Jackiw & Pi solitons
$\xi = -\frac{a}{2} \nabla^2 (\ln \xi)$	$B = -\frac{a'}{2} \nabla^2 (\ln B)$
	$n(\mathbf{r}) = -\frac{\kappa}{2} \nabla^2 (\ln n(\mathbf{r}))$

Table III.I: Analogous equations in fluid mechanics and anyon physics. The symbols are defined in the text.

side also has to be altered too. Alternately, one could have started with the fluid Eq.(3.3.9) and replaced it by

$$\nabla \cdot (B\mathbf{A}) = 0 \, . \tag{3.4.4}$$

In either case, the result is to obtain

$$B = -\frac{a'}{2}\nabla^2(\ln B) \, , \tag{3.4.5}$$

where a' is a dimensional constant. Invoking further Eq.(3.4.3), we get

$$n(\mathbf{r}) = -\frac{\kappa}{2}\nabla^2[\ln n(\mathbf{r})] \, , \tag{3.4.6}$$

where κ is a dimensionless (positive) constant. This is the Liouville equation for the anyon density and the solutions are known. The rotationally symmetric solutions of this equation are basically the same as those given by Eq.(3.3.19), and the corresponding nonsingular vector potential is given by

$$A^i(r) = \frac{a'}{2}\epsilon^{ij}\frac{\hat{r}^j}{r}\left(m - 1 - \frac{2m}{1 + (r_0/r)^{2m}}\right) \, . \tag{3.4.7}$$

This is a self-consistent mean field effective single particle vector potential, which is nonsingular at the origin. Compared with the singular vector potential of a single anyon, $A_\theta = \alpha\frac{\hbar c}{e}\frac{1}{r}$, and $A_r = 0$, the form given by Eq.(3.4.7) is very different, just as in the fluid-mechanics case. Here the mean-field effect on the single particle potential is seen.

The Energy Density Functional

Since we are emphasizing continuum physics, it may be meaningful to ask if Eq.(3.4.6) is "derivable" from variation of an energy density functional and its connection to fluid mechanics. Specifically, consider $n(r)$ to be the smoothed-out continuous number density, satisfying the normalization condition

$$\int n(r)\, d^2r = N\,,$$

where N being the total number of anyons. Further, let $\mathcal{E}[n(r)]$ be the energy density of the system, and is a functional of $n(r)$, satisfying

$$\int \mathcal{E}[n(r)]\, d^2r = E\,,$$

where E being the total energy. Our task is to find the form of $\mathcal{E}[n(r)]$ that yields Eq.(3.4.6) from the constrained variation

$$\delta \int \{\mathcal{E}[n(r)] - \lambda n(r)\}\, d^2r = 0\,. \tag{3.4.8}$$

In the above, the system is of a finite size of radius, say, R, and the variation in $\delta n(r)$ is such that it is zero at the boundary at R. The parameter λ is a Lagrange multiplier. Assume the energy functional is of the form

$$\mathcal{E}[n(r)] = \frac{\hbar c}{r_0}\frac{(\nabla n(r))^2}{n^2(r)}\,, \tag{3.4.9}$$

with r_0 an arbitrary length scale. Substitute it into Eq.(3.4.8) and note that

$$\delta \int \frac{(\nabla n)^2}{n^2} d^2r = -2\int \frac{1}{n}(\nabla^2 \ln n)\, \delta n\, d^2r\,, \tag{3.4.10}$$

the Liouville equation (3.4.7) is therefore obtained with $\kappa = 4\hbar c/\lambda r_0$ (see appendix C). One may now ask if an energy density functional of the type given by Eq.(3.4.10) may be obtained using the analogy with fluid mechanics. For the fluid in a potential-free region, the kinetic energy density is just $\frac{1}{2}\rho_f u^2$. This implies, from the analogy, that the energy density of an anyonic system is proportional to A^2, which may be written as (see Table 3.I) $(\nabla \times \phi)^2$. From Eq.(3.3.1[illegible]), however, ϕ is proportional to $\ln B$, and $\nabla \times \phi$ is proportional to $\nabla \times \ln n$, where Eq.(3.4.3) is invoked. It then follows that the energy density is proportional to $(\nabla \times \ln n(r))^2$, which is exactly the same as Eq.(3.4.9).

The Charge Current Density

Now consider Eq.(3.4.4), i.e. $\nabla \cdot (B\mathbf{A}) = 0$, where $\mathbf{A} = (\nabla \times \phi)$. From Table 2.1 we notice that ϕ may also be expressed as a function of B, i.e.,

$$\phi = \frac{a'}{2} \ln B$$

by replacing ξ by B and ψ by ϕ in Eq.(3.3.18). Therefore, we obtain

$$B\mathbf{A} = \frac{a'}{2} B \nabla \times \ln B = \frac{a'}{2}(\nabla \times B) . \qquad (3.4.11)$$

Note that the statistical field $\mathbf{A}$ is not dynamical. Nevertheless, if we still assume that it obeys Maxwell's equation, then

$$\nabla \times B = \frac{4\pi}{c}\mathbf{J} ,$$

where $\mathbf{J}$ is the charge current density, it follows that

$$B\mathbf{A} \propto \mathbf{J} .$$

Then Eq.(3.4.4) may be regarded as a form of the continuity equation for the current. Moreover, since the field B is itself proportional to $n(r)$, hence the current density

$$\mathbf{J} = c' n(r)\mathbf{A}$$

from Eq.(3.4.11). Here c' is a constant. From this, the Maxwell equation for the vector field becomes (See, for example, Bhaduri, 1988)

$$\nabla^2 \mathbf{A} = \mathbf{J} = c' n(r)\mathbf{A} .$$

It follows that the vector field (of anyon) acquires a mass which is proportional to $\sqrt{n(r)}$. This is exactly the property of a Chern-Simons field. However, we should not confuse this with the Meissner effect of superconductivity, where one studies the response of the the system to an external applied magnetic field (i.e., the photon of the external magnetic field acquires a mass).

3.5 DISCUSSION AND CONCLUDING REMARKS

In summary, we have demonstrated that, when a Dirac Fermion field is coupled with a Chern-Simons gauge field, the zero-mode of the resulting nonlinear equation may have a soliton solution. The density of the field obeys Liouville equation, hence it has the same properties as those for the gauged matter field in Schrödinger system studied by Jackiw and Pi (Jackiw and Pi, 1990, 1991). The Chern-Simons field and the density of the system are solved self-consistently. Since the Lagrangian Eq.(3.2.1) describes an anyon field (Wilczek, 1990), the zero-mode anyon field may be pictured as n anyon-composites of n solitons with $2n$ flux lines and carrying fractional number of charges or as $2n$ vortices on a plane.

We have also shown that the classical vortices may have a soliton-like vorticity solution which satisfies the Liouville equation.

Finally, by analogy to classical fluid mechanics, we are able to find the energy functional of anyons whose number density satisfies the Liouville equation.

We conclude this chapter by the following remark: Anyonic properties arise in two-dimensional *quantum* systems, and there is *no* classical counterpart. Despite of this, we observe a striking resemblance between the velocity vector field of the classical vortices in fluid mechanics and the statistical vector potentials used in anyonic quantum mechanics. This analogy has been used to obtain many of the known equations of anyon physics. This may help in visualizing the collective excitations as vortices in a fluid, but it is not claimed here that the analogy should be taken literally or that anyons are in any sense classical. Indeed, as demonstrated earlier by Chiao *et* *al*, quantized point vortices of two dimensional fluid obey quarter fractional statistics (Chiao *et al*, 1985), indicating that the quantized vortices in the incompressible fluid are anyons.

Chapter 4

The Study Of The Nambu-Jona-Lasinio Vacuum in 3+1 Dimensions

4.1 INTRODUCTION

It is commonly accepted that Quantum Chromodynamics (QCD) is the best candidate for the underlying theory of the strong interactions. The essential features of QCD include the following aspects: asymptotic freedom — at short distance, the quark-gluon interaction coupling strength becomes small; confinement — at large distance, the coupling strength of quark and gluon becomes large, hence quarks and gluons cannot go free; chiral symmetry in the light quark sector — i.e., the theory is invariant under chiral rotation; and the spontaneously breaking of chiral symmetry — the QCD vacuum is not chiral invariant. Understanding of how chiral symmetry breaks spontaneously is essential to the understanding of the lightest hadron. It is known that the QCD vacuum is characterized by nonzero quark- and gluon-condensates (Shifman *et al*, 1979). The light-quark condensate reflects spontaneous chiral symmetry breaking whereas the gluon condensate arises from the breaking of scale-invariance of QCD due to quantum corrections (Shifman *et al*, 1978 and Shuryak, 1988). The melting of the quark-condensate with increasing temperature or density is linked to the restoration of chiral symmetry, while the vanishing of the gluon condensate is connected to quark deconfinement. Because of the nonlinearity and nonperturbativity of the QCD at low energy, it is practically impossible to calculate the hadronic properties from first principles. As a result, one has to look for other ways in order to understand the hadronic spectrum. Attempts have been made to solve QCD numerically, i.e., using computer simulation by lattice gauge theory. However, this method is limited by approximation schemes due to the finiteness of the sample points in the computation, and the predicted results may not be very reliable. Another alternative is to build a suitable model which contains one or more of the essential features of QCD and which is solvable. One of the models is the quark version of the Nambu-Jona-Lasinio model. The model has the features of chiral symmetry in the light quark sector and exhibits the spontaneous breaking of chiral symmetry along with dynamical generation of Fermion mass. Due to the attractive features of the NJL model, we are particularly interested in studying the vacuum properties at finite temperatures and observing how the spontaneously broken chiral symmetry is restored.

We give a short review of the history of the NJL model. In the early 1960s, a Fermion Lagrangian model was proposed by Nambu and Jona-Lasinio (NJL). The model contains four-Fermion interactions and the Lagrangian density is given by

$$\mathcal{L} = \bar{\psi} i \gamma_\mu \partial^\mu \psi + g[(\bar{\psi}\psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2] . \tag{4.1.1}$$

This Lagrangian is invariant under the chiral transformation $\psi \to e^{-i\gamma_5 \theta/2}\psi$ by noting that

$$(\bar{\psi}\psi) \to (\bar{\psi}\psi)\cos\theta - (\bar{\psi} i\gamma_5 \psi)\sin\theta , \tag{4.1.2}$$

$$(\bar{\psi} i\gamma_5 \psi) \to (\bar{\psi} i\gamma_5 \psi)\cos\theta + (\bar{\psi}\psi)\sin\theta . \tag{4.1.3}$$

It was shown that the model can give rise to spontaneous breaking of chiral symmetry with dynamical generation of an effective Fermion mass, analogous to that in the BCS superconductivity theory in condensed matter physics. The breaking of the continuous chiral symmetry is also accompanied by the generation of a massless boson, i.e., the pseudoscalar pion, according to the Goldstone theorem. The Fermion in this model was considered as a nucleon and the pion was described as a bound state of nucleon-antinucleon excitations of the deformed vacuum of the NJL model.

With the discovery of quarks, one finds that it would be more suitable to replace the elementary blocks (i.e., nucleons) in the original NJL model by quarks, and a bound state of quark-antiquark as pion in the chiral limit. If one includes a small current quark mass, more excitations of quark-antiquark as bound states can be obtained and which are identified as mesons, such as η, ω, etc. The connection of the NJL model with QCD may be realized by the proof made by 't Hooft in 1976. He proved that the renormalizable QCD

Lagrangian may effectively be described by a quartic Lagrangian (similar to the NJL model) in the light-quark (i.e., u- and d- quark only) sector, once gluonic degrees of freedom are integrated out ('t Hooft, 1976). The shortcomings of the model are that it is not renormalizable and cannot give rise to confinement. Quantities computed from this model are divergent due to the contact Fermion-Fermion interaction. However, one may consider the model to be an effective one and used only in one-loop approximation with an ultraviolet cutoff on the quark momenta. Indeed, the cutoff introduced as the regularization of the theory serves as an inverse length scale in the problem. Therefore, the introduction of the momentum cutoff plays a double role of providing both confinement and asymptotical freedom, which are the elementary ingredients of QCD. The cutoff should be fixed by experiment.

In the eighties, there had been renewed interest (Ferstl *et al*, 1986, Dhar and Wadia, 1984 and Dhar *et al*, 1985) in the NJL model as an effective QCD Lagrangian in the light-quark sector at low energies and in the large N_c-limit. Much attention was given to the NJL model and related models (Ebert and Volkov, 1983, Volkov, 1984, Ebert and Reinhardt, 1986, Bernard and Meissner, 1988 and Takizawa *et al*, 1990), in their application to quark and nuclear systems (Klevansky and Lemmer, 1990, Meissner *et al*, 1988, da Providencia and de Sousa, 1989 and de Sousa, 1990) and finite-temperature and density effects also have been incorporated (Balin, 1985, Bernard, 1986, and da Providencia *et al*, 1987). The NJL model is also related to the linear (Eguchi, 1976, and Ebert, 1986) and nonlinear (Dhar *et al*, 1984, Meissner *et al*, 1988, Klevansky and Lemmer, 1990) versions of sigma model, which have the same symmetries and physical content as long-wavelength QCD (Cahill and Roberts, 1985). Keeping these connections with QCD in mind, we are especially interested in the study of the NJL vacuum in the chiral symmetry broken phase and the normal phase, both at zero and finite (nonzero) temperatures. The striking similarity of the formation of the energy gap between the ground state and the excited state in both the NJL model and the BCS theory of superconductivity is well known. Based on the analogy with superconductivity, the condensation energy per unit volume of the NJL vacuum may be defined as the energy (or free energy, at finite temperature) density difference between the two phases of the vacuum. We find that this condensation energy per unit volume is a very interesting quantity, and may throw some light on the temperature dependence of the bag constant in the MIT bag model and its scaling behavior in a nuclear medium. We argue that the vanishing of the MIT bag constant due to the melting of the quark condensate does not give rise to quark deconfinement, instead, it reflects only the restoration of chiral symmetry. In the literature, (Bernard *et al*, 1987, and Bailin *et al*, 1985) there have been studies of the temperature dependence of the dynamical mass and the vacuum condensate $<\bar{\psi}\psi>$, but here the condensation free energy and other related quantities, such as entropy and specific heat of the vacuum, are studied. Their behavior near the critical temperature shows striking similarity to that in BCS superconductivity.

There has been considerable recent interest on the modification of quark and gluon condensates in nuclear medium and its applications to nuclear observables (Celenza *et al*, 1992, Brown *et al*, 1991, Brown, 1991 and Cohen *et al*, 1992). We note that the NJL model is an effective model and there is no gluon degree of freedom in this model. Therefore, we can study the quark condensate only. The scaling law of the vacuum condensation energy density in a nuclear medium is examined within our formalism.

Throughout this work, we use the elegant formulation of da Providencia *et al* (da Providencia *et al*, 1987) of the mean-field density matrix to describe the Dirac vacuum, thereby employing the Thomas-Fermi semiclassical method rather than formal field theory. The physics of the condensation energy is transparent in this method. This chapter is organized as follows: section 2 is devoted to a brief review of the many-body formulation introduced by da Providencia *et al* and the calculation of the vacuum condensation energy at zero temperature. In section 3, we study the thermodynamical properties of the vacuum. We extend the formulation of da Providencia *et al* to finite temperature and calculate some thermodynamical quantities. The behavior of entropy and specific heat per unit volume of the vacuum shows clearly that the system undergoes a second-order phase transition, exhibiting a striking similarity to the BCS superconductivity. Section 4 is devoted to the discussion of the scaling behaviour of the chiral condensation energy density in a nuclear medium. We provide a summary in the last section.

4.2 CONDENSATION ENERGY AT ZERO TEMPERATURE

The formalism

For completeness and understanding, we first review the formalism employed by da Providencia *et al* (da Providencia *et al*, 1987). At zero temperature, the vacuum of the system is considered as a many-fermion system in which the negative energy Dirac sea is completely filled according to the Pauli principle (see Fig.4.1(a)).

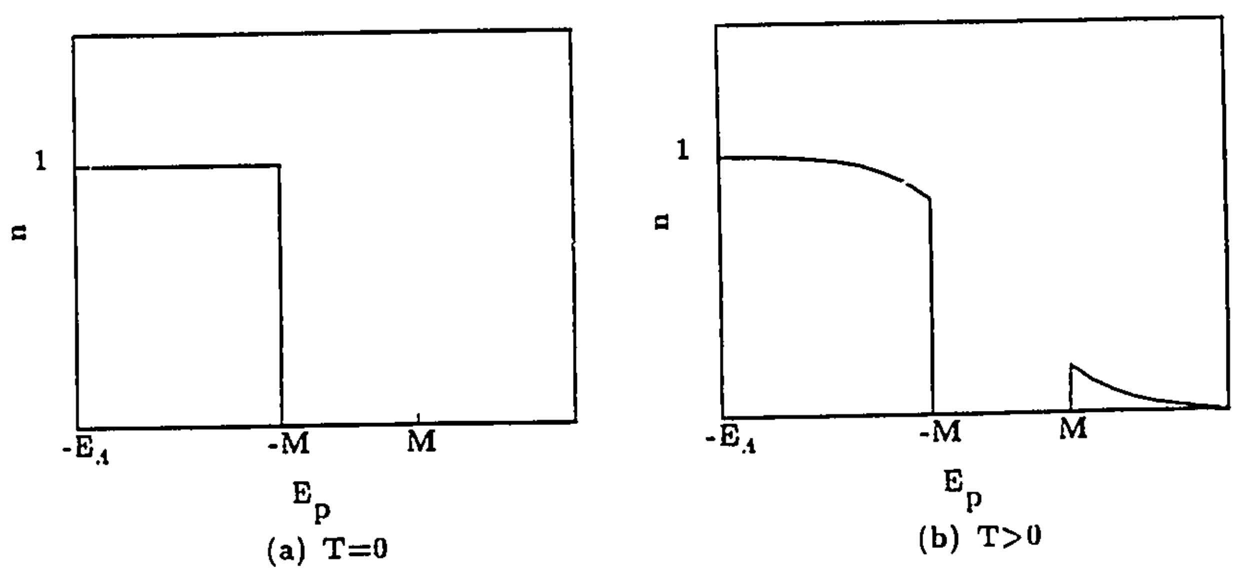


Fig.4.1 Schematic diagrams for the occupancy of the NJL vacuum (a) at $T = 0$ and (b) at finite T. In both, the ultraviolet cut-off at $-E_\Lambda$ remains the same.

As argued in the introduction, an ultraviolet momentum cut-off is necessary to obtain finite answers, so that the number of Fermions in this truncated vacuum is also finite. The Fermions will be taken to be the non-strange quarks (i.e., u and d quarks only) with zero masses in the chiral limit. The NJL model Hamiltonian for this many-body vacuum system is given by (the derivation for the following equation from Eq.(4.1.1) is given in appendix D):

$$\begin{aligned}\mathcal{H} &= \sum_{i=1}^{N} t(i) + \frac{1}{2}\sum_{i\neq j}^{N} V(ij) \qquad (4.2.1)\\ &= \sum_{i=1}^{N} \gamma_5(i)\vec{\sigma}(i)\cdot \mathbf{p}(j) - g\sum_{i\neq j}\delta(\mathbf{x}_i - \mathbf{x}_j)[\beta(i)\beta(j) - \beta(i)\gamma_5(i)\beta(j)\gamma_5(j)] \,,\end{aligned}$$

where $\vec{\sigma}$, γ_5 and β are the usual 4×4 matrices which are defined as follows:

$$\vec{\sigma} = \begin{pmatrix} \vec{\sigma}_2 & 0 \\ 0 & \vec{\sigma}_2 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

and g is the four-Fermion interacting constant. We denote the 2×2 identity matrix by I_2 and the 2×2 Pauli matrices by $\vec{\sigma}_2$. The flavor and color indices are suppressed for simplicity. The effective single-particle Hamiltonian assumes the form within the mean-field approach:

$$h = \gamma_5\vec{\sigma}\cdot\mathbf{p} + \beta M \,, \qquad (4.2.2)$$

where M is an effective mean-field scalar potential (or the dynamical mass), to be determined self-consistently later. The Hartree-Fock vacuum state is defined as having the upper N negative energy states occupied. The state may be described by a Slater determinant $|\Phi>$ with

$$|\Phi> = \prod_{i}^{N} d_i^{\dagger}|0> \,,$$

where $|0>$ is the absolute vacuum and $d^\dagger$ is a negative-energy particle creation operator. The density matrix of state is defined as

$$\rho(\mathbf{x}-\mathbf{x}') = <\Phi|\psi(\mathbf{x})\psi^\dagger(\mathbf{x}')|\Phi> = \sum_{\mathbf{p}}\sum_{s}\psi_{\mathbf{p},s}(\mathbf{x})\psi^\dagger_{\mathbf{p},s}(\mathbf{x}')\,\theta(\Lambda^2-p^2)\,, \tag{4.2.3}$$

where $\psi(\mathbf{x}) = \sum_{\mathbf{p},s} d_{\mathbf{p}}\psi(\mathbf{x})_{\mathbf{p},s}$ and $\psi_{\mathbf{p},s}(\mathbf{x})$ is a stationary negative energy Dirac spinor with momentum $\mathbf{p}$ and spin index s, θ is a standard step function which is defined by

$$\theta(x) = \begin{cases} 1, & x \geq 0; \\ 0, & x < 0, \end{cases}$$

and Λ is the ultraviolet cut-off in momentum, $\Lambda = |\mathbf{p}|_{max}$. Writing $\psi_{\mathbf{p},s} = v_{\mathbf{p},s}e^{i\mathbf{p}\cdot\mathbf{x}}$, the spinor of negative energy $-E_{\mathbf{p}} = -\sqrt{\mathbf{p}^2+M^2}$, $v_{\mathbf{p},s}$, may be obtained by solving a free Dirac equation and is

$$v_{\mathbf{p},s} = \sqrt{\frac{E_{\mathbf{p}}+M}{2E_{\mathbf{p}}}}\begin{pmatrix} \frac{-(\sigma\cdot\mathbf{p})}{E_{\mathbf{p}}+M}\chi_s \\ \chi_s \end{pmatrix}, \tag{4.2.4}$$

where $\chi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $s = \frac{1}{2}$ or $-\frac{1}{2}$. Therefore, the density matrix may be written in the form

$$\rho(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{p}}\rho_{\mathbf{p}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}, \tag{4.2.5}$$

where $\rho_{\mathbf{p}}$, the Wigner transform of $\rho(\mathbf{x},\mathbf{x}')$, is $\mathbf{x}$ independent and defined as

$$\rho_{\mathbf{p}} = \sum_{s} v_{\mathbf{p},s}v^\dagger_{\mathbf{p},s}\theta(\Lambda^2-p^2)\,.$$

A little algebra gives

$$\rho_{\mathbf{p}} = \frac{1}{2}(1-\beta\frac{M}{E_{\mathbf{p}}}-\gamma_5\frac{\vec{\sigma}\cdot\mathbf{p}}{E_{\mathbf{p}}})\theta(\Lambda^2-p^2)\,. \tag{4.2.6}$$

With this density matrix, the vacuum energy of the system may be calculated in the Hartree-Fock approximation from the following expression

$$E_v = \sum_{\mathbf{p}_1} \int d^3x_1 tr_1[(\gamma_5\vec{\sigma}_1 \cdot \mathbf{p}_1)\rho_{\mathbf{p}_1}] + \frac{1}{2}\sum_{\mathbf{p}_1}\sum_{\mathbf{p}_2} \int\int d^3x_1 d^3x_2 tr_1 tr_2[V(12)\rho_{\mathbf{p}_1}\rho_{\mathbf{p}_2}] , \tag{4.2.7}$$

where the interacting potential $V(12)$ is defined in Eq.(4.2.1). In the actual evaluation of E_v (see appendix D), the exchange contribution is also included, but its expression involves the mixed density matrix and is not shown in Eq.(4.2.7). A straightforward calculation then yields the energy per unit volume, i.e., $\mathcal{E} = E_v/V$:

$$\mathcal{E}(\Lambda, M) = -2\mathcal{T}(\Lambda, M) - 4gM^2[\mathcal{V}(\Lambda, M)]^2 + 2g[\Omega(\Lambda)]^2 , \tag{4.2.8}$$

where,

$$\mathcal{T}(\Lambda, M) = \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{E_{\mathbf{p}}}\theta(\Lambda^2 - p^2) , \tag{4.2.9a}$$

$$\mathcal{V}(\Lambda, M) = \sum_{\mathbf{p}} \frac{1}{E_{\mathbf{p}}}\theta(\Lambda^2 - p^2) , \tag{4.2.9b}$$

$$\Omega(\Lambda) = \sum_{\mathbf{p}} \theta(\Lambda^2 - p^2) . \tag{4.2.9c}$$

The last term in Eq.(4.2.8) is the contribution from the interchange term, and is independent of the effective potential M. The minimization of the energy with respect to M gives the following stability condition:

$$M = 4gM\mathcal{V}(\Lambda, M) , \tag{4.2.10}$$

which is called the "gap equation". A trivial solution of the above equation gives $M = 0$, which corresponds to the chiral symmetric Wigner-Weyl phase of the vacuum. We have checked that $\frac{\partial^2\mathcal{E}}{\partial M^2}$ at $M = 0$ is zero for zero current quark mass and is positive for nonzero current quark mass. A nontrivial solution of $M \neq 0$, which corresponds to the chirally broken Goldstone-Nambu phase of the vacuum, is determined by solving the equation

$$1 = 4g \sum_{\mathbf{p}} \frac{1}{E_{\mathbf{p}}} \theta(\Lambda^2 - p^2) . \tag{4.2.11}$$

The generation of the dynamical quark mass M leads to the spontaneous breaking of the chiral symmetry of the NJL vacuum. The nonzero M solution of Eq.(4.2.11) exists only if $g\Lambda^2 > \pi^2$. When $g\Lambda^2 > \pi^2$, the system undergoes a phase transition from a chirally symmetric state to a state of broken chiral symmetry. These states are characterized by the so called order parameter $< \Phi|\bar{\psi}\psi|\Phi >$ which measures the chiral deformation of states. It is obvious that, when $M = 0$, $|\Phi >_0$ is an eigenstate of chirality and

$$< \Phi|\bar{\psi}\psi|\Phi >_0 = 0 .$$

For nonzero M, the state $|\Phi >$ is no longer an eigenstate of chirality and has nonzero order parameter

$$< \Phi|\bar{\psi}\psi|\Phi > = \sum_{\mathbf{p}} tr\beta\rho_{\mathbf{p}} = -\frac{M}{2g} .$$

This completes the review of the formalism given by da Providencia *et al* (da Providencia *et al*, 1987).

The condensation energy density

When the gap equation (4.2.11) is satisfied, the minimum vacuum energy density may be computed from Eq.(4.2.8) and gives:

$$\mathcal{E}_{NG}(\Lambda, M) = -2 \sum_{\mathbf{p}} E_{\mathbf{p}} \theta(\Lambda^2 - p^2) + \frac{M^2}{4g} + 2g\Omega^2 , \tag{4.2.12}$$

in the Nambu-Goldstone phase with $M \neq 0$ denoted by the subscript NG, and

$$\mathcal{E}_{WW}(\Lambda, 0) = -2 \sum_{\mathbf{p}} |\mathbf{p}| \, \theta(\Lambda^2 - p^2) + 2g\Omega^2 \,, \tag{4.2.13}$$

in the Wigner-Weyl phase with $M = 0$ denoted by subscript WW.

We define the condensation energy density of the NJL vacuum as the difference in energy density of the two phases, denoted by $\tilde{B}$,

$$\begin{aligned} \tilde{B} &= \mathcal{E}_{WW}(\Lambda, 0) - \mathcal{E}_{NG}(\Lambda, M) \\ &= 2 \, [\sum_{\mathbf{p}} (E_{\mathbf{p}} - |\mathbf{p}|) \, \theta(\Lambda^2 - p^2) - \frac{M^2}{8g}] \,. \end{aligned} \tag{4.2.14}$$

This is precisely of the form of the condensation energy density for a BCS superconductor (Schrieffer, 1964). In the latter case, M is replaced by the gap parameter Δ. $E_{\mathbf{p}}$ corresponds to the quasi-particle energy $\sqrt{\epsilon_{\mathbf{p}}^2 + \Delta^2}$, with $\epsilon_{\mathbf{p}}$ the nonrelativistical free-electron energy (measured from the Fermi surface) replacing $|\mathbf{p}|$ in Eq.(4.2.14). In this sense, our ultraviolet cut-off is analogous to the Debye cut-off in the BCS superconductor case.

At this point, it is appropriate to examine how $\tilde{B}$, given by Eq.(4.2.14), depends on the number of colors N_c for large N_c. In the NJL model, we find that $\tilde{B}$ has linear dependence on N_c. To examine the N_c and N_f (the number of flavors of massless quarks) dependence, note from Eqs.(4.2.1) to (4.2.9) that the sum over $\mathbf{p}$ is also understood to have the color-flavor degeneracy factor:

$$\sum_{\mathbf{p}} \rightarrow N_c N_f \int \frac{d^3p}{(2\pi)^3} \,. \tag{4.2.15}$$

Now consider Eq.(4.2.8). If we take Λ and M to be independent of N_c and N_f, and g to be proportional to $(N_c N_f)^{-1}$, each term in Eqs.(4.2.1) to (4.2.8) is proportional to $(N_c N_f)$, and so is $\tilde{B}$ given by Eq.(4.2.14). To see why g

should be taken inversely proportional to $(N_c N_f)$, consider the vacuum quark condensate $<\bar{\psi}\psi>$, which in our formalism is given by

$$<\bar{\psi}\psi>=\sum_{\mathbf{p}} tr(\rho_{\mathbf{p}}\beta) = -\sum_{\mathbf{p}} \frac{2M}{E_{\mathbf{p}}}\theta(\Lambda^2 - p^2) , \tag{4.2.16}$$

where we have used Eq.(4.2.6). Note that this $<\bar{\psi}\psi>$ is proportional to $(N_c N_f)$. Using Eqs.(4.2.9) and (4.2.11), we may also write

$$<\bar{\psi}\psi>= -2M\mathcal{V} = -\frac{M}{2g} . \tag{4.2.17}$$

Since M is taken to be independent of N_c and N_f, it follows that $g \propto (N_c N_f)^{-1}$. Therefore, if we identify $\tilde{B}$ as the MIT bag constant (as will be argued later), this gives more reasonable linear dependence on N_c. A numerical estimate of g may be made for the case of $N_c = 3$ and $N_f = 2$. Assume, in this case, that

$$m_\pi^2 = -\frac{m_q}{f_\pi^2} < \bar{u}u + \bar{d}d >= -\frac{m_q}{f_\pi^2} < \bar{\psi}\psi > \tag{4.2.18}$$

Here, m_π is the mass of pion, f_π the pion-decay constant, and $m_q = m_u = m_d$ is the current quark mass assumed to be nonzero. From Eqs.(4.2.17) and (4.2.18), we get

$$g = \frac{M m_q}{2 m_\pi^2 f_\pi^2} . \tag{4.2.19}$$

In our idealized Hamiltonian (4.2.1), the current quark mass m_q is taken to be zero. Therefore, the mass of the Goldstone boson pion m_π is zero, and Eq.(4.2.18) is trivially satisfied. By taking $m_q = 10$ MeV, the constituent quark mass is taken as one third of the nucleon mass, i.e., $M = 330$ MeV, $m_\pi = 140$ MeV and $f_\pi = 93$ MeV, we find that $g \approx 10\ (GeV)^{-2}$. This is only to be taken as an order of magnitude estimate. According to this, since the nonzero solution for mass M of the gap equation (4.2.11) requires that $g\Lambda^2 > \pi^2$, it follows that $\Lambda > \pi/\sqrt{g}$, i.e., $\Lambda > 1$ GeV. We choose $g\Lambda^2 = 11$, and $\Lambda = 1.2$ GeV throughout this work. In Fig.4.2, we plot the numerically calculated vacuum energy density $\mathcal{E}$ against the effective potential M. It is clearly seen that a well-defined minimum appears at $M = 305.6$ MeV, which corresponds to the dynamically generated constituent quark mass. Consequently, the global chiral symmetry of the system is spontaneously broken and a massless Goldstone particle (pion) is generated. Note that the condensation energy density, as defined in Eq.(4.2.3), is a tiny fraction of a percent of $\mathcal{E}(\Lambda, M)$. This is analogous to the situation in a BCS superconductor.

It is interesting to see how the quantity $\tilde{B}^{1/4}$ varies with the ultraviolet momentum cut-off Λ in the NJL model (for $N_f = 2$ and $N_c = 3$). Recall that the coupling constant g in the NJL Hamiltonian is an effective one, which should be chosen judiciously for each choice of Λ. Accordingly, when we vary Λ in the wide range 600 to 6000 MeV, g is adjusted for each choice of Λ so that the self-consistent dynamical mass $M = 300$ MeV in every case. This procedure makes $g\Lambda^2$ vary in the range 10 to 12. The calculated value of $\tilde{B}^{1/4}$ (from Eq.(4.2.3)) is then plotted in Fig.4.3 as a function of Λ. It is also interesting to note that these calculated values are in the range of values $145 - 240$ MeV, which is just the range of MIT bag constant used in bag model phenomenology (Chodos *et al*, 1974).

4.3 VACUUM PROPERTIES AT FINITE-TEMPERATURE

Condensation free energy per unit volume

In the previous section, the self-consistent dynamical mass was that for which the energy density $\mathcal{E}(\Lambda, M)$ of Eq.(4.2.8) was a minimum. At nonzero (finite) temperatures, the relevant quantity to be minimized is the

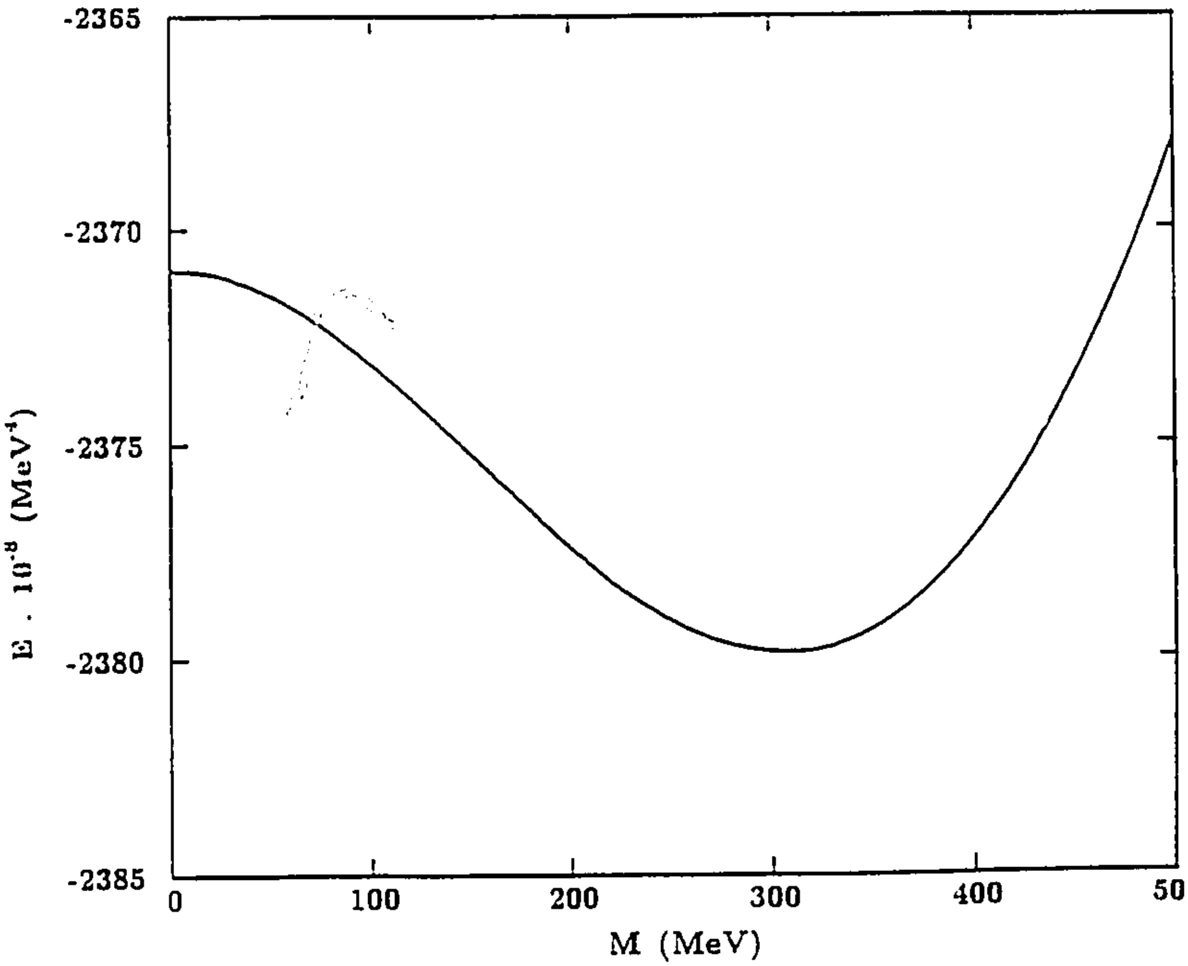


Fig.4.2 Vacuum energy density (Eq.(4.2.8)) vs. dynamic quark mass for $g\Lambda^2 = 11$, $\Lambda = 1200$ MeV and $T = 0$. Note that the minimum at $M \approx 305.6$ MeV is actually quite shallow (see the scale).

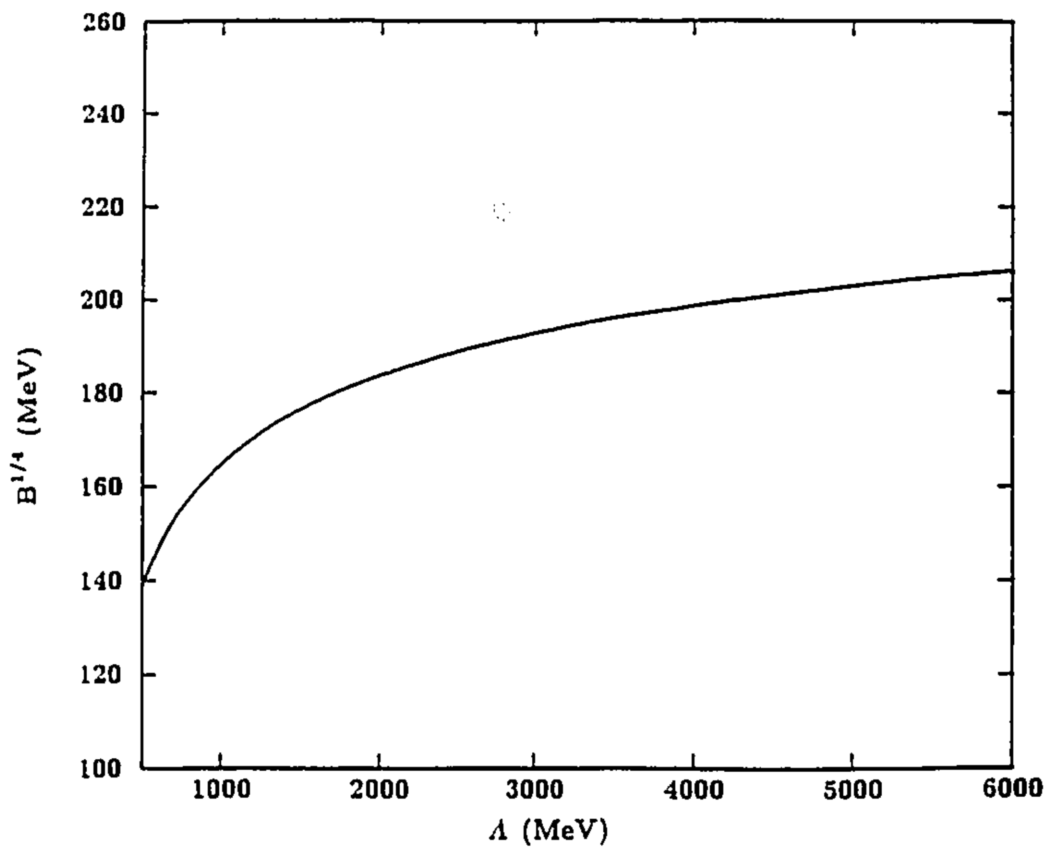


Fig.4.3 The fourth root of the condensation energy ("the bag constant") $\bar{B}^{1/4}$ vs. the ultraviolet momentum cut-off Λ for $M = 300$ MeV and $T = 0$.

Helmholtz free-energy density. Before performing finite temperature calculation, we elucidate the picture of the NJL vacuum at finite temperatures. From Fig.4.1(a), note that when $T = 0$, all negative energy states from $-M$ to $-E_\Lambda = -\sqrt{\Lambda^2 + M^2}$ are fully occupied, while all the positive energy states from M to infinity are empty. For nonzero T, the occupancy of a state is governed by the Fermi distribution function, and some of the particles occupying the negative energy states at $T = 0$ would now be promoted to positive energy states across the mass gap as shown schematically in Fig.4.1(b). Our picture of the NJL vacuum for nonzero T has the same total number of particles as at $T = 0$, but distributed both amongst positive and negative energy states. This does not mean that the vacuum for $T \neq 0$ acquires a nonzero baryon number. The number of particles promoted to the positive energy states creates an equal number of holes in the Dirac sea. These holes may be regarded as antiparticles, and the net baryon number of the NJL vacuum (with $q\bar{q}$ pairs) is still zero. Note that the ultraviolet cut-off at the negative energy, $-E_\Lambda = -\sqrt{\Lambda^2 + M^2}$, is still intact, and is necessary for finite answers, whereas no such cut-off is needed for positive energies.

According to the Fermi-Dirac statistics, the occupation probability of a Fermion for positive energy states $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + M^2}$ is given by the Fermi occupation function

$$n^+ = \frac{1}{e^{\beta(E_{\mathbf{p}} - \mu)} + 1}, \tag{4.3.1}$$

where $\beta = 1/(kT) = 1/T$ is the inverse-temperature, with the Boltzman constant k being 1 for simplicity, and μ denotes the chemical potential of the system. Correspondingly, the Fermi occupation function of a Fermion for negative energy states with $-E_{\mathbf{p}}$ is given by

$$n^- = \frac{1}{e^{-\beta(E_{\mathbf{p}}+\mu)}+1} . \tag{4.3.2}$$

In the above we have taken $E_{\mathbf{p}}$ to be always positive.

For nonzero temperature T, we now have two free parameters M and T while computing the free energy. At zero temperature, the former is fitted to about one-third the nucleon mass. Accordingly, the chemical potential μ is to be seen as a function of the variables (T, M), and is evaluated from the normalization condition that

$$\Omega(\Lambda) = \sum_{\mathbf{p}} (n^- + n^+) \tag{4.3.3}$$

remains a constant. Recall that $\Omega(\Lambda) = \sum_{\mathbf{p}} \theta(\Lambda^2 - p^2)$ at $T = 0$, and 2Ω is the total number of particles in the vacuum (where the factor of 2 is due to spin degeneracy). Throughout, we take Λ to be independent of T and M. The density matrix of the vacuum state at finite temperature is now given by the sum of that for negative energy states and that of positive energy states. From Eq.(4.2.6), the density matrix at finite temperature is given by:

$$\begin{aligned} \rho_{\mathbf{p}} &= \rho_{\mathbf{p}}^- + \rho_{\mathbf{p}}^+ \\ &= I\, n^+ + \frac{1}{2}[I - \beta\frac{M}{E_{\mathbf{p}}} - \gamma_5 \frac{\vec{\sigma}\cdot\mathbf{p}}{E_{\mathbf{p}}}](n^- - n^+) . \end{aligned}$$

By means of this density matrix, it is straightforward to check that the expression for the energy density, Eq.(4.2.8), remains formally the same as that in the case of $T \neq 0$ (see detail in appendix D), namely,

$$\mathcal{E}(\Lambda, M, T) = -2\mathcal{T}(\Lambda, M, T) - 4gM^2[\mathcal{V}(\Lambda, M, T)]^2 + 2g[\Omega(\Lambda)]^2 , \tag{4.3.4}$$

with

$$\mathcal{T}(\Lambda, M, T) = \sum_{\mathbf{p}} \frac{\mathbf{p}^2}{E_{\mathbf{p}}}(n^- - n^+) , \quad (4.3.5a)$$

and

$$\mathcal{V}(\Lambda, M, T) = \sum_{\mathbf{p}} \frac{1}{E_{\mathbf{p}}}(n^- - n^+) . \quad (4.3.5b)$$

In the above equations, an ultraviolet momentum cut-off for the negative energy states is implied. The thermodynamical Helmholtz free-energy per unit volume is defined as

$$\mathcal{F} = \mathcal{E} - \frac{1}{\beta}\mathcal{S} , \quad (4.3.6)$$

where the entropy per unit volume $\mathcal{S}$ is given by

$$\mathcal{S} = -2\sum_{\mathbf{p}}[n^+ \ln n^+ + (1-n^+)\ln(1-n^+) + n^- \ln n^- + (1-n^-)\ln(1-n^-)] . \quad (4.3.7)$$

The overall multiplicative factor of 2 in the above equations is due to the spin degeneracy. For a fixed T, $\mathcal{F}$ is now to be minimized with respect to variations in M. The chemical potential is adjusted for each such variation to obey Eq.(4.3.3), so that $\frac{\partial \Omega}{\partial M} = 0$. It then follows that

$$\frac{\partial \mathcal{F}}{\partial M} = \frac{\partial \mathcal{E}}{\partial M} - \frac{1}{\beta}\frac{\partial \mathcal{S}}{\partial M} . \quad (4.3.8)$$

This leads to the equation (through algebra detailed in Appendix D)

$$2M(1 - 4g\mathcal{V})\frac{\partial}{\partial M}(M\mathcal{V}) = 0 . \quad (4.3.9)$$

Since $\frac{\partial}{\partial M}(M\mathcal{V}) \neq 0$, the gap equation of the same form as that at $T = 0$ follows

$$M = 4gM\mathcal{V}(\Lambda, M, T) . \quad (4.3.10)$$

The $M(T) \neq 0$ solution may be obtained by solving Eqs.(4.3.3) and (4.3.10) numerically in a self-consistent manner. Specifically, for a fixed Λ, assume a trial M at a given T, and calculate μ from Eq.(4.3.3); with this μ, solve Eq.(4.3.10) to obtain $M(T)$. Repeat the cycle till self-consistency is achieved for each T. The computation is done on a DEC VAX8600 and the IMSL program library was utilized. A critical temperature T_c may be found from the gap equation (4.3.10) in the limit of $M(T_c) \to 0$. For $T > T_c$, $M = 0$, and the chemical potential is simply determined by Eq.(4.3.3). We should point out here that our finite temperature gap equation (4.3.10) is not new. The same form was obtained by Bernard *et al* (Bernard *et al*, 1987) by using a standard field theoretic method. However, the advantage of our semi-classical approach is that the significance of the NJL vacuum free-energy difference per unit volume between the two chiral phases and other thermodynamic properties is more apparent.

In Fig.4.4, we plot the free-energy difference per unit volume between the two phases of $M \neq 0$ and $M = 0$, $\Delta\mathcal{F} = [\mathcal{F}(\Lambda, M, T) - \mathcal{F}(\Lambda, 0, T)]$, as a function of M (which is considered as a free parameter) for different temperatures.

As is seen from the diagram, a well defined minimum occurs at $M(T) \neq 0$ for each $T < T_c$ and the critical temperature is found to be $T_c = 213.6$ MeV for $g\Lambda^2 = 11$ and $\Lambda = 1200$ MeV which is the same as the result found from the gap equation by putting $M(T_c) = 0$. When computing $\mathcal{F}(\Lambda, 0, T)$, we calculate the chemical potential for each T with $E_{\mathbf{p}} = |\mathbf{p}|$ to obey Eq.(4.3.3). Fig.4.4 shows clearly that above the critical temperature T_c, there is no minimum in $\Delta\mathcal{F}$ for $M \neq 0$. The dynamically generated mass is destroyed by high temperature of $T > T_c$ and particles become again massless, signaling the restoration of the chiral symmetry.

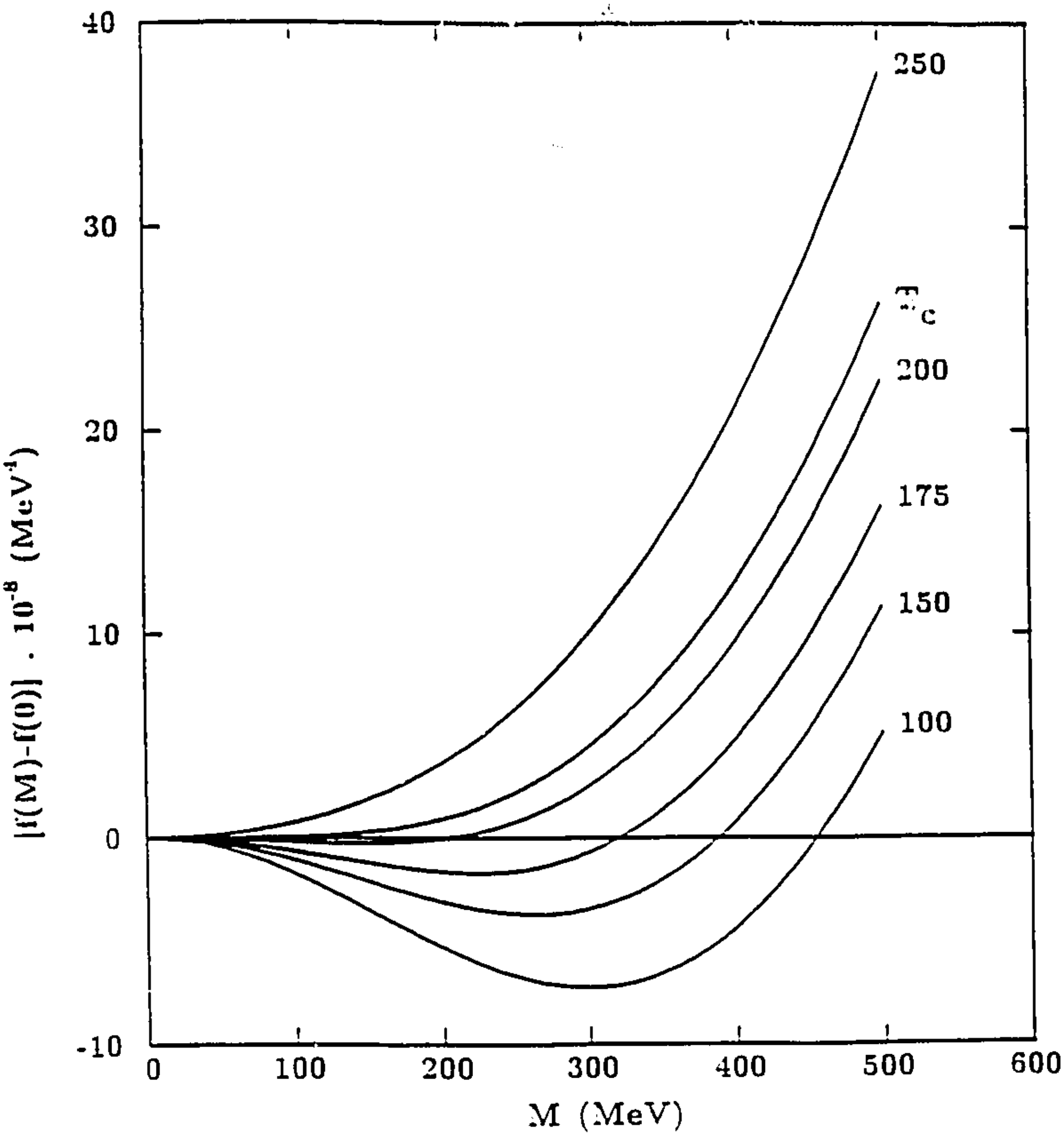


Fig.4.4 Free energy difference per unit volume between the Nambu-Goldstone and the Wigner-Weyl phases against M (a free parameter), for different temperatures. It is found that the chiral symmetry is restored at the critical temperature $T_c = 213.6$ MeV for $g\Lambda^2 = 11$ and $\Lambda = 1200$ MeV.

The dynamically generated quark mass $M(T)$ is usually regarded as the order parameter of the phase transition. We plot its dependence on temperature T in Fig.4.5. Assume that near T_c, $M(T) \propto (T - T_c)^\alpha$, where α is a so-called critical index. A careful numerical analysis yields in this case $\alpha = 0.500 \pm 0.005$, analogous to the case of BCS superconductor, where $\alpha = \frac{1}{2}$.

As in the case of zero-temperature, let us denote the minimum value of the free-energy density at each temperature by $\mathcal{F}_{NG}(\Lambda, M, T)$ for the Nambu-Goldstone phase with $M(T) \neq 0$ and $\mathcal{F}_{WW}(\Lambda, 0, T)$ the Wigner-Weyl phase with $M(T) = 0$, that is

$$\mathcal{F}_{NG}(\Lambda, M, T) = -2 \sum_{\mathbf{p}} E_{\mathbf{p}}(n^{(-)} - n^{(+)}) + \frac{M^2}{4g} + 2g[\Omega(\Lambda)]^2 - \frac{1}{\beta}\mathcal{S} , \qquad (4.3.11)$$

and

$$\mathcal{F}_{WW}(\Lambda, 0, T) = -2 \sum_{\mathbf{p}} |\mathbf{p}| \, (n^{(-)} - n^{(+)}) + 2g[\Omega(\Lambda)]^2 - \frac{1}{\beta}\mathcal{S}_0 , \qquad (4.3.12)$$

where quantities with subscript 0 represent those for the case of $M(T) = 0$. Thus, the temperature dependent vacuum condensation free energy per unit volume, i.e., $\tilde{B}$, is obtained

$$\begin{aligned} \tilde{B}(T) &= \mathcal{F}_{WW}(\Lambda, 0, T) - \mathcal{F}_{NG}(\Lambda, M, T) \\ &= 2 \, [\sum_{\mathbf{p}} (E_{\mathbf{p}} - |\mathbf{p}|)(n^{(-)} - n^{(+)}) - \frac{M^2}{8g}] + \frac{1}{\beta}(\mathcal{S} - \mathcal{S}_0) \, . \qquad (4.3.13) \end{aligned}$$

The quantity $\tilde{B}^{1/4}(T)$ is plotted in Fig.4.5 to compare with the dynamically generated $M(T)$ for parameters $g\Lambda^2 = 11$ and $\Lambda = 1200$ MeV. Two points are worth mentioning. First, the temperature dependences of $M(T)$ and $\tilde{B}^{1/4}(T)$ are very similar, although the former is calculated from the gap equation, and the latter from the vacuum condensation free energy per unit volume. Note that if we identify the vacuum condensation free energy $\tilde{B}$ as the MIT

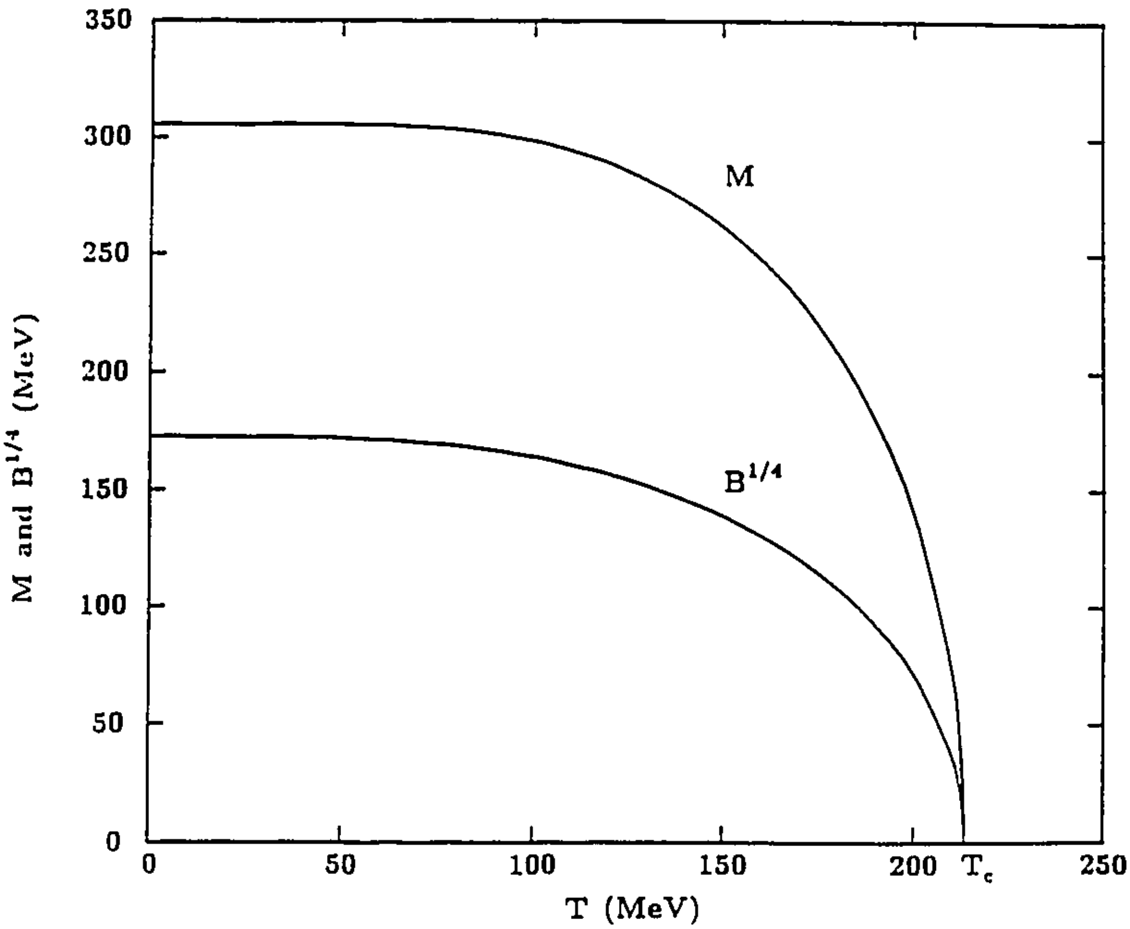


Fig.4.5 Self-consistent dynamical mass $M(T)$ and the fourth root of the condensation free energy density $\tilde{B}^{1/4}(T)$ vs. temperature for $g\Lambda^2 = 11$ and $\Lambda = 1200$ MeV.

bag constant B, this is to be expected in the naive MIT bag model, since the baryon mass is proportional to $B^{1/4}$. The other point is that for small T, $\bar{B}^{1/4}(T)$ varies very slowly with T, in agreement with the observation made by Dey (Dey, 1986). In Fig.4.6, $\bar{B}(T)$ is plotted against T to show that it may be best fitted by the analytic function form

$$\bar{B}(T) \simeq \bar{B}(0)\frac{[1-e^{-\gamma(T_c-T)^2}]}{(1-e^{-\gamma T_c^2})}, \tag{4.3.14}$$

with parameters $\bar{B}(0) = (172.7\ MeV)^4$, $\gamma = 0.1325 \times 10^{-3}\ MeV^{-2}$ and $T_c = 213.6\ MeV$. It is easy to check from the above equation that for $T \approx T_c$, $\bar{B}(T) \propto (T-T_c)^2$, thus $\bar{B}^{1/4}(T)$ has the same index as $M(T)$, which is $\frac{1}{2}$.

Some other thermodynamic quantities

The method we are using is convenient to calculate some other interesting thermodynamical quantities at finite T. For example, the entropy and the specific heat per unit volume denoted by $\mathcal{S}$ and $\mathcal{C}$ respectively. The entropy density of the vacuum system may be calculated directly from Eq.(4.3.7). The numerical results are shown in Fig.4.7 for two different phases. This diagram shows clearly that the vacuum entropy density in the Nambu-Goldstone (chiral broken) phase is smaller than that in the Wigner-Weyl (chiral symmetric) phase. It in turn implies that the lower energy state of the Nambu-Goldstone phase has more order. Having the quantities of free energy density and entropy density, we can compute the specific heat density at constant volume and particle number by the following equation from thermodynamics:

$$\mathcal{C}_v(T) = T\left(\frac{\partial \mathcal{S}}{\partial T}\right)_{V,N} = -T\frac{\partial^2 \mathcal{F}}{\partial T^2}. \tag{4.3.15}$$

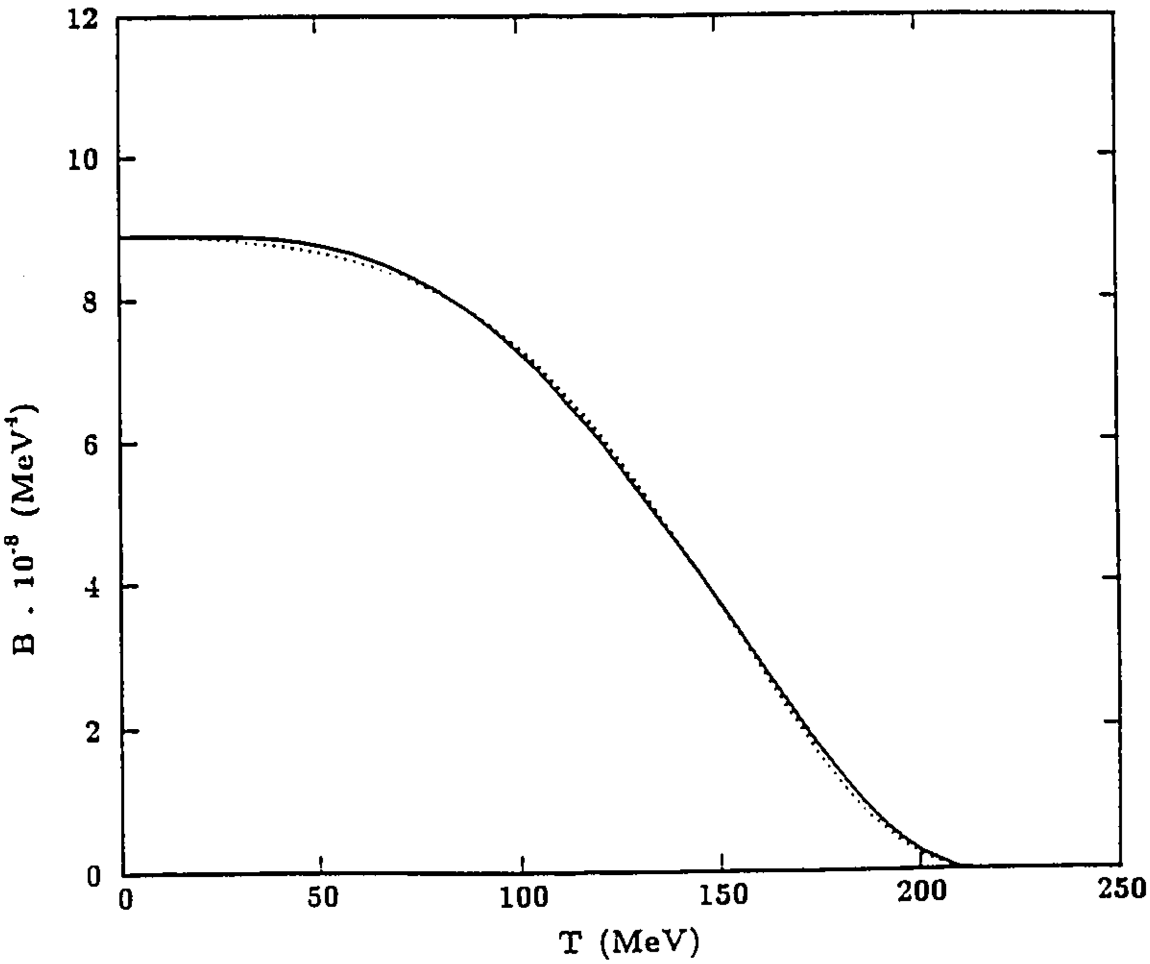


Fig.4.6 The condensation energy ("bag constant") versus temperature for $g\Lambda^2 = 11$ and $\Lambda = 1200$ MeV. $\tilde{B}(T)$ calculated using the analytical form (4.3.12) (dotted curve) is nearly indistinguishable.

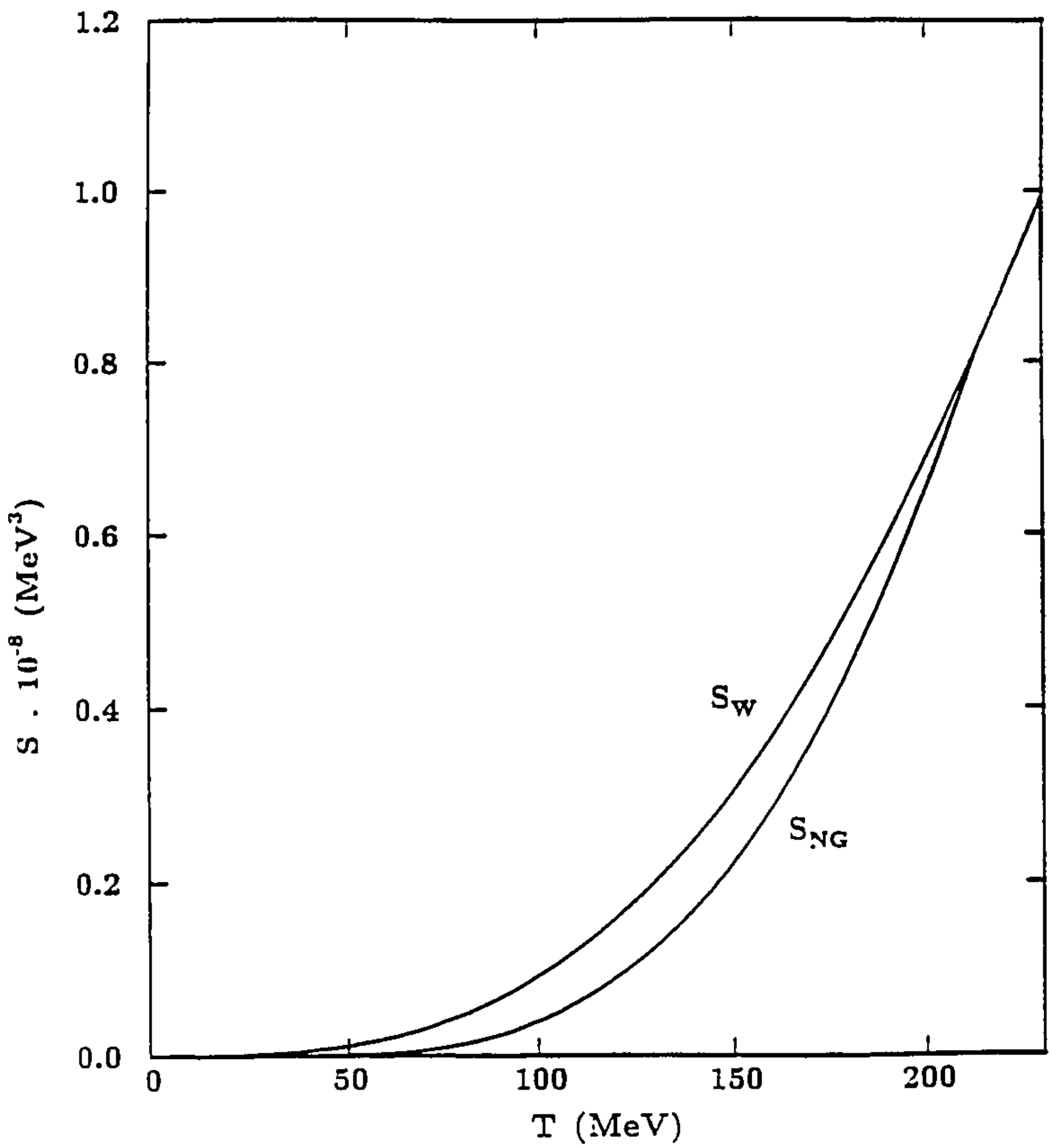


Fig.4.7 Entropy density S versus temperature T in the Nambu-Goldstone and Wigner-Weyl phases for $g\Lambda^2 = 11$ and $\Lambda = 1200$ MeV.

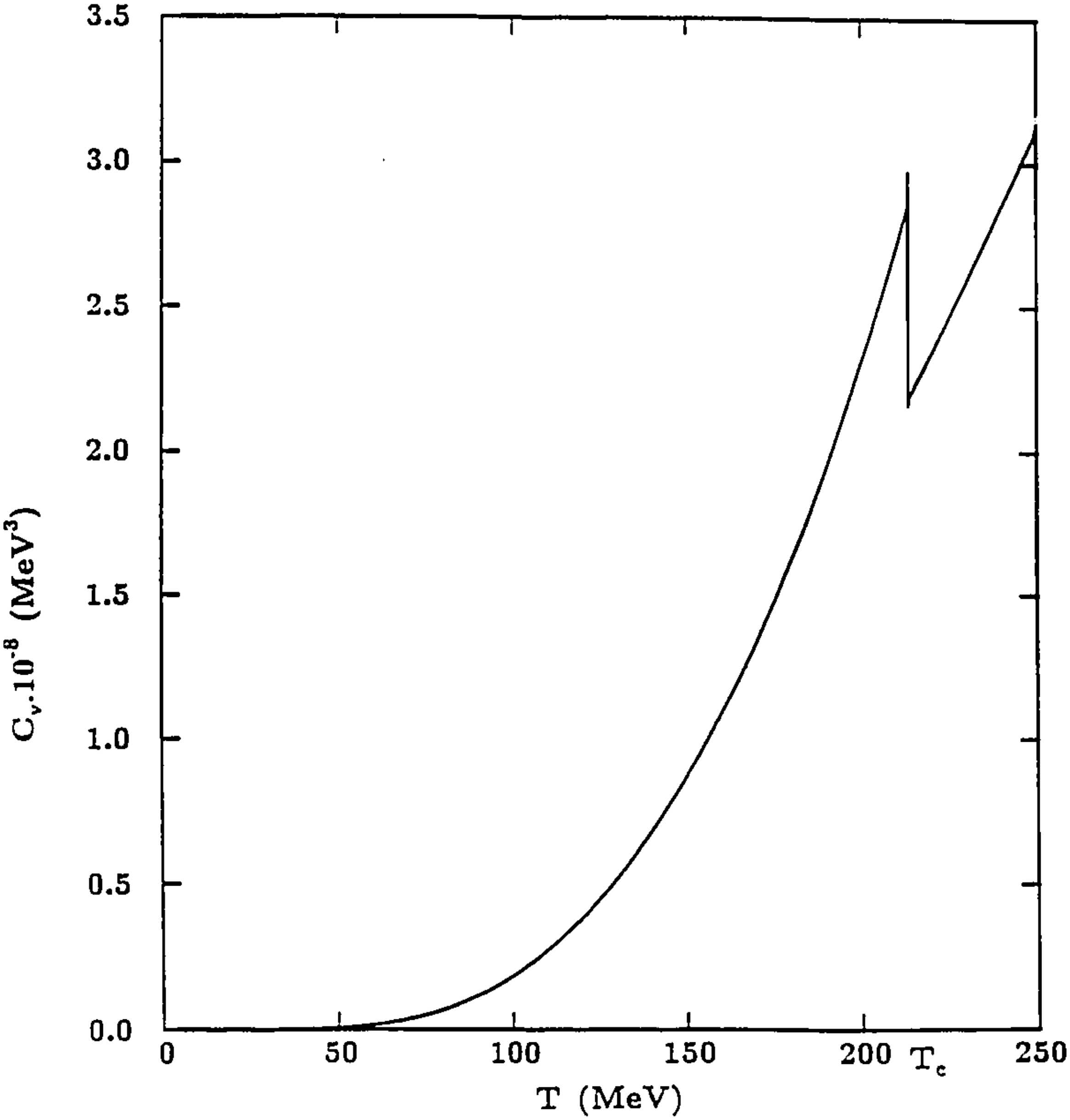


Fig.4.8 The specific heat per unit volume of the vacuum versus temperature for $g\Lambda^2 = 11$ and $\Lambda = 1200$ MeV. For $T < T_c$, the vacuum is in the Nambu-Goldstone phase.

The calculation of the second derivative of the free energy with respect to temperature is also done by numerical methods. This quantity is plotted in Fig.4.8. As we can see that there is a finite jump of $C_v(T)$ at $T = T_c$, and for $T < T_c$, the vacuum is in Nambu-Goldstone phase. This feature again shows the similarity to the nonrelativistic BCS theory of superconductivity, and also tells us that the phase transition at $T = T_c$ is of the second order.

The MIT bag constant

Now we are in a position to identify the calculated vacuum condensation energy (or free energy) $\bar{B}$ as the MIT bag constant B. Such identification is based on the following argument:

In principle, since the NJL model has no confinement, and the naive MIT bag model (Chodos *et al*, 1974, GeGrand *et al*, 1975 and Bardeen *et al*, 1975) has no chiral symmetry, it may seem strange that one can learn anything about the MIT bag constant from the NJL model. Note, however, that these two models have in common the realization of two distinct phases. The MIT bag constant B is defined as the energy difference per unit volume between the perturbative vacuum (inside the hadron) and the QCD vacuum (outside the hadron) where colors cannot propagate. The chiral symmetry is violated in the hadron sector outside the bag with nonzero dynamical mass (this corresponds to the Nambu-Goldstone phase of the NJL model). The deconfined phase within the hadron has manifest chiral symmetry with zero (or light) quark masses (this corresponds to the Wigner-Weyl phase of the NJL model). Outside the bag, the vacuum is at a lower energy state and should be identified with the spontaneously broken chiral phase (Cahill and Roberts, 1985) of QCD. Although the NJL model does not directly result in bag formation and quark confinement, it does produce the two distinct chiral phases appropriate to the inside and the outside of the bag. If confinement and spontaneously broken chiral symmetry were synonymous, it would have been natural to identify the bag constant B with the condensation energy density of the NJL model.

Indeed, the NJL model is shown to be equivalent to the nonlinear σ model (Meissner *et al*, 1988) of Lee-Wick type (Klevansky and Lemmer, 1990 and Lee and Wick, 1984), which can have soliton solution. On the other hand, Cahill and Roberts proved that QCD can also support a soliton solution once the Fermionic degree of freedom is integrated out, and the resulting effective Lagrangian is equivalent to the nonlinear sigma model. Using this effective sigma potential, they obtained the bag constant in a phenomenological bag model (Cahill and Roberts, 1985). We discovered that this bag constant calculated from the sigma model potential by Cahill and Roberts is exactly the vacuum condensation energy per unit volume in the NJL model according to the analysis given by Klevansky and Lemmer.

Obviously, our identification of the MIT bag constant shows only the connection to chiral symmetry and gives no information on the quark-gluon confinement which are believed to be related to the gluon condensate. Therefore, the melting of the MIT bag constant reflects only the restoration of chiral symmetry (as pointed out by Brown, 1991). From the calculation by Lee (Lee, 1989), the gluon condensate remains about half of its zero-temperature value for a temperature as high as 295 MeV. (Recall that the critical temperature of chiral restoration is about 200 MeV.)

For the sake of emphasizing the chiral symmetry, we will call the vacuum condensation energy density (or the MIT bag constant) the chiral condensation energy density in the subsequent sections.

4.4 THE SCALING OF THE CHIRAL CONDENSATION ENERGY DENSITY

There has been much recent interest on the modification of condensates and hadron masses in a nuclear medium, and its applications on nuclear observables (Celenza *et al*, 1992, Brown *et al*, 1991 and Cohen *et al*, 1992). One naturally expects that in dense nuclear medium, physical observables should be greatly reduced than their vacuum values. We may study the scaling behaviour of the chiral condensation energy density in a nuclear medium easily in our formalism. We begin by examining the chiral condensation energy density for the $T = 0$ case (i.e., Eq.(4.2.14)):

$$\Delta\mathcal{E}_\chi = 2\,[\,\sum_p(\epsilon_p - |p|)\theta(\Lambda^2 - \mathrm{p}^2) - \frac{M^2}{8g}]\,, \tag{4.4.1}$$

This quantity may be calculated by replacing $\sum_p$ by $\frac{1}{(2\pi)^3}\int d^3p$. Note that $(M/\Lambda)^2 \approx 0.1 << 1$, neglecting $(M/\Lambda)^4$ and higher power terms, a simple relation for $\Delta\mathcal{E}_\chi$ may be obtained:

$$\Delta\mathcal{E}_\chi \simeq \frac{M^2\Lambda^2}{4\pi^2}\left[1 - \frac{\pi^2}{g\Lambda^2}\right]\,. \tag{4.4.2}$$

For $g\Lambda^2 < \pi^2$, there is no symmetry breaking, and $M = 0$. On the other hand, from Eq.(4.2.17), the quark condensate is given by

$$<\bar{\psi}\psi> = -2M\mathcal{V} = -\frac{M}{2g}\,. \tag{4.4.3}$$

The chiral condensation energy density is thus related to the quark condensate in the NJL model as

$$\Delta\mathcal{E}_\chi \simeq -\frac{M}{2\pi^2} < \bar{\psi}\psi > [g\Lambda^2 - \pi^2] \,, \tag{4.4.4}$$

where $M = M_n/3$ with M_n the nucleon mass, and $g\Lambda^2 > \pi^2$. Assuming that, in a nuclear medium, the chiral condensation energy density has the same form, and the dimensionless quantity $g\Lambda^2$ is a constant, i.e.,

$$g\Lambda^2 = g^*(\Lambda^*)^2 \,, \tag{4.4.5}$$

the ratio of $\Delta\mathcal{E}_\chi$ and $\Delta\mathcal{E}_\chi^*$ is thus given by

$$\frac{(\Delta\mathcal{E}_\chi)^*}{(\Delta\mathcal{E}_\chi)} = \frac{M^* < \bar{\psi}\psi >^*}{M < \bar{\psi}\psi >} \,, \tag{4.4.6}$$

where the asterixes refer to the quantities evaluated in the nuclear medium. There are two kinds of scaling behaviour for the quark condensate in the NJL model used in the literature. One is to assume that the coupling constant g is invariant, i.e., $g = g^*$ (or, $\Lambda = \Lambda^*$) (Cohen *et al*, 1991). Therefore, from Eq.(4.4.3)

$$\frac{< \bar{\psi}\psi >^*}{< \bar{\psi}\psi >} = \frac{M^*}{M} \,, \tag{4.4.7}$$

and hence

$$\frac{(\Delta\mathcal{E}_\chi)^*}{(\Delta\mathcal{E}_\chi)} = (\frac{M^*}{M})^2 \,. \tag{4.4.8}$$

The other is to assume that the momentum cutoff (or the coupling constant (Brown, 1991)) varies linearly with the dynamically generated mass M. Thus, Eq.(4.4.5) implies that $g/g^* = (\Lambda^*)^2/\Lambda^2 = (M^*)^2/M^2$, again from Eq.(4.4.3), giving

$$\frac{< \bar{\psi}\psi >^*}{< \bar{\psi}\psi >} = \frac{(M^*)^3}{M^3} \,, \tag{4.4.9}$$

and

$$\frac{(\Delta\mathcal{E}_\chi)^*}{(\Delta\mathcal{E}_\chi)} = (\frac{M^*}{M})^4 . \qquad (4.4.10)$$

In dense nuclear matte, the ratio of M^*/M is about 0.7 to 0.8 (Siemens and Jensen, 1987). Eq.(4.4.8) shows that the energy density required to restore chiral symmetry in a dense nuclear matter drops to about 50% – 65% of its vacuum value; whereas Eq.(4.4.10) shows that the value drops even lower to 25%-40% of its vacuum value. However, the greater deduction of the vacuum value is somewhat expected. Our finite temperature calculation suggested that the chiral energy density obeys an $M(T)^4$ power law, showing support for the second kind of scaling behaviour (Eq.(4.4.10)). (If the universality of scaling law is correct.) These should have appreciable effect on nuclear observables (Brown, 1991, Brown *et al*, 1991, Brown and Rho, 1991 Dey *et al*, 1985 and Deys, 1986) and the correctness of which scaling scheme will be judged.

4.5 CONCLUDING REMARKS

In summary, we have calculated the vacuum condensation energy (or free energy when $T \neq 0$) per unit volume within the NJL model by using a semi-classical method and the mean-field approximation. We argued that this quantity is identified as the phenomenological MIT bag constant. The temperature dependence of the NJL vacuum condensation free energy per unit volume throws some light on the temperature dependence of the MIT bag constant. We found that the "bag constant" vanishes when the critical temperature T_c is reached. This implies that, when the dynamical quark mass goes to zero, chiral symmetry is restored. The melting of gluon condensate needs much higher energy which is not calculable from the NJL model, a conclusion consistent with that suggested from lattice QCD simulations (Brown *et al*, 1991). We have also extended the formalism of da Providencia *et al* (da Providencia *et al*, 1987) to finite temperatures, yielding the same gap equation as that in finite-temperature field theory. Some thermodynamical quantities are calculated using our method. The thermodynamics of the vacuum in the two phases has been studied, showing all the features of a second order phase transition, a result supported by lattice gauge calculations in the chiral limit (Brown *et al*, 1991 and Karsch *et al*, 1987), and displaying striking similarity with the nonrelativistic BCS theory of superconductivity. Finally, the scaling behaviour of the chiral condensation energy density in a nuclear medium is examined. In view of different schemes used within the NJL model in the literature (Cohen *et al*, 1991 and Brown, 1991), two possible scalings are obtained.

Appendix A

This appendix gives detailed calculations for chapter 1.

First, let us derive the density function $\eta(x) = u^2(x) + v^2(x)$ from Eqs.(1.2.9) and (1.2.12). Note that Eqs.(1.2.9) and (1.2.12) may be rewritten as

$$\frac{d\eta}{\eta} = -m\cos 2\phi \, dx \ , \qquad (A-1)$$

and

$$dx = \frac{2}{\beta}\frac{1}{E - m\sin 2\phi}d\phi \ . \qquad (A-2)$$

Therefore, by the substitution for dx in Eq.(A-1) and direct integration, the relation between η and ϕ is obtained:

$$\ln\eta = \frac{1}{\beta}\ln|E - m\sin 2\phi| + c \ .$$

where c is an integration constant. It is obvious to rewite the above relation as

$$\eta = N|m \sin 2\phi - E|^{\frac{1}{\beta}} , \qquad (A-3)$$

where N is an arbitrary constant related to c.

For the zero-mode, from Eq.(1.2.13)

$$\cos 2\phi_0 = \tanh \beta m x ,$$

hence,

$$\sin 2\phi_0 = \sqrt{1-\cos^2 2\phi_0}$$
$$= \sqrt{1-\tanh^2 \beta m x} = (\cosh \beta m x)^{-1} .$$

From Eq.(A-3), the result of η for the zero-mode gives

$$\eta_0(x) = N(\cosh \beta m x)^{-\frac{1}{\beta}}$$

This is Eq.(1.2.15) in the text.

Similarly, for the non-zero-energy bound state, the solution of $\phi(x)$ from Eq.(1.2.20) may be rewritten as

$$\tan\phi = \frac{1}{E}(m - k\tanh(\frac{k\beta x}{2})) . \qquad (A-4)$$

We assumed $E > 0$ without loss of generality. Thus, $\sin 2\phi$ may be expressed in term of $\tan\phi$:

$$\sin 2\phi = 2\tan\phi\cos^2\phi$$
$$= 2\frac{\tan\phi}{1+\tan^2\phi} ,$$

and note that

$$\begin{aligned} m\sin 2\phi - E &= \frac{2m\tan\phi - E(1+\tan^2\phi)}{1+\tan^2\phi} \\ &= \frac{2mE(m-k\tanh(\frac{k\theta x}{2})) - E(E^2+[m-k\tanh(\frac{k\theta x}{2})])}{E^2+[m-k\tanh(\frac{k\theta x}{2})]^2} \\ &= \frac{E[(m^2-E^2)-k^2\tanh^2(\frac{k\theta x}{2})]}{E^2+[m-k\tanh(\frac{k\theta x}{2})]^2} \\ &= \frac{Ek^2\cosh^{-2}\frac{k\theta x}{2}}{E^2+[m-k\tanh(\frac{k\theta x}{2})]^2}\,. \end{aligned} \qquad (A-5)$$

Based on Eqs.(A-3) and (A-5), the Eq.(1.2.22) is obtained for $E > 0$. For negative E, we should simplely replace E by $|E|$ in the same equation.

Second, we verify the relation given by Eq.(1.3.8), i.e.,

$$-\frac{1}{2}\epsilon_{\mu\nu}\partial^\nu\theta_c = \lambda^2\bar{q}\gamma_\mu q \qquad (A-6)$$

where θ_c is given by Eq.(1.3.3) and q by Eq.(1.3.1) for the finite energy bound state, or θ_c by Eq.(1.3.6) and q by Eq.(1.3.5) for the zero-mode.

The left hand side of Eq.(A-6) may be calculated straightforwardly by noting that $-\frac{1}{2}\epsilon_{10}\partial^0\theta_c(x) = \frac{1}{2}\frac{\partial\theta_c}{\partial t} = 0$ for $\mu = 1$ for a static soliton. Therefore, for the zero-mode, substituting for θ_{0c} from Eq.(1.3.6), we obtain the left hand side of Eq.(A-6) for $\mu = 0$:

$$\begin{aligned} -\frac{1}{2}\epsilon_{01}\partial^1\theta_{0c} &= \frac{1}{2}\frac{d\theta_{0c}}{dx} \\ &= \frac{-1}{2}4\frac{-2me^{-2mx}}{1+e^{-4mx}} = \frac{2m}{\cosh 2mx}\,. \end{aligned} \qquad (A-7)$$

Similarly, for $|E| \neq 0$ bound state solution, substituting for θ_c from Eq.(1.3.3), we obtain

$$\begin{aligned} -\frac{1}{2}\epsilon_{01}\partial^1\theta_c &= -\frac{1}{2}\frac{d\theta_c}{dx} \\ &= \frac{-1}{2}4\frac{-Ek^2}{\cosh^2 kx[E^2+(m-k\tanh kx)^2]}\,. \end{aligned} \qquad (A-8)$$

The right hand side of Eq.(A-6) for $\mu = 1$ may be determined by noting that $q^\dagger\gamma_0\gamma_1 q = q^\dagger\gamma_5 q = 0$. For $\mu = 0$, for the zero-mode, from Eq.(1.3.5)

$$\begin{aligned}\lambda^2\bar{q}_0\gamma_0 q_0 &= \lambda^2 q_0^\dagger q_0 \\ &= \lambda^2\frac{m}{\pi}\frac{1}{\cosh^2 2mx}(e^{2mx}+e^{-2mx}) = \lambda^2\frac{2m}{\pi\cosh 2mx}\,. \end{aligned} \qquad (A-9)$$

For nonzero energy bound state solution, substituting for q from Eq(1.3.1), we have

$$\begin{aligned}\lambda^2\bar{q}\gamma_0 q &= \lambda^2 q^\dagger q \\ &= \lambda^2\frac{N^2k^2E}{\cosh^2 kx[E^2+(m-k\tanh kx)]}\,. \end{aligned} \qquad (A-10)$$

Recall that for the zero-mode, $\lambda^2 = \pi$. Therefore Eq.(A-6) is true for the zero-mode solution. Further, for nonzero energy bound state, we have $\lambda^2 = \frac{2}{N^2}$, thus relation (A-6) is also true for nonzero energy bound state. This completes the proof for Eq.(1.3.8).

We now try to find analytical expression for the bound state energy from the normalization condition (1.2.12).

Note first that Eq.(1.2.12) may be rewritten in the form

$$dx = \frac{2}{\beta}\frac{1}{E - m\sin 2\phi}\,d\phi\,, \qquad (A-11)$$

thus the normalization condition

$$\int_{-\infty}^{+\infty}\eta^2(x)\,dx = 1\,, \qquad (A-12)$$

may be expressed in terms of the integral variable ϕ when we substitute for η from Eq.(1.2.22) and for dx from Eq.(A-11):

$$\int_{\phi^-}^{\phi^+} N^2(m\sin\,2\phi - E)^{\frac{2}{3}}\frac{2}{\beta}\frac{1}{E - m\sin\,2\phi}\,d\phi = 1\,, \qquad (A-13)$$

where $k = \sqrt{m^2 - E^2}$, $\phi^+ = \phi(+\infty) = \arctan\frac{m-k}{E}$, and $\phi^- = \phi(-\infty) = \arctan\frac{m+k}{E}$.

From the above equation, we note that for any β satisfying $n = \frac{2}{\beta} - 1$, with n an integer, an analytic expression for E can always be found. However, no general expression of the whole spectrum can be found. In the following, the bound state energies for $n = 0,\ 1,\ 2,\ 3$ or $\beta = 2,\ 1,\ 2/3,\ 1/2$ are calculated from the Eq.(A-13).

For $\beta = 2$, integration of the LHS of Eq.(A-13) is straightforward, which gives

$$\int_{\phi^-}^{\phi^+} N^2(-1)d\phi = -N^2\Delta\phi \ ,$$

where $\Delta\phi = \phi^+ - \phi^- = -\arctan\frac{k}{E}$. Combining with Eq.(A-13), we obtain the condition

$$\arctan\frac{m-k}{E} - \arctan\frac{m+k}{E} = -\frac{1}{N^2} \ ,$$

this implies

$$E = m\cos\frac{1}{N^2} \ . \qquad (A-14)$$

For $\beta = 1$, the LHS of Eq.(A-13) becomes

$$\begin{aligned}
&\int_{\phi^-}^{\phi^+} N^2(-2)(m\sin 2\phi - E)\, d\phi \\
&= N^2(m\cos 2\phi + 2E\phi)|_{\phi_-}^{\phi^+} \\
&= N^2\left(m\frac{1-\tan^2\phi}{1+\tan^2\phi}|_{\phi^-}^{\phi^+} + 2E\Delta\phi\right) \\
&= 2N^2\left(k - E\arctan\frac{k}{E}\right) \ ,
\end{aligned}$$

Therefore, the eigen-energy relation for $\beta = 1$ is given by

$$k - E\arctan\frac{k}{E} = \frac{1}{2N^2} \ . \qquad (A-15)$$

Similarly, for $\beta = \frac{1}{2}$, the LHS of Eq.(A-13) may be calculated

$$\int_{\phi-}^{\phi+} N^2(m\sin 2\phi - E)^4 4\frac{1}{E - m\sin 2\phi}\,d\phi$$
$$= -4N^2\int_{\phi-}^{\phi+}(m^3\sin^3 2\phi - E^3 - 3Em^2\sin^2 2\phi + 3mE^2\sin 2\phi)\,d\phi$$
$$= -4N^2[m^3(-\frac{1}{2}\cos 2\phi + \frac{1}{6}\cos^3 2\phi) - E^3\phi - 3Em^2(\frac{1}{2}\phi - \frac{1}{8}\sin 2\phi)$$
$$+ 3mE^2(-\frac{1}{2})\cos 2\phi]_{\phi-}^{\phi+}$$
$$= -4N^2[\frac{k}{3E^4m^2}(m^2 - 3E^2)(m^2E^4 + 16m^6 - E^6)$$
$$- m^2k + \frac{3}{2}kE^2 + \frac{E}{2}(2E^2 + 3m^2)\arctan\frac{k}{E}]\,,$$

and the equation for E is obtained for $\beta = \frac{1}{2}$

$$[\frac{k}{3E^4m^2}(m^2 - 3E^2)(m^2E^4 + 16m^6 - E^6)$$
$$- m^2k + \frac{3}{2}kE^2 + \frac{E}{2}(2E^2 + 3m^2)\arctan\frac{k}{E}] = -\frac{1}{4N^2}\,. \qquad (A-16)$$

For $\beta = \frac{2}{3}$, the LHS of Eq.(A-13) gives

$$- 3N^2\int_{\phi-}^{\phi+}(m\sin 2\phi - E)^2 d\phi$$
$$= -3N^2\int_{\phi-}^{\phi+}(m^2\sin^2 2\phi + E^2 - 2mE\sin 2\phi)\,d\phi$$
$$= -3N^2[\frac{m^2}{2}(\phi - \frac{1}{4}\sin 4\phi) + E^2\phi + mE\cos 2\phi]_{\phi-}^{\phi+}$$
$$= \frac{3N^2}{2}[(m^2 + 2E^2)\arctan\frac{k}{E} - 3kE]\,.$$

The final relation for the eigen-energy is thus given by

$$(m^2 + 2E^2)\arctan\frac{k}{E} - 3kE = \frac{2}{3N^2}\,. \qquad (A-17)$$

For the case of $\beta \to \infty$, the expression of the soliton reduces to a step function, which is called an "infinitely thin soliton". The Dirac vector potential behaves like a delta function. These can be checked by examining the solution Eqs.(2.16) and (2.21) in the limit of $\beta \to \infty$: From Eq.(2.26)

$$\theta_c = -(2+\frac{2}{\beta})\arctan[\frac{1}{E}(m - k\tanh\frac{k\beta x}{2})] + \theta_0$$
$$\to \begin{cases} -2\arctan\frac{m-k}{E} + \theta_0, & x > 0; \\ -2\arctan\frac{m+k}{E} + \theta_0, & x < 0, \end{cases} \qquad (A-18)$$

where θ_0 is an integration constant. The above result shows that $\theta_c(x)$ becomes indeed a step function in the $\beta \to \infty$ limit.

From Eq(2.21),

$$V_D = -(\frac{\beta}{2}+1)\frac{k^2 E\cosh^{-2}\frac{k\beta x}{2}}{E^2 + (m - k\tanh\frac{k\beta x}{2})^2}$$
$$\to \begin{cases} 0, & x \neq 0, \\ -\infty, & x = 0. \end{cases} \qquad (A-19)$$

This proves that V_D behaves indeed like a delta function when $\beta \to \infty$.

Appendix B

First, we give some detailed derivations in section 4 of chapter 2.

For Eq.(2.4.5), assuming particle density is cylindrically symmetric, i.e., $\rho(\mathbf{r}) = \rho(r)$, the angular integration in Eq.(2.4.5) can be carried out easily, and the auxiliary function also becomes angle independent, namely, $\phi(\mathbf{r}) = \phi(r)$, as may be seen from the following:

$$\begin{aligned}
\phi &= \int \rho(\mathbf{r}') \ln|\mathbf{r} - \mathbf{r}'| d\mathbf{r}' \\
&= \int \rho(r') \frac{1}{2} \ln(|\mathbf{r} - \mathbf{r}'|^2) d\mathbf{r}' \\
&= \int \int_0^{2\pi} \rho(r') \ln(r^2 + r'^2 - 2rr' \cos\theta) r' dr' d\theta \\
&= \int \rho(r') 2\pi r' \ln r_> dr' \,,
\end{aligned} \qquad (B-2)$$

where

$$r_> = \begin{cases} r & r > r'; \\ r' & r' > r. \end{cases}$$

Therefore, from Eq.(2.4.4), the vector potential **A** may be expressed as (here a scalar may be considered as a vector in the $\hat{z}$ direction):

$$\begin{aligned}\mathbf{A} &= -\alpha \nabla \times \phi(r)\hat{z} \\ &= -\alpha \left(\int^r \rho(r')2\pi r' dr' \right) \nabla \times \hat{z} \ln r \\ &= \alpha \frac{\hat{z} \times \mathbf{r}}{r^2} N(r) \,, \end{aligned} \qquad (B-3)$$

where

$$N(r) = \int^r \rho(r')2\pi r' dr'$$

is just the particle number function, and

$$N(r \to \infty) = N$$

gives the normalization condition.

From the simple expression of **A**, it is easy to express $\mathbf{A} \cdot \mathbf{p}$ and $\mathbf{A}^2$ in terms of $N(r)$, namely,

$$\mathbf{A} \cdot \mathbf{p} = \frac{\alpha N(r)}{r^2} (\hat{z} \times \mathbf{r} \cdot \mathbf{p}) = \frac{\alpha N(r)}{r^2} (\mathbf{r} \times \mathbf{p})_z = \frac{\alpha N(r)}{r^2} l \,, \qquad (B-4)$$

with $l = (\mathbf{r} \times \mathbf{p})_z = r p_\theta$, and

$$\mathbf{A}^2 = \frac{\alpha^2 N^2(r)}{r^4} (\hat{z} \times \mathbf{r}) \cdot (\hat{z} \times \mathbf{r}) = \frac{\alpha^2 N^2(r)}{r^2} \,, \qquad (B-5)$$

where the identity $(\hat{z} \times \mathbf{r}) \cdot (\hat{z} \times \mathbf{r}) = r^2$ is used.

We finally obtain the effective single particle Hamiltonian h of Eq.(2.4.8):

$$\begin{aligned} h &= \frac{1}{2m} (\mathbf{p} + \mathbf{A})^2 + \frac{1}{2} m \omega^2 r^2 \\ &= \frac{1}{2m} [\mathbf{p}^2 + \frac{2\alpha l N(r)}{r^2} + \frac{\alpha^2 N^2(r)}{r^2}] + \frac{1}{2} m \omega^2 r^2 \,. \end{aligned} \qquad (B-6)$$

The relationship between $\rho(r)$ and $\rho_l(r)$ in Eq.(2.4.12) may also be derived from quantum mechanical considerations. Note that with the l-dependence of the effective Hamiltonian Eq.(2.4.8), the Schrödinger equation reads:

$$[\frac{p_r^2}{2m} + \frac{(l+\alpha N(r))^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2]\psi_l(r) = E_l\psi_l(r) \ . \qquad (B-7)$$

Here $\psi_l(r)$ is a wave function for a particle with angular momentum l, or the partial wave function, and E_l is the energy eigenvalue. Recall that the operator $p_r^2 = -\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}$, we can define $\psi_l(r) = \frac{u_l(r)}{\sqrt{r}}$ and the Schrödinger equation (B-7) is simplified to

$$(\frac{p^2}{2m} - \frac{1}{4r^2(2m)} + \frac{(l+\alpha N(r))^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2)u_l = E_l u_l \ . \qquad (B-8)$$

or

$$(\frac{p^2}{2m} + \frac{(l+\alpha N(r)+\frac{1}{2})(l+\alpha N(r)-\frac{1}{2})}{2mr^2} + \frac{1}{2}m\omega^2 r^2)u_l = E_l u_l \ . \qquad (B-9)$$

Here, $p^2 = -\frac{d^2}{dr^2}$ is effectively a one dimensional Laplace operator and l is an integer to assure the singlevaluedness of wave function. The normalization condition should read

$$\sum_l \int \psi_l^2(r)\, dr = \sum_l \int 2\pi u_l^2(r)\, dr = N \ . \qquad (B-10)$$

The particle density may be defined as

$$\rho(r) = \sum_l \psi_l^2(r) = \sum_l \frac{u_l^2(r)}{r} \ , \qquad (B-11)$$

and we define a partial wave density as

$$\rho_l(r) = 2\pi u_l^2(r) \ . \qquad (B-12)$$

Thus $\rho(r)$ and $\rho_l(r)$ are related by

$$\rho(r) = \frac{1}{2\pi r}\sum_l \rho_l(r) . \qquad (B-13)$$

In the classical limit, the summation over discrete angular momenta should be replaced by a continuous integral: i.e., $\sum_l \to \int dl$. Therefore, in the classical limit, the above relation becomes

$$\rho(r) = \frac{1}{2\pi r}\int \rho_l(r) dl . \qquad (B-13a)$$

To obtain the total particle density $\rho(r)$ from the partial wave density $\rho_l(r)$, we need to carry out the integration over l in Eq.(2.4.14), i.e.

$$\rho(r) = \frac{1}{2\pi^2 r}\int_{-l_{max}}^{l_{max}} \left(2m\mu - r^2 - \frac{(l+\alpha N(r))^2}{r^2}\right)^{1/2} d\,l , \qquad (B-14)$$

where l_{max} is determined by the consideration that the above integral is valid only in the classically allowed region, i.e.,

$$2m\mu - r^2 - \frac{(l+\alpha N(r))^2}{r^2} \geq 0 , \qquad (B-15)$$

hence,

$$l_{max} = \sqrt{r^2(2m\mu - r^2)} - \alpha N(r) . \qquad (B-16)$$

We insist that the integral over l must be made symmetrically with respect to $l = 0$, hence,

$$-l_{max} = -\sqrt{r^2(2m\mu - r^2)} + \alpha N(r) . \qquad (B-17)$$

If we consider the effective angular momentum of anyon as $l + \alpha N(r)$, then the l-integral with respect to this quantity is no longer symmetric once $\alpha \neq 0$. This results in explicitly broken time reversal symmetry in the system. In order to carry out the integration with respect to l, we introduce the following quantities which are independent of the integral variable l to simplify the notations:

$$a = \sqrt{2m\mu - r^2}, \quad c = \alpha N(r)\,, \qquad (B-18)$$

and define a new integration variable as x: $x = l + c$. The integration limits are then given by

$$\begin{aligned} x_{max} &= l_{max} + c = a \\ x_{min} &= -l_{max} + c = -a + 2c \end{aligned} \qquad (B-19)$$

Finally, the integral of Eq.(B-14) is ready to be carried out:

$$\begin{aligned} \rho(r) &= \frac{1}{2\pi^2 r^2}\int_{-a+2c}^{a} \sqrt{a^2 - x^2}dx \\ &= \frac{1}{2\pi^2 r^2}\frac{1}{2}[x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a}]_{-a+2c}^{a} \\ &= \frac{1}{4\pi^2 r^2}[a^2(\frac{\pi}{2} + \arcsin\frac{a-2c}{a}) + 2(a-2c)\sqrt{ac - c^2}]\,, \end{aligned} \qquad (B-20)$$

substituting a and c from Eq.(B-18), the above equation reduces to Eq.(2.4.15).

The integral of ΔE in Eq.(2.4.19) with respect to l is calculated similarly such that

$$\Delta E = \int dr \int dl \left(\frac{\alpha N(r) l}{r^2}\rho_l(r)\right) = \int dr \frac{\alpha N(r)}{\pi r^3} I\,, \qquad (B-21)$$

where $I = \int_{-a+2c}^{a} \sqrt{a^2 - x^2}(x - c)dx$. The result is

$$\begin{aligned} I &= \int_{-a+2c}^{a} \sqrt{a^2 - x^2}x dx - c\int_{-a+2c}^{a} \sqrt{a^2 - x^2}dx \\ &= \frac{1}{3}[c(a-c)]^{3/2} - c(2\pi^2 r^2 \rho(r))\,. \end{aligned} \qquad (B-22)$$

Substituting in a and c from Eq.(B-18), the result in equation (2.4.19) can be found.

Having done the above integrations, Eq.(2.4.20) is obvious.

The following is for section 5 in chapter 2.

For the toy test model in section 5, we prove here by an algebraic method that $\dot{\theta}_{ij} = \frac{1}{2}(\dot{\theta}_i + \dot{\theta}_j)$ (i.e., Eq.(2.5.4)).

Note that from the definitions,

$$\begin{aligned} \tan\theta_{ij} &= \frac{y_i - y_j}{x_i - x_j} \\ &= \frac{r_i \sin\theta_i - r_j \sin\theta_j}{r_i \cos\theta_i - r_j \cos\theta_j} \,. \end{aligned}$$

Applying the constraint of particles moving on a unit circle, i.e., the condition

$$r_i = r_j = R = 1$$

to the above relation, we get the simple relation

$$\tan\theta_{ij} = \frac{\sin\theta_i - \sin\theta_j}{\cos\theta_i - \cos\theta_j} \,. \qquad (B-23)$$

Applying time differentiation to both side in the above equation and from the LHS,

$$\begin{aligned} \frac{d}{dt}\tan\theta_{ij} &= \frac{1}{\cos^2\theta_{ij}}\dot{\theta}_{ij} \\ &= [1 + \tan^2\theta_{ij}]\,\dot{\theta}_{ij} \\ &= [1 + \left(\frac{\sin\theta_i - \sin\theta_j}{\cos\theta_i - \cos\theta_j}\right)^2]\,\dot{\theta}_{ij} \\ &= 2\frac{1 - \cos(\theta_i - \theta_j)}{(\cos\theta_i - \cos\theta_j)^2}\,\dot{\theta}_{ij} \,, \end{aligned} \qquad (B-24)$$

a little algebra gives the time derivative of the RHS of Eq.(B-23):

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\sin\theta_i - \sin\theta_j}{\cos\theta_i - \cos\theta_j}\right) \\ &= \frac{(\cos\theta_i\dot{\theta}_i - \cos\theta_j\dot{\theta}_j)(\cos\theta_i - \cos\theta_j) - (\sin\theta_i - \sin\theta_j)(-\sin\theta_i\dot{\theta}_i + \sin\theta_j\dot{\theta}_j)}{(\cos\theta_i - \cos\theta_j)^2} \\ &= \frac{1 - \cos(\theta_i - \theta_j)}{(\cos\theta_i - \cos\theta_j)^2}(\dot{\theta}_i + \dot{\theta}_j) \,. \end{aligned} \qquad (B-25)$$

Equating LHS and RHS, we obtain the simple relation

$$\dot{\theta}_{ij} = \frac{1}{2}(\dot{\theta}_i + \dot{\theta}_j) . \qquad (B-26)$$

This is Eq.(2.5.4) in the text. (In fact, by geometrical method we can prove further that

$$\theta_{ij} = \frac{1}{2}(\theta_i + \theta_j - \pi) .$$

Because we need only the expression of $\dot{\theta}_{ij}$, we will not give the proof for the last relation here.)

Appendix C

First, we give some detail derivations in section 2 of chapter 3.

The formal solution of the vector potential may be obtained by substituting Eq.(3.2.18) into Eq.(3.2.17), i.e.,

$$(\partial_x - ieA_x)\rho^{\frac{1}{2}}e^{ie\omega} = \mp i(\partial_y - ieA_y)\rho^{\frac{1}{2}}e^{ie\omega} . \qquad (C-1)$$

Applying the differential operators on the wave function, we have

$$(\frac{1}{2}\rho^{-\frac{1}{2}}\partial_x\rho + \rho^{\frac{1}{2}}ie\partial_x\omega - ieA_x\rho^{\frac{1}{2}})e^{ie\omega} = \mp i(\frac{1}{2}\rho^{-\frac{1}{2}}\partial_y\rho + \rho^{\frac{1}{2}}ie\partial_y\omega - ieA_y\rho^{\frac{1}{2}})e^{ie\omega} . \qquad (C-2)$$

Comparing the coefficients of the real part and the imaginary part on both side of Eq.(C-2), we finally obtain

$$A_x = \partial_x\omega \pm \frac{1}{2e}\rho^{-1}\partial_y\rho$$

and

$$A_y = \partial_y \omega \mp \frac{1}{2e}\rho^{-1}\partial_x \rho \,.$$

The above component forms of **A** may be written in a compact vector form as follows

$$\mathbf{A} = \nabla\omega \mp \frac{1}{2e}\nabla \times \ln\rho \,, \qquad (C-3)$$

This is just the Eq.(3.2.19) in the text.

To obtain the Liouville equation for the density ρ, from the definition of ϕ, i.e.,

$$\mathbf{A} = \nabla\omega + \nabla \times \phi \,, \qquad (C-4)$$

when compared with Eq.(C-3), we have obviously that

$$\ln\rho = \mp 2e\phi \,. \qquad (C-5)$$

After applying the operator ∇^2 on both sides in the above equation and making use of the Eq.(3.2.15), we get Eq.(3.2.20) in the text:

$$\nabla^2 \ln\rho = \mp\frac{2e^2}{k}\rho \,. \qquad (C-6)$$

This is the Liouville equation, and its solution is well studied.

The charge of the system is obtained by the direct integration for Eq.(3.2.23):

$$Q = e\int \rho \, d^2r \,. \qquad (C-7)$$

Substitute in ρ from the soliton solution Eq.(3.2.21a), we have

$$\begin{aligned} Q &= e\int \frac{4n^2}{\alpha}\frac{1}{r^2[(\frac{r}{r_0})^n + (\frac{r_0}{r})^n]^2}2\pi r dr \\ &= \frac{8\pi e n^2}{\alpha}\int \frac{dx}{x(x^n + x^{-n})^2} \\ &= \frac{8\pi e n^2}{\alpha}\int \frac{x^{2n-1}dx}{(x^{2n}+1)^2} \end{aligned}$$

$$= \frac{8\pi e n^2}{2|n|\alpha} \int \frac{dx^{2n}}{(x^{2n}+1)^2}$$
$$= \frac{4\pi e|n|}{\alpha}[-\frac{1}{x^{2n}+1}]|_0^\infty = \frac{4\pi e|n|}{\alpha} ,$$

where $x = (\frac{r}{r_o})$. The result is independent on r_0. As $\alpha = \frac{e^2}{|k|}$, the charge is given in the form by Eq.(3.2.23) in the text.

It is interesting to see that the charge and the flux are related by the modified Maxwell equation (3.2.2), since the integartion over the entire plane on both side gives

$$\Phi = \int B d^2 r = -\frac{e}{k} \int \rho d^2 r = -\frac{Q}{k} . \qquad (C-8)$$

Thus the flux is given by

$$\Phi = -\frac{|k|}{k}\frac{2\pi}{e} 2|n| , \qquad (C-9)$$

which is a quantized even integer (in units of flux quanta $2\pi/e$).

For section 3, we want to prove Eq.(3.3.7), i.e.,

$$\nabla \times [(\mathbf{u}\cdot\nabla)\mathbf{u}] = -\nabla \times (\mathbf{u} \times \xi) = \nabla \cdot (\xi \mathbf{u}) . \qquad (C-10)$$

The LHS of the first equation may be recast as the component form

$$\epsilon^{ij}\partial_i[(u_l\partial_l)u_j]$$
$$=\epsilon^{ij}\partial_i u_j[(u_l\partial_l)] + \epsilon^{ij}(\partial_l u_j)\partial_i u_l ,$$

where the sum over repeated indexes is implied. In two spatial dimensions, indexes i, j, l run from 1 to 2. Therefore, it is always true that two of the three indexes take the same number (say, $l = i$). Therefore, the second term in above equation may be rewritten as

$$\epsilon^{ij}(\partial_i u_j)(\partial_i u_i) .$$

Recall that the circulation function ξ is defined as $\xi = \nabla \times \mathbf{u} = \epsilon^{ij}\partial_i u_j$, thus it is easy to identify the first term in the expansion of the LHS as $\mathbf{u}\cdot(\nabla\xi)$, and the second term as $\xi\nabla\cdot\mathbf{u}$ due to the above argument. Together, we obtain

$$LHS = \mathbf{u}\cdot(\nabla\xi) + \xi\nabla\cdot\mathbf{u} = \nabla\cdot(\xi\mathbf{u})\,. \qquad (C-11)$$

This is nothing but the right hand side of the second equation in Eq.(C-10).

The proof of the second equation is more straightforward. As in the first case, the component form of the LHS gives

$$-\nabla\times(\mathbf{u}\times\xi) = -\epsilon^{ij}\partial_i(\epsilon_{jl}u^l\xi) = \partial_l(u_l\xi) = \nabla\cdot(\xi\mathbf{u}) = RHS\,.$$

We have used the relation $\epsilon^{ij}\epsilon_{jl} = -\delta_{ij}$ and did not distinguish the upper or lower indexes.

Finally, we give the proof that the energy functional given by Eq.(3.4.9) in section 4 indeed gives rise to the Liouville equation of motion for the particle density. To show this, let us start from the variational equation (3.4.8),

$$\delta\int(E(n) - \lambda n)\; d^2r = 0\,. \qquad (C-12)$$

For the energy density functional given by Eq.(3.4.9), i.e., $E(n) = a(\nabla \ln n)^2$,

$$\delta E(n) = 2a\nabla\ln n\;\nabla(\frac{\delta n}{n})\,. \qquad (C-13)$$

Performing integration by parts over the space, we find that

$$\int \delta E(n) d^2r = 2a\int\frac{1}{n}\nabla^2\ln n\;\delta n\; d^2r\,. \qquad (C-14)$$

Therefore, the Eq.(C-12) gives the steady condition for the density n:

$$\nabla^2\ln n = -\frac{\lambda}{2a}n\,, \qquad (C-15)$$

and this is just the Liouville equation. The constant a is given by $a = (\frac{\hbar c}{r_0})$ and the Lagrangian constant λ is chosen as $\lambda = \frac{4a}{k} = \frac{4\hbar c}{r_0 k}$ in the text.

Appendix D

This appendix gives some detailed calculations for chapter 4.

We first derive Eq.(4.2.1) from Eq.(4.1.1). The Lagrangian is given by

$$\mathcal{L} = \bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi + g[(\bar{\psi}\psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2] . \tag{D-1}$$

The canonical momentum is given by

$$\Pi_0 = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \bar{\psi} i \gamma^0 .$$

The Hamiltonian of the system may be calculated from

$$\begin{aligned}\mathcal{H} &= \Pi_0 \dot{\psi} - \mathcal{L} \\ &= -i\bar{\psi}\vec{\gamma} \cdot \nabla \psi - g[(\bar{\psi}\psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2] .\end{aligned} \tag{D-2}$$

The total energy is given by

$$
\begin{aligned}
E &= \int \mathcal{H}\, d\mathbf{x} \qquad (D-3)\\
&= \int -i\bar{\psi}\vec{\gamma}\cdot\nabla\psi\, d\mathbf{x} - g\int [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2]\, d\mathbf{x}\\
&= \int \psi^\dagger\vec{\alpha}\cdot\mathbf{p}\psi d\mathbf{x} - g\int\int d\mathbf{x}_1 d\mathbf{x}_2\delta(\mathbf{x}_1-\mathbf{x}_2)[(\bar{\psi}\psi)(\mathbf{x}_1)(\bar{\psi}\psi)(\mathbf{x}_2) - (\bar{\psi}\gamma_5\psi)(\mathbf{x}_1)(\bar{\psi}\gamma_5\psi)(\mathbf{x}_2)]\\
&= Tr_1\rho(1)t(1) - gTr_1Tr_2\delta(\mathbf{x}_1-\mathbf{x}_2)[\rho(1)\gamma_0(1)\rho(2)\gamma_0(2) - \rho(1)\gamma_0(1)\gamma_5(1)\rho(2)\gamma_0(2)\gamma_5(2)]\,.
\end{aligned}
$$

Here, the density matrix ρ and the kinetical momentum of a single particle t are defined by

$$\rho(1) = \rho(\mathbf{x}_1) = \psi\psi^\dagger(\mathbf{x}_1)\,,$$

and

$$t(1) = \vec{\alpha}(1)\cdot\mathbf{p}(1)\,,$$

respectively. Since for a N-particle system, the Hamiltonian is expressed as

$$\mathcal{H} = \sum_i t(i) + \frac{1}{2}\sum_{i\neq j} V(ij)\,, \qquad (D-4)$$

we can identify $V(12)$ as

$$V(12) = -2g\delta(\mathbf{x}_1-\mathbf{x}_2)[\beta(1)\beta(2) - \beta(1)\gamma_5(1)\beta(2)\gamma_5(2)]\,, \qquad (D-5)$$

with $\gamma_0 = \beta$. Thus Eq.(4.2.1) follows. Note here we did not include the exchange term.

The expectation value of the Hamiltonian (4.2.1) in the HF vacuum $|\Phi>$ is given by Eq.(4.2.7), ie:

$$E_v = \sum_{\mathbf{p}_1}\int d^3x_1 tr_1[(\gamma_5(1)\vec{\sigma}(1)\cdot\mathbf{p}_1)\rho_{\mathbf{p}_1}] + \frac{1}{2}\sum_{\mathbf{p}_1}\sum_{\mathbf{p}_2}\int\int d^3x_1 d^3x_2 tr_1 tr_2[V(12)\rho_{\mathbf{p}_1}\rho_{\mathbf{p}_2}]\,. \qquad (D-6)$$

Substituting in the interacting potential $V(12)$ from Eq.(D-5) and the density matrix at zero temperature from Eq.(4.2.6), and noting that the density matrix is x-independent, we obtain the vacuum energy per unit volume $\mathcal{E} = E_v / \int d^3x$:

$$\mathcal{E} = \sum_{\mathbf{p}_1} tr_1(\gamma_5(1)\vec{\sigma}(1)\cdot\mathbf{p}_1)\rho_{\mathbf{p}_1} - g\sum_{\mathbf{p}_1}\sum_{\mathbf{p}_2} tr_1 tr_2[\beta(1)\beta(2) - \beta(1)\gamma_5(1)\beta(2)\gamma_5(2)]\rho_{\mathbf{p}_1}\rho_{\mathbf{p}_2} , \qquad (D-7)$$

From the definitions of β, γ_5 and $\vec{\sigma}$, we know that

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} , \quad \vec{\sigma} = \begin{pmatrix} \vec{\sigma}_2 & 0 \\ 0 & \vec{\sigma}_2 \end{pmatrix} .$$

It is easy to check that

$$\gamma_5^2 = I , \quad \vec{\sigma}^2 = 3I , \quad \beta^2 = I , \quad (\vec{\sigma}\cdot\mathbf{p})^2 = p^2 ,$$

$$\vec{\sigma}\gamma_5 = \gamma_5\vec{\sigma} , \quad \beta\gamma_5 = -\gamma_5\beta , \quad \vec{\sigma}\beta = \beta\vec{\sigma}$$

and

$$tr\gamma_5 = tr\beta = tr\vec{\sigma} = tr\gamma_5\vec{\sigma} = tr\beta\vec{\sigma} = tr\beta\gamma_5 = tr\beta\gamma_5\vec{\sigma} = 0 , \quad trI = 4$$

where $I = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}$ and I_2 is a 2×2 unit matrix.

It is now straightforward to calculate the kinetic energy per unit volume $-2\mathcal{T}$ with

$$\begin{aligned} -2\mathcal{T}(\Lambda, M) &= \sum_{\mathbf{p}_1} tr_1\gamma_5(1)\vec{\sigma}(1)\cdot\mathbf{p}_1\rho_{\mathbf{p}_1} \\ &= \sum_{\mathbf{p}_1}\frac{p^2}{E}\theta(\Lambda^2 - p^2) . \qquad (D-8) \end{aligned}$$

The potential energy per unit volume includes the contributions from both the direct term V_{dir} and the exchange term V_{ex}, with

$$\begin{aligned} V_{dir} &= -g[\sum_{\mathbf{p}_1} tr_1(\beta(1)\rho_{\mathbf{p}_1})\sum_{\mathbf{p}_2} tr_2(\beta(2)\rho_{\mathbf{p}_2}) \\ &\quad - \sum_{\mathbf{p}_1} tr_1(\beta(1)\gamma_5(1)\rho_{\mathbf{p}_1})\sum_{\mathbf{p}_2} tr_2(\beta(2)\gamma_5(2)\rho_{\mathbf{p}_2})] \\ &= -g[-\sum_{\mathbf{p}}\frac{2M}{E}\theta(\Lambda^2 - p^2)]^2 - 0 \\ &= -4gM^2\mathcal{V}^2(\Lambda, M) , \qquad (D-9) \end{aligned}$$

and

$$
\begin{aligned}
V_{ex} &= g[\sum_{\mathbf{p}_1}\sum_{\mathbf{p}_2} tr(\beta(1)\rho_{\mathbf{p}_1}\beta(2)\rho_{\mathbf{p}_2}) \\
&- \sum_{\mathbf{p}_1}\sum_{\mathbf{p}_2} tr(\beta(1)\gamma_5(1)\rho_{\mathbf{p}_1}\beta(2)\gamma_5(2)\rho_{\mathbf{p}_2})] \\
&= g[\sum_{\mathbf{p}_1}\sum_{\mathbf{p}_2}(1+\frac{M^2}{E_1E_2}+\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{E_1E_2})\,\theta(\Lambda^2-p_1^2)\,\theta(\Lambda^2-p_2^2) \\
&- \sum_{\mathbf{p}_1}\sum_{\mathbf{p}_2}(-1+\frac{M^2}{E_1E_2}+\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{E_1E_2})\,\theta(\Lambda^2-p_1^2)\,\theta(\Lambda^2-p_2^2)] \\
&= g[\Omega^2(\Lambda)+m^2\mathcal{V}^2(\Lambda,M)-(-\Omega^2(\Lambda)+m^2\mathcal{V}^2(\Lambda,M))] \\
&= 2g\Omega^2(\Lambda)\,, \qquad (D-10)
\end{aligned}
$$

where terms with $\mathbf{p}_1 \cdot \mathbf{p}_2$ give zero contributions. The exchange term is M-independent and, hence, does not play any role in determining the effective potential M. The energy density is obtained in a short form as given by Eq.(4.2.8):

$$
\mathcal{E}(\Lambda, M) = -2\mathcal{T}(\Lambda, M) - 4gM^2\mathcal{V}^2(\Lambda, M) + 2g\Omega^2(\Lambda)\,, \qquad (D-11)
$$

The equilibrium condition requires that

$$
\frac{\partial \mathcal{E}}{\partial M} = -2\frac{\partial \mathcal{T}}{\partial M} - 8gM\mathcal{V}^2 - 8gM^2\mathcal{V}\frac{\partial \mathcal{V}}{\partial M} = 0\,. \qquad (D-12)
$$

Because

$$
\frac{\partial \mathcal{T}}{\partial M} = \frac{\partial}{\partial M}\sum_{\mathbf{p}}\frac{p^2}{E} = \sum_{\mathbf{p}}\frac{-Mp^2}{E^3}\theta(\Lambda^2-p^2)\,, \qquad (D-13)
$$

and

$$
\frac{\partial \mathcal{V}}{\partial M} = \frac{\partial}{\partial M}\sum_{\mathbf{p}}\frac{1}{E} = \sum_{\mathbf{p}}\frac{-M}{E^3}\theta(\Lambda^2-p^2)\,, \qquad (D-14)
$$

we obtain the condition

$$
-2M(1-4g\mathcal{V})\sum_{\mathbf{p}}\frac{p^2}{E^3} = 0\,. \qquad (D-15)
$$

Since $\sum_{\mathbf{p}} \frac{p^2}{E^3} \neq 0$, the gap equation (4.2.10) is obtained

$$M = 4gM\mathcal{V} . \qquad (D-16)$$

We may now calculate the vacuum condensation energy per unit volume using the gap equation (D-16). Since the minimum energy per unit volume may be expressed as

$$\begin{aligned}
\mathcal{E}_{min} &= -2\mathcal{T} - 4gM^2\mathcal{V}^2 + 2g\Omega^2 \\
&= -2\sum_{\mathbf{p}} \frac{p^2}{E}\theta(\Lambda^2 - p^2) - \frac{M^2}{4g} + 2g\Omega^2 \\
&= -2\sum_{\mathbf{p}} \frac{E^2 - M^2}{E}\theta(\Lambda^2 - p^2) - \frac{M^2}{4g} + 2g\Omega^2 \\
&= -2\sum_{\mathbf{p}} E\theta(\Lambda^2 - p^2) + \frac{M^2}{4g} + 2g\Omega^2 , \qquad (D-17)
\end{aligned}$$

which is denoted by $\mathcal{E}_{NG}$ in the Nambu-Goldstone phase (with $M \neq 0$) and by $\mathcal{E}_{WW}$ in the Wigner-Weyl phase (with $M = 0$) respectively. The vacuum condensation energy density (denoted by $\tilde{B}$) is therefore given by (Eq.(4.2.14))

$$\begin{aligned}
\tilde{B} &= \mathcal{E}_{WW} - \mathcal{E}_{NG} \\
&= 2\sum_{\mathbf{p}}(E - |\mathbf{p}|)\,\theta(\Lambda^2 - p^2) - \frac{M^2}{4g} . \qquad (D-18)
\end{aligned}$$

This form is exactly the same as that in the BCS superconducting case.

Finite temperature calculation: The Helmhotz free energy density is defined as

$$\mathcal{F} = \mathcal{E} - TS = \mathcal{E} - \frac{1}{\beta}S , \qquad (D-19)$$

where the internal energy density $\mathcal{E}$ is given by Eq.(4.3.4) and the entropy density given by Eq.(4.3.7). To obtain the gap equation at finite temperature, the Helmhotz free energy has to be minimized, with

$$\frac{\partial\mathcal{F}}{\partial M} = \frac{\partial\mathcal{E}}{\partial M} - \frac{1}{\beta}\frac{\partial S}{\partial M} = 0 . \qquad (D-20)$$

From Eqs.(4.3.4) and (4.3.5), it follows that

$$\frac{\partial \mathcal{E}}{\partial M} = -2\frac{\partial \mathcal{T}}{\partial M} - 8gM^2\mathcal{V}\frac{\partial \mathcal{V}}{\partial M} - 8gM\mathcal{V}^2\,, \qquad (D-21)$$

where

$$\frac{\partial \mathcal{T}}{\partial M} = \sum_{\mathbf{p}} \frac{-p^2 M}{E^3}(n^- - n^+) + \sum_{\mathbf{p}} \frac{p^2}{E}(\frac{\partial n^-}{\partial M} - \frac{\partial n^+}{\partial M})\,, \qquad (D-22)$$

and

$$\frac{\partial \mathcal{V}}{\partial M} = \sum_{\mathbf{p}} \frac{-M}{E^3}(n^- - n^+) + \sum_{\mathbf{p}} \frac{1}{E}(\frac{\partial n^-}{\partial M} - \frac{\partial n^+}{\partial M})\,. \qquad (D-23)$$

From Eq.(4.3.7), we have

$$\begin{aligned}\frac{\partial S}{\partial M} &= -2\frac{\partial}{\partial M}\sum_{\mathbf{p},i}[n^i \ln n^i + (1-n^i)\ln(1-n^i)] \\ &= -2\sum_{\mathbf{p},i}\frac{\partial n^i}{\partial M}\,[\ln n^i - \ln(1-n^i)] \\ &= 2\sum_{\mathbf{p},i}\frac{\partial n^i}{\partial M}\,\ln(\frac{1}{n^i}-1)\,, \end{aligned} \qquad (D-24)$$

where $i = +,-$ and $\ln(\frac{1}{n^{\pm}} - 1) = \ln e^{\pm\beta(E\mp\mu)} = \pm\beta(E \mp \mu)$. Note that $\frac{\partial \Omega}{\partial M} = \sum_{\mathbf{p},i}\frac{\partial n^i}{\partial M} = 0$ due to the conservation of particle number. The expression $\frac{1}{\beta}\frac{\partial S}{M}$ can be simplified as

$$\begin{aligned}\frac{1}{\beta}\frac{\partial S}{\partial M} &= 2\sum_{\mathbf{p}}[(E-\mu)\frac{\partial n^+}{\partial M} - (E+\mu)\frac{\partial n^-}{\partial M}] \\ &= 2\sum_{\mathbf{p}}[E(\frac{\partial n^+}{\partial M} - \frac{\partial n^-}{\partial M}) - -\mu(\frac{\partial n^+}{\partial M} - \frac{\partial n^+}{\partial M})] \\ &= 2\sum_{\mathbf{p}} E(\frac{\partial n^+}{\partial M} - \frac{\partial n^-}{\partial M})\,. \end{aligned} \qquad (D-25)$$

The minimization condition for the free energy per unit volume may then be obtained from the calculation

$$
\begin{aligned}
&\frac{\partial \mathcal{E}}{\partial M} - \frac{1}{\beta}\frac{\partial S}{\partial M} \\
=&2[\sum_{\mathbf{p}} \frac{p^2 M}{E^3}(n^- - n^+) + 4M^2 \mathcal{V} g \sum_{\mathbf{p}} \frac{M}{E^3}(n^- - n^+) \\
&- 4M\mathcal{V}g \sum_{\mathbf{p}} \frac{1}{E}(n^- - n^+)] + 2[-\sum_{\mathbf{p}} \frac{p^2}{E}(\frac{\partial n^-}{\partial M} - \frac{\partial n^+}{\partial M}) \\
&- 4M^2 \mathcal{V} g \sum_{\mathbf{p}} \frac{1}{E}(\frac{\partial n^-}{\partial M} - \frac{\partial n^+}{\partial M})] - 2\sum_{\mathbf{p}}(\frac{\partial n^+}{\partial M} - \frac{\partial n^-}{\partial M}) \\
=&2M(1-4g\mathcal{V})[\sum_{\mathbf{p}} \frac{p^2}{E^3}(n^- - n^+) + 2M^2(1-4\mathcal{V}g)\sum_{\mathbf{p}} \frac{1}{E}(\frac{\partial n^-}{\partial M} - \frac{\partial n^+}{\partial M})] \\
=&2M(1-4\mathcal{V}g)\frac{\partial (M\mathcal{V})}{\partial M} = 0 \,. \qquad (D-26)
\end{aligned}
$$

Since $\frac{\partial(M\mathcal{V})}{\partial M} \neq 0$, we obtain formally the same gap equation as that in the zero temperature case:

$$
\begin{aligned}
M &= M4g\mathcal{V} \\
&= 4gM \sum_{\mathbf{p}} \frac{1}{E}(n^- - n^+) \,. \qquad (D-27)
\end{aligned}
$$

This gives Eq.(4.3.10) in the text. In the finite temperature calculation, the chemical potential μ is determined by particle conservation condition

$$
\begin{aligned}
\Omega(\Lambda) &= \sum_{\mathbf{p}}(n^- + n^+) \\
&= \sum_{\mathbf{p}}[\frac{1}{1+e^{-\beta(E+\mu)}} + \frac{1}{1+e^{\beta(E-\mu)}}] \qquad (D-28)
\end{aligned}
$$

at each temperature and, therefore, is a function of T, i.e., $\mu = \mu(T)$. Here n^+ and n^- are given by Eqs. (4.3.1) and (4.3.2) in the text.

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